The Rank-Width of Edge-Coloured Graphs

Mamadou Moustapha Kanté [a,1] and Michael Rao [b,1]

[a] *Clermont-Université, Université Blaise Pascal, LIMOS, CNRS Complexe Scientifique des Cézeaux 63173 Aubiére Cedex, France*
`mamadou.kante@isima.fr`

[b] *Université Bordeaux 1, LaBRI, CNRS 351 Cours de la libération 33405 Talence Cedex, France.*
`rao@labri.fr`.

Abstract

Clique-width is a complexity measure for graphs (directed or not, with edge-colours or not). *Rank-width* is an equivalent complexity measure for undirected graphs and has good algorithmic and structural properties. It is in particular related to the *vertex-minor* relation. We discuss some possible extensions of the notion of rank-width to directed graphs (with edge-colours or not). A C-coloured graph is a directed graph where the arcs are coloured with colours from the set C. There is not a unique natural notion of rank-width for C-coloured graphs. We define two notions of rank-width for them, both based on a coding of C-coloured graphs by $\mathbb{F}^*$-graphs - edge-coloured graphs where each edge has exactly one colour from a field $\mathbb{F}$ - and named respectively $\mathbb{F}$*-rank-width* and $\mathbb{F}$*-bi-rank-width*. The two notions are equivalent to clique-width. We then present a notion of *vertex-minor* for $\mathbb{F}^*$-graphs and prove that $\mathbb{F}^*$-graphs of bounded $\mathbb{F}$-rank-width are characterised by a list of $\mathbb{F}^*$-graphs to exclude as vertex-minors (this list is finite if $\mathbb{F}$ is finite). A cubic-time algorithm to decide whether an $\mathbb{F}^*$-graph has $\mathbb{F}$-rank-width (resp. $\mathbb{F}$-bi-rank-width) at most k, for fixed k and fixed finite field $\mathbb{F}$, is also given. Graph operations to check MSOL-definable properties on $\mathbb{F}^*$-graphs of bounded $\mathbb{F}$-rank-width (resp. $\mathbb{F}$-bi-rank-width) are presented. A specialisation of all these notions to (directed) graphs without edge colours is presented, which shows that our results generalise the ones in undirected graphs.

Key words: rank-width; clique-width; local complementation; vertex-minor; excluded configuration; 2-structure; sigma-symmetry.

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1 Introduction

In the last three decades have appeared graph complexity measures and associated graph decompositions which have proved their importance in structural as well as algorithmic graph theory. The most known and probably useful ones are *tree-width* [60] - introduced by Robertson and Seymour in their Graph Minors Project [65] which ends up to the proof of the Graph Minor Theorem [64] - and *clique-width* introduced by Courcelle et al. [11,19].

Clique-width is more general than tree-width because every class of graphs of bounded tree-width has bounded clique-width, but the converse is false [19]. In fact many dense graph classes have unbounded tree-width, but bounded clique-width. For instance, distance hereditary graphs have clique-width at most 3. Clique-width is defined in terms of algebraic graph operations - the so-called *VR graph algebra* [11] - that generalise the concatenation of words. These operations allow to construct graphs from basic ones in a simple way, by means of what is called a *clique-width expression*. The clique-width of a graph is roughly the minimum number of basic graphs needed to construct it. Thanks to its definition, it is possible to get uniform constructions for solving in linear time every problem expressible in *monadic second-order logic* (MSOL for short) - provided the clique-width expression of the graph is given - in graph classes of small clique-width [18]. This result is important in complexity theory because many NP-complete problems are expressible in MSOL, *e.g.*, 3-colourability.

But, not all graph classes have bounded clique-width, *e.g.*, planar graphs, chordal graphs (see for instance [54,46,45] for some other examples). Hence, it is important to identify classes of graphs that have bounded clique-width. There exist several possible ways to do that:

(1) Design a polynomial time algorithm that computes the clique-width of a graph and an optimal clique-width expression. But, as expected, it is NP-complete, given (G, k), to check if G has clique-width at most k [24].
(2) What about if k is fixed? It is still open whether there exists a polynomial time algorithm that checks if a given graph has clique-width at most k and constructs an optimal clique-width expression, for fixed $k \geq 4$ (for $k \leq 3$, see the algorithm by Corneil et al. [9]). Recall that this latter problem is linear time for tree-width [2].
(3) It is well-known that graphs of tree-width at most k are characterised by a finite list of graphs to exclude as *minors* [61] and hence, combined with [63], we get the existence of a quadratic algorithm for checking if a graph has tree-width at most k, for fixed k. So, a similar result for clique-width would be welcomed even if such an algorithm would probably not help for constructing an optimal clique-width expression. Unfortunately, no such

result is known for clique-width. Indeed, clique-width is only known to be monotone with respect to the *induced subgraph* relation and this later inclusion is not a *well-quasi-order* on graph classes of bounded clique-width (cycles have clique-width at most 4 and are not well-quasi-ordered by the induced subgraph relation).

Unlike tree-width, clique-width does not have an associated combinatorial decomposition. And, it seems that the combinatorial decomposition associated to tree-width, *tree-decomposition*, make easier the characterisation and the recognition of graphs of tree-width at most k, for fixed k. In fact this combinatorial definition has yielded many structural properties in graph classes of bounded tree-width (see [65]) and was of great help in the proof of the Graph Minor Theorem. Hence, it is relevant to ask for an equivalent combinatorial definition of clique-width.

Oum and Seymour [57] investigated the quest for a polynomial time recognition algorithm for graphs of clique-width at most k, for fixed k. They introduced in this quest the notion of *rank-width* and its associated combinatorial decomposition *rank-decomposition*. They prove that rank-width and clique-width of undirected graphs are equivalent in the sense that a class of undirected graphs has bounded rank-width if and only if it has bounded clique-width. Even if rank-width is not exactly equal to clique-width, it turns out that it shares with clique-width and tree-width their positive properties without their drawbacks. More precisely, rank-width is more general than tree-width because it is equivalent to clique-width, which is more general than tree-width. Of course, deciding if an undirected graph has rank-width at most k when k is part of the input is NP-complete [42]; but, for fixed k there exists a cubic-time algorithm that decides whether the rank-width of an undirected graph is at most k and if so, constructs a rank-decomposition of width at most k [41]. Another advantage of rank-width over clique-width is that it is monotone with respect to the *vertex-minor* relation (recall no such notion, except for induced subgraph relation, is known for clique-width), *i.e.*, if H is a vertex-minor of G, then the rank-width of H is at most the rank-width of G [56].

Furthermore, rank-width is related to the *branch-width* of matroids [34], a complexity measure for matroids. The branch-width of matroids not only generalises the tree-width of graphs in the sense that a graph has bounded tree-width if and only if its associated *cycle matroid* has bounded branch-width, but also shares many structural as well as algorithmic properties with tree-width (see for instance the works by Hliněný et al. [39,40,67] for the algorithmic side and the works by Geelen et al. [34,35,36] for the structural side). For instance, Geelen and his coauthors work on a project aiming at extending techniques of the Graph Minors Project to matroids in order to solve some old and difficult conjectures in matroid theory (see the survey [37]). It is worth noticing that positive answers to that conjectures would imply the Graph Minor Theorem.

It turns out that the branch-width of a binary matroid is one more than the rank-width of its *fundamental graph* [2], and a fundamental graph of a *minor* of a matroid $\mathcal{M}$ is a vertex-minor of a fundamental graph of $\mathcal{M}$ [56]. We can therefore ask to generalising known results on the branch-width of binary matroids to the rank-width of undirected graphs. For instance, the following are generalisations of results in binary matroids: every class of undirected graphs of bounded rank-width is characterised by a finite list of undirected graphs to exclude as vertex-minors [56]; the vertex-minor relation is a well-quasi-order on graph classes of bounded rank-width [59]. Moreover, these structural results combined with the notion of *isotropic systems* introduced by Bouchet [6] have lead to the proof of a weak version of Seese's conjecture [20]. The Seese's conjecture said roughly that graph classes of bounded clique-width are exactly those where MSOL is decidable.

All these structural properties of rank-width have generated interest. First, since clique-width and rank-width of undirected graphs are equivalent, there exists a uniform way to check problems expressible in monadic second-order logic in undirected graphs of small rank-width: transform a rank-decomposition into a clique-width expression [57] and use the uniform algorithms for clique-width [18]. But, since there exist graphs with rank-width k and clique-width at least $2^{\frac{k}{2}}$ [10,58], this method is not *a priori* efficient. Courcelle and Kanté [16] proposed an alternative characterisation of rank-width in terms of graph operations that allow to check MSOL properties by using directly the rank-decomposition and avoiding the translation into a clique-width expression. It is worth noticing that these uniform algorithms for checking MSOL formulas based on algebraic graph operations have hidden constants which are non elementary functions - depending on the formula and the width - that cannot be avoided because of the generality of the method [28]. Since it is well-known that some problems - including problems non expressible in MSOL - admit single exponential algorithms in the tree-width [69] and in the clique-width [38], people have investigated such algorithms for rank-width. For instance, Bui-Xuan et al. [1,4] have proposed the notion of *H-join decomposition* of graphs and a new complexity measure *boolean-width* - equivalent to rank-width - that lead them to get linear time algorithms with runtimes single exponential in the rank-width - provided the rank-decomposition is given - for problems studied in [38,69]. Independently, Hliněný et al. [32] redefine the algebraic characterisation of rank-width proposed in [16] and use Myhill-Nerode logical tools in order to get uniform constructions of algorithms whose runtimes are single exponential in the rank-width. These tools seem promising because they allow to get polynomial time algorithms for problems that were not known to be polynomial in graph classes of bounded rank-width [29,32].

[2] A *fundamental graph* associated to a matroid $\mathcal{M} = (S, \mathcal{I})$ is the bipartite graph $(B, S \setminus B, E)$ where B is a basis of $\mathcal{M}$ and there is an edge between $e \in B$ and $f \in S \setminus B$ if $(B \setminus \{e\}) \cup \{f\}$ is a basis of $\mathcal{M}$.

Due to the many applications of tree-width and clique-width of undirected graphs in various fields, it is relevant to ask similar complexity measures for directed graphs. These directed versions should agree with the corresponding widths on undirected graphs by considering undirected graphs as directed graphs with opposite edges. In the case of tree-width, a natural definition is to take the tree-width of its underlying undirected graph. However, even though this definition yields polynomial time algorithms for MSOL problems (see [14]), it does not help in understanding the structure of directed graphs. In the last years, people have investigated better notions of tree-width and have proposed several ones: *directed tree-width* [44], *D-width* [66], *DAG-width* [55], *Kelly-width* [43] to list some of them. Unfortunately, none of them give rise similar algorithmic and structural properties as the tree-width of undirected graphs [33]. On the other hand, clique-width was originally defined for directed as well as undirected graphs. Furthermore, it can be easily extended to directed graphs with edge-colours (see [14]), and still get the same algorithmic properties as the undirected version. Hence, it is natural to ask for a notion of rank-width for directed graphs (with edge-colours or not) that behaves like the rank-width of undirected graphs. We tackle this question in this paper. This is relevant for two reasons. First, in the case of undirected graphs, clique-width is more general than tree-width and the notion of rank-width shares many structural properties of tree-width, so one can argue whether a notion of tree-width for directed graphs can be really interesting, compared to a similar one for rank-width. Second and not the last, Hliněný et al. in [33] have shown that clique-width is perhaps the only algorithmically useful complexity measure for directed graphs.

As in the case of tree-width, there is not a unique natural way to define a notion of rank-width for directed graphs (with edge-colours or not). Courcelle and Oum suggested in [20] a definition of rank-width for directed graphs as follows: Courcelle [13] described a graph transformation B from (directed) graphs to undirected bipartite graphs so that $f_1(\mathrm{cwd}(B(G))) \leq \mathrm{cwd}(G) \leq f_2(\mathrm{cwd}(B(G)))$, for some functions f_1 and f_2; the rank-width of a (directed) graph is defined as the rank-width of $B(G)$. This definition can be extended to edge-coloured graphs by using a similar coding (see [14, Chapter 6]). This definition gives a cubic-time algorithm that approximates the clique-width of edge-coloured graphs. Another consequence is the proof of a weak version of the Seese's conjecture for edge-coloured graphs [20]. However, this definition suffers from the following drawback: a vertex-minor of $B(G)$ does not always correspond to a coding of an edge-coloured graph and similarly for the notion of *pivot-minor*.

We investigate in this paper a better notion of rank-width for edge-coloured graphs. We are looking for a notion that agrees with the one on undirected graphs and that allows to get similar structural and algorithmic results. The rank-width of an undirected graph is defined as the branch-width of a certain symmetric and submodular function determined by the graph. The definition of this function is based on ranks of matrices and is quite similar to the connectivity function of matroids - a symmetric and submodular function which is the basis of the definition of the branch-width of matroids. And, this leads Oum to adapt the proof techniques by Geelen et al. [34,35] in order to generalise many results on the branch-width of binary matroids to the rank-width of undirected graphs [56,58]. Inspired by the results of Oum, Kanté introduced in [47] two notions of rank-width for directed graphs, namely $GF(4)$*-rank-width* and *bi-rank-width*, the two being also based on ranks of matrices. He proved that these two notions are equivalent to clique-width and derived from the algebraic characterisation of [16] graph operations that approximates - within a factor of two - the two notions of rank-width. These notions have raised some interest, particularly in the algorithmic community. Indeed, Hliněný et al. [29,30,31,33] popularised the bi-rank-width and adapted the Myhill-Nerode type logical tools to get new polynomial time algorithms for directed graph classes of bounded bi-rank-width. So, investigating the structural aspects of $GF(4)$-rank-width and of bi-rank-width is relevant.

We generalise the notions of $GF(4)$-rank-width and of bi-rank-width to edge-coloured graphs and generalise most of the known results on the rank-width of undirected graphs to these notions (answering open questions in [47]). Our notions will be also based on ranks of matrices. For that purposes, we take an injection from the set of colours to a finite field and this allows to represent edge-coloured graphs by matrices (similar to the adjacency matrices of graphs). But, this representation is not enough to get a symmetric and submodular function like the one for undirected graphs. To overcome this difficulty, we will define the notion of σ*-symmetric* matrices, which generalises the notion of symmetric and skew-symmetric matrices. We then explain how to represent edge-coloured graphs by σ-symmetric matrices. We derive from this representation a symmetric and submodular function that will be used to define a first notion of rank-width, called $\mathbb{F}$-rank-width, that generalises the $GF(4)$-rank-width of directed graphs. We study this new complexity measure by generalising most of the known results on the rank-width of undirected graphs [16,41,56]: we define the notions of *vertex-minor* and of *pivot-minor* relations, and prove that the $\mathbb{F}$-rank-width is monotone with respect to these relations; we give characterisations by excluded configurations and by algebraic operations; finally, we propose a cubic-time recognition algorithm for fixed k.

Concerning the notion of bi-rank-width, there is not unfortunately a unique way to extend it to edge-coloured graphs. We will define in this paper a first notion that we call $\mathbb{F}$-*bi-rank-width*. In its definition, we will use the representation of edge-coloured graphs by matrices (not necessarily σ-symmetric) and will use the same idea as the one used in [47] for defining bi-rank-width. $\mathbb{F}$-bi-rank-width generalises bi-rank-width of directed graphs and is monotone with respect to the vertex-minor relation defined for $\mathbb{F}$-rank-width. But, we do not currently know of a characterisation by excluded configurations of edge-coloured graphs of bounded $\mathbb{F}$-bi-rank-width. In Section 6 we will discuss another definition of rank-width that also generalises bi-rank-width.

The paper is organised as follows. In Section 2 we give some preliminary definitions and results. We recall in particular the definitions of clique-width of graphs (directed or not, with edge-colours of not), and of rank-width of undirected graphs. The $\mathbb{F}$-rank-width of edge-coloured graphs is studied in Section 3. We will define the notion of vertex-minor and pivot-minor, and prove that edge-coloured graphs of bounded $\mathbb{F}$-rank-width are characterised by a list of edge-coloured graphs to exclude as vertex-minors (resp. pivot-minors). This set is finite if $\mathbb{F}$ is finite. A cubic-time recognition algorithm, for fixed k and fixed finite field $\mathbb{F}$ and a specialisation to directed graphs are also presented. The $\mathbb{F}$-bi-rank-width is studied in Section 4. We also specialise it to directed graphs. In Section 5 we introduce some algebraic graph operations that generalise the ones in [16]. These operations will be used to characterise exactly the two notions of rank-width. They can be seen as alternatives to clique-width operations for solving MSOL properties. We conclude by some remarks and open questions in Section 6.

This paper is related to a companion paper where the authors introduce a decomposition of edge-coloured graphs on a fixed field [50]. This decomposition plays a role similar to the *split decomposition* [21] for the rank-width of undirected graphs. Particularly we show that the rank-width of an edge-coloured graph is exactly the maximum over the rank-width over all edge-coloured prime graphs in the decomposition, and we give different characterisations of edge-coloured graphs of rank-width one.

2 Preliminaries

For two sets A and B, we let $A \setminus B$ be the set $\{x \in A \mid x \notin B\}$. The power-set of a set V is denoted by 2^V. We sometimes write x to denote the set $\{x\}$. We denote by $\mathbb{N}$ the set containing zero and the positive integers.

We denote by $+$ and $\cdot$ the binary operations of any field and by 0 and 1 the identity elements of $+$ and $\cdot$ respectively. For every prime number p and every positive integer k, we denote by $\mathbb{F}_{p^k}$ the finite field of characteristic p and of order p^k. We recall that they are the only finite fields. For a field $\mathbb{F}$, we let $\mathbb{F}^*$ be the set $\mathbb{F} \setminus \{0\}$. We refer to [52] for our field terminology.

For sets R and C, an (R, C)-*matrix* is a matrix where the rows are indexed by elements in R and columns indexed by elements in C. For an (R, C)-matrix

M, if $X \subseteq R$ and $Y \subseteq C$, we let $M[X,Y]$ be the submatrix of M where the rows and the columns are indexed by X and Y respectively. We let rk be the matrix rank-function (the field will be clear from the context). We denote by M^T the transpose of a matrix M. The order of an (R,C)-matrix is defined as $|R| \times |C|$. We often write $k \times \ell$-matrix to denote a matrix of order $k \times \ell$. For positive integers k and ℓ, we let $O_{k,\ell}$ be the zero $k \times \ell$-matrix and I_k the identity $k \times k$-matrix.

We use the standard graph terminology, see for instance [22]. A *directed graph* G is a couple (V_G, E_G) where V_G is the set of vertices and $E_G \subseteq V_G \times V_G$ is the set of edges. A directed graph G is said to be *oriented* if $(x,y) \in E_G$ implies $(y,x) \notin E_G$, and it is said *undirected* if $(x,y) \in E_G$ implies $(y,x) \in E_G$. An edge between x and y in an undirected graph is denoted by xy (equivalently yx). For a directed graph G, we denote by $G[X]$, called the subgraph [3] of G induced by $X \subseteq V_G$, the directed graph $(X, E_G \cap (X \times X))$; we let G-X be the subgraph $G[V_G \setminus X]$. The degree of a vertex x in an undirected graph G is the cardinal of the set $\{y \mid xy \in E_G\}$. Two directed graphs G and H are *isomorphic* if there exists a bijection $h : V_G \to V_H$ such that $(x,y) \in E_G$ if and only if $(h(x), h(y)) \in E_H$. We call h an *isomorphism* between G and H. All directed graphs are finite and loop-free (*i.e.*, for every $x \in V_G$, $(x,x) \notin E_G$).

A *tree* is an acyclic connected undirected graph. In order to avoid confusions in some lemmas, we will call *nodes* the vertices of trees. The nodes of degree 1 are called *leaves* and the set of leaves in a tree T is denoted by L_T. A *cubic tree* is a tree such that the degree of each node is either 1 or 3. A tree T is *rooted* if it has a distinguished node r, called the *root* of T. For convenience, we will consider a rooted tree as an oriented graph such that the underlying graph is a tree and all the nodes are reachable from the root by a directed path. For a tree T, we let $\overrightarrow{T}$ be the oriented tree obtained from T as follows: pick an edge of T incident with a leaf, subdivide it and root the new tree by considering the new node as the root. For a tree T and an edge e of T, we let T-e denote the graph $(V_T, E_T \setminus \{e\})$.

Let C be a (possibly infinite) set that we call the *colours*. A C*-coloured graph* G is a tuple (V_G, E_G, ℓ_G) where (V_G, E_G) is a directed graph and $\ell_G : E_G \to 2^C \setminus \{\emptyset\}$ is a mapping. Its associated *underlying graph* $\mathfrak{u}(G)$ is the directed graph (V_G, E_G). Two C-coloured graphs G and H are isomorphic if there is an isomorphism h between $\mathfrak{u}(G)$ and $\mathfrak{u}(H)$ such that for every $(x,y) \in E_G$, $\ell_G((x,y)) = \ell_H((h(x), h(y)))$. We call h an *isomorphism* between G and H. We let $\mathscr{G}(C)$ be the class of C-coloured graphs for a fixed colour set C. Even though we authorise infinite colour sets in the definition, most of the results in this article are valid only when the colour set is finite. Remark that an edge-uncoloured graph can be seen as an edge-coloured graph where all the

[3] $G[X]$ is oriented (or undirected) if G is oriented (or undirected).

edges have the same colour.

Remark 2.1 (2-structures and edge-coloured graphs) *A* 2-structure *[23] is a pair (D,R) where D is a finite set and R is an equivalence relation on the set $D_2 = \{(x,y) \mid x,y \in D \text{ and } x \neq y\}$. Every* 2*-structure (D,R) can be seen as a C-coloured graph $G = (D, D_2, \ell)$ where $C := \{[e]_R \mid [e]_R$ is the equivalence class of e with respect to $R\}$ and for every edge e, $\ell(e) := [e]_R$. Equivalently, every C-coloured graph G can be seen as a* 2*-structure (V_G, R) where eRe' if and only if $\ell_G(e) = \ell_G(e')$ and all the non-edges in G are equivalent with respect to R.*

A *parameter* on $\mathscr{G}(C)$ is a function $wd : \mathscr{G}(C) \to \mathbb{N}$ that is invariant under isomorphism. Two parameters on $\mathscr{G}(C)$, say wd and wd', are *equivalent* if there exist two mutually increasing integer functions f and g such that for every edge-coloured graph $G \in \mathscr{G}(C)$, $f(wd'(G)) \leq wd(G) \leq g(wd'(G))$.

We finish these preliminaries by the notions of terms and contexts (see the survey [12] or the book [14] for an introduction to universal algebra). Let $\mathcal{F}$ be a set of function symbols and $\mathcal{C}$ a set of constants. We denote by $T(\mathcal{F},\mathcal{C})$ the set of finite well-formed terms built with $\mathcal{F} \cup \mathcal{C}$. The syntactic tree of a term t in $T(\mathcal{F},\mathcal{C})$ is denoted by $Synt(t)$. Notice that $Synt(t)$ is rooted. If $\mathcal{F}$ is a set of binary function symbols and $\mathcal{C}$ a set of constants, then for every term t in $T(\mathcal{F},\mathcal{C})$, we let $\mathscr{uS}(t)$ be the tree obtained from $\mathscr{u}(Synt(t))$ as follows: let r_1 be a neighbour - in preference a non leaf node - of the root r of $\mathscr{u}(Synt(t))$; forget the orientations of the edges in $\mathscr{u}(Synt(t))$ and contract the edge rr_1 into a single node. If $\mathcal{F}$ is a set of binary and unary function symbols, then for every term t in $T(\mathcal{F},\mathcal{C})$ we associate a term $red(t)$ in $T(\{*\},\{\#\})$ as follows:

$$\begin{aligned} red(t) &= \# && \text{if } t \in \mathcal{C},\\ red(f(t)) &= red(t) && \text{if } f \text{ is unary},\\ red(f(t_1,t_2)) &= *(red(t_1), red(t_2)) && \text{if } f \text{ is binary}. \end{aligned}$$

A *context* is a term in $T(\mathcal{F}, \mathcal{C} \cup \{u\})$ having exactly one single occurrence of the variable u (a nullary symbol). We denote by $Cxt(\mathcal{F},\mathcal{C})$ the set of contexts. We denote by $\mathscr{Id}$ the particular context u. If s is a context and t a term, we let $s \bullet t$ be the term in $T(\mathcal{F},\mathcal{C})$ obtained by substituting t for u in s.

