On the Enumeration and Counting of Minimal Dominating sets in Interval and Permutation Graphs

Mamadou Moustapha Kanté[1], Vincent Limouzy[1], Arnaud Mary[1], Lhouari Nourine[1], and Takeaki Uno[2]

[1] Clermont-Université, Université Blaise Pascal, LIMOS, CNRS, France
[2] National Institute of Informatics, Japan

Abstract. We reduce (in polynomial time) the enumeration of minimal dominating sets in interval and permutation graphs to the enumeration of paths in DAGs. As a consequence, we can enumerate in linear delay, after a polynomial time pre-processing, minimal dominating sets in interval and permutation graphs. We can also count them in polynomial time. This improves considerably upon previously known results on interval graphs, and to our knowledge no output polynomial time algorithm for the enumeration of minimal dominating sets and their counting were known for permutation graphs.

1 Introduction

The Minimum Dominating Set problem is a classic and well-studied graph optimisation problem. A *dominating set* in a graph G is a subset D of its set of vertices such that each vertex is either in D or has a neighbour in D. Computing a minimum dominating set has numerous applications in many areas, *e.g.*, networks, graph theory (see for instance the book [9]). In this paper we are interested in the enumeration of (inclusion-wise) *minimal dominating sets* in interval and permutation graphs. The Minimum Dominating Set problem is known to be tractable in linear time for these two classes [4].

There are two approaches in enumeration algorithms. The *input-sensitive* approach which uses classical worst-case running time analysis, *i.e.*, the running time depends on the length of the input. This approach is usually used in the exact algorithms community, and can be useful for obtaining upper bounds. For instance the best running exponential time algorithm that enumerates all minimal dominating sets in an n-vertex graph runs in time $O(1.7159^n)$ and is at the same time an upper bound to the number of minimal dominating sets in an n-vertex graph [7]. The *output-sensitive* approach measures the time complexity of an enumeration algorithm in the sum of the sizes of the input and output. An algorithm whose running time is bounded by a polynomial depending on the sum of the sizes of the input and output is called an *output-polynomial time* algorithm. In this paper we deal with the output-sensitive approach.

The existence of an output-polynomial time algorithm for the enumeration of minimal dominating sets of graphs is closely related to the well-known *Transversal problem* in hypergraphs. A *transversal* in a hypergraph is a subset of its ground set that intersects every of its hyperedges. The *Transversal problem* asks for an output-polynomial time algorithm for the enumeration of all the (inclusion-wise) minimal transversals of hypergraphs. This is a hard problem and is a long-standing open question (see for instance [5]). However, it is a well-studied problem due to is applications in several areas [5,6,8,13,16], and output-polynomial time algorithms are known to exist when restricted to some hypergraph classes (a summary of some known tractable cases is given in [10]). It is known that the set of minimal dominating sets of a graph is the same as the set of minimal transversals of its *closed neighbourhood hypergraph* [3]. Therefore, whenever the closed neighbourhood hypergraphs of a graph class is in one of the known tractable classes of hypergraphs, there exists an output-polynomial time algorithm for the enumeration of minimal dominating sets of graphs in this graph class. This is the case for instance of degenerate graph classes, line graphs, path-graphs, . . . [10,12]. However, not all closed neighbourhood hypergraphs are in those tractable cases, *e.g.*, closed neighbourhood hypergraphs of split graphs for instance. Indeed, Kanté et al. [11] have shown that there exists an output-polynomial time algorithm for the enumeration of minimal transversals in hypergraphs if and only if there exists one for the enumeration of minimal dominating sets in graphs.

An enumeration algorithm is said to be *linear delay* if it performs a polynomial time pre-processing in the size of the input and such that the delay between two consecutive outputs o_i and o_{i+1} is linear in the size of o_{i+1}. It is clear that linear delay enumeration algorithms are output-polynomial time algorithms. We give linear delay algorithms for the enumeration of minimal dominating sets in interval and permutation graphs, and we use for that the interval and the permutation model respectively. This improves considerably upon the known algorithms on interval graphs (the best known is designed for β-acyclic hypergraphs, and has delay between two consecutive outputs polynomial in the size of the input [5,6]). We can moreover count in polynomial time (in the size of the input) minimal dominating sets. We have no knowledge of such counting results. Liner delay is the best we can hope whenever we want to list the elements of each minimal dominating set. Our techniques can be summarised as follows.

1. We first build in polynomial time a directed acyclic graph (DAG for short) and prove that some paths in this DAG correspond exactly to minimal dominating sets.
2. These paths can be counted in linear time (in the size of the DAG), and can be listed in linear delay with a classical *Depth First Search algorithm*.

Let us describe briefly the construction of the DAG in interval graphs. One first observes that any minimal dominating set in an interval graph is a collection of paths. Secondly, by using the interval model and by ordering the intervals (say their left endpoints from left to right), one can construct any minimal dominating set. In fact if you take a vertical line and those vertices X whose intervals are before that vertical line, then for any minimal dominating set D either $D \cap X$ is a dominating set of the graph induced on X, or the vertex in D just after the vertical line (following the ordering) should be adjacent to those vertices in X not dominated by $D \cap X$. We show that by moving in the right way such a vertical line, we can construct any minimal dominating set by keeping track only of the last two chosen vertices. Following that, we can construct the DAG, the vertices of which will be those pairs (x, y) (with the left endpoint of x before the left endpoint of y) such that x and y can be both in a minimal dominating set, and the arcs are of the form $((x, y), (y, z))$ such that

- there is no vertex with its interval between the right endpoint of y and the left endpoint of z,
- $\{x, y, z\}$ can belong to a minimal dominating set.

We show that the minimal dominating sets of an interval graph are exactly those sets $\{x_{i_1}, \ldots, x_{i_{k+1}}\}$ such that there exists a path $(v_1, \ldots, v_k)$ with $v_j := (x_{i_j}, x_{i_{j+1}})$ with no intervals before the left endpoint of x_{i_1} and after the right endpoint of $x_{i_{k+1}}$.

For the permutation graphs, the construction is of the same flavor, but is more complicated. In fact each minimal dominating set can be still constructed from left to right (by ordering the bottom and top lines from left to right), but we need to keep track of the last three vertices, and following how their segments intersect, we need to keep one or two additional vertices. We postpone the details in Section 4.

Summary. In Section 2 we give some necessary definitions and we deal with interval graphs in Section 3. Permutation graphs are considered in Section 5. Some concluding remarks are given in Section 5.

2 Preliminaries

If A and B are two sets, $A \backslash B$ denotes the set $\{x \in A \mid x \notin B\}$. The power-set of a set V is denoted by 2^V. The size of a set A is denoted $|A|$.

We refer to [1] for graph terminology. The vertex set of a (directed) graph G is denoted by V_G and its edge set (or arc set) by E_G. We only deal with finite and simple (directed) graphs. We denote by n the size of the vertex set of a (directed) graph and by m the size of its edge (or arc) set. An arc from x to y in a directed graph is denoted by (x, y) and an edge between x and y in a graph is denoted by xy.

Let G be a graph. For a vertex x, we denote by $N_G(x)$ the set $\{y \in V_G \mid xy \in E_G\}$, and we let $N_G[x]$ be $N_G(x) \cup \{x\}$. For $X \subseteq V_G$, we write $N_G[X]$ and $N_G(X)$ for respectively $\bigcup_{x \in X} N_G[x]$ and $N_G[X] \setminus X$. We say that a vertex y is a *private neighbour with respect to* $D \subseteq V_G$ of x if $y \in N_G[x] \setminus N_G[D \setminus x]$. (When D is clear from the context, for convenience we will omit the expression "with respect to D".) Note that a private neighbour of a vertex $x \in D$ is either x itself, or a vertex in $V_G \setminus D$, but never a vertex $y \in D \setminus \{x\}$. The set of private neighbours of $x \in D$ is denoted by $P_D(x)$. A subset D of V_G is called an *irredundant set* if for all $x \in D$, we have $P_D(x) \neq \emptyset$. A subset D of V_G is called a *minimal dominating set* if it is an irredundant set, and each vertex in $V_G \setminus D$ has a neighbour in D.

An *intersection graph* is a graph in which each vertex corresponds to a set and two vertices are adjacent if and only if their corresponding sets intersect. The collection of sets in correspondence with the vertices of an intersection graph is called an *intersection model*. A graph is an *interval graph* if it has an intersection model consisting of intervals on a straight line. A graph is a *permutation graph* if it has an intersection model consisting of straight lines between two parallels. We assume without loss of generality that any interval graph (or permutation graph) is given with its intersection model. Indeed, the recognition and a construction of an intersection model can be done in linear time for any interval graph (or permutation graph). See for instance [2] for interval graphs and [15] for permutation graphs.

Given a graph G and a subset $\mathcal{C}$ of 2^{V_G}, we say that an algorithm enumerates $\mathcal{C}$ with *linear delay* if, after a pre-processing that takes time $p(n+m)$ for some polynomial p, it outputs the elements of $\mathcal{C}$ without repetitions, the delay between two consecutive outputs o_i and o_{i+1} being bounded by $O(|o_{i+1}|)$. It is worth noticing that an algorithm which enumerates a subset $\mathcal{C}$ of 2^{V_G} in linear delay outputs the set $\mathcal{C}$ in time $O\left(p(n+m) + \sum_{C \in \mathcal{C}} |C|\right)$ where p is the polynomial bounding the pre-processing time, and is optimal since it does not take asymptotically more time than the size $||\mathcal{C}|| := O\left(\sum_{C \in \mathcal{C}} |C|\right)$ of $\mathcal{C}$.

We finish these preliminaries with the following folklore theorem on the enumeration and the counting of maximal paths in directed acyclic graphs.

