On the Lattice Structure of Betweenness Relations and the Particular Case of the Geodesic One

Mamadou M. Kanté

LIMOS - Université Blaise Pascal

(Based on joint work with Laurent Beaudou and Lhouari Nourine)

Seminar of Discrete Mathematics, Optimisation and Decision, 2013

Metric Betweenness

If (X, d) is a metric space, then z is between x and y if

$$d(x, y) = d(x, z) + d(z, y).$$

In other words z is between x and y if z lies in the line $[x, y]$.

Hans Reichenbach : The direction of time, 1956

Causal Betweenness

Event B is causally between A and C if

$$P(A \cap C) > P(A)P(C)$$
$$P(C|B) > P(C|A)$$
$$P(A|B) > P(A|C)$$
$$P(A \cap C|B) > P(A|B)P(C|B)$$
$$P(B \setminus A)P(B \setminus C) > 0$$

Lattice Betweenness Relations

y is between x and z if $x \leq y \leq z$ or $z \leq y \leq x$.

$$
\begin{aligned}
\iff \quad (x + yz)y &= y + x(y + z) \\
\iff \quad (x + y)(y + z) &= y \\
\iff \quad y(x + z) &= y \\
\iff \quad y + xz &= y
\end{aligned}
$$

Others in Graph Theory and Related

- Everett and Seidman.
  The Hull Number of a Graph, 1985
- Vašek Chvátal in Antimatroids and convexity spaces.
  Antimatroids, Betweenness, Convexity, 2009
- Many others in Graph Theory.
  See Survey by Pelayo, 2004

Betweenness Relations

A **betweenness** on a ground set V is a ternary relation B such that

$$B(x,z,y) \qquad \Longleftrightarrow \qquad B(y,z,x)$$

$B(x,z,y)$ is pronounced "z is between x and y".

A betweenness relation B is also seen as the implicational system Σ_B

$$xz \to y \text{ whenever } y \text{ is between } x \text{ and } z$$

Ex. $B = \{(a,c,b),(b,c,a),(b,a,d),(d,a,b),(a,c,d),(d,c,a)\}$.
$\Sigma_B = \{ab \to c, bd \to a, ad \to c\}$

Examples of Betweenness Relations from Graph Theory

Monophonic path. A vertex z is between x and y if z is in a chordless path between x and y.

$\surd$ Direct implication system and forms an **antimatroid** in chordal graphs (Chvátal'2009).

Examples of Betweenness Relations from Graph Theory

Monophonic path. A vertex z is between x and y if z is in a chordless path between x and y.

$\surd$ Direct implication system and forms an **antimatroid** in chordal graphs (Chvátal'2009).

Geodesic. A vertex z is between x and y if it is in a shortest path between x and y.

$\surd$ A special case of metric betweenness relations.

$\surd$ Applications of geometry to graph theory.

Examples of Betweenness Relations from Graph Theory

Monophonic path. A vertex z is between x and y if z is in a chordless path between x and y.

$\surd$ Direct implication system and forms an **antimatroid** in chordal graphs (Chvátal'2009).

Geodesic. A vertex z is between x and y if it is in a shortest path between x and y.

$\surd$ A special case of metric betweenness relations.

$\surd$ Applications of geometry to graph theory.

Clique. $\Sigma_{C(G)} := \{xy \to V(G) \mid xy \notin E(G)\}$.

$\surd$ X is convex if and only if X is a clique.

$\surd$ We will see a characterization of such convexities.

Plan

Introduction

Lattice of Betweenness Relations

Algorithmic Aspects of Betweenness Relations

Concluding Remarks

From Betweenness Relations to Convexity

$X \subseteq V$ is said **convex** if $X = \bigcup_{x,y \in X} \{z \mid xy \to z \in \Sigma\}$, in other words X closed under Σ.

Ex. $\Sigma = \{ab \to c, bd \to a, ad \to c\}$.
$\{a, b\}$ is not convex, but $\{a, b, c, d\}$ is.