2.1 Clique-Width

We follow the definition of clique-width in [14, Chapter 2]. We will only deal with graphs without loops, even though the definition of clique-width in [14] allows loops. It does not hurt restricting our attention to loopless graphs because removing loops does not change clique-width. Let C be a finite set of colours and k a positive integer. A *C-coloured k-graph* G is a C-coloured graph (V_G, E_G, ℓ_G) equipped with a mapping $\gamma_G : V_G \to \{1, 2, \ldots, k\}$.

Disjoint union. For disjoint C-coloured k-graphs G and H, we let $G \oplus H$ be the C-coloured k-graph $K := (V_G \cup V_H, E_G \cup E_H, \ell_K, \gamma_k)$ where

$$\ell_K((x,y)) := \begin{cases} \ell_G((x,y)) & \text{if } (x,y) \in E_G, \\ \ell_H((x,y)) & \text{if } (x,y) \in E_H, \end{cases}$$

$$\gamma_K(x) := \begin{cases} \gamma_G(x) & \text{if } x \in V_G, \\ \gamma_H(x) & \text{if } x \in V_H. \end{cases}$$

Edge addition. Let $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and $c \in C$. For every C-coloured k-graph G, we let $add^c_{i,j}(G)$ be the C-coloured k-graph $K := (V_G, E_G \cup \{(x,y) \mid x, y \in V_G \text{ and } \gamma_G(x) = i, \gamma_G(y) = j\}, \ell_K, \gamma_G)$ where, for every $(x,y) \in E_K$,

$$\ell_K((x,y)) := \begin{cases} \ell_G((x,y)) & \text{if } \gamma_G(x) \neq i \text{ or } \gamma_G(y) \neq j, \\ \ell_G((x,y)) \cup \{c\} & \text{if } (x,y) \in E_G, \text{ and } \gamma_G(x) = i \text{ and } \gamma_G(y) = j, \\ \{c\} & \text{if } (x,y) \notin E_G, \text{ and } \gamma_G(x) = i \text{ and } \gamma_G(y) = j. \end{cases}$$

Port relabelling. Let $h : \{1, \ldots, k\} \to \{1, \ldots, k\}$ be a mapping. For every C-coloured k-graph G, we let $relab_h(G)$ be the C-coloured k-graph $K := (V_G, E_G, \ell_G, h \circ \gamma_G)$.

Basic graphs. For each $i \in \{1, 2, \ldots, k\}$, we let $\mathbf{i}$ be a constant denoting a C-coloured k-graph with exactly one single vertex and no edge; this unique vertex, let us call it x, is such that $\gamma_{\mathbf{i}}(x) := i$.

Definition 2.2 *We let $\mathcal{F}^C_k := \{\oplus, add^c_{i,j}, relab_h \mid i, j \in \{1, \ldots, k\},\ i \neq j,\ h$ is a mapping from $\{1, \ldots, k\}$ to $\{1, \ldots, k\}\}$. We denote by $\mathcal{C}_k$ the set $\{\mathbf{i} \mid i \in \{1, \ldots, k\}\}$. Each term t in $T(\mathcal{F}^C_k, \mathcal{C}_k)$ evaluates into a C-coloured k-graph $val(t)$. The* clique-width *of a C-coloured graph G, denoted by* $\mathrm{cwd}(G)$*, is the minimum k such that G is isomorphic to $val(t)$ for some term t in $T(\mathcal{F}^C_k, \mathcal{C}_k)$.*

If G is isomorphic to $val(t)$ for some term t in $T(\mathcal{F}^C_k, \mathcal{C}_k)$, then there exists a bijection between V_G and $V_{val(t)}$. Since each vertex in $V_{val(t)}$ is in correspondence with an occurrence of a constant symbol in t, we can assume the existence of a bijection $\mathcal{L}_t$ between V_G and $\mathrm{L}_{u(Synt(t))}$.

Clique-width was first defined for graphs (directed or not) without edge colours. And so, most of the investigations concern only graphs without edge colours (see for instance [11,18,19,24,25,53]). However, many results in the uncoloured version do hold in the edge-coloured version (see [8,14]).

2.2 Rank-Width and Vertex-Minor of Undirected Graphs

Let V be a finite set and $f : 2^V \to \mathbb{N}$ a function. We say that f is *symmetric* if for any $X \subseteq V$, $f(X) = f(V \setminus X)$; f is *submodular* if for any $X, Y \subseteq V$, $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$.

A *layout* of a finite set V is a pair $(T, \mathcal{L})$ of a cubic tree T and a bijective function $\mathcal{L} : V \to \mathrm{L}_T$. For each edge e of T, the connected components of T-e induce a bipartition $(X_e, V \setminus X_e)$ of L_T, and thus a bipartition $(X^e, V \backslash X^e) = (\mathcal{L}^{-1}(X_e), \mathcal{L}^{-1}(V \setminus X_e))$ of V (we will omit the subscript or superscript e when it is clear from the context).

Let $f : 2^V \to \mathbb{N}$ be a symmetric function and $(T, \mathcal{L})$ a layout of V. The *f-width of each edge e of T* is defined as $f(X^e)$ and the *f-width of* $(T, \mathcal{L})$ is the maximum f-width over all edges of T. The *f-width of* V is the minimum f-width over all layouts of V.

Definition 2.3 (Rank-width of undirected graphs [56,57]) *For every undirected graph G, we let M_G be its adjacency (V_G, V_G)-matrix over $\mathbb{F}_2$ where $M_G[x, y] := 1$ if and only if $xy \in E_G$. For every undirected graph G, we let* $\mathrm{cutrk}_G : 2^{V_G} \to \mathbb{N}$ *where* $\mathrm{cutrk}_G(X) := \mathrm{rk}(M_G[X, V_G \setminus X])$*, where* rk *is the matrix rank over $\mathbb{F}_2$. This function is symmetric. The* rank-width *of an undirected graph G, denoted by* $\mathrm{rwd}(G)$*, is the* cutrk_G*-width of V_G.*

Contrary to clique-width, there exists, for fixed k, a cubic-time algorithm that given an undirected graph G, either outputs a layout of V_G of cutrk_G-width at most k, or confirms that the rank-width of G is at least $k+1$ [41]. Rank-width is moreover related to the *vertex-minor* relation.

Definition 2.4 (Local complementation [5,27,56], Vertex-minor [7,56]) *For an undirected graph G and a vertex x of G, the* local complementation at *x, denoted by $G * x$, consists in replacing the subgraph induced on the neighbours of x by its complement. An undirected graph H is a* vertex-minor *of an undirected graph G if H can be obtained from G by applying a sequence of local complementations and deletions of vertices.*

Authors of [5,27,56] also introduced the *pivot* operation on an edge xy, denoted by $G \wedge xy = G * x * y * x = G * y * x * y$. An interesting theorem relating rank-width and the notion of vertex-minor is the following.

Theorem 2.5 ([56]) *For every positive integer k, there exists a finite list $\mathscr{C}_k$ of undirected graphs such that an undirected graph has rank-width at most k if and only if it does not contain as vertex-minor any graph isomorphic to a graph in $\mathscr{C}_k$.*

3 $\mathbb{F}$-Rank-Width of σ-Symmetric $\mathbb{F}^*$-Graphs

We want a notion of rank-width for edge-coloured graphs that generalises the one on undirected graphs and allows to get similar structural and algorithmic results. For that purposes, we will identify each colour by a non-zero element of a field. This representation will allow us to define the rank-width of edge-coloured graphs by using the rank of matrices.

Let $\mathbb{F}$ be a field. An $\mathbb{F}^*$*-graph* G is an $\mathbb{F}^*$-coloured graph where for every edge $(x, y) \in E_G$, we have $\ell_G((x, y)) \in \mathbb{F}^*$, *i.e.*, each edge has exactly one colour in $\mathbb{F}^*$. For every $\mathbb{F}^*$-graph G, one can extend ℓ_G to the mapping $\ell' : V_G \times V_G \to \mathbb{F}$ where for every edge $(x, y) \in E_G$, $\ell'((x, y)) := \ell_G((x, y))$ and for every $(x, y) \notin E_G$, $\ell'((x, y)) := 0$. Therefore, we will consider for all $\mathbb{F}^*$-graphs G that $\ell_G((x, y)) = 0$ for all $(x, y) \notin E_G$. We can represent every $\mathbb{F}^*$-graph G by a (V_G, V_G)-matrix M_G over $\mathbb{F}$ such that $M_G[x, y] := \ell_G((x, y))$ for every $x, y \in V_G$ with $x \neq y$, and $M_G[x, x] := 0$ for every $x \in V_G$.

Let $\sigma : \mathbb{F} \to \mathbb{F}$ be a bijection. We recall that σ is an *involution* if $\sigma(\sigma(a)) = a$ for all $a \in \mathbb{F}$. We call σ a *sesqui-morphism* if σ is an involution, and the mapping $[x \mapsto \sigma(x)/\sigma(1)]$ is an automorphism. It is worth noticing that if $\sigma : \mathbb{F} \to \mathbb{F}$ is a sesqui-morphism, then $\sigma(0) = 0$ and for every $a, b \in \mathbb{F}$, $\sigma(a+b) = \sigma(a)+\sigma(b)$ (*i.e.*, σ is an automorphism for the addition). Moreover, we have the following notable equalities.

Proposition 3.1 *If $\sigma : \mathbb{F} \to \mathbb{F}$ is a sesqui-morphism, then*

$$\begin{aligned}
\sigma(a \cdot b) &= \frac{\sigma(a) \cdot \sigma(b)}{\sigma(1)}, \\
\sigma\left(\frac{a}{b}\right) &= \frac{\sigma(1) \cdot \sigma(a)}{\sigma(b)}, \\
\sigma\left(\frac{a \cdot b}{c}\right) &= \frac{\sigma(a) \cdot \sigma(b)}{\sigma(c)}.
\end{aligned}$$

An $\mathbb{F}^*$-graph G is σ*-symmetric* if $\mathfrak{u}(G)$ is undirected, and for every edge (x, y), $\ell_G((x, y)) = a$ if and only if $\ell_G((y, x)) = \sigma(a)$. Clearly, if G is a σ-symmetric $\mathbb{F}^*$-graph, then $M_G[x, y] = \sigma(M_G[y, x])$. (Matrices verifying this condition are also said to be σ-symmetric[4].) We denote by $\mathscr{S}(\mathbb{F})$ (respectively $\mathscr{S}(\mathbb{F},\sigma)$) the class of $\mathbb{F}^*$-graphs (respectively σ-symmetric $\mathbb{F}^*$-graphs). A nice property of σ-symmetric matrices - that was known to hold in symmetric and skew symmetric matrices - is the following.

Lemma 3.2 *For every σ-symmetric (V,V)-matrix M and for every $X \subseteq V$, we have* $\mathrm{rk}(M[X, V\setminus X]) = \mathrm{rk}(M[V\setminus X, X])$.

Proof. Let $A_1 := M[X, V\setminus X]$ and let M' be the $(V\setminus X, X)$-matrix where $M'[y,x] = \sigma(A_1[x,y])/\sigma(1)$. Since σ is a sesqui-morphism, the mapping $[x \mapsto \sigma(x)/\sigma(1)]$ is an automorphism and then $\mathrm{rk}(M') = \mathrm{rk}((A_1)^T) = rk(A_1)$. But, $M[V\setminus X, X] = \sigma(1)\cdot M'$. Then, $\mathrm{rk}(M[V\setminus X, X]) = \mathrm{rk}(M') = \mathrm{rk}(M[X, V_G\setminus X])$. □

From Lemma 3.2 one can define for σ-symmetric graphs a symmetric function like the cutrk function and derive a rank-width notion for σ-symmetric graphs. Before, let us show how to represent C-coloured graphs by σ-symmetric $\mathbb{F}^*$-graphs.

Let C be a fixed finite colour set. To represent a C-coloured graph, one first takes an injection from $2^C \setminus \{\emptyset\}$ to $\mathbb{F}^*$ for a large enough field $\mathbb{F}$. Hence, any C-coloured graph is an $\mathbb{F}^*$-graph. Notice that this representation is not unique: on one hand, several incomparable fields are possible for $\mathbb{F}$, and on the other hand, the representation depends on the injection from C to $\mathbb{F}^*$. For example, oriented graphs can be represented by an $\mathbb{F}_3^*$-graph or by an $\mathbb{F}_4^*$-graph (see Section 3.4). Since not all $\mathbb{F}^*$-graphs are σ-symmetric for some sesqui-morphism σ, we need to prove that every $\mathbb{F}^*$-graph can be seen as a σ'-symmetric $\mathbb{G}^*$-graph for some field $\mathbb{G}$. If a graph G is not σ-symmetric for any sesqui-morphism σ, that means, for every sesqui-morphism σ, there exist x and y such that $M_G[x,y] \neq \sigma(M_G[y,x])$. So, we need to match every pair $(a,b) \in \mathbb{F}\times\mathbb{F}$ into some element $v(a,b) \in \mathbb{G}$ in such a way that we can define a sesqui-morphism $\sigma' : \mathbb{G} \to \mathbb{G}$ such that $\sigma'(v(a,b)) = v(b,a)$. Since, there are $|\mathbb{F}|^2$ such pairs, $\mathbb{G}$ must have order at least $|\mathbb{F}|^2$. We will prove that every $\mathbb{F}^*$-graph can be seen as a $\widetilde{\sigma}$-symmetric $(\mathbb{F}^2)^*$-graph for some sesqui-morphism $\widetilde{\sigma} : \mathbb{F}^2 \to \mathbb{F}^2$, where $\mathbb{F}^2$ is an algebraic extension of $\mathbb{F}$ of order 2. Let us first make the following observation.

Lemma 3.3 *There exists an element r in $\mathbb{F}^*$ such that the polynomial $X^2 - r(X+1)$ has no root in $\mathbb{F}$.*

Proof. There exist $|\mathbb{F}|-1$ distinct polynomials of the form $X^2 - r(X+1)$, $r \neq 0$. We first notice that 0 or -1 cannot be a root of $X^2 - r(X+1)$, for any

[4] Note that symmetric and skew-symmetric matrices are σ-symmetric.

$r \in \mathbb{F}^*$. Now, two such polynomials cannot have a common root. Assume the contrary and let α be a root of $X^2 - r(X+1)$ and of $X^2 - r'(X+1)$ with $r \neq r'$. Then $(\alpha+1) \cdot (r - r') = 0$, *i.e.*, $r = r'$ since $\alpha \neq -1$, a contradiction. Since -1 and 0 cannot be the roots of any of the polynomials, we have at most $|\mathbb{F}| - 2$ possible roots. Therefore, there exists an element r such that $X^2 - r(X+1)$ has no root in $\mathbb{F}$. $\square$

We can now construct the desired algebraic extension of the finite field $\mathbb{F}$ of characteristic p and order q. Its construction is standard (see the book [52]). Let $r \in \mathbb{F}^*$ be such that $X^2 - r(X+1)$ has no root in $\mathbb{F}$ (such an r exists by Lemma 3.3). We define $\mathbb{F}^2$ to be the field isomorphic to the field $\mathbb{F}[X]/(X^2 - r(X+1))$ (*i.e.* $\mathbb{F}^2$ is the finite field of characteristic p and order q^2). Let $\alpha := X/(X^2 - r(X+1))$. Then every element of $\mathbb{F}^2$ is a polynomial on α of the form $a_0 + a_1\alpha$ where $a_0, a_1 \in \mathbb{F}$. Moreover, α is a root of $X^2 - r(X+1)$ in $\mathbb{F}^2$. We let $\gamma := 1 - r^{-1}\alpha$ and $\tau := r^{-1}\alpha$ be in $\mathbb{F}^2$. Remark that $\alpha = r\tau$ and $1 = \gamma + \tau$.

Lemma 3.4 *We have the following equalities:*

$$\begin{aligned}
\gamma^2 &= (1 + r^{-1})\gamma + r^{-1}\tau, \\
\tau^2 &= r^{-1}\gamma + (1 + r^{-1})\tau, \\
\gamma \cdot \tau &= -(r^{-1}\gamma + r^{-1}\tau).
\end{aligned}$$

Let $\widetilde{f} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}^2$ where, for every $(a, b) \in \mathbb{F} \times \mathbb{F}$, $\widetilde{f}(a, b) := a\gamma + b\tau$. The proof of the following is straightforward.

Lemma 3.5 $\widetilde{f}$ *is a bijection.*

For the sesqui-morphism in $\mathbb{F}^2$, we let $\widetilde{\sigma} : \mathbb{F}^2 \to \mathbb{F}^2$ where $\widetilde{\sigma}(a\gamma + b\tau) := b\gamma + a\tau$. One easily verifies that $\widetilde{\sigma}(\widetilde{\sigma}(\beta)) = \beta$ for all $\beta \in \mathbb{F}^2$.

Lemma 3.6 $\widetilde{\sigma}$ *is an automorphism.*

Proof. An easy computation shows that $\widetilde{\sigma}((a\gamma + b\tau) + (c\gamma + d\tau)) = \widetilde{\sigma}(a\gamma + b\tau) + \widetilde{\sigma}(c\gamma + d\tau)$. For the product, we have.

$$\begin{aligned}
\widetilde{\sigma}((a\gamma + b\tau) \cdot (c\gamma + d\tau)) &= \widetilde{\sigma}(ac\gamma^2 + (ad + bc)\gamma\tau + bd\tau^2) \\
&= ac\widetilde{\sigma}(\gamma^2) + (ad + bc)\widetilde{\sigma}(\gamma\tau) + bd\widetilde{\sigma}(\tau^2)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\sigma}(a\gamma + b\tau) \cdot \widetilde{\sigma}(c\gamma + d\tau) &= (b\gamma + a\tau) \cdot (d\gamma + c\tau) \\
&= bd\gamma^2 + (ad + bc)\gamma\tau + ac\tau^2.
\end{aligned}$$

By Lemma 3.4, $\widetilde{\sigma}(\gamma^2) = \tau^2$, $\widetilde{\sigma}(\tau^2) = \gamma^2$ and $\widetilde{\sigma}(\gamma\tau) = \gamma\tau$. This concludes the proof of the lemma. $\square$

For every $\mathbb{F}^*$-graph G, we let $\widetilde{G}$ be the $(\mathbb{F}^2)^*$-graph $(V_G, E_G \cup \{(y,x) \mid (x,y) \in E_G\}, \ell_{\widetilde{G}})$ where, for every two distinct vertices x and y,

$$\ell_{\widetilde{G}}((x,y)) := \widetilde{f}(\ell_G((x,y)), \ell_G((y,x))).$$

By the definitions of $\widetilde{G}$ and $\widetilde{\sigma}$, and Lemmas 3.4-3.6, we get the following.

Proposition 3.7 *The mapping $[G \mapsto \widetilde{G}]$ from $\mathscr{S}(\mathbb{F})$ to $\mathscr{S}(\mathbb{F}^2, \widetilde{\sigma})$ is a bijection and for every $\mathbb{F}^*$-graph G, $\widetilde{G}$ is $\widetilde{\sigma}$-symmetric. Moreover, for two $\mathbb{F}^*$-graphs G and H, $\widetilde{G}$ and $\widetilde{H}$ are isomorphic if and only if G and H are isomorphic.*

We now summarise the representation of C-coloured graphs - C finite - as $\widetilde{\sigma}$-symmetric $(\mathbb{F}^2)^*$-graph for a large enough field $\mathbb{F}$. Let $\mathscr{C}$ be a class of C-coloured graphs and let $\Pi(\mathscr{C}) \subseteq 2^C$ be the set of subsets of C appearing as colours of edges in graphs of $\mathscr{C}$. Then,

(1) take an injection $i : \Pi(\mathscr{C}) \to \mathbb{F}^*$ for a large enough finite field $\mathbb{F}$,
(2) for every graph G in $\mathscr{C}$, let G' be the $\mathbb{F}^*$-graph obtained from G by replacing each edge colour $A \subseteq C$ by $i(A)$,
(3) take $\widetilde{G'}$ as the representation of G. By Proposition 3.7 $\widetilde{G'}$ is $\widetilde{\sigma}$-symmetric. The rank-width of G will be defined as the rank-width of $\widetilde{G'}$.

Nevertheless, these representations of C-coloured graphs are not unique and two different mappings can give two different rank-width parameters. But, the different parameters are equivalent (see Proposition 3.12).

In the case of an infinite colour set C, our representation does not always work. In fact, we can always take an injection from $\Pi(\mathscr{C})$ to an infinite field $\mathbb{F}$. However, if $\mathbb{F}$ is infinite, a mapping from $\mathscr{S}(\mathbb{F})$ to $\mathscr{S}(\mathbb{G}, \sigma)$ is not always possible with the previous construction. For example, a mapping is possible from $\mathscr{S}(\mathbb{R})$ to $\mathscr{S}(\mathbb{C}, \sigma)$ with $f(a,b) = (1+i)a + (1-i)b$ and $\sigma(a+ib) = a - ib$ (where $a, b \in \mathbb{R}$), but the construction fails for $\mathbb{F} = \mathbb{C}$ since the set of complex numbers is algebraically closed.

From the discussion above, we will focus our attention to σ-symmetric $\mathbb{F}^*$-graphs. The case of $\mathbb{F}^*$-graphs that are not σ-symmetric will be discussed in Section 4.

Along this section, we let $\mathbb{F}$ be a fixed field (of characteristic p, and of order q if $\mathbb{F}$ is finite), and we let $\sigma : \mathbb{F} \to \mathbb{F}$ be a fixed sesqui-morphism. We recall that if G is an $\mathbb{F}^*$-graph, we denote by M_G the (V_G, V_G)-matrix where:

$$M_G[x,y] := \begin{cases} \ell_G((x,y)) & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.8 (Cut-Rank Functions) *The* $\mathbb{F}$-cut-rank *function of a σ-symmetric $\mathbb{F}^*$-graph G is the function* $\mathrm{cutrk}_G^{\mathbb{F}} : 2^{V_G} \to \mathbb{N}$ *where* $\mathrm{cutrk}_G^{\mathbb{F}}(X) = \mathrm{rk}(M_G[X, V_G \setminus X])$ *for all $X \subseteq V_G$.*

Lemma 3.9 *For every σ-symmetric $\mathbb{F}^*$-graph G, the function* $\mathrm{cutrk}_G^{\mathbb{F}}$ *is symmetric and submodular.*

In order to prove the submodularity, we need the *submodular inequality* of the matrix rank-function.

