Fast Algorithms Parameterized by Clique-Width

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Problems

Feedback vertex set

A feedback vertex set is $X \subseteq V(G)$ such that $G \setminus X$ is a forest.

Connected locally checkable properties

Let (σ, ρ) be (co-)finite subsets of $\mathbb{N}$. A connected (σ, ρ)-dominating set is a connected $D \subseteq V(G)$ s.t.

$$|N(x) \cap D| \in \begin{cases} \sigma & \text{if } x \in D \\ \rho & \text{if } x \notin D. \end{cases}$$

Examples. (Total) domination, d-domination, independent set, perfect domination, ...

Hamiltonian Cycle

Find a cycle covering all vertices.

Wanted Algorithms

Objective

Let G be a graph of rank-width k,

- One can compute in time $2^{O(k)} \cdot n^{O(1)}$ a minimum FVS.
- Given (σ, ρ), one can compute in time $2^{O(k \cdot d)} \cdot n^{O(1)}$ an optimum connected (σ, ρ)-dominating set, $d := 1 + \max\{d(\rho), d(\sigma)\}$.

Such algorithms exist when parameterized by tree-width (Bodlaender et al.'2013).

$$d(\mu) := \begin{cases} 0 & \text{if } \mu = \mathbb{N}, \\ \max(\mu) & \text{if } \mu \text{ finite} \\ \max(\mathbb{N} \setminus \mu) & \text{otherwise.} \end{cases}$$

Our Result

Theorem

Let G be a graph of ~~rank-width~~ clique-width k,

- One can compute in time $2^{O(k)} \cdot n^{O(1)}$ a minimum FVS.
- Given (σ, ρ), one can compute in time $2^{O(k \cdot d)} \cdot n^{O(1)}$ an optimum connected (σ, ρ)-dominating set, $d := 1 + \max\{d(\rho), d(\sigma)\}$.

Reminder. $\mathrm{rwd}(G) \leq \mathrm{cwd}(G) \leq 2^{\mathrm{rwd}(G)+1} - 1$.

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Clique-Width

Given G and H, and $lab_G : V(G) \to [k]$, $lab_H : V(H) \to [k]$

1. $\mathbf{1}(x)$ a graph with a single vertex x labeled 1,
2. $G \oplus H$ the disjoint union of G and H,
3. $ren_{i \to j}(G)$ rename all i-vertices into j-vertices (no more vertex labeled i).
4. $add_{i,j}(G))$ add all edges between i-vertices and j-vertices (no parallel edges).

The clique-width of G, $\mathrm{cwd}(G)$, is the minimum k such that G is the value of a term from the above operations.

Importance. MS_1 optimisation problems in FPT polynomial time parameterised by clique-width (Courcelle'93).

Clique-Width versus Rank-Width

- No known FPT polynomial algorithm time for computing CWD

  $f(k) \cdot n^3$ known for RWD.

- RWD is based on ranks of matrices, but graph operations (complicated).

  CWD operations simple and better for algorithmic purposes.

- RWD enjoys several nice structural properties, in particular links with vertex-minor quasi-ordering.

  Whilst CWD only known to be closed under induced subgraph.

Compute an optimum set $D \models P$

Let $\mathcal{C}_k$ a set of characteristics classifying possible solutions.
$tab[s]$ is the optimum value for all D with characteristic s.

1. Compute the $tab[s]$ on $\mathbf{1}(x)$,
2. For each operation, compute tab from tab's of operands.

Assume this gives time $f(k) \cdot n^{O(1)}$, with solution in $tab[s_0]$.

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A connected set satisfying P.

For each s, compute $\mathcal{A}[s]$ which stores the pairs (p, w) s.t. there is D with $p = CC(D)/[k]$ and $w := \mathrm{w}(D)$ is optimum.

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The optimum connected set is $(p, w) \in \mathcal{A}[s_0]$ with $p = I$ and $I \subseteq [k]$ the set of intersected label classes in time

$$f(k) \cdot g(k) \cdot 2^{k \log(k)} \cdot n^{O(1)}.$$

Summary

1. Feedback Vertex Set
2. Representatives for Weighted Partitions
3. Locally Checkable Properties
4. Hamiltonian Cycle

Weighted Partitions

- A weighted partition is (p_0, p, w) with $(p, w) \in \Pi(V \setminus p_0) \times \mathbb{N}$.
- We let acyclic(p, q) holds iff $p \sqcup q$ yields an acyclic forest, ie, $|V| + \#\mathrm{block}(p \sqcup q) - (\#\mathrm{block}(p) + \#\mathrm{block}(q)) = 0$.

