On the Neighbourhood Helly of some Graph Classes and Applications to the Enumeration of Minimal Dominating Sets

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Abstract. We prove that *line graphs* and *path graphs* have bounded neighbourhood Helly. As a consequence, we obtain output-polynomial time algorithms for enumerating the set of *minimal dominating sets* of line graphs and path graphs. Therefore, there exists an output-polynomial time algorithm that enumerates the set of *minimal edge-dominating sets* of any graph.

1 Introduction

A *hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $E(\mathcal{H})$, its set of hyperedges, is a family of subsets of $V(\mathcal{H})$, its set of vertices. A hypergraph is called *Sperner* if there is no hyperedge that is contained in another hyperedge. In [1] Berge defined the notion of k-conformality of hypergraphs. A hypergraph $\mathcal{H}$ is called *k-conformal* if $X \subseteq V(\mathcal{H})$ is contained in a hyperedge of $\mathcal{H}$ whenever each subset of X of size at most k is contained in a hyperedge. The *conformality* of a hypergraph $\mathcal{H}$ is defined as the least k such that $\mathcal{H}$ is k-conformal. An interesting property of the conformality notion is that it leads to an output polynomial time algorithm for the *Transversal problem* in Sperner hypergraphs of bounded conformality [2]. A *transversal* in a hypergraph $\mathcal{H}$ is a subset T of $V(\mathcal{H})$ such that T intersects any hyperedge of $\mathcal{H}$. If we denote by $Tr(\mathcal{H})$ the set of (inclusionwise) minimal transversals, the *Transversal Problem* consists in given a hypergraph $\mathcal{H}$ to compute $Tr(\mathcal{H})$. This problem has applications in graph theory, database theory, data mining, ... (see, e.g., [6,7,8,9]). It is an open question whether we can compute $Tr(\mathcal{H})$ in time $O((\|\mathcal{H}\| + |Tr(\mathcal{H})|)^k)$ for some constant k, where $\|\mathcal{H}\|$ is defined as $|V(\mathcal{H})| + |E(\mathcal{H})|$ (an algorithm achieving such a time is called an output-polynomial time algorithm). The best known algorithm for the *Transversal problem* is the one by Fredman and Khachiyan [10] which runs in time $O(N^{\log(N)})$ where $N = \|\mathcal{H}\| + |Tr(\mathcal{H})|$.

In this paper, we are interested in the conformality of the *closed neighbourhood hypergraphs* of graphs. Let us give some preliminary definitions and notations. A graph is a hypergraph where each hyperedge has size two (and are called *edges*). An edge of a graph is written xy (equivalently yx) instead of $\{x, y\}$. We refer to [4] for graph terminologies not defined in this paper. The *neighbourhood* of a vertex x in a graph G, *i.e.* , $\{y \mid xy \in E(G)\}$, is denoted by $N_G(x)$ and we let $N_G[x]$, the *closed neighbourhood* of x, be $N_G(x) \cup \{x\}$. The *closed neighbourhood hypergraph* $\mathcal{N}(G)$ of a graph G is the hypergraph $(V(G), \{N_G[x] \mid x \in V(G)\})$. A graph is called *$k$-conformal* if $\mathcal{N}(G)$ is k-conformal. The k-conformality of a graph is also known in the literature under the name of *k-neighbourhood Helly* [3,5]. Dually chordal graphs, chordal bipartite graphs, ptolemaic graphs are examples of graphs that have conformality at most 3.

A *cycle of length n* is denoted by C_n. A *claw* is a graph with four vertices isomorphic to the graph $(\{x_1, \ldots, x_4\}, \{x_1x_2, x_1x_3, x_1x_4\})$. A *chordal graph* is a graph without an induced cycle of length greater or equal to 4. The *line graph* of a graph G, denoted by $L(G)$, is the graph with vertex-set $E(G)$ and edge-set $\{ef \mid e, f \in E(G) \text{ and } e \cap f \neq \emptyset\}$.