Proposition 3.10 ([56, Proposition 4.1]) *Let M be an (R,C)-matrix over a field $\mathbb{F}$. Then for all $X_1, Y_1 \subseteq R$ and $X_2, Y_2 \subseteq C$,*

$$\mathrm{rk}(M[X_1,X_2]) + rk(M[Y_1,Y_2]) \geq \mathrm{rk}(M[X_1 \cup Y_1, X_2 \cap Y_2]) + \mathrm{rk}(M[X_1 \cap Y_1, X_2 \cup Y_2]).$$

Proof of Lemma 3.9. The first statement follows from Lemma 3.2. For the second statement, let X and Y be subsets of V_G. We let $A_1 = M_G[X, V_G \setminus X]$ and $A_2 = M_G[Y, V_G \setminus Y]$. We have by definition and Proposition 3.10,

$$\begin{aligned} \mathrm{cutrk}_G^{\mathbb{F}}(X) + \mathrm{cutrk}_G^{\mathbb{F}}(Y) &= \mathrm{rk}(A_1) + \mathrm{rk}(A_2) \\ &\geq \mathrm{rk}(M_G[X \cup Y, (V_G \setminus X) \cap (V_G \setminus Y)]) + \\ &\quad\ \mathrm{rk}(M_G[X \cap Y, (V_G \setminus X) \cup (V_G \setminus Y)]). \end{aligned}$$

Since $(V_G \setminus X) \cap (V_G \setminus Y) = V_G \setminus (X \cup Y)$ and $(V_G \setminus X) \cup (V_G \setminus Y) = V_G \setminus (X \cap Y)$, the second statement holds. $\square$

Definition 3.11 ($\mathbb{F}$-rank-width) *The* $\mathbb{F}$-rank-width *of a σ-symmetric $\mathbb{F}^*$-graph G, denoted by* $\mathrm{rwd}^{\mathbb{F}}(G)$, *is the* $\mathrm{cutrk}_G^{\mathbb{F}}$*-width of V_G.*

If we let σ_1 be the identity automorphism on $\mathbb{F}_2$, then undirected graphs are exactly σ_1-symmetric $\mathbb{F}_2^*$-graphs. Moreover, for every undirected graph G, the functions cutrk_G and $\mathrm{cutrk}_G^{\mathbb{F}_2}$ are equal. Hence, rank-width of undirected graphs and $\mathbb{F}_2$-rank-width coincide.

One can easily check that the $\mathbb{F}$-rank-width of a σ-symmetric $\mathbb{F}^*$-graph is the maximum of the $\mathbb{F}$-rank-width of its connected components. The next proposition compares clique-width and $\mathbb{F}$-rank-width, for finite field $\mathbb{F}$. Its proof, which is an easy adaptation of the one comparing rank-width and clique-width of undirected graphs [57, Proposition 6.3], is given in Appendix because it uses the results in Section 5.

Proposition 3.12 *Let G be a σ-symmetric $\mathbb{F}^*$-graph, with $|\mathbb{F}| = q \in \mathbb{N}$.*

(1) If t is a term in $T(\mathcal{F}_k^{\mathbb{F}}, \mathcal{C}_k)$ isomorphic to G, then $(\mathcal{u}\mathcal{S}(red(t)), \mathcal{L}_t)$ is a layout of V_G of $\mathrm{cutrk}_G^{\mathbb{F}}$-width at most k.
(2) If $(T, \mathcal{L})$ is a layout of V_G of $\mathrm{cutrk}_G^{\mathbb{F}}$-width at most k, then we can construct in time $O(q^k \cdot |V_G|^2)$ a term t in $T(\mathcal{F}_{k'}^{\mathbb{F}}, \mathcal{C}_{k'})$ with $k' \leq 2 \cdot q^k - 1$ such that G is isomorphic to $val(t)$ and $(T, \mathcal{L}) = (\mathcal{u}\mathcal{S}(red(t)), \mathcal{L}_t)$.

In other words, $\mathrm{rwd}^{\mathbb{F}}(G) \leq \mathrm{cwd}(G) \leq 2 \cdot q^{\mathrm{rwd}^{\mathbb{F}}(G)} - 1$.

3.2 Vertex-Minor and Pivot-Minor of σ-Symmetric $\mathbb{F}^*$-Graphs

Definition 3.13 (λ-local complementation) *Let λ in $\mathbb{F}^*$. Let G be an $\mathbb{F}^*$-graph and x a vertex of G. The* λ-local complementation *at x of G is the $\mathbb{F}^*$-graph $G * (x, \lambda)$ represented by the (V_G, V_G)-matrix $M_{G*(x,\lambda)}$ where:*

$$M_{G*(x,\lambda)}[z,t] := \begin{cases} M_G[z,t] + \lambda \cdot M_G[z,x] \cdot M_G[x,t] & \text{if } z \neq t, \\ 0 & \text{otherwise.} \end{cases}$$

An $\mathbb{F}^*$-graph H is locally equivalent *to an $\mathbb{F}^*$-graph G if there exist vertices $x_1, \ldots, x_p$ and $\lambda_1, \ldots, \lambda_p$ in $\mathbb{F}^*$ such that $H = (\cdots((G * (x_1, \lambda_1)) * (x_2, \lambda_2)) * \cdots * (x_p, \lambda_p))$. We call H a* vertex-minor *of G if $H = G'[X]$ for some $X \subseteq V_G$ and G' is locally equivalent to G. Moreover, H is a* proper *vertex-minor of G if $X \subsetneq V_G$.*

One can easily show that for every $\mathbb{F}^*$-graph G and every vertex x of G, the adjacency matrix of $G * (x, \lambda)$ is obtained by modifying the submatrix induced by the neighbours of x. Then for every vertex y of G, $M_G[x,y] = M_{G*(x,\lambda)}[x,y]$. Notice that when $\mathbb{F}$ is the field $\mathbb{F}_2$, this notion of 1-local complementation matches with the local complementation defined by Bouchet in [5].

In this section, we are interested in σ-symmetric graphs, thus we have to restrict ourselves to a subset of λ-local complementations which preserve the σ-symmetry. We say that λ in $\mathbb{F}^*$ is *σ-compatible* if $\sigma(\lambda) = \lambda \cdot \sigma(1)^2$.

Lemma 3.14 *Let G be a σ-symmetric $\mathbb{F}^*$-graph and let $\lambda \in \mathbb{F}^*$ be σ-compatible. Then every λ-local complementation of G is also σ-symmetric.*

Proof. Let $H := G * (x, \lambda)$ for some σ-compatible λ. It is sufficient to prove that $M_H[t, z] = \sigma(M_H[z, t])$ for any $z, t \in V_G,\ z \neq t$.

$$\begin{aligned}
M_H[t, z] &= M_G[t, z] + \lambda \cdot M_G[t, x] \cdot M_G[x, z] \\
&= \sigma(M_G[z, t]) + \lambda \cdot \sigma(M_G[x, t]) \cdot \sigma(M_G[z, x]) \\
&= \sigma(M_G[z, t]) + \lambda \cdot \sigma(1) \cdot \sigma(M_G[z, x] \cdot M_G[x, t]) \\
&= \sigma(M_G[z, t]) + \sigma(\lambda) \cdot \sigma^{-1}(1) \cdot \sigma(M_G[z, x] \cdot M_G[x, t]) \\
&= \sigma(M_G[z, t]) + \sigma(\lambda \cdot M_G[z, x] \cdot M_G[x, t]) \\
&= \sigma(M_G[z, t] + \lambda \cdot M_G[z, x] \cdot M_G[x, t]) \\
&= \sigma(M_H[z, t]). \qquad \square
\end{aligned}$$

Definition 3.15 (σ-locally-equivalent, σ-vertex-minor) *An $\mathbb{F}^*$-graph H is* σ-locally-equivalent *to a σ-symmetric $\mathbb{F}^*$-graph G if if there exist vertices $x_1, \ldots, x_p$ and $\lambda_1, \ldots, \lambda_p$ in $\mathbb{F}^*$, all σ-compatible, such that $H = (\cdots((G * (x_1, \lambda_1)) * (x_2, \lambda_2)) * \cdots * (x_p, \lambda_p))$. We call H a* σ-vertex-minor *of G if $H = G'[X]$ for some $X \subseteq V_G$ and G' is σ-locally-equivalent to G. Moreover, H is a* proper *σ-vertex-minor of G if $X \subsetneq V_G$.*

Observe that if no σ-compatible $\lambda \in \mathbb{F}^*$ exists, H is a σ-vertex-minor of G if and only if H is an induced subgraph of G.

Lemma 3.16 *Let λ be a σ-compatible element in $\mathbb{F}^*$. Let G be a σ-symmetric $\mathbb{F}^*$-graph and x a vertex of G. For every subset X of V_G,*

$$\mathrm{cutrk}^{\mathbb{F}}_{G*(x,\lambda)}(X) = \mathrm{cutrk}^{\mathbb{F}}_{G}(X).$$

Proof. We can assume that $x \in X$ since $\mathrm{cutrk}^{\mathbb{F}}_{G}$ is a symmetric function (Lemma 3.9). For each $y \in X$, the σ-local-complementation at x results in adding a multiple of the row indexed by x to the row indexed by y. Precisely, we obtain $M_{G*(x,\lambda)}[y, V_G \setminus X]$ by adding $\lambda \cdot M_G[y, x] \cdot M_G[x, V_G \setminus X]$ to $M_G[y, V_G \setminus X]$. This operation is repeated for all $y \in X$. In each case, the rank of the matrix does not change. Hence, $\mathrm{cutrk}^{\mathbb{F}}_{G*(x,\lambda)}(X) = \mathrm{cutrk}^{\mathbb{F}}_{G}(X)$. $\square$

Unfortunately, a σ-compatible λ does not always exist. For instance, if the field is $\mathbb{F}_3$ and σ is such that $\sigma(x) = -x$ (see Section 3.4), no σ-compatible λ does exist. We present now another $\mathbb{F}^*$-graph transformation which is defined for every pair $(\mathbb{F}, \sigma)$.

Definition 3.17 (pivot-complementation) *Let G be a σ-symmetric $\mathbb{F}^*$-graph, and x and y two vertices of G such that $\ell_G((x, y)) \neq 0$. The* pivot-complementation at xy *of G is the $\mathbb{F}^*$-graph $G \wedge xy$ represented by the (V_G, V_G)-matrix $M_{G\wedge xy}$ where $M_{G\wedge xy}[z, z] := 0$ for every $z \in V_G$, and for every $z, t \in$*

$V_G \setminus \{x, y\}$ *with* $z \neq t$*:*

$$M_{G\wedge xy}[z,t] := M_G[z,t] - \frac{M_G[z,x]\cdot M_G[y,t]}{M_G[y,x]} - \frac{M_G[z,y]\cdot M_G[x,t]}{M_G[x,y]},$$

$$\begin{aligned}
M_{G\wedge xy}[x,t] &:= \frac{M_G[y,t]}{M_G[y,x]} & M_{G\wedge xy}[y,t] &:= \frac{\sigma(1)\cdot M_G[x,t]}{M_G[x,y]}\\
M_{G\wedge xy}[z,x] &:= \frac{\sigma(1)\cdot M_G[z,y]}{M_G[x,y]} & M_{G\wedge xy}[z,y] &:= \frac{M_G[z,x]}{M_G[y,x]}\\
M_{G\wedge xy}[x,y] &:= -\frac{1}{M_G[y,x]} & M_{G\wedge xy}[y,x] &:= -\frac{\sigma(1)^2}{M_G[x,y]}
\end{aligned}$$

An $\mathbb{F}^*$*-graph* H *is* pivot-equivalent *to an* $\mathbb{F}^*$*-graph* G *if* H *is obtained by applying a sequence of pivot-complementations to* G*. We call* H *a* pivot-minor *of* G *if* $H = G'[X]$ *for some* $X \subseteq V_G$ *and* G' pivot-equivalent *to* G*. Moreover,* H *is a* proper *pivot-minor of* G *if* $X \subsetneq V_G$*.*

It is easy to observe that $G\wedge xy = G\wedge yx$ if $\sigma(1) = 1$. In the case of undirected graphs ($\mathbb{F} = \mathbb{F}_2$), this definition coincides with the pivot-complementation of undirected graphs [56].

Lemma 3.18 *Let* G *be a* σ*-symmetric* $\mathbb{F}^*$*-graph and let* xy *be an edge of* G*. Then* $G\wedge xy$ *is also* σ*-symmetric.*

Proof. Let $z, t \in V_G$, with $z \neq t$. If $\{z,t\}\cap\{x,y\} = \emptyset$, then

$$\begin{aligned}
M_{G\wedge xy}[t,z] &= M_G[t,z] - \frac{M_G[t,x]\cdot M_G[y,z]}{M_G[y,x]} - \frac{M_G[t,y]\cdot M_G[x,z]}{M_G[x,y]}\\
&= \sigma(M_G[z,t]) - \frac{\sigma(M_G[x,t])\cdot\sigma(M_G[z,y])}{\sigma(M_G[x,y])} - \frac{\sigma(M_G[y,t])\cdot\sigma(M_G[z,x])}{\sigma(M_G[y,x])}\\
&= \sigma(M_G[z,t]) - \sigma\left(\frac{M_G[x,t]\cdot M_G[z,y]}{M_G[x,y]}\right) - \sigma\left(\frac{M_G[y,t]\cdot M_G[z,x]}{M_G[y,x]}\right)\\
&= \sigma\left(M_G[z,t]) - \frac{M_G[x,t]\cdot M_G[z,y]}{M_G[x,y]} - \frac{M_G[y,t]\cdot M_G[z,x]}{M_G[y,x]}\right)\\
&= \sigma\left(M_{G\wedge xy}[z,t]\right).
\end{aligned}$$

If $t \neq y$, then:

$$\begin{aligned}
M_{G\wedge xy}[t,x] &= \frac{\sigma(1)\cdot M_G[t,y]}{M_G[x,y]} = \frac{\sigma(1)\cdot\sigma(M_G[y,t])}{\sigma(M_G[y,x])}\\
&= \sigma\left(\frac{M_G[y,t]}{M_G[y,x]}\right) = \sigma\left(M_{G\wedge xy}[x,t]\right).
\end{aligned}$$

Finally:

$$\begin{aligned}M_{G\wedge xy}[y,x] &= -\frac{\sigma(1)^2}{M_G[x,y]} = -\frac{\sigma(1)^2}{\sigma(M_G[y,x])}\\ &= \sigma\left(-\frac{1^2}{M_G[y,x]}\right) = \sigma\left(M_{G\wedge xy}[x,y]\right). \qquad \square\end{aligned}$$

Lemma 3.19 *Let G be a σ-symmetric $\mathbb{F}^*$-graph and xy an edge of G. For every subset X of V_G,*

$$\mathrm{cutrk}^{\mathbb{F}}_{G\wedge xy}(X) = \mathrm{cutrk}^{\mathbb{F}}_{G}(X).$$

Proof. Let $Y := V_G \setminus X$. We can assume w.l.o.g. that $x \in X$. If $y \in X$, then (with $X' := X \setminus \{x,y\}$)

$$\begin{aligned}\mathrm{rk}\left(M_{G\wedge xy}[X,Y]\right) &= \mathrm{rk}\begin{pmatrix}\frac{1}{M_G[y,x]}\cdot M_G[y,Y]\\ \frac{\sigma(1)}{M_G[x,y]}\cdot M_G[x,Y]\\ M_G[X',Y] - \frac{M_G[X',x]\cdot M_G[y,Y]}{M_G[y,x]} - \frac{M_G[X',y]\cdot M_G[x,Y]}{M_G[x,y]}\end{pmatrix}\\ &= \mathrm{rk}\begin{pmatrix}\frac{1}{M_G[y,x]}\cdot M_G[y,Y]\\ \frac{\sigma(1)}{M_G[x,y]}\cdot M_G[x,Y]\\ M_G[X',Y] - \frac{M_G[X',x]\cdot M_G[y,Y]}{M_G[y,x]}\end{pmatrix}\\ &= \mathrm{rk}\begin{pmatrix}\frac{1}{M_G[y,x]}\cdot M_G[y,Y]\\ \frac{\sigma(1)}{M_G[x,y]}\cdot M_G[x,Y]\\ M_G[X',Y]\end{pmatrix} = \mathrm{rk}\begin{pmatrix}M_G[y,Y]\\ M_G[x,Y]\\ M_G[X',Y]\end{pmatrix}\\ &= \mathrm{rk}\left(M_G[X,Y]\right).\end{aligned}$$

If $y \notin X$, then (with $X' := X \setminus \{x\}$ and $Y' := Y \setminus \{y\}$)

$$\begin{aligned}
\mathrm{rk}\left(M_{G \wedge xy}[X,Y]\right) &= \mathrm{rk}\begin{pmatrix} -\frac{1}{M_G[y,x]} & \frac{M_G[y,Y']}{M_G[y,x]} \\ \frac{M_G[X',x]}{M_G[y,x]} & M_G[X',Y'] - \frac{M_G[X',x]\cdot M_G[y,Y']}{M_G[y,x]} - \frac{M_G[X',y]\cdot M_G[x,Y']}{M_G[x,y]} \end{pmatrix} \\
&= \mathrm{rk}\begin{pmatrix} -\frac{1}{M_G[y,x]} & \frac{M_G[y,Y']}{M_G[y,x]} \\ 0 & M_G[X',Y'] - \frac{M_G[X',y]\cdot M_G[x,Y']}{M_G[x,y]} \end{pmatrix} \\
&= \mathrm{rk}\begin{pmatrix} -\frac{1}{M_G[y,x]} & 0 \\ 0 & M_G[X',Y'] - \frac{M_G[X',y]\cdot M_G[x,Y']}{M_G[x,y]} \end{pmatrix} \\
&= \mathrm{rk}\begin{pmatrix} M_G[x,y] & 0 \\ 0 & M_G[X',Y'] - \frac{M_G[X',y]\cdot M_G[x,Y']}{M_G[x,y]} \end{pmatrix} \\
&= \mathrm{rk}\begin{pmatrix} M_G[x,y] & 0 \\ M_G[X',y] & M_G[X',Y'] - \frac{M_G[X',y]\cdot M_G[x,Y']}{M_G[x,y]} \end{pmatrix} \\
&= \mathrm{rk}\begin{pmatrix} M_G[x,y] & M_G[x,Y'] \\ M_G[X',y] & M_G[X',Y'] \end{pmatrix} \\
&= \mathrm{rk}\left(M_G[X,Y]\right). \qquad \square
\end{aligned}$$

Proposition 3.20 *Let G and H be two σ-symmetric $\mathbb{F}^*$-graphs. If H is σ-locally-equivalent (resp. pivot-equivalent) to G, then the $\mathbb{F}$-rank-width of H is equal to the $\mathbb{F}$-rank-width of G. If H is a σ-vertex-minor (resp. pivot-minor) of G, then the $\mathbb{F}$-rank-width of H is at most the $\mathbb{F}$-rank-width of G.*

Proof. The first statement is obvious by Lemmas 3.16 and 3.19. Since taking sub-matrices does not increase the rank, it does not increase the $\mathbb{F}$-rank-width. So, the second statement is true. $\square$

Our goal now is to prove the following which is a generalisation of Theorem 2.5.

Theorem 3.21 *(i) For each positive integer $k \geq 1$, there is a set $\mathscr{C}_k^{(\mathbb{F},\sigma)}$ of σ-symmetric $\mathbb{F}^*$-graphs, each having at most $(6^{k+1}-1)/5$ vertices, such that a σ-symmetric $\mathbb{F}^*$-graph G has $\mathbb{F}$-rank-width at most k if and only if no σ-symmetric $\mathbb{F}^*$-graph in $\mathscr{C}_k^{(\mathbb{F},\sigma)}$ is isomorphic to a pivot-minor of G.*

(ii) Suppose that a σ-compatible $\lambda \in \mathbb{F}^*$ exists. Then for each positive integer $k \geq 1$, there is a set $\mathscr{C}'^{(\mathbb{F},\sigma)}_k$ of σ-symmetric $\mathbb{F}^*$-graphs, each having at most $(6^{k+1}-1)/5$ vertices, such that a σ-symmetric $\mathbb{F}^*$-graph G has $\mathbb{F}$-

rank-width at most k if and only if no σ-symmetric $\mathbb{F}^*$-graph in ${\mathscr{C}'}_k^{(\mathbb{F},\sigma)}$ is isomorphic to a σ-vertex-minor of G.

Note that $\mathscr{C}_k^{(\mathbb{F},\sigma)}$ and ${\mathscr{C}'}_k^{(\mathbb{F},\sigma)}$ are finite if $\mathbb{F}$ is finite. For doing so we adapt the same techniques as in [35,56]. We first prove some inequalities concerning cut-rank functions. All the notions of linear algebra are borrowed from [51].

Proposition 3.22 *Let G be a σ-symmetric $\mathbb{F}^*$-graph, λ a σ-compatible element in $\mathbb{F}^*$ and x a vertex of G. For every subset X of $V_G \setminus \{x\}$,*

$$\mathrm{cutrk}^{\mathbb{F}}_{(G*(x,\lambda))\text{-}x}(X) = \mathrm{rk}\begin{pmatrix} -1 & M_G[x, V_G \setminus (X \cup \{x\})] \\ M_G[X,x] & M_G[X, V_G \setminus (X \cup \{x\})] \end{pmatrix} - 1.$$

Proof. Let X be a subset of $V_G \setminus \{x\}$ and let $Y := V_G \setminus (X \cup \{x\})$. We let J be the matrix $(M_G[z,x] \cdot M_G[x,t])_{z\in X, t\in Y}$. Then,

$$\begin{aligned}\mathrm{cutrk}^{\mathbb{F}}_{(G*(x,\lambda))\text{-}x}(X) &= \mathrm{rk}(M_{G*(x,\lambda)}[X,Y]) \\ &= \mathrm{rk}(M_G[X,Y] + \lambda \cdot J) \\ &= \mathrm{rk}\underbrace{\begin{pmatrix} -1\cdot\lambda^{-1} & M_G[x,Y] \\ 0 & M_G[X,Y] + \lambda\cdot J \end{pmatrix}}_{A} -1.\end{aligned}$$

We now show how to transform the $(\{x\} \cup X, \{x\} \cup Y)$-matrix A by using elementary row operations in order to get the desired equality. For each $z \in X$,

$$-\lambda \cdot M_G[z,x] \cdot A[x, Y \cup \{x\}] = \begin{pmatrix} M_G[z,x] & -\lambda \cdot J[z,Y] \end{pmatrix}.$$

Hence,

$$-\lambda \cdot M_G[z,x] \cdot A[x, Y \cup \{x\}] + A[z, Y \cup \{x\}] = \begin{pmatrix} M_G[z,x] & M_G[z,Y] \end{pmatrix}.$$

Therefore, by adding $-\lambda \cdot M_G[z,x] \cdot A[x, Y \cup \{x\}]$ to each row $A[z, Y \cup \{x\}]$ of A we get the matrix $\begin{pmatrix} -1 & M_G[x,Y] \\ M_G[X,x] & M_G[X,Y] \end{pmatrix}$. This concludes the proof. □

Lemma 3.23 *Let G be a σ-symmetric $\mathbb{F}^*$-graph and x a vertex in V_G. Assume that (X_1, X_2) and (Y_1, Y_2) are partitions of $V_G \setminus \{x\}$. Then,*

$$\mathrm{cutrk}^{\mathbb{F}}_{G\text{-}x}(X_1) + \mathrm{cutrk}^{\mathbb{F}}_{(G*(x,\lambda))\text{-}x}(Y_1) \geq \mathrm{cutrk}^{\mathbb{F}}_G(X_1 \cap Y_1) + \mathrm{cutrk}^{\mathbb{F}}_G(X_2 \cap Y_2) - 1.$$

Proof. We recall that for every vertex z of G, $M_G[z,z] = 0$. Let M' be obtained from M_G by replacing $M_G[x,x]$ by -1. Hence, for every subset X of

V_G, $\mathrm{rk}(M_G[X, V_G \setminus X]) = \mathrm{rk}(M'[X, V_G \setminus X]$. We recall that $Y_2 = V_G \setminus (Y_1 \cup \{x\})$ and $X_2 = V_G \setminus (X_1 \cup \{x\})$. By definition of M',

$$M'[Y_1 \cup \{x\}, Y_2 \cup \{x\}] = \begin{pmatrix} -1 & M_G[x, Y_2] \\ M_G[Y_1, x] & M_G[Y_1, Y_2] \end{pmatrix}.$$

By Proposition 3.22,

$$\mathrm{cutrk}^{\mathbb{F}}_{G\text{-}x}(X_1) + \mathrm{cutrk}^{\mathbb{F}}_{(G*(x,\lambda))\text{-}x}(Y_1) = \mathrm{rk}(M_G[X_1, X_2]) + \mathrm{rk}(M'[Y_1 \cup \{x\}, Y_2 \cup \{x\}]) - 1.$$