Theorem 1 (folklore). *Given a directed acyclic graph D and two disjoint subsets S and P of vertices of D, the enumeration of paths from vertices in S to vertices in P can be done in linear delay. Moreover, counting these paths can be done in linear time in the size of D.*

3 Interval Graphs

We may suppose without loss of generality that in an intersection model of an interval graph all endpoints are pairwise distinct. For an interval graph G, let us denote its interval model by I_G, and for each vertex x of G let $I_G(x)$ be the interval in I_G associated with x. We number the endpoints of the intervals from left to right and we denote by $s(x)$ and $e(x)$ the left and right endpoint of $I_G(x)$ respectively. We can therefore assume that $I_G(x) := [s(x), e(x)]$ and will be viewed as the set of points on the line between $s(x)$ and $e(x)$.

We linearly order the vertices of an interval graph G with the linear order $\preceq$ such that $x \preceq y$ whenever $s(x) \leq s(y)$. We can therefore consider that the vertices of G are enumerated as $x_1, x_2, \ldots, x_n$ with $x_i \preceq x_j$ whenever $i \leq j$. We will in the sequel consider any subset D of V_G as linearly ordered by $\preceq$ and when we write $\{x_{i_1}, \ldots, x_{i_k}\}$, then we consider $x_{i_j} \preceq x_{i_\ell}$ whenever $j \leq \ell$. The proof of the following lemma is straightforward.

Lemma 2. *Let D be an irredundant set of an interval graph G. Then for all distinct vertices x and y in D, the sets $I_G(x) \setminus I_G(y)$ and $I_G(y) \setminus I_G(x)$ are non empty.*

The following is an easy corollary of Lemma 2.

Corollary 3. *Let D be a minimal dominating set of an interval graph G. Then for all distinct vertices x and y in D, we have $e(x) < e(y)$ whenever $x \prec y$.*

For $x \in V_G$, we let $NC(x)$ be the set $\{y \in V_G \mid s(y) > e(x)\}$, and $nc^s(x)$ and $nc^e(y)$ be respectively $\min\{s(y) \mid y \in NC(x)\}$ and $\min\{e(y) \mid y \in NC(x)\}$. Notice that if y is such that $s(y) = nc^s(x)$, then we do not have necessarily $e(y) = nc^e(x)$, and vice-versa. For $D := \{x_{i_1}, \ldots, x_{i_k}\}$ a subset of the vertex set of an interval graph G and $j \leq k$, we denote by D_j the subset $\{x_{i_1}, \ldots, x_{i_j}\}$ of D, and we let $p_D(x_{i_j}) := \min\{e(y) \mid y \in P_{D_j}(x_{i_j})\}$.

Lemma 4. *Let $D := \{x_{i_1}, \ldots, x_{i_k}\}$ be an irredundant set of an interval graph G. For $j \leq k$, let x_{p_j} be such that $e(x_{p_j}) = p_D(x_{i_j})$. Then $x_{p_j} \in P_D(x_{i_j})$.*

Proof. If there exists j' such that $x_{p_j} \in N_G(x_{i_{j'}})$, then $j' > j$, and thus $s(x_{i_j}) < s(x_{i_{j'}})$. Therefore, $s(x_{i_{j'}}) < e(x_{p_j})$, and since $e(x_{p_j}) < e(y)$ for all $y \in P_D(x_{i_j})$, we would have $x_{i_{j'}}$ adjacent to all vertices in $P_D(x_{i_j})$. A contradiction with the fact that D is an irredundant set. □

The following characterises minimal dominating sets in interval graphs, and will be the core of our algorithm.

Proposition 5. *A subset $D := \{x_{i_1}, \ldots, x_{i_k}\}$ of an interval graph G is a minimal dominating set if and only if the following conditions hold.*

1. *For all $\ell \leq n$, we have $s(x_{i_1}) \leq e(x_\ell)$.*
2. *For all $\ell \leq n$, we have $e(x_{i_k}) \geq s(x_\ell)$.*
3. *For all $j \leq k$, we have $j = k$ if $NC(x_{i_j}) = \emptyset$, otherwise we have $e(x_{i_{j+1}}) \geq nc^s(x_{i_j})$.*
4. *For all $1 \leq j < k$, we have $p_D(x_{i_j}) < s(x_{i_{j+1}}) \leq nc^e(x_{i_j})$.*

The following tells us how to compute the private neighbour of a vertex.

Proposition 6. *Let $D := \{x_{i_1}, \ldots, x_{i_k}\}$ be an irredundant set. Then for all $x \in N_G[x_{i_k}]$, we have $x \in P_D(x_{i_k})$ if and only if $e(x_{i_{k-1}}) < s(x)$.*

In order to enumerate in linear delay, and count in polynomial time, the set of minimal dominating sets of an interval graph G, we associate with it a DAG, denoted by $Dag_I(G)$, where the paths from a subset of the sources to a subset of the sinks correspond exactly to minimal dominating sets of G. Let G be an interval graph. The graph $Dag_I(G)$ has vertex set the pairs (x_i, x_j) such that

$$\begin{aligned}
&(V.1) \qquad && x_i \preceq x_j, \\
&(V.2) \qquad && p_{\{x_i\}}(x_i) < s(x_j) \leq nc^e(x_i), \\
&(V.3) \qquad && NC(x_i) \neq \emptyset \text{ and } e(x_j) \geq nc^s(x_i),
\end{aligned}$$

and it has as arc set the set of pairs $((x_i, x_j), (x_j, x_k))$ such that

$$(E.1) \qquad p_{\{x_i, x_j\}}(x_j) < s(x_k) \leq nc^e(x_j).$$

A vertex (x_i, x_j) of $Dag_I(G)$ is called an *initial vertex* if $s(x_i) \leq e(x_\ell)$ for all $1 \leq \ell \leq n$. A vertex (x_i, x_j) of $Dag_I(G)$ is called a *final vertex* if $e(x_j) \geq s(x_\ell)$ for all $1 \leq \ell \leq n$, and then $NC(x_j) = \emptyset$.

Lemma 7. *For every interval graph G, we have the following.*

1. *$Dag_I(G)$ is a DAG.*
2. *If a vertex (x_i, x_j) of $Dag_I(G)$ is an initial vertex (resp. a final vertex), then it is a source (resp. a sink) of $Dag_I(G)$.*
3. *$Dag_I(G)$ can be constructed in time $O(n^3)$.*

Proposition 8. *Let G be an interval graph and let v_1 and v_k be respectively an initial vertex and a final vertex of $Dag_I(G)$. Then $(v_1, v_2, \ldots, v_k)$ is a path of $Dag_I(G)$ if and only if $\{x_{i_1}, \ldots, x_{i_{k+1}}\}$ is a minimal dominating set of G of size greater than or equal to 2 with $v_j := (x_{i_j}, x_{i_{j+1}})$.*

We can now state the main theorem of the section.

Theorem 9. *Let G be an interval graph. Then, after a pre-processing in time $O(n^3)$, one can enumerate in linear delay the minimal dominating sets of G. One can moreover count them in time $O(n^3)$.*

Proof. By Lemma 7 the DAG $Dag_I(G)$ can be constructed in time $O(n^3)$. By Proposition 8, there is a bijection between paths from initial vertices to final vertices in $Dag_I(G)$ and minimal dominating sets of G of size at least 2.

It remains to deal now with minimal dominating sets of size 1. For each x, we can determine in time $O(n)$ if $\{x\}$ is a minimal dominating set. So, let $S := \{x \in V_G \mid \{x\} \text{ is a minimal dominating set of } G\}$. The set S can be constructed at the same time as $Dag_I(G)$. We let G' be the DAG obtained from $Dag_I(G)$ by adding new vertices v_x to $Dag_I(G)$, with in-degree and out-degree 0, for each $x \in S$. Therefore, each such new vertex v_x is a source and a sink at the same time. We define the following subsets of $V_{G'}$.

$$\begin{aligned}
S &:= \{v_x \mid x \in S\} \cup \{v \in V_{Dag_I(G)} \mid v \text{ is an initial vertex}\}, \\
T &:= \{v_x \mid x \in S\} \cup \{v \in V_{Dag_I(G)} \mid v \text{ is a final vertex}\}.
\end{aligned}$$

It is clear now that paths from S to T in G' are in bijection with all the minimal dominating sets of G. Since, paths from S to T in DAGs can be listed in linear delay, and be counted in linear time (see Theorem 1), we are done. □

4 Permutation Graphs

For a permutation graph G let us denote its permutation model by L_G, and for each vertex x of G let $L_G(x)$ be the segment in L_G corresponding to x. We number the endpoints of segments from left to right and we denote by $b(x)$ and $t(x)$ the endpoints of $L_G(x)$ on the bottom line and top line respectively. All endpoints are assumed to be different without loss of generality. We order the vertices of G by their bottom line endpoints, and then the vertices of G are assumed to be enumerated as $x_1, \ldots, x_n$ where $b(x_i) \leq b(x_j)$ whenever $i \leq j$. As in the interval case, we will also consider any subset D of V_G as linearly ordered, and when we write $\{x_{i_1}, \ldots, x_{i_k}\}$, then we consider $i_1 < \ldots < i_k$ and hence $b(x_{i_1}) < \cdots < b(x_{i_k})$. For two vertices x and y of G, we say that $L_G(x) < L_G(y)$ whenever $b(x) < b(y)$ and $t(x) < t(y)$.