The set of convex sets is denoted by $\mathcal{F}_\Sigma$.

From Betweenness Relations to Convexity

$X \subseteq V$ is said **convex** if $X = \bigcup_{x,y \in X} \{z \mid xy \to z \in \Sigma\}$, in other words X closed under Σ.

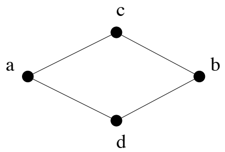
Ex. $\Sigma = \{ab \to c, bd \to a, ad \to c\}$.
$\{a, b\}$ is not convex, but $\{a, b, c, d\}$ is.

The set of convex sets is denoted by $\mathcal{F}_\Sigma$.

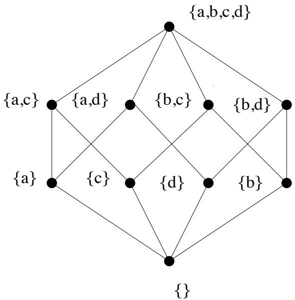
Which geometric properties are satisfied ?

Example

Convex Sets of the Geodesic Betweenness in a Graph



(a) A graph G



(b) Convex sets of G for the geodesic betweenness

Convexity Spaces

Definition (See for instance Kay and Womble' 1971)

$(X, \mathcal{F})$ is a convexity space if

- $\emptyset$ and $X \in \mathcal{F}$,
- $\mathcal{F}$ is closed under intersection

Members of $\mathcal{F}$ are called **convex** sets and is a lattice wrt inclusion.

The **closure** or **convex hull** of a set Y, $\mathcal{F}(Y)$, is the smallest convex set containing it.

Theorem (Monteiro, Portugaliae Mathematica, 1941)

The set of convexity spaces over X is a closure system and forms a lattice when structured under inclusion.

Convexity Spaces
Several studied parameters

- $\surd$ We can define/study **Caratheodory**, **Helly** and **Radon** numbers.
  $h+1 \leq r \leq ch+1$ (Kay&Womble'71)
- $\surd$ Y is a **hull set** if $X = \mathcal{F}(Y)$.
  What is the size of a minimum hull set ?
- $\surd$ **Geodetic sets**, ...

Convexity Spaces
Several studied parameters

$\surd$ We can define/study **Caratheodory**, **Helly** and **Radon** numbers.

$h + 1 \leq r \leq ch + 1$ (Kay&Womble'71)

$\surd$ Y is a **hull set** if $X = \mathcal{F}(Y)$.

What is the size of a minimum hull set ?

$\surd$ **Geodetic sets**, ...

$\star$ **Caratheodory.** $x \in \mathcal{F}(S)$, then $\exists Y \subseteq S$, $|Y| \leq c$ and $x \in \mathcal{F}(Y)$.
$\star$ **Helly.** $\bigcap_{Y \in \mathcal{G} \subseteq \mathcal{F}} Y = \emptyset$, then $\exists \mathcal{H} \subseteq \mathcal{G}$, $|\mathcal{H}| \leq h$ and $\bigcap_{Y \in \mathcal{H}} Y = \emptyset$
$\star$ **Radon.** If $|Y| \geq r$, then $Y = Y_1 \oplus Y_2$ with $\mathcal{F}(Y_1) \cap \mathcal{F}(Y_2) \neq \emptyset$.

Convexity Spaces Associated with Betweenness Relations

(X, F_Σ) is a convexity space for betweenness relation Σ on X.

injective ? $\times$ $\Sigma_1 := \{ad \to b, ad \to c, ac \to b\}$ and $\Sigma_2 := \Sigma_1 \setminus \{ad \to b\}$, then $F_{\Sigma_1} = F_{\Sigma_2}$.

surjective ? $\times$ every convexity space from a betweenness relation contains all singletons.