Since $\mathrm{rk}(M_G[X_1, X_2]) = \mathrm{rk}(M'[X_1, X_2])$, by Proposition 3.10 we get the inequality

$$\begin{aligned} \mathrm{rk}(M_G[X_1, X_2]) + \mathrm{rk}(M'[Y_1 \cup \{x\}, Y_2 \cup \{x\}]) \geq \\ \mathrm{rk}(M'[X_1 \cap Y_1, X_2 \cup Y_2 \cup \{x\}]) + \mathrm{rk}(M'[X_1 \cup Y_1 \cup \{x\}, X_2 \cap Y_2]). \end{aligned}$$

Hence,

$$\mathrm{cutrk}^{\mathbb{F}}_{G\text{-}x}(X_1) + \mathrm{cutrk}^{\mathbb{F}}_{(G*(x,\lambda))\text{-}x}(Y_1) \geq \mathrm{cutrk}^{\mathbb{F}}_{G}(X_1 \cap Y_1) + \mathrm{cutrk}^{\mathbb{F}}_{G}(X_1 \cup Y_1 \cup \{x\}) - 1.$$

By the symmetry of $\mathrm{cutrk}^{\mathbb{F}}_{G}$, we get the desired inequality. $\square$

Proposition 3.24 *Let G be a σ-symmetric $\mathbb{F}^*$-graph and xy an edge of G. For every subset X of $V_G \setminus \{x\}$,*

$$\mathrm{cutrk}^{\mathbb{F}}_{(G \wedge xy)\text{-}x}(X) = \mathrm{rk}\begin{pmatrix} 0 & M_G[x, V_G \setminus (X \cup \{x\})] \\ M_G[X, x] & M_G[X, V_G \setminus (X \cup \{x\})] \end{pmatrix} - 1.$$

Proof. Suppose w.l.o.g. that $y \in X$ (otherwise replace X by $V_G \setminus (X \cup \{x\})$). Let $Y := V_G \setminus (X \cup \{x\})$ and $X' := X \setminus \{y\}$. Then, by elementary row and column operations, we have.

$$\begin{aligned}
\mathrm{cutrk}^{\mathbb{F}}_{(G\wedge xy)\text{-}x}(X) &= \mathrm{rk}\begin{pmatrix} \frac{\sigma(1)}{M_G[x,y]}\cdot M_G[x,Y] \\ M_G[X',Y] - \frac{M_G[X',x]\cdot M_G[y,Y]}{M_G[y,x]} - \frac{M_G[X',y]\cdot M_G[x,Y]}{M_G[x,y]} \end{pmatrix} \\
&= \mathrm{rk}\begin{pmatrix} M_G[y,Y] \\ M_G[X',Y] - \frac{M_G[X',x]\cdot M_G[y,Y]}{M_G[y,x]} \end{pmatrix} \\
&= \mathrm{rk}\begin{pmatrix} M_G[y,x] & M_G[y,Y] \\ 0 & M_G[y,Y] \\ 0 & M_G[X',Y] - \frac{M_G[X',x]\cdot M_G[y,Y]}{M_G[y,x]} \end{pmatrix} - 1 \\
&= \mathrm{rk}\begin{pmatrix} 0 & M_G[y,Y] \\ M_G[y,x] & M_G[y,Y] \\ M_G[X',x] & M_G[X',Y] \end{pmatrix} - 1 \\
&= \mathrm{rk}\begin{pmatrix} 0 & M_G[y,Y] \\ M_G[X,x] & M_G[X,Y] \end{pmatrix} - 1. \qquad \square
\end{aligned}$$

Lemma 3.25 *Let G be a σ-symmetric $\mathbb{F}^*$-graph and xy an edge in V_G. Assume that (X_1, X_2) and (Y_1, Y_2) are partitions of $V_G \setminus \{x\}$. Then*

$$\mathrm{cutrk}^{\mathbb{F}}_{G\text{-}x}(X_1) + \mathrm{cutrk}^{\mathbb{F}}_{(G\wedge xy)\text{-}x}(Y_1) \geq \mathrm{cutrk}^{\mathbb{F}}_{G}(X_1 \cap Y_1) + \mathrm{cutrk}^{\mathbb{F}}_{G}(X_2 \cap Y_2) - 1.$$

Proof. We recall that $Y_2 = V_G \setminus (Y_1 \cup \{x\})$ and $X_2 = V_G \setminus (X_1 \cup \{x\})$. By definition of M_G,

$$M_G[Y_1 \cup \{x\}, Y_2 \cup \{x\}] = \begin{pmatrix} 0 & M_G[x, Y_2] \\ M_G[Y_1, x] & M_G[Y_1, Y_2] \end{pmatrix}.$$

By Proposition 3.24,

$$\mathrm{cutrk}^{\mathbb{F}}_{G\text{-}x}(X_1) + \mathrm{cutrk}^{\mathbb{F}}_{(G\wedge xy)\text{-}x}(Y_1) = \mathrm{rk}(M_G[X_1, X_2]) + \mathrm{rk}(M_G[Y_1 \cup \{x\}, Y_2 \cup \{x\}]) - 1.$$

By Proposition 3.10 we get the inequality

$$\begin{aligned} &\mathrm{rk}(M_G[X_1, X_2]) + \mathrm{rk}(M_G[Y_1 \cup \{x\}, Y_2 \cup \{x\}]) \geq \\ &\mathrm{rk}(M_G[X_1 \cap Y_1, X_2 \cup Y_2 \cup \{x\}]) + \mathrm{rk}(M_G[X_1 \cup Y_1 \cup \{x\}, X_2 \cap Y_2]). \end{aligned}$$

Hence,

$$\mathrm{cutrk}^{\mathbb{F}}_{G\text{-}x}(X_1) + \mathrm{cutrk}^{\mathbb{F}}_{(G\wedge xy)\text{-}x}(Y_1) \geq \mathrm{cutrk}^{\mathbb{F}}_{G}(X_1 \cap Y_1) + \mathrm{cutrk}^{\mathbb{F}}_{G}(X_1 \cup Y_1 \cup \{x\}) - 1.$$

By the symmetry of $\mathrm{cutrk}^{\mathbb{F}}_G$, we get the desired inequality. $\square$

The most important ingredients for proving Theorem 3.21 are Propositions 3.22 and 3.24, and Lemmas 3.23 and 3.25. All the other ingredients are already proved in [35,56] except that they are stated for the connectivity function of matroids in [35] and for undirected graphs in [56]. Their proofs rely only on the fact that the parameter is symmetric, submodular and integer valued. We include them for completeness. We first recall some definitions [35,56].

Let V be a finite set and $f : 2^V \to \mathbb{N}$ a symmetric and submodular function. Let (A, B) be a bipartition of V. A *branching of* B is a triple $(T, r, \mathcal{L})$ where T is a cubic tree with a fixed node $r \in \mathrm{L}_T$ and such that $(T, \mathcal{L})$ is a layout of $B \cup \{x_A\}$ ($x_A \notin B$) with $\mathcal{L}(x_A) = r$. For an edge e of T and a node v of T, we let T_{ev} be the set of nodes in the component of T-e not containing v and we let $Y_{ev} := \mathcal{L}^{-1}(\mathrm{L}_{T_{ev}})$. We say that B is *k-branched* if there exists a branching $(T, r, \mathcal{L})$ such that for each edge e of T, $f(Y_{er}) \leq k$. Remark that if A and B are k-branched, then the f-width of V is at most k.

A subset A of V is called *titanic* with respect to f if for every partition (A_1, A_2, A_3) of A, there is a $i \in \{1, 2, 3\}$ such that $f(A_i) \geq f(A)$ (A_1,A_2 or A_3 may be empty).

Lemma 3.26 ([41, Lemma 3.3]) *Let V be a finite set and $f : 2^V \to \mathbb{N}$ a symmetric and submodular function. Assume that the f-width of V is at most k. Let (A, B) be a bipartition of V such that $f(A) \leq k$. If A is titanic with respect to f, then B is k-branched.*

Let $g : \mathbb{N} \to \mathbb{N}$ be a function. A σ-symmetric $\mathbb{F}^*$-graph G is called (m, g)-*connected* if for every bipartition (A, B) of V_G, $\mathrm{cutrk}^{\mathbb{F}}_G(A) = \ell < m$ implies $|A| \leq g(\ell)$ or $|B| \leq g(\ell)$. This notion will help to bound the sizes of the minimal σ-symmetric $\mathbb{F}^*$-graphs that every σ-symmetric $\mathbb{F}^*$-graph of $\mathbb{F}$-rank-width k must exclude as pivot-minors or σ-vertex-minors.

Lemma 3.27 *Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function with $f(0) = 0$. Let G be an (m, f)-connected σ-symmetric $\mathbb{F}^*$-graph. Then, for every two vertices x and y of G such that $\ell_G((x, y)) \neq 0$, either G-x or $(G \wedge xy)$-x is $(m, 2f)$-connected. Moreover, if a σ-compatible $\lambda \in \mathbb{F}^*$ exists, then for every vertex x of G, either G-x or $(G * (x, \lambda))$-x is $(m, 2f)$-connected.*

Proof. Since $f(0) = 0$, G is connected. Let y be a neighbour of x. Suppose neither G-x nor $(G \wedge xy)$-x is $(m, 2f)$-connected. Then there are bipartitions (A_1, A_2) and (B_1, B_2) of $V_G \setminus \{x\}$ such that $a = \mathrm{cutrk}^{\mathbb{F}}_{G\text{-}x}(A_1) < m$, $b = \mathrm{cutrk}^{\mathbb{F}}_{(G\wedge xy)\text{-}x}(B_1) < m$, and $|A_i| > 2f(a)$, $|B_i| > 2f(b)$ for $i = 1, 2$.

We may assume that $a \geq b$. By Lemma 3.25, we have

$$\mathrm{cutrk}_G^{\mathbb{F}}(A_1 \cap B_1) + \mathrm{cutrk}_G^{\mathbb{F}}(A_2 \cap B_2) \leq a + b + 1.$$

Thus, either $\mathrm{cutrk}_G^{\mathbb{F}}(A_1 \cap B_1) \leq a$ or $\mathrm{cutrk}_G^{\mathbb{F}}(A_2 \cap B_2) \leq b$. So, by hypothesis either $|A_1 \cap B_1| \leq f(a)$ or $|A_2 \cap B_2| \leq f(b)$. Assume that $|A_2 \cap B_2| \leq f(b)$. Similarly, we also have either $|A_2 \cap B_1| \leq f(a)$ or $|A_1 \cap B_2| \leq f(b)$. Since $|A_1 \cap B_2| = |B_2| - |B_2 \cap A_2| > f(b)$, we have $|A_2 \cap B_1| \leq f(a)$. Then $|A_2| = |A_2 \cap B_1| + |A_2 \cap B_2| \leq f(a) + f(b) \leq 2f(a)$. This yields a contradiction.

The proof of the second statement is similar, using Lemma 3.23. □

We let $g(n) = (6^n - 1)/5$. Note that $g(0) = 0,\ g(1) = 1$ and $g(n) = 6g(n-1)+1$ for all $n \geq 1$. We now prove that the minimal σ-symmetric $\mathbb{F}^*$-graphs that have $\mathbb{F}$-rank-width at least $k+1$ are $(k+1, g)$-connected.

Lemma 3.28 *Let $k \geq 1$ and let G be a σ-symmetric $\mathbb{F}^*$-graph of $\mathbb{F}$-rank-width larger than k. If every proper pivot-minor of G has $\mathbb{F}$-rank-width at most k, then G is $(k+1, g)$-connected. Similarly, if a σ-compatible $\lambda \in \mathbb{F}^*$ exists, and every proper σ-vertex-minor of G has $\mathbb{F}$-rank-width at most k, then G is $(k+1, g)$-connected.*

Proof. The proof is similar to the one of [56, Lemma 5.3]. We assume that G is connected since the $\mathbb{F}$-rank-width of G is the maximum of the $\mathbb{F}$-rank-width of its connected components. It is now easy to see that G is $(1, g)$-connected.

Assume that $m \leq k$ and that G is (m, g)-connected but G is not $(m+1, g)$-connected. Then there exists a bipartition (A, B) with $\mathrm{cutrk}_G^{\mathbb{F}}(A) = m$ such that $|A| > g(m)$ and $|B| > g(m)$. Also, either A or B is not k-branched ($\mathrm{rwd}^{\mathbb{F}}(G) > k$). We may assume that B is not k-branched. Let $x \in A$ and $xy \in E_G$.

By Lemma 3.27, either G-x or $(G \wedge xy)$-x is $(m, 2g)$-connected; assume G-x is $(m, 2g)$-connected. Since G-x and $(G \wedge xy)$-x are proper pivot-minors of G, they both have $\mathbb{F}$-rank-width at most k. Let (A_1, A_2, A_3) be a tri-partition of $A \setminus \{x\}$. Since $|A| > g(m) = 6g(m-1) + 1$, there exists an $i \in \{1, 2, 3\}$ such that $|A_i| > 2g(m-1)$. Since G-x is $(m, 2g)$-connected and $|A_i| > 2g(m-1)$,

$$\mathrm{cutrk}_{G\text{-}x}^{\mathbb{F}}(A_i) \geq m \geq \mathrm{cutrk}_{G\text{-}x}^{\mathbb{F}}(A \setminus \{x\}).$$

Therefore, by Lemma 3.26 B is k-branched in G-x. Since B is not k-branched in G, there exists $W \subseteq B$ such that

$$\mathrm{cutrk}_G^{\mathbb{F}}(W) = \mathrm{cutrk}_{G\text{-}x}^{\mathbb{F}}(W) + 1.$$

Thus, the column vectors of $M_G[W, V_G \setminus (W \cup \{x\})]$ do not span $M_G[W, x]$. So, the column vectors of $M_G[W, V_G \setminus (B \cup \{x\})]$ do not span $M_G[W, x]$. Hence, the column vectors of $M_G[B, V_G \setminus (B \cup \{x\})]$ do not span $M_G[B, x]$. Therefore,

$$\mathrm{cutrk}^{\mathbb{F}}_{G\text{-}x}(B) = \mathrm{cutrk}^{\mathbb{F}}_{G}(B) - 1 = m - 1.$$

This implies that $|B| \leq 2g(m-1)$ or $|A \setminus \{x\}| \leq 2g(m-1)$. A contradiction.

The proof of the second statement is similar (replace $G \wedge xy$ by $G * (x, \lambda)$). $\square$

Theorem 3.29 (Size of excluded pivot-minors and σ-vertex-minors) *Let $k \geq 1$ and let G be a σ-symmetric $\mathbb{F}^*$-graph. If G has $\mathbb{F}$-rank-width larger than k but every proper pivot-minor of G has $\mathbb{F}$-rank-width at most k, then $|V_G| \leq (6^{k+1} - 1)/5$.*

Moreover, if a σ-compatible $\lambda \in \mathbb{F}^*$ exists, and if G has $\mathbb{F}$-rank-width larger than k, but every proper σ-vertex-minor of G has $\mathbb{F}$-rank-width at most k, then $|V_G| \leq (6^{k+1} - 1)/5$.

Proof. Let $x \in V_G$. We may assume that $G\text{-}x$ is $(k+1, 2g)$-connected by Lemmas 3.27 and 3.28. Since $G\text{-}x$ has $\mathbb{F}$-rank-width k, there exists a bipartition (A, B) of $V_G \setminus \{x\}$ such that $|A| \geq \frac{1}{3}(|V_G| - 1)$ and $|B| \geq \frac{1}{3}(|V_G| - 1)$ and $\mathrm{cutrk}^{\mathbb{F}}_{G\text{-}x}(A) \leq k$. By $(k+1, 2g)$-connectivity, either $|A| \leq 2g(k)$ or $|B| \leq 2g(k)$. Therefore, $|V_G| - 1 \leq 6g(k)$ and consequently $|V_G| \leq 6g(k) + 1 = g(k+1)$. $\square$

It is surprising that the bound $(6^{k+1} - 1)/5$ does not depend neither on $\mathbb{F}$ nor on σ. But that is because the proof technique is based on the $\mathrm{cutrk}^{\mathbb{F}}_{G}$-width of V_G and neither on $\mathbb{F}$ nor on σ. However, the $\mathbb{F}$-rank-width depends on $\mathbb{F}$ since there is no reason that the rank of a matrix is the same in two different fields. But, as we will see in the following proof of Theorem 3.21, the set of σ-symmetric $\mathbb{F}^*$-graphs to exclude as pivot-minors and σ-vertex-minor depends on $\mathbb{F}$ and σ.

Proof of Theorem 3.21. We show only the proof for the first statement. the other proof is similar. If $k < 0$, we let $\mathscr{C}_k^{(\mathbb{F},\sigma)} = \emptyset$. If $k = 0$, we let $\mathscr{C}_0^{(\mathbb{F},\sigma)} := \{\mathbf{a} \mid a \in \mathbb{F}^*\}$ where $\mathbf{a}$ is the σ-symmetric $\mathbb{F}^*$-graph $(\{x, y\}, \{x \overset{a}{\to} y,\ y \overset{\sigma(a)}{\to} x\})$. It is clear that G has $\mathbb{F}$-rank-width at most 0 if and only if G has no pivot-minor isomorphic to any $\mathbf{a} \in \mathscr{C}_0^{(\mathbb{F},\sigma)}$.

Assume now that $k \geq 1$ and let $\mathscr{C}_k^{(\mathbb{F},\sigma)}$ be the set, up to isomorphism, of σ-symmetric $\mathbb{F}^*$-graphs H such that $\mathrm{rwd}^{\mathbb{F}}(H) > k$ and every proper pivot-minor of H has $\mathbb{F}$-rank-width at most k. By Theorem 3.29, each σ-symmetric $\mathbb{F}^*$-graph in $\mathscr{C}_k^{(\mathbb{F},\sigma)}$ has at most $(6^{k+1} - 1)/5$ vertices.

Let G be a σ-symmetric $\mathbb{F}^*$-graph of $\mathbb{F}$-rank-width at most k. Since every $\mathbb{F}^*$-graph in $\mathscr{C}_k^{(\mathbb{F},\sigma)}$ has $\mathbb{F}$-rank-width larger than k, no $\mathbb{F}^*$-graph in $\mathscr{C}_k^{(\mathbb{F},\sigma)}$ is isomorphic to a pivot-minor of G.

Conversely, assume that the $\mathbb{F}$-rank-width of G is larger than k and let H be a proper pivot-minor of G of minimum size such that $\mathrm{rwd}^{\mathbb{F}}(H) > k$. Then there exists an $\mathbb{F}^*$-graph $H' \in \mathscr{C}_k^{(\mathbb{F},\sigma)}$ isomorphic to H. $\square$

Remark 3.30 *It is worth noticing that the bound* $(6^{k+1}-1)/5$ *on the size of excluded configurations is not tight. For instance, thanks to the characterisation of* σ*-symmetric* $\mathbb{F}^*$*-graphs of* $\mathbb{F}$*-rank-width* 1 *in [49,50], the obstructions for* σ*-symmetric* $\mathbb{F}^*$*-graphs of* $\mathbb{F}$*-rank-width* 1 *by* σ*-vertex-minor (resp. pivot-minor) have at most* 5 *(resp.* 6*) vertices.*

3.3 Recognising $\mathbb{F}$*-Rank-Width at Most* k

The recognition algorithm is an easy corollary of the one by Hliněný and Oum concerning representable matroids [41]. We recall the necessary materials about matroids. We refer to Schrijver [68] for our matroid terminology.

Definition 3.31 (Matroids) *A pair* $\mathcal{M} = (S, \mathcal{I})$ *is called a* matroid *if* S *is a finite set and* $\mathcal{I}$ *is a nonempty collection of subsets of* S *satisfying the following conditions*

(M1) *if* $I \in \mathcal{I}$ *and* $J \subseteq I$*, then* $J \in \mathcal{I}$*,*
(M2) *if* $I, J \in \mathcal{I}$ *and* $|I| < |J|$*, then* $I \cup \{z\} \in \mathcal{I}$ *for some* $z \in J \setminus I$*.*

For $U \subseteq S$*, a subset* B *of* U *is called a* base *of* U *if* B *is an inclusion wise maximal subset of* U *and belongs to* $\mathcal{I}$*. It is easy to see that, if* B_1 *and* B_2 *are bases of* $U \subseteq S$*, then* B_1 *and* B_2 *have the same size. The common size of the bases of a subset* U *of* S *is called the* rank of U*, denoted by* $r_{\mathcal{M}}(U)$*. A set* $B \subseteq S$ *is a base of* $\mathcal{M}$ *if it is a base of* S*.*

Let A *be a* $m \times n$*-matrix. Let* $S := \{1, \ldots, n\}$ *and let* $\mathcal{I}$ *be the collection of all those subsets* I *of* S *such that the columns of* A *with index in* $\mathcal{I}$ *are linearly independent. Then* $\mathcal{M} := (S, \mathcal{I})$ *is a matroid. If* A *has entries in* $\mathbb{F}$*, then* $\mathcal{M}$ *is said* representable over $\mathbb{F}$ *and* A *is called a* representation of $\mathcal{M}$ over $\mathbb{F}$.

If $\mathcal{M} = (S, \mathcal{I})$ *is a matroid, we let* $\lambda_{\mathcal{M}}$ *be defined such that for every subset* U *of* S*,* $\lambda_{\mathcal{M}}(U) = r_{\mathcal{M}}(U) + r_{\mathcal{M}}(S \setminus U) - r_{\mathcal{M}}(S) + 1$ *and call it the* connectivity function of $\mathcal{M}$*. The function* $\lambda_{\mathcal{M}}$ *is symmetric and submodular [68]. The* branch-width of $\mathcal{M}$*, denoted by* $\mathrm{bwd}(\mathcal{M})$*, is the* $\lambda_{\mathcal{M}}$*-width of* S*.*

Let $\mathcal{M} = (S, \mathcal{I})$ *be a matroid and* $\mathcal{P}$ *a partition of* S*. The pair* $(\mathcal{M}, \mathcal{P})$ *is called* *a* partitioned matroid. *A partitioned matroid* $(\mathcal{M},\mathcal{P})$ *is* representable over $\mathbb{F}$ *if* $\mathcal{M}$ *is representable over* $\mathbb{F}$. *For a partitioned matroid* $(\mathcal{M},\mathcal{P})$, *we let* $\lambda_{\mathcal{M}}^{\mathcal{P}}$ *be defined such that for every* $Z \subseteq \mathcal{P}$, *we have* $\lambda_{\mathcal{M}}^{\mathcal{P}}(Z) := \lambda_{\mathcal{M}}(\bigcup_{Y\in Z} Y)$. *The* branch-width *of* $(\mathcal{M},\mathcal{P})$, *denoted by* $\mathrm{bwd}(\mathcal{M},\mathcal{P})$, *is the* $\lambda_{\mathcal{M}}^{\mathcal{P}}$*-width of* $\mathcal{P}$.

Theorem 3.32 ([41]) *Let* $\mathbb{F}$ *be a fixed finite field and* k *be a fixed positive integer. There exists a cubic-time algorithm that takes as input a representable partitioned matroid* $(\mathcal{M},\mathcal{P})$ *over* $\mathbb{F}$ *given with the representation of* $\mathcal{M}$ *over* $\mathbb{F}$ *and outputs a layout of* $\mathcal{P}$ *of* $\lambda_{\mathcal{M}}^{\mathcal{P}}$*-width at most* k *or confirms that the branch-width of* $(\mathcal{M},\mathcal{P})$ *is strictly greater than* k.

