Directed Rank-Width and Displit Decomposition*

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Abstract

Rank-width is a graph complexity measure that has many structural properties. It is known that the rank-width of an undirected graph is the maximum over all induced prime graphs with respect to *split decomposition* and an undirected graph has rank-width at most 1 if and only if it is a distance-hereditary graph. We are interested in an extension of these results to directed graphs. We give several characterizations of directed graphs of rank-width 1 and we prove that the rank-width of a directed graph is the maximum over all induced prime graphs with respect to *displit decomposition*, a new decomposition on directed graphs.

1 Introduction

Rank-width [18, 19] is a graph complexity measure introduced by Oum and Seymour in their investigations on recognition algorithms for undirected graphs of *clique-width* [4] at most k, for fixed k. It is known that a class of graphs has bounded rank-width if and only if it has bounded clique-width [19]. However, rank-width has better algorithmic properties: undirected graphs of rank-width at most k can be recognized by a cubic-time algorithm [13] and are characterized by a finite list of undirected graphs to exclude as *vertex-minors* [18].

Another interesting fact is that rank-width is related to *split decomposition*. The split decomposition, introduced by Cunningham [5], is a generalisation of the well known *modular decomposition* [10, 16]. It was defined on graphs (directed or not), but only the undirected case has been widely studied in literature. Split decomposition of undirected graphs can be computed in linear time [7], and can be used in several problems such as: circle graph recognition [9, 21], parity graph recognition [3, 7], and solving some optimization problems [5, 3, 11, 20]. The rank-width of an undirected graph is the maximum over the rank-width of its induced prime graphs with respect to split decomposition. Moreover, undirected graphs of rank-width at most 1 are exactly *distance hereditary* graphs [18], which are graphs that are *completely decomposable* by the split decomposition.

Despite all these positive results of rank-width on clique-width, clique-width has an undeniable advantage on rank-width: it is defined for undirected as well as directed graphs and its definition can be extended to relational structures. In his investigations for an extension of rank-width to relational structures, Kanté defined in [15] a notion of rank-width for directed graphs, called GF(4)-*rank-width*, and that generalized the rank-width of undirected graphs. He, moreover, generalized two results on undirected graphs: directed graphs of GF(4)-rank-width k can be recognized by a cubic-time algorithm and are also characterized by a finite

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list of directed graphs to exclude as vertex-minors. It is thus natural to ask whether we can generalize all the results known for rank-width of undirected graphs.

In this paper, we are interested in a characterization of directed graphs of GF(4)-rank-width 1, similar to the one for undirected graphs. In the literature, there exist several characterizations of undirected graphs of rank-width 1 that we recall in the following.

Theorem 1 ([1, 12, 18])**.** *Let G be a connected undirected graph. Then the following conditions are equivalent:*

1. *G is completely decomposable by the split decomposition (*i.e.*, every node in the split decomposition tree is degenerated).*
2. *G can be obtained from a single vertex by creating twins or adding pendant vertices.*
3. *G has rank-width* 1*.*
4. *For every $W \subseteq V_G$ with $|W| \geq 4$, $G[W]$ has a non trivial split.*
5. *G is (house, hole, domino, gem)-free.*
6. *G is distance hereditary (*i.e.*, for every $x, y \in V_G$, every chordless path between x and y has the same length).*

The main result of this paper is the extension of Theorem 1 to directed graphs (Theorem 21). We will show in particular that directed graphs of GF(4)-rank-width 1 are obtained by orienting in a certain way distance hereditary graphs and are exactly directed graphs completely decomposable by the *displit decomposition*, a new decomposition that generalizes split decomposition. As a consequence we get that the GF(4)-rank-width of a directed graph is the maximum over the GF(4)-rank-width of its induced prime graphs with respect to displit decomposition.

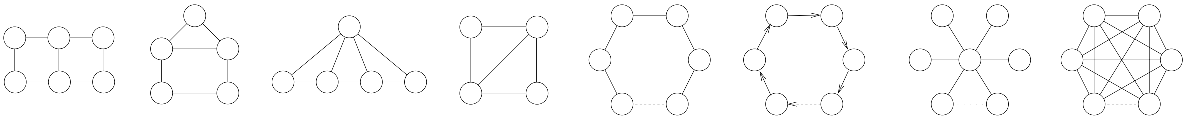
The paper is organized as follows. We give some notations in Section 2 and recall the notion of GF(4)-rank-width in Section 3. In Section 4 we define the notion of displit decomposition and derive some basic properties. In Section 5 we prove our main result. We conclude by a comparison between the split decomposition of directed graphs introduced by Cunningham [5] and the displit decomposition.

2 Preliminaries

When the context is clear we will write u to denote the set $\{u\}$. We denote by 2^V the power-set of a set V and we let be the set of natural integers. A function $f : 2^V \rightarrow$ is said *symmetric* if for any $X \subseteq V$, $f(X) = f(V \backslash X)$; it is said *sub-modular* if for any $X, Y \subseteq V, f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$.

For sets R and C, an (R, C)*-matrix* is a matrix where the rows are indexed by elements in R and columns indexed by elements in C. For an (R, C)-matrix M, if $X \subseteq R$ and $Y \subseteq C$ we let $M[X, Y]$ be the sub-matrix of M where the rows and the columns are indexed by X and Y respectively. If M is an (X, Y)-matrix, M^t denotes the transposed (Y, X)-matrix. A Y*-vector* is an (X, Y)-matrix where $|X| = 1$. The matrix rank function is denoted by rk.

A *directed graph* (or *digraph*) G is a couple (V_G, E_G) where V_G is the set of *vertices* and E_G, the set of *edges*, is a set of ordered pairs (x, y) with $x, y \in V_G$ and $x \neq y$. We consider *undirected graphs* as special cases of directed graphs where $(x, y) \in E_G \Leftrightarrow (y, x) \in E_G$ (edges are denoted xy in this case). Unless otherwise specified, a graph is considered as directed. If G is a digraph and x a vertex of G we denote by $N_G^+(x)$ the set $\{y \mid (x, y) \in E_G\}$, by $N_G^-(x)$ the set $\{y \mid (y, x) \in E_G\}$ and by $N_G(x)$ the set $N_G^+(x) \cup N_G^-(x)$. The *degree* of x is $|N_G(x)|$.



For a graph G we denote by $G[X]$ the sub-graph of G induced by $X \subseteq V_G$ and we let $G-X$ be the sub-graph $G[V_G \backslash X]$. If G is a digraph, let $u(G)$ be the undirected graph obtained from G by forgetting the directions of edges, *i.e.*, $u(G) = (V_G, E_G \cup \{(y,x) \mid (x,y) \in E_G\})$. A digraph G is said *strongly connected* if for every pair $x, y \in V_G$, there is a sequence $x_0 = x, x_1, \ldots x_k = y$ such that $(x_i, x_{i+1}) \in E_G$ for every $i \in \{0, \ldots k-1\}$, and it is said *connected* if $u(G)$ is connected.

A *tree* is an acyclic connected undirected graph. In order to avoid confusions, the vertices of trees will be called *nodes*. The nodes of degree at most 1 in trees are called *leaves* and denoted by L_T. A *sub-cubic* tree is a tree such that the degree of each node is at most 3.

A *layout* of a set V is a pair $(T, \mathcal{L})$ of an undirected tree T and a bijective function $\mathcal{L} : V \to \mathrm{L}_T$. For each edge (u,v) of T we let X_{uv} be the set of leaves reachable from u by a chordless path going through v. Each edge (u,v) of T induces a bipartition $\{X_{uv}, \mathrm{L}_T \backslash X_{uv}\}$ of L_T, and thus a bipartition $\{X^{uv}, V \backslash X^{uv}\} = \{\mathcal{L}^{-1}(X_{uv}), \mathcal{L}^{-1}(\mathrm{L}_T \backslash X_{uv})\}$ of V.