Convexity Spaces Associated with Betweenness Relations

(X, F_Σ) is a convexity space for betweenness relation Σ on X.

injective ? $\times$ $\Sigma_1 := \{ad \to b, ad \to c, ac \to b\}$ and $\Sigma_2 := \Sigma_1 \setminus \{ad \to b\}$, then $F_{\Sigma_1} = F_{\Sigma_2}$.

surjective ? $\times$ every convexity space from a betweenness relation contains all singletons.

We can associate a **canonical**, $\Sigma^c := \{xy \to z \mid z \in \Sigma(x, y)\}$

$$\implies \quad \mathcal{F}_\Sigma = \mathcal{F}_{\Sigma^c}$$

Question

Are convexity spaces from betweenness relations a sublattice of the lattice of all convexity spaces ?

Lattice of Betweenness Relations

$\mathbb{F}_X := \{\mathcal{F}_\Sigma \mid \Sigma \text{ betweenness relation on } X\}$.

Theorem 1 (Beaudou, K., Nourine, 2012)

Given $\mathcal{F}_{\Sigma_1}$ and $\mathcal{F}_{\Sigma_2}$, we have

$$\mathcal{F}_{\Sigma_1} \wedge \mathcal{F}_{\Sigma_2} = \mathcal{F}_{\Sigma_1} \cap \mathcal{F}_{\Sigma_2} = \mathcal{F}_{\Sigma_1 \cup \Sigma_2}$$
$$\mathcal{F}_{\Sigma_1} \vee \mathcal{F}_{\Sigma_2} = \mathcal{F}_{\Sigma_1 \cap \Sigma_2}$$

$\surd$ $\mathbb{F}_X$ is closed under intersection and then is a closure system
$\surd$ Structured under inclusion is a meet-sublattice of the lattice of convexity spaces over X.

Lattice of Betweenness Relations

Proof Ingredients

$\surd$ Canonical betweennesses
$\surd$ The following.

Proposition (Demetrovics et al., 1992)

$\mathcal{F}_\Sigma := 2^X \setminus \bigcup_{ab \to c \in \Sigma} [\{a, b\}, X \setminus c]$ for betweenness relation Σ on X.

Lattice of Betweenness Relations

Proof Ingredients

$\surd$ Canonical betweennesses
$\surd$ The following.

Proposition (Demetrovics et al., 1992)

$\mathcal{F}_\Sigma := 2^X \setminus \bigcup_{ab \to c \in \Sigma} [\{a, b\}, X \setminus c]$ for betweenness relation Σ on X.

$\mathbb{F}_X$ is not a join-sublattice

With $X := \{1, 2, 3, 4\}$, $\Sigma_1 := \{12 \to 4\}$ and $\Sigma_2 := \{23 \to 4\}$

$\mathcal{F}_{\Sigma_1} \cup \mathcal{F}_{\Sigma_2} = 2^X \setminus \{\{1, 2, 3\}\}$ but $\mathcal{F}_{\Sigma_1} \vee \mathcal{F}_{\Sigma_2} = 2^X$.

Poset of Irreducible Elements

Proposition 1 (BKN)

The meet-irreducible are co-atoms and the join are atoms.

Corollary 1 (BKN)

$Bip(\mathbb{F}_X) := (J_{\mathbb{F}_X}, M_{\mathbb{F}_X}, \subseteq)$ where

$$J_{\mathbb{F}_X} := \{F_\perp \cup \{S\} \mid S \in 2^X \setminus F_\perp\} \text{ where } F_\perp = \{\emptyset, X\} \cup \{\{x\} \mid x \in X\}$$
$$M_{\mathbb{F}_X} := \{2^X \setminus [ab, X \setminus \{c\}] \mid a, b, c \in X\}.$$

Corollary 2 (BKN)

$|M_{\mathbb{F}_X}| = \binom{n}{2}(n-2)2^{n-3}$ and $|J_{\mathbb{F}_X}| = 2^n - (n+2)$.

