On the Linear Rank-Width of Graphs
Case of Distance-Hereditary Graphs

Isolde Adler   Mamadou M. Kanté   O-joung Kwon

LIMOS - Université Blaise Pascal
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Introduction

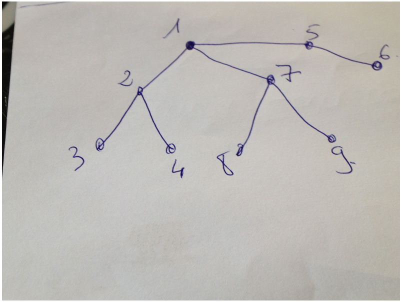
- Linear rank-width not enough studied
  - Thread graphs = linear rank-width 1 (Ganian, 2010)
  - Obstructions for linear rank-width 1 (Adler et al., 2011)
  - Several other characterizations of linear rank-width 1 (Oum-Kwon, 2011 ; Kanté et al., 2012)
- A lot to do : structural as well as in algorithmic graph theory
  - Characterization of bounded linear rank-width via vertex-minor/pivot-minor
  - (Number of) Obstructions for linear rank-width k
  - Tractable problems ? How compared to path-width/rank-width ?

Results

1. linear rank-width and path-width coincide in forests
   $\Longrightarrow$ linear time algorithm for the computation
   $\Longrightarrow$ quasi-linear time for witnessing an optimal layout
2. characterization of linear rank-width of distance-hereditary graphs via split decomposition
   $\Longrightarrow$ Polynomial time algorithm for witnessing an optimal layout

Linear Rank-Width

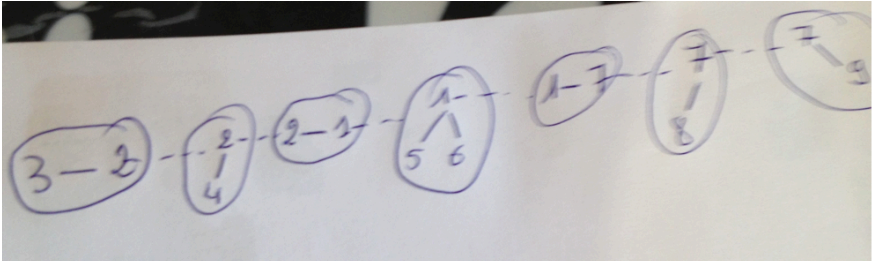
- take any linear ordering $x_1, \ldots, x_n$ of the vertices
- width $= \max\limits_{1 \leq i \leq n-1} \{\mathrm{rk}(A_G[\{x_1, \ldots, x_i\}, -]\}$
- linear rank-width of G, $lrwd(G)$ = minimum over all linear orderings



3, 2, 4, · 1, 8, 7, 9, 5, 6

$$
\mathrm{rk}\begin{pmatrix}
 & 1 & 8 & 7 & 9 & 5 & 6 \\
\hline
3 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$lrwd(G) \leq pwd(G)$



3, 2, 4, 1, 5, 6, 7, 8, 9

$pwd(T) \leq lrwd(T)$

- Take a linear layout $v_1, v_2, \ldots, v_n$ of width $k := lrwd(T)$.
- Clear vertices in this ordering with at most $k + 1$ cops.

Initialisation : Put i cops in vertices $v_1, \ldots, v_i$ such that $X_i := \{v_1, \ldots, v_i\}$ is a basis for $M_i := A_T[X_i, Y_i := V_T \setminus X_i]$.

Inductive step : if X_ℓ is cleared, clear $X_{\ell+1}$ while maintaining the following invariants

- ⋆ each vertex b of a basis B_i of M_i is either occupied or its neighbours in $Y_{\ell+1}$ are occupied,
- ⋆ cops occupy exactly $|B_{\ell+1}|$ vertices

Clearing Step(1)

- Either $v_{\ell+1}$ is linearly independent of B_ℓ in $M_{\ell+1}$ or not.
- Either $v_{\ell+1}$ is occupied by a cop or not after step ℓ.