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For $\mathcal{A}$ and $\mathcal{B}$ sets of weighted partitions

$\mathsf{rmc}(\mathcal{A}) := \{(p_0, p, w) \in \mathcal{A} \mid \forall (p_0, p, w') \in \mathcal{A}, w' \leq w)\}.$

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$$\mathsf{acjoin}(\mathcal{A}, \mathcal{B}) := \mathsf{rmc}\Big(\{(p_0 \setminus V') \cup (q_0 \setminus V) \cup (p_0 \cap q_0), p_{\uparrow(V' \setminus q_0)} \sqcup q_{\uparrow(V \setminus p_0)}, w_1 + w_2) \mid (p_0, p, w_1) \in \mathcal{A}, (q_0, q, w_2) \in \mathcal{B} \text{ and } \mathsf{acyclic}(p_{\uparrow(V' \setminus q_0)}, q_{\uparrow(V \setminus p_0)})\}\Big).$$

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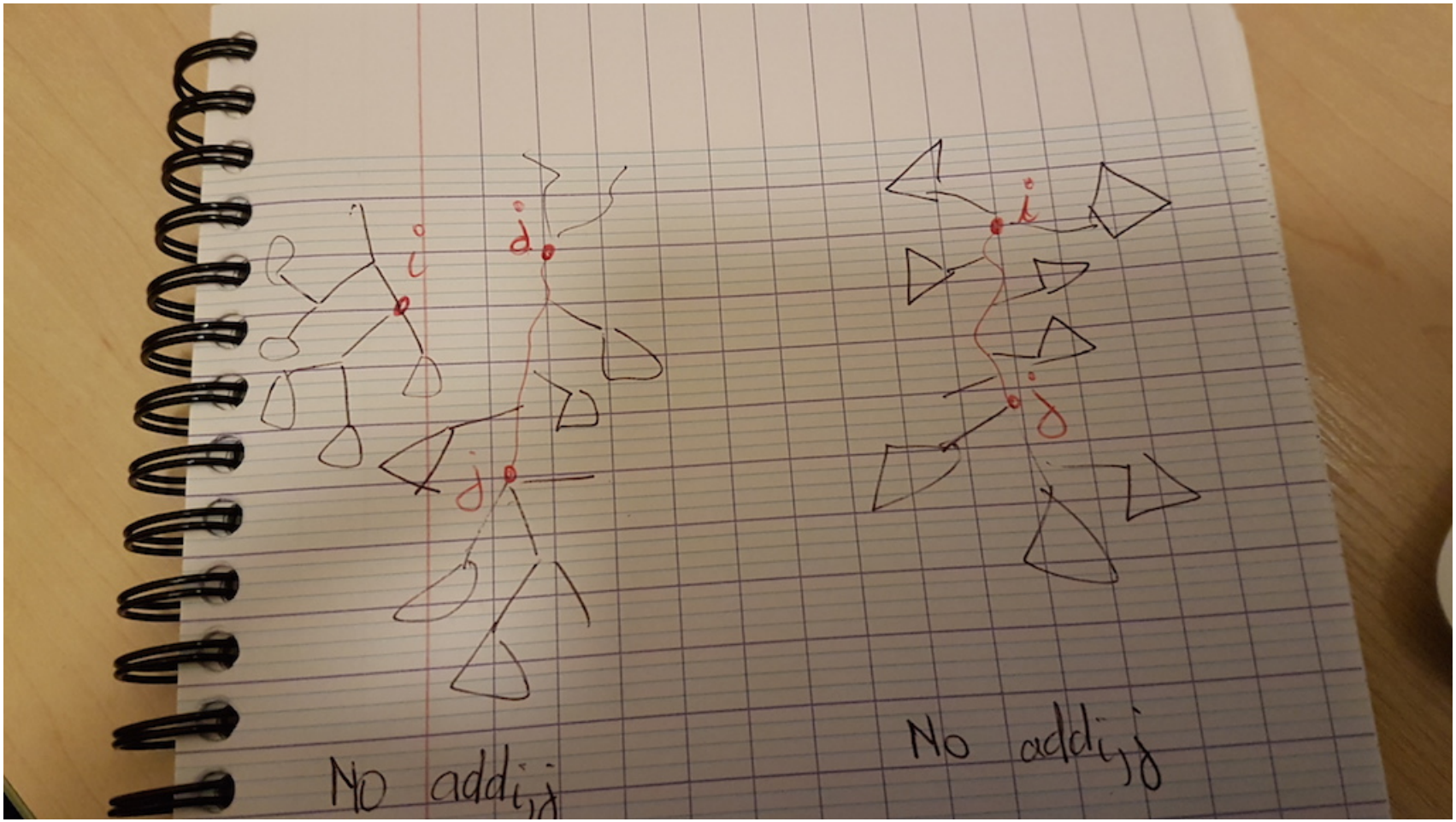
$$\mathsf{proj}(\mathcal{A}, X) := \mathsf{rmc}\Big(\{(p_0 \setminus X, p_{\downarrow(V \setminus X)}, w) \mid (p_0, p, w) \in \mathcal{A} \text{ and } \forall p_i \in p,\ (p_i \setminus X) \neq \emptyset\}\Big).$$