3 Rank-Width of Digraphs

In [15] Kanté defined a notion of rank-width for digraphs named GF(4)-*rank-width*. This notion is based on a function, called *cut-rank function*, that measures how some bipartitions of sets of vertices are connected. The cut-rank function is based on a representation of digraphs by matrices over the field GF(4). We recall that GF(4) has four elements $\{0, 1, \ , \ ^2\}$ with the property that $1 + \ + \ ^2 = 0$ and $\ ^3 = 1$ and is of characteristic 2.

For a digraph G, we denote by M_G the (V_G, V_G)-matrix over GF(4) where:

$$M_G[x,y] = \begin{cases} 0 & \text{if } (x,y) \notin E_G \text{ and } (y,x) \notin E_G \\ & \text{if } (x,y) \in E_G \text{ and } (y,x) \notin E_G \\ ^2 & \text{if } (y,x) \in E_G \text{ and } (x,y) \notin E_G \\ 1 & \text{if } (x,y) \in E_G \text{ and } (y,x) \in E_G. \end{cases}$$

For every subset X of V_G we let $\mathrm{cutrk}_G^{(4)}(X)$, called *cut-rank function*, be $\mathrm{rk}\left(M_G[X, V_G \backslash X]\right)$.

Lemma 2 ([15]). *For every digraph G, the function $\mathrm{cutrk}_G^{(4)}$ is symmetric and sub-modular.*

From now on, we denote by M_G the (V_G, V_G)-matrix over GF(4) of a digraph G.

Definition 3 (GF(4)-Rank-Width). A *sub-cubic layout* of a digraph G is a layout $(T, \mathcal{L})$ of V_G where T is sub-cubic. Let $(T, \mathcal{L})$ be a sub-cubic layout of a digraph G. The GF(4)-*rank-width of an edge* (u,v) *of* T is $\mathrm{cutrk}_G^{(4)}(X^{uv})$. The GF(4)-*rank-width of a sub-cubic layout* $(T, \mathcal{L})$ is the maximum GF(4)-rank-width over all edges of T. The GF(4)-*rank-width of* G, denoted by $\mathrm{rwd}^{(4)}(G)$, is the minimum GF(4)-rank-width over all sub-cubic layouts of G.

Observation 4. *Since* GF(4) *is an extension of* GF(2), *for every undirected graph G we have* $\mathrm{rwd}^{(4)}(G) = \mathrm{rwd}(G)$, *where* $\mathrm{rwd}(G)$ *denotes the rank-width of G.*

4 Displit Decomposition

4.1 Bi-Partitive Families

Two bipartitions $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ of a set V *overlap* if $X_i \cap Y_j \neq \emptyset$ for every $i, j \in \{1, 2\}$.

Definition 5 (Bi-Partitive Family)**.** Let V be a finite set and let $\mathcal{F}$ be a family of bipartitions of V. Then $\mathcal{F}$ is *bi-partitive* if:

- $\{\emptyset, V\} \notin \mathcal{F}$,
- for all $v \in V$, $\{\{v\}, V \backslash \{v\}\} \in \mathcal{F}$ and
- for all $\{X_1, X_2\} \in \mathcal{F}$ and $\{Y_1, Y_2\} \in \mathcal{F}$ such that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ overlap, then $\{X_i \cap Y_j, V \backslash (X_i \cap Y_j)\} \in \mathcal{F}$, for every $i, j \in \{1, 2\}$.

A member $\{X_1, X_2\}$ of a bi-partitive family $\mathcal{F}$ is *trivial* if $|X_1| \leq 1$ or $|X_2| \leq 1$, and is *strong* if there is no $\{Y_1, Y_2\} \in \mathcal{F}$ such that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ overlap.

Bi-partitive families have been studied in [6]. They are very close to partitive families [2, 16] introduced in order to generalize properties of modular decomposition. An example of a bi-partitive family is the family of splits in a strongly connected digraph [5]. The following proposition gives another example of a bi-partitive family.

Proposition 6 (Folklore)**.** *Let* $f : 2^V \rightarrow$ *be a symmetric and sub-modular function and let* $m = \min_{\emptyset \subsetneq X \subsetneq V} f(X)$. *Then the family of minimums* $\mathcal{F} = \{\{X, V \backslash X\} \mid f(X) = m\}$ *is bi-partitive.*

Proof. Let $\{X, V \backslash X\}$ and $\{Y, V \backslash Y\}$ be in $\mathcal{F}$ such that $\{X, V \backslash X\}$ and $\{Y, V \backslash X\}$ overlap. Thus $f(X \cap Y) + f(X \cup Y) \leq 2m$. Since $X \cap Y$ and $X \cup Y$ are non-empty, $f(X \cap Y) \geq m$ and $f(X \cup Y) \geq m$. Thus $f(X \cap Y) = f(X \cup Y) = m$ and $\{X \cap Y, V \backslash (X \cap Y)\}$ and $\{X \cup Y, V \backslash (X \cup Y)\}$ are in $\mathcal{F}$. □

A major result on bi-partitive families, that we recall in the following theorem, is that every bi-partitive family can be represented by a unique labeled tree.

Theorem 7. *Let* $\mathcal{F}$ *be a bi-partitive family on a finite set* V. *Then there is a unique layout* $(T, \mathcal{L})$ *of* V*, called the* representative layout*, such that each internal node of* T *has at least* 3 *neighbors, is marked* `degenerate`*,* `linear` *or* `prime` *and:*

- *For every* $(u, v) \in E_T$*, the bipartition* $\{X^{uv}, V \backslash X^{uv}\}$ *is a strong bipartition in* $\mathcal{F}$ *and there is no other strong bipartition in* $\mathcal{F}$*.*
- *For every internal node* u *of* T*:*
  - *If* u *is* `degenerated`*, then for every* $\emptyset \subsetneq W \subsetneq N_T(u)$*, the bipartition* $\{\cup_{v \in W} X^{uv}, V \backslash \cup_{v \in W} X^{uv}\}$ *is in* $\mathcal{F}$*.*
  - *If* u *is* `linear`*, there is an ordering* $v_1, \ldots, v_k$ *of* $N_T(u)$ *such that for every* $1 \leq i \leq j < k$*, the bipartition* $\{\cup_{\ell \in \{i,\ldots,j\}} X^{uv_\ell}, V \backslash \cup_{\ell \in \{i,\ldots,j\}} X^{uv_\ell}\}$ *is in* $\mathcal{F}$*.*
- *There is no other bipartition in* $\mathcal{F}$*.*

(By convention, an internal node of degree 3 *is always degenerated.)*

Remark 8. Theorem 7 is proved in [6] using a different formalism. It follows also directly from results on partitive families [2, 16] using the simple bijection $f(\mathcal{F}) = \{X \subseteq V \backslash \{v\} \mid \{X, V \backslash X\} \in \mathcal{F}\}$ between bi-partitive families on V and partitive families on $V \backslash \{v\}$, where $v \in V$ is fixed.