For a subset $D := \{x_{i_1}, \ldots, x_{i_{k+1}}\}$ of G, if $k \geq 4$, we let x_D be the vertex $x_{i_r} \in D$ such that $t(x_{i_r}) := \max\{t(x_{i_\ell}) \mid x_{i_\ell} \in D \text{ and } \ell < k-2\}$; if $k \geq 3$, we let

$$A(D) := \begin{cases} D & \text{if } k = 3, \\ \{x_D, x_{i_{k-2}}, x_{i_{k-1}}, x_{i_k}, x_{i_{k+1}}\} & \text{if } k \geq 4. \end{cases}$$

Lemma 10. *Let D be an irredundant set of a permutation graph G. Then $G[D]$ contains neither triangles nor claws. Therefore, for each $x \in D$, $d_{G[D]}(x) \leq 2$.*

Lemma 11. *Let $D' := \{x_{i_1}, x_{i_2}, \ldots, x_{i_{k+1}}\}$ with $k \geq 4$ be a subset of the vertex set of a permutation graph G such that $D := \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ and $A(D')$ are irredundant sets of G. Then for all $l \leq i_{k-3}$, $N_G[x_{i_{k+1}}] \cap P_D(x_{i_l}) = \emptyset$.*

In the next lemmas we show how to construct irredundant sets of a permutation graph G from left to right. Indeed, we characterise exactly the situations where an irredundant set $D := \{x_{i_1}, \ldots, x_{i_k}\}$ can be extended to an irredundant set $D' := D \cup \{x_{i_{k+1}}\}$, and we show that for deciding the extension we need only know $x_{i_{k-2}}, x_{i_{k-1}}, x_{i_k}$, and following the intersections of $L_G(x_{i_{k-2}}), L_G(x_{i_{k-1}})$ and $L_G(x_{i_k})$ we need to know also either $x_{D'}$ or the vertex x_s such that $t(x_s) := \min\{y \in P_D(y)\}$ with y such that $t(y) := \min\{t(x_{i_{k-2}}), t(x_{i_{k-1}}), t(x_{i_k})\}$, or both. The cases summarising the intersections of $L_G(x_{i_{k-2}}), L_G(x_{i_{k-1}})$ and $L_G(x_{i_k})$ are depicted in Fig. 1.

Lemma 12 (Case 1). *Let $D' := \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_{k+1}}\}$ with $k \geq 3$ be a subset of the vertex set of a permutation graph G such that $D := \{x_{i_1}, \ldots, x_{i_k}\}$ is an irredundant set of G and $\{x_{i_{k-2}}, x_{i_{k-1}}, x_{i_k}\}$ corresponds to Case (1) of Fig. 1. Then D' is an irredundant set of G if and only if*

1. *$A(D')$ is an irredundant set of G,*
2. *$t(x_{i_{k+1}}) > \min\{t(y) \mid y \in P_D(x_{i_{k-2}})\}$.*

Lemma 13 (Case 2). *Let $D' := \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_{k+1}}\}$ with $k \geq 3$ be a subset of the vertex set of a permutation graph G such that $D := \{x_{i_1}, \ldots, x_{i_k}\}$ is an irredundant set of G and $\{x_{i_{k-2}}, x_{i_{k-1}}, x_{i_k}\}$ corresponds to Case (2) of Fig. 1. Then D' is an irredundant set of G if and only if $A(D')$ is an irredundant set of G.*

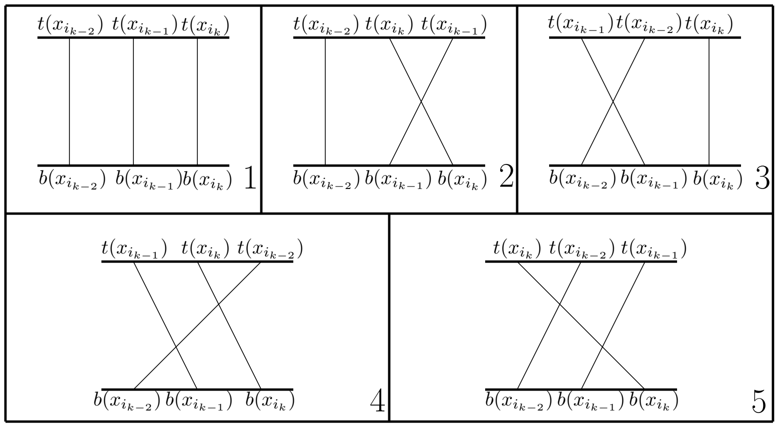

Fig. 1. Different cases following the intersections of $L_G(x_{i_{k-2}}), L_G(x_{i_{k-1}})$ and $L_G(x_{i_k})$.

Lemma 14 (Case 3). *Let $D' := \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_{k+1}}\}$ with $k \geq 3$ be a subset of the vertex set of a permutation graph G such that $D := \{x_{i_1}, \ldots, x_{i_k}\}$ is an irredundant set of G and $\{x_{i_{k-2}}, x_{i_{k-1}}, x_{i_k}\}$ corresponds to Case (3) of Fig. 1. Then D' is an irredundant set of G if and only if*

1. *$A(D')$ is an irredundant set of G,*
2. *$t(x_{i_{k+1}}) > \min\{t(y) \mid y \in P_D(x_{i_{k-1}})\}$.*

Lemma 15 (Case 4). *Let $D' := \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_{k+1}}\}$ with $k \geq 3$ be a subset of the vertex set of a permutation graph G such that $D := \{x_{i_1}, \ldots, x_{i_k}\}$ is an irredundant set of G and $\{x_{i_{k-2}}, x_{i_{k-1}}, x_{i_k}\}$ corresponds to Case (4) of Fig. 1. Then D' is an irredundant set of G if and only if $A(D')$ is an irredundant set of G.*

Lemma 16 (Case 5). *Let $D' := \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_{k+1}}\}$ with $k \geq 3$ be a subset of the vertex set of a permutation graph G such that $D := \{x_{i_1}, \ldots, x_{i_k}\}$ is an irredundant set of G and $\{x_{i_{k-2}}, x_{i_{k-1}}, x_{i_k}\}$ corresponds to Case (5) of Fig. 1. Then D' is an irredundant set of G if and only if*

1. *$A(D') \setminus x_{D'}$ is an irredundant set of G,*
2. *$t(x_{i_{k+1}}) > \min\{t(v) \mid v \in P_D(x_{i_k})\}$.*

The next proposition shows that minimal dominating sets are exactly those irredundant sets $\{x_{i_1}, \ldots, x_{i_k}\}$ with no segments before x_{i_1} (after x_{i_k}), and for each $2 \leq l < k$, there do not exist vertices y with $L_G(x_{i_{l-2}}) < L_G(y)$ and y not in $N_G[D_{l+1}]$.

Proposition 17. *Let $D := \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ be an irredundant set of a permutation graph G. Then D is a minimal dominating set of G if and only if the following conditions are fulfilled*

1. *for each* l, $t(x_l) \geq \min(t(x_{i_1}), t(x_{i_2}), t(x_{i_3}))$ *or* $b(x_l) \geq b(x_1)$,
2. $\{y \mid L_G(x_{i_{k-2}}) < L_G(y)$ *and* $y \notin N_G[D]\} = \emptyset$,
3. *for all* $2 \leq l < k$, *if* $L_G(x_{i_l})$ *does not intersect* $L_G(x_{i_{l-1}})$ *then* $\{y \mid L_G(x_{i_{l-2}}) < L_G(y) < L_G(x_{i_{l-1}})$ *and* $y \notin N_G[D_{l+1}]\} = \emptyset$. *Furthermore,* $\{y \mid L_G(x_{i_{l-1}}) < L_G(y),\ t(y) < t(x_{i_{l+1}}),$ *and* $b(y) < b(x_{i_l})$ *and* $y \notin N_G[D_{l+1}]\} = \emptyset$ *if* $L_G(x_{i_{l+1}})$ *intersects* $L_G(x_{i_l})$.

Proof. Assume that D is a minimal dominating set. Suppose that (1) is false, then there exists x such that $L_G(x) < L_G(x_1)$, $L_G(x) < L_G(x_2)$ and $L_G(x) < L_G(x_3)$. Since D is a dominating set there exists $l \geq 4$ such that $L_G(x_{i_l})$ intersects $L_G(x)$, *i.e.*, $t(x_{i_l}) < t(x)$. But in this case $L_G(x_{i_l})$ intersects $L_G(x_{i_1})$, $L_G(x_{i_2})$ and $L_G(x_{i_3})$ which contradicts Lemma 10. If (2) is not satisfied, there exists x such that $x \notin N_G[D]$ which is in contradiction with D being a dominating set. Now let $2 \leq l < n$ such that $L_G(x_{i_l})$ does not intersect $L_G(x_{i_{l-1}})$. Assume that there exists $x \in \{y \mid L_G(x_{i_{l-2}}) < L_G(y) < L_G(x_{i_{l-1}})$ *and* $y \notin N_G[D_{l+1}]\}$. Then x is not covered by D_{l+1} and since D is a dominating set, there exists $s > l+1$ such that $x \in N_G[x_{i_s}]$. Now since $b(x_{i_s}) > b(x_{i_{l+1}})$, we have that $t(x_{i_s}) < t(x)$ but then $d_{G[D]}(x_s) > 3$ contradicting Lemma 10. Now assume that $L_G(x_{i_{l+1}})$ intersects $L_G(x_{i_l})$ and there exists $x \in \{y \mid L_G(x_{i_{l-1}}) < L_G(y),\ t(y) < t(x_{i_{l+1}}),$ *and* $b(y) < b(x_{i_l})$ *and* $y \notin N_G[D_{l+1}]\}$. Then there exists $s > l+1$ such that $x \in N_G[x_{i_s}]$. Since $b(x_{i_s}) > b(x_{i_{l+1}})$, we have that $t(x_{i_s}) < t(x)$ and then $\{x_{i_s}, x_{i_{l+1}}, x_{i_l}\}$ forms a triangle contradicting Lemma 10.

Let us show now that if (1), (2) and (3) are satisfied, then D is a minimal dominating set. Since D is an irredundant set of G, it remains to show that D is a dominating set. Assume not, *i.e.*, there exists x such that $x \notin N_G[D]$. We know that there exists $y \in D$ such that $L_G(y) < L_G(x)$ otherwise (1) would be violated. So let s such that $b(x_{i_s}) := \max\{b(y) \mid y \in D$ *and* $b(y) < b(x)\}$. We know that $s < k-2$, otherwise (2) would be violated. Now it is sufficient to notice that if $L_G(x_{i_{s+1}})$ intersects $L_G(x_{i_{s+2}})$ then (3) is violated with $l = s+1$, otherwise (3) is violated with $l = s+2$. □

Proposition 18. *Let* $D := \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_{i_{k+1}}\}$ *be a subset of the vertex set of a permutation graph* G *and let* $x \in \{y \mid L_G(x_{i_{k-2}}) < L_G(y)\}$. *Then* $x \in N_G[D]$ *if and only if* $x \in N_G[A(D)]$.

