The competitiveness of randomized algorithms for the Canadian Traveller Problem via linear inequalities

Pierre Bergé[a], Julien Hemery[b], Arpad Rimmel[b], Joanna Tomasik[b,*]

[a]*LRI, Université Paris-Sud, Université Paris-Saclay, 91405 Orsay Cedex, France*
[b]*LRI, CentraleSupélec, Université Paris-Saclay, 91405 Orsay Cedex, France*

Abstract

The Canadian Traveller Problem is a PSPACE-complete optimization problem where a traveller traverses an undirected weighted graph G from source s to target t where some edges E_* are blocked. The traveller does not know which edges are blocked at the beginning. He discovers them when arriving to one of their endpoints. The objective is to minimize the distance traversed to reach t.

Westphal proved that no randomized strategy has a competitive ratio smaller than $|E_*|+1$. We show, using linear algebra techniques, that this bound cannot be attained, especially on a specific class of graphs: apex trees. Indeed, no randomized strategy can be $(|E_*|+1)$-competitive, even on apex trees with only three simple (s,t)-paths.

Keywords: online algorithms, competitive analysis, linear inequality systems

1. Introduction

The *Canadian Traveller Problem* (CTP) generalizes the *Shortest Path Problem* [7]. Given an undirected weighted graph $G = (V, E, \omega)$ and two nodes $s, t \in V$, the objective is to design a strategy which makes a traveller walk from s to t through G on the shortest path possible. However, edges in set E_*, $E_* \subset E$, are blocked. The traveller does not know which edges are blocked when starting his walk. He discovers a blocked edge, also called blockage, when arriving to one of its endpoints. We work on feasible instances: blocked edges E_* do not disconnect s and t in G, so the traveller is ensured to reach t with a finite traversed distance. The CTP has been proven PSPACE-complete [1, 7]. This result is due to uncertainty over blockages. Solutions for the CTP are online algorithms, commonly called strategies in the literature. In one of its variant, the k-*Canadian Traveller Problem* (k-CTP), $|E_*| \leq k$ for a positive integer k.

*Corresponding author
Email address: `Joanna.Tomasik@centralesupelec.fr` (Joanna Tomasik)

State-of-the-art. The competitive ratio of a strategy, which evaluates its quality, is the maximum, over all feasible instances, of the ratio of the distance traversed by the traveller following the strategy and the *optimal offline cost* [3]. The optimal offline cost designates the distance the traveller would traverse if he knew blocked edges from the beginning. It can also be seen as the cost of the shortest path in graph $G\backslash E_*$ which is graph G deprived of edges E_*.

There are two classes of strategies: deterministic and randomized. Westphal [8] proved for the k-CTP that there is no deterministic strategy which achieves a competitive ratio better than $2k + 1$. This ratio is obtained by considering a graph made of $k + 1$ simple (s, t)-paths of similar cost, node-disjoint with exception to nodes s and t, and where k of them are blocked on the edge adjacent to t. In such an instance, the traveller potentially traverses the open path last, which produces the ratio $2k + 1$. In fact, as this instance verifies $|E_*| = k$, this also proves that there is no deterministic strategy with competitive ratio $2\,|E_*|+1$ for CTP. REPOSITION strategy [8], which repeats an attempt to reach t through the shortest (s, t)-path and go back to s after the discovery of a blockage, is optimal, as its competitive ratio is exactly $2\,|E_*| + 1$.

The competitiveness of the randomized strategies is evaluated as the maximal ratio of the mean distance traversed by the traveller following the strategy and the optimal offline cost. Westphal [8] proved that no randomized algorithm can attain a ratio smaller than $k + 1$ for k-CTP. As the proof consists in taking the same instance than in the deterministic case, it implies that any randomized strategy is at best $(|E_*| + 1)$-competitive for CTP. However, the identification of an $(|E_*| + 1)$-competitive randomized strategy for CTP (and a $k+1$-competitive randomized strategy for k-CTP) has not been achieved yet. The randomized strategies proposed in the literature [2, 4] are dedicated to two particular cases, so no conclusion on the competitiveness of randomized strategies in general can be raised. The following table summarizes the state-of-the-art of CTP and k-CTP.

