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Coloration, ensemble indépendant et structure de graphe

Coloring, stable set and structure of graphs

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Chapter 1

Introduction (French)

La volonté de compter les objets apparait naturellement dans l'histoire de l'humanité. Les historiens prouvent que déjà en 3400 av. J.-C., les Sumériens et Mésopotamiens avaient développé un système numérique ainsi que le concept de poids et de mesure. Depuis ce jour, et peut-être même bien avant, les humains continuèrent à enrichir l'idée de compter des choses. Ce qui amena à l'émergence de notions plus abstraites que nous appelons maintenant les Mathématiques. Au fur et à mesure que les Mathématiques ont évolués, des notions plus sophistiqués et complexes ont vu le jour. Une partie fondamentale est l'Arithmétique, que chacun utilise intensément au quotidien. Pour un cerveau humain, calculer une opération simple d'arithmétique peut-être fait en quelques secondes, par exemple la somme de deux petits nombres. Mais dès que des données de grandes tailles sont en jeu, même la plus simple des opérations peut prendre un certain temps. Calculer la somme d'une centaine de nombres, même si chaque étape est facile, peut prendre plusieurs dizaines de secondes. Avec l'agrandissement de la société humaine, le besoin de calculer des choses plus larges émergea. Plusieurs outils ont été développé pour aider à cette tâche. Par exemple, la création du boulier est estimée entre 2700 et 2300 av. J.-C. La question que nos ancêtres se sont posé un jour est la suivante : est-ce que cela peut-il être automatisé? L'idée de faire des machines pour calculer de manière automatique peut avoir un impact gigantesque sur la vie humaine pour les raisons suivantes. Si une machine peut calculer de manière à ce qu'aucune erreur ne soit faite, cela veut dire qu'une machine donnerait toujours la bonne réponse. De plus, si une machine peut calculer avec une très grande vitesse, elle peut alors donner la bonne réponse à chaque fois et beaucoup plus vite qu'un être humain. Pour avoir un aperçu de la puissance de calcul des ordinateurs de notre ère, intéressons nous à ce simple fait. Une *opération en virgule flottante* est un calcul qui fait intervenir au moins deux nombres réels (un nombre qui peut être décrit avec une virgule, par exemple 1,567). Par exemple, multiplier 1,545 par 143,75482 est considéré comme étant une opération en virgule flottante. Même l'esprit le plus vif aurait besoin d'au moins une seconde pour calculer la précédente opération. Le super-ordinateur plus performant enregistré à ce jour est capable de faire 93.000.000.000.000.000 opérations en virgule flottante **par seconde!** Cependant, même si cette performance est incroyable, multiplier des nombres entre eux, même à une très grande vitesse, n'est pas suffisant pour envoyer des fusées dans l'espace, calculer le plus court chemin sur un GPS ou encore contrôler le chaîne de production d'une usine. Ce dont un ordinateur a besoin pour maximiser l'utilité de sa grande puissance de calcul est une série d'opérations qu'il doit suivre pas à pas. C'est ce qu'on appel un algorithme. Avec des données en entrée, un ordinateur suivant un algorithme va appliquer les règles contenues dans l'algorithme aux données et retourner le résultat en sortie. Par exemple, "étant donné deux nombres x et y, multiplier x par y et retourner le résultat", est un exemple simple de ce qu'est un algorithme.

Bien évidemment, de nos jours il existe des algorithmes bien plus sophistiqués. Prenons un exemple plus avancé que la multiplication de deux nombres. Étant donné une carte et la longueur de chaque section de route, on souhaite calculer le plus court chemin entre deux points. Comme le problème est cette fois plus compliqué, et comme nous sommes de bons scientifiques, une bonne idée serait d'abstraire ce problème avec un modèle qui encode toutes les informations dont nous avons besoin, et de résoudre le problème sur ce modèle abstrait. Un modèle abstrait nous débarrasse de la réalité et devient un pure objet mathématique qui ouvre alors les portes vers toutes les mathématiques pour nous aider à le résoudre. Dans ce contexte, un choix naturel pour un modèle serait de dessiner sur un papier blanc un point pour chaque intersection de route, de dessiner une ligne entre toute paire de points reliés par une section de route et d'écrire le long de cette ligne le nombre correspondant à la longueur de la section qu'elle représente. Maintenant en gardant uniquement notre papier blanc contenant des points, des lignes et des valeurs, nous avons toutes les informations nécessaires pour calculer le plus court chemin entre n'importe quelle paire de points. Notez que même si nous n'avons pas donné d'algorithme qui répond à la question posée, nous avons un modèle des données qui encode uniquement ce qui est réellement essentiel. Voir Figure 1.1 pour un exemple d'un tel modèle. Étant donné un modèle qui contient toutes les données utiles pour notre problème, que pouvons nous dire dessus et comment utiliser ses propriétés intéressantes pour nous aider à fournir une solution à notre question? C'est ce genre de questions qui définit en grande partie l'Informatique Théorique. D'un côté, extraire les propriétés mathématiques des modèles, et de l'autre essayer d'utiliser ces propriétés pour développer des algorithmes sophistiqués. Bien sûr, n'importe laquelle de ces branches est un domaine à part entière des Mathématiques et de l'Informatique. Dans cette thèse, je vais exposer un état de l'art et des résultats nouveaux concernant des problèmes liés à ces deux branches.

Maintenant que nous savons tous ce vers quoi nous souhaitons aller, soyons plus formel vis à vis de ces concepts. Le modèle présenté plus haut est appelé un *graphe*. Chaque point est appelé un sommet et chaque ligne entre deux sommets, droite ou non cela n'a pas d'importance, est appelée une arête. Deux sommets liés par une arête sont dit adjacents. Un voisin d'un sommet v est n'importe quel sommet u qui est adjacent à v. Le degré d'un sommet est le nombre de voisins qu'il a. Notez qu'un

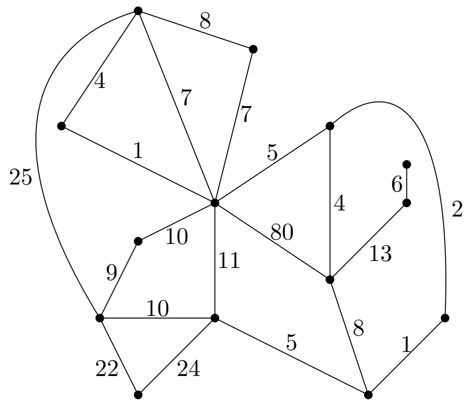


Figure 1.1: An example of a map model.

graphe n'est pas un modèle géométrique, dans le sense où nous n'avons pas de coordonnées sur les sommets, et les arêtes détiennent uniquement l'information si oui ou non deux sommets sont adjacents. Les graphes sont des outils très puissants permettant de modéliser de nombreux problèmes, du plus théorique au plus appliqué. Nous allons donner un aperçu de deux problèmes canoniques de théorie des graphes en lien direct avec plusieurs résultats présentés dans ce manuscrit.

Supposons que l'on nous fournisse un ensemble de produits chimique que nous devons stocker dans des entrepôts. Certains d'entre eux ne peuvent pas être stockés ensemble sans prendre le risque de générer une dangereuse réaction chimique. Ouvrir un entrepôt chimique est très onéreux. Nous souhaitons donc minimiser le nombre d'entrepôt à ouvrir. Nous pouvons modéliser ce problème de la manière suivante. Construisons un graphe où chaque sommet correspond à un produit chimique et pour chaque paire de sommets, mettons une arête si et seulement si les produits sont incompatibles. La traduction de notre but, qui est de minimiser le nombre d'entrepôts, peut être formulée de la manière suivante. Nous voulons attribuer à chaque sommet une couleur tel que pour toute paire de sommets adjacents les couleurs soient différentes, et nous souhaitons minimiser le nombre de couleurs utilisées. À la fin de notre procédé, le nombre de couleurs est le nombre d'entrepôts à ouvrir et une couleur est équivalente à un type d'entrepôt où tous les sommets de cette couleur seront stockés. Voir Figure 1.2 pour un exemple.

Le problème de coloration peut être énoncé de la manière suivante. Pour tout entier $k \geq 1$, une *k-coloration* d'un graphe G est une affection d'au plus k couleurs aux sommets de G. Plus formellement, c'est une fonction $c : V(G) \to \{1, \dots, k\}$. Une *k-coloration propre* est une k-coloration satisfaisant $c(u) \neq c(v)$ pour toute paire de sommets adjacents u et v. Un graphe est dit *k-colorable* si il admet une k-coloration propre. Il est alors naturel de définir le nombre minimum de couleurs nécessaires pour colorer proprement le graphe. Le *nombre chromatique* d'un graphe G, noté $\chi(G)$, est le plus petit entier k tel que G soit k-colorable. Donc, pour résoudre notre problème de produits chimique de manière optimale, nous devons trouver le nombre chromatique

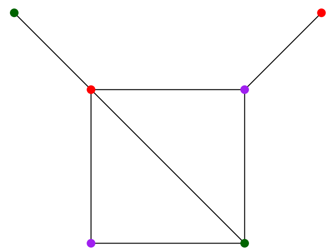


Figure 1.2: Example of an optimal coloring.

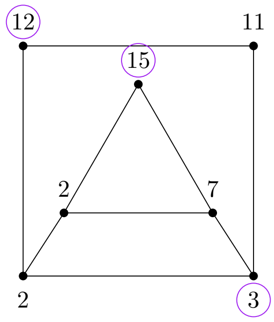


Figure 1.3: Example of an optimal set S. Picked vertices are circled in purple.

de notre modèle.

Le deuxième problème classique qui est étudié dans ce manuscrit peut être énoncé d'un point de vue pratique de la manière suivante. Supposons que nous ayons un ensemble possible d'emplacements où nous pouvons ouvrir un restaurant de notre chaîne. Bien sûr, on ne peut pas ouvrir deux restaurants trop proches l'un de l'autre, ce qui aurait pour effet de diviser la clientèle. Chaque emplacement a un bénéfice estimé. Nous souhaitons ouvrir des restaurants tel que le profit soit maximiser tout en respectant la contrainte que deux restaurants ne peuvent pas être trop proches. On peut modéliser cela par un graphe. Pour chaque emplacement sur notre carte, mettons un sommet et associons à chaque sommet le nombre représentant son bénéfice estimé. Mettons une arête entre deux emplacements (qui sont maintenant des sommets) dès qu'ils sont trop proches. Ce que l'on souhaite trouver maintenant est un ensemble de sommets S dans notre graphe tel que tous les sommets de S soient deux à deux non-adjacents *et* qui maximise la somme des profits estimés sur tous les sommets de S. Voir Figure 1.3 pour un exemple. Nous allons expliquer maintenant ce problème en des termes de théorie des graphes.

Un *ensemble indépendant* est un sous-ensemble de sommets $S \subseteq V(G)$ deux à deux non-adjacents. Le *cardinal maximum d'un ensemble indépendant* d'un graphe G est noté $\alpha(G)$. Le problème d'*Ensemble Indépendant Maximum* est le problème consistant à trouver l'ensemble indépendant de cardinal maximum pour un graphe donné. Soit G un graphe, la version pondérée de ce problème est définie par une fonction de poids sur les sommets de G, $w : V(G) \to \mathbb{Q}$ qui attribue à chaque sommet v un poids $w(v)$.

Le problème d'*Ensemble Indépendant de Poids Maximum* est maintenant de trouver un ensemble indépendant de poids maximum, que l'on note par $\alpha_w(G)$. Notons que si $w(v) = 1$ for tout sommet $v \in V(G)$, ce problème est équivalent à la version non pondérée.

Le problème de coloration de graphe et d'ensemble indépendant de poids maximum sont tous deux des problèmes *difficiles*. Mais que veut dire exactement difficile dans notre contexte? Si vous essayez de trouver une solution optimale à l'un des problèmes précédents sur un graphe avec plus de trente sommets, vous allez probablement passer au moins quelques heures pour trouver la bonne solution. En Informatique Théorique, il y a une classification des problèmes en fonction de leur difficulté. C'est une notion très importante en Informatique car cela peut donner une *idée* si oui ou non un problème spécifique peut être résolu efficacement sur un ordinateur. Un problème est dit décidable en temps polynomial si étant donné une entrée de taille n, le nombre d'opérations élémentaires nécessaires[1] pour trouver une solution est borné par un polynôme en n. La classe de tous les problèmes décidables en temps polynomial est noté P. Par exemple, calculer le plus court chemin entre deux sommets u et v est un problème qui est dans la classe P.

D'un autre côté, il y a des problèmes pour lesquels une solution peut être vérifiée en temps polynomial mais pour lesquels il n'y a pas, à l'heure actuelle, d'algorithme polynomial capable d'en trouver une solution. Par exemple, étant donné un graphe, est-il possible de colorer proprement ses sommets en utilisant au plus k couleurs? Vérifier si une solution donnée est valide est facile, cependant dans le cas général, nous n'avons pas d'algorithme polynomial permettant de résoudre ce problème. Le problème de satisfaisabilité est un problème canonique pour lequel on ne sait pas si il existe un algorithme polynomial mais une solution peut être vérifiée rapidement. Ce problème, noté SAT, est un problème de décision qui demande si il existe une interprétation d'un ensemble de variables booléennes qui satisfait une expression booléenne donnée. Nous n'irons pas plus loin dans les détails de ce problème, mais il est important de retenir que ce problème est standard et nous ne savons pas si un jour nous pourrons le résoudre de manière efficace ou non. La classe de problèmes qui sont au moins aussi difficile[2] que le problème SAT est appelée la classe des problèmes NP-Difficiles. De plus, la classe des problèmes NP-Difficiles *et* pour lesquels il est possible de vérifier une solution en temps polynomial est appelée la classe des problèmes NP-Complets. Cook en 1971[21] prouva que le problème SAT est NP-Complet et que tout autre problème dans NP peut être réduit au problème SAT en temps polynomial. Le problème de k-coloration est NP-Complet et le problème qui consiste à trouver un ensemble indépendant de poids au moins k est également NP-Complet. En d'autres termes, ces problèmes sont difficiles. Dans ce manuscrit nous présentons des avancées liées au problème de coloration, au problème d'indépendant de poids maximum et la réfutation d'une conjecture de théorie des graphes en lien

[1]Une opération élémentaire peut être une opération arithmétique, ou vérifier si deux sommets sont adjacents, etc ...

[2]Plus formellement, un problème est NP-Difficile si l'on peut transformer en temps polynomial une instance de SAT en une instance de notre problème.

avec les cliques et les ensembles indépendants.

