Linear rank-width of distance-hereditary graphs II. Vertex-minor obstructions

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Abstract

In the companion paper [Linear rank-width of distance-hereditary graphs I. A polynomial-time algorithm, Algorithmica 78(1):342–377, 2017], we presented a characterization of the linear rank-width of distance-hereditary graphs, from which we derived an algorithm to compute it in polynomial time. In this paper, we investigate structural properties of distance-hereditary graphs based on this characterization.

First, we prove that for a fixed tree T, every distance-hereditary graph of sufficiently large linear rank-width contains a vertex-minor isomorphic to T. We extend this property to bigger graph classes, namely, classes of graphs whose prime induced subgraphs have bounded linear rank-width. Here, prime graphs are graphs containing no splits. We conjecture that for every tree T, every graph of sufficiently large linear rank-width contains a vertex-minor isomorphic to T. Our result implies that it is sufficient to prove this conjecture for prime graphs.

For a class Φ of graphs closed under taking vertex-minors, a graph G is called a *vertex-minor obstruction* for Φ if $G \notin \Phi$ but all of its proper vertex-minors are contained in Φ. Secondly, we provide, for each $k \geqslant 2$, a set of distance-hereditary graphs that contains all distance-hereditary vertex-minor obstructions for graphs of linear rank-width at most k. Also, we give a simpler way to obtain the known vertex-minor obstructions for graphs of linear rank-width at most 1.

1 Introduction

Linear rank-width is a linear-type width parameter of graphs motivated by the rank-width of graphs [33]. The *vertex-minor* relation is a graph containment relation which was introduced by Bouchet [7, 8, 10, 9, 11] in his studies of circle graphs and 4-regular Eulerian digraphs. The vertex-minor relation has an important role in the theory of (linear) rank-width [29, 32, 30, 25, 31] as

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(linear) rank-width does not increase when taking vertex-minors of a graph. We provide concise definitions in Section 2.

The problem of computing linear rank-width has been discussed recently. Kashyap [26] proved that it is NP-hard to compute matroid path-width on binary matroids. Proposition 3.1 in [32] shows that the problem of determining the linear rank-width of a bipartite graph is equivalent to the problem of determining the path-width of a binary matroid, and from this relation, we can show that computing linear rank-width is NP-hard in general. Adler and the authors of this paper [3] proved that the linear rank-width of distance-hereditary graphs, which are graphs of rank-width 1, can be computed in time $\mathcal{O}(n^2 \log n)$ where n is the number of vertices in an input graph. Jeong, Kim, and Oum [24] showed that there is a constructive algorithm to test whether a given graph has linear rank-width at most k in time $f(k) \cdot n^3$ for some function f. Using this, they also proved that for every fixed integer w, there is a polynomial-time algorithm to compute linear rank-width on graphs of rank-width w.

In this paper, we focus on structural aspects of linear rank-width. The first result of the Graph Minor series papers is that for a fixed tree T, every graph of sufficiently large path-width contains a minor isomorphic to T [34], and this was later used by Blumensath and Courcelle [6] to define a hierarchy of *incidence graphs* based on *monadic second-order transductions*. In order to obtain a similar hierarchy for graphs, still based on monadic second-order transductions, Courcelle [14] asked whether for a fixed tree T, every bipartite graph of sufficiently large linear rank-width contains a vertex-minor isomorphic to T. We conjecture that it is true for any graph.

Conjecture 1.1. *For every fixed tree T, there is an integer $f(T)$ such that every graph of linear rank-width at least $f(T)$ contains a vertex-minor isomorphic to T.*

Recently, Kwon and Oum [28] claimed that for any positive integers m, n, if T is the disjoint union of m copies of $K_{1,n}$, then such a function exists. However, it remains open in general.

We show that Conjecture 1.1 is true if and only if it is true in prime graphs with respect to *split decompositions* [16]. A *split* in a graph is a vertex partition (A, B) such that $|A|$, $|B| \geqslant 2$ and the set of edges joining A and B induces a complete bipartite subgraph. *Prime graphs* are graphs without splits and they form, with complete graphs and stars, the basic graphs in the theory of canonical split decompositions developed by Cunningham [16]. They are also considered when studying the rank-width of graphs because the rank-width of a graph is the maximum rank-width over all its prime induced subgraphs.

We prove the following.

Theorem 1.2. *Let p be a positive integer and let T be a tree. Let G be a graph such that every prime induced subgraph of G has linear rank-width at most p. If G has linear rank-width at least $40(p + 2)|V(T)|$, then G contains a vertex-minor isomorphic to T.*

A graph G is *distance-hereditary* if for every connected induced subgraph H of G and two vertices v and w in H, the distance between v and w in H is the same as their distance in G. It is known that every prime induced subgraph of a distance-hereditary graph has size at most 3 [10]. Together with this fact, our result implies that Conjecture 1.1 is also true for distance-hereditary graphs.

To prove Theorem 1.2, we essentially prove that for a fixed tree T, every graph admitting a canonical split decomposition whose decomposition tree has sufficiently large path-width contains a vertex-minor isomorphic to T. Combining with a relation between the linear rank-width of a graph

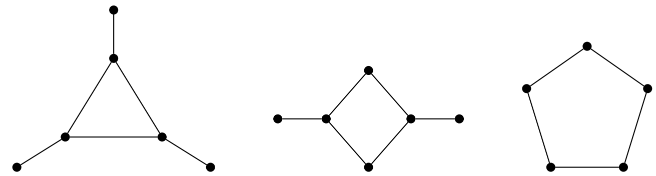


Figure 1: The three vertex-minor obstructions for graphs of linear rank-width at most 1. The first two graphs are distance-hereditary.

and the path-width of its canonical split decomposition, we obtain Theorem 1.2. We will obtain such a relation in Section 4. The vertex-minor relation cannot be replaced with the induced subgraph relation because there is a cograph admitting a canonical split decomposition whose decomposition tree has sufficiently large path-width [13, 22], but cographs have no P_4 as an induced subgraph.

In the second part, we investigate the set of distance-hereditary vertex-minor obstructions for graphs of bounded linear rank-width. A graph is a *vertex-minor obstruction* for graphs of linear rank-width k if it has linear rank-width $k+1$ and every proper vertex-minor has linear rank-width at most k. Robertson and Seymour [36] showed that for every infinite sequence $G_1, G_2, \dots$ of graphs, there exist G_i and G_j with $i < j$ such that G_i is isomorphic to a minor of G_j. In other words, graphs are *well-quasi-ordered* under the minor relation. Interestingly, this property implies that for any proper class $\mathcal{C}$ of graphs closed under taking minors, the set of minor obstructions for $\mathcal{C}$ is finite.

Motivated by the Graph Minor Theorem [36] and its special case on tree-width [35], Oum [29, 31] showed that for every infinite sequence G_1, $G_2, \dots$ of graphs of bounded rank-width, there exist G_i and G_j with $i < j$ such that G_i is isomorphic to a vertex-minor of G_j. We can obtain the following as a corollary.

Theorem 1.3 (Oum [29]). *For every class $\mathcal{C}$ of graphs with bounded rank-width that is closed under taking vertex-minors, there is a finite list of graphs G_1, $G_2, \dots, G_m$ such that a graph is in $\mathcal{C}$ if and only if it has no vertex-minor isomorphic to G_i for some $i \in \{1, 2, \dots, m\}$.*

Theorem 1.3 implies that for every integer k, the class of all graphs of (linear) rank-width at most k can be characterized by a finite list of vertex-minor obstructions. However, it does not give any explicit number of necessary vertex-minor obstructions or bound on the size of such graphs. Oum [32] proved that for each k, the size of a vertex-minor obstruction for graphs of rank-width at most k is at most $(6^{k+1} - 1)/5$. For linear rank-width, obtaining such an upper bound on the size of vertex-minor obstructions remains an open problem. Jeong, Kwon, and Oum [25] showed that the number of vertex-minor obstructions for linear rank-width at most k is at least $2^{\Omega(3^k)}$.

Adler, Farley, and Proskurowski [1] obtained the set of all three vertex-minor obstructions for graphs of linear rank-width at most 1, depicted in Figure 1, two of which are distance-hereditary. In this paper, we construct a set of graphs containing all vertex-minor obstructions for graphs of linear rank-width at most k that are distance-hereditary. This is an analogous result to the characterization of acyclic minor obstructions for graphs of path-width at most k, investigated by Takahashi, Ueno, and Kajitani [37], and Ellis, Sudborough, and Turner [20]. As a similar work, Koutsonas, Thilikos, and Yamazaki [27] characterized matroid obstructions for bounded matroid path-width that are cycle matroids of outerplanar graphs.

Lastly, we obtain simpler proofs of known characterizations of graphs of linear rank-width at most 1 [1, 12].

The paper is organized as follows. Section 2 provides some preliminary concepts, including linear rank-width and vertex-minors. In Section 3, we introduce necessary notions regarding split decompositions, and restate the structural characterization of linear rank-width on distance-hereditary graphs. Section 4 presents a relation between the linear rank-width of a graph whose prime induced subgraphs have bounded linear rank-width and the path-width of its decomposition tree. From this, we prove Theorem 1.2 in Section 5. In Section 6, we provide a way to generate all vertex-minor obstructions for graphs of bounded linear rank-width that are distance-hereditary graphs. Section 7 presents simpler proofs for known characterizations of the graphs of linear rank-width at most 1.

2 Preliminaries

In this paper, graphs are finite, simple and undirected. Our graph terminology is standard, see for instance [19]. Let G be a graph. We denote the vertex set of G by $V(G)$ and the edge set by $E(G)$. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X, and let $G-X := G[V(G)\setminus X]$. For $v \in V(G)$, we write $G-x$ for $G-\{x\}$. For $F \subseteq E(G)$, let $G-F := (V(G), E(G)\setminus F)$. Similarly, for $e \in E(G)$, we write $G-e$ for $G-\{e\}$. For a vertex x of G, let $N_G(x)$ be the set of *neighbors* of x in G and we call $|N_G(x)|$ the *degree* of x in G. Two vertices x and y are *twins* if $N_G(x)\setminus\{y\} = N_G(y)\setminus\{x\}$. An edge e of a connected graph G is a *cut-edge* if $G-e$ is disconnected. A vertex v in a connected graph G is a *cut vertex* if $G-v$ is disconnected. A connected graph is 2*-connected* if it has at least 3 vertices and has no cut vertices.

A *tree* is a connected graph containing no cycles. A vertex of degree one in a tree is called a *leaf*. A *subcubic tree* is a tree with maximum degree at most three, and a *path* is a tree with maximum degree at most two. The *length* of a path is the number of its edges. A *star* is a tree with a distinguished vertex, called its *center*, adjacent to all other vertices. A *complete graph* is a graph with all possible edges. A graph G is called *distance-hereditary* if for every pair of two vertices x and y of G the distance of x and y in G equals the distance of x and y in any connected induced subgraph containing both x and y [4]. It is well-known that a graph is distance-hereditary if and only if it can be obtained from a single vertex by repeatedly adding a vertex of degree one, or creating a twin of a vertex in the graph [23]. An induced cycle of length at least 5 is not distance-hereditary.

A subset F of the edge set of G is called a *matching* if no two edges in F share an end vertex.

For an edge e of a graph G, we denote by G/e the graph obtained by contracting e. A graph H is a *minor* of a graph G if H is obtained from a subgraph of G by contractions of edges.

For a positive integer n, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

2.1 Linear rank-width

For sets R and C, an (R, C)*-matrix* is a matrix whose rows and columns are indexed by R and C, respectively. For an (R, C)-matrix M and subsets $X \subseteq R$ and $Y \subseteq C$, let $M[X, Y]$ be the submatrix of M whose rows and columns are indexed by X and Y, respectively.

Let G be a graph. We denote by A_G the *adjacency matrix* of G over the binary field; that is, for $v, w \in V(G)$, $A_G[v, w] = 1$ if v is adjacent to w, and $A_G[v, w] = 0$, otherwise. For a graph G, let $\mathrm{cutrk}^*_G : 2^{V(G)} \times 2^{V(G)} \to \mathbb{Z}$ be the function such that $\mathrm{cutrk}^*_G(X, Y) := \mathrm{rank}(A_G[X, Y])$ for all $X, Y \subseteq V(G)$, where rank is computed over the binary field. The *cut-rank function* of G is the function $\mathrm{cutrk}_G : 2^{V(G)} \to \mathbb{Z}$ where for each $X \subseteq V(G)$,

$$\mathrm{cutrk}_G(X) := \mathrm{cutrk}^*_G(X, V(G)\backslash X).$$

An ordering $(x_1, \ldots, x_n)$ of the vertex set $V(G)$ is called a *linear layout* of G. If $|V(G)| \geqslant 2$, then the *width* of a linear layout $(x_1, \ldots, x_n)$ of G is defined as

$$\max_{1 \leqslant i \leqslant n-1} \{\mathrm{cutrk}_G(\{x_1, \ldots, x_i\})\},$$

and if $|V(G)| = 1$, then the width is defined to be 0. The *linear rank-width* of G, denoted by $\mathrm{lrw}(G)$, is defined as the minimum width over all linear layouts of G.

Caterpillars and complete graphs have linear rank-width at most 1. Ganian [21] gave a characterization of graphs of linear rank-width at most 1, and called them *thread graphs*. Adler and Kanté [2] showed that linear rank-width and path-width coincide on forests, and therefore, there is a linear-time algorithm to compute the linear rank-width of forests. It is easy to see that the linear rank-width of a graph is the maximum over the linear rank-widths of its connected components.

For a linear layout L of a graph G and two vertices v and w, we denote by $v \leqslant_L w$ if $v = w$ or v appears before w in the linear layout. For two orderings $(v_1, v_2, \ldots, v_n)$ and $(w_1, w_2, \ldots, w_m)$, let

$$(v_1, v_2, \ldots, v_n) \oplus (w_1, w_2, \ldots, w_m) := (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m).$$

2.2 Vertex-minors

For a graph G and a vertex x of G, the *local complementation at x in G* is an operation to replace the subgraph induced by the set of neighbors of x with its complement. The resulting graph is denoted by $G * x$. If a graph H can be obtained from G by applying a sequence of local complementations, then G and H are called *locally equivalent*. A graph H is called a *vertex-minor* of a graph G if H can be obtained from G by applying a sequence of local complementations and deletions of vertices. Bouchet [11] observed that local complementations do not change the cut-rank function. This directly implies that every vertex-minor H of G satisfies that $\mathrm{lrw}(H) \leqslant \mathrm{lrw}(G)$.

Lemma 2.1 (Bouchet [11]; See Corollary 2)**.** *Let G be a graph and let x be a vertex of G. Then for every subset X of $V(G)$, we have* $\mathrm{cutrk}_G(X) = \mathrm{cutrk}_{G*x}(X)$*.*

For an edge xy of G, let $W_1 := N_G(x) \cap N_G(y)$, $W_2 := (N_G(x)\backslash N_G(y))\backslash\{y\}$, and $W_3 := (N_G(y)\backslash N_G(x))\backslash\{x\}$. The *pivoting on xy* of G, denoted by $G \wedge xy$, is the operation to flip the adjacencies between distinct sets W_i and W_j, and swap the vertices x and y. Flipping the adjacency between two vertices v and w is an operation that adds an edge if there was no edge between v and w, and removes an edge, otherwise. It is known that $G \wedge xy = G * x * y * x = G * y * x * y$ [32, Proposition 2.1]. See Figure 2 for an example. A graph H is called a *pivot-minor* of a graph G if H can be obtained from G by applying a sequence of pivotings on edges and deletions of vertices. Observe that every pivot-minor of a graph is a vertex-minor of the graph, because of the relation $G \wedge xy = G * x * y * x$.