Needed Information



Dynamic Programming Table

$s : [k] \to \{1, 2, -2\}$ and $V_s^+ := \{x_{i_1}, \ldots, x_{i_p}\}$ with $\{i_1, \ldots, i_p\} := s^{-1}(2)$.

$(p_0, p, w) \in \mathcal{A}[s]$ if there is $F \subseteq G$ and $E_s^0 \subseteq \{v_0\} \times V(F)$

1. The set $p_0 \subseteq s^{-1}(-2)$ and $p \in \Pi(s^{-1}(\{1, 2\}) \cup \{v_0\})$.
2. The set $s^{-1}(1) = \{i \in [k] \mid |V(F) \cap lab_G^{-1}(i)| = 1\}$ and $s^{-1}(2) \cup (s^{-1}(-2) \setminus p_0) = \{i \in [k] \mid |V(F) \cap lab_G^{-1}(i)| \geq 2\}$.
3. The graph $F^+ := (V(F) \cup \{v_0\} \cup V_s^+, E(F) \cup E_s^0 \cup E_s^+)$ is a forest where $E_s^+ := \bigcup_{1 \leq j \leq p} x_{i_j} \times (V(F) \cap lab_G^{-1}(i_j))$.
4. Each component C of $(V(F) \cup \{v_0\}, E(F) \cup E_s^0)$ intersects $s^{-1}(\{1, 2\}) \cup \{v_0\}$.
5. The partition p is precisely $CC(F^+)_{\downarrow s^{-1}(\{1,2\}) \cup \{\{v_0\}\}}$.

$G = \mathbf{1}(x)$

$$\mathcal{A}_G[s] := \begin{cases} \{(\emptyset, \{\{1, v_0\}\}, \mathrm{w}(x)), (\emptyset, \{\{1\}, \{v_0\}\}, \mathrm{w}(x))\} & \text{if } s(1) = 1, \\ \{(\{1\}, \{\{v_0\}\}, 0)\} & \text{if } s(1) = -2, \\ \emptyset & \text{if } s(1) = 2. \end{cases}$$

Argue its correctness (and assuming at least two vertices in FVS).

$G = add_{i,j}(H)$

Let $(p_0, p, w) \in \mathcal{A}_H[s']$

- If $p_0 \cap \{i, j\} \neq \emptyset$, then $(p_0, p, w) \in \mathcal{A}_G[s']$.
- Otherwise, glue blocks containing i and j, respectively.
  - $s'(i), s'(j) \in \{1, 2\}$ (otherwise contains a cycle).
  - Let s such that $s(\ell) := s'(\ell)$ for $\ell \notin \{i, j\}$ and

$$(s'(i), s'(j)) := \begin{cases} (1,1) & \text{if } s(i) = s(j) = 1 \\ (2,1) & \text{if } (s(i), s(j)) = (-2, 1) \\ (1,2) & \text{if } (s(i), s(j)) = (1, -2). \end{cases}$$

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Add to $\mathcal{A}_G[s]$

$\mathsf{proj}(s^{-1}(-2), \mathsf{acjoin}(\{(p_0, p, w)\}, \{([k] \setminus \{i, j\}, \{\{i, j\}\}, 0)\})$.