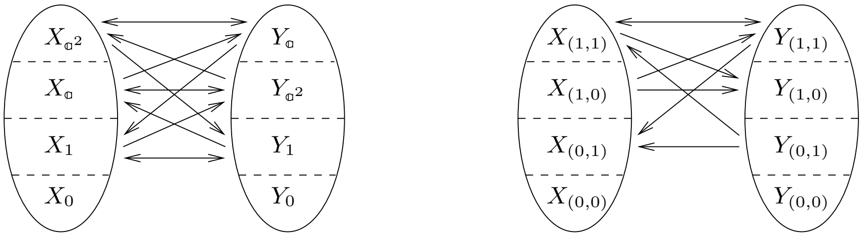


Figure 1: Schematic view of a displit (left) and a Cunningham's split (right).

Remark 9. If $\mathcal{F}$ is a bi-partitive family with the additional property:

- for all $\{X_1, X_2\} \in \mathcal{F}$ and $\{Y_1, Y_2\} \in \mathcal{F}$ such that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ overlap, $\{X_1 \Delta Y_1, X_1 \Delta Y_2\} \in \mathcal{F}$,

then $\mathcal{F}$ is said to be *strongly bi-partitive*. The representative layout of a strongly bi-partitive family has no `linear` node. Cunningham showed that the family of splits in a connected undirected graph is strongly bi-partitive [5]. Another example is the family of bi-joins in an undirected graph [17].

4.2 Displits

Definition 10 (Displit)**.** Let G be a digraph. A bipartition $\{X_1, X_2\}$ of V_G is a *displit* if $X_1 \neq \emptyset$, $X_2 \neq \emptyset$ and $\mathrm{cutrk}_G^{(4)}(X_1) \leq 1$.

Figure 1 shows a comparison between displits and splits on digraphs. A graph G is *degenerated* if every bipartition of V_G is a displit, and G is *prime* if every displit in G is trivial. Finally G is *linear* if there is an ordering $x_1, \ldots, x_n$ of it's vertices such that the family of displits in G is $\{\{\{x_i, \ldots, x_j\}, V_G \backslash \{x_i, \ldots, x_j\}\} \mid 1 \leq i \leq j < n\}$. By convention, a graph with at most 3 vertices is only degenerated.

By Proposition 6, the family of displits in a connected digraph is bi-partitive. By Theorem 7, this family can be represented by a unique labeled layout, that we call *displit decomposition*.

Observation 11. *If $\{X_1, X_2\}$ is a displit in G, then $\{X_1, X_2\}$ is a split in $\mathbb{u}(G)$. The converse is not necessarily true.*

4.3 Quotient Graphs

Let $(T, \mathcal{L})$ be a displit decomposition of a connected digraph G and let u be an internal node of T. We recall that for every node v in $N_T(u)$, X_{uv} is the set of leaves reachable from u by a chordless path going through v. The set $\{X^{uv} = \mathcal{L}^{-1}(X_{uv}) \mid v \in N_T(u)\}$ is a proper partition of V_G, and for every $v \in N_T(u)$, $\{X^{uv}, V_G \backslash X^{uv}\}$ is a displit.

For every $v \in N_T(u)$, we choose a vertex x_v in X^{uv} such that x_v is adjacent to a vertex in $V_G \backslash X^{uv}$. Such a x_v always exists since G is connected. Let $C(u)$ be the graph of vertex set $N_T(u)$ and of edge set $\{(v, w) \mid (x_v, x_w) \in E_G\}$. It is worth noticing that $C(u)$ is isomorphic to $G[\{x_v \mid v \in N_T(u)\}]$, and that $C(u)$ is not unique for a node u. Then we will consider $C(u)$ as an induced sub-graph of G. We now prove or state some technical lemmas.

Lemma 12. *Let $\{X, Y\}$ be a displit in G, and let $x \in X$ and $y \in Y$ such that x is adjacent to y. Let $\{X', Y'\}$ be a bipartition of V_G with $Y' \subseteq Y$. Then $\mathrm{cutrk}_G^{(4)}(Y') = \mathrm{cutrk}_{G'}^{(4)}(Y')$, where $G' = G[Y \cup \{x\}]$.*

Proof. Obviously $\text{cutrk}^{(4)}_{G'}(Y') \leq \text{cutrk}^{(4)}_{G}(Y')$. By definition of displits, there is an X-vector A and a Y-vector B such that $M_G[X,Y] = A^t \cdot B$. Since x is adjacent to a vertex in Y, $A[x] \neq 0$. Thus $M_G[X,Y'] = A[x]^{-1} \cdot A^t \cdot M_G[\{x\},Y']$. Therefore, $\text{rk}(M_G[X' \backslash (X \backslash \{x\}), Y']) = \text{rk}(M_G[X',Y'])$ since all rows in $M_G[X,Y']$ are generated by the row $M_G[\{x\},Y']$. □

Lemma 13. *Let $(T,\mathcal{L})$ be a displit decomposition of a digraph G and let u be a node of T. If u is prime (resp. degenerated, linear), then $C(u)$ is prime (resp. degenerated, linear).*

Proof. Let $\{X,Y\}$ be a bi-partition of $V_{C(u)}$, let $X' = \cup_{v\in X} X^{uv}$ and let $Y' = V_G \backslash X'$. We show that $\{X,Y\}$ is a displit in $C(u)$ if and only if $\{X',Y'\}$ is a displit in G. Trivially, if $\{X',Y'\}$ is a displit in G, then $\{X,Y\}$ is a displit in $C(u)$.

Now suppose that $\{X,Y\}$ is a displit in $C(u)$. $\{X',Y'\}$ does not overlap $\{X^{uv}, V_G \backslash X^{uv}\}$ for every $v \in N_T(u)$. We apply $|N_T(u)|$ times Lemma 12, for all $\{X^{uv}, V_G\backslash X^{uv}\}$. Thus $\{X',Y'\}$ is a displit if and only if $\{X,Y\}$ is a displit. □

The following lemmas give characterization of degenerate and linear graphs. (Proofs are given in Appendix.)

Lemma 14. *If G is degenerated with at least 4 vertices, then either $u(G)$ is a star, or G is C_3' where each of the 3 vertices is substituted by a complete graph (maybe with 0 vertex).*

Lemma 15. *If G is linear and has at least 4 vertices, then there is an ordering $(x_1,\ldots,x_n)$ of vertices of V_G, and a function $f : V_G \to \{0,1,2\}$ such that for all $j > i$:*

- *$(x_i,x_j) \in E_G$ if $f(x_i) \equiv f(x_j) \pmod 3$ or $f(x_i) \equiv f(x_j)+1 \pmod 3$,*
- *$(x_j,x_i) \in E_G$ if $f(x_i) \equiv f(x_j)-1 \pmod 3$ or $f(x_i) \equiv f(x_j)+1 \pmod 3$,*
- *there are no other edges in the graph.*

Theorem 16. *Let G be a connected digraph with at least 3 vertices, and let $(T,\mathcal{L})$ be its displit decomposition. Then $\text{rwd}^{(4)}(G) = \max\{\text{rwd}^{(4)}(C(u)) \mid u \in V_T \backslash L_T\}$.*

Proof. Let $m = \max\{\text{rwd}^{(4)}(C(u)) \mid u \in V_T\backslash L_T\}$. Obviously $m \leq \text{rwd}^{(4)}(G)$ (since $C(u)$ is an induced sub-graph of G). For every $u \in V_T \backslash L_T$, let $(T_u, \mathcal{L}_u)$ be a sub-cubic layout of $C(u)$ of GF(4)-rank-width at most m. We suppose w.l.o.g. that the T_u are pairwise disjoint. We construct a sub-cubic layout $(T',\mathcal{L}')$ of G of GF(4)-rank-width at most m. Let T' be the union of all T_u (for $u \in V_T\backslash L_T$), after the identification of the vertices u in T_v and v in T_u for every $(u,v) \in E_{T-L_T}$, and after contraction of every vertex of degree 2. For all $x \in V_G$, let $\mathcal{L}'(x) = \mathcal{L}_u(\mathcal{L}(x))$ where $\{u\} = N_T(\mathcal{L}(x))$.