	Deterministic strategies	
	CTP	**k-CTP**
Result	REPOSITION strategy is optimal and $(2(\|E_*\|) + 1)$-competitive	REPOSITION strategy is optimal and $(2k + 1)$-competitive
	Randomized strategies	
	CTP	**k-CTP**
Result	Any randomized strategy A is c_A-competitive with $c_A \geq \|E_*\| + 1$	Any randomized strategy A is c_A-competitive with $c_A \geq k + 1$
Open Question	Can we find a strategy which is $(\|E_*\| + 1)$-competitive ?	Can we find a strategy which is $(k + 1)$-competitive ?

Table 1: State-of-the-art and open questions for the CTP and k-CTP in any graph [2, 8]

Contributions and paper organization. In Section 2, we introduce the notation and give the definition of CTP and the competitive ratio. In Section 3, we put in place ε-apex trees which are a subfamily of apex trees, already evoked in [4]. We explain why the optimal randomized strategy over these instances consist in a randomized variant of REPOSITION strategy. This restrains the study of randomized strategies over ε-apex trees to randomized REPOSITION strategy, where the distribution used to select paths needs, however, to be clarified.

In Section 4, we determine the competitiveness of randomized strategies over a set $\mathcal{R}$ of five road maps (G_α, E_*) containing an ε-apex tree G_α we designed. Farkas' lemma [5] enables to prove that, for any strategy A, its competitive ratio cannot be smaller or equal to $|E_*| + 1$ on all road maps of set $\mathcal{R}$. Finally, no randomized strategy can drop below ratio $|E_*|+1$, even on ε-apex trees with only three simple paths (as graph G_α). This gives an answer to the open question of the CTP in Table 1. In Section 5, we conclude this article by presenting what can be done for future research.

2. Preliminaries

A traveller traverses an undirected weighted graph $G = (V, E, \omega)$, $n = |V|$ and $m = |E|$. He starts his walk at source $s \in V$. His objective is to reach target $t \in V$ with a minimum cost (also called distance), which is the sum of the weights of edges traversed. Set E_* contains blocked edges, which means that when the traveller reaches an endpoint of one of these edges, he discovers that he cannot pass through it. A pair (G, E_*) is called a *road map*. From now on, we suppose that any road map (G, E_*) is feasible, *i.e.* s and t are connected in graph $G \backslash E_*$. We study the competitiveness of randomized strategies for the CTP, which means that $|E_*|$ is unbounded.

We remind the definition of the competitive ratio introduced in [3]. Let $\omega_A(G, E_*)$ be the distance traversed by the traveller guided by a given strategy A on graph G from source s to target t with blocked edges E_*. The shortest (s,t)-path in $G \backslash E_*$ is called the *optimal offline path* of map (G, E_*) and its cost, noted $\omega_{\min}(G, E_*)$, is the optimal offline cost of map (G, E_*). Strategy A is c_A-competitive if, for any road map (G, E_*):

$$\omega_A(G, E_*) \leq c_A \omega_{\min}(G, E_*) + \eta,$$

where η is constant. For randomized strategies, it becomes:

$$\mathbb{E}[\omega_A(G, E_*)] \leq c_A \omega_{\min}(G, E_*) + \eta.$$

Eventually, we recall the description of REPOSITION, as we work on its randomized variants in the remainder. REPOSITION is the optimal deterministic strategy from the competitive analysis point of view because its competitive ratio is $2|E_*| + 1$. Starting at source s, the traveller computes the shortest (s,t)-path in G and traverses it. If he is blocked on this path, he returns to s and restarts the process over graph G deprived of discovered blocked edges as many times as necessary until reaching t. REPOSITION can be randomized by the use of a certain probability distribution to select an (s,t)-path to be traversed. Bender *et al.* [2] built a randomized REPOSITION which is $(k+1)$-competitive over graphs made up uniquely of node-disjoint (s,t)-paths.

3. Randomized strategies for apex trees

Recent works studied the competitiveness of randomized strategies over apex trees [**?** 4]. An apex tree is a graph composed of a tree rooted in t and node-disjoint paths that connect s to nodes of the tree. As optimal strategies have been established for graphs with exclusively node-disjoint (s,t)-paths [2, 8], the question of the competitiveness of randomized strategies over apex trees, which represent a more general family of graphs, is of interest. Demaine *et al.* proved that for apex trees in which all simple (s,t)-paths have the same cost, there is a $(|E_*|+1)$-competitive randomized strategy [4]. The open question is whether a $(|E_*|+1)$-competitive randomized strategy for apex trees with arbitrary costs exists. A weaker but also significant result consists in finding a randomized strategy with competitive ratio $k+1$ for the k-CTP targeting apex trees.