Let $\mathcal{F}$ be a family of subsets of some ground set. A graph G is an *intersection graph of $\mathcal{F}$* if there exists a bijection between $V(G)$ and $\mathcal{F}$ and such that there exists an edge between x and y if and only if their corresponding images in $\mathcal{F}$ intersect. A *path graph* is an intersection graph of paths in a tree. Path graphs constitute a subclass of chordal graphs [11].

We let $Min(\mathcal{N}(G))$ be the hypergraph obtained from $\mathcal{N}(G)$ by removing those hyperedges that contain a hyperedge. It is clear that $Min(\mathcal{N}(G))$ is Sperner. Notice that the conformality of a hypergraph $\mathcal{H}$ may be different from that of $Min(\mathcal{H})$. In this paper, we prove the following.

Theorem 1. *Line graphs are* 6*-conformal, path graphs and* $(C_4, C_5, claw)$*-free graphs are* 3*-conformal. Moreover, if we let* $\mathcal{ML} := \{Min(\mathcal{N}(G)) \mid G \text{ is a line graph}\}$ *and* $\mathcal{MP} := \{Min(\mathcal{N}(G)) \mid G \text{ is a path graph or a } (C_4, C_5, claw)\}$*-free, then* $\mathcal{ML}$ *and* $\mathcal{MP}$ *have conformality bounded by* 6 *and* 3 *respectively.*

A subset D of the vertex-set of a graph G is called a *dominating set* if every vertex in $V(G) \setminus D$ is adjacent to a vertex in D. We denote by $\mathcal{D}(G)$ the set of (inclusionwise) minimal dominating sets of a graph G. The computation of $\mathcal{D}(G)$ of every graph G, known as the *DOM problem*, in output-polynomial time is a hard task and it was known for a while that $\mathcal{D}(G) = Tr(\mathcal{N}(G))$ for every graph G. Therefore, an output polynomial-time algorithm for the *Transversal problem* is also an output-polynomial time algorithm for the *DOM problem*. The authors have proved in [14] that the other direction also holds, *i.e.* , an output-polynomial time algorithm for *DOM problem* is also an output-polynomial time algorithm for the *Transversal problem*.

Output-polynomial time algorithms for the *DOM problem* are only known for few graph classes (see [13,14] for some of them). As a corollary of Theorem 1, we obtain output-polynomial time algorithms for the *DOM problem* in line graphs, path graphs and (C_4, C_5, claw)-free graphs, and to our knowledge this was not known.

Theorem 2.
1. *For every line graph G, one can compute $\mathcal{D}(G)$ in time $O(\|G\|^5 \cdot |\mathcal{D}(G)|^6)$.*
2. *For every path graph or $(C_4, C_5, claw)$-free graph G, one can compute $\mathcal{D}(G)$ in time $O(\|G\|^2 \cdot |\mathcal{D}(G)|^3)$.*

A subset F of the edge-set of G is called an *edge-dominating set* if every edge in $E(G) \setminus D$ is incident to an edge in D. We denote by $\mathcal{ED}(G))$ the set of (inclusionwise) minimal edge-dominating sets of a graph G. It was open whether an output-polynomial time algorithm for computing $\mathcal{ED}(G)$ exists. It is well established that D is a dominating set of $L(G)$ if and only if D is an edge-dominating set of G. As a corollary of Theorem 2 we obtain the following theorem.

Theorem 3. *For every graph G, one can compute $\mathcal{ED}(G)$ in time $O(\|L(G)\|^5 \cdot |\mathcal{ED}(G)|^6)$.*

2 Some remarks on the k-conformality

Definition 1. *Let $\mathcal{H}$ be a hypergraph and let k be a positive integer. A k-*bad-set* in $\mathcal{H}$ is a subset S of $V(\mathcal{H})$ with $|S| > k$ and such that:*

- *for all subsets $S' \subseteq S$ of size k, there exists $e \in E(\mathcal{H})$ such that $S' \subseteq e$,*
- *for all hyperedges $e \in E(G)$, $S \nsubseteq e$.*

A graph G is said to have a k-bad-set if $\mathcal{N}(G)$ has one.

Remark that a k-bad-set is also a k'-bad-set for every $k' \leq k$. The proof of the following is immediate from the definition of k-conformality.