2.3 Path-width

A *path decomposition* of a graph G is a pair $(P, \mathcal{B})$, where P is a path and $\mathcal{B} = (B_t)_{t \in V(P)}$ is a family of vertex subsets of G such that

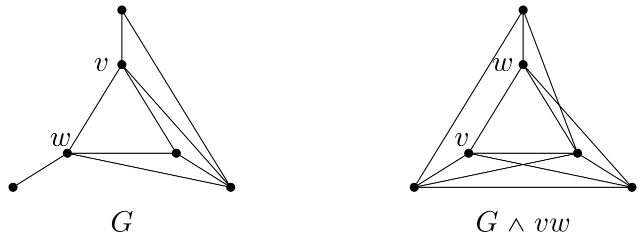


Figure 2: An example of pivoting.

1. for every $v \in V(G)$ there exists $t \in V(P)$ such that $v \in B_t$,
2. for every $uv \in E(G)$ there exists $t \in V(P)$ such that $\{u, v\} \subseteq B_t$,
3. for every $v \in V(G)$, the set $\{t \in V(P) : v \in B_t\}$ induces a subpath of P.

The *width* of a path decomposition $(P, \mathcal{B})$ is defined as $\max\{|B_t| : t \in V(P)\} - 1$. The *path-width* of G, denoted by $\mathrm{pw}(G)$, is defined as the minimum width over all path-decompositions of G.

It is well known that if H is a minor of G, then $\mathrm{pw}(H) \leqslant \mathrm{pw}(G)$. Robertson and Seymour [34] first proved that for a fixed tree T, every graph of sufficiently large path-width contains a minor isomorphic to T. Bienstock, Robertson, Seymour, and Thomas [5] optimized the necessary function, and Diestel [18] later provided a short proof of it.

Theorem 2.2 (Bienstock, Robertson, Seymour, and Thomas [5]; Diestel [18])**.** *For every forest F, every graph with path-width at least $|V(F)| - 1$ has a minor isomorphic to F.*

We recall the following theorem which characterizes the path-width of trees and is used for computing their path-width in linear time.

Theorem 2.3 (Ellis, Sudborough, and Turner [20]; Takahashi, Ueno, and Kajitani [37])**.** *Let T be a tree and let k be a positive integer. The following are equivalent.*

(1) T has path-width at most k.

(2) For every node x of T, at most two of the subtrees of $T - x$ have path-width k and all other subtrees of $T - x$ have path-width at most $k - 1$.

(3) T has a path P such that for each node v of P and each connected component T' of $T - v$ not containing a node of P, $\mathrm{pw}(T') \leqslant k - 1$.

3 Linear rank-width of distance-hereditary graphs

In this section, we recall the characterization of the linear rank-width of distance-hereditary graphs investigated by Adler and the authors of this paper [3]. For this characterization, we need to introduce split decompositions and the new notion of limbs introduced in [3]. We will follow the definition for split decompositions used by Bouchet [10].

A *split* in a connected graph G is a vertex partition (X, Y) of G such that $|X|, |Y| \geqslant 2$ and $\mathrm{cutrk}_G(X) = 1$. *Prime graphs* are connected graphs that do not have a split. Note that every

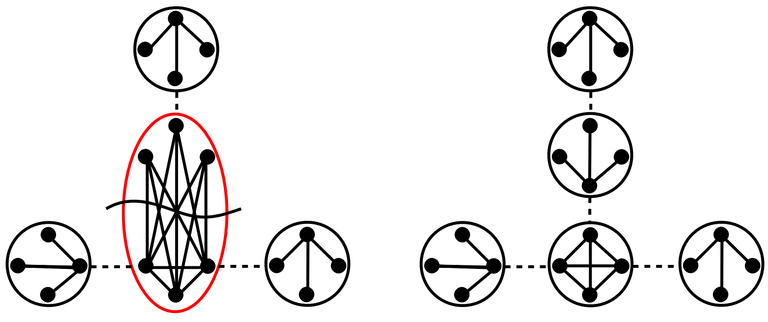


Figure 3: An example of replacing a bag B with its simple decomposition. Circles indicate bags and dotted edges indicate marked edges. When we replace a bag B with its simple decomposition, other marked edges are still marked edges.

connected graph with at most 3 vertices is a prime graph, by definition. Also, one can observe that every connected graph on 4 vertices admits a split, and it is not a prime graph.

A *marked graph* is a connected graph D with a matching $M(D)$ where every edge in $M(D)$ is a cut-edge. Every edge in $M(D)$ is called a marked edge, and the end vertices of marked edges are called *marked vertices*. The connected components of $D - M(D)$ are called *bags* of D. The edges in $E(D)\backslash M(D)$ are called *unmarked edges*, and the vertices that are not marked are called *unmarked vertices*.

If (X, Y) is a split in a graph G, then we construct a new marked graph D such that

- $V(D) = V(G) \cup \{x', y'\}$ for two distinct new vertices $x', y' \notin V(G)$,
- $E(D) = E(G[X]) \cup E(G[Y]) \cup \{x'y'\} \cup E'$ where

$$\begin{aligned} E' := \{x'x : x \in X \text{ and there exists } y \in Y \text{ such that } xy \in E(G)\} \cup \\ \{y'y : y \in Y \text{ and there exists } x \in X \text{ such that } xy \in E(G)\}, \end{aligned}$$

- $x'y'$ is a marked edge, and all edges in E' are unmarked edges.

The marked graph D is called a *simple decomposition of* G. See Figure 3 for an example.

A *split decomposition* of a connected graph G is a marked graph D defined inductively to be either G or a marked graph defined from a split decomposition D' of G by replacing a bag with its simple decomposition. For a marked edge xy of a marked graph D, the *recomposition of* D *along* xy is the marked graph $(D \wedge xy) - \{x, y\}$. For a split decomposition D, let $\mathcal{G}[D]$ denote the graph obtained from D by recomposing all marked edges. Note that if D is a split decomposition of G, then $\mathcal{G}[D] = G$.

Since each marked edge of a split decomposition D is a cut-edge and all marked edges form a matching, if we contract all unmarked edges in D, then we obtain a tree. We call it the *decomposition tree of* G *associated with* D and denote it by T_D. To distinguish the vertices of T_D from the vertices of G or D, the vertices of T_D will be called *nodes*. For a node v of T_D, we write $\mathsf{bag}_D(v)$ to denote the bag of D with which it is in correspondence, and for a bag B of D, we write $\mathsf{node}_D(B)$ to denote the node of T_D with which it is in correspondence. Two bags of D are called *adjacent bags* if their corresponding nodes in T_D are adjacent. A sequence of bags $B_1 - B_2 - \cdots - B_m$ is called a path of bags if for each $i \in [m-1]$, B_i and B_{i+1} are adjacent bags, and all of $B_1, B_2, \ldots, B_m$ are pairwise distinct. Clearly, for two bags B and B', there is a unique path of bags from B to B', which corresponds to the path from $\mathsf{node}_D(B)$ to $\mathsf{node}_D(B')$ in T_D. We denote by $\mathrm{dist}_D(B, B')$ the distance from $\mathsf{node}_D(B)$ to $\mathsf{node}_D(B')$ in T_D; in other words, it is one less than the number of bags in the unique path of bags from B to B' in D.

3.1 Canonical split decompositions and local complementations

A split decomposition is called *canonical* if each bag is either a prime graph, a star, or a complete graph, and every recomposition of a marked edge in D results in a split decomposition without the same property. The following is due to Cunningham and Edmonds [15], and Dahlhaus [17].

Theorem 3.1 (Cunningham and Edmonds [15]; Dahlhaus [17]). *Every connected graph G has a unique canonical split decomposition, up to isomorphism, and it can be computed in time $\mathcal{O}(|V(G)|+|E(G)|)$.*

A bag is called a *prime bag* if it is a prime graph on at least 5 vertices, and a bag is called a *complete bag* or a *star bag* if it is a complete graph or a star, respectively.

Let D be a split decomposition of a connected graph G with bags that are either a prime graph, a complete graph or a star. The *type of a bag* of D is either P, K, or S depending on whether it is a prime graph, a complete graph, or a star, respectively. The *type of a marked edge uv* is AB where A and B are the types of the bags containing u and v respectively. If $A = S$ or $B = S$, then we can replace S by S_p or S_c depending on whether the end vertex of the marked edge is a leaf or the center of the star, respectively. Bouchet characterized canonical split decompositions in terms of the types of marked edges.

Theorem 3.2 (Bouchet [10]). *Let D be a split decomposition of a connected graph whose bags are either a prime graph, a complete graph, or a star. Then D is a canonical split decomposition if and only if it has no marked edge of type KK or S_pS_c.*

We will use the following characterizations of trees and of distance-hereditary graphs.

Theorem 3.3 (Bouchet [10]).

(1) A connected graph is distance-hereditary if and only if every bag of its canonical split decomposition is of type K or S.

(2) A connected graph is a tree if and only if every bag of its canonical split decomposition is a star bag whose center is an unmarked vertex.

We now relate the split decompositions of a graph and the ones of its locally equivalent graphs. Let D be a split decomposition of a connected graph. A vertex v of D *represents* an unmarked vertex x (or is a *representative* of x) if either $v = x$ or there is a path of even length from v to x in D starting with a marked edge such that marked edges and unmarked edges appear alternately in the path. Two unmarked vertices x and y are *linked* in D if there is a path from x to y in D such that unmarked edges and marked edges appear alternately in the path. Linkedness of unmarked vertices exactly represents the adjacency relation between those vertices in the original graph.

Lemma 3.4 (Adler, Kanté, and Kwon [3]). *Let D be a split decomposition of a connected graph G. Let v' and w' be two vertices in a same bag of D, and let v and w be two unmarked vertices of D represented by v' and w', respectively. The following are equivalent.*

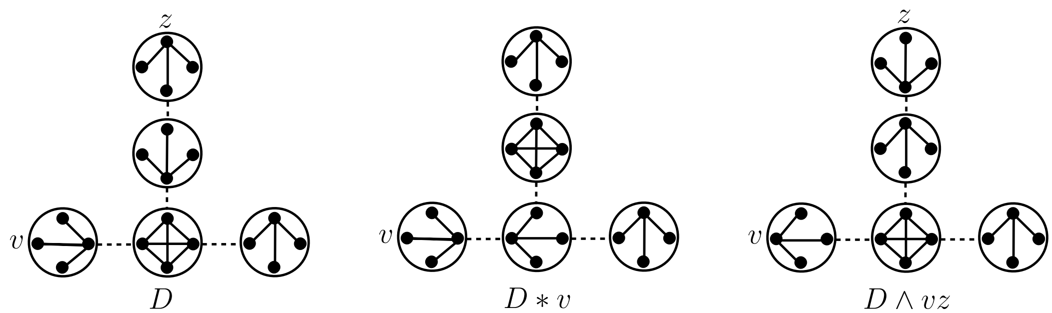


Figure 4: Examples of local complementation and pivoting in a split decomposition.

1. v and w are linked in D.

2. $vw \in E(G)$.

3. $v'w' \in E(D)$.

A *local complementation* at an unmarked vertex x in a split decomposition D, denoted by $D * x$, is the operation to replace each bag B containing a representative w of x with $B * w$. Bouchet observed that $D * x$ is a split decomposition of $\mathcal{G}[D] * x$, and $M(D) = M(D * x)$. To see why $D * x$ is a split decomposition of $\mathcal{G}[D] * x$, let us consider the bag containing x as a root bag R. For a bag B, if B contains no representative of x, then it is easy to see that for any unmarked vertex contained in the sub-decomposition rooted at B, it is not adjacent to x in the original graph, and therefore, this part should be the same in the split decomposition of $\mathcal{G}[D] * x$. Assume that a bag B' contains a representative of x. By the definition of representativity, there is a unique vertex in B' that is a representative of x, say u. Let v, w be two neighbors of u in B'. Note that every vertex represented by v in D is adjacent to every vertex represented by w in D if and only if v is adjacent to w by Lemma 3.4. Observe that after applying local complementation at x, the adjacency relations between the set of vertices represented by v and the set of vertices represented by w are changed, and therefore, the adjacency relation between v and w in the split decomposition of $\mathcal{G}[D] * x$ should be different from their adjacency in D. It means that $B' * u$ is a correct shape of the bag in the split decomposition of $\mathcal{G}[D] * x$. A formal proof of this fact can be found in Bouchet [10, Section 4].

Two split decompositions D and D' are *locally equivalent* if D can be obtained from D' by applying a sequence of local complementations at unmarked vertices. As expected, this local complementation also preserves the property that the split decomposition is canonical.

Lemma 3.5 (Bouchet [10])**.** *Let D be the canonical split decomposition of a connected graph G. If x is a vertex of G, then $D * x$ is the canonical split decomposition of $G * x$.*

Let x and y be linked unmarked vertices in a split decomposition D, and let P be the path in D linking x and y such that unmarked edges and marked edges appear alternately in the path. Note that if B is a bag of type S containing an unmarked edge of P, then the center of B is a representative of either x or y. The *pivoting on xy of D*, denoted by $D \wedge xy$, is the split decomposition obtained as follows: for each bag B containing an unmarked edge of P, if $v, w \in V(B)$ represent respectively x and y in D, then we replace B with $B \wedge vw$. It is worth noticing that by Lemma 3.4, we have $vw \in E(B)$, hence $B \wedge vw$ is well-defined.

Lemma 3.6 (Adler, Kanté, and Kwon [3]). *Let D be a split decomposition of a connected graph G. If $xy \in E(G)$, then $D \wedge xy = D * x * y * x$.*

3.2 Removing vertices

Let G be a distance-hereditary graph and let D be its split decomposition. Let S be a vertex set of G. We explain how we transform D into a split decomposition of $G - S$. Note that the split decomposition obtained from D by removing vertices in S is not necessarily a split decomposition because the resulting marked graph may have bags of size at most 2. In this case, we need to recompose a marked edge incident with each bag of size at most 2 unless the resulting marked graph has at most two vertices.

Suppose that D is canonical. We frequently consider connected components T of $D - V(B)$, for a bag B of D. This will be used to define limbs in the next subsection. For a bag B of D and a connected component T of $D - V(B)$, let us denote by $\zeta_b(D, B, T)$ and $\zeta_c(D, B, T)$ the end vertices of the marked edge in D linking B and T that are in $V(B)$ and in $V(T)$ respectively. Subscripts b and c stand for bag and component, respectively. We always treat T as a canonical split decomposition and regard $\zeta_c(D, B, T)$ as an unmarked vertex.

3.3 Limbs and characterization of linear rank-width

To present the characterization of the linear rank-width of distance-hereditary graphs, we need the new notion called limbs [3]. For an unmarked vertex y in D and a bag B of D containing a marked vertex representing y, let T be the connected component of $D - V(B)$ containing y, and let $v := \zeta_c(D, B, T)$ and $w := \zeta_b(D, B, T)$. We define the *limb* $\mathcal{L} := \mathcal{L}_D[B, y]$ with respect to B and y as follows:

1. if B is of type K, then $\mathcal{L} := T * v - v$,
2. if B is of type S and w is a leaf, then $\mathcal{L} := T - v$,
3. if B is of type S and w is the center, then $\mathcal{L} := T \wedge vy - v$.