We can now derive our recognition algorithm from Theorem 3.32. For a set X, we let X' be a disjoint copy of it defined as $\{x' \mid x \in X\}$. For an $\mathbb{F}^*$-graph G, we let $\mathcal{M}_G$ be the matroid on $V_G \cup V_G'$ represented by the $(V_G, V_G \cup V_G')$-matrix (recall that I_n denotes the identity square matrix of size n):

$$\begin{array}{cc} & \begin{array}{cc} V_G & V_G' \end{array} \\ V_G & \begin{pmatrix} I_{|V_G|} & M_G \end{pmatrix} \end{array}$$

For each $x \in V$, we let $P_x := \{x, x'\}$ and we let $\Pi(G) := \{P_x \mid x \in V_G\}$.

Proposition 3.33 *Let* G *be an* $\mathbb{F}^*$*-graph. For every* $X \subseteq V_G$, $\lambda_{\mathcal{M}_G}^{\Pi(G)}(P) = \mathrm{rk}(M_G[X, V_G \setminus X]) + \mathrm{rk}(M_G[V_G \setminus X, X]) + 1$ *where* $P := \{P_x \mid x \in X\}$.

Proof. For $X \subseteq V_G$ and $P := \{P_x \mid x \in X\}$, we have

$$\begin{aligned}
\lambda_{\mathcal{M}_G}^{\Pi(G)}(P) &= r_{\mathcal{M}_G}(X \cup X') + r_{\mathcal{M}_G}((V_G \setminus X) \cup (V_G \setminus X)') - r_{\mathcal{M}_G}(V_G \cup V_G') + 1 \\
&= rk \begin{pmatrix} 0 & M_G[V_G \setminus X, X] \\ I_{|X|} & M_G[X, X] \end{pmatrix} + rk \begin{pmatrix} 0 & M_G[X, V_G \setminus X] \\ I_{|V_G|-|X|} & M_G[V_G \setminus X, V_G \setminus X] \end{pmatrix} - |V_G| + 1 \\
&= |X| + \mathrm{rk}(M_G[V_G \setminus X, X]) + |V_G - X| + \mathrm{rk}(M_G[X, V_G \setminus X]) - |V_G| + 1 \\
&= \mathrm{rk}(M_G[X, V_G \setminus X]) + \mathrm{rk}(M_G[V_G \setminus X, X]) + 1. \qquad \square
\end{aligned}$$

Since when G is σ-symmetric, we have $\mathrm{rk}(M_G[X, V_G \setminus X]) = \mathrm{rk}(M_G[V_G \setminus X, X]) = \mathrm{cutrk}_G^{\mathbb{F}}(X)$, we get the following as corollaries of Proposition 3.33.

Corollary 3.34 *Let* G *be a* σ*-symmetric* $\mathbb{F}^*$*-graph. For every* $X \subseteq V_G$, $\lambda_{\mathcal{M}_G}^{\Pi(G)}(P) = 2 \cdot \mathrm{cutrk}_G^{\mathbb{F}}(X) + 1$ *where* $P := \{P_x \mid x \in X\}$.

Corollary 3.35 *Let* G *be a* σ*-symmetric* $\mathbb{F}^*$*-graph and let* $p : V_G \to \Pi(G)$ *be the bijective function such that* $p(x) = P_x$. *If* $(T, \mathcal{L})$ *is a layout of* $\Pi(G)$ *of* $\lambda_{\mathcal{M}_G}^{\Pi(G)}$*-width* $2k+1$, *then* $(T, \mathcal{L} \circ p)$ *is a layout of* V_G *of* $\mathrm{cutrk}_G^{\mathbb{F}}$*-width* k.

Conversely, if $(T, \mathcal{L})$ *is a layout of* V_G *of* $\mathrm{cutrk}_G^{\mathbb{F}}$*-width* k*, then* $(T, \mathcal{L} \circ p^{-1})$ *is a layout of* $\Pi(G)$ *of* $\lambda_{\mathcal{M}_G}^{\Pi(G)}$*-width* $2k+1$*.*

Theorem 3.36 (Checking $\mathbb{F}$-Rank-Width at most k) *For fixed* k *and a fixed finite field* $\mathbb{F}$*, there exists a cubic-time algorithm that, for a* σ*-symmetric* $\mathbb{F}^*$*-graph* G*, either outputs a layout of* V_G *of* $\mathrm{cutrk}_G^{\mathbb{F}}$*-width at most* k *or confirms that the* $\mathbb{F}$*-rank-width of* G *is larger than* k*.*

Proof. Let k be fixed and let $\mathcal{A}$ be the algorithm constructed in Theorem 3.32 for $2k+1$. Let G be a σ-symmetric $\mathbb{F}^*$-graph. We run the algorithm $\mathcal{A}$ with input $(\mathcal{M}_G, \Pi(G))$. If it confirms that $\mathrm{bwd}(\mathcal{M}_G, \Pi(G)) > 2k+1$, then the $\mathbb{F}$-rank-width of G is greater than k (Corollary 3.34). If it outputs a layout of $\Pi(G)$ of $\lambda_{\mathcal{M}_G}^{\Pi(G)}$-width at most $2k+1$, we can transform it into a layout of V_G of $\mathrm{cutrk}_G^{\mathbb{F}}$-width at most k by Corollary 3.35. The fact that the algorithm $\mathcal{A}$ runs in cubic-time concludes the proof. $\square$

3.4 Specialisations to Graphs

We specialise in this section the $\mathbb{F}$-rank-width to directed and oriented graphs. As we already said, for undirected graphs the $\mathbb{F}_2$-rank-width matches with the rank-width.

Directed Graphs [47]. We recall that the adjacency matrix of a directed graph G is the (V_G, V_G)-matrix M_G over $\mathbb{F}_2$ where $M_G[x,y] := 1$ if and only if $(x,y) \in E_G$. This matrix is not symmetric except when G is undirected. In particular, $\mathrm{rk}(M_G[X, V_G \setminus X])$ is *a priori* different from $\mathrm{rk}(M_G[V_G \setminus X, X])$. The quest for finding another representation of directed graphs by matrices where $\mathrm{rk}(M_G[X, V_G \setminus X]) = \mathrm{rk}(M_G[V_G \setminus X, X])$ motivates the definition of σ-symmetry. We now give this representation.

We recall that $\mathbb{F}_4$ is the finite field of order four. We let $\{0, 1, \eth, \eth^2\}$ be its elements with the property that $1 + \eth + \eth^2 = 0$ and $\eth^3 = 1$. Moreover, it is of characteristic 2. We let $\sigma_4 : \mathbb{F}_4 \to \mathbb{F}_4$ be the automorphism where $\sigma_4(\eth) = \eth^2$ and $\sigma_4(\eth^2) = \eth$. It is clearly a sesqui-morphism.

For every directed graph G, let $\widetilde{G} := (V_G, E_G \cup \{(y,x) | (x,y) \in E_G\}, \ell_{\widetilde{G}})$ be the $\mathbb{F}_4{}^*$-graph where for every pair of vertices (x, y):

$$\ell_{\widetilde{G}}((x,y)) := \begin{cases} 1 & \text{if } (x,y) \in E_G \text{ and } (y,x) \in E_G, \\ \eth & (x,y) \in E_G \text{ and } (y,x) \notin E_G, \\ \eth^2 & (y,x) \in E_G \text{ and } (x,y) \notin E_G, \\ 0 & \text{otherwise.} \end{cases}$$

One easily verifies that $\widetilde{G}$ is σ_4-symmetric and is actually the one constructed in Section 3. We define the *rank-width* of a directed graph G, denoted by $\mathrm{rwd}^{\mathbb{F}_4}(G)$, as the $\mathbb{F}_4$-rank-width of $\widetilde{G}$.

Remark 3.37 *Let G be an undirected graph. We denote by $\overleftrightarrow{G}$ the directed graph obtained from G by replacing each edge xy in G by two opposite ones. By the definition of $\overleftrightarrow{G}$ we have $M_G = M_{\overleftrightarrow{G}}$. Then* $\mathrm{rwd}^{\mathbb{F}_4}(\overleftrightarrow{G}) = \mathrm{rwd}(G)$ *since $\mathbb{F}_4$ is an extension of $\mathbb{F}_2$.*

We now specialise the notion of vertex-minor. We recall that given a sesqui-morphism $\sigma : \mathbb{F} \to \mathbb{F}$, an element λ of $\mathbb{F}^*$ is said σ-compatible if $\sigma(\lambda) = \lambda \cdot \sigma(1)^2$. Since $\sigma_4(1) = 1$, 1 is σ_4-compatible and is the only one. We then denote $G * (v, 1)$ by $G * v$, and say that a directed graph H is a *vertex-minor* of a directed graph G if $\widetilde{H}$ is a σ_4-vertex-minor of $\widetilde{G}$. One easily verifies that if a directed graph H is obtained from a directed graph G by applying a 1-local-complementation at x, then H is obtained from G by modifying the subgraph induced on the neighbours of x as shown on Table 1. Figure 1 gives an example of a 1-local complementation. Thanks to the characterisation of directed graphs of rank-width 1 in [49] (see Remark 3.30), we can compute - with the help of a computer program - the set of obstructions for directed graphs of rank-width 1 with respect to vertex-minor (resp. pivot-minor). These two sets are depicted in Figures 2 and 3 respectively.

G	$G * x$
$z \perp t$	$z \leftrightarrow t$
$z \to t$	$z \leftarrow t$
$z \leftarrow t$	$z \to t$
$z \leftrightarrow t$	$z \perp t$

(a)

G	$G * x$
$z \perp t$	$z \to t$
$z \to t$	$z \perp t$
$z \leftarrow t$	$z \leftrightarrow t$
$z \leftrightarrow t$	$z \leftarrow t$

(b)

Table 1
We use the following notations: $x \to y$ means $\ell_G((x,y)) = \eth$, $x \leftarrow y$ means $\ell_G((x,y)) = \eth^2$, $x \leftrightarrow y$ means $\ell_G((x,y)) = 1$, and $z \perp t$ means $\ell_G((x,y) = 0)$.
(a) Uniform Case: $z \leftarrow x \to t$ or $z \to x \leftarrow t$ or $z \leftrightarrow x \leftrightarrow t$.
(b) Non Uniform Case: $z \leftarrow x \leftarrow t$ or $z \to x \leftrightarrow t$ or $z \leftrightarrow x \to t$.

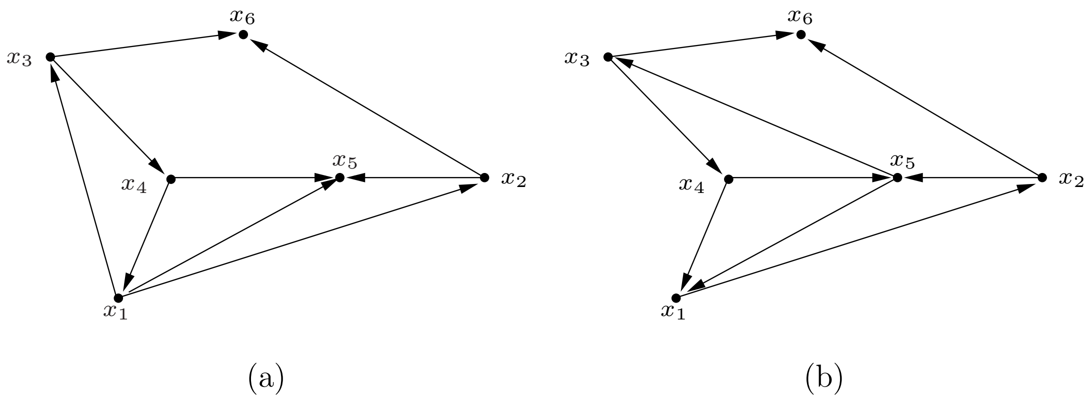

Figure 1. (a) A directed graph G. (b) The directed graph $G * x_4$.

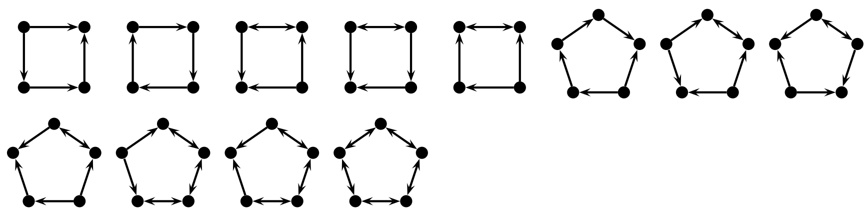

Figure 2. Vertex-minor exclusions for directed graphs of rank-width 1.

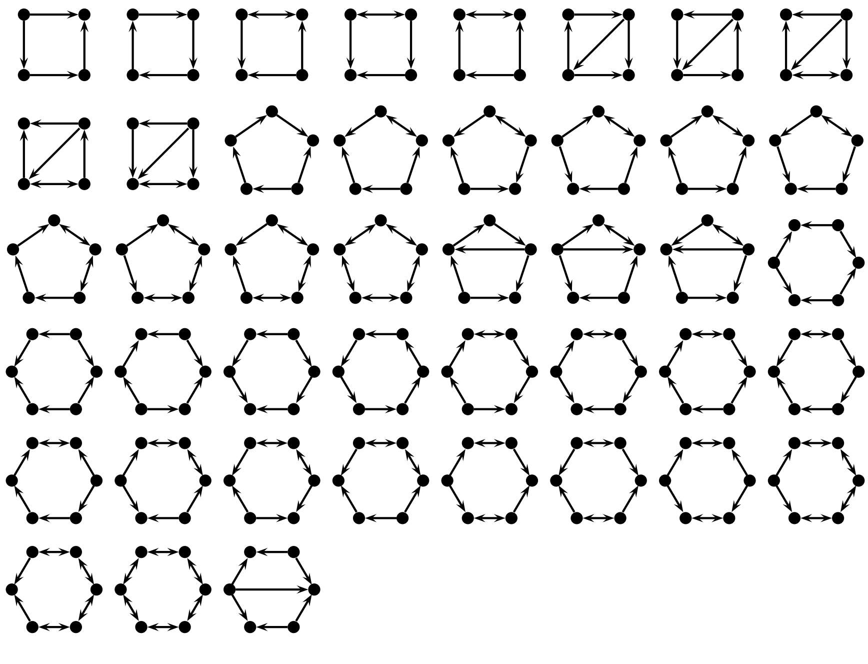

Figure 3. Pivot-minor exclusions for directed graphs of rank-width 1.

Moreover, as in the undirected case, we have $G \wedge xy = G \wedge yx = G * x * y * x = G * y * x * y$. As corollaries of Theorems 3.21 and 3.36 we get the following.

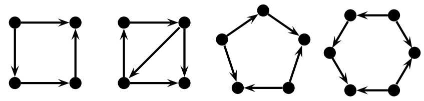

Figure 4. Pivot-minor exclusions for oriented graphs of $\mathbb{F}_3$-rank-width 1.

Theorem 3.38 *For each positive integer k, there is a finite list $\mathscr{C}_k$ of directed graphs having at most $(6^{k+1}-1)/5$ vertices such that a directed graph G has rank-width at most k if and only if no directed graph in $\mathscr{C}_k$ is isomorphic to a vertex-minor of G.*

Theorem 3.39 *For fixed k, there exists a cubic-time algorithm that, for a directed graph G, either outputs a layout of V_G of $\mathrm{cutrk}_G^{\mathbb{F}_4}$-width at most k or confirms that the rank-width of G is larger than k.*

Oriented Graphs. We can define another parameter in the case of oriented graphs. Let $G=(V,A)$ be an oriented graph, and let $\widetilde{G} := (V,E,\ell)$ be the $\mathbb{F}_3^*$-graph such that $E = A \cup A'$ where $A' = \{(y,x) | (x,y) \in A\}$, $\ell((x,y)) := 1$ if $(x,y) \in A$ and $\ell((x,y)) := -1$ if $(x,y) \in A'$. Clearly, $\widetilde{G}$ is a σ_3-symmetric $\mathbb{F}_3^*$-graph, with $\sigma_3(x) := -x$. Moreover, one can show immediately that σ_3 is a sesqui-morphism. Note that there is no σ_3-compatible λ in $\mathbb{F}_3^*$, thus no σ_3-vertex-minor is defined on σ_3-symmetric $\mathbb{F}_3^*$-graphs. Nevertheless, oriented graphs of $\mathbb{F}_3$-rank-width k are characterised by a finite set of oriented graphs $\mathscr{C}_k^{(\mathbb{F}_3,\sigma_3)}$ of forbidden pivot-minors (whereas sets $\mathscr{C}_k^{(\mathbb{F}_4,\sigma_4)}$ and $\mathscr{C}'^{(\mathbb{F}_4,\sigma_4)}_k$ contains directed graphs). Again, with the help of [49], we can compute the set of obstructions - depicted in Figure 4 - for oriented graphs of $\mathbb{F}_3$-rank-width 1 with respect to pivot-minor relation.

$\mathbb{F}_3$-rank-width and $\mathbb{F}_4$-rank-width of oriented graphs are two equivalent parameters, since they are both equivalent to clique-width. But these two parameters are not equal. In one hand, tournaments of $\mathbb{F}_3$-rank-width 1 are exactly tournaments completely decomposable by *bi-join decomposition* (see [50]), and a cut $\{X,Y\}$ in a tournament has $\mathbb{F}_4$-rank 1 if and only if X or Y is a module. Since there are tournaments completely decomposable by bi-join and prime with respect to the *modular decomposition* (see [3]), there are oriented graphs of $\mathbb{F}_3$-rank-width 1 and $\mathbb{F}_4$-rank-width at least 2. On the other hand, the graph in Figure 5 (right) has $\mathbb{F}_4$-rank-width 2 and $\mathbb{F}_3$-rank-width 3.

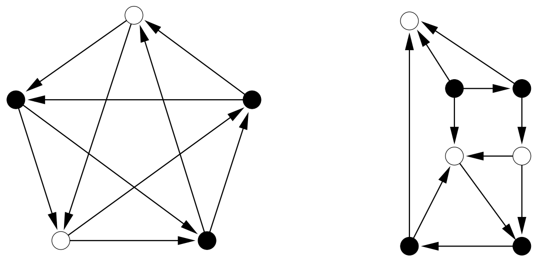

Figure 5. Left: an oriented graph of $\mathbb{F}_3$-rank-width 1 and $\mathbb{F}_4$-rank-width 2 (white/black vertices give a cut of $\mathbb{F}_3$-rank-width 1). Right: an oriented graph of $\mathbb{F}_3$-rank-width 3 and $\mathbb{F}_4$-rank-width 2 (white/black vertices give a cut of $\mathbb{F}_4$-rank-width 2).

4 The Second Notion of Rank-Width: $\mathbb{F}$-Bi-Rank-Width

4.1 Definitions and Comparisons to Other Parameters

Recall that if G is an $\mathbb{F}^*$-graph, we denote by M_G the (V_G, V_G)-matrix over $\mathbb{F}$ where $M_G[x,y] := \ell_G((x,y))$ if $x \neq y$, and $M_G[x,x] = 0$. As for the notion of $\mathbb{F}$-rank-width, we use matrix rank functions for the notion of $\mathbb{F}$-bi-rank-width. Its definition is similar to the one of bi-rank-width.

Definition 4.1 (Bi-Cut-Rank Function) *For an $\mathbb{F}^*$-graph G, we let* $\mathrm{bicutrk}_G^{\mathbb{F}} : 2^{V_G} \to \mathbb{N}$ *where* $\mathrm{bicutrk}_G^{\mathbb{F}}(X) = \mathrm{rk}(M_G[X, V_G \setminus X]) + \mathrm{rk}(M_G[V_G \setminus X, X])$ *for all $X \subseteq V_G$.*

Lemma 4.2 *For every $\mathbb{F}^*$-graph G, the function* $\mathrm{bicutrk}_G^{\mathbb{F}}$ *is symmetric and submodular.*

Proof. Let X and Y be subsets of V_G. We let $A_1 := M_G[X, V_G \setminus X]$, $A_2 := M_G[V_G \setminus X, X]$, $B_1 =: M_G[Y, V_G \setminus Y]$ and $B_2 := M_G[V_G \setminus Y, Y]$. By definition,

$$\mathrm{bicutrk}_G^{\mathbb{F}}(X) = \mathrm{rk}(A_1) + \mathrm{rk}(A_2) = \mathrm{rk}(A_2) + \mathrm{rk}(A_1) = \mathrm{bicutrk}_G^{\mathbb{F}}(V_G \setminus X).$$

For the submodularity, we have by definition,

$$\mathrm{bicutrk}_G^{\mathbb{F}}(X) + \mathrm{bicutrk}_G^{\mathbb{F}}(Y) = \mathrm{rk}(A_1) + \mathrm{rk}(A_2) + \mathrm{rk}(B_1) + \mathrm{rk}(B_2).$$

By Proposition 3.10,

$$\mathrm{rk}(A_1) + \mathrm{rk}(B_1) \geq \mathrm{rk}(M_G[X \cup Y, (V_G \setminus X) \cap (V_G \setminus Y)]) + \mathrm{rk}(M_G[X \cap Y, (V_G \setminus X) \cup (V_G \setminus Y)])$$

and

$$\mathrm{rk}(A_2) + \mathrm{rk}(B_2) \geq \mathrm{rk}(M_G[(V_G \setminus X) \cup (V_G \setminus Y), X \cap Y]) + \mathrm{rk}(M_G[(V_G \setminus X) \cap (V_G \setminus Y), X \cup Y]).$$

Since $(V_G \setminus X) \cap (V_G \setminus Y) = V_G \setminus (X \cup Y)$ and $(V_G \setminus X) \cup (V_G \setminus Y) = V_G \setminus (X \cap Y)$ the second statement holds. $\square$

Definition 4.3 ($\mathbb{F}$-bi-rank-width) *The $\mathbb{F}$-bi-rank-width of an $\mathbb{F}^*$-graph, denoted by* $\mathrm{brwd}^{\mathbb{F}}(G)$*, is the* $\mathrm{bicutrk}_G^{\mathbb{F}}$*-width of V_G.*

We now compare $\mathbb{F}$-bi-rank-width - $\mathbb{F}$ finite - with clique-width and $\mathbb{F}$-rank-width. The proof of the following is given in Appendix.

Proposition 4.4 *Let G be an $\mathbb{F}^*$-graph, with $|\mathbb{F}| = q \in \mathbb{N}$.*

(1) If t is a term in $T(\mathcal{F}_k^{\mathbb{F}}, \mathcal{C}_k)$ isomorphic to G, then $(\mathcal{uS}(red(t)), \mathcal{L}_t)$ is a layout of V_G of $\mathrm{bicutrk}_G^{\mathbb{F}}$*-width at most $2k$.*
(2) If $(T, \mathcal{L})$ is a layout of V_G of $\mathrm{bicutrk}_G^{\mathbb{F}}$*-width at most k, then we can construct in time $O(q^k \cdot |V_G|^2)$ a term t in $T(\mathcal{F}_{k'}^{\mathbb{F}}, \mathcal{C}_{k'})$ with $k' \leq 2 \cdot q^k - 1$ such that G is isomorphic to $val(t)$, and $(\mathcal{uS}(red(t)), \mathcal{L}_t) = (T, \mathcal{L})$.*

In other words, $\frac{1}{2}\,\mathrm{brwd}^{\mathbb{F}}(G) \leq \mathrm{cwd}(G) \leq 2 \cdot q^{\mathrm{brwd}^{\mathbb{F}}(G)} - 1$.

The mapping $[G \mapsto \widetilde{G}]$ from $\mathscr{S}(\mathbb{F})$ to $\mathscr{S}(\mathbb{F}^2, \widetilde{\sigma})$ constructed in Section 3 is such that for every $x, y \in V_G$, $M_{\widetilde{G}}[x, y] = \gamma \cdot M_G[x, y] + \tau \cdot M_G[y, x]$ for some fixed $\gamma, \tau \in (\mathbb{F}^2)^*$ with $\gamma, \tau \notin \mathbb{F}$.