Example of $Bip(\mathbb{F}_X)$

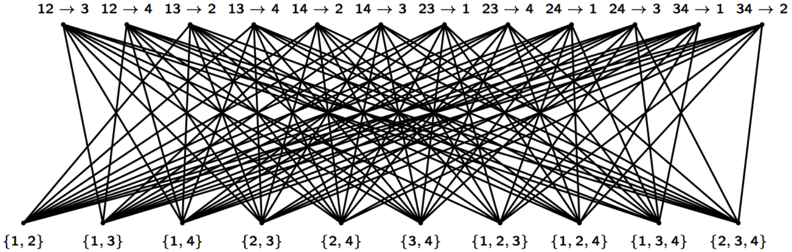


Figure : Irreducible poset for $n = 4$

Plan

Introduction

Lattice of Betweenness Relations

Algorithmic Aspects of Betweenness Relations
- Chordal Graphs
- Other Graph Classes : Logic and Graph Operations

Concluding Remarks

Algorithmic Aspects of Betweenness Relations

Hull Number (HN)

Input. A betweenness relation Σ
Output. Compute a hull set of minimum size

$\surd$ HN is NP-complete even on the geodesic betweenness of bipartite graphs (Araujo et al., 2011)
$\surd$ Its complexity is open for the geodesic betweenness in several graph classes.

Optimal Cover Problem (OCP)

Input. A betweenness relation Σ
Output. Compute a betweenness relation $\Sigma' \equiv \Sigma$ of minimum size

Remarks

$\surd$ OCP = computation **hydra number**.
$\surd$ NP-complete for general convexity spaces (Maier, 1980), but open for betweenness relations.

Geodesic Betweenness in Graphs

Its complexity was open for more than 25 years in the case of chordal graphs.

Theorem (KN, 2012)

- *One can compute in time $O(m + n)$ a hull set of minimum size in distance-hereditary graphs (anecdotic).*
- *One can compute in time $O(n^3)$ a hull set of minimum size in chordal graphs.*

A graph is **distance-hereditary** if distances are preserved in connected induced subgraphs.

A graph is **chordal** if it does not contain cycles of length ≥ 4.

Plan

Introduction

Lattice of Betweenness Relations

Algorithmic Aspects of Betweenness Relations
- Chordal Graphs
- Other Graph Classes : Logic and Graph Operations

Concluding Remarks

Chordal Graphs

A vertex is **simplicial** if its neighbourhood is a clique.

A **perfect elimination ordering** of G is an ordering $(x_1, \ldots, x_n)$ such that x_i is simplicial in $G[\{x_i, \ldots, x_n\}]$.

We borrow ideas from Database Theory and use the following results by Dirac'61, Fulkerson-Gross'65 and Tarjan-Lueker'76.

Theorem 1

(i) Every chordal graph has at least two simplicial vertices.

(ii) G is chordal iff it has a perfect elimination ordering.

(iii) A perfect elimination ordering of a chordal graph can be computed in time $O(n + m)$.

Functional Dependencies

A **functional dependency** on a ground set V is a pair (X, y), written $X \to y$, with X the **premise** and y the **conclusion**.

An **implicational system** is set of functional dependencies on V.

F is **closed** if $y \in F$ whenever $X \subseteq F$, for all $X \to y \in \Sigma$.

The **closure** of X, $\Sigma(X)$, is the smallest closed set containing X.

A **key** is an inclusionwise minimal set X such that $\Sigma(X) = V$.

Betweenness Relations as Implicational Systems

A Betweenness relation is an implicational system with premises of size 2.

To every graph G, we associate the implicational system

$$\Sigma_G := \bigcup_{x,y \in V} \{xy \rightarrow z \mid z \text{in a shortest path between } x \text{ and } y\}.$$

Fact 1

K is a minimum key of Σ_G iff K is a minimum hull set of G.

The Algorithm

A vertex x is an **extreme point** in Σ if x is not a conclusion.

Ex. Simplicial vertices are extreme points in Σ_G.

Algorithm

1. Construct $\Sigma := \Sigma_G$ and take a perfect elimination ordering $(x_1, \ldots, x_n)$.
2. For each i, decide whether to put x_i in the key and let $\Sigma := \Sigma \setminus x_i$.
3. The remaining implicational system, if exists, is with premises of size 1. Compute a key and add it to the already computed one.
4. Return the computed key.