It is not hard to see that $(T',\mathcal{L}')$ is a sub-cubic layout of G. Moreover, by Lemma 12, in T' every edge has GF(4)-rank-width at most m. □

4.4 Decomposition Algorithm

It is known that the split decomposition of an undirected graph can be computed in linear time [7], and the split decomposition of a digraph in time $O(m\log(n))$ [14]. We present here a simple $O(nm)$ algorithm to compute the displit decomposition of a digraph. This algorithm is a simple adaptation of [9]. Due to space limitation, we present only the main lines, stated in the following two lemmas without proofs.

Lemma 17. *Let x and y be two vertices of a connected digraph G. We can compute in time $O(n+m)$ a non trivial displit $\{X,Y\}$ such that $x \in X$ and $y \in Y$ (if it exists).*

Lemma 18. *Given a digraph G, we can compute in time $O(nm)$ a family $\mathcal{F}$ of non overlapping displits such that for every displit $\{X, Y\}$ in G, either $\{X, Y\} \in \mathcal{F}$, or there is a bipartition $\{X', Y'\} \in \mathcal{F}$ such that $\{X, Y\}$ and $\{X', Y'\}$ overlap.*

The family constructed in the previous lemma contains obviously all strong displits in G. A final $O(nm)$ procedure finds every non-strong displits in $\mathcal{F}$. This leads to the following theorem.

Theorem 19. *The displit decomposition of every digraph can be computed in time $O(nm)$.*

5 Digraphs of GF(4)-rank-width 1

In [15] Kanté defined a notion of *vertex-minor* for digraphs that extended the one for undirected graphs. He also characterized the class of digraphs of GF(4)-rank-width at most k in the following.

Theorem 20 ([15])**.** *For each k, there is a finite list $\mathcal{C}_k$ of digraphs having at most $(6^{k+1}-1)/5$ vertices such that a digraph G has* GF(4)*-rank-width at most k if and only if no digraph in $\mathcal{C}_k$ is isomorphic to a vertex-minor of G.*

When $k = 1$, the digraphs to exclude as vertex-minors have at most 7 vertices. However, we do not know any polynomial-time algorithm that checks whether a given graph is a vertex-minor of another. We will give in this section several characterizations of digraphs of GF(4)-rank-width 1. As a consequence we get an algorithm for recognizing digraphs of GF(4)-rank-width 1.

A vertex x of a digraph G is a *pendant vertex* of another vertex y if y is the only neighbor of x in G. Two vertices x and y of a digraph G are called *dtwins* if x and y verify one of the following exclusive conditions ($A = N^+_{G-y}(x),\ B = N^-_{G-y}(x)$):

1. $N^+_{G-x}(y) = A,\ N^-_{G-x}(y) = B$ or,
2. $N^+_{G-x}(y) = B,\ N^-_{G-x}(y) = (B \backslash A) \cup (A \backslash B)$ or,
3. $N^+_{G-x}(y) = (A \backslash B) \cup (B \backslash A),\ N^-_{G-x}(y) = A$.

We say that a digraph is *completely decomposable by the displit decomposition* if every node in the displit decomposition is degenerate or linear. The main result of this paper is the following theorem, analogous to Theorem 1.

Theorem 21. *Let G be a connected digraph with at least* 2 *vertices. Then the following conditions are equivalent:*

1. *G is completely decomposable by the displit decomposition.*
2. *G can be obtained from a single vertex by creating dtwins or adding pendant vertices.*
3. *G has* GF(4)*-rank-width* 1.
4. *For every $W \subseteq V$ with $|W| \geq 4$, $G[W]$ has a non-trivial displit.*
5. *$u(G)$ is distance-hereditary and for every $W \subseteq V$ with $|W| \leq 5$, $\mathrm{rwd}^{(4)}(G[W]) \leq 1$.*

Condition 5 gives a characterization of digraphs of GF(4)-rank-width 1 by forbidden induced sub-graphs: a digraph has GF(4)-rank-width 1 if and only if it is $(\mathcal{H}, \mathcal{C})$-free, where $\mathcal{H}$ is the set of digraphs G such that $u(G)$ is a house, a gem, a domino or a hole ($C_k, k \geq 5$), and $\mathcal{C}$ is the set of connected digraphs G with at most 5 vertices such that $\mathrm{rwd}^{(4)}(G) > 1$

and for every $x \in V_G$, $\mathrm{rwd}^{(4)}(G-x) \leq 1$. A computer check shows that $\mathcal{C}$ contains 78 graphs (Figure 3 in Appendix).

Before proving Theorem 21, let us state and prove two technical lemmas. The following is immediate from the definitions.

Proposition 22. *Let x and y be two vertices of a digraph G. Then $\{x, y\}$ is a displit if and only if x and y are dtwins or x is a pendant vertex of y or y is a pendant vertex of x.*

The following proposition is a straightforward adaptation of [18, Proposition 7.1].

Proposition 23. *Let x and y be dtwins of a digraph G such that $G-x$ has at least one edge. Then,* $\mathrm{rwd}^{(4)}(G-x) = \mathrm{rwd}^{(4)}(G)$

Proof. By definition of GF(4)-rank-width we have $\mathrm{rwd}^{(4)}(G-x) \leq \mathrm{rwd}^{(4)}(G)$. We will prove that $\mathrm{rwd}^{(4)}(G-x) \geq \mathrm{rwd}^{(4)}(G)$. Let $(T, \mathcal{L})$ be a sub-cubic layout of GF(4)-rank-width $k = \mathrm{rwd}^{(4)}(G-x)$ of $G-x$. By definition, there is a bijection $\mathcal{L}$ between V_{G-x} and L_T. Let $v = \mathcal{L}(y)$ and let $u \in V_T$ such that $uv \in E_T$. Let T' be obtained from T as follows: $V_{T'}$ is the set $V_T \cup \{u', w\}$ (where u' and w are two new nodes) and $E_{T'}$ the set $(E_T \backslash \{uv\}) \cup \{uu', u'v, u'w\}$. We let $\mathcal{L}' : V_G \to \mathrm{L}_{T'}$ be such that $\mathcal{L}'(x) = w$ and for every $z \in V_G \backslash x$, $\mathcal{L}'(z) = \mathcal{L}(z)$.

It is clear that $(T', \mathcal{L}')$ is a sub-cubic layout of G. We claim that the GF(4)-rank-width of $(T', \mathcal{L}')$ is equal to the GF(4)-rank-width of $(T, \mathcal{L})$.

It is clear that the GF(4)-rank-width of the edges $u'v$ and $u'w$ are at most 1. Since x and y are dtwins, the GF(4)-rank-width of the edge uu' is at most 1 (Proposition 22). Moreover, the other edges of T' are in T, then their GF(4)-rank-width in $(T', \mathcal{L}')$ is equal to their GF(4)-rank-width in $(T, \mathcal{L})$ (Lemma 12). Since $G-x$ has at least one edge we have $\mathrm{rwd}^{(4)}(G-x) \geq 1$. Therefore $\mathrm{rwd}^{(4)}(G-x) \geq \mathrm{rwd}^{(4)}(G)$. □

We can now begin the proof of Theorem 21.