Let G be a permutation graph and let $\perp$ be a non vertex of G. Thanks to Lemmas 12-16, and Propositions 17 and 18, the DAG to which some paths correspond to the minimal dominating sets of G will have as vertices those quintuplets $(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5})$ in $(V_G \cup \perp)^5$ that correspond to Cases (1)-(5) of Fig. 1, and the arcs describe the construction of minimal dominating sets from left to right (and thanks to Proposition 18 the arcs can be constructed in polynomial time).

We denote by Initial(G) those quintuplets $(x_i, x_j, x_k, \perp, x_s) \in (V_G \cup \perp)^5$ such that $A := \{x_i, x_j, x_k\}$ is an irredundant set of G, and one of the following conditions hold

(I.1) $t(x_s) := \min\{t(y) \mid y \in P_A(x_i)\}$ if A corresponds to Case (1) of Fig. 1 and $\{y \mid L_G(y) < L_G(x_i)\} = \emptyset$.

(I.2) $x_s = \perp$ if A corresponds to Case (2) of Fig. 1 and $\{y \mid t(y) < t(x_k),\ b(y) < b(x_j),\ \textit{and}\ y \notin N_G[A]\} = \emptyset$.
(I.3) $t(x_s) := \min\{t(y) \mid y \in P_A(x_j)\}$ if A corresponds to Case (3) of Fig. 1 and $\{y \mid t(y) < t(x_j)\ \textit{and}\ b(y) < b(x_i)\} = \emptyset$.
(I.4) $x_s = \perp$ if A corresponds to Case (4) of Fig. 1 and $\{y \mid t(y) < t(x_j),\ b(y) < b(x_i),\ \textit{and}\ y \notin N_G[A]\} = \emptyset$.
(I.5) $t(x_s) := \min\{t(y) \mid y \in P_A(x_k)\}$ if A corresponds to Case (5) of Fig. 1 and $\{y \mid t(y) < t(x_k)\ \textit{and}\ b(y) < b(x_i)\} = \emptyset$.

We denote by Regular(G) those quintuplets $(x_i, x_j, x_k, x_r, x_s) \in (V_G \cup \perp)^5$ such that $A := \{x_i, x_j, x_k\}$ is an irredundant set of G, and one of the following conditions hold

(R.1) $r < i$ and $x_s = \perp$ if A corresponds to Case (2) or (4) of Fig. 1.
(R.2) $r < i$ and $x_s \in N_G[x_i] \setminus N_G[\{x_r, x_j, x_k\}]$ if A corresponds to Case (1) of Fig. 1.
(R.3) $r < i$ and $x_s \in N_G[x_j] \setminus N_G[\{x_r, x_i, x_k\}]$ if A corresponds to Case (3) of Fig. 1.
(R.4) $x_r = \perp$ and $x_s \in N_G[x_k] \setminus N_G[\{x_r, x_i, x_j\}]$ if A corresponds to Case (5) of Fig. 1.

For $v := (x_i, x_j, x_k, x_r, x_s)$ in Initial(G) $\cup$ Regular(G) we let

$$A(v) := \begin{cases} \{x_r, x_i, x_j, x_k\} & \text{if } x_r \neq \perp, \\ \{x_i, x_j, x_k\} & \text{otherwise.} \end{cases}$$

We let $Dag_P(G)$ be the DAG with vertex set Initial(G) $\cup$ Regular(G) and such that there is an arc (v_1, v_2) with $v_1 := (x_i, x_j, x_k, x_r, x_s)$ and $v_2 := (x_j, x_k, x_l, x_{r'}, x_{s'})$ if the following conditions are satisfied.

(A.1) If $x_{r'} \neq \perp$, then $t(x_{r'}) = \max(t(x_r), t(x_i))$ if $x_r \neq \perp$, otherwise $t(x_{r'}) = x_i$.
(A.2) $A := A(v_1) \cup \{x_l\}$ is an irredundant set of G.
(A.3) $t(x_l) > t(x_s)$ if $x_s \neq \perp$.
(A.4) If $L_G(x_k)$ does not intersect $L_G(x_j)$ then $\{y \mid L_G(x_i) < L_G(y) < L_G(x_j)\ \textit{and}\ y \notin N_G[A]\} = \emptyset$. Furthermore, $\{y \mid L_G(x_j) < L_G(y),\ t(y) < t(x_l),\ \textit{and}\ b(y) < b(x_k)\ \textit{and}\ y \notin N_G[A]\} = \emptyset$ if $L_G(x_l)$ intersects $L_G(x_k)$.
(A.5) If $x_{s'} \neq \perp$ then
  (A.5.1) $x_s = x_{s'}$ if $x_s \neq \perp$ and $\min\{t(x_i), t(x_j), t(x_k)\} = \min\{t(x_j), t(x_k), t(x_l)\}$.
  (A.5.2) $t(x_{s'}) = \min\{t(x) \mid x \in P_A(y)\}$ where y is the vertex such that $t(y) = \min\{t(x_j), t(x_k), t(x_l)\}$ otherwise.

A vertex v of $Dag_P(G)$ is called an *initial vertex* if it belongs to Initial(G) and it is called a *final vertex* if $\{y \mid L_G(x_i) < L_G(y)\ \textit{and}\ y \notin N_G[A(v)]\} = \emptyset$. The set of final vertices is denoted by Final(G).

Proposition 19. *Let G a be permutation graph. A subset $D := \{x_{i_1}, x_{i_2}, ..., x_{i_{k+2}}\}$ of V_G is a minimal dominating set of G of size greater than or equal to three, if and only if there exists a path $(v_1, ..., v_k)$ of $Dag_P(G)$ where $v_j := (x_{i_j}, x_{i_{j+1}}, x_{i_{j+2}}, x_{r_j}, x_{s_j})$, and $v_1 \in$ Initial(G) and $v_k \in$ Final(G).*

Theorem 20. *Let G be a permutation graph. Then, after a pre-processing in time $O(n^8)$, one can enumerate in linear delay the minimal dominating sets of G. One can moreover count them in time $O(n^8)$.*

Proof. By Proposition 19 there is a bijection between minimal dominating sets of G of size greater than or equal to 3, and paths from $\textsc{Initial}(G)$ to $\textsc{Final}(G)$ in $Dag_P(G)$. Let $S_1 := \{x \in V_G \mid \{x\}$ is a minimal dominating of $G\}$, and let $S_2 := \{\{x, y\} \mid \{x, y\}$ is a minimal dominating set of $G\}$. Clearly S_1 and S_2 can be constructed in time $O(n^3)$. We let G' be the DAG obtained from $Dag_P(G)$ by adding new vertices v_x and v_{xy}, with in-degree and out-degree 0, for each $x \in S_1$ and each $\{x, y\} \in S_2$. We let

$$S := \textsc{Initial}(G) \cup \{v_x \mid x \in S_1\} \cup \{v_{x,y} \mid \{x, y\} \in S_2\},$$
$$T := \textsc{Final}(G) \cup \{v_x \mid x \in S_1\} \cup \{v_{x,y} \mid \{x, y\} \in S_2\}.$$

It is now clear that paths from S to T in G' corresponds to minimal dominating sets of G, and since such paths can be listed in linear delay in DAGs, and be counted in linear time (see Theorem 1), it remains to show that $Dag_P(G)$ can be constructed in time $O(n^8)$.

First notice that the number of vertices of $Dag_P(G)$ is bounded by n^5. Furthermore, for a quintuplet $(x_i, x_j, x_k, x_r, x_s)$, we can check in linear time if it is a vertex of $Dag_P(G)$. Then the time complexity for the creation of $V_{Dag_P(G)}$ is bounded by $O(n^6)$. Now we analyse the time complexity to compute the neighbourhood of a vertex of $Dag_P(G)$. For $v := (x_i, x_j, x_k, x_r, x_s) \in V_{Dag_P(G)}$, we need to check if (v, w) is an arc for all $w = (x_j, x_k, x_l, x_{r'}, x_{s'})$ where $t(x_{r'}) = \max(t(x_r), t(x_i))$. There is at most n^2 candidates for w, and we can check if there is an arc from v to w in linear time. So the total time complexity to create $Dag_P(G)$ is bounded by $O(n^8)$. □

5 Conclusion

If we want to list a subset $\mathcal{C} \subseteq 2^{V_G}$ of a graph G, by outputting each element of each $C \in \mathcal{C}$, then the size of $\mathcal{C}$ defined as $\sum_{C \in \mathcal{C}} |C|$ is a lower bound. We have proposed linear delay algorithms for the enumeration of minimal dominating sets in interval and permutation graphs the running times of which match the above lower bound. Our techniques allow also a polynomial time algorithm (in the sizes of the graphs) for counting minimal dominating sets. The techniques used in this paper can be adapted to list with linear delay (and also count in polynomial time) the *minimal connected dominating sets* and *minimal total dominating sets* in interval and permutation graphs [14]. The results presented here and in [14] can be extended to *trapezoid graphs*, but the proofs are more tricky. It is not known whether one can enumerate minimal dominating sets in *circle graphs*, can we adapt some of our techniques to them?

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A Proof of Theorem 1

In this appendix, we recall the algorithms that enumerate (and count) maximal paths in a DAG. The following algorithm lists with linear delay all paths in a dag G from S to P. It can be used to also count all paths from S to P, by commenting the call to the function `All-Maximal-Path`$(G, s, p, \emptyset)$, and uncommenting the call to the function `Count-Maximal-Path`(G, s, p). It is straightforward to check that the algorithms use linear space in the size of the input (we can give a reference to the graph), and perform the desired outputs.