To verify invariants, we need :

⋆ Clear $v_{\ell+1}$ and put a cop either on it or on its neighbours in $Y_{\ell+1}$ if it is linearly independent of B_ℓ in $M_{\ell+1}$.

⋆ Free cops in B_ℓ that are not in the "chosen" basis of $M_{\ell+1}$.

To do so, construct B-basic trees

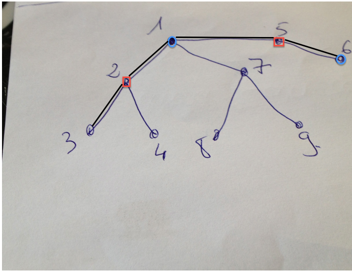
B-basic tree

For (X, Y) a cut, B a basis of $A_T[X, Y]$ and $x \in X$

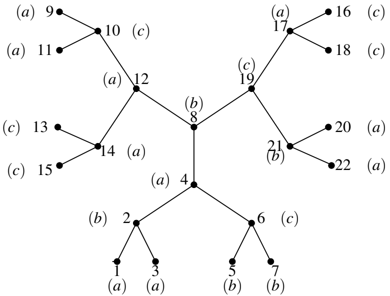
- Take $B' \subseteq B$ spanning x.
- Let $T' := T[B' \cup x \cup (N(B' \cup x) \cap Y)]$.

Properties

- T' is connected and leaves are from X.
- Vertices in $N(B' \cup x) \cap Y$ have degree 2.
- $|N(B' \cup x) \cap Y| = |B'|$.

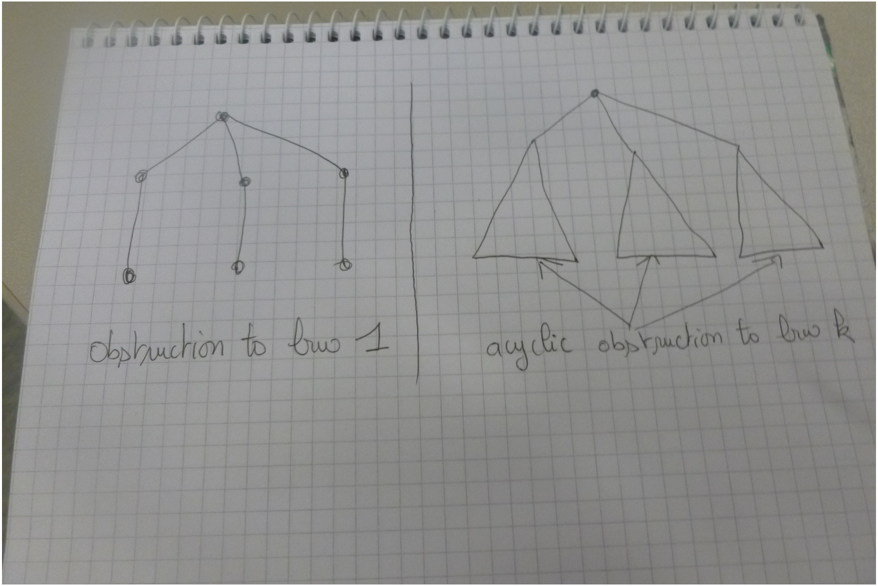


A strategy



(a) 9
10 (c)
(a) 11
(a) 12
(c) 13
14 (a)
(c) 15
(b) 8
(c) 19
(a) 17
16 (c)
18 (c)
20 (a)
21 (b)
22 (a)
(a) 4
(b) 2
6 (c)
1 (a)
3 (a)
5 (b)
7 (b)

A consequence : acyclic obstructions

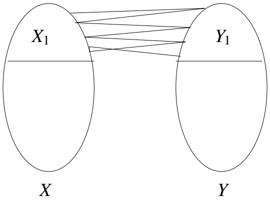


Sommaire

Distance-Hereditary Graphs

- Distances are preserved when taking connected induced subgraphs
- Trees, cographs are distance-hereditary graphs
- Distance-hereditary graphs = graphs of rank-width 1
- Several other characterizations :
  - (House,Hole,Domino,Gem)-free graphs
  - perfect elimination ordering : removal of pendant vertices and/or false/true twins
  - completely decomposable by split decomposition