Les algorithmes avancés sur des problèmes difficiles sont possibles grâce à la connaissance de la structure des données que nous avons en entrée. Par exemple, trouver l'ensemble indépendant de poids maximum peut être fait en temps polynomial dans des classes de graphes particulière en utilisant les connaissances que nous avons sur leurs structures. D'un autre côté, en ne sachant rien de spécial sur la structure des graphes en entrée, il est peu probable pour que l'on soit capable de fournir un algorithme efficace pour ce problème. Donc, en théorie des graphes, décrire la structure des objets que l'on manipule est d'une importance capitale et est un domaine de la théorie des graphes à part entière. La théorie structurelle des graphes a pour but de prouver des théorèmes décrivant les propriétés des graphes. Par exemple, le théorème de Kuratowski [62] décrit complètement lorsqu'il est possible de dessiner un graphe sur le plan sans croisement d'arête. Même si ceci est considéré comme un travail purement théorique, l'impacte que ce genre de résultats a sur des problèmes plus appliqués de théorie des graphes est très important. Dans le Chapitre 4 et Chapitre 6 nous traitons, respectivement, d'une généralisation du problème de coloration et d'une conjecture lié à des graphes particuliers. Les résultats présentés dans ces deux chapitres ne sont pas d'une nature algorithmique. Ils sont théoriques et améliore la connaissance autour de certaines classes de graphes.

1.1 Contenu du manuscrit

Nous allons donner un aperçu de ce qui est présenté dans ce manuscrit. Les sujets principaux sont la coloration, la coloration par liste, les ensemble indépendant de poids maximum et les graphes normaux.

Le Chapitre 3 est dédié au problème de k-coloration dans les graphes. Nous commençons par une brève histoire de la coloration de graphe dans la Section 3.1 et présentons quelques résultats connus concernant une classe de graphes très importante, les *graphes parfaits*, qui ont un lien très étroit avec le problème de coloration. Puis nous expliquons pourquoi les classes de graphes interdisant des chemins comme sous-graphes induits (les graphes P_ℓ-free) sont importants pour le problème de k-coloration et présentons un résumé des résultats marquant concernant la k-coloration des graphes P_ℓ-free. Ensuite, nous présentons dans la Section 3.2 la structure des graphes (P_6, bull)-free. Enfin, nous exposons dans la Section 3.3 un algorithme polynomial pour la 4-coloration des graphes (P_6, bull)-free et pour la k-coloration des graphes (P_6, bull, gem)-free.

Dans le Chapitre 4 nous nous intéressons au problème de coloration par liste, qui est une généralisation du problème de coloration. Dans la Section 4.1 nous expliquons comment le problème de coloration peut être généralisé au problème de coloration par liste et pourquoi la classe des graphes sans griffe est importante pour ce problème. En Section 4.2 nous décrivons la structure des graphes parfaits sans griffe. Nous utilisons cette description pour prouver, en Section 4.3, que n'importe quel graphe par-fait sans griffe ayant des cliques de taille au plus 4, a un nombre chromatique égal au nombre chromatique par liste.

Le Chapitre 5 est dédié au problème d'indépendant de poids maximum. La Section 5.1 décrit le contexte et explique pourquoi la classe des graphe P_ℓ-free est intéressante vis à vis de ce problème. Puis en Section 5.2 nous fournissons une description structurelle des graphes sans taureau que nous utiliserons pour nos algorithmes. Finalement, les sections 5.3 et 5.4 sont dédiées à la présentation d'un algorithme polynomial pour le problème d'indépendant de poids maximum dans les graphes (P_6, bull)-free et (P_7, bull)-free. Les techniques utilisées en Section 5.3 et Section 5.4 sont différentes.

Le Chapitre 6 traite de la réfutation de la conjecture des graphes normaux. La Section 6.1 commence par décrire l'origine des graphes normaux, expose le contexte et donne un aperçu de ce qui est connu vis à vis de cette classe de graphes. Dans la Section 6.2 nous décrivons la philosophie de notre outils principal, la *Méthode Probabiliste*, et fournissons également un exemple d'utilisation de cette méthode en explicitant la preuve d'un célèbre théorème d'Erdős. Enfin, en Section 6.3 nous décrivons la structure de notre graphe aléatoire et donnons la preuve de notre théorème faisant appel à un lemma clef dont la preuve est décrite en Section 6.4.

Chapter 2

Introduction

2.1 Context

The notion of counting elements appeared in the story of humanity naturally. As early as 3400 BC, historians have proof that the Sumerians in Mesopotamia developed a numeral system and the concept of weights and measures. From this day, and maybe even earlier, humans continued to develop the concept of counting things, which eventually led to more abstract notions that now fall under what we call Mathematics. As Mathematics evolved, more sophisticated and complex notions arose. One fundamental one is Arithmetic, that everybody still uses intensively for the day to day life. For a human mind, calculating simple arithmetic operations can be done in a few seconds, for instance, the sum of two small numbers. But as soon as more data is involved, even the simplest operations can take more time. Calculating the sum of a hundred numbers, even though each step is easy, can take several dozen of seconds. As human society grew, the need to calculate larger things emerged. Several tools were manufactured to help deal with such a task. For example, the creation of the abacus is estimated between 2700 and 2300 BC. Of course what our ancestor eventually had in mind is the great following question: can it be automated? The idea to make machines calculate automatically can have tremendous impact on human life for the following reasons. If the machine calculating process can be done such that no mistake can ever happens, it means that a machine would never be wrong. In addition, if a machine can calculate with great speed, it can give the right answer every time way faster than what a human could do. To get a grasp at how the most advanced computers are good at calculating nowadays, here is a simple fact. A *floating point operation* is a calculation that involves at least two real numbers (a number that can be written with a comma, such as 1.567). For instance, multiplying 1.545 with 143.75482 is considered as a floating point operation. Even the swiftest mind needs at least a second to compute the previous operation. The best super computer registered as of today is capable of doing $93,000,000,000,000,000$ floating point operations **per second**! However, even though this is an incredible performance, multiplying numbers alone, even at great speed, is not sufficient to send rockets into space, to compute a shortest path on a GPS, or to control an industry process line. What a computer needs to use its calculating speed at maximum power is a series of operations that it needs to follow step by step. This is what is called an algorithm. Given data as an input, a computer following an algorithm will apply the rules contained in the algorithm to the data and give the result as an output. For instance, "given two numbers x and y, multiply x and y and give the result", is a simple example of what is an algorithm.

Of course nowadays there exist very deep and sophisticated algorithms. Let us pick a more advanced example than multiplying two numbers. Given a map and the length of every road section, we would like to compute the shortest path between two points. Since the problem here is more complicated, and as we are good scientists, a good idea would be to abstract this problem with a model that encodes all the information we need, and then solve the problem on the abstract model. An abstract model rids us of the reality and becomes a pure mathematical object, which then opens the door to all of mathematics to help us solve the problem. In this context, a natural choice for a model would be to draw a point for every intersection of roads on a blank sheet of paper, draw a line between every pair of points connected with a road, and finally on every line, write the number corresponding to the length of the road section it represents. Now we can only keep our blank sheet of paper and we have all the information we need to compute a shortest path between any pair of points. Note that even though we didn't give any algorithm to answer the question, we provided a model of the data that encodes only what is really needed. See Figure 2.1 for an example of such a model. Given a model that holds all the interesting data of our problem, what can we say about it and how can we use any interesting fact to help us solve it? This is for the most part, what *Theoretical Computer Science* is about. On the one hand, extract properties from mathematical models, and on the other hand, try to use those properties to develop clever algorithms. Of course, any of those two branches is a whole field of Mathematics/Computer Science of its own. In this thesis, I am going to give an overview and results related to those two branches.

Now that we all know what we are aiming for, let us get more formal on those concepts. The model presented above is called a *graph*. Each point is called a *vertex* and any line between two vertices, straight or not, it does not matter, is called an *edge*. Two vertices linked by an edge are called *adjacent*. The *neighbor* of a vertex v is any vertex u that is adjacent to v. The *degree* of a vertex is the number of neighbors it has. Note that a graph is not a geometric model, in the sense that we do not have coordinates on vertices, and edges are only there to keep the information whether or not two vertices are adjacent. Graphs are very powerful tools able to model many different problem, from the most theoretic to the most applied ones. We will give an overview of two canonical graph theoretic problems on which several results presented in this manuscript are based.

Assume that we are given a set of chemical products that we need to store in warehouses. Some of them cannot be stored in the same warehouse without taking the risk of seeing a dangerous chemical reaction happening. Opening a chemical

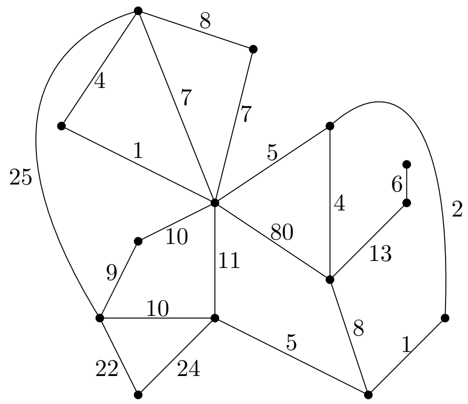


Figure 2.1: An example of a map model.

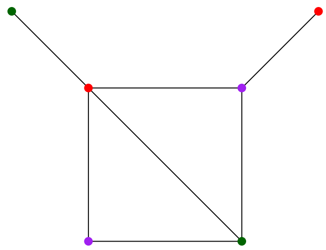


Figure 2.2: Example of an optimal coloring.

warehouse is very expensive, so we want to minimize the number of warehouses that needs to be opened. We can model this problem with the following. Construct a graph where each vertex correspond to a chemical product, and for any pair of vertices, put an edge if and only if the products are incompatible. Now the translation of our goal, which is to minimize the number of warehouses, can be put as follows. On each vertex we assign a color such that any pair of adjacent vertices have a different color, and we want to minimize the number of used colors. In the end, the number of colors is the number of warehouses we need to open, and a color is equivalent to a type of warehouse where all vertices of this specific color will be stored. See Figure 2.2 for an example.

The coloring problem can be stated as follows. For any integer $k \geq 1$, a *k-coloring* of a graph G is an assignment of at most k colors to the vertices of G. More formally it is a mapping $c : V(G) \to \{1, \ldots, k\}$. A *proper k-coloring* is a k-coloring satisfying $c(u) \neq c(v)$ for any two adjacent vertices u and v. A graph is said to be *k-colorable* if it admits a proper k-coloring. It is then natural to define the minimum number of colors needed to properly color the graph. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable. Hence, to solve our chemical products problem optimally, we need to find the chromatic number of our model.

The second classical problem that is studied in this manuscript can be stated in a practical way as follows. Assume that we are given a set of possible locations where

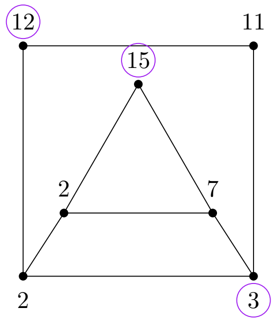


Figure 2.3: Example of an optimal set S. Picked vertices are circled in purple.

we can open one restaurant of our restaurant chain. Of course we cannot open two restaurants too close to each other, this would only split the customer mass. Each of the locations has an estimated benefit. We want to open restaurants such that it maximizes our profit while satisfying the constraint that no two restaurants can be too close. We can model this by a graph. For every location on our map, put a vertex and associate with each vertex a number representing its estimated profit. Put an edge between two locations (which are now vertices) whenever they are too close to each other. What we want to do now is to find a set of vertices S in our graph such that all vertices in S are pairwise non-adjacent *and* that maximizes the sum of the estimated profit on all vertices of S. See Figure 2.3 for an example. Let us explain in a graph theoretical way what this optimization problem is.

A *stable set* is a subset of vertices $S \subseteq V(G)$ such that any two vertices in S are non-adjacent. The *stability number* of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of a stable set contained in G. The *Maximum Stable Set Problem*, shortened *MSS*, is the problem of finding the stable set of maximum cardinality in a given graph. Let G be a graph, the weighted version of this problem is defined by a weight function on the vertices of G, $w : V(G) \to \mathbb{Q}$ that assigns to each vertex v a weight $w(v)$. The *Maximum Weight Stable Set Problem*, shortened *MWSS*, is now to find the stable set of maximum weight, that we denote by $\alpha_w(G)$. Note that if $w(v) = 1$ for every $v \in V(G)$, this is equivalent to the non weighted version.

The graph coloring problem, and the Maximum Stable Set Problem are both *hard* problems. But what does hard mean exactly in our context? If you try to find an optimal solution to one of the problems stated above on a graph with more than thirty vertices, you might end up spending a few hours to find the correct solution. In Theoretical Computer Science, there is a classification of problems according to their difficulty. This is a very important concept in Computer Science since it can give an *idea* on whether or not a specific problem might be solved efficiently on a computer. A problem is said to be *polynomial-time solvable* if given an input of size n, the number of basic operations[1] needed to find a solution is bounded by a polynomial in n. The class

[1] A basic operation can either be an arithmetic operation or checking if two vertices are adjacent, etc ...

of all polynomial-time solvable problems is denoted by P. For example, computing the shortest path between two vertices u and v is a problem that is in this class P.

On the other hand, there are problems for which a given solution can be verified in polynomial time but there is presently no polynomial-time algorithm known that can find a solution. For example, given a graph, is it possible to properly color its vertices by using no more than k colors? Verifying a given solution is easy, however in the general case, we do not have a polynomial-time algorithm that can solve this problem. The satisfiability problem is a canonical problem for which we do not know if there exists a polynomial-time algorithm that can solve it but a given solution can be checked quickly. This problem, shortened as *SAT*, is a decision problem which asks if there exists an interpretation of a set of boolean variables that satisfies a given boolean formula. We will not go deeper into the details on this problem, but what is important to remember is that this problem is the standard problem for which we do not know yet if one day we will be able to solve it efficiently. The class of problems that are at least as hard[2] as the SAT problem is called the *NP-Hard* class. Moreover, the class of problems that are NP-Hard *and* for which it is possible to check in polynomial time if a given solution is valid is called the *NP-Complete* class. Cook in 1971 [21], proved that the SAT problem is NP-Complete and that for any other problem in NP, there exists a polynomial reduction to the SAT problem. The k-coloring problem is NP-Complete and the problem of finding a stable set of weight at least k is also NP-Complete. In other words, these problems are difficult. In this manuscript we will present some advances concerning coloring problems, the MWSS problem and a disproof of a graph theory conjecture related to clique and stable sets.