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Remark. No edge between i-vertices and j-vertices prior to $add_{i,j}$ (Irredundant expression).

$G = G_1 \oplus G_2$

Let $(p_{01}, p_1, w_1) \in \mathcal{A}_{G_1}[s_1]$ and $(p_{02}, p_2, w_2) \in \mathcal{A}_{G_2}[s_2]$

Let s such that for each $i \in [k]$

$$s(i) = \begin{cases} s_1(i) = s_2(i) = -2 & \text{if } i \in p_{01} \cap p_{02} \\ s_2(i) & \text{if } i \in p_{01}, \\ s_1(i) & \text{if } i \in p_{02}, \\ 2 & \text{if } i \in [k] \setminus (p_{01} \cup p_{02}) \text{ and } s_1(i), s_2(i) \in \{1, 2\} \\ -2 & \text{if } i \in [k] \setminus (p_{01} \cup p_{02}) \text{ and } (s_1(i), s_2(i)) \in \\ & \{(1, -2), (-2, 1), (1, 1), (-2, -2)\} \end{cases}$$

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$\mathsf{acjoin}(\mathsf{proj}(s^{-1}(-2), \{(p_{01}, p_1, w_1)\}), \mathsf{proj}(s^{-1}(-2), \{(p_{02}, p_2, w_2)\}))$.

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- If $i \in p_0$, then $(p_0 \setminus \{i\}, p, w) \in \mathcal{A}_G[s']$,
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- If $i \notin p_0$, $j \in p_0$, $s'(i) \in \{1, 2\}$, then add to $\mathcal{A}_G[s]$ with $s(j) = s'(i)$ $\mathsf{proj}(\{i\}, \mathsf{acjoin}(\{(p_0, p, w)\}, \{([k] \setminus \{i, j\}, \{\{i, j\}\}, 0)\}))$.

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- Now, $i, j \notin p_0$, then
  - if $s'(i), s'(j) \in \{1, 2\}$, then add to $\mathcal{A}_G[s]$ with $s(j) = 2$ $\mathsf{proj}(\{i\}, \mathsf{acjoin}(\{(p_0, p, w)\}, \{([k] \setminus \{i, j\}, \{\{i, j\}\}, 0)\}))$.

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  - otherwise $s'(i), s'(j) \in \{1, -2\}$, add to $\mathcal{A}_G[s]$ with $s(j) = -2$ $\mathsf{proj}(\{i, j\}, \{(p_0, p, w)\})$.

Summary

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2. Representatives for Weighted Partitions
3. Locally Checkable Properties
4. Hamiltonian Cycle

Ac-Representation

$\mathcal{A}$, $\mathcal{A}'$ sets of weighted partitions, and $(q_0, q, 0)$.

$$\textbf{ac-opt}(\mathcal{A}, (q_0, q, 0)) := \max\{w \mid (q_0, p, w) \in \mathcal{A},\ p \sqcup q = \{V \setminus q_0\} \text{ and } \textsf{acyclic}(p, q)\}.$$

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$\mathcal{A}'$ ac-represents $\mathcal{A}$ if $\textbf{ac-opt}(\mathcal{A}, (q_0, q, 0)) = \textbf{ac-opt}(\mathcal{A}', (q_0, q, 0))$.

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Proposition

The operators rmc, proj and acjoin preserve ac-representation.

Computing an Ac-Representative

$V' := V \cup \{v_0\}$ and cuts are tri-partitions (U, V_1, V_2) with $v_0 \in V_1$

$$M[(p_0, p), (q_0, q)] := \begin{cases} 0 & \text{if } p_0 \neq q_0 \text{ or } p \sqcup q \neq \{V' \setminus p_0\}, \\ \alpha^{2|V' \setminus p_0| - (\#\mathrm{block}(p) + \#\mathrm{block}(q))} & \text{otherwise.} \end{cases}$$

$$C[(p_0, p), (U, V_1, V_2)] := \begin{cases} 0 & \text{if } p_0 \neq U \text{ or } p \not\sqsubseteq (V_1, V_2), \\ \alpha^{|V' \setminus U| - \#\mathrm{block}(p)} & \text{otherwise.} \end{cases}$$

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Theorem

We have $M = C \cdot C^t$. Moreover, there exists an algorithm ac-reduce *that given a set of weighted partitions $\mathcal{A}$, outputs in time $|\mathcal{A}| \cdot 3^{(\omega-1)|V|} \cdot |V|^{O(1)}$ a maximum ac-generator $\mathcal{A}'$ of size $\leq (|V| + 1) \cdot 3^{|V|}$ that ac-represents $\mathcal{A}$.*

Back to FVS Algorithm

Apply ac-reduce after computing $\mathcal{A}_G[s]$.