While T is a canonical split decomposition, $\mathcal{L}$ may not be a canonical split decomposition, because deleting v may create a bag of size 2. We analyze the cases when such a bag appears, and describe how to transform it into a canonical split decomposition. Suppose that a bag B' of size 2 appears in $\mathcal{L}$. If B' has no adjacent bags in $\mathcal{L}$, then B' itself is a canonical split decomposition. We may assume that there is a bag adjacent to B'.

1. (B' has one adjacent bag B_1.)
   If $v_1 \in V(B_1)$ is the marked vertex adjacent to a vertex of B' and r is the unmarked vertex of B' in $\mathcal{L}$, then we remove the bag B' and replace v_1 with r. In other words, we recompose along the marked edge connecting B' and B_1.
2. (B' has two adjacent bags B_1 and B_2.)
   If $v_1 \in V(B_1)$ and $v_2 \in V(B_2)$ are the two marked vertices that are adjacent to the two marked vertices of B', then we remove B' and add a marked edge v_1v_2. If the new marked edge v_1v_2 is of type KK or S_pS_c, then by recomposing along v_1v_2, we finally transform the limb into a canonical split decomposition.

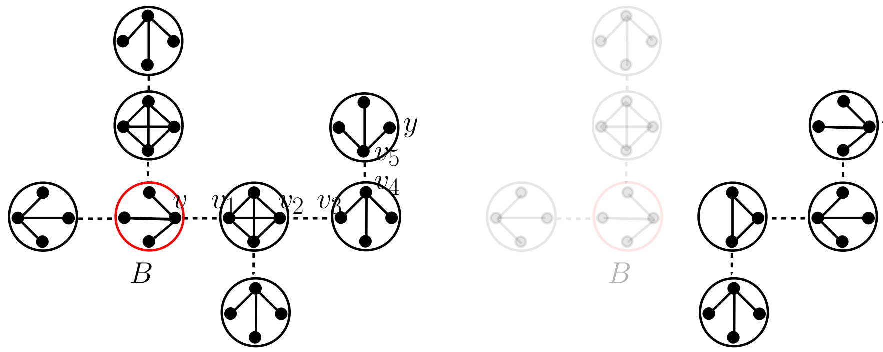


Figure 5: An example of a limb $\mathcal{L}_D[B, y]$. Let B_3 be the bag containing y and let $B - B_1 - B_2 - B_3$ be the path in D from B to B_3. Let vv_1, v_2v_3 and v_4v_5 the marked edges between, respectively, B and B_1, B_1 and B_2, and B_2 and B_3. Let T be the connected component of $D - V(B)$ containing y. Then $\mathcal{L}_D[B, y]$ is $T \wedge v_1y - v_1$. The bags of $\mathcal{L}_D[B, y]$ corresponding to B_1, B_2 and B_3 are respectively obtained by doing a pivoting on v_1v_2, v_3v_4 and yv_5.

Let $\mathcal{LC}_D[B, y]$ be the canonical split decomposition obtained from $\mathcal{L}_D[B, y]$ and we call it the *canonical limb*. Let $\mathcal{LG}_D[B, y]$ be the graph obtained from $\mathcal{L}_D[B, y]$ by recomposing all marked edges. For a bag B of D and a connected component T of $D - V(B)$, we define $f_D(B, T)$ as the linear rank-width of $\mathcal{LG}_D[B, y]$ for some unmarked vertex $y \in V(T)$. It was shown that $f_D(B, T)$ does not depend on the choice of y.

Proposition 3.7 (Adler, Kanté, and Kwon; Proposition 3.4 of [3])**.** *Let B be a bag of D and let y be an unmarked vertex of D represented by a vertex w in B. Let $x \in V(\mathcal{G}[D])$. If an unmarked vertex y' is represented by w in $D * x$, then $\mathcal{LG}_D[B, y]$ is locally equivalent to $\mathcal{LG}_{D*x}[(D * x)[V(B)], y']$. Therefore, $f_D(B, T) = f_{D*x}((D * x)[V(B)], T_x)$ where T and T_x are the components of $D - V(B)$ and $(D * x) - V(B)$ containing y, respectively.*

As a variant of Theorem 2.3, distance-hereditary graphs of bounded linear rank-width can be characterized using limbs.

Theorem 3.8 (Adler, Kanté, and Kwon [3])**.** *Let k be a positive integer and let D be the canonical split decomposition of a connected distance-hereditary graph G. Then the following are equivalent.*

(1) G has linear rank-width at most k.

(2) For each bag B of D, $D - V(B)$ has at most two connected components T such that $f_D(B, T) = k$, and every other connected component T' of $D - V(B)$ satisfies that $f_D(B, T') \leqslant k - 1$.

(3) T_D has a path P such that for each node v of P and each connected component H of $D - V(\mathsf{bag}_D(v))$ containing no bags $\mathsf{bag}_D(w)$ with $w \in V(P)$, $f_D(\mathsf{bag}_D(v), H) \leqslant k - 1$.

4 Path-width of decomposition trees

To prove Theorem 1.2, we derive a relation between the linear rank-width of a graph whose prime induced subgraphs have bounded linear rank-width and the path-width of its decomposition tree.

Proposition 4.1. *Let p be a positive integer. Let G be a connected graph whose prime induced subgraphs have linear rank-width at most p, and let D be the canonical split decomposition of G, and let T_D be the decomposition tree of G associated with D. Then* $\mathrm{lrw}(G) \leqslant 2(p+2)(\mathrm{pw}(T_D)+1)$.

We prove Proposition 4.1 by induction on the path-width of T_D. If its path-width is 0, then it consists of one node, and the result directly follows from the given condition that every prime induced subgraph has linear rank-width at most p. Note that complete graphs and stars have linear rank-width at most 1. We assume that the path-width of T_D is at least 1. Using Lemma 2.3, T contains a path P such that for each node v of P and each connected component T' of $T - v$ not containing a node of P, $\mathrm{pw}(T') \leqslant k - 1$. So, by induction, we can obtain an upper bound of the linear rank-width of split decompositions corresponding to such components T'. From this, we will obtain an upper bound of the linear rank-width of the whole graph.

We need the following lemma. We point out that Lemma 4.2 does not require D to be a canonical split decomposition, and this relaxation will be useful for an easier argument in the main proof.

Lemma 4.2. *Let k and p be positive integers. Let B be a bag of a split decomposition D with two unmarked vertices x and y such that for every connected component H of $D - V(B)$, $\mathrm{lrw}(\mathcal{G}[H]) \leqslant k$. If B has a linear layout of width at most p whose first and last vertices are x and y respectively, then $\mathcal{G}[D]$ has a linear layout of width at most $2p + k$ whose first and last vertices are x and y respectively.*

Proof. Let $G := \mathcal{G}[D]$, and let $L_B := (w_1, w_2, \ldots, w_m)$ be a linear layout of B of width at most p such that $x = w_1$ and $y = w_m$. For each $j \in [m]$,

1. if w_j is an unmarked vertex, then let $L_j := (w_j)$, and
2. if $w_j = \zeta_b(D, B, H)$ for some connected component H of $D - V(B)$, then let L_j be a linear layout of $\mathcal{G}[H] - \zeta_c(D, B, H)$ having width at most k.

We define $L := L_1 \oplus L_2 \oplus \cdots \oplus L_m$. We observe that L is a linear layout of G. For each $j \in [m]$, we choose an unmarked vertex y_j represented by w_j. If w_j is an unmarked vertex, then $y_j = w_j$.

We claim that L has width at most $2p+k$. It is sufficient to prove that for every $w \in V(G) \backslash \{x, y\}$, $\mathrm{cutrk}_G(\{v : v \leqslant_L w\}) \leqslant 2p + k$. Let $w \in V(G) \backslash \{x, y\}$ and let $S_w := \{v : v \leqslant_L w\}$ and $T_w := V(G) \backslash S_w$.

Let H_j be a connected component of $D - V(B)$ such that $\zeta_b(D, B, H_j) = w_j$. Observe that if all vertices in $V(H_j) \cap V(G)$ are contained in S_w, then all vertices in $V(H_j) \cap V(G)$ that have a neighbor in T_w have exactly the same set of neighbors in T_w, which is $N_G(y_j) \cap T_w$. Therefore, when we compute the rank of the matrix $A(G)[S_w, T_w]$, we can replace all vertices in $V(H_j) \cap V(G)$ with y_j. The same observation holds for connected components fully contained in T_w. Also, for two distinct connected components H_{j_1}, H_{j_2} of $D - V(B)$ where all vertices of $V(H_{j_1}) \cap V(G)$ are contained in S_w and all vertices of $V(H_{j_2}) \cap V(G)$ are contained in T_w, y_1 and y_2 are adjacent in G if and only if $\zeta_b(D, B, H_{j_1})$ is adjacent to $\zeta_b(D, B, H_{j_2})$ in B. This is an implication of Lemma 3.4.

Having it, we can observe that if w is an unmarked vertex in B, then

$$\mathrm{cutrk}_G(S_w) = \mathrm{cutrk}_B(\{v : v \leqslant_{L_B} w\}) \leqslant p.$$

Thus, we may assume that w is contained in some connected component H of $D - V(B)$. Let $j \in [m]$ such that $\zeta_b(D, B, H) = w_j$.

Note that H is the unique component of $D - V(B)$ possibly intersecting both S_w and T_w. Since all vertices of $V(H) \cap V(G)$ having a neighbor in $V(G)\backslash V(H)$ have the same neighborhood in $V(G)\backslash V(H)$ (that is, $(V(H) \cap V(G), V(G)\backslash V(H))$ is a split), we have

(1) $\mathrm{cutrk}^*_G(S_w, T_w\backslash V(H)) \leqslant \max\{\mathrm{cutrk}_B(\{v : v \leqslant_{L_B} w_{j-1}\}), \mathrm{cutrk}_B(\{v : v \leqslant_{L_B} w_j\})\} \leqslant p$.

(2) $\mathrm{cutrk}^*_G(S_w\backslash V(H), T_w) \leqslant \max\{\mathrm{cutrk}_B(\{v : v \leqslant_{L_B} w_{j-1}\}), \mathrm{cutrk}_B(\{v : v \leqslant_{L_B} w_j\})\} \leqslant p$.

(3) $\mathrm{cutrk}^*_G(S_w \cap V(H), T_w \cap V(H)) \leqslant k$.

Therefore, we have

$$\begin{aligned}\mathrm{cutrk}_G(S_w) &\leqslant \mathrm{cutrk}^*_G(S_w, T_w\backslash V(H)) + \mathrm{cutrk}^*_G(S_w\backslash V(H), T_w)\\ &\quad + \mathrm{cutrk}^*_G(S_w \cap V(H), T_w \cap V(H))\\ &\leqslant p + p + k \leqslant 2p + k.\end{aligned}$$

We conclude that L is a linear layout of G of width at most $2p + k$ whose first and last vertices are x and y, respectively. □

Proof of Proposition 4.1. We prove it by induction on $k := \mathrm{pw}(T_D)$. If $k = 0$, then T_D consists of one node, and G is either a prime graph, a complete graph, or a star. Note that complete graphs and stars have linear rank-width at most 1. Thus, we have $\mathrm{lrw}(G) \leqslant p \leqslant 2(p + 2)$. We may assume that $k \geqslant 1$.

Since $\mathrm{pw}(T_D) = k \geqslant 1$, by Theorem 2.3, there exists a path $P := v_1 v_2 \cdots v_n$ in T_D such that for each node v in P and each connected component T of $T_D - v$ not intersecting P, $\mathrm{pw}(T) \leqslant k - 1$. For each $i \in [n]$, let $B_i := \mathsf{bag}_D(v_i)$. By induction hypothesis, for each $i \in [n]$ and each connected component H of $D - V(B_i)$ not intersecting $\bigcup_{1 \leqslant j \leqslant n} V(B_j)$, we have $\mathrm{lrw}(\mathcal{G}[H]) \leqslant 2(p + 2)k$.

Now, let us modify the given canonical split decomposition by two additional unmarked vertices so that we can easily apply Lemma 4.2. For each $i \in [n]$, let L_{B_i} be a linear layout of B_i of width at most p. First, we add a twin of the first vertex of L_{B_1} in B_1 such that the added vertex is unmarked. Similarly, we add a twin of the last vertex of L_{B_n} in B_n such that the added vertex is unmarked. Let x_1 be the vertex added to B_1 and y_n be the vertex added to B_n. It is not difficult to see that B_1 has a linear layout of width at most p whose first vertex is x_1, and B_n has a linear layout of width at most p whose last vertex is y_n.

Assume for a moment that $n \geqslant 2$. For each $i \in [n - 1]$, let y_i and x_{i+1} be the marked vertices of B_i and B_{i+1}, respectively, such that $y_i x_{i+1}$ is the marked edge connecting B_i and B_{i+1}. If y_i is not the end vertex of L_{B_i}, then we reorder L_{B_i} so that y_i is the end vertex. Similarly, if x_{i+1} is not the first vertex of $L_{B_{i+1}}$, then we reorder $L_{B_{i+1}}$ so that x_{i+1} is the first vertex. Until now, the width of each L_{B_i} may increase by at most 2. This is because the rank of a matrix increase by at most 1 when we move one element in the column indices (resp. the row indices) to the row indices (resp. the column indices).

Note that the resulting decomposition is not necessarily canonical, as we may add a twin of a vertex in a prime graph. But this is not a problem when we apply Lemma 4.2. By the above modification, we know that for each $i \in [n]$, there is a linear layout of B_i of width at most $p + 2$ whose first and last vertices are x_i and y_i, respectively.

We define the following sub-decompositions. See Figure 6 for an illustration. If $n = 1$, then let $D_1 := D$. Otherwise,

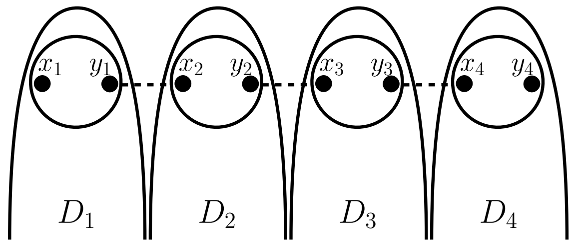


Figure 6: The sequence of sub-decompositions $D_1, \ldots, D_n$ in Proposition 4.1.

1. let D_1 be the connected component of $D - V(B_2)$ containing B_1,
2. let D_n be the connected component of $D - V(B_{n-1})$ containing B_n, and
3. for each $i \in \{2, 3, \ldots, n-1\}$, let D_i be the connected component of $D - (V(B_{i-1}) \cup V(B_{i+1}))$ containing B_i.

We regard the vertices x_i and y_i as unmarked vertices of D_i.

Recall that $\mathrm{pw}(T) \leqslant k-1$ for every node v of P and every connected component T of $T_D - v$ not intersecting P. Therefore, $\mathrm{lrw}(\mathcal{G}[H]) \leqslant 2(p+2)k$, for each connected component H of $D_i - V(B_i)$, by induction hypothesis. Thus, by Lemma 4.2, $\mathcal{G}[D_i]$ has a linear layout L_i of width at most $2(p+2) + 2(p+2)k = 2(p+2)(k+1)$ whose first and last vertices are x_i and y_i, respectively. For each $i \in [n]$, let L_i' be the linear layout obtained from L_i by removing x_i and y_i. Then it is not hard to check that

$$L_1' \oplus L_2' \oplus \cdots \oplus L_n'$$

is a linear layout of G having width at most $2(p+2)(k+1)$. We conclude that $\mathrm{lrw}(G) \leqslant 2(p+2)(\mathrm{pw}(T_D)+1)$. □

For distance-hereditary graphs, the following establishes a lower bound and the tight upper bound of linear rank-width with respect to the path-width of their canonical split decompositions.