Advanced algorithms on difficult problems are only possible thanks to the knowledge of the data structure we have in input. For example, computing the maximum-weight stable set can be done efficiently in specific classes of graphs thanks to the precise structure of these classes. On the other hand, when we do not know anything on the structure of the graph we have in input, it is unlikely that we will be able to produce an efficient algorithm for this problem. Hence, in graph theory, describing the structure of the object we handle is very important and is a subfield of its own. Structural graph theory aims at proving theorems describing properties of graphs. For example, the theorem of Kuratowski [62] fully describes when it is possible to draw a graph on the plane without any edge crossing. Even though this can be considered a purely theoretical work, it has a great impact on more applied graph theory problems. In Chapter 4 and Chapter 6 we deal with respectively, a generalization of the coloring problem and a conjecture linked to a specific graph class. The results presented in these two chapters are not of an algorithmic aspect. They are theoretical and improve the knowledge around two specific classes of graphs.

[2]More formally, a problem is NP-Hard if one can transform in polynomial time an instance of SAT to an instance of our problem.

2.2 Outline of the manuscript

We will give a general outline of what is present in this manuscript. The main topics discussed are coloring, list coloring, maximum-weighted stable set and normal graphs.

Chapter 3 is dedicated to the k-coloring problem in graphs. We start with a short history of graph coloring in Section 3.1 and then present some known results regarding a very important graph class, *perfect graphs*, which is closely linked to the coloring problem. Then we explain why graph classes forbidding induced paths, P_ℓ-free graphs, are important for the k-coloring problem and present a summary of important results regarding the k-coloring problem in P_ℓ-free graphs. Afterwards, we present in Section 3.2 the structure of (P_6,bull)-free graphs. Finally we expose in Section 3.3 a polynomial-time algorithm for the 4-coloring problem in (P_6,bull)-free graphs and a polynomial-time algorithm for the k-coloring problem in (P_6, bull, gem)-free graphs.

In Chapter 4 we deal with the list coloring problem, which is a generalization of the coloring problem. In Section 4.1 we explain how the coloring problem can be generalized to the list coloring problem and why the class of claw-free graphs is important for this problem. Then in Section 4.2 we describe the structure of claw-free *perfect* graphs. We use these descriptions to prove in Section 4.3 that any claw-free perfect graph with clique number bounded by 4 has its chromatic number equal to its choice number.

Chapter 5 is dedicated to the Maximum Weight Stable Set problem. Section 5.1 describes the context and why P_ℓ-free graphs are interesting regarding this problem. Then in Section 5.2 we give structure properties of bull-free graphs that will be used in our algorithms. Finally, Section 5.3 and 5.4 are dedicated to the description of a polynomial-time algorithm for the MWSS problem in respectively (P_6, bull)-free graphs and (P_7, bull)-free graphs. The techniques used in Section 5.3 and Section 5.4 are different.

Chapter 6 deals with a disproof of the Normal Graph Conjecture. First, in Section 6.1 we start by describing the origins of normal graphs and expose the context and what is known around this specific graph class. In Section 6.2 we describe the philosophy of our main tool used in our disproof, the Probabilistic Method, and also provide a pedagogical example of how it was used in a famous proof of Erdős. Finally in Section 6.3 we describe the structure of our random graph and in Section 6.4 provide the proof of our key lemma.

2.3 Definitions

We will define in this subsection classical elementary graph theoretic concepts.

A finite simple *graph*, denoted by $G = (V, E)$, is an ordered pair consisting of a finite set V, called the vertices, and E, the set of edges which are 2-element subsets of V. An edge $\{u, v\}$ is also denoted by uv. To refer specifically to the set of vertices and edges of a graph G, we respectively denote this by $V(G)$ and $E(G)$. Given a vertex v, the *neighborhood* of v, denoted by $N(v)$, is the set of all vertices adjacent to v. The *closed neighborhood*, denoted by $N[v]$, is defined by $N[v] = N(v) \cup \{v\}$. Similarly, for any subset of vertices $S \subseteq V(G)$, we define $N(S) = (\bigcup_{v \in S} N(v)) \setminus S$ and $N[S] = N(S) \cup S$.

The *maximum degree* of a graph G, denoted by $\Delta(G)$, is the maximum degree among all vertices of G. The *complement* of a graph G, denoted by $\overline{G}$, refers to the graph on the same vertex set and with edge set $\binom{V(G)}{2} \setminus E(G)$. The *clique number* of a graph G, denoted by $\omega(G)$, is the maximum cardinality of a *clique*, contained in G, which is a subset $K \subseteq V(G)$ of vertices such that any two vertices in K are adjacent. A clique is also called a *complete graph* and is denoted by K_n where $n \geq 1$ is the number of vertices. The complete graph on three vertices, K_3, is also called a *triangle*. A *stable set* is a subset of vertices $S \subseteq V(G)$ such that any two vertices in S are non-adjacent. The *stability number* of a graph G, denoted by $\alpha(G)$, is the maximum cardinality of a stable set contained in G. Given a graph G, the *induced subgraph* H on the vertex set $S \subseteq V(G)$, denoted by $G[S]$ is the graph on vertex set S and whose edges set is $E(H) = \{uv \in E(G) \mid u, v \in S\}$. Given two graphs G and G', we say that G is *isomorphic* to G' if there exists a bijection f from $V(G)$ to $V(G')$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in G'. Given a family $\mathcal{H}$ of graphs, a graph G is said to be *$\mathcal{H}$-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{H}$. When $\mathcal{H}$ has only one element H, we say that G is *H-free*.

2.4 Preliminaries

Modular decomposition

We say that a vertex v is *complete* to S if v is adjacent to every vertex in S, and that v is *anticomplete* to S if v has no neighbor in S. For two sets $S, T \subseteq V(G)$ we say that S is complete to T if every vertex of S is adjacent to every vertex of T, and we say that S is anticomplete to T if no vertex of S is adjacent to any vertex of T. A *homogeneous set* is a set $S \subseteq V(G)$ such that for every vertex v in $V(G) \setminus S$, either v is complete to S or anticomplete to S, see Figure 2.4. A homogeneous set is said to be proper if it contains at least two vertices and is different from $V(G)$. A *prime graph* is a graph that has no proper homogeneous set. A *module*[3] is a homogeneous set S such that every homogeneous set S' satisfies either $S' \subseteq S$ or $S \subseteq S'$ or $S \cap S' = \emptyset$. In particular $V(G)$ is a module and every singleton $\{v\}$ $(v \in V(G))$ is a module. Given a graph G and a partition P of its vertex set where each partition class is a module of G, the *quotient graph*, denoted by G/P is defined as the subgraph of G induced by picking one vertex from each partition class. The theory of modular decomposition (the study of the

[3]Note that in the literature one might find a different, but still close, definition of what is a module and a homogeneous set.

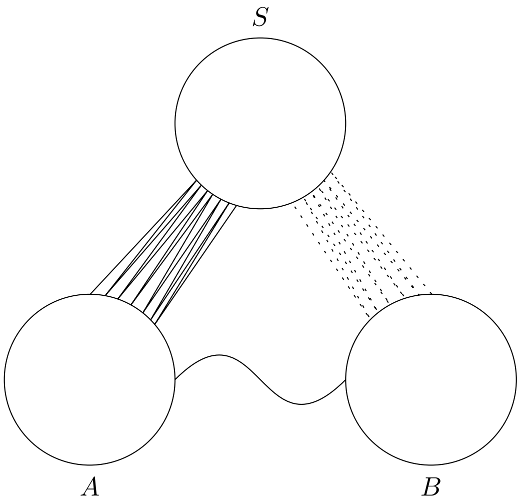


Figure 2.4: Homogeneous set S, with A complete to S and B anticomplete to S.

modules of a graph) is a rich one, starting from the seminal work of Gallai [34]. We mention here only the results we will use. A subset of vertices $S \subseteq V(G)$ is a *maximal module* if $S \neq V(G)$ and there is no module S' such that $S \subsetneq S' \subsetneq V(G)$.

- Any graph G has at most $2|V(G)|$ modules, and they can be produced by an algorithm of time complexity $O(|V(G)| + |E(G)|)$ [89].
- If both G and $\overline{G}$ are connected, then G has at least four maximal modules and they form a partition of $V(G)$ (called a *modular partition*), and every homogeneous set of G different from $V(G)$ is included in a maximal module; moreover, the induced subgraph G' of G obtained by picking one vertex from each maximal module of G is a prime graph.

A homogeneous set is a generalization of a connected component in the sense that, in the connected component, every vertex has the same set of non-neighbors outside of the component. In a homogeneous set, every vertex has the same set of non-neighbors and neighbors outside of the homogeneous set.

Gallai defined a recursive algorithm to compute the modular decomposition of any graph G. This algorithm takes any graph G in input and outputs the modular decomposition tree, denoted by $T(G)$, which totally encodes the modular decomposition. A modular decomposition tree contains three types of nodes. Prime nodes, series nodes and parallel nodes. A prime node represents the fact that the graph G is connected and so is its complement. It means that we can find a modular partition of its vertex set P such that the quotient graph G/P is a prime graph. A serial node implies that the quotient graph induced by the label of its children is a complete graph. Finally, a parallel node implies that the quotient graph induced by the label of its children is a stable set. More formally, the recursive algorithm $\mathcal{A}$ defined by Gallai is described in Algorithm 1.

Algorithm 1 $\mathcal{A}(G)$

Input : A graph G
Output : The modular decomposition tree of G

procedure $\mathcal{A}(G)$
  if $|V(G)| = 1$ **then**
    return $V(G)$
  else if G is disconnected **then**
    partition G into components $M_1, \ldots, M_k$
    create a parallel node R with label $V(G)$
  else if $\overline{G}$ is disconnected **then**
    partition G into co-components $M_1, \ldots, M_k$
    create a serial node R with label $V(G)$
  else
    partition G into maximal modules $M_1, \ldots, M_k$
    create a prime node R with label $V(G)$
  end if
  for all $i \in \{1, \ldots, k\}$ **do**
    set the node $\mathcal{A}(G[M_i])$ as child of R
  end for
  return R
end procedure

This algorithm produces the modular decomposition tree of the graph G. In other words, it is a tree which totally encodes the relation (full adjacency or full non-adjacency) between any pair of modules of G and recursively on any subgraph of G induced by a module. See Figure 2.5 for an example of a modular decomposition.

Clique-width

Informally, the clique-width is an integer which measures the complexity of constructing G through a sequence of certain operations. More precisely, the *clique-width* of a graph G, denoted by $cw(G)$, first introduced in [23], is defined as the minimum number of labels needed to construct G by using the following four operations (see Figure 2.6 for an example):

- Create a vertex v labeled by integer i.
- Make the disjoint union of two labeled graphs.
- Join by an edge all vertices with label i to all vertices with label j for two labels $i \neq j$.
- Relabel all vertices of label i by label j.

A *c-expression* for a graph G of clique-width c is a sequence of the above four operations that generates G and uses at most c different labels. The clique-width is

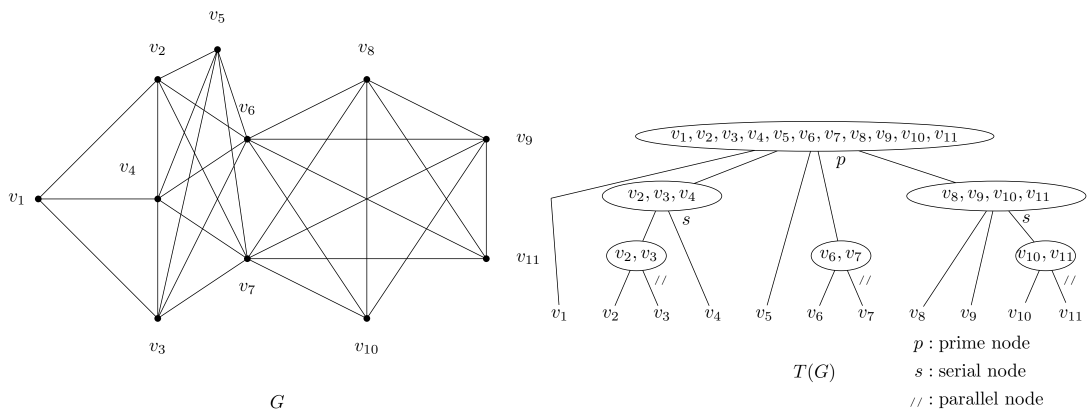


Figure 2.5: A graph G and its modular decomposition tree $T(G)$.

a graph parameter that has been widely studied. A famous result involves the class of P_4-free graphs, also known as co-graphs. In fact, co-graphs can be defined as the graphs having a clique-width of at most 2, as proved in [25]. It is shown in [32] that it is NP-hard to compute the clique-width of a graph G. On the other hand Oum and Seymour [80], provide an algorithm that, given a graph G and a fixed integer c, outputs a c'-expression in $\mathcal{O}(n^9 \log n)$, where $c' = 2^{3c+2} - 1$ or a witness that G has clique-width at least $c+1$. This was later improved by Oum [79] with a complexity of $\mathcal{O}(n^3)$ where $c' = 8^c - 1$. Courcelle, Makowsky and Rotics proved a meta-theorem which has been used in several occasions to prove the existence of polynomial-time algorithm deciding the k-coloring problem or solving the maximum-weight stable set in certain classes of graphs. They proved that if a class of graph has bounded clique-width c, and for any graph in this class it is possible to find a c-expression in at most $f(G)$ computational steps, then it is possible to find the maximum-weight stable set or decide the k-colorability of G for fixed k, in at most $f(G)$ computational steps. This result is incredibly useful as, in certain cases, it basically narrows down the problem to only showing that a certain class of graphs has bounded clique-width. More formally, their theorem is as follows.

THEOREM 2.1 [24]

If a class of graphs $\mathcal{C}$ has bounded clique-width c, and there is a polynomial f such that for every graph G in $\mathcal{C}$ with n vertices and m edges a c-expression can be found in time $\mathcal{O}(f(n,m))$, then for fixed k the k-coloring problem or the MWSS problem can be solved in time $\mathcal{O}(f(n,m))$ for every graph G in $\mathcal{C}$.

Moreover, in order to upper bound the clique-width of a graph G, it suffices to consider only the prime induced subgraphs of G.

THEOREM 2.2 [24, 25]

The clique-width of a graph is the maximum of the clique-width of its prime induced subgraphs.

