Title:	Minimal Dominating Set Enumeration
Name:	Mamadou Moustapha Kanté, Lhouari Nourine
Affil./Addr.	Clermont-Université, Université Blaise Pascal, LIMOS, CNRS, France.
Keywords:	Enumeration; Transversal hypergraph; Dominating set
SumOriWork:	2011-2014; Kanté, Limouzy, Mary, Nourine, Uno

Minimal Dominating Set Enumeration

Mamadou Moustapha Kanté, Lhouari Nourine

Clermont-Université, Université Blaise Pascal, LIMOS, CNRS, France.

Years aud Authors of Summarized Original Work

2011-2014; Kanté, Limouzy, Mary, Nourine, Uno

Keywords

Enumeration; Transversal hypergraph; Dominating set

Problem Definition

Let G be a graph on n vertices and m edges. An edge is written xy (equivalently yx). A *dominating set* in G is a set of vertices D such that every vertex of G is either in D or is adjacent to some vertex of D. It is said to be *minimal* if it does not contain any other dominating set as a proper subset. For every vertex x let $N[x]$ be $\{x\} \cup \{y \mid xy \in E\}$, and for every $S \subseteq V$ let $N[S] := \bigcup_{x \in S} N[x]$. For $S \subseteq V$ and $x \in S$ we call any $y \in N[x] \setminus N[S \setminus x]$ a *private neighbor of x with respect to S*. The set of minimal dominating sets of G is denoted by $\mathcal{D}(G)$. We are interested in an output-polynomial algorithm for enumerating $\mathcal{D}(G)$, *i.e.*, listing, without repetitions, all the elements of $\mathcal{D}(G)$ in time bounded by $p(n + m, \sum\limits_{D \in \mathcal{D}(G)} |D|)$ (Dom-Enum for short).

It is easy to see that Dom-Enum is a special case of Hypergraph Dualization. Let $\mathcal{N}(G)$, called the *closed neighborhood hypergraph*, be the hypergraph with hyper-edges $\{N[x] \mid x \in V\}$. It is easy to see that D is a dominating set of G if and only if D is a transversal of $\mathcal{N}(G)$. Hence, Dom-Enum is a special case of Hypergraph Dualization. For several graph classes the closed neighborhood hypergraphs are subclasses of hypergraph classes where an output-polynomial algorithm is known for Hypergraph Dualization, *e.g.* minor-closed classes of graphs, graphs of bounded degree, graphs of bounded conformality, graphs of bounded degeneracy, graphs of logarithmic degeneracy, [11; 12; 19]. So, Dom-Enum seems more tractable than Hypergraph Dualization since there exist families of hypergraphs that are not closed neighborhoods of graphs [1].

Key Results

Contrary to several special cases of HYPERGRAPH DUALIZATION in graphs, (*e.g.* enumeration of maximal independent sets, enumeration of spanning forests, etc.) DOM-ENUM is equivalent to TRANS-ENUM. Indeed, it is proved in [13] that with every hypergraph $\mathcal{H}$ one can associate a co-bipartite graph $\mathcal{B}(\mathcal{H})$ such that every minimal dominating set of $\mathcal{B}(\mathcal{H})$ is either a transversal of $\mathcal{H}$ or has size at most 2. A consequence is that there exists a polynomial delay polynomial space algorithm for TRANS-ENUM if and only if there exists one for DOM-ENUM, even in co-bipartite graphs. The reduction is moreover asymptotically tight (with respect to polynomial delay reductions as defined in [19]) in the sense that there exist hypergraphs $\mathcal{H}$ such that for every graph G we cannot have $tr(\mathcal{H}) = \mathcal{D}(G)$ [13]. This intriguing result has the advantage of bringing tools from graph structural theory to tackle the difficult and widely open problem HYPERGRAPH DUALIZATION. Furthermore, until recently the most graph classes where DOM-ENUM is known to be tractable were those for which closed neighborhood hypergraphs were subclasses of some of the tractable hypergraph classes for HYPERGRAPH DUALIZATION. We will give examples of graph classes where graph theory helps a lot to solve DOM-ENUM, and sometimes allows to introduce new techniques for the enumeration.