Algorithm 1: List-Count-Path(S, P)

```
Input: A DAG G, and two subsets of its vertex set S and P.
Output: List all paths (with linear delay) from S to P.
begin
    Let s and p be new vertices
    Add (s, x) for all x ∈ S
    Add (x, p) for all x ∈ P
    Let (s = x_1, ..., x_n = p) be a topological ordering of G
    // The following loop deletes all vertices not in a path from S to P
    Mark all vertices in P ∪ {p}
    for i = n − 1 to 1 do
        for each neighbour y of x_i do
            if y is marked then
                mark x_i
            end
        end
    end
    Delete all vertices which are not marked
    All-Maximal-Path(G, s, p, ∅) // lists paths with linear delay
    // Count-Maximal-Path(G, s, p) // count paths
end
```

B Proofs from Section 3

Proof (of Proposition 5). Suppose that D is a minimal dominating set of G. If (1) does not hold, then there exists $\ell \leq n$ such that $s(x_{i_1}) > e(x_\ell)$. Since for all $1 < j \leq k$, $s(x_{i_j}) > s(x_{i_1})$, we can conclude that $N_G[x_\ell] \cap D = \emptyset$, a contradiction. If (2) is not satisfied, then there exists $\ell \leq n$ such that $e(x_{i_k}) < s(x_\ell)$. By Corollary 3, we have that $e(x_{i_k}) > e(x_{i_\ell})$ for all $1 \leq \ell < k$, we can again conclude in this case that $N_G[x_\ell] \cap D = \emptyset$, which yields a contradiction.

We now show (3). Let $j \leq k$ such that $NC(x_{i_j}) = \emptyset$. Since $x_{i_j} \preceq x_{i_k}$, we have $s(x_{i_j}) \leq s(x_{i_k})$. Therefore, $N_G[x_{i_k}] \subseteq N_G[x_{i_j}] \cup NC(x_{i_j})$, and since $NC(x_{i_j}) = \emptyset$, we have $N_G[x_{i_k}] \subseteq N_G[x_{i_j}]$. So, assuming that $j \neq k$ would

Algorithm 2: All-Maximal-Path(G, x, p, D)

```
Input: A DAG G with one single sink p, a vertex x and a subset D.
Output: Lists all paths from x to p with linear delay and appends D.
begin
    if x = p then
        output(D)
    end
    else
        for each neighbour y of x do
            All-Maximal-Path(G, y, p, D ∪ {x})
        end
    end
end
```

Algorithm 3: Count-Maximal-Path(G, s, p)

```
Input: A DAG G with one single source s and one single sink p.
Output: The number of paths from s to p
begin
    Initialise Count[x] to 0 for every vertex in G, and Count[p] = 1
    for i = n to 1 do
        for each neighbour y of x_i do
            Count[x_i] = Count[x_i] + Count[y]
        end
    end
    Output Count[s]
end
```

contradict the fact that D is a minimal dominating set. Hence, $j = k$ whenever $NC(x_{i_j}) = \emptyset$. Assume now that $NC(x_{i_j}) \neq \emptyset$. Then $j < k$. If $e(x_{i_{j+1}}) < nc^s(x_{i_j})$, then $N_G[x_{i_{j+1}}] \cap NC(x_{i_j}) = \emptyset$. Since $s(x_{i_j}) < s(x_{i_{j+1}})$, we can therefore conclude that $N_G[x_{i_{j+1}}] \subseteq N_G[x_{i_j}]$. But again this contradicts the fact that D is a minimal dominating set.

It remains to show (4). Let $1 \leq j < k$. Because D is a minimal dominating set $P_D(x_{i_j})$ is not empty, and therefore $p_D(x_{i_j})$ exists. Moreover, by (3) $NC(x_{i_j})$ is not empty, and therefore $nc^e(x_{i_j})$ is well-defined. So, the inequality in (4) is well-defined. Assume that (4) does not hold. Then either $p_D(x_{i_j}) \geq s(x_{i_{j+1}})$ or $s(x_{i_{j+1}}) > nc^e(x_{i_j})$. Assume first that $p_D(x_{i_j}) \geq s(x_{i_{j+1}})$ and let x_{p_j} be such that $e(x_{p_j}) = p_D(x_{i_j})$. If $e(x_{p_j}) \leq e(x_{i_j})$, then we would have $x_{p_j} \in N_G[x_{i_{j+1}}]$ which would contradict Lemma 4. If $e(x_{p_j}) > e(x_{i_j})$, then either $x_{p_j} \in N_G[x_{i_{j+1}}]$, which would contradict Lemma 4, or since $s(x_{i_j}) < s(x_{i_{j+1}})$, we will have $I_G(x_{i_{j+1}}) \subseteq I_G(x_{i_j})$ which contradicts Lemma 2. We can thus conclude that $p_D(x_{i_j}) < s(x_{i_{j+1}})$. Assume now that $s(x_{i_{j+1}}) > nc^e(x_{i_j})$ and let $x_{e_j} \in NC(x_{i_j})$ be such that $e(x_{e_j}) = nc^e(x_{i_j})$. Since $x_{e_j} \in NC(x_{i_j})$, we have $e(x_{i_j}) < s(x_{e_j})$, and by Corollary 3, we also have $e(x_{i_{j'}}) < s(x_{e_j})$ for all $j' < j$. Therefore,

$N_G[x_{e_j}] \cap \{x_{i_1}, \ldots, x_{i_j}\} = \emptyset$. Moreover, since $e(x_{e_j}) < s(x_{i_{j+1}})$ and $s(x_{i_{j+1}}) < s(x_{i_r})$ for all $r > j+1$, we can also conclude that $N_G[x_{e_j}] \cap \{x_{i_{j+1}}, \ldots, x_{i_k}\} = \emptyset$. Therefore, $N_G[x_{e_j}] \cap D = \emptyset$, which yields a contradiction with the fact that D is a dominating set.

We now assume that D satisfies Conditions (1) to (4). We first prove that D is a dominating set. Let x in $V_G \setminus D$. By (1) and (2) we know that $s(x_{i_1}) < e(x)$ and $e(x_{i_k}) > s(x)$. So let $r := \max\{j \leq k \mid s(x_{i_j}) \leq e(x)\}$. Notice that r is well-defined by (1). If $N_G[x] \cap \{x_{i_1}, \ldots, x_{i_r}\} \neq \emptyset$, then $N_G[x] \cap D \neq \emptyset$. So, suppose that $N_G[x] \cap \{x_{i_1}, \ldots, x_{i_r}\} = \emptyset$. Notice that by (2) we have necessarily $r < k$. Now, if $r < k$, then by (3) $NC(x_{i_r}) \neq \emptyset$ and $x \in NC(x_{i_r})$. So, $nc^e(x_{i_r}) \leq e(x)$. By definition of r, we have $s(x_{i_{r+1}}) > e(x)$, *i.e.*, $s(x_{i_{r+1}}) > nc^e(x_{i_r})$, which contradicts (4).

We now prove that D is minimal. We first prove that $P_{D_j}(x_{i_j}) \neq \emptyset$ for all $1 \leq j \leq k$. By definition of D_j, it is clear that $P_{D_1}(x_{i_1}) \neq \emptyset$. So, let $\ell > 1$ be the maximum $\leq k$ such that $P_{D_\ell}(x_{i_\ell}) \neq \emptyset$. If $\ell = k$, then we are done. So assume that $\ell < k$. By (3) $NC(x_{i_\ell}) \neq \emptyset$ and let $x_{s_\ell} \in NC(x_{i_\ell})$ be such that $s(x_{s_\ell}) = nc^s(x_{i_\ell})$. By (3) we have $e(x_{i_{\ell+1}}) \geq s(x_{s_\ell})$ and by (4) $s(x_{i_{\ell+1}}) \leq nc^e(x_{i_\ell})$. And since $nc^e(x_{i_\ell}) \leq e(x_{s_\ell})$, we can conclude that $I_G(x_{s_\ell}) \cap I_G(x_{i_{\ell+1}}) \neq \emptyset$, and thus $x_{s_\ell} \in N_G[x_{i_{\ell+1}}] \setminus N_G[x_{i_\ell}]$. Assume there exists $j < \ell$ such that $x_{s_\ell} \in N_G[x_{i_j}]$, *i.e.*, $e(x_{i_j}) > e(x_{i_\ell})$, and let j' be the maximum among such js. Since $s(x_{i_{j'+1}}) > s(x_{i_{j'}})$, we will have $I_G[x_{i_{j'+1}}] \subseteq I_G[x_{i_{j'}}]$, and this contradicts (3) because we would have $e(x_{i_{j'+1}}) < nc^s(x_{i_{j'}})$. Therefore, $x_{s_\ell} \in P_{D_{\ell+1}}(x_{i_{\ell+1}})$ since for all $j \leq \ell$, $x_{s_\ell} \notin N_G[x_{i_j}]$.

It remains now to prove that $P_D(x_{i_j}) \neq \emptyset$ for all $1 \leq j \leq k$. Since $P_{D_k}(x_{i_k}) \neq \emptyset$, we can conclude that $P_D(x_{i_k}) \neq \emptyset$. So let $1 \leq j < k$, and let x_{p_j} be such that $e(x_\ell^j) = p_D(x_{i_j})$. By (4), we have $s(x_{i_{j+1}}) > e(x_{p_j})$, *i.e.*, $x_{p_j} \notin N_G[x_{i_{j+1}}]$. And since $s(x_{i_r}) > s(x_{i_{j+1}})$ for all $r > j+1$, we can conclude that $x_{p_j} \notin \cup_{j+1 \leq r \leq k} N_G[x_{i_r}]$. Therefore, $x_{p_j} \in P_D(x_{i_j})$. □

Proof (of Proposition 6). Let $x \in N_G[x_{i_k}]$ be in $P_D(x_{i_k})$. Then $e(x) \geq s(x_{x_k})$ and $x \notin N_G[x_{i_{k-1}}]$. Therefore, $e(x_{i_{k-1}}) < s(x)$ since by definition $s(x_{i_{k-1}}) \leq s(x_{i_k})$.