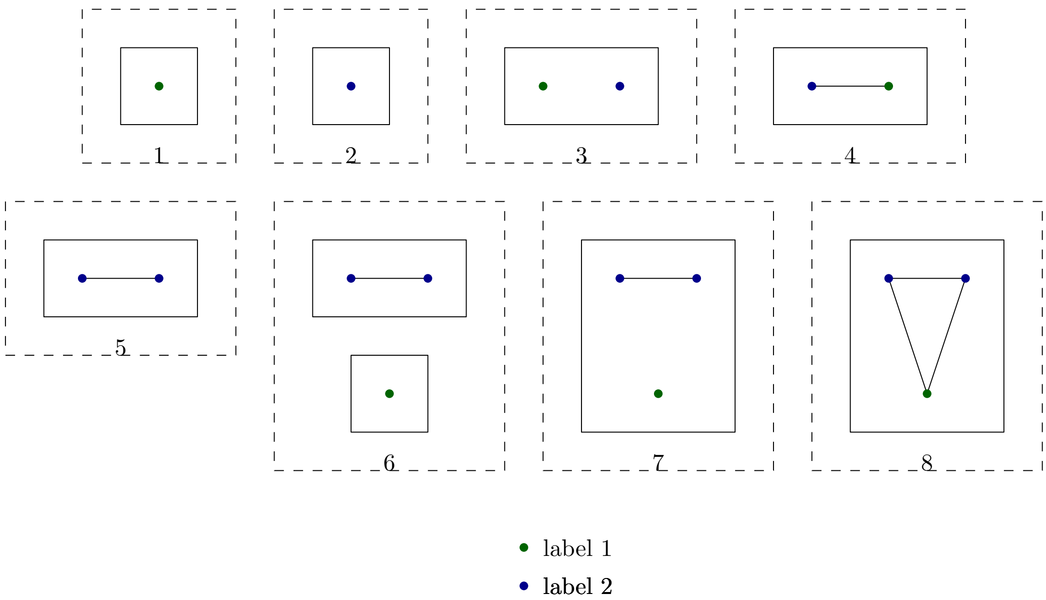


Figure 2.6: Creating a triangle in 8 operations.

Chapter 3

Graph Coloring

3.1 Context and motivations

In 1852, Francis Guthrie, a South African mathematician and botanist, while coloring a map of the counties of England in a way such that no two counties sharing a border would receive the same color (see Figure 3.1), noticed in his example that four colors were required. He conjectured from this in 1852 that four colors would be sufficient to color any map as described above. This map coloring problem is the origin of graph coloring. In fact, this exact problem can be seen as a graph coloring problem as follows. For every region of the map put exactly one point in its center. Add an edge between two points whenever the corresponding regions share a border. Finally, ask to assign a color to every vertex such that any two adjacent vertices get different colors. What is the minimum number of color needed to color such a graph? A *planar graph* is a graph that can be drawn on the plane in such a way that no edges cross each other. The Four Color Conjecture asked the following, any planar graph can be colored with four colors. This problem, solved since 1977 by Appel and Haken [4], started a very important field in graph theory.

Formally, the coloring problem can be stated as follows. For any integer $k \geq 1$, a *k-coloring* of a graph G is an assignment of at most k colors to the vertices of G. More formally it is a mapping $c : V(G) \to \{1, \ldots, k\}$. A *proper k-coloring* is a k-coloring satisfying $c(u) \neq c(v)$ for any two adjacent vertices u and v. A graph is said to be *k-colorable* if it admits a proper k-coloring. It is then natural to define the minimum number of colors needed to properly color the graph. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable.

More generally, graph coloring is a way to formalize a conflict problem in discrete mathematics. Whenever two elements are in conflict, put an edge between them and ask for them to not have the same color. From this setting, natural questions arise. Can we provide the upper and lower bounds on the number of colors needed to respect all the constraints? Is it possible to find a color for every element with a fast algorithm? Rough upper and lower bounds can be obtained easily. Remark that in a clique of size

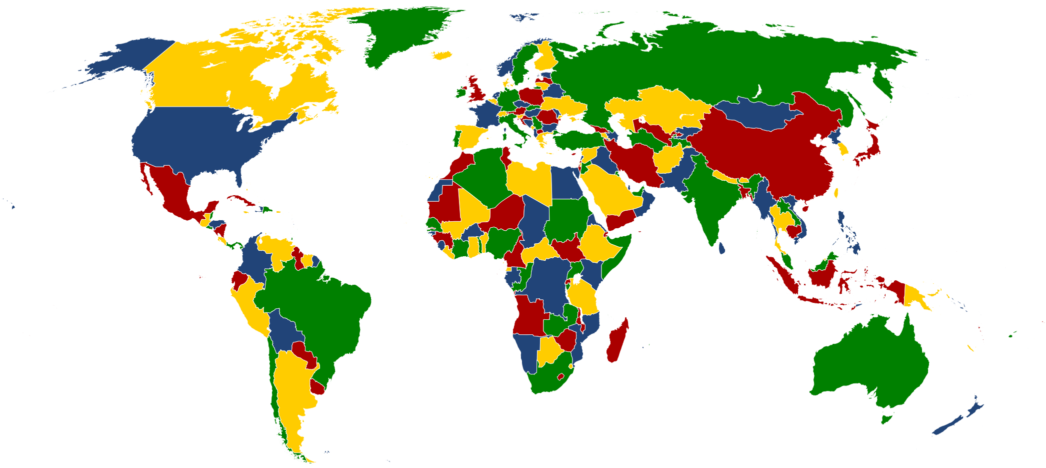


Figure 3.1: A four coloring of the world map

n, the number of colors needed is n. Hence we obtain the following lower bound on the chromatic number, $\omega(G) \leq \chi(G)$. It can be proved by induction on the number of vertices of G that $\chi(G) \leq \Delta(G)+1$. Pick a vertex v, color $G \setminus \{v\}$ by induction and assign to v a color not present in its neighborhood. Hence we obtain the trivial inequality:

$$\omega(G) \leq \chi(G) \leq \Delta(G)+1.$$

Coloring properly a graph is a way of grouping together vertices that can have the same color. An ideal way of achieving this is to group them in stable sets of maximum size as follows. Pick a stable set of maximum cardinality S and assign to every vertex of S the same color. Delete S from the graph and repeat this until there is no more vertex in the graph. This procedure gives the following other lower bound:

$$\frac{|V(G)|}{\alpha(G)} \leq \chi(G).$$

It is quite natural to ask how the chromatic number behaves whenever we forbid big sets of pairwise adjacent vertices? It is easy to see that $\chi(K_n) = n$. One could wonder if we can upper bound the chromatic number in terms of the clique number. Even though it might appear to be counter intuitive, this is false. In fact, the chromatic number can be arbitrarily larger than the clique number. Mycielski [75] provided a famous iterative construction of a family of graphs for which (see Figure 3.2 for an example), given any integer $k \geq 1$ there exists a graph G in this family such that $\omega(G) \leq 2$ and $\chi(G) = k$.

Mycielski graphs are defined inductively as follows. The first Mycielski graph M_1 is the single vertex graph. The second Mycielski graph M_2 is isomorphic to K_2. For $k \geq 2$, let $V(M_k) = \{v_1, v_2, \ldots, v_n\}$ where n is the total number of vertices. The $k+1^{\text{th}}$ Mycielski graph, M_{k+1}, is obtained from M_k by doing the following operations:

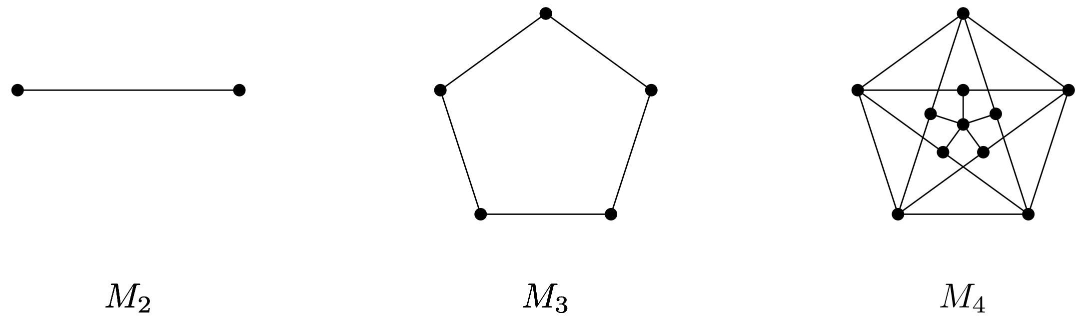


Figure 3.2: The first three Mycielski graphs.

1. Create a copy w_i of every vertex v_i and add an additional vertex z;
2. For each copied vertex w_i, put an edge between w_i and every neighbor of v_i;
3. Add an edge between z and every copied vertex w_i.

In other words, M_{k+1} is defined by $V(M_{k+1}) = V(M_k) \cup \{w_1, \dots, w_n, z\}$ and $E(M_{k+1}) = E(M_k) \cup \{w_i v_j \mid v_i v_j \in E(M_k)\} \cup \{w_i z \mid 1 \leq i \leq n\}$.

THEOREM 3.1 [75]

The Mycielski graph M_k, $k \geq 1$, is triangle-free and has chromatic number k.

Proof. First, let us show that any Mycielski graph is triangle-free. The first Mycielski graph is the single vertex graph. We proceed by induction on $k \geq 2$. The base case, M_2, is obviously triangle-free since it is isomorphic to K_2. Let us show that M_{k+1} is triangle-free. The set W of copied vertices is a stable set of M_{k+1}. The vertex z is only adjacent to vertices of W. So z is not contained in any triangle. If there is a triangle T in M_{k+1}, two vertices of T must be in $V(M_k)$ and the third vertex is in W. Let $V(T) = \{w_i, v_j, v_k\}$. Since w_i is adjacent to v_j and v_k, it follows from the definition of M_{k+1} that v_i, v_j and v_k are pairwise adjacent. Hence, $\{v_i, v_j, v_k\}$ induces a triangle in M_k, a contradiction.

It remains to show that $\chi(M_k) = k$ for every $k \geq 1$. The first Mycielski graph is the single vertex graph and has chromatic number 1. We proceed by induction on $k \geq 2$. The base case M_2 has chromatic number 2 since it is isomorphic to K_2. Let us prove that $\chi(M_{k+1}) = k + 1$. By the induction hypothesis we can color the vertices of $V(M_k)$ with k colors. Now assign to every vertex w_i the same color as v_i and assign an additional color to z. We obtain that $\chi(M_{k+1}) \leq k + 1$. It suffices to show now that $\chi(M_{k+1}) \geq k + 1$. By the induction hypothesis, k different colors appear in M_k. Furthermore, for every color c, there exists a vertex v_i of M_k, for some $i \in \{1, \dots, n\}$ depending on c, whose neighborhood in M_k contains all other $k - 1$ colors, otherwise we could recolor all vertices colored c in M_k and reach a contradiction on the chromatic number of M_k. Since w_i and v_i have the same neighborhood in M_k, it follows that k different colors appear in W. Finally, z being adjacent to every vertices of W,

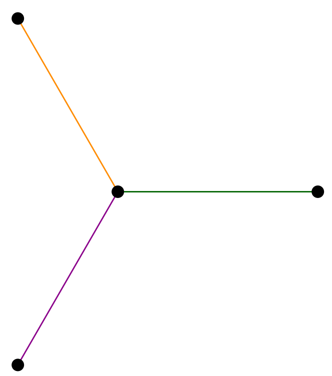


Figure 3.3: Edge coloring example.

needs an additional color. Which gives that $\chi(M_{k+1}) \geq k+1$, and the conclusion follows. $\square$

One could wonder what specific graphs G are such that $\omega(G) = \chi(G)$? And which graphs G are such that $\omega(G) < \chi(G)$? We will discuss this matter in the next subsection.

Edge coloring of a graph is the analogous version of the vertex coloring, applied to the edge set. A *k-edge-coloring* of a graph G is an assignment of k colors to the edges of G, i.e. a mapping $C : E(G) \to \{1, \dots, k\}$. Similarly, a *proper k-edge-coloring* is a k-coloring of the edges verifying $c(e_1) \neq c(e_2)$ for any two edges e_1 and e_2 sharing at least one common vertex (see Figure 3.3). The *chromatic index* of a graph G, denoted by $\chi'(G)$, is the smallest integer k such that G is k-edge-colorable. A trivial lower bound on the chromatic index is given by the maximum degree. Given a graph G, $\Delta(G) \leq \chi'(G)$. In fact, the chromatic index of simple graphs cannot be far from the maximum degree. In 1964, Vizing [93] proved the following theorem.

THEOREM 3.2 [93]

Let G be a simple graph, then $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$.

A *multigraph* is a graph that can have multiple edges between pair of vertices. The *multiplicity* of a graph G, denoted by $\mu(G)$ is the maximum number of edges in any bundle of parallel edges. In a multigraph,the chromatic index is linked to both the maximum degree and the multiplicity. Vizing proved the following more general theorem.

THEOREM 3.3 [93]

Let G be a multigraph, then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$.

Remark that Theorem 3.2 is a special case of Theorem 3.3 where $\mu(G) = 1$. Given a graph H, the *line-graph* of H, denoted by $\mathcal{L}(H)$, is the graph whose vertices are the edges of H and whose edges are the pairs of adjacents edges of H, see Figure 3.4.

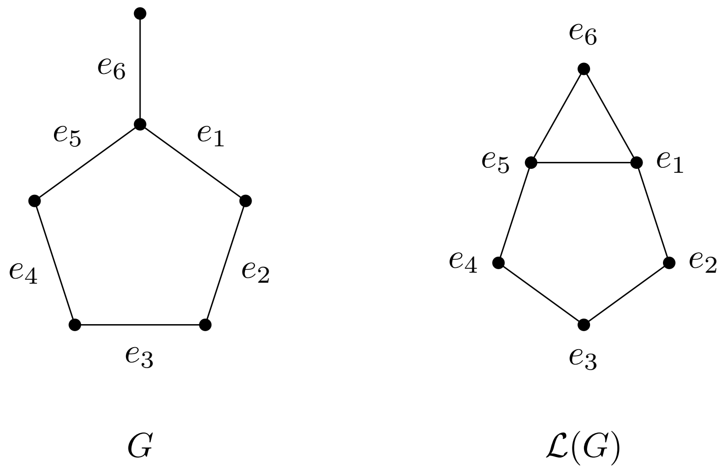


Figure 3.4: A graph G and its line-graph, $\mathcal{L}(G)$.

Edge coloring can be restated in terms of line-graphs. Given a graph G, coloring the edges of G is equivalent to color the vertices of its line-graph $\mathcal{L}(G)$.