Correctness

$$\Sigma' := \Sigma \setminus x_1 \setminus \ldots \setminus x_i.$$

Lemma 1

If x_{i+1} is an extreme point in Σ', then any key of Σ' is of the form $K \cup \{x_{i+1}\}$ where K is a key of $\Sigma' \setminus x_{i+1}$ defined as

$$\{zy \to t \in \Sigma' \mid z, t, y \neq \Sigma'(\{x_{i+1}\})\} \qquad \bigcup$$

$$\{y \to z \mid yx \to z \in \Sigma' \text{ and } x \in \Sigma'(\{x_{i+1}\}),\ y, z \notin \Sigma'(\{x_{i+1}\})\}.$$

Remove from $\Sigma \setminus x_1 \setminus \ldots \setminus x_i$ all those vertices that can be obtained from x_{i+1} to get $\Sigma \setminus x_1 \setminus \ldots \setminus x_i \setminus x_{i+1}$.

Correctness

$$\Sigma' := \Sigma \setminus x_1 \setminus \ldots \setminus x_i.$$

Lemma 2

If x_{i+1} is not an extreme point in Σ', then it appears as a conclusion only in functional dependencies with premises of size 1. Define $\Sigma' \setminus x_{i+1}$ as

$$\Sigma' \setminus \{zx_{i+1} \to y \in \Sigma'\}) \cup \left(\{tz \to y \mid zx_{i+1} \to y, t \to x_{i+1} \in \Sigma'\}\right)$$

A minimum key in $\Sigma' \setminus x_{i+1}$ is a minimum key in Σ'. Conversely, to any minimum key in Σ', one can associate a minimum key in $\Sigma' \setminus x_{i+1}$.

We cannot decide whether to put x_{i+1} in a key, however we can replace it safely from $\Sigma \setminus x_1 \setminus \ldots \setminus x_i$.

Time Complexity

Proposition 1

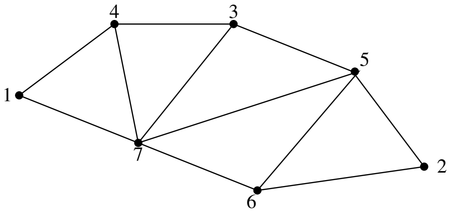
For every graph G, Σ_G can be computed in time at most $O(n^3)$.

Proposition 2

If Σ is an implicational system on V with premises of size 1, then a minimum key of Σ can be computed in time $O(|V| + |\Sigma|)$.

Example

A chordal graph G and its associated implicational system.



$12 \to 567$

$13 \to 47$

$15 \to 6$

$16 \to 7$

$23 \to 5$

$24 \to 3567$

$27 \to 56$

$36 \to 57$

$45 \to 37$

$46 \to 7$

Example

1 is an extreme point in Σ and set $K := \{1\}$

Σ	$\Sigma \setminus 1$
$12 \to 567$	$2 \to 567$
$13 \to 47$	$3 \to 47$
$15 \to 6$	$5 \to 6$
$16 \to 7$	$6 \to 7$
$23 \to 5$	$23 \to 5$
$24 \to 3567$	$24 \to 3567$
$27 \to 56$	$27 \to 56$
$36 \to 57$	$36 \to 57$
$45 \to 37$	$45 \to 37$
$46 \to 7$	$46 \to 7$

Example

2 is an extreme point in $\Sigma \setminus 1$ and set $K := \{1, 2\}$

Σ	$\Sigma \setminus 1$	$\Sigma \setminus 1 \setminus 2$
$12 \to 567$	$2 \to 567$	
$13 \to 47$	$3 \to 47$	
$15 \to 6$	$5 \to 6$	
$16 \to 7$	$6 \to 7$	
$23 \to 5$	$23 \to 5$	$3 \to 4$
$24 \to 3567$	$24 \to 3567$	$4 \to 3$
$27 \to 56$	$27 \to 56$	
$36 \to 57$	$36 \to 57$	
$45 \to 37$	$45 \to 37$	
$46 \to 7$	$46 \to 7$	