Assume now that $x \in N_G[x_{i_k}]$ is such that $e(x_{i_{k-1}}) < s(x)$. By Corollary 3, we know that $e(x_{i_r}) < e(x_{i_{k-1}})$ for all $r < k-1$. Hence, $s(x) > e(x_{i_j})$ for all $j < k$. Therefore, $N_G[x] \cap D = x_{i_k}$, *i.e.*, $x \in P_D(x_{i_k})$. □

Proof (of Lemma 7). We first prove (1). Assume that there exists a circuit $(v_1, \ldots, v_k,\ v_{k+1} = v_1)$ with $v_j := (x_{i_j}, x_{i_{j+1}})$ for $1 \leq j \leq k$. Then, we would have $x_{i_1} \prec x_{i_2} \prec x_{i_3} \prec \cdots \prec x_{i_k} \prec x_{i_{k+1}} = x_{i_1}$ which is a contradiction with the fact that $\preceq$ is a linear order.

We now prove (2). If (x_i, x_j) is an initial vertex, then $s(x_i) \leq e(x_\ell)$ for all $1 \leq \ell \leq n$. Assume there exists (x_ℓ, x_i) such that $((x_\ell, x_i), (x_i, x_\ell))$ is an edge.

We would have $s(x_\ell) \leq s(x_i)$, and therefore $p_{\{x_\ell\}}(x_\ell) < s(x_i) \leq nc^e(x_\ell)$, and thus there exists x_s such that $s(x_i) > e(x_s)$, a contradiction.

Assume now that (x_i, x_j) is a final vertex. Since, $e(x_j) \geq s(x_\ell)$ for all $1 \leq i \leq n$, we have $NC(x_j) = \emptyset$. And thus there cannot exist a vertex (x_j, x_k). So, (x_i, x_j) is a sink.

It remains to prove (3). For each vertex x_i of G, one can compute $NC(x_i)$ in time $O(n)$ and compute $p_{\{x_i\}}(x_i)$, $nc^e(x_i)$ and $nc^s(x_i)$ in time $O(n \cdot \log(n))$. Since we can decide in constant time if $x_i \preceq x_j$, we can therefore construct the set of vertices of $Dag_I(G)$ in $O(n^2)$ since each of the conditions can be checked in constant time whenever $p_{\{x_i\}}(x_i)$, $nc^e(x_i)$ and $nc^s(x_i)$ are known. For each pair (x_i, x_j) by Proposition 6, one can compute $p_{\{x_i,x_j\}}(x_j)$ in time $O(|N_G[x_j]|)$. Therefore, one can compute the edges of $Dag_I(G)$ in time $O(n^3)$. ☐

Proof (of Proposition 8). By Proposition 5 it is enough to prove that $(v_1, \ldots, v_k)$ is a path of $Dag_I(G)$ if and only if Conditions (1)-(4) are satisfied (provided we assume D of size at least 2).

Assume first that $(v_1, \ldots, v_k)$ is a path of $Dag_I(G)$. Since v_1 is an initial vertex, then $s(x_{i_1}) \leq e(x_\ell)$ for all $1 \leq \ell \leq n$. Similarly, since v_k is a final vertex $e(x_{i_{k+1}}) \geq s(x_\ell)$ for all $1 \leq \ell \leq n$, and moreover $NC(x_{i_{k+1}}) = \emptyset$. Now, since for each j, $(x_{i_j}, x_{i_{j+1}})$ is a vertex of $Dag_I(G)$, then by (V.3) we have $e(x_{i_{j+1}}) \geq nc^s(x_{i_j})$. It remains now to check (4). Since (x_{i_1}, x_{i_2}) is a vertex of $Dag_I(G)$, by (V.2) $p_{\{x_{i_1}\}}(x_{i_1}) < s(x_{i_2}) \leq nc^e(x_{i_1})$ and then (4) is satisfied for $j = 1$. Let j be the maximum $< k$ such that (4) is satisfied and let us prove it for $j+1 < k$. Since (v_{j-1}, v_j) is an arc, then by (E.1) $p_{\{x_{i_{j-1}}, x_{i_j}\}}(x_{i_j}) < s(x_{i_{j+1}}) \leq nc^e(x_{i_j})$. But, by Proposition 6, $x \in P_{D_j}(x_{i_j})$ if and only if $e(x_{i_{j-1}}) < s(x)$, *i.e.*, $x \in P_{D_j}(x_{i_j})$ if and only if $x \in P_{\{x_{i_{j-1}}, x_{i_j}\}}(x_{i_j})$. Therefore, we can conclude that $p_D(x_{i_j}) < s(x_{i_{j+1}}) \leq nc^e(x_{i_j})$. Since now the four conditions of Proposition 5 are satisfied, we can conclude that D is a minimal dominating set of G.

Now assume that $D := \{x_{i_1}, \ldots, x_{i_{k+1}}\}$ is a minimal dominating set G of size at least 2. Then the four conditions of Proposition 5 are satisfied. For each $1 \leq j \leq k$, let $v_j := (x_{i_j}, x_{i_{j+1}})$. It is clear that each v_j is a vertex of $Dag_I(G)$ by Conditions (3) and (4). By Conditions (1), (2) and (3) one can conclude that v_1 is an initial vertex and v_k is a final vertex. To conclude it is enough to prove that (v_j, v_{j+1}) is an arc for all $1 \leq j < k$. But, by Condition (4) for all $1 \leq j < k$, we have $p_D(x_{i_j}) < s(x_{i_{j+1}}) \leq nc^e(x_{i_j})$, and by Proposition 6 we have $P_{\{x_{i_{j-1}}, x_{i_j}\}}(x_{i_j}) = P_{D_j}(x_{i_j})$, *i.e.*, $p_{\{x_{i_{j-1}}, x_{i_j}\}}(x_{i_j}) < s(x_{i_{j+1}}) \leq nc^e(x_{i_j})$. Therefore, (v_j, v_{j+1}) is an arc of $Dag_I(G)$. ☐

C Proofs from Section 4

Proof (of Lemma 10). Assume that $G[D]$ contains a claw $\{v, x, y, z\}$ with centre v and assume w.l.o.g that $L_G(x) < L_G(y) < L_G(z)$. Then we claim that $P_D(y) = \emptyset$. Indeed $u \in P_D(y)$ only if $(b(x) < b(u) < b(y)$ and $t(y) < t(u) < t(z))$ or

$(b(y) < b(u) < b(z)$ and $t(x) < t(u) < t(y))$ and in both cases $L_G(u)$ intersects $L_G(v)$. Now assume that $G[D]$ contains a triangle $\{x, y, z\}$ and assume w.l.o.g that $b(x) < b(y) < b(z)$. Then we claim that $P_D(y) = \emptyset$. Indeed let $u \in N_G(y)$, then either $(b(u) < b(y)$ and $t(y) < t(u))$ but in this case $L_G(u)$ intersects $L_G(z)$ or $(b(u) > b(y)$ and $t(y) > t(u))$ and in this case $L_G(u)$ intersects $L_G(x)$. □

Proof (of Lemma 11). Let $l \leq i_{k-3}$ and let $x \in P_D(x_{i_l})$. Let $\{t_1, t_2, t_3\} := \{t(x_{i_{k-2}}), t(x_{i_{k-1}}), t(x_{i_k})\}$ with $t_1 < t_2 < t_3$ and let y_i be such that $t(y_i) = t_i$. First notice that $t(x_{i_{k+1}}) > t_1$, otherwise $G[\{x_{i_{k-2}}, x_{i_{k-1}}, x_{i_k}, x_{i_{k+1}}\}]$ would induce either a claw or contain a triangle contradicting Lemma 10. Similarly, $t(x_{i_j}) < t_3$ for all $j \leq k-3$. Observe moreover that $b(x) < b(y_3)$, otherwise since $t(x_{i_l}) < t_3$, we would have $x \in N_G(y_3)$, a contradiction.

Assume first that $t(x) > t(x_{i_l})$, then $b(x) < b(x_{i_l})$ and thus $t(x_{i_l}) < t_1$, otherwise $L_G(x)$ and $L_G(y_1)$ would intersect. If $x \in N_G(x_{i_{k+1}})$, then $t(x_{i_{k+1}}) < t_1$ because $b(x_{i_{k+1}}) > b(x)$ and this would yield a contradiction with the observation that $t(x_{i_{k+1}}) < t_1$. Suppose now that $t(x) < t(x_{i_l})$, then $t(x_{i_{k+1}}) > t(x_{D'})$. Indeed if $t(x_{i_{k+1}}) < t(x_{D'})$, then we would have $t(x_{D'}), t(x_{i_{k+1}}) > t_2$ because otherwise $\{y_3, y_2, x_{D'}, x_{i_{k+1}}\}$ would induce either a claw with centre $x_{i_{k+1}}$ or contain a triangle. Now if $t(x_{D'}), t(x_{i_{k+1}}) > t_2$, then $\{y_1, y_2, x_{D'}, x_{i_{k+1}}\}$ would induce either a claw with centre $x_{D'}$ or contain a triangle. Assuming that $t(x_{i_{k+1}}) > t(x_{D'})$ we have $t(x) < t(x_{i_l}) < t(x_{D'}) < t(x_{i_{k+1}})$ and $b(x) < b(y_3) < b(x_{i_{k+1}})$, *i.e.*, $L_G(x) < L_G(x_{i_{k+1}})$. □

Proof (of Lemma 12). Assume first that D' is an irredundant set of G. Then since the irredundance property is closed under inclusion, (1) holds. Assume now that $t(x_{i_{k+1}}) \leq \min\{t(y) \mid y \in P_D(x_{i_{k-2}})\}$. By Lemma 10 we have $t(x_{i_{k+1}}) > t(x_{i_{k-2}})$. Moreover, $x \in P_D(x_{i_{k-2}})$ only if $(b(x) < b(x_{i_{k-2}})$ and $t(x_{i_{k-2}}) < t(x) < t(x_{i_{k-1}}))$ or $(b(x_{i_{k-2}}) < b(x) < b(x_{i_{k-1}})$ and $t(x) < t(x_{i_{k-2}}))$. Now, if $t(x_{i_{k+1}}) \leq \min\{t(y) \mid y \in P_D(x_{i_{k-2}})\}$, then $x \in P_D(x_{i_{k-2}})$ implies that $(b(x) < b(x_{i_{k-2}})$ and $t(x_{i_{k-2}}) < t(x) < t(x_{i_{k-1}}))$. But in this case we would have $P_D(x_{i_{k-2}}) \subseteq N_G(x_{i_{k+1}})$, contradicting the irredundancy of D' since $P_{D'}(x_{i_{k-2}}) \subseteq P_D(x_{i_{k-2}})$.