Theorem

There is an algorithm that, given an n-vertex graph G and an irredundant clique-width k-expression of G, computes a minimum feedback vertex set in time $3^{(4+\omega)k} \cdot n^{O(1)}$.

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$G = G_1 \oplus G_2$. $\mathcal{A}_G[s]$ is updated from 3^{2k} entries, each of size 2^{2k}, i.e. we update $\mathcal{A}_G[s]$ in time $3^{(\omega+3)\cdot k}$, and $\mathcal{A}_G$ in time $3^{(\omega+4)\cdot k} \cdot k^{O(1)}$.

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Characteristics

G k-labeled, and $D \subseteq V(G)$ and $R' \in \{0, 1, \ldots, d\}^k$.

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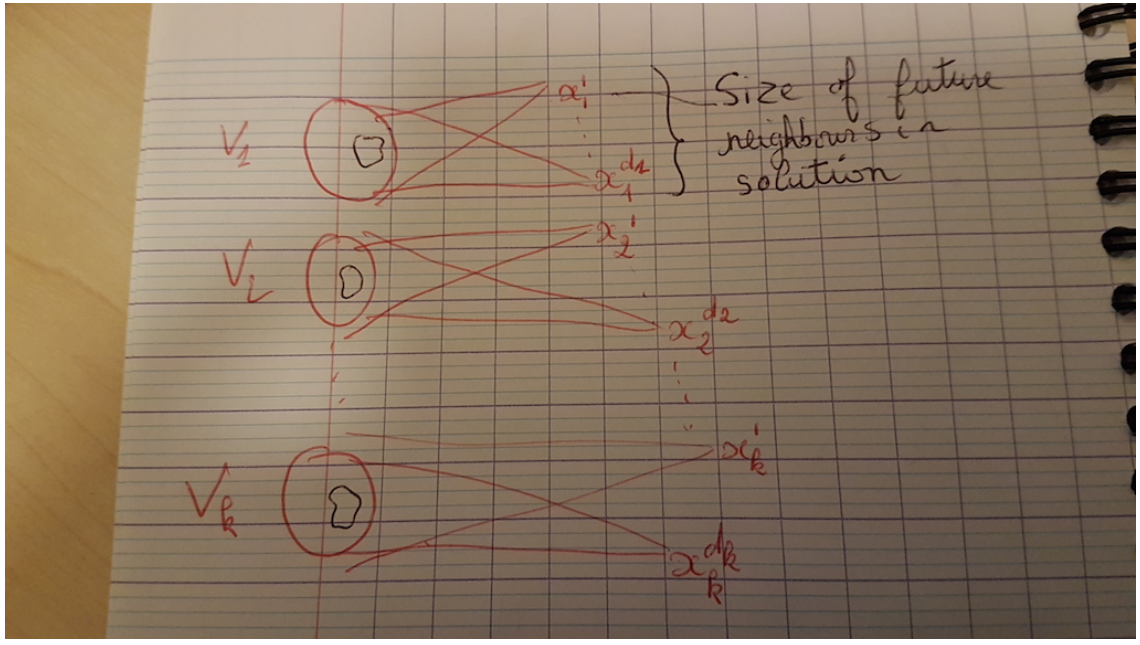

Dynamic Programming Table

$$\begin{aligned}\mathcal{D}_G[R, R'] := \mathsf{rmc}\Big(\{(p_0, p, w) \mid\ & \exists D \subseteq V(G), r(D) = R, \mathrm{w}(D) = w,\\ & p_0 = \{i \in [k] \mid r_i(D) = 0\},\\ & D \cup V^+(R')\ (\sigma, \rho)\text{-dominates } V(G) \text{ in } \mathsf{f}_{R'}(G),\\ & p = CC(G[D])_{\downarrow [k] \setminus p_0}, \text{ and } G[D] \text{ is connected if } R' = \{0\}^k, \text{ otherwise}\\ & \forall C \in CC(G[D]), \exists x \in C \text{ with } lab_G(x) = j \text{ and } R'_j \neq 0\}\Big).\end{aligned}$$