We specify a subfamily of apex trees called ε-apex trees (ε-ATs). An ε-AT is composed of a tree rooted in t whose all edges are of weight ε. Leaves of this tree are connected to s with node-disjoint paths of arbitrary cost (Figure 1). We suppose that the traveller traverses an ε-AT G with blocked edges in E_* belonging to the tree rooted in t (their weight is ε). Let $\mathcal{P}$ be the set of simple (s,t)-paths of G. There is a bijective relation between paths in $\mathcal{P}$ and the leaves of the tree: for any leaf of the tree, there is exactly one simple (s,t)-path passing through it. We call the *memory* of the traveller the ordered set:

$$\mathcal{M} = \{(e_a^*, Q_a), (e_b^*, Q_b), \ldots, (e_z^*, Q_z)\},$$

which indicates the blocked edges that the traveller discovered successively (e_a^* then e_b^*, etc.) and the simple (s,t)-path he was traversing at each revelation (he was traversing Q_a when he discovered blockage e_a^*). The most competitive way to traverse an ε-AT is to follow the randomized REPOSITION strategy with the adequate discrete random variable X which, given the memory $\mathcal{M}$ of the traveller, assigns to remaining open paths in $\mathcal{P}$ of the graph a probability to be chosen. In short:

1. Draw an open (s,t)-path $Q \in \mathcal{P}$ according to random variable X.
2. If the traveller discovers at node v of Q a blocked edge e^*, append pair (e^*, Q) to memory $\mathcal{M}$, go back to s on the shortest (v,s)-path and restart the process, otherwise terminate.

Indeed, the traveller has no alternative: because of the structure of an ε-AT, each time the traveller meets a blockage, the only way for him to reach t with minimum distance is to make a detour via node s. For this reason, the randomized REPOSITION strategy is the best for ε-ATs.

Consequently, the optimal randomized strategy over ε-ATs is determined by the optimality of the distribution of X. We study in the next section the

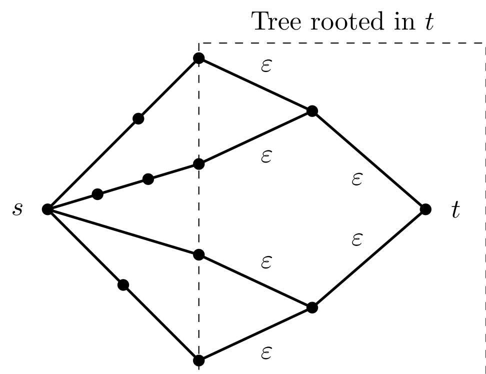


Figure 1: An example of ε-AT with four simple (s, t)-paths

competitiveness of randomized strategies over an ε-AT G_α. We prove that, even with ε-ATs of only three simple paths, no random variable X makes the randomized REPOSITION have competitive ratio $|E_*| + 1$.

4. No randomized strategy can be $(|E_*| + 1)$-competitive

We prove that any randomized strategy A is c_A-competitive with $c_A > |E_*| + 1$. We design an ε-AT G_α which depends on parameter $\sqrt{2} \leq \alpha < \frac{3}{2}$. We propose a road atlas composed of maps with graph G_α and sets of blocked edges of cardinality one or two. We build the inequality system $Bx \leq d$ such that it has a nonnegative solution iff there is a strategy which is $(|E_*| + 1)$-competitive over $\mathcal{R}$. Thanks to Farkas' lemma [5, 6], we prove that no nonnegative solution to this system exists.

Proposition 1 (Farkas' lemma, Proposition 6.4.3 in [6]). *Let $B \in \mathbb{R}^{m \times l}$ be a matrix and $d \in \mathbb{R}^m$ be a vector. The system $Bx \leq d$ has a nonnegative solution iff every nonnegative vector $y \in \mathbb{R}^m$ with $y^T B \geq \mathbf{0^T}$ also satisfies $y^T d \geq 0$.*

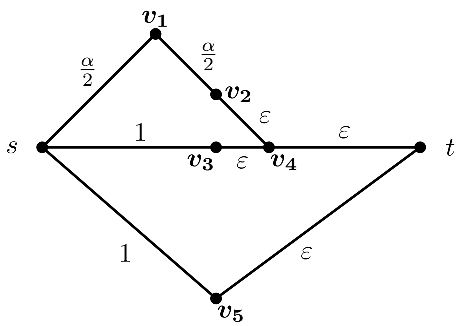
Keeping this lemma in mind, we build a system of linear inequalities such that if there is a nonnegative $y \in \mathbb{R}^m$ satisfying $y^T B \geq 0$ and $y^T d < 0$, then there is no nonnegative vector x such that $Bx \leq d$.