Proof of Theorem 21. $1 \to 2$). By induction on $|V_G|$. It is trivial if $|V_G| \leq 2$. Otherwise let $(T, \mathcal{L})$ be the displit decomposition of G, and let u be a leaf in $T - L_T$. If u is degenerated, let $\{v, w\} \subseteq N_T(u) \cap L_T$. Otherwise u is linear and has at least 4 neighbors. Let $v_1, \ldots v_k$ be its ordering. If $N_T(u) \backslash L_T \subseteq \{v_2, \ldots, v_{k-1}\}$, take $v = v_1$ and $w = v_k$. Otherwise take $v = v_2$ and $w = v_3$. In all cases, $\{\mathcal{L}^{-1}(\{v, w\}), V_G \backslash \mathcal{L}^{-1}(\{v, w\})\}$ is a displit. By Proposition 22, either $x = \mathcal{L}^{-1}(v)$ and $y = \mathcal{L}^{-1}(w)$ are dtwins, or one is a pendant vertex of the other. If x and y are dtwins or x is a pendant vertex of y we let $G' = G - x$, otherwise $G' = G - y$. By induction G' is obtained from a single vertex by creating dtwins or adding pendant vertices.

$2 \to 3$). By induction on $|V_G|$. It is trivial if $|V_G| \leq 2$. Otherwise let $x \in V_G$ be the last added vertex. If x is a pendant vertex, let $\{y\} = N_G(x)$, otherwise let y be the dtwin of x. By induction, $\mathrm{rwd}^{(4)}(G-x) = 1$. Using Proposition 23, $\mathrm{rwd}^{(4)}(G) = 1$.

$3 \to 4$). If $\mathrm{rwd}^{(4)}(G) \leq 1$, then for every $W \subseteq V_G$, $\mathrm{rwd}^{(4)}(G[W]) \leq 1$. When $|W| \geq 4$, a sub-cubic layout of $G[W]$ has an edge (u, v) such that $\{X^{uv}, V \backslash X^{uv}\}$ is non-trivial, and thus $G[W]$ has a non-trivial displit.

$4 \to 1$). Suppose that G is not completely decomposable. Then the displit decomposition of G has a prime node u. By definition of a representative layout, the degree of u is at least 4. By Lemma 13, the quotient graph $C(u)$ is prime and is an induced sub-graph of G with at least 4 vertices.

$3 \to 5$). By Observation 11, $\mathrm{rwd}(u(G)) = 1$ since the layout of GF(4)-rank-width 1 for G is a layout of rank-width 1 for $u(G)$. Thus by Theorem 1, $u(G)$ is distance hereditary. Moreover, for every $W \subseteq V$, we have $\mathrm{rwd}^{(4)}(G[W]) \leq 1$.

5 → 3). Due to space limitation we will give only a sketch of the proof. (For the complete proof see Appendix A.3.) Suppose that G is a digraph such that $\mathrm{rwd}^{(4)}(G) > 1$ and such that $u(G)$ is distance hereditary. Let W be a minimal subset of V_G such that $\mathrm{rwd}^{(4)}(G[W]) > 1$. Working on the split decomposition of $u(G[W])$, one can show successively that:

- $u(G[W])$ has no pendant vertex,
- if $u(G[W])$ has a false twin, then $G[W]$ has at most 4 vertices,
- if $u(G[W])$ has no false twin and no pendant vertex, then $u(G)$ is complete, and
- if $u(G[W])$ is complete, then $G[W]$ has at most 5 vertices.

Thus there is a $W \subseteq V_G$ of size at most 5 such that $\mathrm{rwd}^{(4)}(G[W]) > 1$. □

As a corollary of Theorems 19 and 21, we get an algorithm for recognizing digraphs of GF(4)-rank-width 1.

Corollary 24. *Digraphs of* GF(4)*-rank-width* 1 *can be recognized in time* $O(nm)$.

6 Concluding Remarks

Differences with Cunningham's split decomposition of digraphs. Cunningham shows that the family of splits in a strongly connected digraph is bi-partitive. He also gives a characterization of degenerate and linear graphs for the split decomposition: a graph is degenerate for the split decomposition if and only if it is complete or is a star, and is linear if and only if it is a *circle of transitive tournaments* (CTT) [5].

The displit decomposition and the split decomposition of digraphs are both generalization of the split decomposition of undirected graphs. A first difference is that for the displit decomposition the graph has only to be connected.

The quotient graphs of the displit decomposition are induced sub-graphs of the original graph; this is not necessarily true for the split decomposition of digraphs.

Finally, the split decomposition and the displit decomposition are mutually exclusive. For all $k \geq 3$ the graph C'_k is linear for the split decomposition (and thus completely decomposable) since it is a CTT, but it is prime for the displit decomposition since $u(C'_k)$ is prime for the split decomposition. In the other hand, we can construct an infinite family of graphs linear for the displit decomposition and prime for the split decomposition (Lemma 36 in Appendix).

Links between bi-rank-width and Cunningham's split decomposition. Kanté defined another digraph parameter called *bi-rank-with*, and showed relations between GF(4)-rank-width and bi-rank-width [15]. A strongly connected digraph is completely decomposable by Cunningham's split decomposition if and only if it has bi-rank-width 2. It is open to find another characterization for digraphs of bi-rank-width 2.

Generalization to 2-structures. A 2-*structure* is a complete digraph with labels on edges. We mention that GF(4)-rank-width and displit decomposition can be generalized to 2-structures over finite fields. For a field , we obtain a decomposition for 2-structures over with a characterization theorem similar to Theorem 21.

An interesting case is GF(3), which gives a decomposition theory for oriented graphs (*i.e.*, directed anti-symmetric graph). The set of exclusion for oriented graphs of GF(3)-rank-width 1 is given in Figure 4.

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A Appendix

A digraph G is *complete* if for every pair of vertices x, y, (x, y) and (y, x) are edges of G; a vertex x of G is *universal* if for every vertex $y \neq x$, (x, y) and (y, x) are edges of G. In a complete digraph every vertex is a universal vertex.

Before continuing, let us state and prove some technical lemmas.

Lemma 25. *Let G be a digraph with a universal vertex $y \in V_G$, and let $\{X, Y\}$ be a bipartition of V_G with $y \in Y$. Then $\{X, Y\}$ is a displit in G if and only if X is a module of $G - y$.*

Proof. If $\{X, Y\}$ is a displit, then by definition, there exist an X-vector A and a Y-vector B such that $M_G[X, Y] = A^t \cdot B$. For every $z \in Y$, we have $M_G[X, \{z\}] = A^t \cdot B[z]$. Since y is a universal vertex, then $M_G[X, \{y\}] = (1 \cdots 1)^t = A^t \cdot B[y]$, *i.e.*, $A^t = B[y]^{-1} \cdot (1 \cdots 1)^t$. Therefore, for every $z \in Y$, z does not distinguish any two vertices of X, *i.e.*, X is a module in $G - y$.

Conversely, if X is a module in $G - y$, then $M_G[X, Y] = (1 \cdots 1)^t \cdot B$ for some Y-vector B. And, $\{X, Y\}$ is a displit in G. □

We define now an operation for digraphs such that $u(G)$ is complete. This operation will permit to transform a digraph into a digraph with a universal vertex, and thus will make Lemma 25 applicable.