Proposition 4.5 *For every $\mathbb{F}^*$-graph G and every subset X of V_G, we have*

$$M_{\widetilde{G}}[X, V_G \setminus X] = \gamma \cdot M_G[X, V_G \setminus X] + \tau \cdot M_G^T[V_G \setminus X, X].$$

Lemma 4.6 ([51,52]) *(i) Let M be a matrix over $\mathbb{F}$. If the rank of M over $\mathbb{F}$ is k, then the rank of M over any finite extension of M is k.*
(ii) If A and B are two matrices over $\mathbb{F}$, then $\mathrm{rk}(A + B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$ and $\mathrm{rk}(A \cdot B) \leq \min\{\mathrm{rk}(A), \mathrm{rk}(B)\}$. If $a \in \mathbb{F}^*$, then $\mathrm{rk}(a \cdot A) = \mathrm{rk}(A)$.

Proposition 4.7 *Let G be an $\mathbb{F}^*$-graph. Then*

(1) $\mathrm{rwd}^{\mathbb{F}^2}(\widetilde{G}) \leq \mathrm{brwd}^{\mathbb{F}}(G) \leq 4 \cdot \mathrm{rwd}^{\mathbb{F}^2}(\widetilde{G})$.
(2) If G is σ-symmetric for some sesqui-morphism $\sigma : \mathbb{F} \to \mathbb{F}$, then $\mathrm{brwd}^{\mathbb{F}}(G) = 2 \cdot \mathrm{rwd}^{\mathbb{F}}(G)$.

Proof. It is sufficient to compare $\mathrm{cutrk}_{\widetilde{G}}^{\mathbb{F}^2}(X)$ and $\mathrm{bicutrk}_G^{\mathbb{F}}(X)$ for every subset X of V_G.

(1) From Lemma 4.6 and Proposition 4.5 we have.

$$\begin{aligned}\mathrm{rk}(M_{\widetilde{G}}[X, V_G \setminus X]) &\leq \mathrm{rk}(M_G[X, V_G \setminus X]) + \mathrm{rk}(M_G[V_G \setminus X, X]) \\ &= \mathrm{bicutrk}_G^{\mathbb{F}}(X).\end{aligned}$$

We now prove that $\mathrm{bicutrk}^{\mathbb{F}}_G(X) \leq 4 \cdot \mathrm{cutrk}^{\mathbb{F}^2}_{\widetilde{G}}(X)$. Let $M_1 := M_G[X, V_G \setminus X]$ and $M_2 := M^T_G[V_G \setminus X, X]$. We recall that each entry of $M_{\widetilde{G}}$ is of the form $a \cdot \gamma + b \cdot \tau$ for a unique pair $(a, b) \in \mathbb{F} \times \mathbb{F}$. Let π_1 and π_2 be mappings from $\mathbb{F}^2$ to $\mathbb{F}$ such that:

$$\pi_1(a \cdot \gamma + b \cdot \tau) = a,$$
$$\pi_2(a \cdot \gamma + b \cdot \tau) = b.$$

Clearly, $M_1 = \pi_1(M_{\widetilde{G}}[X, V_G \setminus X])$ and $M_2 = \pi_2(M_{\widetilde{G}}[V_G \setminus X, X])$. It is also straightforward to verify that π_1 and π_2 are homomorphisms with respect to the addition. Moreover, for every $c \in \mathbb{F}, \delta \in \mathbb{F}^2$ and $i \in \{1, 2\}$, $\pi_i(c\cdot\delta) = c\cdot\pi_i(\delta)$.

We let $v_1, \ldots, v_k$ be a column-basis of $M_{\widetilde{G}}[X, V_G \setminus X]$. Then for each column-vector v in $M_{\widetilde{G}}[X, V_G \setminus X]$, $v = \sum_{i\leq k} \alpha_i \cdot v_i$ where $\alpha_i \in \mathbb{F}^2$. Then we have for $j \in \{1, 2\}$,

$$\begin{aligned}
\pi_j(v) &= \sum_{i\leq k} \pi_j(\alpha_i \cdot v_i) \\
&= \sum_{i\leq k} \pi_j\left(\alpha_i \cdot (\pi_1(v_i) \cdot \gamma + \pi_2(v_i) \cdot \tau)\right) \\
&= \sum_{i\leq k} \pi_j\left(\alpha_i \cdot \gamma \cdot \pi_1(v_i) + \alpha_i \cdot \tau \cdot \pi_2(v_i)\right) \\
&= \sum_{i\leq k} \pi_j(\alpha_i \cdot \gamma) \cdot \pi_1(v_i) + \pi_j(\alpha_i \cdot \tau) \cdot \pi_2(v_i)
\end{aligned}$$

Thus, every column-vector of M_j is a linear combination of $2k$ vectors $\pi_1(v_i)$ and $\pi_2(v_i)$ for $i \in \{1, \ldots, k\}$, *i.e.*, $\mathrm{rk}(M_j) \leq 2k$. Therefore, $\mathrm{bicutrk}^{\mathbb{F}}_G(X) = \mathrm{rk}(M_1) + \mathrm{rk}(M_2) \leq 4 \cdot \mathrm{cutrk}^{\mathbb{F}^2}_{\widetilde{G}}(X)$.

(2) Assume now that G is σ-symmetric. By definition of $\mathrm{bicutrk}^{\mathbb{F}}_G$, we have $\mathrm{bicutrk}^{\mathbb{F}}_G(X) = \mathrm{rk}(M_G[X, V_G \setminus X]) + \mathrm{rk}(M_G[V_G \setminus X, X])$. But since G is σ-symmetric, by Lemma 3.9, we have $\mathrm{rk}(M_G[X, V_G \setminus X]) = \mathrm{rk}(M_G[V_G \setminus X, X])$. We can then conclude that $\mathrm{bicutrk}^{\mathbb{F}}_G(X) = 2 \cdot \mathrm{cutrk}^{\mathbb{F}}_G(X)$. $\square$

The notion of λ-local complementation defined in Section 3.2 also preserves the $\mathbb{F}$-bi-rank-width.

Lemma 4.8 *Let G be an $\mathbb{F}^*$-graph and λ an element in $\mathbb{F}^*$. If $G * (x, \lambda)$ is the λ-local complementation of G at x, then for every subset X of V_G, we have $\mathrm{bicutrk}^{\mathbb{F}}_{G*(x,\lambda)}(X) = \mathrm{bicutrk}^{\mathbb{F}}_G(X)$.*

Proof. Assume by symmetry that x is in X. Let y be a neighbour of x in X. If we apply a λ-local complementation at x, we obtain $M_{G*(x,\lambda)}[y, V_G \setminus X]$ by adding $\lambda \cdot M_G[y,x] \cdot M_G[x, V_G \setminus X]$ to $M_G[y, V_G \setminus X]$. Therefore, $\mathrm{rk}(M_{G*(x,\lambda)}[X, V_G \setminus X]) = \mathrm{rk}(M_G[X, V_G \setminus X])$. Similarly, we obtain $M_{G*(x,\lambda)}[V_G \setminus X, y]$ by adding to $M_G[V_G \setminus X, y]$ the column $\lambda \cdot M_G[V_G \setminus X, x] \cdot M_G[x, y]$. Again, $\mathrm{rk}(M_{G*(x,\lambda)}[V_G \setminus X, X]) = \mathrm{rk}(M_G[V_G \setminus X, X])$. We can thus conclude that $\mathrm{bicutrk}^{\mathbb{F}}_{G*(x,\lambda)}(X) = \mathrm{bicutrk}^{\mathbb{F}}_G(X)$. □

Corollary 4.9 *Let G and H be two $\mathbb{F}^*$-graphs. If H is locally equivalent to G, then the $\mathbb{F}$-bi-rank-width of H is equal to the $\mathbb{F}$-bi-rank-width of G. If H is a vertex-minor of G, then the $\mathbb{F}$-bi-rank-width of H is at most the $\mathbb{F}$-bi-rank-width of G.*

Note that the pivot-complementation defined in Section 3.2 is not well defined in the case of $\mathbb{F}^*$-graphs that are not σ-symmetric. Currently, we do not have a characterisation of $\mathbb{F}^*$-graphs of bounded $\mathbb{F}$-bi-rank-width as the one in Theorem 3.21. We leave it as an open question. Indeed, our notion of vertex-minor is not a well-quasi-order on $\mathbb{F}^*$-graphs of bounded $\mathbb{F}$-bi-rank-width (see Remark 4.13).

4.2 Recognising $\mathbb{F}$-Bi-Rank-Width at Most k

We recall that if G is an $\mathbb{F}^*$-graph, we denote by $(\mathcal{M}_G, \Pi(G))$ the partitioned matroid represented over $\mathbb{F}$ where $\Pi(G) := \{P_x \mid x \in V_G\}$ with $P_x := \{x, x'\}$ and $\mathcal{M}_G$ is the matroid represented by the $(V_G, V_G \cup V'_G)$-matrix over $\mathbb{F}$ (V'_G is an isomorphic copy of V_G)

$$\begin{array}{cc} & \begin{array}{cc} V_G & V'_G \end{array} \\ V_G & \begin{pmatrix} I_{|V_G|} & M_G \end{pmatrix} \end{array}.$$

The following are corollaries of Proposition 3.33.

Corollary 4.10 *Let G be an $\mathbb{F}^*$-graph. For every $X \subseteq V_G$, $\lambda^{\Pi(G)}_{\mathcal{M}_G}(P) = \mathrm{bicutrk}^{\mathbb{F}}_G(X) + 1$ where $P := \{P_x \mid x \in X\}$.*

Corollary 4.11 *Let G be an $\mathbb{F}^*$-graph and let $p : V_G \to \Pi(G)$ be the bijective function such that $p(x) = P_x$. If $(T, \mathcal{L})$ is a layout of $\Pi(G)$ of $\lambda^{\Pi(G)}_{\mathcal{M}_G}$-width $k+1$, then $(T, \mathcal{L} \circ p)$ is a layout of V_G of $\mathrm{bicutrk}^{\mathbb{F}}_G$-width k. Conversely, if $(T, \mathcal{L})$ is a layout of V_G of $\mathrm{bicutrk}^{\mathbb{F}}_G$-width k, then $(T, \mathcal{L} \circ p^{-1})$ is a layout of $\Pi(G)$ of $\lambda^{\Pi(G)}_{\mathcal{M}_G}$-width $k+1$.*

Theorem 4.12 (Checking $\mathbb{F}$-Bi-Rank-Width at most k) *For a fixed finite field $\mathbb{F}$ and a fixed integer k, there exists a cubic-time algorithm that, for an $\mathbb{F}^*$-graph G, either outputs a layout of V_G of $\mathrm{bicutrk}^{\mathbb{F}}_G$-width at most k or confirms that the $\mathbb{F}$-bi-rank-width of G is larger than k.*

Proof. Let k be fixed and let $\mathcal{A}$ be the algorithm constructed in Theorem 3.32 for $k+1$. Let G be an $\mathbb{F}^*$-graph. We run the algorithm $\mathcal{A}$ with input $(\mathcal{M}_G, \Pi(G))$. If it confirms that $\mathrm{bwd}(\mathcal{M}_G, \Pi(G)) > k+1$, then the $\mathbb{F}$-bi-rank-width of G is greater than k (Corollary 4.10). If it outputs a layout of $\Pi(G)$ of $\lambda_{\mathcal{M}_G}^{\Pi(G)}$-width at most $k+1$, we can transform it into a layout of V_G of $\mathrm{bicutrk}_G^{\mathbb{F}}$-width at most k by Corollary 4.11. The fact that the algorithm $\mathcal{A}$ runs in cubic-time concludes the proof. □

4.3 A Specialisation to Graphs

The *bi-rank-width* of a directed graph G, denoted by $\mathrm{brwd}(G)$, is its $\mathbb{F}_2$-bi-rank-width [47]. One easily checks that if G is undirected, *i.e.*, if E_G is symmetric, then $\mathrm{brwd}(G) = 2 \cdot \mathrm{rwd}(G)$. It is worth noticing that the paper [30] has used the term *signed rank-width* to denote the bi-rank-width of *signed graphs*, which are bipartite directed graphs.

A directed graph is *strongly connected* if for every pair (x, y) of vertices, there is a directed path from x to y. Clearly in a strongly connected graph G, for every $\emptyset \subsetneq X \subsetneq V_G$, we have $\mathrm{bicutrk}_G^{\mathbb{F}}(X) \geq 2$. It is easy to show that strongly connected graphs of bi-rank-width 2 are exactly the graphs completely decomposable by Cunningham's split decomposition of directed graphs [21].

The 1-local complementation of a directed graph seen as an $\mathbb{F}_2^*$-graph is the one defined by Bouchet [5] and Fon-Der-Flaass [27]. One easily verifies that if H is obtained by applying a 1-local complementation at x to G, then $(z, t) \in E_H$ if and only if:

- $(z, t) \notin E_G$, $(z, x) \in E_G$ and $(x, t) \in E_G$ or,
- $(z, t) \in E_G$, and either $(z, x) \notin E_G$ or $(x, t) \notin E_G$.

Figures 6 illustrates a 1-local complementation of a directed graph seen as an $\mathbb{F}_2^*$-graph.

The 1-local complementation of a directed graph seen as an $\mathbb{F}_2^*$-graph can be different from the one when we consider it as a σ_4-symmetric $\mathbb{F}_4^*$-graph (see Section 3.4). Figures 7 and 8 illustrate this observation. We leave open the question of finding a notion of vertex-minor for directed graphs, that not only lets invariant $\mathbb{F}_4$-rank-width and $\mathbb{F}_2$-bi-rank-width, but also is independent of the representation.

Remark 4.13 *Directed graphs of bounded bi-rank-width are not well-quasi-ordered by the vertex-minor relation. In fact the class $\mathcal{EC}$ of directed even cycles such that each vertex has either in-degree* 2 *or out-degree* 2 *are of bounded bi-rank-width and are not well-quasi-ordered by the vertex-minor relation since*

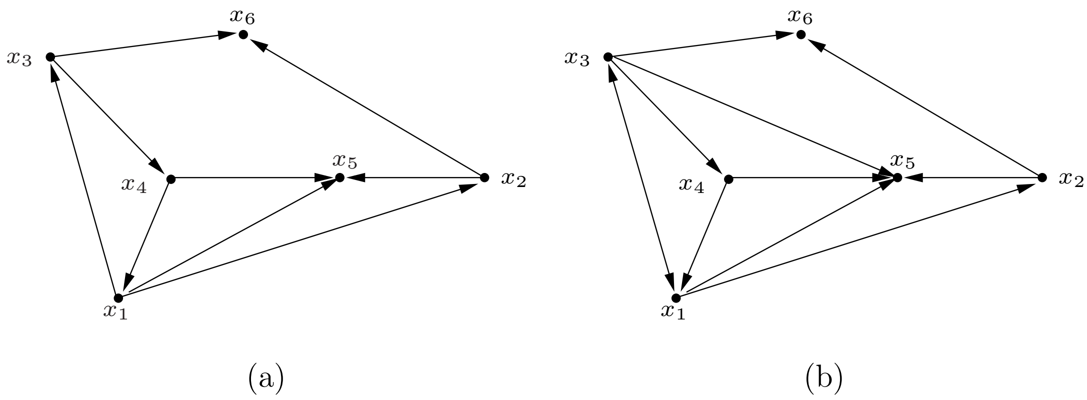

Figure 6. (a) A directed graph seen as an $\mathbb{F}_2^*$-graph. (b) Its 1-local complementation at x_4.

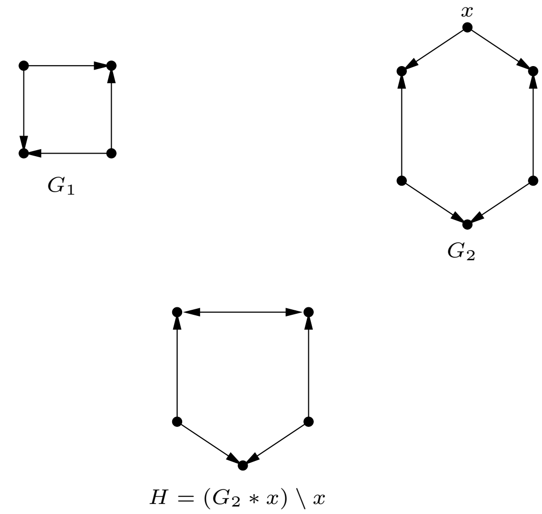

Figure 7. G_1 and G_2 are graphs in $\mathcal{EC}$. Each graph in $\mathcal{EC}$, seen as an $\mathbb{F}_2^*$-graph, is isomorphic to its 1-local complementations. This is not the case if we consider them as a σ_4-symmetric $\mathbb{F}_4^*$-graph. For instance, H is a vertex-minor of G_2 seen as a σ_4-symmetric $\mathbb{F}_4^*$-graph.

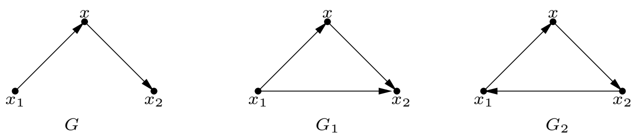

Figure 8. G_1 is a vertex-minor of G seen as an $\mathbb{F}_2^*$-graph and the only one locally equivalent to it, and G_2 is a vertex-minor of G seen as a σ_4-symmetric $\mathbb{F}_4^*$-graph. G_1 is not isomorphic to G_2.

none of them is a vertex-minor of another. In fact each of them is isomorphic to its 1*-local complementations. Figure 7 illustrates such cycles.*

5 Algebraic Graph Operations for $\mathbb{F}$-Rank-Width and $\mathbb{F}$-Bi-Rank-Width

Courcelle and the first author gave in [16] graph operations that characterise exactly the notion of rank-width of undirected graphs. These operations are interesting because they allow to check monadic second-order properties on undirected graph classes of bounded rank-width without using clique-width operations. We give in Section 5.1 graph operations, that generalise the ones in [16] and that characterise exactly $\mathbb{F}$-rank-width. A specialisation that allows to characterise exactly $\mathbb{F}$-bi-rank-width is then presented in Section 5.2. We let $\mathbb{F}$ be a fixed finite field along this section. For a fixed positive integer k, we let $\mathbb{F}^k$ be the set of row vectors of length k. If T is a rooted tree and u a node of T, we let T_u be the sub-tree of T rooted at u.

5.1 Graph Operations Characterising $\mathbb{F}$-Rank-Width

We let $\sigma : \mathbb{F} \to \mathbb{F}$ be a fixed sesqui-morphism. If $u := (u_1, \ldots, u_k) \in \mathbb{F}^k$, we let $\sigma(u)$ be $(\sigma(u_1), \ldots, \sigma(u_k))$. Similarly, if $M = (m_{i,j})$ is a matrix, we let $\sigma(M)$ be the matrix $(\sigma(m_{i,j}))$. In this section we deal with σ-symmetric $\mathbb{F}^*$-graphs.

An $\mathbb{F}^k$-*colouring* of a σ-symmetric $\mathbb{F}^*$-graph G is a mapping $\gamma_G : V_G \to \mathbb{F}^k$ with no constraint on the values of γ_G for adjacent vertices. An $\mathbb{F}^k$-*coloured σ-symmetric $\mathbb{F}^*$-graph* G is a tuple $(V_G, E_G, \ell_G, \gamma_G)$ where (V_G, E_G, ℓ_G) is a σ-symmetric $\mathbb{F}^*$-graph and γ_G is an $\mathbb{F}^k$-colouring of (V_G, E_G, ℓ_G). Notice that an $\mathbb{F}^k$-coloured σ-symmetric $\mathbb{F}^*$-graph has not only its edges coloured with colours from $\mathbb{F}$, but also its vertices with colours from $\mathbb{F}^k$. With an $\mathbb{F}^k$-coloured σ-symmetric $\mathbb{F}^*$-graph G, we associate the $(V_G \times \{1, 2, \ldots, k\})$-matrix Γ_G, the row vectors of which are the vectors $\gamma_G(x)$ in $\mathbb{F}^k$ for x in V_G.

The following is a binary graph operation that combines several operations consisting in adding coloured edges between its disjoint arguments and re-colour them independently.

Definition 5.1 (Bilinear Products) *Let k, ℓ and m be positive integers and let M, N and P be $k \times \ell$, $k \times m$ and $\ell \times m$ matrices, respectively, over $\mathbb{F}$. For an $\mathbb{F}^k$-coloured σ-symmetric $\mathbb{F}^*$-graph G and an $\mathbb{F}^\ell$-coloured σ-symmetric $\mathbb{F}^*$-graph H, we let $G \otimes_{M,N,P} H$ be the $\mathbb{F}^m$-coloured σ-symmetric $\mathbb{F}^*$-graph*

$K := (V_G \cup V_H, E_G \cup E_H \cup E', \ell_K, \gamma_K)$ *where:*

$$E' := \{xy \mid x \in V_G,\ y \in V_H \text{ and } \gamma_G(x) \cdot M \cdot \sigma(\gamma_H(y))^T \neq 0\},$$

$$\ell_K((x,y)) := \begin{cases} \ell_G((x,y)) & \text{if } x, y \in V_G, \\ \ell_H((x,y)) & \text{if } x, y \in V_H, \\ \gamma_G(x) \cdot M \cdot \sigma(\gamma_H(y))^T & \text{if } x \in V_G,\ y \in V_H, \\ \sigma\left(\gamma_G(y) \cdot M \cdot \sigma(\gamma_H(x))^T\right) & \text{if } y \in V_G,\ x \in V_H. \end{cases}$$

$$\gamma_K(x) := \begin{cases} \gamma_G(x) \cdot N & \text{if } x \in V_G, \\ \gamma_H(x) \cdot P & \text{if } x \in V_H. \end{cases}$$

Definition 5.2 (Constants) *For each $u \in \mathbb{F}^k$, we let $\mathbf{u}$ be a constant denoting an $\mathbb{F}^k$-coloured σ-symmetric $\mathbb{F}^*$-graph with exactly one vertex and no edge; this unique vertex is coloured by u.*

We denote by $\mathcal{C}_n^{\mathbb{F}}$ the set $\{\mathbf{u} \mid u \in \mathbb{F}^1 \cup \cdots \cup \mathbb{F}^n\}$. We let $\mathcal{R}_n^{(\mathbb{F},\sigma)}$ be the set of bilinear products $\otimes_{M,N,P}$ where M, N and P are respectively $k \times \ell$, $k \times m$ and $\ell \times m$ matrices for $k, \ell, m \leq n$. Each term t in $T(\mathcal{R}_n^{(\mathbb{F},\sigma)}, \mathcal{C}_n^{\mathbb{F}})$ defines, up to isomorphism, a σ-symmetric $\mathbb{F}^*$-graph $val(t)$. We write by abuse of notation $G = val(t)$ to say that G is isomorphic to $val(t)$.

Similarly to terms in $T(\mathcal{F}_k^{\mathbb{F}}, \mathcal{C}_k)$, if $G = val(t)$ for some term t in $T(\mathcal{R}_n^{(\mathbb{F},\sigma)}, \mathcal{C}_n^{\mathbb{F}})$, then there exists a bijection between V_G and leaves in $_u(Synt(t))$ that we still denote by $\mathcal{L}_t$.

One easily verifies that the operations $\otimes_{M,N,P}$ can be defined in terms of the disjoint union and quantifier-free operations. The following is thus a corollary of results in [11,17].

Theorem 5.3 *Let k be a fixed integer. For each monadic second-order property φ, there exists an algorithm that checks for every term $t \in T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$, in time $O(f(k) \cdot |t|)$, for some function f, if the σ-symmetric $\mathbb{F}^*$-graph defined by this term, up to isomorphism, satisfies φ.*

The principal result of this section is the following.

Theorem 5.4 *Let G be a σ-symmetric $\mathbb{F}^*$-graph.*

(1) If $G = val(t)$ for some term t in $T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$, then $(_u\mathcal{S}(t), \mathcal{L}_t)$ is a layout of V_G of $\mathrm{cutrk}_G^{\mathbb{F}}$-width at most k.