It is widely known now that every monadic second-order formula can be checked in polynomial time in graph classes of bounded *clique-width* [3; 20]. Courcelle proved in [2] that one can also enumerate, with linear delay linear space, the solutions of every monadic second-order formula. Since one can express in monadic second-order logic that a subset D of vertices is a minimal dominating set, DOM-ENUM has a linear delay linear space in graph classes of bounded clique-width. The algorithm by Courcelle is quite ingenious: it firsts constructs a DAG some sub-trees of which correspond to the positive runs of the tree-automata associated with the formula on the given graph and then enumerate these sub-trees.

Many graph classes do not have bounded clique-width (interval graphs, permutation graphs, unit-disk graphs, etc.) and many such graph classes have nice structures that helped in the past for solving combinatorial problems, *e.g.* the clique-tree of chordal graphs, permutation models, etc. For some of these graph classes structural results can help to solve DOM-ENUM.

A common tool in enumeration area is the *parsimonious reduction.* One wants to enumerate a set of objects $\mathcal{O}$ and instead constructs a bijective function $b : \mathcal{O} \to \mathcal{T}$ such that there is an efficient algorithm to enumerate $\mathcal{T}$. For instance it is proved in [11; 13] that every minimal dominating set D of a split graph G can be characterized by $D \cap C(G)$ where $C(G)$ is the clique of G. A consequence is that in a split graph G there is a bijection between $\mathcal{D}(G)$ and the set $\{S \subseteq C(G) \mid \forall x \in S,\ x \text{ has a private neighbor}\}$, and since this later set is an independent system, DOM-ENUM in split graphs admits a linear delay polynomial space algorithm.

One can obtain other parsimonious reductions using graph structures. For instance, it is easy to check that every minimal dominating set in an interval graph is a collection of paths. Moreover, using the interval model (and ordering intervals from their left endpoints) every minimal dominating set can be constructed greedily by keeping track of the last two chosen vertices. Indeed it is proved in [14] that with every interval graph G one can associate a DAG the maximal paths of which are in bijection with the minimal dominating sets of G. The nodes of the DAG are pairs (x, y) such that $x < y$ and such that x and y can be both in a minimal dominating set, and the arcs are $((x, y), (y, z))$ such that (1) $\{x, y, z\}$ can be in a minimal dominating set, (2) there is no vertex between y and z that is not dominated by y or z; sources are pairs

(x, y) where every interval before x is dominated by x, and sinks are pairs (x, y) where every interval after y is dominated by y. This reduction to maximal paths of a DAG can be adapted to several other graph classes having a linear structure similar to the interval model, *e.g.* permutation graphs, circular-arc graphs [14]. In general, if for every graph G in a graph class $\mathcal{C}$ one can associate an ordering of the vertices such that for every subset $S \subseteq V$ the possible ways to extend S into a minimal dominating set depends only on the last k vertices of S, for some fixed constant k depending only on $\mathcal{C}$, then for every $G \in \mathcal{C}$ the enumeration of $\mathcal{D}(G)$ can be reduced to the enumeration of paths in a DAG as for interval graphs and thus DOM-ENUM is tractable in $\mathcal{C}$ [19]. This seems for instance to be the case for d-trapezoid graphs.

Parsimonious reductions between graph classes can be also defined. For instance, the *completion* of a graph G, *i.e.* the set of edges that can be added to G without changing $\mathcal{D}(G)$ are characterized in [11; 13]; this characterization lead the authors to prove that the completion of every P_6-free chordal graph is a split graph, which results in a linear delay polynomial space algorithm for DOM-ENUM in P_6-free chordal graph.

The techniques developed by the HYPERGRAPH DUALIZATION community combined with graph structural theory can give rise to new tractable cases of DOM-ENUM. For instance, the main drawback of Berge's algorithm is that at some level computed transversals are not necessarily subsets of solutions and this prevents from obtaining an output-polynomial algorithm since the computed set may be arbitrary large compared to the solution set [21]. One way to overcome this difficulty consists in choosing some levels $l_1, \ldots, l_k$ of Berge's algorithm such that every computed set at level l_j is a subset of a solution at level l_{j+1}. A difficulty with that scheme is to compute all the descendants in level l_{j+1} of a transversal in level l_j. This idea combined with the structure of minimal dominating sets in line graphs is used to derive a polynomial delay polynomial space algorithm for DOM-ENUM in line graphs [15]. A consequence is that there is a polynomial delay polynomial space algorithm to list the set of *minimal edge dominating sets* in graphs.