Let G be a digraph such that $u(G)$ is complete, and let $W \subseteq V_G$. Let $G * W = (V_G, E')$ where $E' \cap W^2 = E \cap W^2$, $E' \cap (V \setminus W)^2 = E \cap (V \setminus W)^2$, and for every $(x, y) \in W \times (V \setminus W)$:

- if $(x, y) \in E$ and $(y, x) \in E$ then $(y, x) \in E'$,
- if $(x, y) \in E$ and $(y, x) \notin E$ then $(x, y) \in E'$ and $(y, x) \in E'$,
- if $(x, y) \notin E$ and $(y, x) \in E$ then $(x, y) \in E'$.

We make the following easy observations.

Proposition 26. *Let G such that $u(G)$ is complete.*

(1) $G * A * A = G * (V \setminus A)$.

(2) $G * A * A * A = G$, and thus $G * A * (V \setminus A) = G$.

(3) Let $x \in V_G$ and let $B = N^-_G(x) \setminus N^+_G(x)$. Then $N^+_{G*B}(x) = V \setminus \{x\}$.

(4) Let $x \in V_G$, let $A = N^+_G(x) \setminus N^-_G(x)$ and let $B = N^-_G(x) \setminus N^+_G(x)$. Then x is universal in $G * B * A * A$.

Lemma 27. *Let G such that $u(G)$ is complete, and let $W \subseteq V_G$. $\{X, Y\}$ is a displit in G if and only if $\{X, Y\}$ is a displit in $G * W$.*

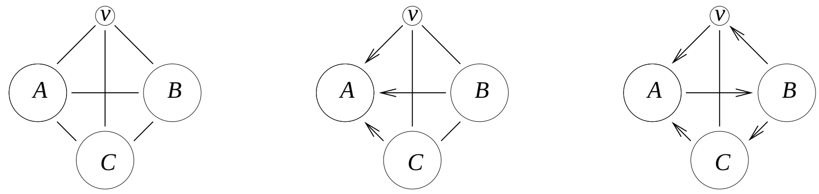
Proof. If $\{X, Y\}$ is a displit in G, then $M_G[X, Y] = A^t \cdot B$ for an X-vector A and a Y-vector B. Then it is not hard to show that $M_{G*W}[X, Y] = A'^t \cdot B'$, with $A'[x] = \ ^2 \cdot A[v]$ if $x \in W$ and $A'[v] = A[v]$ if $x \notin W$, and $B'[y] = \ \cdot B[y]$ if $y \in W$ and $B'[y] = B[y]$ if $y \notin W$. The other way follows by Proposition 26 (2). □

A.1 Proof of Lemma 14

If G is degenerated, then $u(G)$ is degenerated for the split decomposition. Thus either $u(G)$ is a star or $u(G)$ is complete [5]. Note that every graph such that $u(G)$ is a star is degenerated for the displit decomposition.

We now suppose that $u(G)$ is complete. By Lemma 25, if G has a universal vertex x, then $G-x$ is degenerated for the modular decomposition, *i.e.*, every subset of vertices is a module. A graph degenerated for the modular decomposition is either complete or edgeless [8]. Since $u(G)$ is complete, G is thus complete.

Suppose now that G has no universal vertex. Take an arbitrary vertex $x \in V_G$ and let $A = N_G^+(x) \setminus N_G^-(x)$, $B = N_G^-(x) \setminus N_G^+(x)$ and $C = N_G^+(x) \cap N_G^-(x)$. Let $G' = G * B$ and $G'' = G' * A * A$. By Proposition 26 (4), x is universal in G'', thus by Lemma 25, G'' is complete. Therefore, in G, A dominates B, B dominates C and C dominates A. This is summarized in the following figure.



$G'' = G' * A * A$ $\quad$ $G' = G * B$ $\quad$ G

Thus G is a C_3' in where we have substituted the vertices by complete graphs of size $|A|$, $|B|$ and $|C|+1$.

A.2 Proof of Lemma 15

Suppose that G is linear with at least 4 vertices. Let $(x_1, \ldots, x_n)$ be the linear ordering. Then for every $j \geq i$, $\{\{x_i, \ldots x_j\}, V \setminus \{x_i, \ldots x_j\}\}$ is a split in $u(G)$. Since the family of splits in undirected graphs is strongly bi-partitive: $u(G)$ is degenerated, *i.e.*, G is either a star or is complete. If $u(G)$ is a star, then by Lemma 14, G is degenerated. Thus $u(G)$ is complete.

Let $A = N_G^+(x_n) \setminus N_G^-(x_n)$ and $B = N_G^-(x_n) \setminus N_G^+(x_n)$. Let $G'' = G * B * A * A$. By Proposition 26 (4), x_n is universal in G''. By Lemma 25, G is linear if and only if $G'' - x_n$ is linear for the modular decomposition, *i.e.*, $G'' - x_n$ is a transitive tournament [8]. Finally it is not hard to show that $G'' - x_n$ is a transitive tournament if and only if G has the desired form.

A.3 Proof of Theorem 21 $(5 \to 3)$

We start by two technical Propositions.

Proposition 28. *Let G be a connected digraph such that $u(G)$ is distance hereditary and* $\mathrm{rwd}^{(4)}(G) = 1$*. If $\{X, Y\}$ is a strong split in $u(G)$, then $\{X, Y\}$ is a displit in G.*

Proof. Since $\{X, Y\}$ is a strong split in $u(G)$, there is no displit $\{X', Y'\}$ such that $\{X', Y'\}$ and $\{X, Y\}$ overlap. Suppose that $\{X, Y\}$ is not a displit in G and let $(T, \mathcal{L})$ be a sub-cubic layout of G of GF(4)-rank-width 1. Each edge uv of T induces a bipartition $\{X^{uv}, V_G \backslash X^{uv}\}$ of V_G.

Since $\{X^{uv}, V_G \backslash X^{uv}\}$ is a split, it does not overlap $\{X, Y\}$. If $X^{uv} \subsetneq X$ or $X^{uv} \subsetneq Y$, we orient the edge from v to u (we remove the edge (u, v)), otherwise we orient it from u to v (we remove the edge (v, u)). Let now w be an internal node in T of out-degree zero. Such a node always exists and w cuts V_G into three sets $\{A, B, C\}$. For every $W \in \{A, B, C\}$, $\{W, V_G \setminus W\}$ is a displit, and $W \subseteq X$ or $W \subseteq Y$. Thus there is a $W \in \{A, B, C\}$ such that $\{W, V_G \setminus W\} = \{X, Y\}$. Contradiction. □

Proposition 29. *Let X, Y_1 and Y_2 such that $X \cap (Y_1 \cup Y_2) = \emptyset$, $\{X, Y_1\}$ is a displit in $G[X \cup Y_1]$ and $\{X, Y_2\}$ is a displit in $G[X \cup Y_2]$. If there is an edge $xy \in (X \times Y_1 \cap Y_2) \cap E_{u(G)}$, then $\{X, Y_1 \cup Y_2\}$ is a displit in $G[X \cup Y_1 \cup Y_2]$.*

Proof. If $\{X, Y_i\}$, for $i \in \{1, 2\}$, is a displit in $G[X \cup Y_i]$, then $M_G[X, Y_i] = A_i^t \cdot B_i$ for X-vectors A_i and Y_i-vectors B_i. Since $M_G[X, \{y\}] \neq (0 \cdots 0)^t$, $B_i[y] \neq 0$. Moreover, $M_G[X, \{y\}] = A_1^t \cdot B_1[y] = A_2^t \cdot B_2[y]$, *i.e.*, $A_2^t = \alpha \cdot A_1^t$ for $\alpha = B_1[y]/B_2[y]$. We let C be the $(Y_1 \cup Y_2)$-vector where $C[z] = B_1[z]$ if $z \in Y_1$ and $C[z] = \alpha \cdot B_2[z]$ if $z \in Y_2 \backslash Y_1$. It is easy to verify that $M_G[X, Y_1 \cup Y_2] = A_1^t \cdot C$.