Definition 2. *Let k be a positive integer. A hypergraph is k-conformal if and only if it has no k-bad-set.*

A *minimal k-bad-set* in a hypergraph $\mathcal{H}$ is a k-bad-set in $\mathcal{H}$ of size $k+1$. Notice that, for every vertex x of a minimal k-bad-set S, there exists a hyperedge e which contains all S, but x.

Proposition 1 *Let k be a positive integer. A hypergraph is k-conformal if and only if it has no minimal k'-bad-set for every $k' \geq k$.*

Proof. The first direction is straigtforward since a k'-bad-set with $k' \geq k$ is also a k-bad-set. Now assume that a hypergraph $\mathcal{H}$ has no minimal k'-bad-set with $k' \geq k$ but has a non minimal k-bad-set S. Assume that S is minimal with respect to inclusion. Then, for every $x \in S$, $S \setminus \{x\}$ is not a k-bad-set and since S is a k-bad-set, there exists $e \in E(\mathcal{H})$ such that $S \setminus \{x\} \subseteq e$. Therefore, S is a minimal k'-bad-set with $k' := |S| - 1 \geq k$. This is a contradiction with the assumption that $\mathcal{H}$ has no k'-bad-set with $k' \geq k$. □

3 Line graphs

For a graph G we denote by $G(F)$, for $F \subseteq E(G)$, the graph with vertex-set $\{x \in V(G) \mid x \text{ is incident with an edge in } F\}$ and F as edge-set. A *clique* is a graph with pairwise adjacent vertices (it is denoted K_n if it has n vertices). A *tree* is an acyclic (without induced cycle) connected graph.

A *vertex cover* in a graph G is a subset S of $V(G)$ such that every edge of G intersects S and a *matching* is a subset M of $E(G)$ such that for every $e, f \in M$ we have $e \cap f = \emptyset$. We denote by $\tau(G)$ and $\nu(G)$, respectively, the maximum size of a vertex cover and of a matching. If e is an edge of a graph G, we define the closed neighbourhood of e as $N_{L(G)}[e]$.

Lemma 1 *Let $k \geq 3$ be a positive integer. Let G be a graph and let S be a subset of $E(G)$. If S is a k-bad-set of $L(G)$, then $\nu(G(S)) = 2$.*

Proof. Let $S \subseteq E(G)$ be a k-bad-set of $L(G)$ and assume that $\nu(G(S)) \geq 3$. Let M be a maximum matching of $G(S)$. Let S' be a subset of M with $|S'| = 3$ (such a subset of M exists since $\nu(G(S)) \geq 3$). It is easy to see that no edge of $E(G)$ can be incident to all edges in S' and then any subset of S of size k and containing S' is not included in the closed neighbourhood of an edge of G, contradicting the fact that S is a k-bad-set of $L(G)$. Therefore, $\nu(G(S)) \geq 2$. Assume now that $\nu(G(S)) = 1$. This implies that there exists an edge $e \in S$ incident to all edges in S, which contradicts again the fact that S is a k-bad-set. We can thus conclude that $\nu(G(S)) = 2$. □

Lemma 2 *Let $k \geq 3$ be a positive integer. Let G be a bipartite graph and let S be a subset of $E(G)$. If S is a k-bad-set of $L(G)$, then $\tau(G(S)) = 2$.*

Proof. By Lemma 1, we have $\nu(G(S)) = 2$. By König's Theorem, we have that $\tau(G(S)) = \nu(G(S)) = 2$. □

Proposition 2 *Line graphs are 6-conformal.*

Proof. Let G be a graph and assume that $L(G)$ has a 6-bad-set $S \subseteq E(G)$. By Lemma 1, we have $\nu(G(S))) = 2$. Let $\{x_1x_2, x_3x_4\}$ be a maximum matching of $G(S)$. By definition of a maximum matching, we know that every edge of S intersects $\{x_1, x_2, x_3, x_4\}$. For $i \in \{1, \ldots, 4\}$, we let P_i be the set $\{e \in S \mid e \cap \{x_1, \ldots, x_4\} = \{x_i\}\}$. One of P_1 or P_2 must be empty. Otherwise let $e_1 \in P_1$ and $e_2 \in P_2$, then $\{e_1, e_2, x_3x_4, x_1x_2\}$ would not be in a closed neighbourhood, contradicting the fact that S is a 6-bad-set. Similarly, one of P_3 or P_4 is empty. Therefore, at most two sets among $P_1, \ldots, P_4$ are non empty. We identify two cases.