Plan

Introduction

Lattice of Betweenness Relations

Algorithmic Aspects of Betweenness Relations

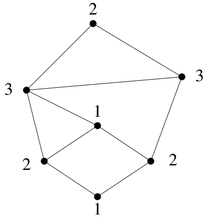
- Chordal Graphs
- Other Graph Classes : Logic and Graph Operations

Concluding Remarks

Clique-Width

A complexity measure based on Graph Grammars.

A k-**graph** = each vertex has *exactly one* colour in $\{1, \ldots, k\}$.



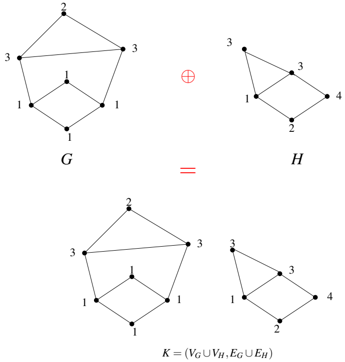
3-graphe

Clique-Width

A complexity measure based on Graph Grammars.

A k-**graph** = each vertex has *exactly one* colour in $\{1, \ldots, k\}$.

$G \oplus H$ = disjoint union of k-graphs.



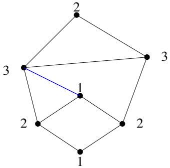
Clique-Width

A complexity measure based on Graph Grammars.

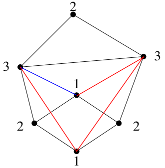
A k-**graph** = each vertex has *exactly one* colour in $\{1, \ldots, k\}$.

$G \oplus H$ = disjoint union of k-graphs.

$add_{i,j}(G)$ = addition of edges between i-vertices and j-vertices.



G



$add_{1,3}(G)$

Clique-Width

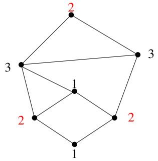
A complexity measure based on Graph Grammars.

A k-**graph** = each vertex has *exactly one* colour in $\{1, \ldots, k\}$.

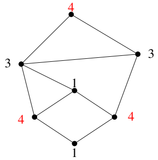
$G \oplus H$ = disjoint union of k-graphs.

$add_{i,j}(G)$ = addition of edges between i-vertices and j-vertices.

$ren_{i \to j}(G)$ = recolour i-vertices into j-vertices.



G



$ren_{2 \to 4}(G)$

Clique-Width

A complexity measure based on Graph Grammars.

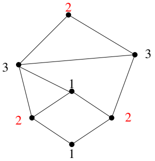
A k-**graph** = each vertex has *exactly one* colour in $\{1, \ldots, k\}$.

$G \oplus H$ = disjoint union of k-graphs.

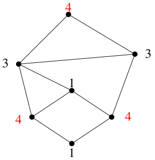
$add_{i,j}(G)$ = addition of edges between i-vertices and j-vertices.

$ren_{i \to j}(G)$ = recolour i-vertices into j-vertices.

i = a graph with one vertex coloured i.



G

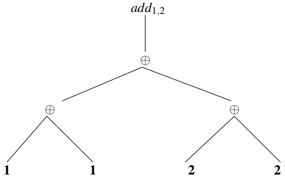


$ren_{2 \to 4}(G)$

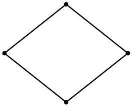
Clique-Width

- $F_k = \{\oplus, add_{i,j}, ren_{i \to j} \mid i, j \in [k]\}$.
- $C_k = \{\mathbf{i} \mid i \in [k]\}$

A term t in $T(F_k, C_k)$ defines a graph $val(t)$.