Another famous technique in enumeration area is the *back tracking*. Start from the empty set, and in each iteration choose a vertex x and partition the problem into two sub-problems: the enumeration of minimal dominating sets containing x, and the enumeration of those not containing x; at each step we have a set X to include in the solution and a set Y not to include. If at each step one can solve the EXTENSION PROBLEM, *i.e.* whether there is a minimal dominating set containing X and not intersecting Y, then DOM-ENUM admits a polynomial delay polynomial space algorithm. However, the EXTENSION PROBLEM is NP-complete in general [19] and even in split graphs [16]. But, sometimes structure helps. For instance, in split graphs whenever $X \cup Y \subseteq C(G)$, the EXTENSION PROBLEM is polynomial [11; 13] and was the key for the linear delay algorithm. Another special case of the EXTENSION PROBLEM is proved to be polynomial in chordal graphs using the *clique tree* of chordal graphs and is also the key to prove that DOM-ENUM in chordal graphs admits a polynomial delay polynomial space algorithm [16]. The algorithm uses deeply the clique tree and is a nested combination of several enumeration algorithms.

Open Problems

1. The first major challenge is to find an output-polynomial algorithm for DOM-ENUM, even in co-bipartite graphs. One way to address this problem is to understand the structure of minimal dominating sets in a graph. Failing to solve this problem,

can graphs help to improve the quasi-polynomial time algorithm by Fredman and Khachiyan [7]?
2. Until now if the techniques used to solve Dom-Enum in many graph classes are well-known, deep structural theory of graphs is not used and the used graph structures are more or less ad-hoc. Can we unify all these results and obtain at the same time new positive results. Indeed, there are several well-studied graph classes where the status of Dom-Enum is still open: bipartite graphs, unit-disk graphs, graphs of bounded expansion to cite a few. Are developed tools sufficient to address these graph classes?
3. There are several well-studied variants of the dominating set problem, in particular *total dominating set* and *connected dominating set* (see the monographs [9; 10]). It is proved in [13] that the enumeration of minimal total dominating sets and minimal connected dominating sets in split graphs is equivalent to Hypergraph Dualization. This is somehow surprising and we do not yet understand why such small variations make the problem difficult even in split graphs. Can we explain this situation?
4. From [13] we know that the enumeration of minimal connected dominating sets is harder than Hypergraph Dualization. Are both problems equivalent? Can we find a graph class $\mathcal{C}$ where each graph in $\mathcal{C}$ has a non-exponential number of minimal connected dominating sets, but minimum connected dominating set is NP-complete? Notice that if a class of graphs $\mathcal{C}$ has a polynomially bounded number of minimal separators, then the enumeration of minimal connected dominating sets can be reduced to Dom-Enum [13].
5. A related question to Dom-Enum is a tight bound for the number of minimal dominating sets in graphs. The best upper bound is $O(1.7159^n)$ and the best lower bound is $15^{n/6}$ [6]. For several graph classes tight bounds were obtained [4; 8]. Prove that $15^{n/6}$ is the upper bound or find the tight bound.
6. Another related subject to Dom-Enum is the counting of (minimal) dominating sets in time polynomial in the input graph. If the counting of dominating sets is a #P-hard problem and have been investigated in the past [5; 17; 18], not so much is known for the counting of minimal dominating sets, one can cite few examples: graphs of bounded clique-width [2], and interval, permutation and circular-arc graphs [14]. If we define for G the *minimal domination polynomial* $MD(G, x)$ that is the generating function of its minimal dominating sets, for which graph classes this polynomial can be computed? Does it have a (linear) recursive definition? For which values x can we evaluate it?