3.1.1 Perfect graphs

The birth of perfect graphs takes root in the work of Shannon in [86] concerning zero error capacity of a noisy channel. A *perfect graph* is a graph G such that every induced subgraph H of G satisfies $\chi(H) = \omega(H)$. It is Claude Berge, motivated by the Shannon capacity, who initiated the study of perfect graphs. One remarkable moment in the history of perfect graphs is in 1961 when Claude Berge [7] formulated two very famous conjectures (both of them proved by now) about perfect graphs. The first one is the following.

- A graph G is perfect if and only if its complement is perfect.

This was proved by Lovász [65] using the so called Replication Lemma which we restate here for its self interest. Let G be a graph and v a vertex of G. We say that G' is obtained from G by replicating v if G' is obtained by adding a new vertex v' adjacent to v and to all the neighbors of v in G (v' is also called a *twin* of v).

Lemma 3.4 [65] Replication Lemma

If G is a perfect graph and G' is obtained from G by replicating a vertex v of G, then G' is perfect.

Let $\ell \geq 3$ be an integer, the *cycle* on ℓ vertices is the graph C_ℓ with $V(C_\ell) = \{v_1, \ldots, v_\ell\}$ and $E(C_\ell) = \{v_1v_2, v_2v_3, \ldots, v_{\ell-1}v_\ell, v_\ell v_1\}$. A *hole* of a graph G is an induced subgraph of G which is isomorphic to a cycle on at least four vertices. An *antihole* of a graph G is an induced subgraph of G whose complement is a hole in $\overline{G}$. An *odd hole* is a hole on an odd number of vertices and an *odd antihole* is an antihole on

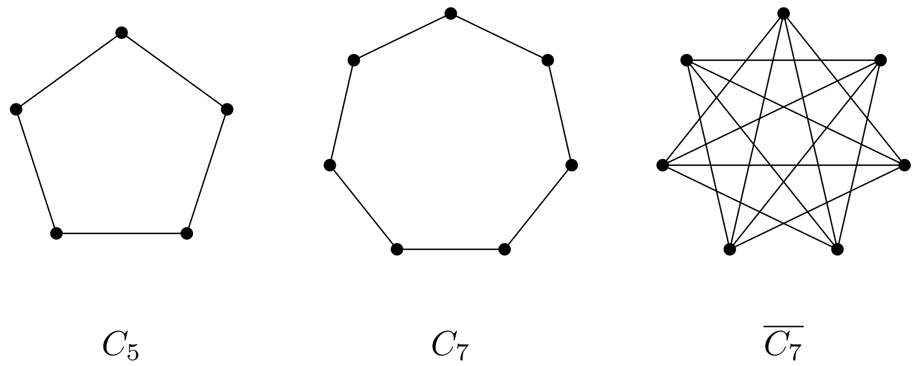


Figure 3.5: From left to right: C_5, C_7, $\overline{C_7}$.

an odd number of vertices. See Figure 3.5 for some small examples of odd holes and odd antiholes. A *Berge graph* is a graph that does not contain any odd hole nor odd antihole. The second conjecture, which is the most famous of the two, can be stated as follows.

- A graph is perfect if and only if it is Berge.

This was proved in 2002 by Chudnovsky, Robertson, Seymour and Thomas[18].

THEOREM 3.5 [18]

A graph is perfect if and only if it is Berge.

A polynomial-time algorithm computing the maximum-weight stable set in any perfect graph was designed by Grötchel, Lovász and Schrijver [40] in 1981.

THEOREM 3.6 [40]

The maximum-weight stable set problem can be solved in polynomial time in the class of perfect graphs.

Their algorithm relies on *semi-definite programming* and more precisely on the ellipsoid method. Let $\mathcal{C}$ be a subclass of perfect graphs for which there exists an algorithm that computes a maximum-weight stable set and a maximum-weight clique in $\mathcal{O}(n^k)$ for any graph of $\mathcal{C}$. Gröstchel, Lovász and Schrijver manufactured an algorithm that given a graph of $\mathcal{C}$, computes an optimal coloring for it by using the maximum weighted stable set and clique algorithm as a black box:

THEOREM 3.7 [41]

There exists an algorithm of complexity $\mathcal{O}(n^{k+2})$ whose input is a graph from $\mathcal{C}$ and whose output is an optimal colouring of G.

We refer the reader to the survey of Nicolas Trotignon [90] for a more in-depth overview regarding perfect graphs.

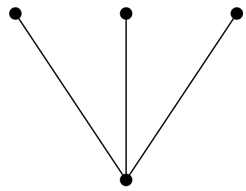


Figure 3.6: The claw graph.

However, this is not a practical algorithm as it relies on the *ellipsoid* method. This algorithm is not considered *combinatorial* in the sense that a part of it relies on semi-definite programming and not on structural aspects of graphs. There is no formal definition of what a combinatorial algorithm is, but one could say that it is an algorithm using purely graph theoretic approaches such as graph decomposition and graph searches (note that linear programming is also considered combinatorial). Hence, one of the famous and still open problem concerning perfect graphs is the following. Does there exist a purely combinatorial algorithm coloring optimally any perfect graph? Several authors managed to answer in the affirmative for a few subclasses of perfect graphs. A class of graphs $\mathcal{G}$ is *hereditary* if for every $G \in \mathcal{G}$ and every induced subgraph H of G, H is also in $\mathcal{G}$. It is common to consider graph classes defined by forbidden induced subgraphs, we then talk about the class of $\mathcal{F}$-free graphs for a given family $\mathcal{F}$ of forbidden graphs. The *bull* is the graph G with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_2v_3, v_1v_3, v_1v_4, v_2v_5\}$ (see Figure 3.8). For instance, de Figueiredo and Maffray provided algorithms solving different optimization problems in bull-free perfect graphs, including the MWSS problem and the coloring problem. Later, Penev [82] improved the complexity of these algorithms and proved the following theorem.

THEOREM 3.8 [82]

The k-coloring problem can be solved in time $\mathcal{O}(n^8)$ in the class of bull-free perfect graphs.

Although it is known that the k-coloring problem, and even computing the chromatic number, of a perfect graph can be done in polynomial time, what about combinatorial coloring algorithms in various subclasses of perfect graphs? The *claw* is the graph composed of three pairwise non-adjacent vertices S and an additional one complete to S (see Figure 3.6). Given two integer p and q, the *complete bipartite*, denoted by $K_{p,q}$, is the bipartite graph with vertex set $V(K_{p,q}) = \{X \cup Y\}$ where $|X| = p$, $|Y| = q$ and there is all possible edges between X and Y.

The claw is isomorphic to the graph $K_{1,3}$. Hsu produced a polynomial-time algorithm to compute an optimal coloring for any claw-free perfect graph.

THEOREM 3.9 [50]

An optimal proper coloring of any claw-free perfect graph can be computed in time $\mathcal{O}(n^4)$.

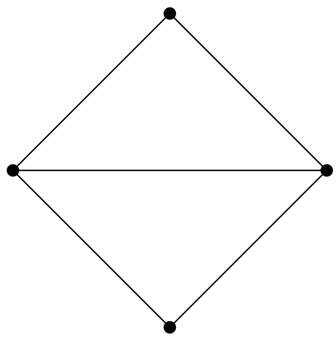


Figure 3.7: The diamond.

The *diamond* is the graph isomorphic to K_4 minus an edge (see Figure 3.7). It is also noted $K_4 - e$. Tucker provided a polynomial time k-coloring algorithm for any positive integer k.

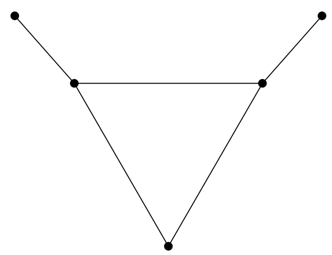


Figure 3.8: The bull.

Theorem 3.10 [91]

Any perfect diamond-free graphs G can be optimally colored in time $\mathcal{O}(\omega(G)n^2)$.

A *chordal graph* is a graph without any hole or antihole. A *weakly chordal graph* is a graph without any C_k or $\overline{C_k}$ for $k \geq 5$. Hayward et al. provided a coloring algorithm for the class of weakly chordal graphs.

Theorem 3.11 [44]

There exists an algorithm which, given any weakly chordal graph G on n vertices and m edges, returns a coloring of G with $\omega(G)$ colors in time $\mathcal{O}(n^4 + n^3m)$.

The *square* is the graph isomorphic to C_4. Chudnovsky et al. proved the following theorem, tightening even more what is left to do.

Theorem 3.12 [15]

There exists an algorithm which, given any square-free perfect graph G on n vertices, returns a coloring of G with $\omega(G)$ colors in time $\mathcal{O}(n^9)$.

The following section surveys some of the well known results concerning coloring in the class of P_ℓ-free graphs and explains why this is an interesting class for the k-coloring problem.

3.1.2 P_ℓ-free graphs

One starting point concerning this problem is the following theorem of Holyer.

Theorem 3.13 [48]

For any fixed $k \geq 3$, the k-coloring problem is NP-Complete for the class of line-graphs.

In fact, Holyer proved that deciding if a graph whose all vertices are exactly of degree 3, is 3 or 4-edge colorable is NP-Complete. Its proof involves a reduction from the 3-SAT problem. It is easy to see that that line-graphs are included in the class of claw-free graphs. Hence, we can easily deduce the following observation.

P_6

Figure 3.9: The P_6 graph.

OBSERVATION 3.14

For any fixed $k \geq 3$, the k-coloring problem is NP-Complete for the class of H-free graphs where H contains the claw.

Another important theorem is due to Kaminski and Lozin.

THEOREM 3.15 [53]

For every $k, g \geq 3$, the problem of k-coloring graphs with girth at least g is NP-Complete.

Let $\ell \geq 1$ be an integer, the *path* on ℓ vertices is the graph P_ℓ with $V(P_\ell) = \{v_1, \ldots, v_\ell\}$ and $E(P_\ell) = \{v_1v_2, v_2v_3, \ldots, v_{\ell-1}v_\ell\}$ (see Figure 3.9). When considering a cycle C or a path P, a *chord*, is an edge not included in the edge set of C or P whose endpoints are in the vertex set of C or P. From the previous results stated above, we can deduce the following observation.

COROLLARY 3.16

For any fixed $k \geq 3$ and graph H which is not a disjoint union of paths, deciding whether an H-free graph is k-colorable is NP-Complete.

Hence, among H-free graphs where H is any given graph, the only graph classes worth taking a look at concerning the k-colorability problem are the one forbidding induced paths. Many results came to light during the last few decades. Here are some of the most important ones. Kral' et al. proved the following:

THEOREM 3.17 [61]

Given a graph H and an integer k. Determining if the chromatic number of a H-free graph is at most k is polynomial-time solvable if H is an induced subgraph of P_4 or of $P_3 \cup K_1$, and NP-Complete for any other H.

Corneil et al. settled the coloring problem for any P_4-free graph.

THEOREM 3.18 [22]

Computing the chromatic number of any P_4-free graph can be done in polynomial time.

A generalization of vertex coloring, called here *k-restricted-coloring* is defined as follows. Assign to each vertex a list of colors which is a subset of $\{1, \ldots, k\}$. Is it possible to find a proper coloring of the vertices such that each vertex picks a color from its authorized colors list. The case of P_5-free graphs has been settled by Hoàng et al.

THEOREM 3.19 [46]

The k-restricted-coloring problem can be solved in polynomial time in any P_5-free graph.

Randerath and Schiermeyer [83] proved that the 3-coloring problem can be decided in polynomial time in the class of P_6-free graphs. This was later improved by Broersma et al. [13]. This has been generalized to the class of P_7-free graphs by Bonomo et al.

THEOREM 3.20 [9]

For any P_7-free graph, the 3-coloring problem can be decided in polynomial time.

On the other hand, Huang settled complexity results when either $k \geq 4$ and $\ell \geq 7$ or $k \geq 5$ and $\ell \geq 6$.

THEOREM 3.21 [51]

The k-coloring problem is NP-Complete for P_ℓ-free graphs when either $k \geq 4$ and $\ell \geq 7$ or $k \geq 5$ and $\ell \geq 6$.

Which leads us to the fact that the open cases are when $k = 3$ and $\ell \geq 8$ and when $k = 4$ and $\ell = 6$ since their complexity status is still unknown. The following table sums up the state of the art concerning the k-coloring of P_ℓ-free graphs.

$\ell \backslash k$	≤ 2	3	4	≥ 5
≤ 4	P	P	P	P
5	P	P	P	P
6	P	P	**?**	NPC
7	P	P	NPC	NPC
≥ 8	P	**?**	NPC	NPC

A few results are known in subclasses of P_6-free graphs. Hell et al. [45] proved that deciding whether a (P_6, C_4)-free graph is 4-colorable can be done in polynomial time and gave the full list of forbidden induced subgraphs characterizing the 4-colorable (P_6, C_4)-free graphs. Chudnovsky et al. [16] gave a polynomial-time algorithm that decides if a (P_6, C_5)-free graph is 4-colorable. Another interesting result is that of Brause et al. [12] who gave a polynomial-time algorithm that decides if a (P_6, bull, Z_2)-free graph or a (P_6, bull, kite)-free graph is 4-colorable, where Z_2 and the kite are the graphs depicted in Figure 3.10. A natural generalization of the two previous graph classes is the class of (P_6, bull)-free graphs. A (P_6, bull)-free graph, can have a Z_2 or a kite graph as an induced subgraph. Section 3.2 and 3.3 will be dedicated to the study of this class of graphs.

Simplifications of one of our coloring procedure rely on the following theorem.

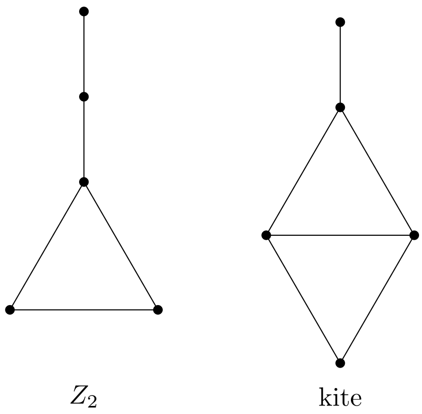


Figure 3.10: The graph Z_2 and the kite.

THEOREM 3.22

Every P_4-free graph G satisfies the following two properties:

- *If G has at least two vertices, then it has a pair of twins [22].*
- *G has clique-width at most 2 [25].*

Finally, we need the following result of Brandstädt et al. [10]. They established that (P_6, K_3)-free graphs have bounded clique-width and that a c-expression can be computed efficiently.