t

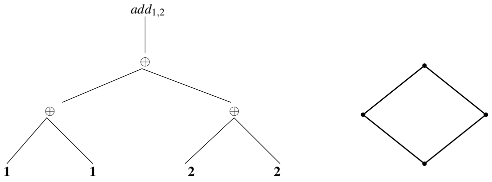


$val(t)$

Clique-Width

- $F_k = \{\oplus, add_{i,j}, ren_{i\to j} \mid i,j \in [k]\}$.
- $C_k = \{\mathbf{i} \mid i \in [k]\}$

A term t in $T(F_k, C_k)$ defines a graph $val(t)$.



t $val(t)$

$$\text{cwd}(G) := \min\{k \mid G = val(t), t \in T(F_k, C_k)\}$$

Monadic Second-Order Logic

A k-graph is the relational structure $\langle V_G, edg_G, (p_{iG})_{i \in [k]} \rangle$.

Atomic Formulas. $x \in X$, $edg(x, y)$, $p_i(x)$, $x = y$.

MSO formulas. Boolean combinations and element/set quantifications.

Ex. $\forall X(x \in X \ \wedge \ \forall z, t(z \in X \ \wedge \ edg(z, t) \implies t \in X) \implies y \in X)$.

Monadic Second-Order Logic

A k-graph is the relational structure $\langle V_G, edg_G, (p_{iG})_{i\in[k]}\rangle$.

Atomic Formulas. $x \in X$, $edg(x,y)$, $p_i(x)$, $x = y$.

MSO formulas. Boolean combinations and element/set quantifications.

Ex. $\forall X(x \in X \ \wedge \ \forall z,t(z \in X \ \wedge \ edg(z,t) \implies t \in X) \implies y \in X)$.

MSO optimisation. Find a tuple $(Z_1,\ldots,Z_q)$ of $(2^{V_G})^q$ such that

$$\sum_{1\leq i\leq q} |Z_i| = opt\left\{\sum_{1\leq i\leq q} \left|W_j\right| \mid G \models \varphi(W_1,\ldots,W_q)\right\}.$$

MSO and Clique-Width

Theorem 2 (Courcelle,Makowski,Rotics'00 and Oum'05)

Every MSO optimisation problem can be solved in time $O(f(k) \cdot n^3)$ in graphs of clique-width at most k. If clique-width expression is given, it can be solved in time $O(g(k) \cdot n)$.

MSO definability of Hull Set

Proposition 3

If there exists an MSO formula $\varphi(x, z, y)$ stating that z is in a shortest path between x and y, then there exists an MSO formula stating that X is a hull set.

$$Cl(X) \equiv \forall x, y(x \in X \;\wedge\; y \in X \implies \neg\exists z(\varphi(x, z, y))),$$

$$CH(X, Y) \equiv Cl(Y) \wedge X \subseteq Y \wedge \forall Z(X \subseteq Z \wedge Z \subseteq Y \implies \neg Cl(Z))$$

$$HullSet(X) \equiv \forall Z\big(Z \subsetneq V \implies \neg CH(X, Z)\big)$$

Hull Number of DH Graphs

G is distance-hereditary iff chordless paths are shortest paths.

There exists an MSO formula stating that z is in a chordless path between x and y in a graph.

Distance-Hereditary graphs have clique-width at most 3 and clique-width expressions can be computed in time $O(n + m)$.

Combine Theorem 2 and Proposition 3.

Plan

Introduction

Lattice of Betweenness Relations

Algorithmic Aspects of Betweenness Relations
- Chordal Graphs
- Other Graph Classes : Logic and Graph Operations

Concluding Remarks

Concluding Remarks

Conjecture. NP-complete in planar graphs, but polynomial in bounded degree and clique-width bounded graphs.

Techniques for DH graphs can be used for other betweenness relations (triangle paths, monophonic paths, etc.) to compute a minimum hull set in clique-width bounded graphs.

Betweenness relations give **dependence graphs** and allow to MSO define any betweenness relation. Characterise those of bounded clique-width.

Dichotomy. Find a sharp line between tractable and intractable cases. Can the lattice structure of betweenness relations can help ?

Thank you!!