□

From now on, we let G be a connected digraph such that:

- $u(G)$ is distance hereditary,
- $\mathrm{rwd}^{(4)}(G) > 1$ and
- for every $x \in V_G$, $\mathrm{rwd}^{(4)}(G - x) \leq 1$.

In particular G has at least 4 vertices. Let $(T, \mathcal{L})$ be the split decomposition of $u(G)$.

Claim 30. *G is prime w.r.t. the displit decomposition (*i.e.*, there is no displit $\{X, Y\}$ of G such that $|X| > 1$ and $|Y| > 1$).*

Proof. Suppose that G is not prime and let $\{X, Y\}$ be a non-trivial displit in G. Let $xy \in (X \times Y) \cap E_{u(G)}$. Let $G_1 = G[X \cup \{y\}]$ and $G_2 = G[Y \cup \{x\}]$. By hypothesis, $\mathrm{rwd}^{(4)}(G_1) \leq 1$ and $\mathrm{rwd}^{(4)}(G_2) \leq 1$.

Let $(T_i, \mathcal{L}_i)$ be a sub-cubic layout of G_i of GF(4)-rank-width 1, for $i \in \{1, 2\}$. Let T be the tree $(V_{T_1} \cup V_{T_2}, E_{T_1} \cup E_{T_2} \cup \{uv\})$ where $\mathcal{L}_1(y) = v$ and $\mathcal{L}_2(x) = u$. Let $\mathcal{L} : V_G \to \mathrm{L}_T$ be the bijection such that $\mathcal{L}(x) = \mathcal{L}_1(x)$ if $x \in X$ and $\mathcal{L}(y) = \mathcal{L}_2(y)$ if $y \in Y$. It is not hard to see that $(T, \mathcal{L})$ is a sub-cubic layout of G. Moreover, by Lemma 12, the GF(4)-rank-width of $(T, \mathcal{L})$ is 1. Contradiction. □

Claim 31. *$u(G)$ has no pendant vertex.*

Proof. Suppose that x is a pendant vertex of y in $u(G)$. Then $\{\{x, y\}, V_G \setminus \{x, y\}\}$ is a non-trivial displit. Contradiction with Claim 30. □

Claim 32. *If $u(G)$ has a false twin, then G has at most 4 vertices.*

Proof. Let x and y be two false twins in $u(G)$. By Claim 31, $|N_G(x)| > 1$ (similarly for y).

Assume for every $z, t \in N_G(x)$, $\{\{z, t\}, \{x, y\}\}$ is a displit in $G[\{x, y, z, t\}]$. Then by Proposition 29, $\{\{x, y\}, N_G(x)\}$ is a displit in $G[\{x, y\} \cup N_G(x)]$, and thus $\{\{x, y\}, V_G \backslash \{x, y\}\}$ is a non-trivial displit in G. Contradiction with Claim 30.

Let $z, t \in N_G(x)$ such that $\{\{z, t\}, \{x, y\}\}$ is not a displit. Moreover, since y is not adjacent to x, $\{\{x, z\}, \{y, t\}\}$ and $\{\{x, t\}, \{y, z\}\}$ are not displits in $G[\{x, y, z, t\}]$. Thus $\mathrm{rwd}^{(4)}(G[\{x, y, z, t\}]) > 1$. Therefore, $V_G = \{x, y, z, t\}$. □

Claim 33. *If $u(G)$ has no pendant vertex and no false twin, then $u(G)$ is complete.*

Proof. Suppose that $u(G)$ is not complete. Thus the split decomposition tree $(T, \mathcal{L})$ of $u(G)$ has at least two internal nodes.

Let u be an internal node adjacent to two leaves u_1 and u_2. We know that u is a clique node, otherwise there will be a false twin or a pendant vertex in $u(G)$. Since $u(G)$ is not complete, u is adjacent to an internal node v. By the properties of split decomposition tree [5], v must be a star node, and since $u(G)$ has no pendant vertex and no false twin, v has to be adjacent to an internal node w.

$B = V_G \setminus (X^{vu} \cup X^{vw})$. Note that $|X^{vu}| \geq 2$ and $|X^{vw}| \geq 2$. Suppose that $|X^{vw}| > 2$ and let $c, c' \in X^{vw} \setminus \{y\}$ such that $x \in X^{vu} \cap N_G(B \cup X^{vw})$ and $y \in N_G(X^{vu}) \setminus X^{vu}$. Then $\{X^{vu}, (V_G \setminus X^{vu}) \setminus \{c\}\}$ is a strong split in $u(G - c)$, and $\{X^{vu}, (V_G \setminus X^{vu}) \setminus \{c'\}\}$ is a strong split in $u(G - c')$. By Proposition 28, $\{X^{vu}, (V_G \setminus X^{vu}) \setminus \{c\}\}$ is a displit in $G - c$ and $\{X^{vu}, (V_G \setminus X^{vu}) \setminus \{c'\}\}$ is a displit in $G - c'$, and by Proposition 29 $\{X^{vu}, V_G \setminus X^{vu}\}$ is a non-trivial displit in G. Contradiction.

Thus $|X^{vw}| = 2$ and w is a complete node. Now u and w play a symmetric role, and with the previous argument $|X^{vu}| = 2$. Suppose w.l.o.g. that $N_G(X^{vu}) \cap B \neq \emptyset$, and let $y \in B \cap N_G(X^{vu})$, $x \in X^{vu} \cap N_G(b)$ and $X^{vw} = \{c, c'\}$. Then $\{X^{vu}, (V_G \setminus X^{vu}) \setminus \{c\}\}$ is a strong split in $u(G - c)$, and $\{X^{vu}, (V_G \setminus X^{vu}) \setminus \{c'\}\}$ is a strong split in $u(G - c')$. By Proposition 28, $\{X^{vu}, (V_G \setminus X^{vu}) \setminus \{c\}\}$ is a displit in $G - c$ and $\{X^{vu}, (V_G \setminus X^{vu}) \setminus \{c'\}\}$ is a displit in $G - c'$. Now Proposition 29 can apply with xy, and $\{X^{vu}, V_G \setminus X^{vu}\}$ is a non-trivial displit of G. Contradiction. □

Claim 34. *If $u(G)$ is complete, then G has at most 5 vertices.*

Proof. Let $x \in V_G$. Let $A = N_G^+(x) \setminus N_G^-(x)$ and $B = N_G^-(x) \setminus N_G^+(x)$. Let $G'' = G * B * A * A$. By Proposition 26 (4), x is universal in G''. By Lemma 25, G is prime if and only if $G'' - x$ is prime for the modular decomposition. If a graph H is prime for the modular decomposition, and $|V_H| > 4$, then there is a $W' \subsetneq V_H$ such that $|W| \geq 3$ and $H[W]$ is prime [8]. Thus if $|V_G| > 5$, there is a $W \subsetneq V_G \setminus \{x\}$ such that $|W| \geq 3$ and $G''[W]$ is prime for the modular decomposition. By Lemma 25, $G[W \cup \{x\}]$ is prime for the displit decomposition. Contradiction with $\mathrm{rwd}^{(4)}(G[W \cup \{x\}]) \leq 1$. □

To summarize, we get the following lemma.