Proposition 4.3. *Let D be the canonical split decomposition of a connected distance-hereditary graph G. Then* $\frac{1}{2}\mathrm{pw}(T_D) \leqslant \mathrm{lrw}(G) \leqslant \mathrm{pw}(T_D) + 1$.

The upper bound part is tight. For instance, every complete graph with at least two vertices has linear rank-width 1 and the path-width of its decomposition tree has path-width 0. Also, for each odd integer $k = 2n+1$ with $n \geqslant 1$, every complete binary tree of height k (each path from a leaf to the root has distance k) has linear rank-width $\lceil k/2 \rceil = n+1$, and its decomposition tree has path-width $\lceil (k-1)/2 \rceil = n$. (Note that the linear rank-width and the path-width of a tree are the same [2].) We will need the following lemmas.

Lemma 4.4. *Let G be a graph and let $uv \in E(G)$. Then* $\mathrm{pw}(G) \leqslant \mathrm{pw}(G/uv) + 1$.

Proof. Let $(P, \mathcal{B})$ be an optimal path-decomposition of G/uv, and let z be the contracted vertex in G/uv. It is not hard to check that a new path-decomposition obtained by removing z and adding u and v in each bag containing z is a path-decomposition of G. We conclude that $\mathrm{pw}(G) \leqslant \mathrm{pw}(G/uv) + 1$. □

Lemma 4.5. *Let G be a graph. Let u be a vertex of degree 2 in G such that v_1, v_2 are the neighbors of u in G and $v_1v_2 \notin E(G)$. Then* $\mathrm{pw}(G) \leqslant \mathrm{pw}(G/uv_1/uv_2) + 1$.

Proof. Let w be the contracted vertex in $G/uv_1/uv_2$, and let $(P, \mathcal{B})$ be an optimal path-decomposition of $G/uv_1/uv_2$ of width $t := \mathrm{pw}(G/uv_1/uv_2)$. We may assume that no two adjacent bags in $(P, \mathcal{B})$ are equal.

We obtain a path-decomposition $(P, \mathcal{B}')$ from $(P, \mathcal{B})$ by replacing w with v_1 and v_2 in all bags containing w. Note that $(P, \mathcal{B}')$ is a path-decomposition of $G - u$. Since no two adjacent bags in $(P, \mathcal{B})$ are equal, no two adjacent bags in $(P, \mathcal{B}')$ are equal. We explain how to add u in the current decomposition.

We first assume that there are two adjacent bags B_1 and B_2 in $(P, \mathcal{B}')$ containing both v_1 and v_2, respectively. We obtain a path-decomposition $(P', \mathcal{B}'')$ from $(P, \mathcal{B}')$ by subdividing the edge between B_1 and B_2, and adding a new bag $B' = (B_1 \cap B_2) \cup \{u\}$. Since B_1 and B_2 are not the same, $|B_1 \cap B_2| \leqslant t + 1$ and therefore, $|B'| \leqslant t + 2$. Thus, $(P', \mathcal{B}'')$ is a path-decomposition of G of width at most $t + 1$, and $\mathrm{pw}(G) \leqslant \mathrm{pw}(G/uv_1/uv_2) + 1$.

Now we may assume that there is only one bag B in $(P, \mathcal{B}')$ containing both v_1 and v_2. In this case, since $v_1v_2 \notin E(G)$, we can obtain a path decomposition of G by replacing this bag B with a sequence of two bags B_1 and B_2, where $B_1 := B \backslash \{v_2\} \cup \{u\}$ and $B_2 := B \backslash \{v_1\} \cup \{u\}$. This implies that $\mathrm{pw}(G) \leqslant \mathrm{pw}(G/uv_1/uv_2) + 1$. □

We are now ready to prove Proposition 4.3. We need the split decomposition characterization of graphs of linear rank-width at most 1 proved by Bui-Xuan, Kanté, and Limouzy [12] for the base case, which can be easily obtained by Theorem 3.8. We give a proof of this characterization in Theorem 7.1.

Proof of Proposition 4.3. (1) Let us first prove that $\mathrm{pw}(T_D) \leqslant 2\,\mathrm{lrw}(G)$ by induction on the linear rank-width of G. Let $k := \mathrm{lrw}(G)$. If $k = 0$, then G consists of a vertex, and $\mathrm{pw}(T_D) = 0$. If $k = 1$, then by Theorem 7.1, T_D is a path and we have $\mathrm{pw}(T_D) \leqslant 1 \leqslant 2k$. Thus, we may assume that $k \geqslant 2$. By Theorem 3.8, there exists a path P in T_D such that

- for every node v in P and every connected component H of $D - V(\mathsf{bag}_D(v))$ containing no bag in $\{\mathsf{bag}_D(w) \mid w \in V(P)\}$, $f_D(\mathsf{bag}_D(v), H) \leqslant k - 1$.

Let v be a node of P and C be a connected component of $D - V(\mathsf{bag}_D(v))$ containing no bag $\mathsf{bag}_D(w)$ with $w \in V(P)$. Let y be an unmarked vertex of C represented by $\zeta_c(D, \mathsf{bag}_D(v), C)$, and let $L := \mathcal{LC}_D[V(\mathsf{bag}_D(v)), y]$. By induction hypothesis, the decomposition tree T_L of L has path-width at most $2k - 2$. We claim that $\mathrm{pw}(T_C) \leqslant 2k - 1$, where T_C is the decomposition tree of C. By the definition of canonical limbs, either $T_L = T_C$ or T_L is obtained from T_C using one of the following operations:

1. Removing a node of degree 1.

2. Removing a node of degree 2 with its neighbors v_1, v_2 and adding an edge v_1v_2.

3. Removing a node of degree 2 with its neighbors v_1, v_2 and identifying v_1 and v_2.

The first two cases can be regarded as contracting one edge. So, $\mathrm{pw}(T_C) \leqslant \mathrm{pw}(T_L) + 1 \leqslant (2k - 2) + 1 = 2k - 1$ by Lemma 4.4. The last case corresponds to contracting two edges incident with a vertex of degree 2. By Lemma 4.5, $\mathrm{pw}(T_C) \leqslant \mathrm{pw}(T_L) + 1 \leqslant 2k - 1$.

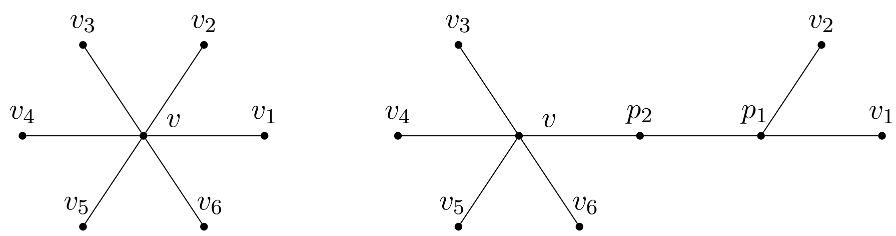


Figure 7: Splitting an edge in Lemma 5.1.

Therefore, for each node v of P and each connected component T' of $T_D - v$ not containing a node of P we have that $\mathrm{pw}(T') \leqslant 2k-1$. By Theorem 2.3, T_D has path-width at most $2k$, as required.

(2) We prove that $\mathrm{lrw}(G) \leqslant \mathrm{pw}(T_D)+1$ by induction on the path-width of T_D. Let $k := \mathrm{pw}(T_D)$ If $k = 0$, then T_D consists of one node. Since G is distance-hereditary, G should be a star or a complete graph, and therefore, we have $\mathrm{lrw}(G) \leqslant 1 = \mathrm{pw}(T_D) + 1$. We may assume that $k \geqslant 1$.

By Theorem 2.3, there exists a path $P = v_0 v_1 \cdots v_n v_{n+1}$ in T_D such that for every node v in P and every connected component F of $T_D - v$ containing no nodes of P, $\mathrm{pw}(F) \leqslant k-1$. Let v be a node of P and let C be a connected component of $D - V(\mathsf{bag}_D(v))$ containing no bags $\mathsf{bag}_D(w)$ with $w \in V(P)$. By induction hypothesis, $\mathcal{G}[C]$ has linear rank-width at most $(k-1)+1 = k$. By the definition of limbs, we conclude that $f_D(\mathsf{bag}_D(v), C) \leqslant k$. Thus, by Theorem 3.8, we conclude that $\mathrm{lrw}(G) \leqslant k+1$. □

We could not confirm that the lower bound in Proposition 4.3 is tight. We leave the following as an open question.

Question 1. *Let D be the canonical split decomposition of a connected distance-hereditary graph G. Is it true that* $\mathrm{pw}(T_D) \leqslant \mathrm{lrw}(G)$*?*

5 Containing a tree as a vertex-minor

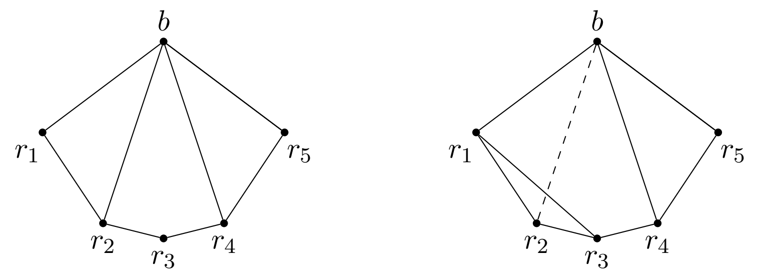
In this section, we prove our first main result.

Theorem 1.2. *Let p be a positive integer and let T be a tree. Let G be a graph such that every prime induced subgraph of G has linear rank-width at most p. If* $\mathrm{lrw}(G) \geqslant 40(p+2)|V(T)|$*, then G contains a vertex-minor isomorphic to T.*

To prove it, we observe that the decomposition tree of the canonical split decomposition of G has large path-width using Theorem 4.1. The main argument of this section is that if G admits a canonical split decomposition whose decomposition tree has sufficiently large path-width, then G contains a vertex-minor isomorphic to T.

We first prove that every tree is a vertex-minor of some subcubic tree having slightly more vertices. For a tree T, we denote by $\phi(T)$ the sum of the degrees of vertices of T whose degrees are at least 4. Every subcubic tree T satisfies that $\phi(T) = 0$.

Lemma 5.1. *Let k be a positive integer and let T be a tree with $\phi(T) = k$. Then T is a pivot-minor of a tree T' with $\phi(T') = k-1$ and $|V(T')| = |V(T)| + 2$.*



$G[\{b, r_1, r_2, \ldots, r_5\}]$ $\qquad$ $G[\{b, r_1, r_2, \ldots, r_5\}] * r_1 * r_2$

Figure 8: Reducing from $G[\{b, r_1, r_2, \ldots, r_s\}]$ in Lemma 5.3.

Proof. Since $\phi(T) \geqslant 1$, T has a vertex of degree at least 4. Let $v \in V(T)$ be a vertex of degree at least 4, and let $v_1, v_2, \ldots, v_m$ be its neighbors. We obtain T' from T by replacing the edge vv_1 with the path $vp_2p_1v_1$, removing vv_2 and adding an edge between p_1 and v_2. It is easy to verify that $(T' \wedge p_1p_2) - \{p_1, p_2\} = T$. We depict this procedure in Figure 7. We observe that p_1 and p_2 are vertices of degree at most 3 in T', and the degree of v in T' is one less than the degree of v in T. Therefore, we have $\phi(T') = k - 1$. $\square$

Lemma 5.2. *Every tree T is a pivot-minor of a subcubic tree T' with $|V(T')| \leqslant 5|V(T)|$.*

Proof. By Lemma 5.1, T is a pivot-minor of a subcubic tree T' with $|V(T')| \leqslant |V(T)| + 2\phi(T)$. Since $\phi(T) \leqslant 2|E(T)| \leqslant 2|V(T)|$, we conclude that $|V(T')| \leqslant |V(T)| + 2\phi(T) \leqslant 5|V(T)|$. $\square$

We recall that by (2) of Theorem 3.3, a connected graph is a tree if and only if every bag of its canonical split decomposition is a star bag whose center is an unmarked vertex. The basic strategy is to extract the canonical split decomposition of a subcubic tree from the canonical split decomposition of G. To do this, we will obtain a star from each prime bag, without changing too much the shape of the obtained canonical split decomposition. Lemma 5.4 describes how to obtain a star from a prime graph as a vertex-minor, without applying local complementations at some special vertices, which will correspond to marked vertices.

We observe that every prime graph on at least 5 vertices is 2-connected. This is because if a connected graph G contains a cut vertex v and $T_1, T_2, \ldots, T_m$ are connected components of $G - v$ and T_1 has smallest number of vertices, then $\left(V(T_1) \cup \{v\}, \bigcup_{j \in \{2,\ldots,m\}} V(T_j)\right)$ is a split of G. We use this observation in Lemma 5.4.

Lemma 5.3. *Let abc be an induced path in a 2-connected graph G. By applying local complementations at vertices in $V(G) \backslash \{a, b\}$, we can obtain G' locally equivalent to G such that $G'[\{a, b, c\}]$ is a triangle.*

Proof. As b is not a cut vertex of G, there is a path from a to c in $G - b$. Let $r_1r_2 \cdots r_s$ be the shortest path from $c = r_1$ to $a = r_s$ in $G - b$. Note that $s \geqslant 3$ as a is not adjacent to c. See Figure 8 for an illustration.

We prove by induction on s that there exists a graph G', obtained from G by applying local complementations only at vertices in $\{r_1, r_2, \ldots, r_{s-1}\}$ and such that $G'[\{a, b, c\}]$ is a triangle. We illustrate this procedure in Figure 8. Assume that $s = 3$. If b is adjacent to r_2, then we remove this edge by applying a local complementation at $c = r_1$. And then we apply a local complementation at r_2 to create an edge between a and c. Then abc becomes a triangle.

We assume that $s \geqslant 4$. Similarly, if b is adjacent to r_2, then we remove this edge by applying a local complementation at $c = r_1$, and then we apply a local complementation at r_2 to create an edge between c and r_3. If b is not adjacent to r_2, then we apply a local complementation at r_2 to create an edge between c and r_3. Let G_1 be the resulting graph. Then $r_1 r_3 r_4 \cdots r_s$ is an induced path in $G_1 - b$. Thus, by induction hypothesis, we can obtain G_2 locally equivalent to G_1 by applying local complementations only at vertices in $\{r_1, r_3, \ldots, r_{s-1}\}$ such that $G_2[\{a, b, c\}]$ is a triangle. □

Lemma 5.4. *Let G be a prime graph on at least 5 vertices, and let $a, b, c \in V(G)$. By applying local complementations at vertices in $V(G)\backslash\{a, b\}$, we can obtain G' locally equivalent to G such that acb is an induced path of G'.*

Proof. We first create a triangle or an induced path of length 2 on $\{a, b, c\}$ by applying local complementations at vertices in $V(G)\backslash\{a, b, c\}$. For this argument, a, b, c are symmetric. Without loss of generality, we assume that the distance between a and b is at most the distance between a and c or between b and c. Let $P = p_1 p_2 \cdots p_m$ be a shortest path from $a = p_1$ to $b = p_m$ in G. By the distance property, $c \notin V(P)$. We define

$$G_1 := \begin{cases} G * p_2 * p_3 * \cdots * p_{m-1} & \text{if } m \geqslant 3, \\ G & \text{otherwise.} \end{cases}$$

It is not difficult to observe that a and b are adjacent in G_1. Now, we take a shortest path $Q = q_1 q_2 \cdots q_n$ from $c = q_1$ to $q_n \in \{a, b\}$ in G_1. We define

$$G_2 := \begin{cases} G_1 * q_2 * q_3 * \cdots * q_{n-1} & \text{if } n \geqslant 3, \\ G_1 & \text{otherwise.} \end{cases}$$

We observe that c has a neighbor on $\{a, b\}$ in G_2. Furthermore, if a and b are not adjacent in G_2, it means that the last local complementation removed this edge, and it implies that c should be adjacent to both a and b in G_2. Therefore, either $G_2[\{a, b, c\}]$ is a triangle or an induced path of length 2.