Assume now that (1) and (2) are true and let us show that $P_{D'}(x_{i_j}) \neq \emptyset$ for all $1 \leq j \leq k+1$. We may assume that $k \geq 4$ since otherwise $A(D') = D'$ and then we are done by assumption. So, $x_{D'}$ exists. Let x be such that $t(x) = \min\{t(y) \mid y \in P_D(x_{i_{k-2}})\}$. By definition $b(x) < b(x_{i_{k-1}}) < b(x_{i_{k+1}})$, and by (2) $t(x) < t(x_{i_{k+1}})$. Therefore, $L_G(x) < L_G(x_{i_{k+1}})$, *i.e.*, $x \notin N_G(x_{i_{k+1}})$, and hence $P_{D'}(x_{i_{k-2}}) \neq \emptyset$. Let $k-1 \leq j \leq k+1$ and let $x \in P_{A(D')}(x_{i_j})$. If there exists $s \leq k-3$ such that $x \in N_G(x_{i_s})$, then $t(x_{i_s}) > t(x)$ (because if $x \in P_{A(D')}(x_{i_j})$, then $L_G(x_{i_{k-2}}) < L_G(x)$), and thus $t(x_{D'}) > t(x)$ (because by definition $t(x_{D'}) > t(x_{i_s})$ for all $s \leq k-3$). Since $b(x_{D'}) < b(x_{i_{k-2}})$, then we would have $x \in N_G(x_{D'})$, which contradicts the irredundancy of $A(D')$. The fact that $P_D(x_{i_j}) \cap N_G[x_{i_{k+2}}] = \emptyset$ for all $j \leq k-3$ concludes the proof. □

Proof (of Lemma 13). We may assume that $k \geq 4$ otherwise $A(D') = D'$ and we are done. It is also clear that if D' is an irredundant set of G, then so is $A(D')$. Suppose now that $A(D')$ is an irrendundant set of G and let us show that D' is. By Lemma 11, we know that for all $j < k-2$, $P_{D'}(x_{i_j}) = P_D(x_{i_j}) \neq \emptyset$. Note first that $t(x_{i_{k+1}}) > t(x_{i_k})$, otherwise $\{x_{i_{k-1}}, x_{i_k}, x_{i_{k+1}}\}$ would form a triangle, contradicting Lemma 10. Moreover, for each $x \in P_D(x_{i_{k-2}})$, we have $t(x) < t(x_{i_k})$ and $b(x) < b(x_{i_{k-1}}) < b(x_{i_{k+1}})$ and then $x \in P_{D'}(x_{i_{k-2}})$. Let $k-1 \leq j \leq k+1$ and let us show that $P_{D'}(x_{i_j}) \neq \emptyset$. Let $x \in P_{A(D')}(x_{i_j})$. Thus we have $b(x) > b(x_{i_{k-2}})$ and then $t(x) > t(x_{D'})$ otherwise $L_G(x_{D'})$ would intersect $L_G(x)$. Now since for all $y \in D \setminus A(D')$ we have $b(y) < b(x_{i_{k-2}})$ and $t(y) < t(x_{D'})$, $L_G(y)$ cannot intersect $L_G(x)$, and then $x \in P_{D'}(x_{i_j})$. □

Proof (of Lemma 14). Let x_s be such that $t(x_s) = \min\{t(y) \mid y \in P_D(x_{i_{k-1}})\}$. If D' is irredundant, then $A(D')$ is clearly irredundant. Since $x \in P_D(x_{i_{k-1}})$ only if ($b(x) < b(x_{i_{k-2}})$ and $t(x_{i_{k-1}}) < t(x) < t(x_{i_{k-2}})$) or ($b(x_{i_{k-2}}) < b(x) < b(x_{i_{k-1}})$ and $t(x_{i_{k-2}}) < t(x) < t(x_{i_k})$)), $t(x_{i_{k+1}}) < t(x_s)$ would imply that $P_D(x_{i_{k-1}}) \subseteq N_G[x_{i_{k+1}}]\cap$, contradicting the fact that $P_{D'}(x_{i_{k-1}}) \neq \emptyset$.

Now assume that D' satisfies (1) and (2). We may assume that $k \geq 4$ since otherwise $A(D') = D'$ and then we are done by assumption. So, $x_{D'}$ exists. Since D is irredundant, for all $l < k-2$, $t(x_{i_l}) < t(x_{i_{k-2}})$, otherwise $\{x_{i_l}, x_{i_{k-2}}, x_{i_{k-1}}\}$ would form a triangle, contradicting Lemma 10. Therefore, $P_{D'}(x_{i_k}) = P_{A(D')}(x_{i_k})$ since $x \in P_{A(D')}(x_{i_k})$ only if $b(x) > b(x_{i_{k-1}})$ and $t(x) > t(x_{i_{k-2}})$. Now, observe that $t(x_{i_{k+1}}) > t(x_{D'})$, otherwise, $\{x_{D'}, x_{i_{k-2}}, x_{i_k}, x_{i_{k+1}}\}$ would form a claw with centre $x_{i_{k+1}}$, contradicting Lemma 10. Therefore, $x \in P_{A(D')}(x_{i_{k+1}})$ only if $b(x) > b(x_{i_{k-1}})$ and $t(x) > t(x_{i_{k-2}}) > t(x_{D'})$. Hence, $P_{D'}(x_{i_{k+1}}) \neq \emptyset$ because $b(x_{i_l}) < b(x_{i_{k-2}})$ and $t(x_{i_l}) < t(x_{D'})$ for all $l < k-2$. By assumptions, $x_s \in P_D(x_{i_{k-1}}) \setminus N_G(x_{i_{k+1}})$ and therefore $P_{D'}(x_{i_{k-1}}) \neq \emptyset$. It remains to prove that $P_{D'}(x_{i_{k-2}}) \neq \emptyset$. By definition of $A(D')$, $x \in P_{A(D')}(x_{i_{k-2}})$ only if $t(x) > t(x_{D'})$ and $b(x) > b(x_{i_{k-2}})$ (notice that $t(x) < t(x_{i_{k-2}})$ also). Since for all $l < k-2$, $b(x_{i_l}) < b(x_{i_{k-2}})$ and $t(x_{i_l}) \leq t(x_{D'})$, we have $N_G(x_{i_l}) \cap P_{A(D')}(x_{i_{k-2}}) \neq \emptyset$, *i.e.*, $P_{D'}(x_{i_{k-2}}) \neq \emptyset$. □

Proof (of Lemma 15). We may assume that $k \geq 4$, otherwise $A(D') = D'$ and we are done. It is clear that $A(D')$ is an irredundant of G if D' is because $A(D') \subseteq D'$. Now assume that $A(D')$ is an irredundant set of G. Since by Lemma 11, $P'_D(x_{i_l}) = P_D(x_{i_l})$ for all $l < k-2$, we just prove that $P_{D'}(x_{i_j}) \neq \emptyset$ for all $k-2 \leq j \leq k+1$. One first notices that $t(x_{D'}) < t(x_{i_{k-2}})$, otherwise $\{x_{D'}, x_{i_{k-2}}, x_{i_k}\}$ forms a triangle, contradicting Lemma 10. Similarly $t(x_{i_{k+1}}) > t(x_{i_{k-2}})$, otherwise either $\{x_{i_k}, x_{i_{k+1}}, x_{i_{k-2}}\}$ induces a triangle or $\{x_{i_{k+1}}, x_{i_{k-2}}, x_{i_{k-1}}, x_{i_k}\}$ induces a claw (with $x_{i_{k-2}}$ as the centre) contradicting also Lemma 10. Therefore, $P_{D'}(x_{i_{k+1}}) \neq \emptyset$. One can also notice that if $x \in P_{A(D')}(x_{i_k})$, then $b(x_{i_{k-1}}) < b(x) < b(x_{i_k})$ and $t(x_{i_{k+1}}) > t(x) > t(x_{i_{k-2}})$. Since for all $l < k-2$, $t(x) < t(x_{D'}) < t(x_{i_{k-2}})$, then $N_G[x_{i_l}] \cap P_{A(D')}(x_{i_k}) \neq \emptyset$

which implies that $P_{D'}(x_{i_k}) \neq \emptyset$. One can easily check that $P_{A(D')}(x_{i_{k-1}}) = P_{A(D')\setminus\{x_{i_{k+1}}\}}(x_{i_{k-1}})$. Therefore, we have $P_{D'}(x_{i_{k-1}}) = P_D(x_{i_{k-1}}) \neq \emptyset$. It remains now to prove that $P_{D'}(x_{i_{k-2}}) \neq \emptyset$. Indeed, $x \in P_{A(D')}(x_{i_{k-2}})$ implies that $t(x) > t_(x_{D'})$ and $b(x) > b(x_{i_{k-2}})$. Since for all $l < k-2$, $b(x_{i_l}) < b(x_{i_{k-2}})$ and $t(x_{i_l}) < t(x_{D'})$, then $P_{A(D')}(x_{i_{k-2}}) \cap N_G[x_{i_l}] \neq \emptyset$, *i.e.*, $P_{D'}(x_{i_{k-1}}) \neq \emptyset$. Therefore D' is an irredundant set of G. □

Proof (of Lemma 16). Let $A := A(D')\backslash x_{D'}$ (notice that if $k = 3$ then $A = A(D')$ since $x_{D'}$ does not exist). Assume first that D' is an irredundant set of G. Then A is also an irredundant set of G and $t(x_{i_{k+1}}) > \min\{t(v) \mid v \in P_D(x_{i_k})\}$, otherwise since $b(x) < b(x_{i_{k+1}})$ for all $x \in P_{D'}(x_{i_k})$, $L_G(x_{i_{k+1}})$ would intersect $L_G(x)$ for all $x \in P_{D'}(x_{i_k})$ contradicting the fact that D' is an irredundant set of G.