Theorem 2. *There is no randomized strategy with competitive ratio $|E_*| + 1$ on ε-ATs even with three simple (s, t)-paths.*

Proof. We start by introducing in Figure 2 graph G_α which is an ε-AT composed of three simple (s, t)-paths, noted Q_a, Q_b, Q_c.

We focus on a road atlas $\mathcal{R}$ made for G_α composed of five road maps, where only edges with weight $\varepsilon \ll 1$ can be blocked. First, we put two road maps into set *mcalr*, each one containing one blocked edge which is either (v_4, t) or (v_5, t):

$$\{(G_\alpha, E_*) : |E_*| = 1, E_* \subset \{((v_4, t), (v_5, t)\}\} \subset \mathcal{R}.$$



(a) Graph G_α

Path	Sequence of nodes
Q_a	$s \to v_1 \to v_2 \to v_4 \to t$
Q_b	$s \to v_3 \to v_4 \to t$
Q_c	$s \to v_5 \to t$

(b) Paths Q_a, Q_b, and Q_c

Figure 2: Graph G_α and its three simple paths Q_a, Q_b, and Q_c

Second, we put three road maps into $\mathcal{R}$, each one containing graph G_α and two blocked edges among (v_2, v_4), (v_3, v_4), and (v_5, t):

$$\{(G_\alpha, E_*) : |E_*| = 2, E_* \subset \{(v_2, v_4), (v_3, v_4), (v_5, t)\}\} \subset \mathcal{R}.$$

In the remainder of the proof, we neglect ε involved in calculations, as if ε tends to zero (weights in a CTP instance must be positive, this is why ε replaces zero). We make parameter α be in the interval $\left[\sqrt{2}, \frac{3}{2}\right[$.

Let A be a randomized REPOSITION strategy. We note p_a, p_b, and p_c the probabilities for the traveller to choose path Q_a, Q_b, and Q_c at departures with strategy A, respectively. They obviously fulfil $p_a + p_b + p_c = 1$. Let $p\left(Q_b | \left(v_2, v_4\right), Q_a\right)$ the probability to select path Q_b after discovering blocage (v_2, v_4) on path Q_a. In other words, set $\{(v_2, v_4), Q_a\}$ is the memory of the traveller. We define similarly probabilities $p\left(Q_c | \left(v_2, v_4\right), Q_a\right)$, $p\left(Q_a | \left(v_3, v_4\right), Q_b\right)$, $p\left(Q_c | \left(v_3, v_4\right), Q_b\right)$, $p\left(Q_a | \left(v_5, t\right), Q_c\right)$ and $p\left(Q_b | \left(v_5, t\right), Q_c\right)$. Note that the sum of probabilities with the same condition is equal to 1, for example,

$$p\left(Q_b | \left(v_2, v_4\right), Q_a\right) + p\left(Q_c | \left(v_2, v_4\right), Q_a\right) = 1.$$

In Table 2, we define six variables $x_{\cdot,\cdot}$ resulting from the conditional probabilities presented above, arranged in a vector $\mathbf{x}_A = (x_{a,b}\ x_{a,c}\ x_{b,a}\ x_{b,c}\ x_{c,a}\ x_{c,b})^T$.

Variable	Definition
$x_{a,b}$	$p\left(Q_b | \left(v_2, v_4\right), Q_a\right) p_a$
$x_{a,c}$	$p\left(Q_c | \left(v_2, v_4\right), Q_a\right) p_a$
$x_{b,a}$	$p\left(Q_a | \left(v_3, v_4\right), Q_b\right) p_b$
$x_{b,c}$	$p\left(Q_c | \left(v_3, v_4\right), Q_b\right) p_b$
$x_{c,a}$	$p\left(Q_a | \left(v_5, t\right), Q_c\right) p_c$
$x_{c,b}$	$p\left(Q_b | \left(v_5, t\right), Q_c\right) p_c$