We do not want to apply local complementations at a, b to create the required induced path. If acb is already an induced path, then we are done. If $G_2[\{a, b, c\}]$ is a triangle, then we apply a local complementation at c. Therefore, we may assume that abc or bac is an induced path. Note that G_2 is still prime by Lemma 2.1, and therefore G_2 is 2-connected.

Assume without loss of generality that abc is an induced path in G_2 as the case bac is an induced path is symmetric. We apply Lemma 5.3. Then by applying local complementations at vertices in $V(G)\backslash\{a, b\}$, we can obtain G_3 locally equivalent to G_2 such that $G_3[\{a, b, c\}]$ is a triangle. By applying a local complementation at c, we obtain the required path. A symmetric argument holds when bac is an induced path in G_2. This terminates the proof of the lemma. □

Starting from a split decomposition whose decomposition tree is a subdivision of a huge binary tree, we will extract a split decomposition of some fixed binary tree. To do this, we need to explain how we sequentially transform each bag into a star whose center is unmarked. Lemma 5.5 deals

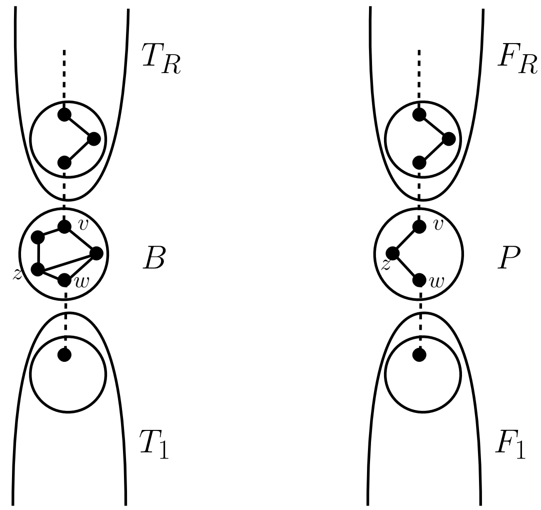


Figure 9: An example application of Lemma 5.5.

with the case when a bag has two neighbor bags, and Lemma 5.6 deals with the case when a bag has three neighbor bags.

A canonical split decomposition D is *rooted* if we distinguish a leaf bag and call it the *root* of D. Let D be a rooted canonical split decomposition with root bag R. A bag B is a *descendant* of a bag B' if B' is on the path of bags from R to B in D, and in this case, we also say that B' is an *ascendant* of B. If B is a descendant of B' and B and B' are adjacent bags, then we call B a *child* of B' and B' the *parent* of B. A bag in D is called a *non-root bag* if it is not the root bag.

Lemma 5.5. *Let D be a rooted canonical split decomposition of a connected graph with root bag R and let B be a non-root bag of D such that*

- *$D - V(B)$ has exactly two connected components T_1 and T_R where T_R contains R,*
- *the parent of B is a star and $\zeta_c(D, B, T_R)$ is a leaf.*

Then by possibly applying local complementations at unmarked vertices of D contained in $V(T_1) \cup V(B)$ and deleting some unmarked vertices in B, we can transform D into a canonical split decomposition D' containing a bag P such that

1. *$D' - V(P)$ consists of exactly two connected components F_R and F_1,*
2. *$F_R = T_R$ or $F_R = T_R * \zeta_c(D, B, T_R)$,*
3. *F_1 is locally equivalent to T_1, and*
4. *P is a star bag whose center is unmarked.*

Proof. Let $v := \zeta_b(D, B, T_R)$ and $w := \zeta_b(D, B, T_1)$. Let y be an unmarked vertex in D represented by w. See Figure 9 for the setting.

First assume that B is a star bag. Since $\zeta_c(D, B, T_R)$ is a leaf, v is not the center of B because D is a canonical split decomposition and every canonical split decomposition has no marked edge of type S_pS_c by Theorem 3.2. If its center is unmarked, then we are done. We may assume that the center of B is w. Since $|V(B)| \geqslant 3$, B contains at least one unmarked vertex, which is adjacent to w. We choose an unmarked leaf vertex z in B. We observe that y is linked to z, that is, $yz \in E(G)$. Then in $D \wedge yz$, z becomes the center of a star, and T_R does not change. Also, T_1 is changed to the decomposition obtained from T_1 by pivoting yz' where $z' = \zeta_c(D, B, T_1)$. Thus, the resulting decomposition satisfies the required property. If B is a complete bag, then we choose an unmarked vertex in B, and apply a local complementation at this vertex. Then the resulting decomposition satisfies the required property.

Now, suppose that B is a prime bag. Choose an unmarked vertex z of B that is adjacent to w. Since a prime graph with at least 5 vertices is 2-connected, there is always an unmarked vertex adjacent to w. Note that y and z are linked.

Let B_1 be the child of B. If B_1 is a star bag whose center is adjacent to B, then by pivoting yz we transform B_1 into a star bag having $\zeta_c(D, B, T_1)$ as a leaf. If B_1 is a complete bag, then we apply a local complementation at y. In the resulting decomposition, either B_1 is a prime bag or $\zeta_c(D, B, T_1)$ is a leaf of a star bag. Let B' be the bag modified from B in the resulting decomposition. Note that B' is still a prime graph by Lemma 2.1.

We apply Lemma 5.4 with $(a, b, c) = (v, w, z)$. By Lemma 5.4, we can modify B' into an induced path vzw by only applying local complementations at unmarked vertices in B' and removing all unmarked vertices in B' except z. Note that the marked edges incident with B' are still marked edges that cannot be recomposed, as both have types S_pS_p or S_pP. Let D' be the modified decomposition and let P be the new bag in D' modified from B'. Then $D' - V(P)$ has two connected components F_R and F_1 where

- $F_R = T_R$ or $F_R = T_R * \zeta_c(D, B, T_R)$,
- F_1 is locally equivalent to T_1, and
- P is a star whose center is unmarked,

as required. □

Lemma 5.6. *Let D be a rooted canonical split decomposition of a connected graph with root bag R and let B be a non-root bag of D such that*

- *$D - V(B)$ has exactly three connected components T_1, T_2, and T_R where T_R contains R,*
- *the distance from $\mathsf{node}_D(B)$ to $\mathsf{node}_D(R)$ is at least 3 in T_D,*
- *the parent P_1 of B and its parent P_2 satisfy that $\mathsf{node}_D(P_1)$ and $\mathsf{node}_D(P_2)$ have degree 2 in T_D,*
- *P_1 and P_2 are stars whose centers are unmarked, and*
- *for each $i \in \{1, 2\}$, the child B_i of B in T_i satisfies that $\mathsf{node}_D(B_i)$ has degree 2 in T_D.*

Then by possibly applying local complementations at unmarked vertices of D contained in $V(T_1) \cup V(T_2) \cup V(B) \cup V(P_1) \cup V(P_2)$ and deleting some unmarked vertices in $V(T_1) \cup V(T_2) \cup V(B) \cup V(P_1) \cup V(P_2)$ and recomposing some marked edges, we can transform D into a canonical split decomposition D' containing a bag P such that

1. *$D' - V(P)$ consists of exactly three connected components F_1, F_2, and F_R,*

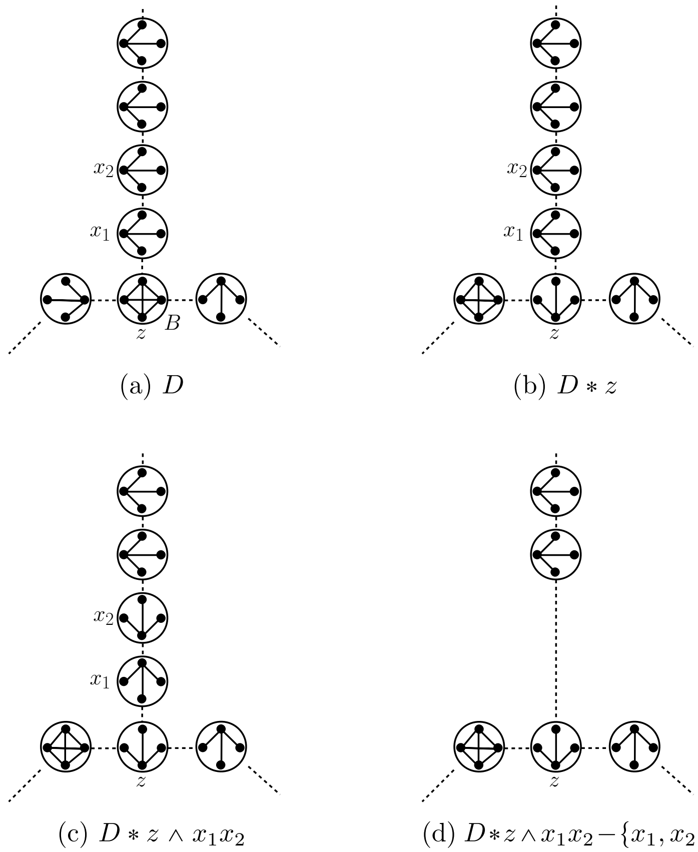


Figure 10: When B is a complete bag and has an unmarked vertex in Lemma 5.6.

2. $F_R = T_R - (V(P_1) \cup V(P_2))$,

3. for each $i \in \{1, 2\}$, F_i *is locally equivalent to* T_i *or* $T_i - V(B_i)$, *and*

4. P *is a star bag whose center is unmarked.*

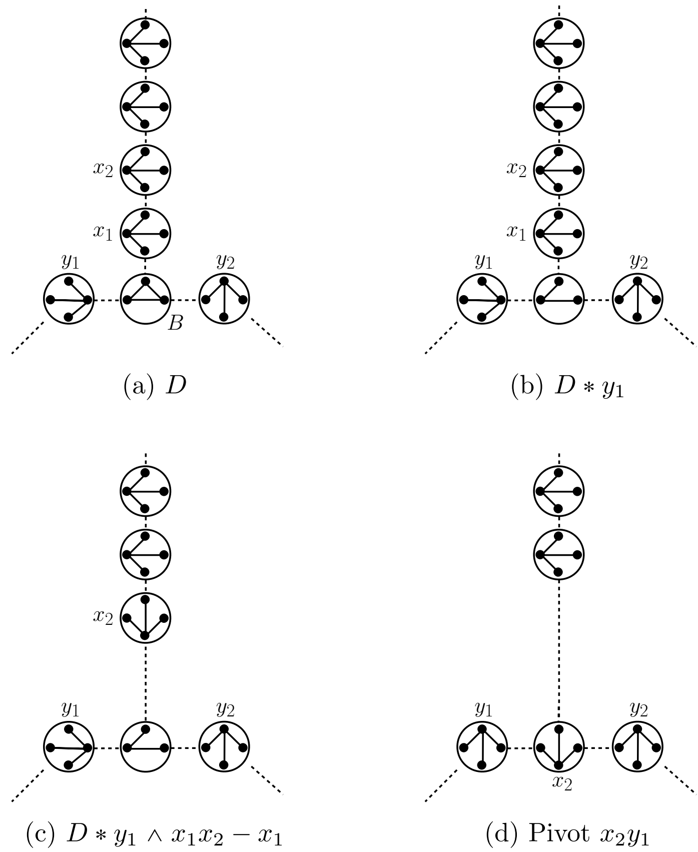
Proof. For each $i \in \{1, 2\}$, let x_i be the center of P_i, and let $v := \zeta_b(D, B, T_R)$, and for each $i \in \{1, 2\}$, let $v_i := \zeta_b(D, B, T_i)$, and y_i be an unmarked vertex represented by v_i.

We first deal with an easier case.

Case 1. *B is either a star or a complete graph, and has an unmarked vertex.*

The case when B is a complete graph is depicted in Figure 10. We first transform B into a star whose center is unmarked. Let z be an unmarked vertex in B.

Assume that B is a star. Since $\zeta_c(D, B, T_R)$ is a leaf of a star, v is not the center of B because D is a canonical split decomposition. We may assume that the center of B is either v_1 or v_2. By symmetry, we may assume that it is v_1. In this case, y_1 and z are linked in D. Thus, B becomes a star whose center is z in $D \wedge y_1z$. If B is a complete bag, then we apply a local complementation at z. Then B becomes a star whose center is z. Note that in any case, T_R does not change by this local complementation as $\zeta_c(D, B, T_R)$ is a leaf of a star, and T_i becomes a split decomposition locally equivalent to T_i.



(a) D (b) $D * y_1$

(c) $D * y_1 \wedge x_1x_2 - x_1$ (d) Pivot x_2y_1

Figure 11: When B is a complete bag and has no unmarked vertices in Lemma 5.6.

Let D_1 be the resulting decomposition. Lastly, we transform D_1 into a split decomposition D_2 as follows:

1. We pivot x_1x_2 and then remove all unmarked vertices contained in P_1 and P_2.

2. We recompose marked edges incident with P_1 and P_2. Equivalently, we remove all vertices in P_1 and P_2 in the decomposition, and add a new marked edge between v and the marked vertex in the parent of P_2 that is adjacent to P_2.

Note that D_2 is canonical, as the new marked edge has the same type as before. Thus, we obtained a required decomposition.

Now, we may assume that either B is a prime bag, or $|V(B)| = 3$.

Case 2. $|V(B)| = 3$.

An example case is depicted in Figure 11.

Since $|V(B)| = 3$, B is either a star or a complete graph. We first modify B into a star whose center is v_1. First assume that B is a star. Since $\zeta_c(D, B, T_R)$ is a leaf of a star, v is not the center of B because D is a canonical split decomposition. Thus, the center of B is either v_1 or v_2. We may assume that the center of B is v_2; otherwise, we already have that B is a star whose center is v_1. Since v_1 is adjacent to v_2, y_1 and y_2 are linked in D. Then B becomes a star whose center is v_1 in $D \wedge y_1y_2$. If B is a complete bag, then we apply a local complementation at y_1. Then B becomes a star whose center is v_1. Note that T_R does not change by this local complementation as $\zeta_c(D, B, T_R)$ is a leaf of a star and the center of the parent of B is unmarked. Let D_1 be the resulting decomposition.

Let w be the marked vertex in P_2 that is adjacent to P_1. We transform D_1 into a split decomposition D_2 as follows:

1. We pivot x_1x_2.
2. We delete the vertices of $V(P_1)$, and add a marked edge between v and w.
3. We recompose the new marked edge vw (it is of type S_pS_c).