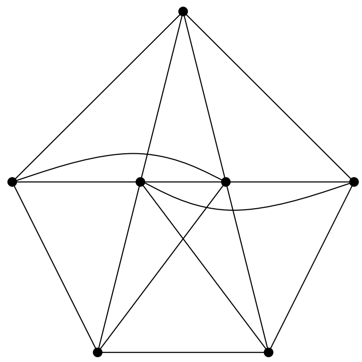
THEOREM 3.23 [10]

The class of (P_6, K_3)-free graphs has bounded clique-width c, and a c-expression can be found in time $O(|V(G)|^2)$ for every graph G in this class.

This chapter aims at providing a polynomial-time algorithm for the 4-coloring problem of (P_6, bull)-free graphs and a polynomial-time algorithm for the k-coloring problem of (P_6, bull, gem)-free graphs (see Figure 3.14 for the gem). In order to do this, we describe the structure of these graphs in the following section.

3.2 Structure of (P_6, bull)-free graphs

The goal of this section is to give a structural description of (P_6, bull)-free graphs that suits well with the coloring problem. A *quasi-prime graph* G is a graph for which every proper homogeneous set of G is a clique. In a quasi-prime graph, every proper homogeneous set consists of pairwise twins.



double-wheel

Figure 3.11: The double-wheel graph.

3.2.1 General structure

The *double-wheel* graph is the graph G with $V(G) = \{v_1, v_2, \dots, v_5, a, b\}$ such that $\{v_1, v_2, \dots, v_5\}$ induces a C_5 and a, b two adjacent vertices complete to the C_5. See Figure 3.11.

LEMMA 3.24

In order to decide in polynomial time the 4-colorability of a (P_6, bull)-free graph it suffices to prove it for the (P_6, bull)-free graphs G that satisfy the following properties:

(a) G is connected and $\overline{G}$ is connected.

(b) G is quasi-prime.

(c) G is K_5-free and double-wheel-free.

Proof. Assume that we want to determine whether a (P_6, bull)-free graph G is 4-colorable.

(a) If G is not connected we can examine each component of G separately. Now suppose that $\overline{G}$ is not connected. So $V(G)$ can be partitioned into two non-empty sets V_1 and V_2 that are complete to each other. It follows that $\chi(G) = \chi(G[V_1]) + \chi(G[V_2])$. A necessary condition for G to be 4-colorable is that $G[V_i]$ is 3-colorable for each $i = 1, 2$. Using the algorithms from [13] or [83] we can test whether $G[V_1]$ and $G[V_2]$ are 3-colorable. If any of them is not 3-colorable we declare that G is not 4-colorable and stop. If each $G[V_i]$ is 3-colorable, we can determine the value of $\chi(G[V_i])$ by further testing whether $G[V_i]$ is either edgeless or bipartite. Hence we can determine if G is 4-colorable (and if it is, give a 4-coloring) in polynomial time.

(b) Suppose that G is not quasi-prime. So G has a homogeneous set X that is not a clique and $X \neq V(G)$. Since G and $\overline{G}$ are connected X is included in a maximal module. Hence let us consider any maximal module M of G that is not a clique. We know that $M \neq V(G)$, so the set $N(M)$ is not empty, and $N(M)$ is complete to M. So a necessary condition for G to be 4-colorable is that $G[M]$ is 3-colorable. Using the algorithms from [13] or [83] we can determine whether $G[M]$ is 3-colorable or not. If it is not we declare that G is not 4-colorable and stop. If $G[M]$ is 3-colorable, we can determine the value of $\chi(G[M])$ by further testing whether $G[M]$ is either edgeless or bipartite. Then we build a new graph G' from G by removing M and adding a clique K_M of size $\chi(G[M])$ with edges from every vertex of K_M to every vertex in $N(M)$ and no other edge. Thus K_M is a homogeneous set in G', with the same neighborhood as M in G. We observe that:

$$G' \text{ is } P_6\text{-free and bull-free.} \tag{3.1}$$

Proof: If G' has an induced subgraph H that is either a P_6 or a bull, then H must contain a vertex v from K_M (because $G' \setminus K_M = G \setminus M$), and H does not contain two vertices from K_M since H has no twins. Then, replacing v with any vertex from M yields an induced P_6 or bull in G, a contradiction. So (3.1) holds.

We repeat this operation for every maximal module of G that is not a clique. Hence we obtain a graph G'' where every such module M has been replaced with a clique K_M, and, by the same argument as in (3.1), G'' is P_6-free and bull-free. For convenience we set $K_L = L$ whenever L is a maximal module of G that is a clique. We observe that:

$$G'' \text{ is quasi-prime.} \tag{3.2}$$

Proof: Suppose that G'' has a homogeneous set Y'' that is not a clique, and $Y'' \neq V(G'')$. Let $A'' = N_{G''}(Y'')$ and $B'' = V(G'') \setminus (Y'' \cup A'')$. For each vertex $x \in V(G'')$ let M_x be the maximal module of G such that $x \in K_{M_x}$. Let $Y = \bigcup_{x \in Y''} M_x$, $A = \bigcup_{x \in A''} M_x$ and $B = \bigcup_{x \in B''} M_x$. In G the set Y is complete to A and anticomplete to B, and $V(G) = Y \cup A \cup B$. So Y is a homogeneous set of G, and $Y \neq V(G)$, so there is a maximal module L of G such that $Y \subseteq L$. But this implies $Y'' \subseteq K_L$, a contradiction. So (3.2) holds.

$$G \text{ is 4-colorable if and only if } G'' \text{ is 4-colorable.} \tag{3.3}$$

Proof: Let c be a 4-coloring of G. For each maximal module M of G we have $|c(M)| \geq \chi(G[M])$. So we can assign to the vertices of K_M distinct colors from the set $c(M)$. Doing this for every M yields a 4-coloring of G''. Conversely, let c'' be a 4-coloring of G''. For every maximal module M of G, consider a $\chi(G[M])$-coloring of M and assign to each class of this coloring one color from the set $c''(K_M)$ (a different color for each class). Doing this for every M yields a 4-coloring of G. So (3.3) holds.

The operations performed to construct G'' can be done in polynomial time using modular decomposition [89] and the algorithms from [13, 83]. Since the maximal

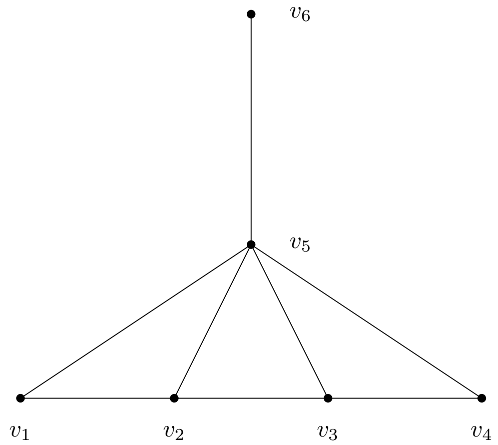


Figure 3.12: The broom.

modules of G form a partition of $V(G)$ their number is $O(|V(G)|)$. So we can ensure that property (b) holds through a polynomial time reduction.

(c) One can decide in polynomial time whether G contains K_5 or the double wheel, and if it does we stop since these two graphs are not 4-colorable. □

The complexity of testing if a P_6-free graph on n vertices is 3-colorable is $O(n^{\alpha+2})$ in [83] (where α is the exponent given by the fast matrix multiplication, $\alpha < 2.36$) and seems to be $O(n^6)$ in [13] using the special dominating set argument from [92]. Hence, by using the algorithm from [83], the total complexity of the reduction steps described in the preceding lemma is $O(n^6)$.

3.2.2 Brooms and magnets

In this subsection we prove that if a quasi-prime (P_6, bull)-free graph G contains certain special graphs (called "magnets"), then the 4-colorability of G can be solved in polynomial time using a reduction to the 2-list coloring problem.

We first show that if a (P_6, bull)-free graph G contains a certain graph which we call a broom, then either G is not quasi-prime, or the broom can be extended to subgraphs that will be convenient to us.

A *broom* is a graph with six vertices $v_1, \dots, v_6$ and edges v_1v_2, v_2v_3, v_3v_4 and v_5v_i for each $i \in \{1, 2, 3, 4, 6\}$. See Figure 3.12.

Let F_0 be the graph with seven vertices $v_1, \dots, v_7$ and edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_6v_i for all $i \in \{1, \dots, 5\}$, and v_7v_1, v_7v_2, v_7v_3, v_7v_4. See Figure 3.13.

The following lemma is an extension of Lemma 2 from [28].

LEMMA 3.25

In a bull-free graph G, let $\{v_1, \dots, v_6\}$ be a 6-tuple that induces a broom, with edges v_1v_2, v_2v_3, v_3v_4 and v_5v_i for each $i \in \{1, 2, 3, 4, 6\}$. Then one of the following holds:

- *G has a proper homogeneous set that contains $\{v_1, v_2, v_3, v_4\}$.*
- *There is a vertex z in $V(G) \setminus \{v_1, \dots, v_6\}$ that is complete to $\{v_1, v_4, v_6\}$ and anticomplete to $\{v_2, v_3, v_5\}$.*
- *There are two non-adjacent vertices z, t in $V(G) \setminus \{v_1, \dots, v_6\}$ such that z is complete to $\{v_1, v_4, v_5, v_6\}$ and anticomplete to $\{v_2, v_3\}$ and t is complete to $\{v_1, v_2, v_3, v_4\}$ and anticomplete to $\{v_5, v_6\}$ (and so $\{v_1, \dots, v_5, z, t\}$ induces an F_0).*

Proof. Let us assume that the second and third outcome of the lemma do not occur. Let $P = \{v_1, v_2, v_3, v_4\}$ and $R = V(G) \setminus P$. We classify the vertices of R as follows; let:

- $A = \{x \in R \mid x \text{ is complete to } P \cup \{v_6\}\}$.
- $B = \{x \in R \mid x \text{ is complete to } P \text{ and not adjacent to } v_6\}$.
- $F = \{x \in R \mid x \text{ is anticomplete to } P\}$.
- $X = \{x \in R \setminus F \mid N(x) \cap P \text{ is included in either } \{v_1, v_3\} \text{ or } \{v_2, v_4\}\}$.
- $Y = \{x \in R \setminus (A \cup B) \mid x \text{ is complete to } \{v_1, v_2\} \text{ or to } \{v_3, v_4\}\}$.
- $Z = \{x \in R \mid N(x) \cap P = \{v_1, v_4\}\}$.

Note that $v_5 \in A$ and $v_6 \in F$. We claim that:

$$\text{The sets } A, B, F, X, Y, Z \text{ form a partition of } R. \tag{3.4}$$

Proof: Clearly these sets are pairwise disjoint. Suppose that there is a vertex z in $R \setminus (A \cup B \cup F \cup X \cup Y \cup Z)$. Since z is not in F, it has a neighbor in P, and up to symmetry we may assume that z has a neighbor in $\{v_1, v_2\}$, and since z is not in $Y \cup A \cup B$ it has exactly one neighbor in $\{v_1, v_2\}$. Now since z is not in X, it must also have a neighbor in $\{v_3, v_4\}$, and similarly it has exactly one neighbor in $\{v_3, v_4\}$. Since z is not in $X \cup Z$, it must be that $N(z) \cap P = \{v_2, v_3\}$; but then $P \cup \{z\}$ induces a bull. So (3.4) holds.

$$F \text{ is anticomplete to } Y. \tag{3.5}$$

Proof: Suppose that there are adjacent vertices $f \in F$ and $y \in Y$. Up to symmetry y is complete to $\{v_1, v_2\}$. Then y must be adjacent to v_3, for otherwise $\{f, y, v_1, v_2, v_3\}$ induces a bull, and then to v_4, for otherwise $\{f, y, v_2, v_3, v_4\}$ induces a bull. But then y should be in $A \cup B$, not in Y. So (3.5) holds.

$$A \cup B \text{ is complete to } X. \tag{3.6}$$

Proof: Suppose that there are non-adjacent vertices $a \in A \cup B$ and $x \in X$. Up to symmetry x has exactly one neighbor in $\{v_1, v_2\}$. Then x must be adjacent to v_4, for otherwise $\{x, v_1, v_2, a, v_4\}$ induces a bull. So x is not adjacent to v_3 and, by a symmetric argument, x must be adjacent to v_1. But then x should be in Z, not in X. So (3.6) holds.

$$A \text{ is complete to } Y \cup Z. \tag{3.7}$$

Proof: Suppose that there are non-adjacent vertices a and $y \in Y \cup Z$. Suppose that $y \in Y$, say y is complete to $\{v_1, v_2\}$. By (3.5), y is not adjacent to v_6. Then y must be adjacent to v_3, for otherwise $\{y, v_2, v_3, a, v_6\}$ induces a bull, and then to v_4, for otherwise $\{y, v_3, v_4, a, v_6\}$ induces a bull. But then y should be in $A \cup B$, not in Y. Now suppose that $y \in Z$. Then y is adjacent to v_6, for otherwise $\{y, v_1, v_2, a, v_6\}$ induces a bull. But then we obtain the second outcome of the lemma, a contradiction. So (3.7) holds.

Let B' be the set of vertices b in B for which there exists in $\overline{G}$ a chordless path b_0-b_1-$\cdots$-b_k ($k \geq 1$) such that $b_0 \in Y \cup Z$, $b_1, \ldots, b_k \in B$ and $b_k = b$. Such a path will be called a *B'-path* for b.

$$B \setminus B' \text{ is complete to } Y \cup Z \cup B'. \tag{3.8}$$

This follows directly from the definition of B'.

$$A \text{ is complete to } B'. \tag{3.9}$$

Proof: Consider any $a \in A$ and $b \in B'$. Let b_0-b_1-$\cdots$-b_k be a B'-path for b, as above. By (3.7), a is adjacent to b_0. Pick any v_h in $P \cap N(b_0)$. First suppose that b_0 is not adjacent to v_6. Then for each $i \geq 1$ and by induction, a is adjacent to b_i, for otherwise $\{b_i, v_h, b_{i-1}, a, v_6\}$ induces a bull. Hence a is adjacent to b. Now suppose that b_0 is adjacent to v_6; by (3.5), this means that $b_0 \in Z$. Then a must be adjacent to b_1, for otherwise we obtain the third outcome of the lemma (where b_0, b_1 play the role of z, t). Then for each $i \geq 2$ and by induction, a is adjacent to b_i, for otherwise $\{b_i, b_{i-2}, v_6, a, b_{i-1}\}$ induces a bull. Hence a is adjacent to b. So (3.9) holds.