Case 1. Two sets among $P_1, \ldots, P_4$ are non empty. Assume without loss of generality that they are P_1 and P_3. Let $e_1 \in P_1$ and $e_2 \in P_3$. Let S' be a subset of S of size 6 that contains $\{e, e', x_1x_2, x_3x_4\}$. Since S is a 6-bad-set, there exists an edge whose closed neighbourhood contains S', and the only possible one is x_1x_3. Moreover, there must exist an edge e that is neither incident to x_1 nor to x_3, otherwise S would not be a 6-bad-set (the closed neighbourhood of x_1x_3 would contain S). Let again S' be a subset of S of size 6 and that contains $\{x_1x_2, x_3x_4, e_1, e_2, e\}$. But, the subset $\{x_1x_2, x_3x_4, e_1, e_2, e\}$ of S cannot be contained in the closed neighbourhood of any edge in $E(G)$. So, we can conclude that at most one set among $P_1, \ldots, P_4$ is non empty.

Case 2. One set among $P_1, \ldots, P_4$ is non empty, say P_1 is this set. Let $e \in P_1$. Then, the set $\{x_1x_2, x_3x_4, e\}$ must be included in the closed neighbourhood of some edge. The only two possible such edges are x_1x_3 and x_1x_4. Assume without loss of generality that $x_1x_3 \in E(G)$. Since S is a 6-bad-set of $L(G)$ there exists an edge e in S which is not in the closed neighbourhood of x_1x_3. Since P_2, P_3 and P_4 are empty, that edge must be x_2x_4. Then, $\{e, x_1x_2, x_3x_4, x_2x_4\}$ must be included in a closed neighbourhood of an edge, and that edge is clearly x_1x_4. Again, since S is a 6-bad-set, there must exist an edge not in the neighboorhood of x_1x_4, and the only possible choice is x_2x_3. Therefore, $\{e, x_1x_2, x_3x_4, x_2x_4, x_2x_3\}$ is included in S and since it has size 5, it is included in a closed neighbourhood of an edge. But no such edge exists. So, we can conclude that P_1 is also empty.

From the two cases above, we have that P_i is empty for all $i \in \{1, \ldots, 4\}$, and then $V(G(S)) = \{x_1, x_2, x_3, x_4\}$. Since S is a 6-bad-set, its size must be at least 7, which contradicts the fact that $|V(G(S))| = 4$ since the number of edges in a graph with four vertices is at most 6. This completes the proof. □

Since the line graph of K_4 is 6-conformal and not 5-conformal, the bound from Proposition 2 is tight. Therefore, line graphs of graphs that contain K_4 as induced subgraphs have conformality 6. By adapting the proof of Proposition 2, we can prove the following.

Proposition 3 *Line graphs of K_4-free graphs are 5-conformal.*

Proof. If we define the sets P_i, $i \in \{1, \ldots, 4\}$ as in the proof of Proposition 2 and follow the same proof with S being a 5-bad-set and not a 6-bad-set, the only way for a line graph to have a 5-bad-set S is when all P_i's are empty. But, in this case $|V(G(S))| = 4$ and since a 5-bad-set must have at least 6 edges, we conclude that the only possible 5-bad-set for a line graph $L(G)$ is that G contains K_4. □

Even if the bound 6 is optimal in the class of line graphs, it is not at all in the class of line graphs of bipartite graphs.

Proposition 4 *Line graphs of bipartite graphs are 4-conformal.*

Proof. Let G be a bipartite graph and let S be a 4-bad-set of $L(G)$. By definition of a 4-bad-set, $|S| \geq 5$. From Lemma 2, $\tau(G(S)) = 2$. Let $\{x, y\}$ be a maximum vertex cover of $G(S)$. Then $xy \notin E(G)$, otherwise the edge xy will be incident to all edges of S, contradicting the fact that S is a 4-bad-set of $L(G)$. We identify three cases.