Observe that the bag B' in D_2 obtained by merging B and P_2 is a star whose center is v_1, and it contains an unmarked vertex x_2. Moreover, D_2 is canonical. Lastly, we pivot y_1x_2. Then B' becomes a star whose center is x_2. Note that the connected components of $D_2 - V(B')$ are respectively $T_R - (V(P_1) \cup V(P_2))$ and F_1 and F_2 such that F_i is locally equivalent to T_i for $i \in \{1, 2\}$.

Now, it remains to show the lemma when B is a prime bag. We reduce this case to ***Case 1*** or ***Case 2*** by applying Lemma 5.4. Note that in the previous cases, we deduce that F_i is locally equivalent to T_i for each $i \in \{1, 2\}$. But when we transform B into a star bag, we may merge B with one of its child bags.

Case 3. *B is a prime bag.*

Note that applying a local complementation at an unmarked vertex in B does not change the fact that y_1 is represented by v_1. This is because the alternating path from y_1 to v_1 does not change when we apply a local complementation at an unmarked vertex in B.

We apply Lemma 5.4 with $(a, b, c) = (v, v_2, v_1)$ so that B is transformed into an indued path vv_1v_2. Note that applying a local complementation at v_1 can be simulated by applying a local complementation at y_1. Since B is a prime graph on at least 5 vertices, by Lemma 5.4, we can modify B into an induced path vv_1v_2 by only applying local complementations at unmarked vertices in B and y_1. Then we remove all the other vertices of B.

Note that the marked edge connecting B and P_1 is still a valid marked edge as $\zeta_c(D, B, T_R)$ is a leaf of a star. However, for $i \in \{1, 2\}$, the marked edge incident with v_i and $\zeta_c(D, B, T_i)$ may have type S_pS_c. In this case, we recompose this marked edge so that the resulting decomposition is canonical.

Let D_1 be the modified decomposition. Since both $\mathsf{node}_D(P_1)$ and $\mathsf{node}_D(P_2)$ have degree 2 in T_D, the bag B' of D_1 modified from B still has 3 adjacent bags in D_1. As B' is a star bag of D_1, we can reduce the remaining steps to ***Case 1*** or ***Case 2*** depending on the size of B', from which we can construct the required canonical split decomposition. □

We are ready to prove the main result of the section. We note that for a graph H, any subdivision of H contains a vertex-minor isomorphic to H. We will use this fact. For a tree T, let $\eta(T)$ be the tree obtained from T by replacing each edge with a path of length 4.

Proof of Theorem 1.2. Let $t := |V(T)|$ and suppose that $\mathrm{lrw}(G) \geqslant 40(p+2)t$. By Lemma 5.2, there exists a subcubic tree T' such that T is a vertex-minor of T' and $|V(T')| \leqslant 5t$. We consider the tree

$\eta(T')$ which is the tree obtained from T' by replacing each edge with a path of length 4. Observe that $|V(\eta(T'))| \leqslant 20t$.

Let D be the canonical split decomposition of G and let T_D be the decomposition tree of D. Since $\mathrm{lrw}(G) \geqslant 40(p+2)t$, by Proposition 4.1, $\mathrm{pw}(T_D) \geqslant 20t-1$. Since $|V(\eta(T'))| \leqslant 20t$, from Theorem 2.2, T_D contains a minor isomorphic to $\eta(T')$. Since the maximum degree of $\eta(T')$ is at most 3, T_D contains a subgraph T_1 that is isomorphic to a subdivision of $\eta(T')$. Let $D_1 := D[\bigcup_{v \in V(T_1)} V(\mathsf{bag}_D(v))]$. Observe that D_1 is not necessarily a decomposition of an induced subgraph of G, as the unmarked vertex which was a marked vertex before does not correspond to a real vertex of G.

(Preprocess 1) To make it as a decomposition of an induced subgraph of G, we obtain a new decomposition D_2 from D_1 as follows: For every unmarked vertex x of D_1 that was a marked vertex in D, there is a vertex $y \in V(G)$ represented by x in D. We choose such a vertex and replace x with y. We can observe that D_2 is a canonical split decomposition of an induced subgraph of G, and T_{D_2} is isomorphic to T_{D_1}.

(Preprocess 2) We choose a leaf bag R_2 of D_2 and regard it as the root of D_2. We first transform R_2 into a star where the marked vertex in R_2 is a leaf by applying local complementations. Let v be the marked vertex of R_2, and v' be a neighbor of v in R_2, and w be an unmarked vertex of D_2 represented by v. If R_2 is a star whose center is unmarked, then we do nothing. If R_2 is a star whose center is v, then we pivot $v'w$. If R_2 is a complete bag, then we apply local complementation at v'. Then R_2 becomes a star whose center is unmarked.

Assume that R_2 is a prime bag and let C be the child of R_2. If C is a star whose center c is adjacent to v, then we do a pivot at $v'w$ to turn C into a star with c as a leaf. If C is a complete graph, then we apply a local complementation at w. The bag modified from C is either a prime graph or a star whose leaf is adjacent to v. Let R_2' be the resulting bag from R_2.

Now, we choose one more unmarked vertex v'' in R_2' adjacent to v. Such a vertex exists as R_2' is 2-connected. Applying Lemma 5.4 to R_2' with $(a,b,c) = (v,v',v'')$, there exists a sequence $x_1, x_2, \ldots, x_\ell$ of vertices in $V(R_2') \backslash \{v, v'\}$ such that $vv''v'$ is an induced path of $R_2' * x_1 * x_2 * \cdots * x_\ell$. We apply this sequence of local complementations and then remove all vertices in R_2' except v, v', and v''. By the previous procedure, the resulting decomposition is canonical and the bag modified from R_2' is a star whose center is unmarked.

Let D_3 be the resulting decomposition, and R_3 be the root bag that is modified from R_2. Note that T_{D_3} is isomorphic to T_{D_2}. Now we describe the main steps to find T as a vertex-minor.

As T_{D_3} is isomorphic to a subdivision of $\eta(T')$, there is a subdivision mapping g from T' to T_{D_3} such that for each edge e of T', $g(e)$ is a path of length at least 4. Note that $g(V(T'))$ is exactly the union of the set of all leaves and the set of all vertices of degree at least 3 in T_{D_3}.

A bag B in a rooted canonical split decomposition is *good* if every bag on the path from B to the root bag is a star whose center is unmarked. Let $B_1, B_2, \ldots, B_m$ be an ordering of bags in $\{\mathsf{bag}_{D_3}(v) : v \in g(V(T'))\}$ such that

- for each $i \in \{2, 3, \ldots, m\}$, every ascendant bag of B_i in the set $\{\mathsf{bag}_{D_3}(v) : v \in g(V(T'))\}$ is contained in $\{B_1, B_2, \ldots, B_{i-1}\}$.

Such an ordering can be found using BFS. Clearly, $B_1 = R_3$. For each $i \in \{2, 3, \ldots, m\}$, let $F(B_i)$ be the bag B in $\{B_1, B_2, \ldots, B_{i-1}\}$ such that B is an ascendant bag of B_i, and B is closest to B_i. We will construct below a sequence $F_1, F_2, \ldots, F_m$ of rooted canonical split decompositions such that $\mathsf{node}_{D_3}(B_j) \in V(T_{F_i})$ for $1 \leqslant i, j \leqslant m$, and for convenience we keep continuing calling B_j the bag $\mathsf{bag}_{F_i}(\mathsf{node}_{D_3}(B_j))$.

We construct a sequence of canonical split decompositions $F_1, F_2, \ldots, F_m$ such that

- $D_3 = F_1$,
- for each $i \in [m-1]$, $\mathcal{G}[F_{i+1}]$ is a vertex-minor of $\mathcal{G}[F_i]$,
- in each F_i with $i \in [m]$,
  - $B_1, B_2, \ldots, B_i$ are good,
  - when $i \geqslant 2$, for $B \in \{B_2, \ldots, B_i\}$, $\mathrm{dist}_{F_i}(B, F(B)) \geqslant 1$,
  - for $B \in \{B_{i+1}, B_{i+2}, \ldots, B_m\}$, if $F(B) \in \{B_1, B_2, \ldots, B_i\}$, then $\mathrm{dist}_{F_i}(B, F(B)) \geqslant 3$, and otherwise $\mathrm{dist}_{F_i}(B, F(B)) \geqslant 4$.

By (Preprocess 2), $B_1 = R_3$ is good in D_3. Thus, $F_1 = D_3$ is indeed a sequence satisfying the above conditions.

We describe how we can construct F_{i+1} from F_i satisfying the above three conditions. Let us consider the bag B_{i+1}. First assume that B_{i+1} is good. Then $F_{i+1} = F_i$ satisfies the conditions as well (because $\mathrm{dist}_{F_i}(B_{i+1}, F(B_{i+1})) \geqslant 3 \geqslant 1$), so, we can set $F_{i+1} = F_i$. Thus, we may assume that B_{i+1} is not good in F_i.

As $F(B_i) \in \{B_1, B_2, \ldots, B_i\}$, we know that $\mathrm{dist}_{F_i}(B_{i+1}, F(B_{i+1})) \geqslant 3$. Let $F(B_{i+1}) = U_1 - U_2 - \cdots - U_y = B_{i+1}$ be the path of bags in F_i from $F(B_{i+1})$ to B_{i+1}, where $y \geqslant 4$.

We recursively apply Lemma 5.5 to $U_2, U_3, \ldots, U_{y-1}$ so that the bag modified from each of $U_2, U_3, \ldots, U_{y-1}$ is a star whose center is unmarked. Note that when we apply Lemma 5.5 to $U_2, U_3, \ldots, U_{y-1}$, the decomposition tree does not change.

Next we apply Lemma 5.6 to B_{i+1} so that the bag modified from B_{i+1} is a star whose center is unmarked. When we apply Lemma 5.6 to B_{i+1}, some child bags of B_{i+1} may be merged with B_{i+1}. Thus if U is a bag with $F(U) = B_{i+1}$, then the value $\mathrm{dist}_{F_i}(U, B_{i+1})$ may decrease by at most 1.

Let F_{i+1} be the resulting decomposition. We can verify that in F_{i+1},

- $B_1, B_2, \ldots, B_{i+1}$ are good,
- for $B \in \{B_2, \ldots, B_{i+1}\}$, $\mathrm{dist}_{F_{i+1}}(B, F(B)) \geqslant 1$,
- for $B \in \{B_{i+2}, \ldots, B_m\}$, if $F(B) \in \{B_1, B_2, \ldots, B_{i+1}\}$, then $\mathrm{dist}_{F_{i+1}}(B, F(B)) \geqslant 3$, and otherwise $\mathrm{dist}_{F_{i+1}}(B, F(B)) \geqslant 4$.

Thus, we can find such a sequence $F_1, F_2, \ldots, F_m$.

Let $D_4 := F_m$. Note that T_{D_4} is isomorphic to a subdivision of T', and every bag of D_4 is a star whose center is unmarked. Therefore, $\mathcal{G}[D_4]$ is isomorphic to a tree that can be obtained from a subdivision of T' by adding some leaves, and in particular, $\mathcal{G}[D_4]$ contains an induced subgraph isomorphic to a subdivision of T'. Thus, G contains a vertex-minor isomorphic to T', and also contains a vertex-minor isomorphic to T, as required. □

6 Distance-hereditary vertex-minor obstructions for graphs of bounded linear rank-width

In this section, we describe a way to generate all vertex-minor obstructions for graphs of bounded linear rank-width that are distance-hereditary graphs. It generalizes the constructions developed by Jeong, Kwon, and Oum [25].

For a distance-hereditary graph G, a connected distance-hereditary graph G' is a *one-vertex DH-extension* of G if $G = G' - v$ for some vertex $v \in V(G')$. For convenience, if G' is a *one-vertex DH-extension* of G, and D and D' are canonical split decompositions of G and G' respectively, then D' is also called a *one-vertex DH-extension* of D.

Let D_1, D_2 and D_3 be three canonical split decompositions. For each $i \in \{1,2,3\}$, let D_i' be a one-vertex DH extension of D_i with a new unmarked vertex w_i and such that w_i is not contained in a star bag centered at w_i. Furthermore, we choose an unmarked vertex z_i linked to w_i. Let B be a complete graph or a star, on three vertices v_1, v_2, v_3. For each $i \in \{1,2,3\}$, let D_i'' be a split decomposition such that

1. if B is a complete graph, then $D_i'' := D_i' * w_i$,
2. if B is a star with center v_i, then $D_i'' := D_i' \wedge w_i z_i$,
3. if B is a star with v_i a leaf, then $D_i'' := D_i'$.

Let $\mathcal{N}(D_1, D_2, D_3, K)$ be the set of all possible canonical split decompositions obtained from the disjoint union of such D_1'', D_2'', D_3'' and a complete bag B on three vertices v_1, v_2, v_3, by adding the marked edges v_1w_1, v_2w_2, and v_3w_3. For $i \in \{1,2,3\}$, let $\mathcal{N}(D_1, D_2, D_3, (S,i))$ be the set of all possible canonical split decompositions obtained from the disjoint union of such D_1'', D_2'', D_3'' and a star bag B on three vertices v_1, v_2, v_3 whose center is v_i, by adding the marked edges v_1w_1, v_2w_2, and v_3w_3.

For a set $\mathcal{D}$ of canonical split decompositions, let

$$\Delta(\mathcal{D}) := \left(\bigcup_{D_1,D_2,D_3 \in \mathcal{D}} \mathcal{N}(D_1, D_2, D_3, K) \right) \cup \left(\bigcup_{D_1,D_2,D_3 \in \mathcal{D}, i \in \{1,2,3\}} \mathcal{N}(D_1, D_2, D_3, (S,i)) \right),$$
$$\mathcal{D}^+ := \mathcal{D} \cup \{D' : D' \text{ is a one vertex DH-extension of } D \in \mathcal{D}\}.$$

For each non-negative integer k, we recursively construct the set Ψ_k of canonical split decompositions as follows.

1. $\Psi_0 := \{K_2\}$ (K_2 is the canonical split decomposition of itself.)
2. For $k \geqslant 0$, let $\Psi_{k+1} := \Delta(\Psi_k^+)$.

We prove the following.

Theorem 6.1. *Let k be a non-negative integer. Every distance-hereditary graph of linear rank-width at least $k+1$ contains a vertex-minor isomorphic to a graph whose canonical split decomposition is isomorphic to a decomposition in Ψ_k.*

We prove some intermediate lemma.

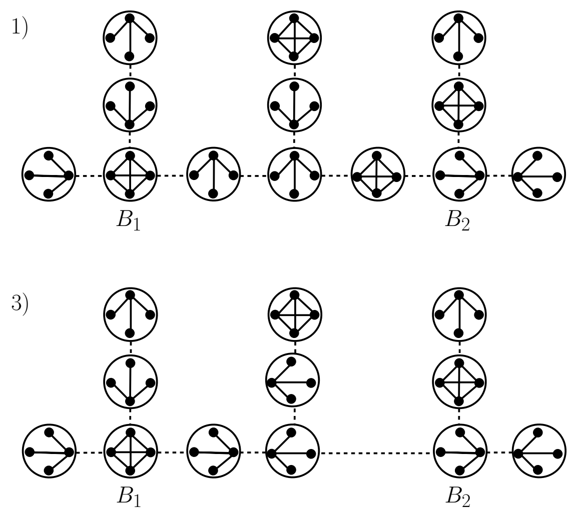


Figure 12: A shorten procedure described in Lemma 6.2.