Let F' be the set of vertices in the components of F that have a neighbor in $X \cup Z$.

$$A \cup (B \setminus B') \text{ is complete to } F'. \tag{3.10}$$

Proof: Consider any $a \in A \cup (B \setminus B')$ and $f \in F'$. By the definition of F' there is a chordless path f_0-$\cdots$-f_k with $f_0 \in X \cup Z$, $f_1, \ldots, f_k \in F'$ and $f_k = f$. By (3.6), (3.7) and (3.8), a is adjacent to f_0. Since $f_0 \in X \cup Z$, there are non-adjacent vertices $v, v' \in P$ such that f_0 is adjacent to v and not to v'. Then a is adjacent to f_1, for otherwise $\{f_1, f_0, v, a, v'\}$ induces a bull. Then for each $i \geq 2$ and by induction, a is adjacent to f_i, for otherwise $\{f_i, f_{i-1}, f_{i-2}, a, v'\}$ induces a bull. Hence a is adjacent to f. So (3.10) holds.

$$B' \text{ is anticomplete to } F \setminus F'. \tag{3.11}$$

Proof: Consider any $b \in B'$ and $f \in F \setminus F'$, and take a B'-path b_0-$\cdots$-b_k for b as above. Vertex b_0 is not adjacent to f by (3.5) (if $b_0 \in Y$) or by the definition of F' (if

$b_0 \in Z$). There exist two adjacent vertices v_j, v_{j+1} of P such that b_0 is adjacent to exactly one of them. Vertex f is not adjacent to b_1, for otherwise $\{b_0, v_j, v_{j+1}, b_1, f\}$ induces a bull. Then for each $i \geq 2$ and by induction, f is not adjacent to b_i, for otherwise $\{f, b_i, b_{i-2}, v, b_{i-1}\}$ induces a bull, where v is any vertex in $P \cap N(b_0)$. Hence f is not adjacent to b. So (3.11) is proved.

Now let $H = P \cup X \cup Y \cup Z \cup F' \cup B'$. By (3.4), $V(G)$ is partitioned into the three sets H, $A \cup (B \setminus B')$ and $F \setminus F'$. It follows from the definition of the sets and Claims (3.6)–(3.11) that H is complete to $A \cup (B \setminus B')$ and anticomplete to $F \setminus F'$, and we know that $A \cup (B \setminus B') \neq \emptyset$ since $v_5 \in A$. So H is a homogeneous set that contains $\{v_1, v_2, v_3, v_4\}$, and it is proper since it does not contain v_5. □

We recall the variant of the coloring problem known as *list coloring*, which is defined as follows. Every vertex v of a graph G has a list $L(v)$ of allowed colors; then we want to know whether the graph admits a coloring c such that $c(v) \in L(v)$ for all v. When all lists have size at most 2 we call it a 2-list coloring problem; it is known that such a problem can be solved in linear time in the size of the input (the number of lists), as it is reducible to the 2-satisfiability of Boolean formulas, see [5].

Let us say that a subgraph F of G is a *magnet* if every vertex x in $G \setminus F$ has two neighbors $u, v \in V(F)$ such that $uv \in E(F)$.

LEMMA 3.26

If a graph G contains a magnet of bounded size, the 4-coloring problem can be solved on G in linear time.

Proof. Let F be a magnet in G. We try every 4-coloring of F. Since F has bounded size there is a bounded number of possibilities. We try to extend the coloring to the rest of the graph as a list coloring problem. Every vertex v in $G \setminus F$ has a list $L(v)$ of available colors, namely the set $\{1, 2, 3, 4\}$ minus the colors assigned to the neighbors of v in F. Since F is a magnet every list has size at most 2. So coloring $G \setminus F$ is a 2-list coloring problem, which can be solved in linear time by reducing it to the 2-satisfiability problem. □

In a graph G, let $\sim$ be the relation defined on the set $E(G)$ by putting $e \sim f$ if and only if e and f have a common vertex and $e \cup f$ induces a P_3 in G. We say that G is *P_3-connected* if it is connected and for any two edges $e, f \in E(G)$ there is a sequence e_0, $e_1, \ldots, e_k$ of edges of G such that $e_0 = e$, $e_k = f$, and for all $i \in \{0, \ldots, k-1\}$ $e_i \sim e_{i+1}$. (In other words, G is P_3-connected if it is connected and $E(G)$ is the unique class of the equivalence closure of $\sim$.)

LEMMA 3.27

Let G be a bull-free graph and let F be a P_3-connected induced subgraph of G. Suppose that there are adjacent vertices x, y in $G \setminus F$ such that x is anticomplete to F, and y has two adjacent neighbors a, b in F. Then y is complete to F.

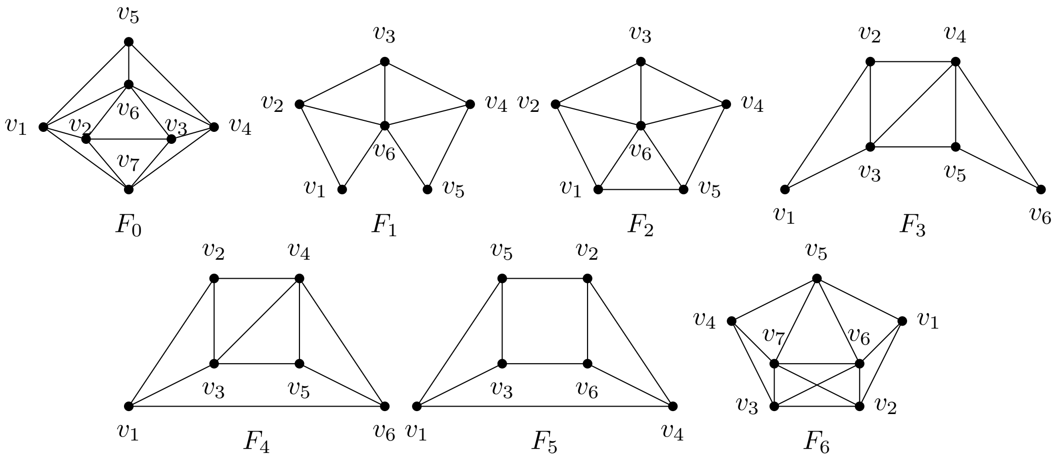


Figure 3.13: Magnets

Proof. Let a, b be two adjacent neighbors of y in F. Suppose that y has a non-neighbor c in F. Since F is P_3-connected, there is a sequence e_0, e_1, …, e_k of edges of F such that $e_0 = \{a, b\}$, e_k contains c, and for all $i \in \{0, \dots, k-1\}$ the edges e_i and e_{i+1} have a common vertex and $e_i \cup e_{i+1}$ induces a P_3. Then there is an integer i such that y is complete to the two ends of e_i and not complete to the two ends of e_{i+1}, say $e_i = uv$ and $e_{i+1} = vw$; but then $\{x, y, u, v, w\}$ induces a bull, a contradiction. □

We define six more graphs as follows (see Figure 3.13):

- Let F_1 be the graph with vertices $v_1, \dots, v_6$ and edges v_1v_2, v_2v_3, v_3v_4, v_4v_5 and v_6v_i for all $i \in \{1, \dots, 5\}$.
- Let F_2 be the graph obtained from F_1 by adding the edge v_1v_5.
- Let F_3 be the graph with vertices $v_1, \dots, v_6$ and edges v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4, v_3v_5, v_4v_5, v_4v_6 and v_5v_6.
- Let F_4 be the graph obtained from F_3 by adding the edge v_1v_6.
- Let $F_5 = \overline{C_6}$.
- Let F_6 be the graph with vertices $v_1, \dots, v_7$ and edges v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_6v_1, v_6v_2, v_6v_3, v_6v_5, v_7v_2, v_7v_3, v_7v_4, v_7v_5 and v_7v_6.

The *gem* is the graph with vertices $v_1, \dots, v_5$ and edges v_1v_2, v_2v_3, v_3v_4 and v_5v_i for all $i \in \{1, 2, 3, 4\}$ (see Figure 3.14). Recall the graph F_0 is defined at the beginning of this subsection. It is easy to check that each of F_3, F_4, F_5, F_6 and F_0 is P_3-connected.

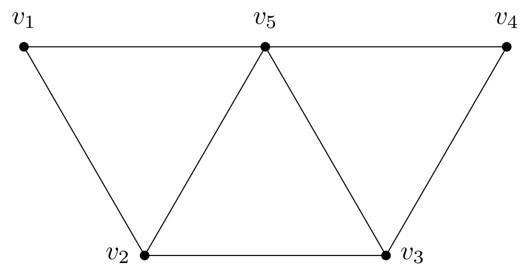


Figure 3.14: The gem graph.

LEMMA 3.28

Let G be a quasi-prime bull-free graph that contains no K_5 and no double wheel. Let F be an induced subgraph of G. Then:

- *If F is (isomorphic to) F_0, then F is a magnet in G.*
- *If G is F_0-free, and F induces a gem, with vertices $v_1, \dots, v_5$ and edges v_1v_2, v_2v_3, v_3v_4 and v_5v_i for each $i \in \{1, \dots, 4\}$, then either F is a magnet or some vertex in $G \setminus F$ is complete to $\{v_1, v_4\}$ and anticomplete to $\{v_2, v_3, v_5\}$.*
- *If G is F_0-free, and F is (isomorphic to) any of $F_1, \dots, F_6$, then F is a magnet in G.*

Proof. We use the same notation as in the definition of $F_0, F_1, \dots, F_6$.

First suppose that F is isomorphic to F_0. Suppose that F is not a magnet, so there is a vertex z in $G \setminus F$ such that $N_F(z)$ is a stable set. We claim that every such vertex satisfies $N_F(z) = \emptyset$. For suppose not. If z is adjacent to v_1, then it is also adjacent to v_3, for otherwise $\{z, v_1, v_5, v_6, v_3\}$ induces a bull, and to v_4, for otherwise $\{z, v_1, v_2, v_6, v_4\}$ induces a bull; but then $N_F(z)$ is not a stable set. So z is not adjacent to v_1, and, by a similar argument (not using v_7), z is not adjacent to any of v_2, v_3, v_4 or v_5. Then z is not adjacent to v_7, for otherwise $\{z, v_7, v_3, v_4, v_5\}$ induces a bull, and also not adjacent to v_6, for otherwise $\{z, v_6, v_5, v_4, v_7\}$ induces a bull. So the claim holds. Since G is connected, there are adjacent vertices x, y in $G \setminus F$ such that $N_F(x) = \emptyset$ and $N_F(y) \neq \emptyset$. By the same proof as for the claim, $N_F(y)$ is not a stable set. Since F_0 is P_3-connected, Lemma 3.27 implies that y is complete to $V(F)$. But then $(V(F) \setminus \{v_7\}) \cup \{y\}$ induces a double wheel, a contradiction. This proves the first item of the lemma.

Now we prove the second item of the lemma. Let F have vertices $v_1, \dots, v_5$ and edges v_1v_2, v_2v_3, v_3v_4 and v_5v_i for each $i \in \{1, \dots, 4\}$. Suppose that F is not a magnet; so there is a vertex y such that $N_F(y)$ is a stable set. First suppose that $N_F(y) \neq \emptyset$. If y is adjacent to v_5, then $F \cup \{y\}$ induces broom. By Lemma 3.25 and since G is quasi-prime (so G cannot have a homogeneous set that contains the four vertices of a P_4) and G contains no F_0, there is a vertex z complete to $\{v_1, v_4\}$ and anticomplete to $\{v_2, v_3, v_5\}$, and so the desired result holds. Now suppose that y is not adjacent to v_5; so, up to symmetry, y has exactly one neighbor in $\{v_1, v_2\}$. Then y is adjacent to v_4, for otherwise $\{y, v_1, v_2, v_5, v_4\}$ induces a bull, so y has exactly one neighbor in $\{v_3, v_4\}$, and by symmetry y is adjacent to v_1. So the desired result holds. Now suppose that $N_F(y) = \emptyset$. Since G is connected there is an edge uv such that $N_F(u) = \emptyset$ and $N_F(v) \neq \emptyset$. By the preceding argument we may assume that $N_F(v)$ is not a stable set. Suppose that v is adjacent to v_5. Up to symmetry, v is also adjacent to a vertex $w \in \{v_1, v_2\}$. Then v is adjacent to v_4, for otherwise $\{u, v, w, v_5, v_4\}$ induces a bull, and, by symmetry, to v_1, and also to v_2, for otherwise $\{u, v, v_4, v_5, v_2\}$ induces a bull, and, by symmetry, to v_3. Hence $\{u, v, v_1, v_2, v_3, v_4\}$ induces a broom, so by Lemma 3.25 there is a vertex y complete to $\{v_1, v_4\}$ and anticomplete to $\{v_2, v_3, v\}$, and so the desired result holds. Now suppose that v is not adjacent to v_5. Then v is adjacent to two adjacent vertices in $\{v_1, v_2, v_3, v_4\}$, and since $G[\{v_1, v_2, v_3, v_4\}]$ is P_3-connected Lemma 3.27 implies that v is complete to $\{v_1, v_2, v_3, v_4\}$, so $\{u, v, v_1, v_2, v_3, v_4\}$ induces a broom again and we can conclude as above.

Now we prove the third item of the lemma. First let $F = F_1$, with the same notation as in the definition. Suppose that F is not a magnet. In particular the gem induced by $F \setminus \{v_5\}$ is not a magnet, so, by the second item of this lemma, there is a vertex z complete to $\{v_1, v_4\}$ and anticomplete to $\{v_2, v_3, v_6\}$. If z is not adjacent to v_5, then $\{z, v_4, v_5, v_6, v_2\}$ induces a bull. If z is adjacent to v_5, then $\{v_1, z, v_5, v_4, v_3\}$ induces a bull, a contradiction.

Now let $F = F_2$, with the same notation as in the definition. Suppose that F is not a magnet. For each $i \in \{1, \ldots, 5\}$ the gem induced by $F \setminus \{v_i\}$ is not a magnet, so, by the second item of this lemma, there is a vertex z_i complete to $\{v_{i-1}, v_{i+1}\}$ and anticomplete to $\{v_{i-2}, v_{i+2}, v_6\}$. Then z_i is adjacent to v_i, for otherwise $\{z_i, v_{i-1}, v_i, v_6, v_{i+2}\}$ induces a bull. The vertices $z_1, \ldots, z_5$ are pairwise distinct because the sets $N_F(z_i)$ are pairwise different. Since G contains no K_5, the set $\{z_1, \ldots, z_5\}$ is not a clique, so, up to symmetry, z_1 is non-adjacent to either z_2 or z_3. If z_1 is not adjacent to z_2, then $\{z_1, v_2, z_2, v_3, v_4\}$ induces a bull. If z_1 is not adjacent to z_3, then $\{z_1, v_5, v_6, v_4, z_3\}$ induces a bull, a contradiction.

