Linear Rank-Width and Linear Clique-Width of Trees

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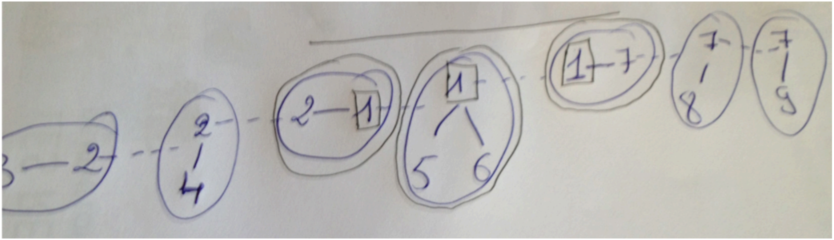
Graph Parameters

- Two famous graph parameters : tree-width and clique-width, and equivalent/variant complexity measures
- Tree-width and path-width play an important role in the Graph Minors Project
- Clique-width is important in complexity theory and its equivalent graph parameter rank-width has many structural properties and is linked to a Matroid Minors Project conducted by Geelen et al.
- Path-width has many structural characterisations, while linear rank-width : a little is known
- Characterisation of linear rank-width of trees

Path-width(1)



Path-Width of G

$$\mathrm{wd}(P, B) := \max\{|B_t| \mid t \in V(P)\} - 1$$

$$\mathrm{pwd}(G) := \min\{\mathrm{wd}(P, B) \mid (P, B) \text{ path decomposition of } G\}.$$

Path-width(2)

- Disjoint union of caterpillars = path-width 1
- $\text{pwd}(T_h) = \lceil h/2 \rceil$
- $\text{pwd}(G) \leq \text{twd}(G) \cdot \log(n)$
- Computation of the path-width of $TWD(\leq k)$ in polynomial time, even linear for trees
- Trees are obstructions to bounded path-width

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- A characterisation by cops and robber game

Path-width(3) : invisible robber game

- k cops and 1 invisible robber
- cops move by helicopter
- robber moves through paths not containing cops (she can identify cops positions)
- cops win if they have a strategy to catch the robber (land a helicoper on the robber position)
- minimum number of cops

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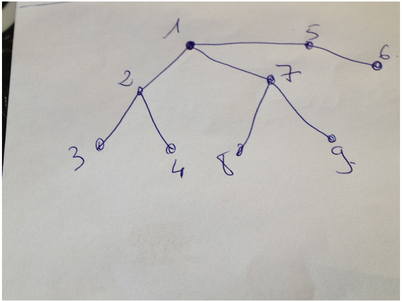
Linear Rank-Width

- take any linear ordering $x_1, \ldots, x_n$ of the vertices
- width $= \max_{1 \leq i \leq n-1} \{\mathrm{rk}(A_G[\{x_1, \ldots, x_i\}, -]\}$
- linear rank-width of G, $lrwd(G)$ = minimum over all linear orderings



3, 2, 4, · 1, 8, 7, 9, 5, 6

$$\mathrm{rk}\begin{pmatrix} & 1 & 8 & 7 & 9 & 5 & 6 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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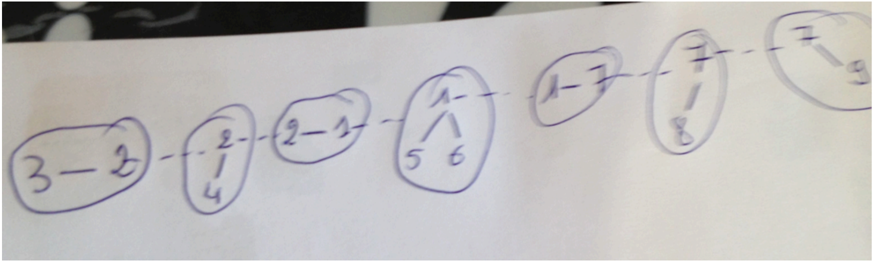
$lrwd(G) \leq pwd(G)$



3, 2, 4, 1, 5, 6, 7, 8, 9

$pwd(T) \leq lrwd(T)$

- Take a linear layout $v_1, v_2, \ldots, v_n$ of width $k := lrwd(T)$.
- Clear vertices in this ordering with at most $k+1$ cops.

Initialisation : Put i cops in vertices $v_1, \ldots, v_i$ such that $X_i := \{v_1, \ldots, v_i\}$ is a basis for $M_i := A_T[X_i, Y_i := V_T \setminus X_i]$.