Lemma 6.2. *Let D be the canonical split decomposition of a connected distance-hereditary graph containing two distinct bags B_1 and B_2, and for each $i \in \{1, 2\}$, let T_i be the connected component of $D - V(B_i)$ such that T_i contains B_{3-i}. If*

- *$\zeta_b(D, B_1, T_1)$ is not the center of a star and*
- *B_2 is a star bag and $\zeta_b(D, B_2, T_2)$ is a leaf of B_2,*

then there exists a canonical split decomposition D' such that

1. *$\mathcal{G}[D]$ has $\mathcal{G}[D']$ as a vertex-minor,*
2. *$D[V(T_2)\backslash V(T_1)] = D'[V(T_2)\backslash V(T_1)]$,*
3. *$D[V(T_1)\backslash V(T_2)] = D'[V(T_1)\backslash V(T_2)]$, and*
4. *either B_1 and B_2 are adjacent in D', or there is a path of bags $B_1 - B - B_2$ in D' such that $|V(B)| = 3$ and B is a star bag whose center is unmarked.*

Proof. If B_1 and B_2 are adjacent bags in D, then we are done. We assume that B_1 and B_2 are not adjacent. Let $B_1 = U_1 - U_2 - \cdots - U_m = B_2$ be the path of bags in D. Also, let $P = p_1 p_2 \ldots p_\ell$ be the shortest path from $\zeta_b(D, B_1, T_1) = p_1$ to $\zeta_b(D, B_2, T_2) = p_\ell$ in D. Note that $\ell \geqslant 4$ as $m \geqslant 3$.

Suppose that there exists a bag U_i containing exactly two consecutive vertices p_j, p_{j+1} of P. In this case, we remove U_i and remove all the connected components of $D - V(U_i)$ that contain neither B_1 nor B_2, and add a marked edge $p_{j-1}p_{j+2}$. This procedure corresponds to removing all unmarked vertices in the removed sub-decomposition. Since this operation does not change the parts $D[V(T_2)\backslash V(T_1)]$ and $D[V(T_1)\backslash V(T_2)]$, applying this operation consecutively, we may assume that for each $i \in \{2, 3, \ldots, m-1\}$, U_i contains three consecutive vertices of P. In other words, U_i is a star whose center is adjacent to neither a vertex of U_{i-1} nor to a vertex of U_{i+1}. See 2) of Figure 12.

Suppose that $m \geqslant 4$. Note that U_2 contains p_2, p_3, p_4 and U_3 contains p_5, p_6, p_7. Take two unmarked vertices x_3 and x_6 of D that are represented by p_3 and p_6, respectively. Observe that

x_3 and x_6 are linked in D. Let $D' := D \wedge x_3x_6$. Notice that $D'[V(U_2)]$ and $D'[V(U_3)]$ are stars whose centers are adjacent to each other. Moreover, $D'[V(T_2)\backslash V(T_1)] = D[V(T_2)\backslash V(T_1)]$ and similarly, $D'[V(T_1)\backslash V(T_2)] = D[V(T_1)\backslash V(T_2)]$. For each $i \in \{2, 3\}$, we delete from D', U_i and all the connected components of $D' - V(U_i)$, except the two connected components containing B_1 and B_2 respectively, and add the marked edge p_1p_8. See 3) and 4) of Figure 12. By the assumption that p_1 is not the center of B_1, the marked edge incident with B_1 is of type S_pS_p or KS_p. Therefore, the resulting decomposition is a canonical split decomposition satisfying the conditions (1), (2), (3), and the number of bags containing P is decreased by two.

Applying this procedure recursively, at the end, we obtain a canonical split decomposition such that either B_1 and B_2 are adjacent, or there is a path of bags $B_1 - B - B_2$ such that B is a star bag whose center is adjacent to neither B_1 nor B_2. In the latter case, we remove all unmarked leaves of B, and remove all connected components of $D - V(B)$ containing neither B_1 nor B_2, and replace the center of B with an unmarked vertex represented by it. Then we obtain the required decomposition. □

The next proposition says how we can replace limbs having linear rank-width $k \geqslant 1$ into a canonical split decomposition in Ψ^+_{k-1} using Lemma 6.2. In this proposition, we sometimes remove unmarked vertices from a given split decomposition, to take a split decomposition of the graph obtained by removing the corresponding vertices. We described this operation in Section 3.2.

Proposition 6.3. *Let D and A be the canonical split decompositions of some connected distance-hereditary graphs. Let B be a star bag of D, v be a leaf of B, T be a connected component of $D - V(B)$ such that $\zeta_b(D, B, T) = v$, and let w be an unmarked vertex of D represented by v. If $\mathcal{LG}_D[B, w]$ has a vertex-minor that is either $\mathcal{G}[A]$ or a one-vertex DH extension of $\mathcal{G}[A]$, then there exists a canonical split decomposition D', such that*

1. *D' is a vertex-minor of D,*
2. *either $D' - V(T) = D - V(T)$ or $D' - V(T) = (D - V(T)) * v$, and*
3. *for some unmarked vertex w' of D' represented by v, $\mathcal{LC}_{D'}[B, w']$ is either A or a one-vertex DH-extension of A.*

Proof. Suppose that $\mathcal{LG}_D[B, w]$ has a vertex-minor that is either $\mathcal{G}[A]$ or a one-vertex DH extension of $\mathcal{G}[A]$. It means that there exist a sequence $x_1, x_2, \ldots, x_m$ of vertices of $\mathcal{LG}_D[B, w]$ and $S \subseteq V(\mathcal{LG}_D[B, w])$ such that $(\mathcal{LG}_D[B, w] * x_1 * x_2 * \ldots * x_m) - S$ is either $\mathcal{G}[A]$ or a one-vertex DH-extension of $\mathcal{G}[A]$. So, there exists $Q \subseteq V(\mathcal{L}_D[B, w])$ such that the graph obtained from $(\mathcal{L}_D[B, w] * x_1 * x_2 * \ldots * x_m)[Q]$ by recomposing all marked edges is either $\mathcal{G}[A]$ or a one-vertex DH-extension of $\mathcal{G}[A]$. As v is a leaf of B, $\mathcal{L}_D[B, w]$ is an induced subgraph of D. Thus, we have

$$(\mathcal{L}_D[B, w] * x_1 * x_2 * \ldots * x_m)[Q] = (D * x_1 * x_2 * \ldots * x_m)[Q].$$

Let $D_1 = D * x_1 * x_2 * \ldots * x_m$. Note that $D[V(B)] = D_1[V(B)]$ as v is a leaf of B, and $\{x_1, x_2, \ldots, x_m\} \subseteq V(T)$.

We choose a bag B' in D_1 such that

1. B' has a vertex of Q, and
2. $\text{dist}_{D_1}(B, B')$ is minimum.

Let us check that all the hypothesis of Lemma 6.2 with $(B_1, B_2) = (B', B)$ are satisfied. Let T_1 be the connected component of $D_1 - V(B')$ containing B and let T_2 be the connected component of $D_1 - V(B)$ containing B'. Let $y := \zeta_b(D_1, B', T_1)$. From the choice of B', we have $y \notin Q$; otherwise, there exists an unmarked vertex represented by y, and all the vertices on the path from y to it should be contained in Q, as Q induces a connected graph. In particular, the bag in T_1 containing a vertex adjacent to a marked vertex in B' should contain a vertex of Q, and this contradicts to the minimality of the distance between B and B'. In addition, y is not the center of a star bag because $D_1[Q]$ is connected and B' has at least two vertices of Q. Therefore, the bags B and B' satisfy the hypothesis of Lemma 6.2 with $(B_1, B_2) = (B', B)$.

By applying Lemma 6.2 on B and B', there exists a canonical split decomposition D_2 such that

1. $\mathcal{G}[D_1]$ has $\mathcal{G}[D_2]$ as a vertex-minor,
2. $D_1[V(T_2)\backslash V(T_1)] = D_2[V(T_2)\backslash V(T_1)]$,
3. $D_1[V(T_1)\backslash V(T_2)] = D_2[V(T_1)\backslash V(T_2)]$,
4. either B and B' are adjacent in D_2, or there exists a path of bags $B - B_s - B'$ in D_2 such that $|V(B_s)| = 3$ and B_s is a star bag whose center is unmarked.

We obtain D_3 from D_2 by removing the vertices of $V(T_2)\backslash V(T_1)$ that are not contained in $Q \cup \{y\}$, and then recomposing all new recomposable marked edges. Since recomposable marked edges only appeared in the part $V(T_2)\backslash V(T_1)$, we have $D_3[V(T_1)\backslash V(T_2)] = D_2[V(T_1)\backslash V(T_2)]$. Furthermore, the bag B_s still exists in D_3 if it exists in D_2. This is because

- the bag B' contains at least two vertices of Q in D_2, and thus B' remains as a bag of same type in D_3, and
- the type of the marked edge connecting B' and B_s does not change when recompositions are applied.

Let B_2 be the bag of D_3 containing y. We divide into cases depending on whether B and B_2 are adjacent or not.

Case 1. *B and B_2 are adjacent in D_3.*

In this case, D_3 itself is the desired decomposition D'. Choose an unmarked vertex z in D_3 represented by v. Then $\mathcal{LC}_{D_3}[B, z]$ is the same as the split decomposition obtained from $(\mathcal{L}_D[B, w] * x_1 * x_2 * \ldots * x_m)[Q]$ by recomposing all recomposable marked edges, which is either $\mathcal{G}[A]$ or a one-vertex DH-extension of $\mathcal{G}[A]$.

Case 2. *There exists a path of bags $B - B_s - B_2$ such that $|V(B_s)| = 3$ and B_s is a star bag whose center is unmarked.*

Let c be the center of B_s, and let c_1 and c_2 be two leaves of B_s that are adjacent to y and v, respectively. Choose an unmarked vertex z of D_3 represented by c_1, and let $H := \mathcal{LC}_{D_3}[B_s, z]$. By construction, H is either A or a one-vertex DH-extension of A.

If $H = A$, then we can regard $\mathcal{LC}_{D_3}[B, c]$ as a one-vertex DH-extension of A with the new vertex c. Therefore, we may assume that H is a one-vertex DH-extension of A. Let a be the newly added vertex a in H.

We would like to remove the extended vertex a from H, and then add c to H so that we obtain a new one-vertex extension of A which contains c. But this is not always possible because the operation of removing a may disconnect the remaining part of H from c. We first deal with this special case.

Assume that B_2 is a star whose center is an unmarked vertex in D_3. In this case this center should be z. We obtain a new decomposition D_4 by applying a local complementation at c, removing c and recomposing a marked edge incident with B_s. Note that D_4 is exactly the decomposition obtained from the disjoint union of the two connected components of $D_3 - V(B_s)$ by adding a marked edge yv, and thus it is canonical. Also, z is represented by v in D_4, and we have $\mathcal{LC}_{D_4}[B, z] = H$. Thus, D_4 is the required decomposition.

Now we assume that c is linked to at least two vertices of H in D_3. Since H is a one vertex DH-extension of A and A was chosen as a canonical split decomposition of a connected graph, $\mathcal{G}[H] - a$ is connected. So, if we define D_4 as the canonical split decomposition obtained from $D_3 - a$, then D_4 is connected and $\mathcal{LC}_{D_4}[B, c]$ can be regarded as a one vertex DH-extension of A. Therefore, D_4 is the required decomposition. □

Proof of Theorem 6.1. We prove it by induction on k. If $k = 0$, then $\text{lrw}(G) \geqslant 1$ and G has an edge. Thus, we may assume that $k \geqslant 1$.

Let D be the canonical split decomposition of G. Since G has linear rank-width at least $k+1$, by Theorem 3.8, there exists a bag B in D with three connected components T_1, T_2, T_3 of $D - V(B)$ such that $f_D(B, T_i) \geqslant k$ for each $i \in \{1, 2, 3\}$.

We remove all connected components of $D - V(B)$ other than T_1, T_2, T_3, and for each marked vertex w in B that was adjacent to some removed component, we choose a vertex w' in D represented by B and replace w with w'. Note that the resulting decomposition is a canonical split decomposition of an induced subgraph of G.

Now, if B is a star whose center is unmarked, then we apply a local complementation at this vertex, and otherwise, we change nothing. Then we obtain a new decomposition by removing all unmarked vertices in B. Let us denote by D' this canonical split decomposition and denote by B' the bag modified from B, and denote by T_1', T_2', T_3' the decompositions modified from T_1, T_2, T_3, respectively.

For each $i \in \{1, 2, 3\}$, let $v_i := \zeta_b(D', B', T_i')$ and $w_i := \zeta_c(D', B', T_i')$, and z_i be an unmarked vertex of D' represented by v_i in D'.

We define a new decomposition D_1 as follows. If B' is a star bag centered at v_3, then let $D_1 := D'$. If B is a complete bag, then let $D_1 := D' * z_3$. If B is a star bag centered at $v_i \in \{v_1, v_2\}$, then let $D_1 := D * z_i * z_3$. One easily checks that $D_1[\{v_1, v_2, v_3\}]$ is a star centered at v_3. Let $B^1 := D_1[\{v_1, v_2, v_3\}]$ and, for $j \in \{1, 2, 3\}$, let $T_j^1 := D_1[V(T_j')]$. Note that z_i is still represented by v_i.

Since v_1 and v_2 are leaves of B^1, for each $i \in \{1, 2\}$, $\mathcal{L}_{D_1}[B^1, z_i] = T_i^1 - w_i$ and by the induction hypothesis, there exists a canonical split decomposition F_i in Ψ_{k-1} such that $\mathcal{LG}_{D_1}[B^1, z_i]$ has a vertex-minor isomorphic to $\mathcal{G}[F_i]$. By applying Proposition 6.3 to T_1^1 and T_2^1, we can obtain a canonical split decomposition D_2 satisfying that

1. $D_2[V(B^1)] = D_1[V(B^1)]$,
2. $D_2[V(T_3)]$ is either T_3' or $T_3' * w_3$ (because T_3' may be affected by applying local complementation at v_1 or v_2 when applying Proposition 6.3), and
3. for each $i \in \{1, 2\}$, $\mathcal{LC}_{D_2}[D_2[V(B^1)], z_i^2]$ is isomorphic to a canonical split decomposition in Ψ_{k-1}^+ for some unmarked vertex z_i^2 of D_2 represented by v_i.

Let $B^2 := D_2[V(B^1)]$. For each $i \in \{1,2\}$, let T_i^2 be the connected component of $D_2 - V(B^2)$ containing z_i^2, and $w_i^2 := \zeta_c(D_2, B^2, T_i^2)$. Let $w_3^2 := w_3$, $z_3^2 := z_3$, and $T_3^2 := D_2[V(T_3)]$.

Now, we want to transform B^2 into a star whose center is v_2 by applying local complementations at z_3^2 and z_2^2. We can verify that

1. $(D_2 * z_3^2 * z_2^2)[V(B^2)]$ is a star whose center is v_2,
2. $(D_2 * z_3^2 * z_2^2)[V(T_1^2)] = T_1^2 * w_1^2 * w_1^2 = T_1^2$,
3. $(D_2 * z_3^2 * z_2^2)[V(T_2^2)] = T_2^2 * w_2^2 * z_2^2$,
4. $(D_2 * z_3^2 * z_2^2)[V(T_3^2)] = T_3^2 * z_3^2 * w_3^2$.