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Solution. $\mathbf{opt}_R\ \mathbf{opt}\{w \mid (p_0, p, w) \in \mathcal{D}_G[R, \{0\}^k], p := [k] \setminus p_0\}.$

$G = \mathbf{1}(x)$

$$tab_G[R, R'] := \begin{cases} \emptyset & \text{if } R_1 \notin \{0,1\} \text{ or } R_i \neq 0 \text{ for } i \neq 1, \\ \{([k] \setminus \{1\}, \{\{1\}\}, \mathrm{w}(x)\} & \text{if } (R' = \{0\}^k \text{ or } R'_1 \neq 0),\ R_1 = 1, \text{ and} \\ & \{x\} \cup V^+(R')\ (\sigma,\rho)\text{-dominates } \{x\}, \\ \{([k], \{\{\emptyset\}\}, 0\} & \text{if } R_1 = 0,\ V^+(R')\ (\sigma,\rho)\text{-dominates } \{x\}, \\ \emptyset & \text{in all other cases.} \end{cases}$$

$G = ren_{i \to j}(H)$

- If ($R_i \neq 0$ or $R'_i \neq R'_j$), then $tab_G[R, R'] = \emptyset$,
- Otherwise $tab_G[R, R'] :=$ reduce(rmc($\mathcal{A}$))

$$\mathcal{A} := tab_H[R, R'] \cup \mathsf{join}(\mathcal{A}', \{([k], \{\{\emptyset\}\}, 0)\})$$

with

$$\mathcal{A}' := \bigcup_{\substack{\forall \ell \notin \{i,j\}, S_\ell = R_\ell, S_i \neq 0 \\ R_j = \min\{S_i + S_j, d\}}} \mathsf{proj}(\{i\}, \mathsf{join}(tab_H[S, R'], \{([k] \setminus \{i,j\}, \{\{i,j\}\}, 0)\})).$$

$G = add_{i,j}(H)$

$S'_\ell := R'_\ell$ for all $\ell \notin \{i, j\}$, and $S'_i := R'_i - R_j$ and $S'_j := R'_j - R_i$.

- If $S'_i < 0$ or $S'_j < 0$, then $tab_G[R, R'] := \emptyset$.
- If $R_i = 0$ or $R_j = 0$, then $tab_G[R, R'] := tab_H[R, S']$.
- Otherwise, $tab_G[R, R'] := \mathsf{rmc}(\mathcal{A})$ with

$$\mathcal{A} := \mathsf{join}(tab_H[R, S'], \{([k] \setminus \{i, j\}, \{\{i, j\}\}, 0)\})$$

$G = G_1 \oplus G_2$

- If $R' = \{0\}^k$, then $tab_G[R, R'] = \mathsf{rmc}(\mathcal{A}_1 \cup \mathcal{A}_2)$

$$\mathcal{A}_1 := \begin{cases} tab_{G_1}[R, \{0\}^k] & \text{if } tab_{G_2}[\{0\}^k, \{0\}^k] \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\mathcal{A}_2 := \begin{cases} tab_{G_2}[R, \{0\}^k] & \text{if } tab_{G_1}[\{0\}^k, \{0\}^k] \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

- Otherwise,

$$\mathcal{A} := \bigcup_{\substack{S,S' \in \{0,1,\ldots,d\}^k \\ R_i = \min\{S_i + S'_i, d\}}} \mathsf{join}(tab_{G_1}[S, R'], tab_{G_2}[S', R']).$$

Theorem

There is an algorithm that, given an n-vertex graph G and an irredundant clique-width k-expression of G, computes an optimum connected (σ, ρ)-dominating set in time $(d+1)^{3k} \cdot 2^{(1+\omega)k} \cdot n^{O(1)}$.

The update at renaming and addition of edges is done from at most d^2 tables from tab_H, each having at most 2^k entries.

For a disjoint union, $tab_G[R, R']$ is updated from $(d+1)^k$ entries from tab_{G_1} and tab_{G_2}. Each such join needs $2^{(1+\omega)k}$, i.e., the update of tab_G in time $(d+1)^{3k} \cdot 2^{(1+\omega)k}$.