Cross-References

Hypergraph Dualization
Polynomial Time Algorithms for Dualization; Solvable Cases
Reverse Search
Counting Problems

Recommended Reading

1. Andreas Brandstädt, Van Bang Le and Jeremy P. Spinrad: Graph Classes a Survey. SIAM Monographs on Discrete Mathematics and Applications, Philadelphia (1999)

2. Bruno Courcelle: Linear Delay Enumeration and Monadic Second-Order Logic. Discrete Applied Mathematics, 157, 2675–2700 (2009)
3. Bruno Courcelle, B., Janos A. Makowsky, Udi Rotics: Linear Time Solvable Optimization Problems on Graphs of Bounded Clique-Width. Theory of Computing Systems 33(2), 125–150 (2000)
4. Jean-François Couturier, Pinar Heggernes, Pim van 't Hof, Dieter Kratsch: Minimal dominating sets in graph classes: Combinatorial bounds and enumeration. Theor. Comput. Sci. 487: 82-94 (2013)
5. Klaus Dohmen, Peter Tittmann: Domination Reliability. Electr. J. Comb. 19(1): P15 (2012)
6. Fedor V. Fomin, Fabrizio Grandoni, Artem V. Pyatkin, Alexey A. Stepanov: Combinatorial bounds via measure and conquer: Bounding minimal dominating sets and applications. ACM Transactions on Algorithms 5(1) (2008)
7. Michael L. Fredman, Leonid Khachiyan: On the Complexity of Dualization of Monotone Disjunctive Normal Forms. J. Algorithms 21(3): 618-628 (1996)
8. Petr A. Golovach, Pinar Heggernes, Mamadou Moustapha Kanté, Dieter Kratsch, Yngve Villanger:Minimal Dominating Sets in Interval Graphs and Trees. Submitted (2014)
9. Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater: Fundamentals of Domination in Graphs, volume 208 of Pure and Applied Mathematics. Marcel Dekker (1998)
10. Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater: Domination in Graphs: Advanced Topics, volume 209 of Pure and Applied Mathematics. Marcel Dekker, New York (1998)
11. Mamadou Moustapha Kanté, Vincent Limouzy, Arnaud Mary, Lhouari Nourine: Enumeration of Minimal Dominating Sets and Variants. FCT 2011: 298-309
12. Mamadou Moustapha Kanté, Vincent Limouzy, Arnaud Mary, Lhouari Nourine: On the Neighbourhood Helly of Some Graph Classes and Applications to the Enumeration of Minimal Dominating Sets. ISAAC 2012: 289-298
13. Mamamdou Moustapha Kanté, Vincent Limouzy, Arnaud Mary, Lhouari Nourine: On the Enumeration of Minimal Dominating Sets and Related Notions. In revision, 2014.
14. Mamadou Moustapha Kanté, Vincent Limouzy, Arnaud Mary, Lhouari Nourine, Takeaki Uno: On the Enumeration and Counting of Minimal Dominating sets in Interval and Permutation Graphs. ISAAC 2013: 339-349
15. Mamadou Moustapha Kanté, Vincent Limouzy, Arnaud Mary, Lhouari Nourine, Takeaki Uno: Polynomial Delay Algorithm for Listing Minimal Edge Dominating sets in Graphs. CoRR abs/1404.3501 (2014)
16. Mamadou Moustapha Kanté, Vincent Limouzy, Arnaud Mary, Lhouari Nourine, Takeaki Uno: A Polynomial Delay Algorithm for Enumerating Minimal Dominating Sets in Chordal Graphs. CoRR abs/1404.3501 (2014)
17. Shuji Kijima, Yoshio Okamoto, Takeaki Uno: Dominating Set Counting in Graph Classes. COCOON 2011: 13-24
18. Tomer Kotek, James Preen, Frank Simon, Peter Tittmann, Martin Trinks: Recurrence Relations and Splitting Formulas for the Domination Polynomial. Electr. J. Comb. 19(3): P47 (2012)
19. Arnaud Mary. *Énumération des Dominants Minimaux d'un graphe.* PhD thesis, Université Blaise Pascal, 2013.
20. Sang-il Oum, Paul D. Seymour: Approximating clique-width and branch-width. J. Comb. Theory, Ser. B 96(4): 514-528 (2006)
21. Ken Takata: A Worst-Case Analysis of the Sequential Method to List the Minimal Hitting Sets of a Hypergraph. SIAM J. Discrete Math. 21(4): 936-946 (2007)