We apply Proposition 6.3 to $D_2 * z_3^2 * z_2^2$ and obtain a canonical split decomposition D_3 so that

1. $D_3[V(B^2)] = (D_2 * z_3^2 * z_2^2)[V(B^2)]$ and $D_3[V(T_1^2)] = (D_2 * z_3^2 * z_2^2)[V(T_1^2)]$,
2. $D_3[V(T_2^2)]$ is either $(D_2 * z_3^2 * z_2^2)[V(T_2^2)]$ or $(D_2 * z_3^2 * z_2^2)[V(T_2^2)] * w_2^2$, and
3. $\mathcal{LC}_{D_3}[D_3[V(B^2)], z_3^3]$ is isomorphic to a canonical split decomposition in Ψ_{k-1}^+ for some unmarked vertex z_3^3 of D_3 represented by v_3.

Let $B^3 := D_3[V(B^2)]$. Let T_3^3 be the connected component of $D_3 - V(B^3)$ containing z_3^3, and $w_3^3 := \zeta_c(D_3, B^3, T_3^3)$. Note that $T_3^3 - w_3^3 \in \Psi_{k-1}^+$ and for $i \in \{1,2\}$, z_i^2 is still represented by v_i in D_3. We define $T_1^3 := D_3[V(T_1^2)]$, $T_2^3 := D_3[V(T_2^2)]$ and define $w_1^3, w_2^3, z_1^3, z_2^3$ as the same as $w_1^2, w_2^2, z_1^2, z_2^2$, respectively.

Now we claim that $D_3 \in \Psi_k$ or $D_3 * z_2^3 \in \Psi_k$. We observe two cases depending on whether T_2^3 is equal to $(D_2 * z_3^2 * z_2^2)[V(T_2^2)]$ or to $(D_2 * z_3^2 * z_2^2)[V(T_2^2)] * w_2^2$.

Case 1. $T_2^3 = (D_2 * z_3^2 * z_2^2)[V(T_2^2)]$.

We observe that B^3 is a star whose center is v_2, and the three connected components of $D_3 - V(B^3)$ are T_1^2, $T_2^2 * w_2^2 * z_2^2$, and T_3^3. In this case, $D_3 * z_2^2 \in \Psi_k$ because

1. $(D_3 * z_2^2)[V(B^3)]$ is a complete bag, and
2. the three components of $D_3 * z_2^2 - V(B^3)$ are $T_1^2 * w_1^2$, $T_2^2 * w_2^2$, and $T_3^3 * w_3^3$,

and the limbs of $D_3 * z_2^2$ with respect to B^3 are $T_1^2 - w_1^2$, $T_2^2 - w_2^2$, and $T_3^3 - w_3^3$, which are contained in Ψ_{k-1}^+.

Case 2. $T_2^3 = (D_2 * z_3^2 * z_2^2)[V(T_2^2)] * w_2^2$.

We observe that B^3 is a star centered at v_2, and the three components of $D_3 - V(B^3)$ are T_1^2, $T_2^2 * w_2^2 * z_2^2 * w_2^2 = T_2^2 \wedge w_2^2 z_2^2$, and T_3^3. We can see that $D_3 \in \Psi_k$ because the limbs with respect to B^3 are $T_1^2 - w_1^2$, $T_2^2 - w_2^2$, and $T_3^3 - w_3^3$, which are contained in Ψ_{k-1}^+.

We conclude that G has a vertex-minor isomorphic to $\mathcal{G}[D_3]$ where $D_3 \in \Psi_k$, as required. □

In order to prove that Ψ_k is a minimal set of canonical split decompositions of distance-hereditary vertex-minor obstructions for linear rank-width at most k, we need to prove that for every $D \in \Psi_k$, $\mathcal{G}[D]$ has linear rank-width $k+1$ and all its proper vertex-minors have linear rank-width at most k. However, while $\mathrm{lrw}(\mathcal{G}[D]) = k+1$ for all $D \in \Psi_k$, they are not minimal with respect to having linear rank-width $k+1$. For instance for many canonical split decompositions D

in Ψ_1, $\mathcal{G}[D]$ is not a vertex-minor obstruction for linear rank-width 1 as it contains either α_1 or γ_1 as a proper vertex-minor (see Section 7). We guess that the following set Φ_k would form a minimal set of distance-hereditary vertex-minor obstructions, but we leave it as an open problem.

1. $\Phi_0 := \{K_2\}$.
2. For $k \geqslant 0$, let $\Phi_{k+1} := \Delta(\Phi_k)$.

Our intuition is supported by the following.

Proposition 6.4. *Let k be a non-negative integer and let $D \in \Phi_k$. Then* $\operatorname{lrw}(\mathcal{G}[D]) = k+1$ *and every proper vertex-minor of $\mathcal{G}[D]$ has linear rank-width at most k.*

We need the following two lemmas.

Lemma 6.5. *Let $D \in \Phi_k$ and v be an unmarked vertex in D. Then $D * v \in \Phi_k$.*

Proof. We proceed by induction on k. We may assume that $k \geqslant 1$. By the construction, there exists a bag B of D such that the three limbs D_1, D_2, D_3 in D corresponding to the bag B are contained in Φ_{k-1}.

Let $B' := B$ or $B' := B * v'$ be a bag of $D * v$ depending on whether v has a representative v' in B. Let D'_1, D'_2 and D'_3 be the three limbs of $D * v$ corresponding to the bag B' such that D'_i and D_i came from the same component of $D - V(B)$. One checks by Proposition 3.7 that D'_i is locally equivalent to D_i. So by the induction hypothesis, $D'_i \in \Phi_{k-1}$. Furthermore, $D * v$ is the canonical split decomposition obtained from D'_i following the construction of Φ_k. Therefore, $D * v \in \Phi_k$. □

Lemma 6.6 (Bouchet [9])**.** *Let G be a graph, v be a vertex of G and w be an arbitrary neighbor of v. Then every proper vertex-minor obtained from G by deleting v is locally equivalent to either $G - v$, $G * v - v$, or $G \wedge vw - v$.*

Proof of Proposition 6.4. By construction, it is not hard to prove by induction with the help of Theorem 3.8 that $\operatorname{lrw}(\mathcal{G}[D]) = k + 1$ for every split decomposition $D \in \Phi_k$. For the second statement, by Lemmas 6.5 and 6.6, it is sufficient to show that if $D \in \Phi_k$ and v is an unmarked vertex of D, then $\mathcal{G}[D] - v$ has linear rank-width at most k. We use induction on k to prove it. We may assume that $k \geqslant 1$. Let B be the bag of D such that $D - V(B)$ has exactly three limbs that are contained in Φ_{k-1}. Clearly there is no other bag having the same property. Since B has no unmarked vertices, v is contained in one of the limbs D', and by induction hypothesis, $\mathcal{G}[D'] - v$ has linear rank-width at most $k - 1$. Therefore, by Theorem 3.8, $\mathcal{G}[D] - v$ has linear rank-width at most k. □

We finish by pointing out that it is proved in [25] that the number of distance-hereditary vertex-minor obstructions for linear rank-width k is at least $2^{\Omega(3^k)}$. One can easily check by induction that the number of graphs in Ψ_k is bounded by $2^{O(3^k)}$. Therefore, we can conclude that the number of distance-hereditary vertex-minor obstructions for linear rank-width k is equal to $2^{\Theta(3^k)}$.

7 Simpler proofs for the characterizations of graphs of linear rank-width at most 1

In this section, we obtain simpler proofs for known characterizations of the graphs of linear rank-width at most 1 using Theorem 3.8. Theorem 7.1 was originally proved by Bui-Xuan, Kanté, and Limouzy [12].

Theorem 7.1 (Bui-Xuan, Kanté, and Limouzy [12])**.** *Let G be a connected graph and let D be the canonical split decomposition of G. The following two are equivalent.*

(1) G has linear rank-width at most 1.

(2) G is distance-hereditary and T_D is a path.

Proof. We first prove that (2) implies (1). Let $T_D := u_1 u_2 \cdots u_m$. For each $1 \leqslant i \leqslant m$, we take any ordering L_i of unmarked vertices in $\mathsf{bag}_D(u_i)$. Since G is distance-hereditary, by Theorem 3.3, each bag of D is a complete graph or a star. Thus, we can easily check that $L_1 \oplus L_2 \oplus \ldots \oplus L_m$ is a linear layout of G having width at most 1.

We prove that (1) implies (2). Suppose that G has linear rank-width at most 1. From the known fact that a connected graph has rank-width at most 1 if and only if it is distance-hereditary [32], G is distance-hereditary. Suppose that T_D is not a path. Then there exists a bag B of D such that B has at least three neighbor bags in D. Thus, $D - V(B)$ has at least three components T where $f_D(B, T) \geqslant 1$. By Theorem 3.8, G has linear rank-width at least 2, which is a contradiction. □

From Theorem 7.1, we have a linear-time algorithm to recognize the graphs of linear rank-width at most 1.

Theorem 7.2. *For a given graph G, we can test whether G has linear rank-width at most 1 or not in time $\mathcal{O}(|V(G)| + |E(G)|)$.*

Proof. We first compute the canonical split decomposition D of each connected component of G using the algorithm from Theorem 3.1. It takes $\mathcal{O}(|V(G)| + |E(G)|)$ time. Furthermore, this algorithm outputs the type of each bag together. Note that each bag of a canonical split decomposition of a connected distance-hereditary graph is either a complete graph or a star by Theorem 3.3. Thus, if there is a prime bag, then we answer that G has linear rank-width more than 1.

Additionally, we check whether T_D is a path or not. By Theorem 7.1, if T_D is a path and each bag is not prime, then we conclude that G has linear rank-width at most 1, and otherwise, G has linear rank-width at least 2. □

The list of induced subgraph obstructions for graphs of linear rank-width at most 1 was characterized by Adler, Farley, and Proskurowski [1]. The obstructions consist of the known obstructions for distance-hereditary graphs [4], and the set Ω_T of the induced subgraph obstructions for graphs of linear rank-width at most 1 that are distance-hereditary. See Figure 13 for the list of obstructions $\alpha_i, \beta_j, \gamma_k$ in Ω_T where $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 6$, $1 \leqslant k \leqslant 4$. This set Ω_T can be obtained from Theorem 7.1 in a much easier way than the previous result.

Recall that a graph H is called a *pivot-minor* of a graph G if H can be obtained from G by applying a sequence of pivotings on edges and deletions of vertices.

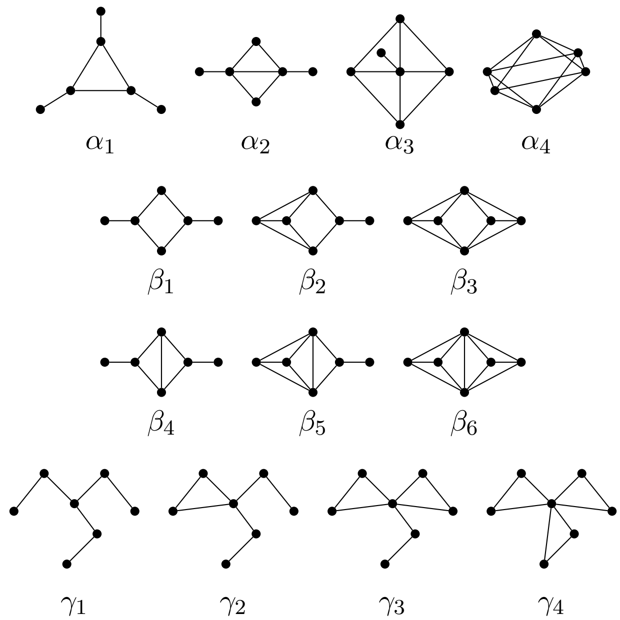


Figure 13: The induced subgraph obstructions for graphs of linear rank-width at most 1 that are distance-hereditary.

type of B	type of v_1w_1	type of v_2w_2	type of v_3w_3	induced subgraph
A complete bag	KS_p	KS_p	KS_p	α_1
	KS_c	KS_p	KS_p	α_2
	KS_c	KS_c	KS_p	α_3
	KS_c	KS_c	KS_c	α_4
A star bag	S_cS_c	S_pS_p	S_pS_p	β_1
with center at v_1	S_cS_c	S_pS_p	S_pK	β_2
	S_cS_c	S_pK	S_pK	β_3
	S_cK	S_pS_p	S_pS_p	β_4
	S_cK	S_pS_p	S_pK	β_5
	S_cK	S_pK	S_pK	β_6
A star bag	S_pS_p	S_pS_p	S_pS_p	γ_1
with center at	S_pK	S_pS_p	S_pS_p	γ_2
a vertex	S_pK	S_pK	S_pS_p	γ_3
other than v_i	S_pK	S_pK	S_pK	γ_4

Table 1: Summary of all cases in Theorem 7.3

Theorem 7.3 (Adler, Farley, and Proskurowski [1]). *Let G be a connected graph. The following are equivalent.*

1. *G has linear rank-width at most* 1.

2. *G is distance-hereditary and G has no induced subgraph isomorphic to a graph in*

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \gamma_1, \gamma_2, \gamma_3, \gamma_4\}.$$

3. *G has no pivot-minor isomorphic to a graph in $\{C_5, C_6, \alpha_1, \alpha_2, \beta_1, \beta_3, \beta_4, \beta_6\}$.*

4. *G has no vertex-minor isomorphic to a graph in $\{C_5, \alpha_1, \beta_1\}$.*

Proof. By Lemma 2.1, ((1) → (4)) is clear as C_5, α_1 and β_1 have linear rank-width 2. We can easily confirm the directions ((4) → (3) → (2)); see [1]. We add a proof for ((2) → (1)).

Suppose that G has linear rank-width at least 2 and it is distance-hereditary. Let D be the canonical split decomposition of G. By Theorem 7.1, T_D is not a path. Thus there exists a bag B of D such that $D - V(B)$ has at least three connected components T_1, T_2, T_3. For each $i \in \{1, 2, 3\}$, let $v_i := \zeta_b(D, B, T_i)$ and $w_i := \zeta_c(D, B, T_i)$. We have three cases; B is a complete bag, or B is a star bag with the center at one of v_1, v_2, v_3, or B is a star bag with the center at a vertex of $V(B)\backslash\{v_1, v_2, v_3\}$.

If B is a complete bag, then G has an induced subgraph isomorphic to one of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ depending on the types of the marked edges v_iw_i. If B is a star bag with the center at one of v_1, v_2, v_3, then G has an induced subgraph isomorphic to one of $\beta_1, \beta_2, \ldots, \beta_6$. Finally, if B is a star bag with the center at a vertex of $V(B)\backslash\{v_1, v_2, v_3\}$, then G has an induced subgraph isomorphic to one of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$. We summarize all the cases in Table 1. □

8 Conclusion

In this paper we used the characterization of the linear rank-width of distance-hereditary graphs given in [3] to prove that Question 1.1 is true if and only if it is true in prime graphs. Also, for each non-negative integer k, we compute a set of distance-hereditary graphs such that every distance-hereditary graph of linear rank-width at least $k + 1$ contains a vertex-minor isomorphic to one of the graphs in the set.

Computing an upper bound on the size of vertex-minor obstructions for graphs of bounded linear rank-width is a challenging open question. Until now only a bound on obstructions for graphs of bounded rank-width is known [32]. Secondly, resolving Question 1.1 in all graphs seems to require new techniques. We currently do not have any idea on how to reduce any graph of small rank-width but large linear rank-width into a distance-hereditary graph whose decomposition tree has large path-width. One might start with graphs of rank-width 2.

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