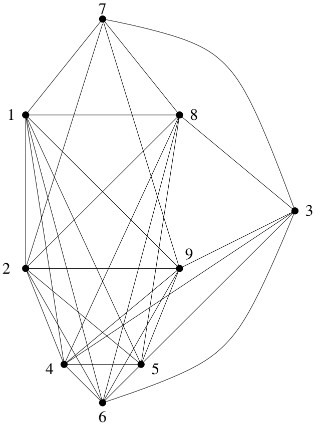
Split Decomposition



$|X|, |Y| \geq 2$

- prime graph = graph without split
- A strong split is a split that does not overlap any other split
- split decomposition = iteratively splitting wrt strong splits
  - each block is either prime or a clique or a star
  - no splitting of cliques and stars
- distance-hereditary = each block is a clique or a star

Split Decomposition



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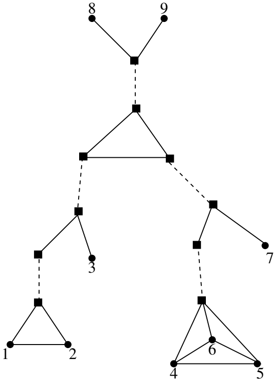


S_G

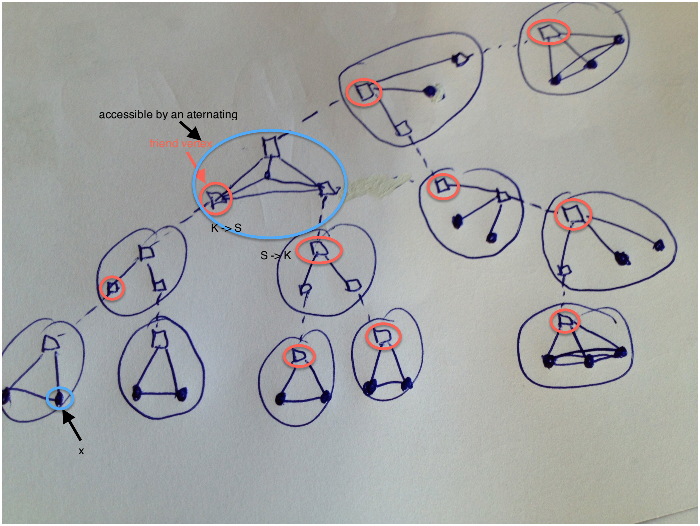
Split Decompositions and Local Complementations

- Local complementation at x in G is the graph $G * x$ where $zt \in E(G * x)$ iff

$$\begin{cases} z \text{ or } t \notin N(x) \text{ and } zt \in E(G) \\ z, t \in N(x) \text{ and } zt \notin E(G). \end{cases}$$

- Local complementations do not change (linear) rank-width
- Local complementations do not change the shape of the split decomposition
  - prime blocks remain prime
  - some cliques become stars and some stars cliques
- Pivot at $xy \in E(G) = G \wedge xy := G * x * y * x = G * y * x * y$

Split Decompositions and Local Complementations



Limbs in Split Decompositions

- Let D be the split decomposition of a distance-hereditary graph G
- Given a bag B, a connected component T of $D \setminus V(B)$ and a vertex $y \in V(G) \cap V(T)$,
  - let v be the vertex of T neighbor of a vertex in B
  - let w in B be the neighbor of v
- The limb $\mathcal{L}[D, B, y]$ is the decomposition
  - $T * v \setminus v$ if B is a clique
  - $T \setminus v$ if B is a star and w is a leaf
  - $T \wedge vy \setminus v$ if B is a star and w is the centre
- The graph associated with $\mathcal{L}[D, B, y]$ is denoted by $\widehat{\mathcal{L}}[D, B, y]$

Properties of Limbs

- Graphs associated with limbs are connected
- $\hat{\mathcal{L}}[D, B, y]$ and $\hat{\mathcal{L}}[D, B, x]$ are locally equivalent
- $\hat{\mathcal{L}}[D, B, y]$ and $\hat{\mathcal{L}}[D * x, T * x, y']$ locally equivalent for every $x \in V(G)$