Table 2: Definition of variables in $\mathbf{x}_A$

We suppose that the competitive ratio of strategy A is $|E_*|+1$ and produce the consequence of this assumption for each road map from $\mathcal{R}$. For road map $(G_\alpha, \{(v_2,v_4),(v_3,v_4)\})$, the optimal offline path is Q_a with cost α. If the traveller chooses Q_a (he does this with the probability $x_{a,b}+x_{a,c}$), he reaches t without discovering any blockage so the competitive ratio is 1. If he first chooses path Q_b or Q_c and then Q_a (probability $x_{b,a}+x_{c,a}$), the competitive ratio is $\frac{2+\alpha}{\alpha}$. If the traveller traverses path Q_a after trying both Q_b, Q_c (probability $x_{b,c}+x_{c,b}$), the competitive ratio is $\frac{4+\alpha}{\alpha}$. Vector $\mathbf{x}_A$ thus fulfils:

$$(x_{a,b}+x_{a,c}) + (x_{b,a}+x_{c,a})\frac{2+\alpha}{\alpha} + (x_{b,c}+x_{c,b})\frac{4+\alpha}{\alpha} \leq 3.$$

Similar linear inequalities can be written for all other road maps in $\mathcal{R}$ and vector $\mathbf{x}_A$ is a solution of an inequality system $B'\mathbf{x} \leq d'$, with $\mathbf{x} \geq \mathbf{0}$:

$$\begin{pmatrix} 1 & 1 & \frac{2+\alpha}{\alpha} & \frac{4+\alpha}{\alpha} & \frac{2+\alpha}{\alpha} & \frac{4+\alpha}{\alpha} \\ 2\alpha+1 & 2\alpha+3 & 1 & 1 & 2\alpha+3 & 3 \\ 2\alpha+3 & 2\alpha+1 & 2\alpha+3 & 3 & 1 & 1 \\ \alpha+2 & \alpha+2 & 3 & 3 & 1 & 1 \\ \alpha & \alpha & 1 & 1 & 2+\alpha & 3 \end{pmatrix} \begin{pmatrix} x_{a,b} \\ x_{a,c} \\ x_{b,a} \\ x_{b,c} \\ x_{c,a} \\ x_{c,b} \end{pmatrix} \leq \begin{pmatrix} 3 \\ 3 \\ 3 \\ 2 \\ 2 \end{pmatrix}.$$

Then, we write this system of inequalities in the canonical form and eliminate one redundant variable: we take $x_{c,b} = 1 - \sum_{i,j \neq c,b} x_{i,j}$. However, we must preserve the condition $x_{c,b} \geq 0$ which is equivalent to $\sum_{i,j\neq c,b} x_{i,j} \leq 1$. This also ensures that the sum of variables in $\mathbf{x}_A$ does not exceed 1. Finally, as strategy A is $(|E_*|+1)$-competitive on road atlas $\mathcal{R}$, vector $\mathbf{x}_A^c = (x_{a,b}\ x_{a,c}\ x_{b,a}\ x_{b,c}\ x_{c,a})^T$ is a solution of the canonical system $B\mathbf{x} \leq d$, $\mathbf{x} \geq \mathbf{0}$ (with $B \in \mathbb{R}^{6\times 5}$, $d \in \mathbb{R}^6$):

$$\begin{pmatrix} -\frac{4}{\alpha} & -\frac{4}{\alpha} & -\frac{2}{\alpha} & 0 & -\frac{2}{\alpha} \\ 2(\alpha-1) & 2\alpha & -2 & -2 & 2\alpha \\ 2(\alpha+1) & 2\alpha & 2(\alpha+1) & 2 & 0 \\ \alpha+1 & \alpha+1 & 2 & 2 & 0 \\ \alpha-3 & \alpha-3 & -2 & -2 & \alpha-1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_{a,b} \\ x_{a,c} \\ x_{b,a} \\ x_{b,c} \\ x_{c,a} \end{pmatrix} \leq \begin{pmatrix} 2-\frac{4}{\alpha} \\ 0 \\ 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

We define vector $\mathbf{y} \in \mathbb{R}^6$ such that $\mathbf{y}^T = \begin{pmatrix} \alpha(\alpha-1) & 0 & 0 & \alpha+1 & 2 & F(\alpha,\delta) \end{pmatrix}$ where $F(\alpha,\delta) = -2\alpha^2+5\alpha-3-\delta(\alpha-1)$. Polynomial $-2\alpha^2+5\alpha-3$ is positive for $1 < \alpha < \frac{3}{2}$. We set $\delta > 0$ small enough to guarantee $F(\alpha,\delta) > 0$. Therefore, vector y is nonnegative.