Summary

1. Feedback Vertex Set
2. Representatives for Weighted Partitions
3. Locally Checkable Properties
4. Hamiltonian Cycle

Algorithm

- Consider all partitions of $V(G)$ into paths.
- A partition $\mathcal{P}$ is represented by the multiset $\{(i,j,\ell) \mid$ there are exactly ℓ paths between an i-vertex and an j-vertex$\}$.
- The number of such partitions is bounded by n^{k^2}.

$G = ren_{i\to j}(H)$. Repeat $\mathcal{P} \setminus \{(i,\ell,p),(j,\ell,p')\} \cup \{(j,\ell,p+p')\}$ until no (i,ℓ,p).

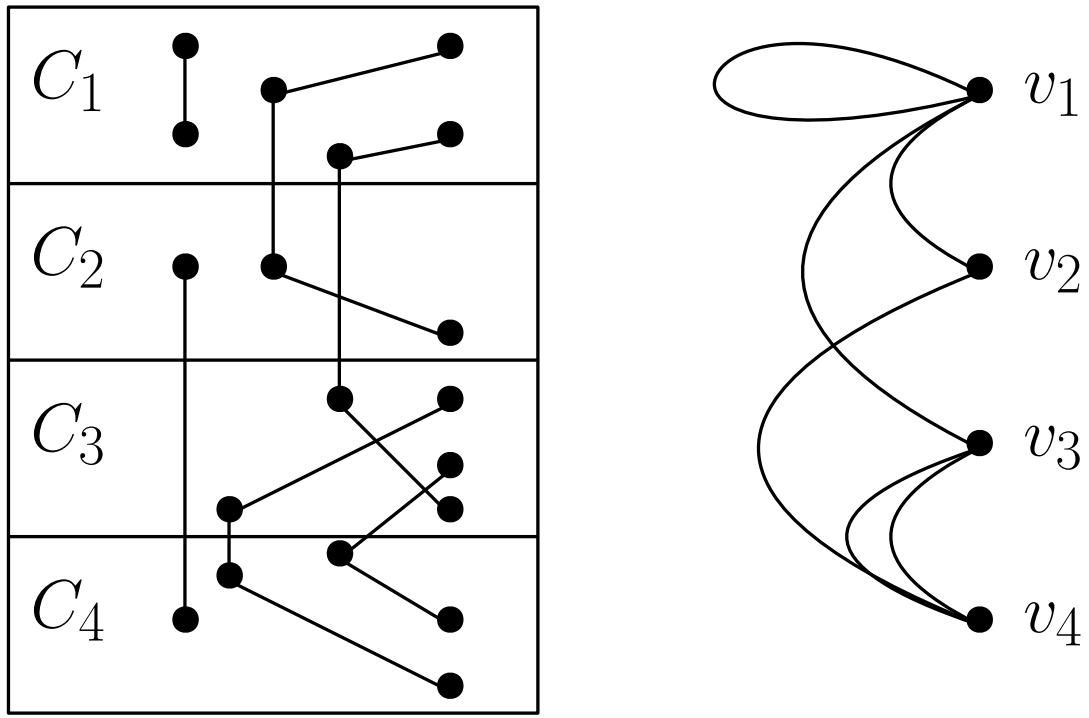
$G = G_1 \oplus G_2$. Take disjoint unions of partitions.

$G = add_{i,j}(H)$. Add as a possible partition (connect some paths)

$$\mathcal{P} \setminus \{(t,i,n_1),(j,\ell,n_2),(t,\ell,n_3)\} \cup \{(t,i,n_1-1),(j,\ell,n_2-2),(t,\ell,n_3+1)\}.$$

Characteristics

- For each partition $\mathcal{P}$, compute the multigraph $Aux(\mathcal{P})$ on k vertices where ℓ edges between v_i and v_j if $(i, j, \ell) \in \mathcal{P}$.
- $\mathcal{P} \equiv \mathcal{Q}$ if $Aux(\mathcal{P})$ and $Aux(\mathcal{Q})$ have same degree sequence and same connected components.



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Reduction. Take in each equivalence class one representative.

Proposition

The operations in the algorithm preserve representativity, i.e., whenever $\mathcal{P}$ can be extended in the future into a Hamiltonian cycle, then does $\mathcal{Q}$ if $\mathcal{P} \equiv \mathcal{Q}$.

Time complexity. There are $n^{O(k)}$ degree sequences and for each at most $2^{k \log(k)}$ possible partitions.
Tight under ETH.

Thank You