Case 1. x has only one neighbour x' in $G(S)$. Let S' be a subset of S of size 4 that contains the edge xx'. Since S is a 4-bad-set of $L(G)$, there must exist an edge e of $E(G)$ whose closed neighbourhood contains S'. Since xy is not an edge, e must be yx'. But, in this case the closed neighbourhood of yx' will contain S, contradicting the fact that S is a 4-bad-set of $L(G)$.

Case 2. y has only one neighbour y' in $G(S)$. This case is similar to Case 1 (replace x by y and x' by y').

Case 3. Each of x and y has at least two neighbours in $G(S)$. Let x_1 and x_2 (resp. y_1 and y_2) be two neighbours of x (resp. y) in $G(S)$. Let $S' := \{xx_1, xx_2, yy_1, yy_2\}$ be a subset of S. There must exist an edge e in $E(G)$ whose closed neighbourhood contains S'. One can easily check that the only possible choice for e is xy which is a contradiction with the fact that $xy \notin E(G)$. Since there is no edge whose closed neighbourhood contains S', we have a contradiction with the fact that S is a 4-bad-set of $L(G)$. This concludes the proof. $\square$

The bound in Proposition 4 is tight because the line graph of the cycle C_4 is 4-conformal but not 3-conformal. One easily checks that if a graph contains C_4 as an induced cycle, then its line graph is 4-conformal, but not 3-conformal.

4 Path graphs

We now prove that path graphs are 3-conformal. A *clique tree* of a graph G is a tree T whose vertices are in bijection with the (inclusionwise) maximal cliques of G and such that those maximal cliques that contain a vertex x induce a subtree of T, which we will denote by T^x. Observe that G is the intersection graph of these subtrees. It is well-known that a graph is chordal if and only if it has a clique-tree [11] and path graphs are exactly those chordal graphs where for every vertex x, T^x is a path (It is worth noticing that path graphs can be recognised in polynomial time [12]).

A *rooted tree* is a tree with a distinguished vertex called its *root*. In a rooted tree T we define the partial order $\preceq_T$ where $x \preceq_T y$ if and only if the path from the root to x goes through y. For v a vertex of a rooted tree T, we let T_v be the subtree of T rooted at v and induced by the vertices in $\{x \in V(T) \mid x \preceq_T v\}$.

Proposition 5 *Path graphs are* 3*-conformal.*

Proof. Let G be a path graph and let T be its clique-tree. Assume that G has a minimal $(k-1)$-bad-set $S := \{x_1, x_2, ..., x_k\}$ with $4k > 3$. Since S is a minimal $(k-1)$-bad-set, there exists a vertex x such that $\{x_1, x_2, ..., x_{k-1}\} \subseteq N_G[x]$. Let $T^x = (t_1, t_2, \ldots, t_\ell)$. Since, $\{x_1, x_2, ..., x_{k-1}\} \subseteq N_G[x]$, each subtree T^{x_i}, for $i \in \{1, \ldots, k-1\}$, must intersect T^x and $P_{x_i} := T^{x_i} \cap T^x$ forms a sub-path of T^x. Let $s_i := \min\{j \mid t_j \in P_{x_i}\}$ and $e_i := \max\{j \mid t_j \in P_{x_i}\}$ for $i \in \{1, \ldots, k-1\}$. Assume without loss of generality that the vertices $x_1, \ldots, x_{k-1}$ are ordered such that $i < j \Longrightarrow s_i \leq s_j$. Since S is a $(k-1)$-bad-set, we know that T^{x_k} does not intersect T^x. We let t_r be the unique vertex of T^x such that every path from t_r to any vertex of T^{x_k} intersects T^x only on t_r. We root T at t_r.We identify two cases(see Figures 4).