$\Longrightarrow$

- Choice of D not important (we can replace D by $D * x$)
- Choice of y not important

$$f(D, B, T) = \text{linear rank-width of some } \hat{\mathcal{L}}[D, B, y]$$

Characterizing Linear Rank-Width k

Theorem

Let k be a positive integer and let D be the split decomposition of a distance-hereditary graph G. Then $\mathrm{lrwd}(G) \leq k$ if and only if for each bag B of D, D has at most two components T of $D \setminus V(B)$ such that $f(D, B, T) = k$, and for all the other components T' of $D \setminus V(B)$, $f(D, B, T') \leq k - 1$.

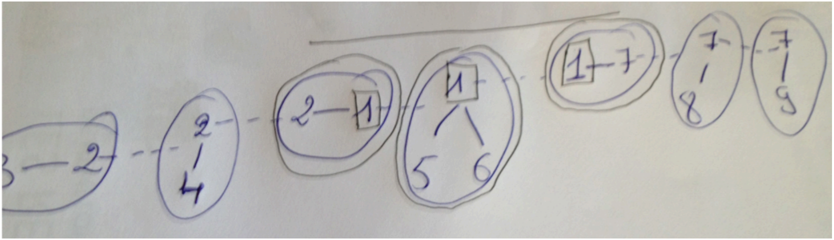
- Similar to the characterization of path-width of trees
- Gives a polynomial time algorithm for constructing an optimal layout in the same spirit as the one for path-width of trees

Conclusion

- Can we extend the idea for distance-hereditary graphs to graphs of bounded rank-width ?
  - Probably YES when prime blocks are « simple »
  - What should play the role of split decomposition for bounded rank-width in general ?
- We can also characterize the linear clique-width of forests
  - If a path of length at least 3 exists, then $\text{lcwd}(T) = \text{pwd}(T) + 2$
  - Otherwise, $\text{lcwd}(T) \in \{\text{pwd}(T) + 1, \text{pwd}(T) + 2\}$ depending on whether T is connected
- Can we have a similar characterization for the linear clique-width of cographs ?
  - cographs are completely decomposable wrt modular decomposition

THANK YOU

Path-width(1)



Path-Width of G

$$\mathrm{wd}(P,B) := \max\{|B_t| \mid t \in V(P)\} - 1$$

$$\mathrm{pwd}(G) := \min\{\mathrm{wd}(P,B) \mid (P,B) \text{ path decomposition of } G\}.$$

Path-width(2)

- Disjoint union of caterpillars = path-width 1
- $\text{pwd}(T_h) = \lceil h/2 \rceil$
- $\text{pwd}(G) \leq \text{twd}(G) \cdot \log(n)$
- Computation of the path-width of $TWD(\leq k)$ in polynomial time, even linear for trees
- Trees are obstructions to bounded path-width

Path-width(2)

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- Computation of the path-width of $TWD(\leq k)$ in polynomial time, even linear for trees
- Trees are obstructions to bounded path-width
- A characterisation by cops and robber game

Path-width(3) : invisible robber game

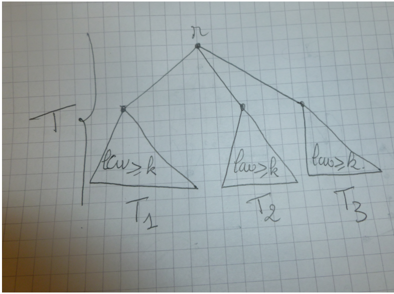
- k cops and 1 invisible robber
- cops move by helicopter
- robber moves through paths not containing cops (she can identify cops positions)
- cops win if they have a strategy to catch the robber (land a helicoper on the robber position)
- minimum number of cops

Linear Clique-Width of Trees

Lemma

If T_1, T_2, T_3 have linear clique-width at least k, then T has linear clique-width $\geq k + 1$.

$\Longrightarrow$



Proposition

If T is a disjoint union of stars, then $lcw(T) = pw(T) + 2$, otherwise $lcw(T) = pw(T) + 1$.