(2) Any layout $(T, \mathcal{L})$ of V_G, of $\mathrm{cutrk}_G^{\mathbb{F}}$-width at most k, can be transformed, in time $O(k^2 \cdot |V_G|^2)$, into a term t in $T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$ such that $G = val(t)$ and $(_u\mathcal{S}(t), \mathcal{L}_t) = (T, \mathcal{L})$.

In other words, G has $\mathbb{F}$-rank-width at most k if and only if it is isomorphic to $val(t)$ for some term t in $T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$.

As a corollary of Theorem 5.4, we get the following.

Theorem 5.5 ([16]) *Let $\sigma_1 : \mathbb{F}_2 \to \mathbb{F}_2$ be the identity automorphism. An undirected graph has rank-width at most k if and only if it is isomorphic to $val(t)$ for some term t in $T(\mathcal{R}_k^{(\mathbb{F}_2,\sigma_1)}, \mathcal{C}_k^{\mathbb{F}_2})$.*

The rest of the section is devoted to the proof of Theorem 5.4. We refer to Section 2 for the definition of a context. We recall however that we denote by Id the particular context u.

Lemma 5.6 *If $K = G \otimes_{M,N,P} H$, then $M_K[V_G, V_H] = \Gamma_G \cdot M \cdot \sigma(\Gamma_H)^T$ and $\Gamma_K = \begin{pmatrix} \Gamma_G \cdot N \\ \Gamma_H \cdot P \end{pmatrix}$. Moreover, K is isomorphic to $H \otimes_{M',P,N} G$ where $M' = \frac{1}{\sigma(1)^2} \cdot \sigma(M)^T$.*

Lemma 5.7 *Let $t = c \bullet t'$ where $t' \in T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$ and $c \in Cxt(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}}) \setminus \{\mathit{Id}\}$. If $G = val(t)$ and $H = val(t')$, then*

$$\begin{aligned} M_G[V_H, V_G \setminus V_H] &= \Gamma_H \cdot B, \\ \Gamma_{G[V_H]} &= \Gamma_H \cdot C. \end{aligned}$$

for some matrices B and C.

Proof. We prove it by induction on the structure of c. We identify two cases (the two other cases are similar by symmetry and Lemma 5.6).

Case 1 $c = \mathit{Id} \otimes_{M,N,P} t''$, *i.e.*, $t'' \in T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$. Then, $G = H \otimes_{M,N,P} K$ where $K = val(t'')$. By Lemma 5.6,

$$\begin{aligned} M_G[V_H, V_G \setminus V_H] &= \Gamma_H \cdot M \cdot \sigma(\Gamma_K)^T, \\ \Gamma_{G[H]} &= \Gamma_H \cdot N. \end{aligned}$$

We let $B = M \cdot \sigma(\Gamma_K)^T$ and $C = N$.

Case 2 $c = c' \otimes_{M,N,P} t''$, *i.e.*, $t'' \in T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$. We let $G' = val(c' \bullet t')$ and $K = val(t'')$. Hence, $G = G' \otimes_{M,N,P} K$. By definition and Lemma 5.6,

$$M_G[V_H, V_G \setminus V_H] = \begin{pmatrix} M_{G'}[V_H, V_{G'} \setminus V_H] & (\Gamma_{G'} \cdot M \cdot \sigma(\Gamma_K)^T)[V_H, V_K] \end{pmatrix}$$

By inductive hypothesis, $M_{G'}[V_H, V_{G'} \setminus V_H] = \Gamma_H \cdot B'$ for some matrix B'. Moreover, $(\Gamma_{G'} \cdot M \cdot \sigma(\Gamma_K)^T)[V_H, V_K] = \Gamma_{G'[V_H]} \cdot M \cdot \sigma(\Gamma_K)^T$. But by inductive hypothesis, $\Gamma_{G'[V_H]} = \Gamma_H \cdot C'$ for some matrix C'. Then, $M_G[V_H, V_G \setminus V_H] = \Gamma_H \cdot B$ where $B = \begin{pmatrix} B' & C' \cdot M \cdot \sigma(\Gamma_K)^T \end{pmatrix}$. Moreover, $\Gamma_{G[H]} = \Gamma_H \cdot C$ where $C = C' \cdot N$ since $\Gamma_{G[V_H]} = \Gamma_{G'[V_H]} \cdot N$. $\square$

Let V be a subset of V_G. A subset X of V is called a *vertex-basis of* $M_G[V, V_G \setminus V]$ if either $M_G[V, V_G \setminus V]$ is a zero-matrix and $|X| = 1$, or $\{M_G[x, V_G \setminus V] \mid x \in X\}$ is linearly independent and generates the row space of $M_G[V, V_G \setminus V]$.

Proof of Theorem 5.4. (1) Assume $G = val(t)$ for some term t in $T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$. Hence, $({}_u\mathcal{S}(t), \mathcal{L}_t)$ is a layout of V_G. In order to prove that the $\text{cutrk}_G^{\mathbb{F}}$-width of $({}_u\mathcal{S}(t), \mathcal{L}_t)$ is at most k, it is sufficient to prove that for each subgraph H of G associated to a sub-term t' of t, $\text{cutrk}_G^{\mathbb{F}}(V_H) \leq k$. However, we have proved in Lemma 5.7 that $M_G[V_H, V_G \setminus V_H] = \Gamma_H \cdot B$ for some matrix B. And since each such H is $\mathbb{F}^\ell$-coloured for some $\ell \leq k$, we are done.

(2) Let $(T, \mathcal{L})$ be a layout of V_G of $\text{cutrk}_G^{\mathbb{F}}$-width at most k. For each node u of $\overrightarrow{T}$, we let G_u be the subgraph of G induced by the vertices that are in correspondence with the leaves of the sub-tree of $\overrightarrow{T}$ rooted at u. It is clear that $G_r = G$ (r is the root of $\overrightarrow{T}$). We let $r(u)$ be $\max\{1, \text{cutrk}_G^{\mathbb{F}}(V_{G_u})\}$.

We will construct inductively, bottom-up, for each node u of $\overrightarrow{T}$ a term t_u in $T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$ and a vertex-basis X_u of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ such that:

(i) ${}_u(Synt(t_u)) = \overrightarrow{T}_u$ and $val(t_u)$ is an $\mathbb{F}^{r(u)}$-coloured σ-symmetric $\mathbb{F}^*$-graph isomorphic to G_u,
(ii) $M_G[V_{G_u}, V_G \setminus V_{G_u}] = \Gamma_{val(t_u)} \cdot M_G[X_u, V_G \setminus V_{G_u}]$.

If we prove the two statements (i) and (ii), we have proved that $val(t_r)$ is isomorphic to G and that $({}_u\mathcal{S}(t), \mathcal{L}_t) = (T, \mathcal{L})$.

If u is a leaf, then we let $X_u = \{\mathcal{L}^{-1}(u)\}$. If u is a leaf and $\mathcal{L}^{-1}(u)$ has a neighbour in V_G, we let $t_u = \mathbf{1}$. If u is a leaf and $\mathcal{L}^{-1}(u)$ has not a neighbour in V_G, we let $t_u = \mathbf{0}$. It is clear that t_u and X_u verify statements (i) and (ii) above.

We now show how to construct t_u and X_u, for each internal node u, from t_{u_i} and X_{u_i}, for $i = 1, 2$, where u_1 and u_2 are the children of u in $\overrightarrow{T}$. We let $r(u_1) := h$ and $r(u_2) := \ell$. We let $X_{u_1} := \{x_1, \ldots, x_h\}$ and $X_{u_2} := \{y_1, \ldots, y_\ell\}$. We let $M := \frac{1}{\sigma(1)} \cdot M_G[X_{u_1}, X_{u_2}]$, and $H := val(t_{u_1})$ and $K := val(t_{u_2})$.

Claim 5.8 $M_G[V_{G_{u_1}}, V_{G_{u_2}}] = \Gamma_H \cdot M \cdot \sigma(\Gamma_K)^T$.

Proof of Claim 5.8. Let $x \in V_{G_{u_1}}$ and $y \in V_{G_{u_2}}$. By inductive Properties (i) and (ii), $M_G[x, V_G \setminus V_{G_{u_1}}] = \gamma_H(x) \cdot M_G[X_{u_1}, V_G \setminus V_{G_{u_1}}]$ and $M_G[y, V_G \setminus V_{G_{u_2}}] =$

$\gamma_K(y) \cdot M_G[X_{u_2}, V_G \setminus V_{G_{u_2}}]$. Hence, $\gamma_H(x) \cdot M = \frac{1}{\sigma(1)} \cdot M_G[x, X_{u_2}]$. Therefore,

$$\begin{aligned}
\gamma_H(x) \cdot M \cdot \sigma(\gamma_K(y))^T &= \frac{1}{\sigma(1)} \cdot M_G[x, X_{u_2}] \cdot \sigma(\gamma_K(y))^T \\
&= \frac{1}{\sigma(1)} \cdot \sigma(M_G[X_{u_2}, x])^T \cdot \sigma(\gamma_K(y))^T \\
&= \frac{1}{\sigma(1)} \cdot \sigma(\gamma_K(y)) \cdot \sigma(M_G[X_{u_2}, x]) \\
&= \sigma(\gamma_K(y) \cdot M_G[X_{u_2}, x]) \\
&= \sigma(M_G[y, x]) = M_G[x, y]. \qquad \square
\end{aligned}$$

It remains now to find a vertex-basis X_u of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ and matrices N and P such that $M_G[V_{G_u}, V_G \setminus V_{G_u}] = \begin{pmatrix} \Gamma_H \cdot N \\ \Gamma_K \cdot P \end{pmatrix} \cdot M_G[X_u, V_G \setminus V_{G_u}]$.

It is straightforward to show that $\{M_G[z, V_G \setminus V_{G_u}] \mid z \in X_{u_1} \cup X_{u_2}\}$ generates the row space of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$. Therefore, we can find a vertex-basis X_u of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ which is a subset $X_{u_1} \cup X_{u_2}$. That means, for each $z \in X_{u_1} \cup X_{u_2}$, there exists a row vector b_z in $\mathbb{F}^{r(u)}$ such that $M_G[z, V_G \setminus V_{G_u}] = b_z \cdot M_G[X_u, V_G \setminus V_{G_u}]$. We let $t_u = t_{u_1} \otimes_{M,N,P} t_{u_2}$ where:

$$N := \begin{pmatrix} b_{x_1}, & \cdots & , b_{x_h} \end{pmatrix}^T \qquad\qquad P := \begin{pmatrix} b_{y_1}, & \cdots & , b_{y_h} \end{pmatrix}^T$$

It is clear that ${}_{u}(Synt(t_u)) = \overrightarrow{T}_u$ and $val(t_u)$ is an $\mathbb{F}^{r(u)}$-coloured σ-symmetric $\mathbb{F}^*$-graph. From Claim 5.8 $val(t_u)$ is isomorphic to G_u. It then remains to show that $\Gamma_H \cdot N \cdot M_G[X_u, V_G \setminus V_{G_u}] = M_G[V_{G_{u_1}}, V_G \setminus V_{G_u}]$ and $\Gamma_K \cdot P \cdot M_G[X_u, V_G \setminus V_{G_u}] = M_G[V_{G_{u_2}}, V_G \setminus V_{G_u}]$. For that it is sufficient to prove, for each y in $V_G \setminus V_{G_u}$, that $M_G[X_{u_1}, y] = N \cdot M_G[X_u, y]$ and $M_G[X_{u_2}, y] = P \cdot M_G[X_u, y]$. But, this is a simple computation by the definitions of N, P and X_u.

We now discuss the time complexity of the construction of t_r. The construction of t_u and X_u for leaves u takes clearly constant time. If u is an internal node, then the construction of the matrix M takes time $O(k^2)$, and if we know the vertex basis X_u and the row vectors b_z for each vertex $z \in X_{u_1} \cup X_{u_2}$, we can construct N and P in time $O(k^2)$. It remains to explain the construction of X_u and the row vectors b_z in $O(k^2 \cdot n')$ with $n' = |V_G \setminus V_{G_u}|$. For that, we transform the $2k \times n'$-matrix $M_G[X_{u_1} \cup X_{u_2}, V_G \setminus V_{G_u}]$ in row echelon form. This can be done in time $O(k^2 \cdot n')$ (see [51]). At the end of this algorithm we can identify X_u (the indexes of the non zero rows) and compute b_z for each $z \in X_{u_1} \cup X_{u_2}$. Since $\overrightarrow{T}$ has $2|V_G| - 1$ nodes and $n' \leq |V_G|$, we are done. $\square$

Graph operations in $\mathcal{R}_n^{(\mathbb{F},\sigma)}$ are specialised in order to characterise exactly $\mathbb{F}$-bi-rank-width. Let k_1 and k_2 be positive integers. An $\mathbb{F}^{k_1,k_2}$-*bicolouring* of an $\mathbb{F}^*$-graph G is a couple of mappings $\gamma_G^+ : V_G \to \mathbb{F}^{k_1}$ and $\gamma_G^- : V_G \to \mathbb{F}^{k_2}$. An $\mathbb{F}^{k_1,k_2}$-*bicoloured* $\mathbb{F}^*$-*graph* is a tuple $(V_G, E_G, \ell_G, \gamma_G^+, \gamma_G^-)$ where (V_G, E_G, ℓ_G) is a $\mathbb{F}^*$-graph and (γ_G^+, γ_G^-) is an $\mathbb{F}^{k_1,k_2}$-bicolouring. With an $\mathbb{F}^{k_1,k_2}$-bicoloured $\mathbb{F}^*$-graph G we associate the $(V_G, \{1,\ldots,k_1\})$ and $(V_G, \{1,\ldots,k_2\})$-matrices Γ_G^+ and Γ_G^-, the row vectors of which are respectively $\gamma_G^+(x)$ and $\gamma_G^-(x)$ for x in V_G.

Definition 5.9 *Let $k_1, k_2, \ell_1, \ell_2, m_1$ and m_2 be positive integers. Let M_1, M_2, N_1, N_2, P_1 and P_2 be respectively $k_1 \times \ell_1$, $k_2 \times \ell_2$, $k_1 \times m_1$, $k_2 \times m_2$, $\ell_1 \times m_1$ and $\ell_2 \times m_2$-matrices. For an $\mathbb{F}^{k_1,k_2}$-bicoloured $\mathbb{F}^*$-graph G and an $\mathbb{F}^{\ell_1,\ell_2}$-bicoloured $\mathbb{F}^*$-graph H, we let $G \otimes_{M_1,M_2,N_1,N_2,P_1,P_2} H$ be the $\mathbb{F}^{m_1,m_2}$-bicoloured $\mathbb{F}^*$-graph $K := (V_G \cup V_H, E_G \cup E_H \cup E_1 \cup E_2, \ell_K, \gamma_K^+, \gamma_k^-)$ where:*

$$E_1 := \{(x,y) \mid x \in V_G,\ y \in V_H \text{ and } \gamma_G^+(x) \cdot M_1 \cdot (\gamma_H^-(y))^T \neq 0\},$$
$$E_2 := \{(y,x) \mid x \in V_G,\ y \in V_H \text{ and } \gamma_G^-(x) \cdot M_2 \cdot (\gamma_H^+(y))^T \neq 0\},$$
$$\ell_K((x,y)) := \begin{cases} \ell_G((x,y)) & \text{if } x,y \in V_G, \\ \ell_H((x,y)) & \text{if } x,y \in V_H, \\ \gamma_G^+(x) \cdot M_1 \cdot (\gamma_H^-(y))^T & \text{if } x \in V_G \text{ and } y \in V_H, \\ \gamma_G^-(y) \cdot M_2 \cdot (\gamma_H^+(x))^T & \text{if } y \in V_G \text{ and } x \in V_H, \end{cases}$$
$$\gamma_K^+(x) := \begin{cases} \gamma_G^+(x) \cdot N_1 & \text{if } x \in V_G, \\ \gamma_H^+(x) \cdot P_1 & \text{if } x \in V_H, \end{cases}$$
$$\gamma_K^-(x) := \begin{cases} \gamma_G^-(x) \cdot N_2 & \text{if } x \in V_G, \\ \gamma_H^-(x) \cdot P_2 & \text{if } x \in V_H. \end{cases}$$

Definition 5.10 *For each pair $(u,v) \in \mathbb{F}^{k_1} \times \mathbb{F}^{k_2}$, we let $\mathbf{u} \cdot \mathbf{v}$ be the constant denoting an $\mathbb{F}^{k_1,k_2}$-bicoloured $\mathbb{F}^*$-graph with exactly one vertex and no edge; this single vertex is coloured by (u,v).*

We let $\mathcal{BC}_n^{\mathbb{F}}$ be the set $\{\mathbf{u} \cdot \mathbf{v} \mid (u,v) \in \mathbb{F}^{k_1} \times \mathbb{F}^{k_2}$ and $k_1 + k_2 \leq n\}$. We denote by $\mathcal{BR}_n^{\mathbb{F}}$ the set of all operations $\otimes_{M_1,M_2,N_1,N_2,P_1,P_2}$ where M_1, M_2, N_1, N_2, P_1 and P_2 are respectively $k_1 \times \ell_1$, $k_2 \times \ell_2$, $k_1 \times m_1$, $k_2 \times m_2$, $\ell_1 \times m_1$ and $\ell_2 \times m_2$-matrices and $k_1 + k_2$, $\ell_1 + \ell_2$ and $m_1 + m_2 \leq n$. Every term t in $T(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}})$ defines, up to isomorphism, an $\mathbb{F}^*$-graph denoted by $val(t)$. If G is isomorphic to $val(t)$, we still denote by $\mathcal{L}_t$ the bijection between V_G and the leaves of $Synt(t)$.

The operations in $\mathcal{BR}_n^{\mathbb{F}}$ can be defined in terms of disjoint union and quantifier-free operations. Therefore, Theorem 5.3 is still true if we replace $\mathcal{R}_n^{(\mathbb{F},\sigma)}$ by

$\mathcal{BR}_n^{\mathbb{F}}$.

Theorem 5.11 *Let G be an $\mathbb{F}^*$-graph.*

(1) If $G = val(t)$ for some term t in $T(\mathcal{BR}_k^{\mathbb{F}}, \mathcal{BC}_k^{\mathbb{F}})$, then $(\mathcal{uS}(t), \mathcal{L}_t)$ is a layout of V_G of $\mathrm{bicutrk}_G^{\mathbb{F}}$-width at most k.
(2) Any layout $(T, \mathcal{L})$ of V_G, of $\mathrm{bicutrk}_G^{\mathbb{F}}$-width at most k, can be transformed, in time $O(k^2 \cdot |V_G|^2)$, into a term t in $T(\mathcal{BR}_k^{\mathbb{F}}, \mathcal{BC}_k^{\mathbb{F}})$ such that $G = val(t)$ and $(\mathcal{uS}(t), \mathcal{L}_t) = (T, \mathcal{L})$.

In other words, G has $\mathbb{F}$-bi-rank-width at most k if and only if it is isomorphic to $val(t)$ for some term t in $T(\mathcal{BR}_k^{\mathbb{F}}, \mathcal{BC}_k^{\mathbb{F}})$.

The proof is similar to the one of Theorem 5.4.

Lemma 5.12 *If $K = G \otimes_{M_1,M_2,N_1,N_2,P_1,P2} H$, then*

$$M_K[V_G, V_H] = \Gamma_G^+ \cdot M_1 \cdot (\Gamma_H^-)^T, \qquad M_K[V_H, V_G] = \left(\Gamma_G^- \cdot M_2 \cdot (\Gamma_H^+)^T\right)^T,$$
$$\Gamma_K^+ = \begin{pmatrix} \Gamma_G^+ \cdot N_1 \\ \Gamma_H^+ \cdot P_1 \end{pmatrix}, \qquad \Gamma_K^- = \begin{pmatrix} \Gamma_G^- \cdot N_2 \\ \Gamma_H^- \cdot P_2 \end{pmatrix}.$$

Moreover, K is isomorphic to $H \otimes_{(M_2)^T,(M_1)^T,P_1,P_2,N_1,N_2} G$.

Lemma 5.13 *Let $t = c \bullet t'$ where $t' \in T(\mathcal{BR}_k^{\mathbb{F}}, \mathcal{BC}_k^{\mathbb{F}})$ and $c \in Cxt(\mathcal{BR}_k^{\mathbb{F}}, \mathcal{BC}_k^{\mathbb{F}}) \setminus \{\mathcal{Id}\}$. If $G = val(t)$ and $H = val(t')$, then $M_G[V_H, V_G \setminus V_H] = \Gamma_H^+ \cdot B_1$ and $M_G[V_G \setminus V_H, V_H] = (\Gamma_H^- \cdot B_2)^T$ for some matrices B_1 and B_2.*

Proof. We prove it by induction on the structure of c, by showing in addition that $\Gamma_{G[V_H]}^+ = \Gamma_H^+ \cdot C_1$ and $\Gamma_{G[V_H]}^- = \Gamma_H^- \cdot C_2$ for some matrices C_1 and C_2. We identify two cases (the two other cases are similar by symmetry and Lemma 5.12).

Case 1 $c = \mathcal{Id} \otimes_{M_1,M_2,N_1,N_2,P_1,P_2} t''$. We let $K = val(t'')$. Then $G = H \otimes_{M_1,M_2,N_1,N_2,P_1,P_2} K$. By Lemma 5.12,

$$M_G[V_H, V_G \setminus V_H] = \Gamma_H^+ \cdot M_1 \cdot (\Gamma_K^-)^T, \quad M_G[V_G \setminus V_H, V_H] = \left(\Gamma_H^- \cdot M_2 \cdot (\Gamma_K^+)^T\right)^T,$$
$$\Gamma_{G[V_H]}^+ = \Gamma_H^+ \cdot N_1, \qquad \Gamma_{G[V_H]}^- = \Gamma_H^- \cdot N_2.$$

We let $B_1 = M_1 \cdot (\Gamma_K^-)^T$, $B_2 = M_2 \cdot (\Gamma_K^+)^T$, $C_1 = N_1$ and $C_2 = N_2$.

Case 2 $c = c' \otimes_{M,M',N,P} t''$ where $c' \in Cxt(\mathcal{BR}_k^{\mathbb{F}}, \mathcal{BC}_k^{\mathbb{F}}) \setminus \{\mathcal{Id}\}$. We let $K = val(t'')$ and $G' = val(c' \bullet t')$. Hence $G = G' \otimes_{M_1,M_2,N_1,N_2,P_1,P_2} K$. By Lemma

5.12,

$$M_G[V_H, V_G \setminus V_H] = \Big(M_{G'}[V_H, V_{G'} \setminus V_H] \quad \Gamma^+_{G'[V_H]} \cdot M_1 \cdot (\Gamma^-_K)^T \Big),$$
$$M_G[V_G \setminus V_H, V_H] = \Big(M_{G'}[V_{G'} \setminus V_H, V_H] \quad \big(\Gamma^-_{G'[V_H]} \cdot M_2 \cdot (\Gamma^+_K)^T\big)^T \Big)$$

By inductive hypothesis, $M_{G'}[V_H, V_{G'} \setminus V_H] = \Gamma^+_H \cdot B_1'$ and $M_{G'}[V_{G'} \setminus V_H, V_H] = (\Gamma^-_H \cdot B_2')^T$. Moreover, $\Gamma^+_{G'[V_H]} = \Gamma^+_H \cdot C_1'$ and $\Gamma^-_{G'[V_H]} = \Gamma^-_H \cdot C_2'$. Therefore, letting

$$B_1 = \Big(B_1' \;\; C_1' \cdot M_1 \cdot (\Gamma^-_K)^T \Big), \qquad B_2 = \Big(B_2' \;\; C_2' \cdot M_2 \cdot (\Gamma^+_K)^T \Big),$$
$$C_1 = C_1' \cdot N_1, \qquad C_2 = C_2' \cdot N_2$$

concludes the proof. □

Proof of Theorem 5.11. (1) Let G be isomorphic to $val(t)$ for some term t in $T(\mathcal{BR}^{\mathbb{F}}_k, \mathcal{BC}^{\mathbb{F}}_k)$. Thus, $({}_u\mathcal{S}(t), \mathcal{L}_t)$ is a layout of V_G. In order to prove that the $\mathrm{bicutrk}^{\mathbb{F}}_G$-width of $({}_u\mathcal{S}(t), \mathcal{L}_t)$ is at most k, it is sufficient to prove that for each subgraph H of G associated to a sub-term t' of t, $\mathrm{bicutrk}^{\mathbb{F}}_G(V_H) \leq k$. But, we have proved in Lemma 5.13 that $M_G[V_H, V_G \setminus V_H] = \Gamma^+_H \cdot B_1$ and $M_G[V_G \setminus V_H, V_H] = \Big(\Gamma^-_H \cdot B_2\Big)^T$ for some matrices B_1 and B_2. And since each such H is $\mathbb{F}^{k_1,k_2}$-coloured where $k_1 + k_2 \leq k$, we are done.