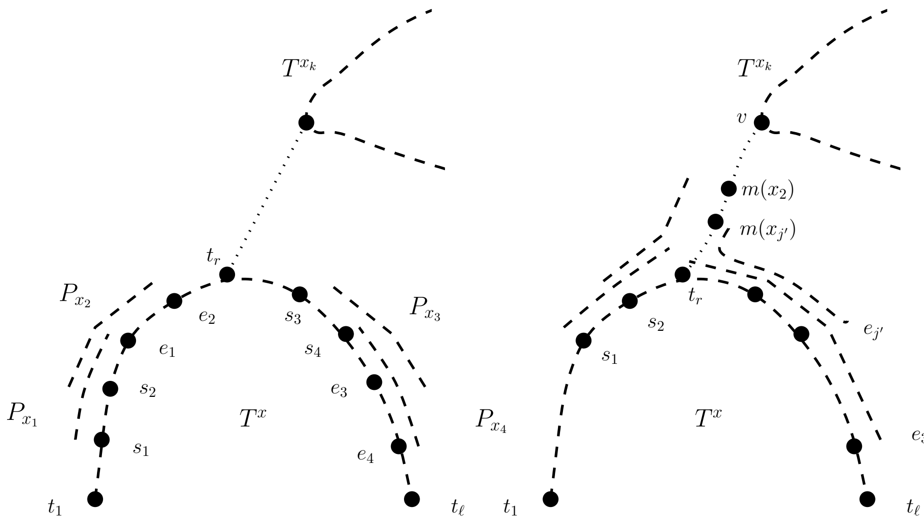

Fig. 1. The first case of the proof of the Proposition 5, is illustrated on the left part. Case two is described on the opposite side.

Case 1. For all $j \in \{1, \ldots, k-1\}$, T^{x_j} does not contain t_r. Assume first that for all $i \in \{1, \ldots, k-1\}$, $e_i < r$, and let $e_j := \min\{e_i \mid i \in \{1, \ldots, k-1\}\}$. Since S is a $(k-1)$-bad-set, there exists a vertex z such that $\{x_j, x_k\} \subseteq N_G[z]$. But in this case T^z would intersect T^{x_j} for every $x_j \in S$ which leads to a contradiction. Similarly, if for all $j \in \{1, \ldots, k-1\}$, we have $s_j > r$, any vertex who is adjacent to x_{k-1} and x_k would also be adjacent to $x_1, \ldots, x_{k-2}$, yielding a contradiction. We can therefore assume that $S \setminus \{x_k\} = S_1 \cup S_2$, with $S_1, S_2 \neq \emptyset$ and such that $e_j < r$ for every $x_j \in S_1$ and $s_j > r$ for every $x_j \in S_2$. Let us choose $x_i \in S_1$ and $x_j \in S_2$. Then since S is a $(k-1)$-bad-set, there must exist a vertex z such that $\{x_i, x_j, x_k\} \subseteq N_G[z]$, but no path in T can intersect at the same time the three paths T^{x_i}, T^{x_j} and T^{x_k}, which yields again a contradiction.

Case 2. There is at least one vertex $x_j \in S \setminus \{x_k\}$ such that $t_r \in T^{x_j}$ (ie. $s_j \leq r$ and $e_j \geq r$). Note that in this case, for every vertex $x_i \in S \setminus \{x_k, x_j\}$, we have $t_r \in T^{x_i}$ otherwise, every vertex whose neighborhood contains x_k and x_i would be adjacent to x_j, which is in contradiction with the fact that S is a minimal $(k-1)$-bad-set. Hence, for every $i \in \{1, \ldots, k-1\}$, $s_i \leq r \leq e_i$. Let v be the vertex of T_{t_r} which is the greatest vertex of T^{x_k} (greatest with respect to $\preceq_T$) and let P be the path between t_r and v. Note that for all $i \in \{1, \ldots, k-1\}$, T^i must intersect P on at least one different vertex from t_r (which implies that $s_i = t_r$ or $e_i = t_r$). Otherwise, for any vertex z such that $\{x_k, x_i\} \in N_G[z]$, T^z would contain t_r and hence S would be included in $N_G[z]$. For every $i \in \{1, \ldots, k-1\}$, let $m(x_i)$ be $\max_{\preceq_T}\{x \in T \mid x \in P \cap T^{x_i}\}$ ($m(x_i)$ is the greatest vertex, with respect to $\preceq_T$, of $P \cap T^{x_j}$). Let j' be such that $m(x_{j'}) := \max_{\preceq_T}\{m(x_i) \mid i \in \{1, \ldots, k-1\}\}$.