First, we check that $\mathbf{y}^T B \geq \mathbf{0^T}$. For this purpose, we note as $\mathbf{B}_1, \ldots, \mathbf{B}_6$ the column vectors of matrix B. We have, indeed

$$\begin{cases} \mathbf{y}^T\mathbf{B}_1 = -\frac{4}{\alpha}\alpha(\alpha-1) + (1+\alpha)^2 + 2(\alpha-3) + F(\alpha,\delta) = \alpha^2 - 2 + F(\alpha,\delta) \geq 0 \\ \mathbf{y}^T\mathbf{B}_2 = -\frac{4}{\alpha}\alpha(\alpha-1) + (1+\alpha)^2 + 2(\alpha-3) + F(\alpha,\delta) = \alpha^2 - 2 + F(\alpha,\delta) \geq 0 \\ \mathbf{y}^T\mathbf{B}_3 = -\frac{2}{\alpha}\alpha(\alpha-1) + 2\alpha(1+\alpha) - 4 + F(\alpha,\delta) = F(\alpha,\delta) \geq 0 \\ \mathbf{y}^T\mathbf{B}_4 = 2(1+\alpha) - 4 + F(\alpha,\delta) = 2(\alpha-1) + F(\alpha,\delta) \geq 0 \\ \mathbf{y}^T\mathbf{B}_5 = -\frac{2}{\alpha}\alpha(\alpha-1) + 2(\alpha-1) + F(\alpha,\delta) = F(\alpha,\delta) \geq 0 \end{cases}$$

Eventually, we obtain that $y^T d < 0$.

$$\begin{aligned}\mathbf{y}^T d &= \alpha(\alpha-1)(2-\tfrac{4}{\alpha})+1+\alpha-2+F(\alpha,\delta)\\ &= 2\alpha^2-6\alpha+4+1+\alpha-2-2\alpha^2+5\alpha-3-\delta(\alpha-1) = -\delta(\alpha-1)\end{aligned}$$

Farkas' lemma yields a contradiction: no probability vector $\mathbf{x}_A$ is a solution of our system $B\mathbf{x} \leq d$. So, no randomized strategy is $(|E_*|+1)$-competitive. ■

5. Conclusion

Apex trees, and more particularly ε-ATs represent a family of graphs for which the competitiveness of randomized strategies over it is a challenging question. We proved, by constructing a system of linear inequalities and applying Farkas' lemma, that even on a very small ε-AT G_α with three simple (s,t)-paths, there is no randomized strategy with competitive ratio $|E_*|+1$. More generally, this leads to the conclusion that no randomized strategy has a competitive ratio $|E_*|+1$, which is the lower bound established by Westphal.

Even if we know now that no randomized strategy can be $(|E_*|+1)$-competitive, the open question for the parameterized variant k-CTP in Table 1 remains unanswered. Our technique seem appropriate to determine whether there exists (or there does not exist) a $(k+1)$-competitive strategy over ε-ATs. Future research could also focus on the evolution of the optimal competitive ratio, as function of $|E_*|$, on ε-ATs. Identifying a new lower bound of the competitive ratio, larger than $|E_*|+1$, would be a significant breakthrough.

References

[1] A. Bar-Noy and B. Schieber. The Canadian Traveller Problem. In *Proc. of ACM/SIAM SODA*, pages 261–270, 1991.

[2] M. Bender and St. Westphal. An optimal randomized online algorithm for the k-Canadian Traveller Problem on node-disjoint paths. *J. Comb. Optim.*, 30(1):87–96, 2015.

[3] A. Borodin and R. El-Yaniv. *Online computation and competitive analysis.* Cambridge University Press, 1998.

[4] E. D. Demaine, Y. Huang, C.-S. Liao, and K. Sadakane. Canadians Should Travel Randomly. *Proc. of ICALP*, pages 380–391, 2014.

[5] J. Farkas. Uber die theorie der einfachen ungleichungen. *J. fur die Reine und Angewandte Math.*, 124:1–27, 1902.

[6] J. Matousek and B. Gärtner. *Understanding and Using Linear Programming.* Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2006.

[7] Ch. Papadimitriou and M. Yannakakis. Shortest paths without a map. *Theor. Comput. Sci.*, 84(1):127–150, 1991.

[8] St. Westphal. A note on the k-Canadian Traveller Problem. *Inform. Proces. Lett.*, 106(3):87–89, 2008.

[9] Y. Xu, M. Hu, B. Su, B. Zhu, and Z. Zhu. The Canadian traveller problem and its competitive analysis. *J. Comb. Optim.*, 18(2):195–205, 2009.