Lemma 35. *Let G be a graph such that $u(G)$ is distance hereditary, and $\mathrm{rwd}^{(4)}(G) > 1$. Then there exists $W \subseteq V_G$ such that $\mathrm{rwd}^{(4)}(G[W]) > 1$ and $u(G[W])$ is either a C_4, a diamond, a K_4 or a K_5.*

Proof. Let $W \subseteq V_G$ be a set of minimum size such that $\mathrm{rwd}^{(4)}(G[W]) > 1$. Obviously $|W| \geq 4$, and for every $x \in W$, $\mathrm{rwd}^{(4)}(G[W \setminus \{x\}]) \leq 1$. Thus $G[W]$ is connected, otherwise $G[W]$ has a non trivial displit. By Claim 31, $u(G[W])$ has no pendant vertex. By Claim 32, if $u(G[W])$ has a false twin, then $|W| \leq 4$ and thus $u(G[W])$ is either C_4, a diamond or a K_4. Finally, by Claims 33 and 34, if $u(G[W])$ has no pendant vertex and no false twin, then $u(G[W])$ is complete of size at most 5. □

A.4 Lemma 36

For $k \geq 2$, let G_k be the digraph of vertex set $\{y_0, \ldots, y_{2k}\}$ and of edge set E_G such that:

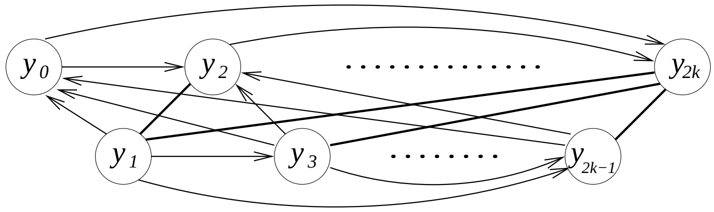


Figure 2: Graph G_k, linear for the displit decomposition and prime for the split decomposition.

- $(y_{2i}, y_{2j}) \in E_{G_k}$ for every $0 \leq i < j \leq k$,
- $(y_{2i+1}, y_{2j+1}) \in E_{G_k}$ for every $0 \leq i < j < k$,
- $(y_{2j+1}, y_{2i}) \in E_{G_k}$ for every $0 \leq i \leq j < k$,
- $(y_{2i}, y_{2j+1}) \in E_{G_k}$ and $(y_{2j+1}, y_{2i}) \in E_{G_k}$ for every $0 \leq j < i \leq k$.

Lemma 36. *G_k is prime for split decomposition and linear for displit decomposition.*

Proof. Obviously G_k respects condition of Lemma 15 with $f(y_i) = 1$ if i is even and $f(y_i) = 0$ if i is odd.

Suppose that G_k has a non-trivial split $\{X, Y\}$. W.l.o.g. $y_0 \in Y$. We suppose that X and Y are respectively partitionned into $\{X_{(i,j)}\}_{i,j\in\{0,1\}}$ and $\{Y_{(i,j)}\}_{i,j\in\{0,1\}}$ as in Figure 1.

Since $u(G_k)$ is complete, then $y_0 \notin Y_{(0,0)}$. Suppose that $y_0 \in Y_{(1,1)}$. Then $X_{(1,1)}$ is empty, and there is no $(x, y) \in X \times Y$ such that $\{(x, y), (y, x)\} \subseteq E_G$. The graph $(V_G \setminus \{y_0\}, E_{G_k - y_0} \cap \{(y, x) \mid (x, y) \in E_{G_k - y_0}\})$ is connected, thus either $V_G \setminus \{y_0\} \subseteq X$ or $V_G \setminus \{y_0\} \subseteq Y$. Contradiction.

Suppose now that $y_0 \in Y_{(1,0)}$. Then $X_{(0,0)}$ and $X_{(1,0)}$ are empty. Moreover for every even i, $y_i \in Y$. Since $\{X, Y\}$ is non trivial, there is a $p < q$ such that $y_{2p+1} \in X$ and $y_{2q+1} \in X$. But $\{\{y_{2p}, y_{2q}\}, \{y_{2p+1}, y_{2q+1}\}\}$ is not a split in $G_k[\{y_{2p}, y_{2p+1}, y_{2q}, y_{2q+1}\}]$. Contradiction. The last case $y_0 \in Y_{(0,1)}$ is similar to the case $y_0 \in Y_{(1,0)}$. □

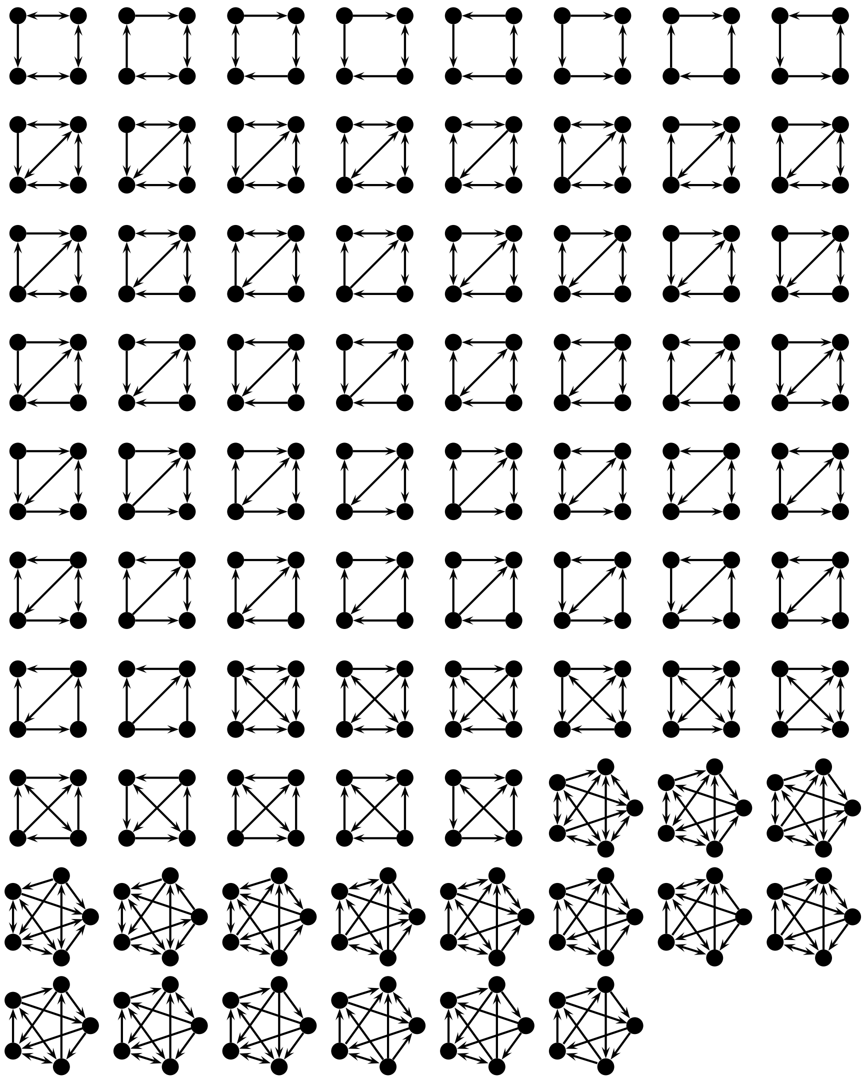


Figure 3: Exclusions for directed graphs of GF(4)-rank-width 1.

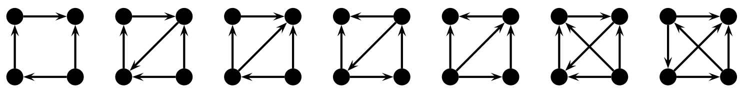


Figure 4: Exclusions for oriented graphs of GF(3)-rank-width 1.