Then any closed neighborhood that contains x_k and $x_{j'}$ will also contain S, which yields a contradiction. □

5 (C_4, C_5, claw)-free graphs

Now we show that (C_4, C_5, claw)-free graphs are 3-conformal.

Proposition 6 *$(C_4, C_5, claw)$-free graphs are 3-conformal.*

Proof. Let G be a (C_4, C_5, claw)-free graph and assume it is not 3-conformal. Then, there exists a k-bad-set S with $|S| > k$ for $k \geq 3$. Since S is a k-bad-set, the subgraph induced by S is not a clique and therefore there exists x_1 and x_2 such that $x_1x_2 \notin E(G)$. Let $x_3 \in S \setminus \{x_1, x_2\}$. Then, x_3x_1 or x_3x_2 is an edge, otherwise since S is k-bad-set for $k \geq 3$, there exists z adjacent to x_1, x_2 and x_3 and this will induce a claw in G (which is claw-free). Assume therefore that $x_3x_1 \in E(G)$.

Since S is a k-bad-set, there exists z and z' such that z is adjacent to x_1 and x_2 and not to x_3, and z' is adjacent to x_2 and x_3 and not to x_1. If $x_2x_3 \notin E(G)$ and $zz' \notin E(G)$ then $\{z, x_1, x_3, z', x_2\}$ induces a C_5 which yields a contradiction (G is C_5-free). If $x_2x_3 \notin E(G)$ and $zz' \in E(G)$, then $\{z, x_1, x_3, z'\}$ induces a C_4 which is a contradiction (G is C_4-free). And if $x_2x_3 \in E(G)$, then $\{z, x_1, x_3, x_2\}$ induces a C_4 which is again a contradiction. We can therefore conclude that no k-bad-set for $k \geq 3$ exists and hence G is 3-conformal. □

6 Proofs of Theorems

We can now prove Theorems 1, 2 and 3.

Proof (Proof of Theorem 1). The first part of the theorem follows from Propositions 2, 5 and 6. For the second part, one easily checks that if we replace in the arguments "there exists z such that $S' \subseteq N_G[z]$" by "there exists a closed neighbourhood $N_G[z] \supseteq S'$" the same arguments follow. So, the second statement is also true. □

It is clear that Theorems 2 and 3 follow from Theorem 1, and Theorem 4 and Proposition 7 stated below.

Theorem 4 ([2]). *Let $\mathcal{H}$ be a k-conformal Sperner hypergraph. Then one can compute $Tr(\mathcal{H})$ in time $O(\|H\|^{k-1} \cdot |Tr(\mathcal{H})|^k)$.*

Proposition 7 (Folklore) *Let G be a graph and let D be a subset of $E(G)$. Then D is a dominating set of $L(G)$ if and only if D is an edge-dominating set of G.*

7 Conclusion

We have proven that line graphs, path graphs and (C_4, C_5, claw)-free graphs have bounded conformality. A direct consequence, using the result by Boros et al. in [2] is that we can enumerate minimal dominating sets in output-polynomial time in line graphs, path graphs and (C_4, C_5, claw)-free graphs. Path graphs was one of the maximal subclasses of chordal graphs where no output-polynomial time algorithm for the *DOM problem* was known. *Chordal domination perfect graphs*, which form a subclass of chordal graphs, do not have bounded conformality and therefore we cannot expect using the algorithm by Boros et al. to get an output-polynomial time algorithm for the *DOM problem* in chordal graphs. Notice that chordal claw-free graphs is a maximal subclass of chordal domination perfect graphs and have conformality at most 3 by Proposition 6. We leave open the quest for an output-polynomial time algorithm for the *DOM problem* in chordal graphs, or at least in its other subclasses such as chordal domination perfect graphs.

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