Inductive step : if X_ℓ is cleared, clear $X_{\ell+1}$ while maintaining the following invariants

- ⋆ each vertex b of a basis B_i of M_i is either occupied or its neighbours in $Y_{\ell+1}$ are occupied,
- ⋆ cops occupy exactly $|B_{\ell+1}|$ vertices

Clearing Step(1)

- Either $v_{\ell+1}$ is linearly independent of B_ℓ in $M_{\ell+1}$ or not.
- Either $v_{\ell+1}$ is occupied by a cop or not after step ℓ.

To verify invariants, we need :

- ⋆ Clear $v_{\ell+1}$ and put a cop either on it or on its neighbours in $Y_{\ell+1}$ if it is linearly independent of B_ℓ in $M_{\ell+1}$.
- ⋆ Free cops in B_ℓ that are not in the "chosen" basis of $M_{\ell+1}$.

To do so, construct B-basic trees

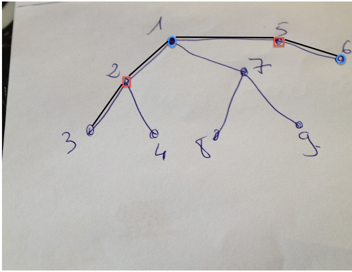
B-basic tree

For (X, Y) a cut, B a basis of $A_T[X, Y]$ and $x \in X$

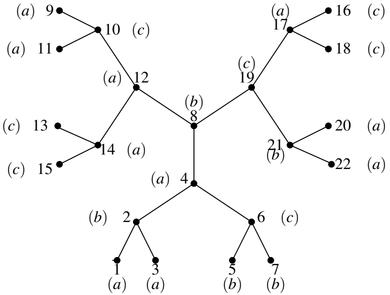
- Take $B' \subseteq B$ spanning x.
- Let $T' := T[B' \cup x \cup (N(B' \cup x) \cap Y)]$.

Properties

- T' is connected and leaves are from X.
- Vertices in $N(B' \cup x) \cap Y$ have degree 2.
- $|N(B' \cup x) \cap Y| = |B'|$.

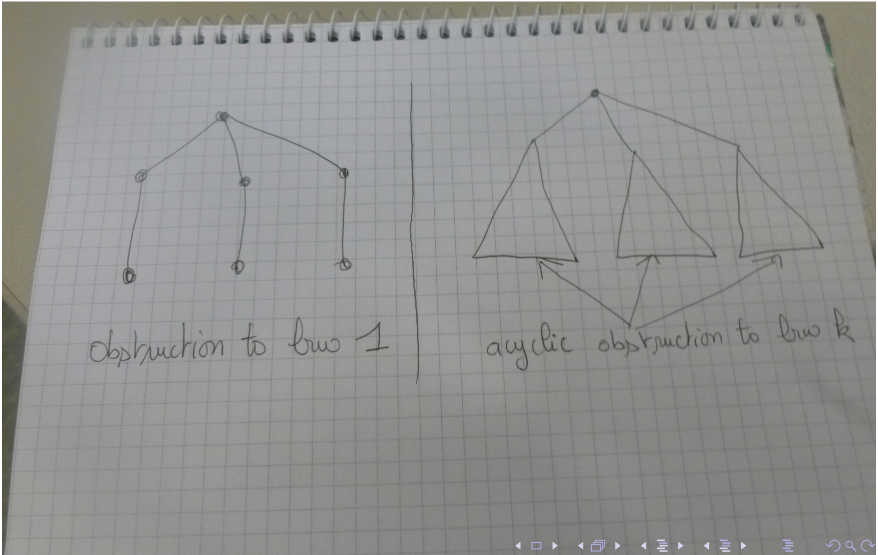


A strategy



(a) 9
10 (c)
(a) 11
(a) 12
(c) 13
14 (a)
(c) 15
(b) 8
(a) 17
16 (c)
18 (c)
(c) 19
20 (a)
21 (b)
22 (a)
(a) 4
(b) 2
6 (c)
1 (a)
3 (a)
5 (b)
7 (b)

A consequence : acyclic obstructions



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Linear Clique-Width of Trees

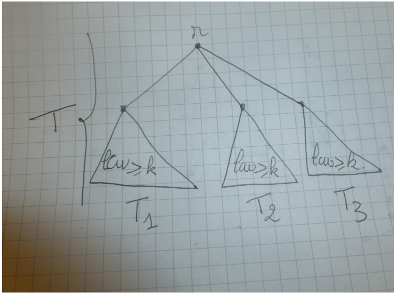
Lemma

If T_1, T_2, T_3 have linear clique-width at least k, then T has linear clique-width $\geq k+1$.

$\Longrightarrow$

Proposition

If T is a disjoint union of stars, then $lcw(T) = pw(T) + 2$, otherwise $lcw(T) = pw(T) + 1$.



Thank you ! !