(2) For the second statement, let $(T, \mathcal{L})$ be a layout of V_G of $\mathrm{bicutrk}^{\mathbb{F}}_G$-width at most k. For each node u of $\overrightarrow{T}$, we let G_u be the subgraph of G induced by the vertices that are in correspondence with the leaves of the sub-tree of $\overrightarrow{T}$ rooted at u. We let $r_1(u)$ be $\max\{1, \mathrm{rk}(M_G[V_{G_u}, V_G \setminus V_{G_u}])\}$ and $r_2(u)$ be $\max\{1, \mathrm{rk}(M_G[V_G \setminus V_{G_u}, V_{G_u}])\}$.

We will construct, bottom-up, for each node u of $\overrightarrow{T}$ a term t_u and vertex-bases X^+_u and X^-_u of, respectively, $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ and $(M_G[V_G \setminus V_{G_u}, V_{G_u}])^T$ such that:

(i) ${}_u(Synt(t)) = \overrightarrow{T}_u$ and $val(t_u)$ is an $\mathbb{F}^{r_1(u),r_2(u)}$-bicoloured $\mathbb{F}^*$-graph isomorphic to G_u,
(ii) $M_G[V_{G_u}, V_G \setminus V_{G_u}] = \Gamma^+_{val(t_u)} \cdot M_G[X^+_u, V_G \setminus V_{G_u}]$ and $M_G[V_G \setminus V_{G_u}, V_{G_u}] = M_G[V_G \setminus V_{G_u}, X^-_u] \cdot (\Gamma^-_{val(t_u)})^T$.

One clearly notices that statements (i) and (ii) imply statement (2) we want to prove.

Let u be a leaf. We let $X^+_u := X^-_u := \{\mathcal{L}^{-1}(u)\}$. If $\mathcal{L}^{-1}(u)$ has in-neighbours, we let $c_1 := 1$, otherwise $c_1 := 0$; if it has out-neighbours, we let $c_2 := 1$, otherwise $c_2 := 0$. We let $t_u := \mathbf{c_1} \cdot \mathbf{c_2}$. X_u and t_u verify clearly statements (i) and (ii).

Let us now construct t_u, and X_u^+ and X_u^- for each internal node u. Since u has two children u_1 and u_2, we have already constructed t_{u_i}, $X_{u_i}^+$ and $X_{u_i}^-$, for $i = 1, 2$, verifying the statements (i) and (ii). We let $r_1(u_1) := h$, $r_2(u_1) = h'$, $r_1(u_2) = \ell$ and $r_2(u_2) := \ell'$. We let $X_{u_1}^+ := \{x_{i_1}, \ldots, x_{i_h}\}$, $X_{u_1}^- := \{x_{j_1}, \ldots, x_{j_{h'}}\}$, $X_{u_2}^+ := \{y_{t_1}, \ldots, y_{t_\ell}\}$ and $X_{u_2}^- := \{y_{s_1}, \ldots, y_{s_{\ell'}}\}$. We let $M_1 := M_G[X_{u_1}^+, X_{u_2}^-]$ and $M_2 := (M_G[X_{u_2}^+, X_{u_1}^-])^T$, and $H = val(t_{u_1})$ and $K = val(t_{u_2})$.

Claim 5.14 $M_G[V_{G_{u_1}}, V_{G_{u_2}}] = \Gamma_H^+ \cdot M_1 \cdot (\Gamma_K^-)^T$ *and* $M_G[V_{G_{u_2}}, V_{G_{u_1}}] = \left(\Gamma_H^- \cdot M_2 \cdot (\Gamma_K^+)^T\right)^T$.

Proof of Claim 5.14. Let $x \in V_{G_{u_1}}$ and $y \in V_{G_{u_2}}$. By inductive hypothesis,

$$\begin{aligned}
M_G[x, V_G \setminus V_{G_{u_1}}] &= \gamma_H^+(x) \cdot M_G[X_{u_1}^+, V_G \setminus V_{G_{u_1}}],\\
M_G[V_G \setminus V_{G_{u_1}}, x] &= M_G[V_G \setminus V_{G_{u_1}}, X_{u_1}^-] \cdot (\gamma_H^-(x))^T\\
M_G[y, V_G \setminus V_{G_{u_2}}] &= \gamma_K^+(y) \cdot M_G[X_{u_2}^+, V_G \setminus V_{G_{u_2}}],\\
M_G[V_G \setminus V_{G_{u_2}}, y] &= M_G[V_G \setminus V_{G_{u_2}}, X_{u_2}^-] \cdot (\gamma_K^-(y))^T.
\end{aligned}$$

Hence,

$$\gamma_H^+(x) \cdot M_1 \cdot (\gamma_K^-(y))^T = M_G[x, X_{u_2}^-] \cdot (\gamma_K^-(y))^T = M_G[x, y],$$

and

$$\begin{aligned}
\gamma_H^-(x) \cdot M_2 \cdot (\gamma_K^+(y))^T &= \gamma_H^-(x) \cdot (M_G[X_{u_2}^+, X_{u_1}^-])^T \cdot (\gamma_K^+(y))^T\\
&= (M_G[X_{u_2}^+, X_{u_1}^-] \cdot (\gamma_H^-(x))^T)^T \cdot (\gamma_K^+(y))^T\\
&= (M_G[X_{u_2}^+, x])^T \cdot (\gamma_K^+(y))^T\\
&= \left(\gamma_K^+(y) \cdot M_G[X_{u_2}^+, x]\right)^T = M_G[y, x]. \qquad \square
\end{aligned}$$

It remains now to find vertex-bases X_u^+ and X_u^- of, respectively, $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ and $(M_G[V_G \setminus V_{G_u}, V_{G_u}])^T$, and matrices N_1, N_2, P_1 and P_2 such that

$$\begin{aligned}
M_G[V_{G_u}, V_G \setminus V_{G_u}] &= \begin{pmatrix} \Gamma_H^+ \cdot N_1 \\ \Gamma_K^+ \cdot P_1 \end{pmatrix} \cdot M_G[X_u^+, V_G \setminus V_{G_u}], \text{ and}\\
M_G[V_G \setminus V_{G_u}, V_{G_u}] &= M_G[V_G \setminus V_{G_u}, X_u^-] \cdot \begin{pmatrix} \Gamma_H^- \cdot N_2 \\ \Gamma_K^- \cdot P_2 \end{pmatrix}^T.
\end{aligned}$$

It is straightforward to show that $\{M_G[z, V_G \setminus V_{G_u}] \mid z \in X_{u_1}^+ \cup X_{u_2}^+\}$ generates the row space of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$. Similarly, $\{(M_G[V_G \setminus V_{G_u}, z])^T \mid$

$z \in X_{u_1}^- \cup X_{u_2}^-\}$ generates the row space of $(M_G[V_G \setminus V_{G_u}, V_{G_u}])^T$. Therefore, we can find vertex-bases $X_u^+ \subseteq X_{u_1}^+ \cup X_{u_2}^+$ and $X_u^- \subseteq X_{u_1}^- \cup X_{u_2}^-$ of, respectively, $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ and $(M_G[V_G \setminus V_{G_u}, V_{G_u}])^T$. That means, for each $z \in X_{u_1}^+ \cup X_{u_2}^+$, there exists a row vector b_z such that $M_G[z, V_G \setminus V_{G_u}] = b_z \cdot M_G[X_u^+, V_G \setminus V_{G_u}]$. Similarly, for each $z' \in X_{u_1}^- \cup X_{u_2}^-$, there exists a row vector b'_z such that $M_G[V_G \setminus V_{G_u}, z] = b'_z \cdot M_G[V_G \setminus V_{G_u}, X_u^-]$. We let $t_u = t_{u_1} \otimes_{M_1,M_2,N_1,N_2,P_1,P_2} t_{u_2}$ where:

$$N_1 := \left(b_{x_{i_1}}, \cdots, b_{x_{i_h}}\right)^T \qquad P_1 := \left(b_{y_{t_1}}, \cdots, b_{y_{t_\ell}}\right)^T$$
$$N_2 := \left(b'_{x_{j_1}}, \cdots, b'_{x_{j_{h'}}}\right)^T \qquad P_2 := \left(b'_{y_{s_1}}, \cdots, b'_{y_{s_{\ell'}}}\right)^T$$

We have clearly ${}_u(Synt(t)) = \overrightarrow{T}_u$ and $val(t_u)$ is an $\mathbb{F}^{r_1(u),r_2(u)}$-bicoloured $\mathbb{F}^*$-graph. From Claim 5.14 $val(t_u)$ is isomorphic to G_u. Hence, we need just show that $\Gamma_H^+ \cdot N_1 \cdot M_G[X_u^+, V_G \setminus V_{G_u}] = M_G[V_{G_{u_1}}, V_G \setminus V_{G_u}]$ and $\Gamma_K^+ \cdot P_1 \cdot M_G[X_u^+, V_G \setminus V_{G_u}] = M_G[V_{G_{u_2}}, V_G \setminus V_{G_u}]$, and $M_G[V_G \setminus V_{G_u}, X_u^-] \cdot (\Gamma_H^- \cdot N_2)^T = M_G[V_G \setminus V_{G_u}, V_{G_{u_1}}]$ and $M_G[V_G \setminus V_{G_u}, X_u^-] \cdot (\Gamma_K^- \cdot P_2)^T = M_G[V_G \setminus V_{G_u}, V_{G_{u_2}}]$. But, this is an easy computation by the definitions of N_1, N_2, P_1 and P_2, and X_u^+ and X_u^-.

As in the proof of Theorem 5.4(2), the time complexity of the construction of t_u, for each node u of $\overrightarrow{T}$, is dominated by the construction of the vertex-bases X_u^+ and X_u^-. But again, this can be done in time $O(k^2 \cdot n')$ where $n' = |V_G \setminus V_{G_u}|$ since it is enough to transform each matrix $M_G[X_{u_1}^+ \cup X_{u_2}^+, V_G \setminus V_{G_u}]$ and $(M_G[V_G \setminus V_{G_u}, X_{u_1}^- \cup X_{u_2}^-])^T$ in row echelon form. The fact that $\overrightarrow{T}$ has $2|V_G| - 1$ nodes concludes the proof. □

6 Conclusion

Unlike clique-width, there is not a unique way to define a notion of rank-width for edge-coloured graphs. Based on the works by Kanté, we have introduced two notions of rank-width, namely $\mathbb{F}$-rank-width and $\mathbb{F}$-bi-rank-width, for $\mathbb{F}^*$-graphs and explain how to use them to handle edge-coloured graphs with edge colours from a finite set C. If the specialisation of $\mathbb{F}$-bi-rank-width to directed graphs called bi-rank-width has during the last years raised many interests, due to the works by Hliněný et al. [29,31,32,33], it suffers from lack of a structure theorem. On the other side, the $\mathbb{F}$-rank-width shares many structural properties with the rank-width of undirected graphs. Indeed, this paper combined with [50] unifies many results concerning undirected graphs of bounded rank-width and allows their generalisation to directed graphs, and more generally to $\mathbb{F}^*$-graphs of bounded $\mathbb{F}$-rank-width. This a first step towards a structure theory for directed graphs and $\mathbb{F}^*$-graphs like the one derived from the Graph Minors Project. Moreover, the graph operations given in Section 5 allow to apply in a straightforward way the tools by Hliněný et al. [29,31,32,33] to $\mathbb{F}$-rank-width and therefore to $\mathbb{F}_4$-rank-width of directed graphs.

We now discuss about another notion of rank-width for edge-coloured graphs that generalises bi-rank-width. We will imitate what is done in [16, Section 5]. Let G be an edge-coloured graph with colours from a finite set C. For each colour a, we let G_a be the directed graph obtained from G by keeping only edges coloured with a. Then, we define the symmetric and submodular function $bcutrk_G : 2^{V_G} \to \mathbb{N}$ with $bcutrk_G(X) := \sum_{a\in C} \mathrm{bicutrk}^{\mathbb{F}_2}_{G_a}(X)$. Therefore, the *b-rank-width* of G, $bisrwd(G)$, is defined as the $bcutrk_G$-width of V_G. This notion of b-rank-width will still be equivalent to clique-width and we can still adapt the algorithm by Hliněný and Oum [41] to derive a cubic-time recognition algorithm for graphs of b-rank-width at most k, for fixed k. By adapting the algebraic operations in Section 5.2, we will be able to find an algebraic characterisation. Moreover, one easily verifies that if H_a is a 1-local complementation of G_a, then $bisrwd(G') = bisrwd(G)$, where G' is obtained from G by replacing G_a by H_a, for $a \in C$. Hence, one can define a notion of vertex-minor such that the b-rank-width is monotone with respect to it. However, as for $\mathbb{F}$-bi-rank-width, the tools used in Section 3.2 do not seem to be adaptable for obtaining a characterisation like the one in Theorem 2.5. We leave as an open question the quest for such characterisations for b-rank-width and $\mathbb{F}$-bi-rank-width.

We conclude now this paper by stating some of the many open questions

(1) Oum [56] conjectured that if a class of undirected graphs exclude a bipartite circle graph as pivot-minor, then it has bounded rank-width. A similar conjecture can be done for σ-symmetric $\mathbb{F}^*$-graphs.
(2) Kanté has proved in [48] that the pivot-minor relation is a well-quasi-order in σ-symmetric $\mathbb{F}^*$-graph classes of bounded $\mathbb{F}$-rank-width and has related the $\mathbb{F}$-rank-width to the branch-width of $\mathbb{F}$-representable matroids. Is it true that the pivot-minor relation is a well-quasi-order on σ-symmetric $\mathbb{F}^*$-graphs? A positive answer would imply a similar conjecture for $\mathbb{F}$-representable matroids.
(3) It is still open whether we can check if a fixed σ-symmetric $\mathbb{F}^*$-graph H is a is a pivot-minor of a given σ-symmetric $\mathbb{F}^*$-graph G. Courcelle and Oum [20] have proved that this problem is polynomial in undirected graph classes of bounded rank-width. Results from [48] imply also that it is polynomial in σ-symmetric $\mathbb{F}^*$-graph classes of bounded $\mathbb{F}$-rank-width.
(4) Recently, some authors investigated the clique-width of multigraphs [15] or weighted graphs [26]. These graphs can be seen as $\mathbb{N}$-graphs. It is easy to verify that the rank-width is not equivalent to the clique-width when C is infinite. It would be interesting to investigate the rank-width over an

infinite field $\mathbb{G}$, and in particular its algorithmic aspects: the recognition of $\mathbb{G}^*$-graphs of bounded rank-width, and the property checking on $\mathbb{G}^*$-graphs of bounded rank-width.

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A Proofs of Propositions 3.12 and 4.4

If $R \subseteq \{1, \ldots, k\} \times \{1, \ldots, k\} \times \mathbb{F}$, we let $\bigcirc_R$ be the composition of the functions $add_{i,j}^a$ with $(i, j, a) \in R$. This notation is non ambiguous because $add_{i,j}^a \circ add_{k,l}^b = add_{k,l}^b \circ add_{i,j}^a$.

Proof of Proposition 3.12. (1) Assume $G = val(t)$ for some term t in $T(\mathcal{F}_k^{\mathbb{F}}, \mathcal{C}_k)$. In order to prove that $(\mathcal{uS}(red(t)), \mathcal{L}_t)$ is a layout of V_G of $\mathrm{cutrk}_G^{\mathbb{F}}$-width at most k, it is enough to prove that for every subgraph H of G that is a value of a sub-term t' of t, $\mathrm{cutrk}_G^{\mathbb{F}}(V_H) \leq k$. But, by the definition of operations in $\mathcal{F}_k^{\mathbb{F}}$, the sub-matrix $M_G[V_H, V_G \setminus V_H]$ has at most k distinct rows. Thus, $\mathrm{cutrk}_G^{\mathbb{F}}(V_H) = \mathrm{rk}(M_G[V_H, V_G \setminus V_H]) \leq k$.

(2) Assume $(T, \mathcal{L})$ is a layout of V_G of $\mathrm{cutrk}_G^{\mathbb{F}}$-width k. Then, by Theorem 5.4 we can construct in time $O(k^2 \cdot |V_G|^2)$ a term t in $T(\mathcal{R}_k^{(\mathbb{F},\sigma)}, \mathcal{C}_k^{\mathbb{F}})$ such that $G = val(t)$ and $(T, \mathcal{L}) = (\mathcal{uS}(t), \mathcal{L}_t)$. We will construct inductively, on the size

of t, a term t' in $T(\mathcal{F}^{\mathbb{F}}_{k'}, \mathcal{C}_{k'})$ with $k' \leq 2 \cdot q^k - 1$ such $G = val(t')$. We let $\beta : \mathbb{F}^k \to \{1, \ldots, q^k\}$ be a bijective function that enumerates the set of vectors in $\mathbb{F}^k$ with $\beta(O_{1,k}) = 1$. We let $\{2', \ldots, (q^k)'\}$ be a disjoint copy of the set $\{2, \ldots, q^k\}$. If $t = \mathbf{u}$, then we let $t' := \beta(\mathbf{u})$. Suppose then that $t = t_1 \otimes_{M,N,P} t_2$. Then, we let

$$t' := relab_{g'}(relab_g((\bigcirc_{R'})((\bigcirc_R)(t'_1 \oplus relab_h(t'_2)))))$$

where

- $R := \{(\beta(u), \beta(v)', \lambda) \mid u \cdot M \cdot \sigma(v)^T = \lambda\}$,
- $R' := \{(\beta(v)', \beta(u), \sigma(\lambda)) \mid u \cdot M \cdot \sigma(v)^T = \lambda\}$,
- $h : \{1, \ldots, q^k\} \to \{1\} \cup \{2', \ldots, (q^k)'\}$ is such that $h(1) = 1$ and $h(i) := i'$,
- $g : \{1, \ldots, q^k\} \to \{1, \ldots, q^k\}$ is such that $g(i) := \beta(\beta^{-1}(i) \cdot N)$,
- $g' : \{1\} \cup \{2', \cdots, (q^k)'\} \to \{1, \ldots, q^k\}$ is such that $g'(1) := 1$ and $g'(i') := \beta(\beta^{-1}(i) \cdot P)$.

It is a straightforward induction to check that $G = val(t')$ and that $(T, \mathcal{L}) = (\mathcal{uS}(red(t)), \mathcal{L}_t)$. $\square$

Proof of Proposition 4.4. (1) Assume $G = val(t)$ for some term t in $T(\mathcal{F}^{\mathbb{F}}_k, \mathcal{C}_k)$. In order to prove that $(\mathcal{uS}(red(t)), \mathcal{L}_t)$ is a layout of V_G of $\text{bicutrk}^{\mathbb{F}}_G$-width at most k, it is enough to prove that for every subgraph H of G that is a value of a sub-term t' of t, $\text{bicutrk}^{\mathbb{F}}_G(V_H) \leq 2k$. But, by the definition of operations in $\mathcal{F}^{\mathbb{F}}_k$, the sub-matrices $M_G[V_H, V_G \setminus V_H]$ and $M_G[V_G \setminus V_H, V_H]$ have at most k distinct rows. Thus, $\text{bicutrk}^{\mathbb{F}}_G(V_H) = \text{rk}(M_G[V_H, V_G \setminus V_H]) + \text{rk}(M_G[V_G \setminus V_H, V_H]) \leq 2k$.

(2) Assume $(T, \mathcal{L})$ is a layout of V_G of $\text{bicutrk}^{\mathbb{F}}_G$-width k. Then, by Theorem 5.4 we can construct in time $O(k^2 \cdot |V_G|^2)$ a term t in $T(\mathcal{BR}^{\mathbb{F}}_k, \mathcal{BC}^{\mathbb{F}}_k)$ such that $G = val(t)$ and $(T, \mathcal{L}) = (\mathcal{uS}(t), \mathcal{L}_t)$. We will construct inductively, on the size of t, a term t' in $T(\mathcal{F}^{\mathbb{F}}_{k'}, \mathcal{C}_{k'})$ with $k' \leq 2 \cdot q^k - 1$ such $G = val(t')$. For each pair (k_1, k_2) with $k_1 + k_2 \leq k$, we let $\alpha_{k_1,k_2} : \mathbb{F}^{k_1} \times \mathbb{F}^{k_2} \to \{1, \ldots, q^{k_1+k_2}\}$ be a bijective function that enumerates the set of pairs of vectors in $\mathbb{F}^{k_1} \times \mathbb{F}^{k_2}$ with $\alpha_{k_1,k_2}((O_{1,k_1}, O_{1,k_2})) = 1$. We let $\{2', \ldots, (q^k)'\}$ be a disjoint copy of the set $\{2, \ldots, q^k\}$.

If $t = \mathbf{u} \cdot \mathbf{v}$, then we let $t' := \alpha_{\mathbf{1},\mathbf{1}}((\mathbf{u}, \mathbf{v}))$. Suppose now that $t = t_1 \otimes_{M_1,M_2,N_1,N_2,P_1,P_2} t_2$ with $M_1, M_2, N_1, N_2, P_1, P_2$ being respectively $k_1 \times \ell_1$, $k_2 \times \ell_2$, $k_1 \times k'_1$, $k_2 \times k'_2$, $\ell_1 \times k'_1$ and $\ell_2 \times k'_2$-matrices. Then, we let

$$t' := relab_{g'}(relab_g((\bigcirc_{R'})((\bigcirc_R)(t'_1 \oplus relab_h(t'_2)))))$$

where

- $R := \{(i, j', c) \mid i = \alpha_{k_1,k_2}((u_1, u_2)), j = \alpha_{\ell_1,\ell_2}((v_1, v_2))$ and $u_1 \cdot M_1 \cdot v_1^T = c\}$,
- $R' := \{(j', i, c) \mid j = \alpha_{\ell_1,\ell_2}((v_1, v_2)), i = \alpha_{k_1,k_2}((u_1, u_2)),$ and $u_2 \cdot M_2 \cdot v_2^T = c\}$,
- $h : \{1, \ldots, q^k\} \to \{1\} \cup \{2', \ldots, (q^k)'\}$ is such that $h(1) = 1$ and $h(i) := i'$,
- $g : \{1, \ldots, q^k\} \to \{1, \ldots, q^k\}$ is such that if $\alpha^{-1}_{k_1,k_2}(i) = (u_1, u_2)$, then $g(i) := \alpha_{k'_1,k'_2}((u_1 \cdot N_1, u_2 \cdot N_2))$,
- $g' : \{1\} \cup \{2', \cdots, (q^k)'\} \to \{1, \ldots, q^k\}$ is such that $g'(1) := 1$ and if $\alpha^{-1}_{\ell_1,\ell_2}(i) = (v_1, v_2)$, then $g'(i') := \alpha_{k'_1,k'_2}((v_1 \cdot P_1, v_2 \cdot P_2))$.

An easy induction shows that $G = val(t')$ and that $(T, \mathcal{L}) = ({}_u\mathcal{S}(red(t)), \mathcal{L}_t)$. $\square$