Now assume that Conditions (1) and (2) are satisfied and let us prove that D' is an irredundant set of G. Since by Lemma 11 $P_{D'}(x_{i_l}) = P_D(x_{i_l})$ for all $l < k-2$, we just have to prove that $P_{D'}(x_{i_j}) \neq \emptyset$ for all $k-2 \leq j \leq k+1$. Note first that Condition (2) guarantees that $P_{D'}(x_{i_k}) \neq \emptyset$. Let us show that $P_{D'}(x_{i_{k-2}}) \neq \emptyset$ and let $x \in P_D(x_{i_{k-2}})$. Remark that $t(x_{i_{k+1}}) > t(x_{i_k})$ since otherwise $\{x_{i_k}, x_{i_{k-2}}, x_{i_{k+1}}\}$ would form a triangle contradicting Lemma 10. Now one can easily check that that $t(x) < t(x_{i_k})$ and then $L_G(x_{i_{k+1}})$ cannot intersect $L_G(x)$ and then $x \in P_{D'}(x_{i_{k-2}})$. Now let $j \in \{k-1, k+1\}$ and let $x \in P_A(x_{i_j})$. Then we have $t(x) > t(x_{i_{k-2}})$. Furthermore, for all $y \in D \setminus A$, we have $t(y) < t(x_{i_{k-2}})$ since otherwise $\{y, x_{i_{k-2}}, x_{i_k}\}$ would form a triangle. Thus $L_G(y)$ cannot intersect $L_G(x)$ and then $x \in P_{D'}(x_{i_j})$. □

Proof (of Proposition 18). We may assume that $k \geq 4$, otherwise the proposition is trivially true because in this case $A(D) = D$. So, x_D exists. Clearly, if $x \in N_G[A(D)]$, then $x \in N_G[D]$. So assume that $x \in N_G[D] \setminus N_G[A(D)]$ and let $l < k-2$ such that $x \in N_G[x_{i_l}]$. Since $b(x) > b(x_{i_{k-2}})$ and since $b(x_{i_l}) < b(x_{i_{k-2}})$, we have $t(x_{i_l}) > t(x)$. But now we have $b(x_{D'}) < b(x)$ and $t(x_{D'}) > t(x_{i_l}) > t(x)$ and then $x \in N_G[x_{D'}]$ contradicting that $x \notin N_G[A(D)]$. □

Proof (of Proposition 19). Suppose first that there exists a path $(v_1, ..., v_k)$ of $Dag_P(G)$ with $v_j := (x_{i_j}, x_{i_{j+1}}, x_{i_{j+2}}, x_{r_j}, x_{s_j})$, and v_1 and v_k respectively an initial and a final vertex, and let $D := \{x_{i_1}, x_{i_2}, ..., x_{i_{k+2}}\}$. We first show by induction on $3 \leq \ell \leq k+2$ that

(i) $D_\ell := \{x_{i_1}, \ldots, x_{i_\ell}\}$ is an irredundant set of G,
(ii) $x_{r_{\ell-3}} = x_{D_\ell}$ if x_{D_ℓ} is defined,
(iii) $x_{s_{\ell-2}}$ (whenever different from $\perp$) corresponds to $\min\{t(z) \mid z \in P_{D_\ell}(y)\}$ where $y \in D_\ell$ is such that $t(y) = \min\{t(x_{i_{\ell-2}}), t(x_{i_{\ell-1}}), t(x_{i_\ell})\}$.

Since v_1 is an initial vertex, D_3 is an irredundant set of G by definition, and (ii) and (iii) are trivially satisfied. Assume that (i)-(iii) are satisfied for some ℓ and let us prove them for $\ell+1$. Since there exists an arc $(v_{\ell-2}, v_{\ell-1})$, we know that $A(v_{\ell-2}) \cup \{x_{i_{\ell+1}}\}$ is an irredundant set of G, and one easily checks that it corresponds to $A(D_{\ell+1})$ (or to $A(D_{\ell+1}) \setminus x_{D_{\ell+1}}$) if $\{x_{i_{\ell-2}}, x_{i_{\ell-1}}, x_{i_\ell}\}$ corresponds to Cases (1)-(4) (or Case (5)) of Fig. 1. By Lemmas 12-16 we can conclude that $D_{\ell+1}$ is an irredundant set of G since by inductive hypothesis D_ℓ is an irredundant set of G, and by (A.3) we have $t(x_{i_{\ell+1}}) > t(x_{s_{\ell-2}})$ in Cases (1), (3) and (5) of Fig. 1. It is an easy computation to check (with (A.5)) that $x_{s_{i_{\ell-1}}}$ corresponds to $\min\{t(z) \mid z \in P_{D_\ell}(y)\}$ where $y \in D_\ell$ is such that $t(y) = \min\{t(x_{i_{\ell-1}}), t(x_{i_\ell}), t(x_{i_{\ell+1}})\}$, and that $x_{r_{\ell-2}} = x_{D_{\ell+1}}$ if $x_{D_{\ell+1}}$ is defined (by (A.1)).

It remains now to show that D is a minimal dominating set (it is enough to check the three conditions in Proposition 17). Since v_1 is an initial vertex of $Dag_P(G)$, we can verify that (1) holds. Since v_k is a final vertex, we have $\{y \mid L_G(x_{i_k}) < L_G(y) \textit{ and } y \notin N_G[A(v_k)]\} = \emptyset$. One checks with (A.1) that this implies $\{y \mid L_G(x_{i_k}) < L_G(y) \textit{ and } y \notin N_G[A(v_k) \cup \{x_{i_{k-1}}]\} = \emptyset$, which implies by Proposition 18 that $\{y \mid L_G(x_{i_k}) < L_G(y) \textit{ and } y \notin N_G[D]\} = \emptyset$ and then (2) holds. It remains to show Condition (3). Since v_1 is an initial vertex, we know that (3) holds for $l = 2$. Now let $2 < l < k$ and let $A := A(v_{l-2}) \cup \{x_{i_{l+1}}\}$ (which is well-defined). Since there is an arc (v_{l-2}, v_{l-1}), we have $\{y \mid L_G(x_{i_{l-2}}) < L_G(y) < L_G(x_{i_{l-1}}) \textit{ and } y \notin N_G[A]\} = \emptyset$ if $L_G(x_{i_l})$ does not intersect $L_G(x_{i_{l-1}})$ and then by Proposition 18 we have $\{y \mid L_G(x_{i_{l-2}}) < L_G(y) < L_G(x_{i_{l-1}}) \textit{ and } y \notin N_G[D_{l+1}]\} = \emptyset$. If $L_G(x_{i_{l+1}})$ intersects $L_G(x_{i_l})$ then $\{y \mid L_G(x_{i_{l-1}}) < L_G(y),\ t(v) < t(x_{i_{l+1}}),\ \textit{and } b(y) < b(x_{i_l}) \textit{ and } y \notin N_G[A]\} = \emptyset$ and then, again by Proposition 18, $\{y \mid L_G(x_{i_{l-1}}) < L_G(y),\ t(y) < t(x_{i_{l+1}}),\ \textit{and } b(y) < b(x_{i_l}) \textit{ and } y \notin N_G[D_{l+1}]\} = \emptyset$. Therefore (3) is satisfied.

Now assume that $D := \{x_{i_1}, x_{i_2}, ..., x_{i_{k+2}}\}$ is a minimal dominating set of G. For $1 \le j \le k$ let $v_j := (x_{i_j}, x_{i_{j+1}}, x_{i_{j+2}}, x_{r_j}, x_{s_j})$ such that

$$x_{r_j} := \begin{cases} \bot & \text{if } j = 1 \text{ or } \{x_{i_j}, x_{i_{j+1}}, x_{i_{j+2}}\} \text{ corresponds to Case (5) of Fig. 1,} \\ x_{D_{j+3}} & \text{otherwise.} \end{cases}$$

$$x_{s_j} := \begin{cases} \bot & \text{if } \{x_{i_j}, x_{i_{j+1}}, x_{i_{j+2}}\} \text{ corresponds to Case (2)} \\ & \text{and (4) of Fig. 1,} \\ \min\{t(x) \mid x \in P_{D_{j+2}}(y)\} & \text{otherwise with} \\ & t(y) = \min\{t(x_{i_j}), t(x_{i_{j+1}}), t(x_{i_{j+2}})\}. \end{cases}$$

We claim that $(v_1, v_2, \ldots, v_k)$ is a path in $Dag_P(G)$. First notice by Proposition 17(1) that v_1 is an initial vertex, and by (2) v_k is a final vertex. Let us show now that for all $1 \le j < k$, (v_j, v_{j+1}) is an arc. One easily checks by definition that $x_{r_{j+1}} = \max\{t(x_{r_j}), t(x_{i_j})\}$, and then (A.1) is verified. Now, since D is a minimal dominating set, $A(v_j) \cup \{x_{i_{j+1}}\}$ is clearly an irredundant set of G, and then (A.2) is also satisfied. By Lemmas 12-16 (A.3) is verified, and by Propositions 17 and 18 (A.4) is also satisfied. Now if $x_{s_{j+1}} \neq \bot$ and $\min\{t(x_{i_j}), t(x_{i_{j+1}}), t(x_{i_{j+2}})\} =$

$\min\{t(x_{i_{j+1}}), t(x_{i_{j+2}}), t(x_{i_{j+3}})\}$, then we clearly have $x_{s_{j+1}} = x_{s_j}$ by definition, and then (A.5.1) is checked. Otherwise, By Proposition 18 and by definition of x_{s_j} (A.5.2) is trivially satisfied. $\square$