Now let $F = F_3$. (When F is F_4, F_5 or F_6 the proof is similar and we omit the details.) Suppose that there is a vertex z in $G \setminus F$ such that $N_F(z)$ is a stable set. We claim that every such vertex satisfies $N_F(z) = \emptyset$. For suppose not. If z is adjacent to v_1, then z is adjacent to v_5, for otherwise $\{z, v_1, v_2, v_3, v_5\}$ induces a bull; but then $\{z, v_5, v_6, v_4, v_2\}$ induces a bull. So, and by symmetry, z has no neighbor in $\{v_1, v_6\}$. If z has a neighbor in $\{v_2, v_3\}$, then $\{z, v_2, v_3, v_4, v_6\}$ induces a bull. So, and by symmetry, z has no neighbor in $\{v_2, v_3, v_4, v_5\}$. Thus the claim holds. (The same claim holds when F is F_4, F_5 or F_6 and we omit the details.) Since G is connected, there are adjacent vertices x, y in $G \setminus F$ such that $N_F(x) = \emptyset$ and $N_F(y) \neq \emptyset$. By the same argument as for the claim, $N_F(y)$ is not a stable set. Since F is P_3-connected, Lemma 3.27 implies that y is complete to F. Note that v_1-v_3-v_4-v_6 is an induced P_4 in F. (When $F = F_4$, use the P_4 v_2-v_1-v_6-v_5; when $F = F_5$ use any P_4 of F; when $F = F_6$ use the P_4 v_1-v_2-v_3-v_4.) Then $\{v_1, v_3, v_4, v_6, y, x\}$ induces a broom, so, since G is quasi-prime and contains no F_0, Lemma 3.25 implies the existence of a vertex z that is complete to $\{v_1, v_6, x\}$ and anticomplete to $\{v_3, v_4, y\}$. Clearly, $z \notin V(F) \cup \{x, y\}$. Then z is not adjacent to v_2, for otherwise $\{x, z, v_1, v_2, v_4\}$ induces a bull; and similarly z is not adjacent to v_5; but then $\{z, v_1, v_2, y, v_5\}$ induces a bull. (A similar contradiction occurs when F is F_4, F_5 or F_6 and we omit the details.) □

One can test in polynomial time whether a graph contains any of F_0, $F_1, \dots, F_6$. It follows from Lemmas 3.26 and 3.28 that if G is a quasi-prime (P_6, bull)-free graph that contains no K_5 and no double wheel and contains any of F_0, $F_1, \dots, F_6$, then the 4-colorability of G can be decided in polynomial time. Therefore we will assume that G contains none of F_0, $F_1, \dots, F_6$.

3.2.3 When there is no gem

This section is dedicated to give a detailed structure of G when we add the additional constraint of forbidding the gem graph. We prove the following theorem that will allow us to use known results concerning the k-coloring problem and graphs of bounded clique-width.

THEOREM 3.29

Let G be a prime (P_6, bull, gem)-free graph that contains a C_5. Then G is triangle-free.

Proof. Since G contains a C_5, there are five disjoint subsets $U_1, \dots, U_5$ of $V(G)$ such that the following properties hold for each $i \in \{1, \dots, 5\}$, with subscripts modulo 5:

- U_i is anticomplete to $U_{i-2} \cup U_{i+2}$;
- U_i contains a vertex that is complete to $U_{i-1} \cup U_{i+1}$.

Let $U = U_1 \cup \cdots \cup U_5$ and $R = V(G) \setminus U$. We choose these sets so that the set U is maximal with the above properties. For each $i \in \{1, \dots, 5\}$ let u_i be a vertex in U_i that is complete to $U_{i-1} \cup U_{i+1}$. We claim that:

$$\text{Each of } U_1, \dots, U_5 \text{ is a stable set.} \tag{3.12}$$

Proof: Suppose on the contrary and up to symmetry that U_1 is not a stable set. So $G[U_1]$ has a component X of size at least 2. Since G is prime, X is not a homogeneous set, so there is a vertex $z \in V(G) \setminus X$ and two vertices $x, y \in X$ such that z is adjacent to y and not to x, and since X is connected we may choose x and y adjacent. Remark that z is not in U_1 for otherwise it would be in X and z is not in $U_3 \cup U_4$ because $U_3 \cup U_4$ is anticomplete to U_1. Suppose that z is adjacent to u_2. Then z is adjacent to u_5, for otherwise $\{u_5, x, u_2, z, y\}$ induces a gem. Then z has no neighbor v in U_3, for otherwise $\{v, z, y, x, u_2\}$ induces a gem, and by symmetry z has no neighbor in U_4. Because z is anticomplete to $U_3 \cup U_4$ it is not in $U_2 \cup U_5$. But then the 5-tuple $(U_1 \cup \{z\}$, U_2, U_3, U_4, $U_5)$ contradicts the maximality of U (since u_2 and u_5 are complete to $U_1 \cup \{z\}$). So z is not adjacent to u_2, and, by symmetry, z is not adjacent to u_5. Then z is adjacent to u_3, for otherwise $\{z, y, x, u_2, u_3\}$ induces a bull. By symmetry z is adjacent to u_4. But now $\{u_2, u_3, z, u_4, u_5\}$ induces a bull. So (3.12) holds.

It follows easily from the definition of the sets $U_1, \dots, U_5$ and (3.12) that $G[U]$ contains no triangle. Moreover:

$$\text{There is no triangle } \{x, y, z\} \text{ with } x, y \in U \text{ and } z \in R. \tag{3.13}$$

Proof: Suppose the contrary. By (3.12) and up to symmetry, let $x \in U_1$ and $y \in U_2$. Then z is adjacent to exactly one of u_3, u_5, for otherwise $\{u_5, x, y, z, u_3\}$ induces a bull or a gem. Up to symmetry we may assume that z is adjacent to u_3 and not to u_5. Then z has no neighbor $v \in U_4$, for otherwise $\{x, y, u_3, v, z\}$ induces a gem; and z is adjacent to u_1, for otherwise $\{u_1, y, z, u_3, u_4\}$ induces a bull; and z has no neighbor $v \in U_5$, for otherwise $\{v, u_1, y, u_3, z\}$ induces a gem. It follows that the 5-tuple $(U_1, U_2 \cup \{z\}, U_3, U_4, U_5)$ contradicts the maximality of U (since u_1 and u_3 are complete to $U_2 \cup \{z\}$). So (3.13) holds.

$$\text{There is no triangle } \{x, y, z\} \text{ with } x \in U \text{ and } y, z \in R. \tag{3.14}$$

Proof: Suppose the contrary. Up to symmetry, let $x \in U_1$. Let X be the component of $N(x)$ that contains y, z. Since G is prime, X is not a homogeneous set, so there is a vertex t with a neighbor and a non-neighbor in X, and since X is connected and up to relabelling we may assume that t is adjacent to y and not to z. Vertex t is not adjacent to x, by the definition of X. By (3.13), y and z have no neighbor in $\{u_2, u_5\}$. Then t is adjacent to u_2, for otherwise $\{t, y, z, x, u_2\}$ induces a bull, and by symmetry t is adjacent to u_5. If t is adjacent to u_3, then it is also adjacent to u_4, for otherwise $\{x, u_2, t, u_3, u_4\}$ induces a bull; but then $\{u_2, u_3, u_4, u_5, t\}$ induces a gem. So t is not adjacent to u_3, and, by symmetry, t is not adjacent to u_4. If y is adjacent to u_3, then z is adjacent to u_3, for otherwise $\{u_3, y, z, x, u_5\}$ induces a bull; but then $\{u_3, y, z, u_4, t\}$ induces a bull. So y is not adjacent to u_3, and also not to u_4 by symmetry, and similarly z has no neighbor in $\{u_3, u_4\}$. But then u_3-u_4-u_5-t-y-z is an induced P_6, a contradiction. So (3.14) holds.

$$\text{There is no triangle } \{x, y, z\} \text{ with } x, y, z \in R. \tag{3.15}$$

Proof: Suppose there is a such a triangle. Since G is prime it is connected, so there is a shortest path P from U to a triangle $T = \{x, y, z\} \subseteq R$. Let $P = p_0\text{-}\cdots\text{-}p_k$, with $p_0 \in U$, $p_1, \dots, p_k \in R$, and $p_k = x$, and $k \geq 1$. We may assume that $p_0 \in U_1$. We observe that y is not adjacent to p_{k-1}, for otherwise $\{x, y, p_{k-1}\}$ is a triangle and $P \setminus p_k$ is a shorter path than P; and y has no neighbor p_i in $P \setminus \{p_k, p_{k-1}\}$, for otherwise p_0-$\cdots$-p_i is a shorter path than P from U to T. Likewise, z has no neighbor in $P \setminus p_k$. Moreover there is no edge between $P \setminus \{p_0, p_1\}$ and U for otherwise there is a path strictly shorter than P between U and T. By (3.13) p_1 has no neighbor in $\{u_2, u_5\}$ and has at most one neighbor in $\{u_3, u_4\}$; by symmetry we may assume that p_1 is not adjacent to u_4. If $k \geq 3$, then u_4-u_5-p_0-p_1-p_2-p_3 is an induced P_6. If $k = 2$, then u_4-u_5-p_0-p_1-p_2-y is an induced P_6. So $k = 1$. Then p_1 is adjacent to u_3, for otherwise u_3-u_4-u_5-p_0-p_1-y is an induced P_6. Let X be the component of $N(p_1)$ that contains y, z. Since G is prime, X is not a homogeneous set, so there is a vertex t with a neighbor and a non-neighbor in X, and since X is connected and up to relabelling we may assume that t is adjacent to y and not to z. Vertex t is not adjacent to x, by the definition of X. Then t is adjacent to p_0, for otherwise $\{t, y, z, p_1, p_0\}$ induces a bull, and t is adjacent to u_3, for otherwise $\{t, y, z, p_1, u_3\}$ induces a bull. By (3.13), t has no neighbor in $\{u_4, u_5\}$. Then u_5-u_4-u_3-t-y-z is an induced P_6. So (3.15) holds.

Claims (3.12)–(3.15) imply the theorem. □

3.2.4 When there is a gem

Since the previous section treated the case when G is gem-free, we still need to describe what happens when G contains a gem. Suppose that $v_1, \ldots, v_5$ are five vertices that induce a gem with edges v_1v_2, v_2v_3, v_3v_4 and v_5v_i for each $i \in \{1, 2, 3, 4\}$. We can define the following sets. Let $S = \{v_1, \ldots, v_5\}$ and let:

- $V_i = \{x \in V(G) \mid N_S(x) \setminus \{v_i\} = N_S(v_i)\}$ for each $i \in \{1, \ldots, 5\}$.
- $X = \{x \in V(G) \mid x$ is complete to $\{v_1, v_4\}$ and anticomplete to $\{v_2, v_3\}\}$.
- $W = \{x \in V(G) \mid x$ is anticomplete to $\{v_1, v_2, v_3, v_4\}$ and has a neighbor in $V_5\}$.
- $Z = \{x \in V(G) \mid x$ is anticomplete to $\{v_1, v_2, v_3, v_4\} \cup V_5\}$.
- $Z_1 = \{x \in V(G) \mid x$ is in any component of Z that has a neighbor in $W\}$.
- $Z_0 = Z \setminus Z_1$.

We note that constructing these sets can be done in time $O(n^2)$ by scanning adjacency lists. We are now ready to prove the following structural result knowing that G contains a gem. See Figure 3.15 for a summary of the structural description in this case.

THEOREM 3.30

Let G be a (P_6, bull)-free graph. Assume that G is quasi-prime, contains no K_5, no double wheel and no F_0, F_1, …, F_6, and that G contains a gem induced by $\{v_1, \ldots, v_5\}$. Let S, V_i ($i = 1, \ldots, 5$), X, W, Z, Z_0 and Z_1 be the sets defined as above. Then the following holds:

(a) X is not empty.

(b) X is anticomplete to $V_2 \cup V_3 \cup V_5$ and complete to $V_1 \cup V_4$.

(c) $V(G) = \bigcup_{i=1}^{5} V_i \cup W \cup X \cup Z$.

(d) V_5 is complete to $V_1 \cup \cdots \cup V_4$.

(e) W is complete to X and anticomplete to $V_1 \cup \cdots \cup V_4$.

(f) Z is anticomplete to $V_1 \cup \cdots \cup V_4$.

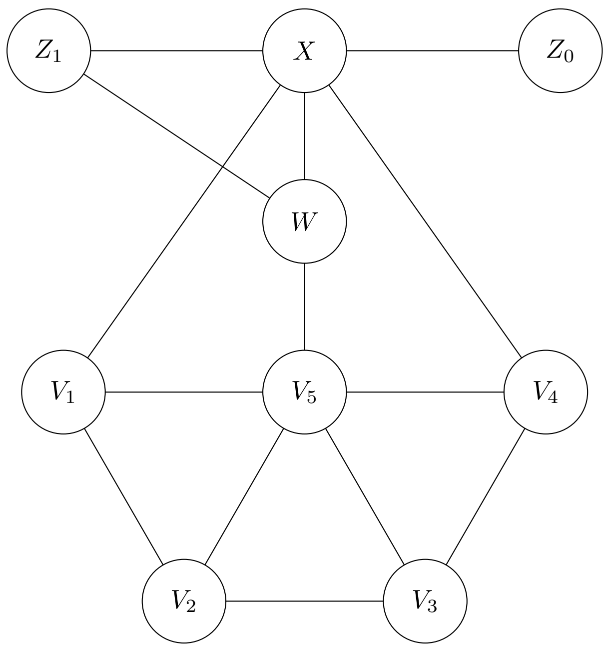


Figure 3.15: The partition of $V(G)$ as in Theorem 3.30. A line between two sets represents partial or complete adjacency. No line means that the two sets are anticomplete to each other.

(g) Z_1 *is complete to* X.

(h) *Every component of* X *is homogeneous and is a clique.*

(i) *Every component of* Z_0 *is homogeneous and is a clique.*

(j) X *is a homogeneous set in* $G \setminus Z_0$.

(k) *If* $Z_1 \neq \emptyset$, *then there is a vertex* w^* *in* W *such that* $Z_1 \cap N(w^*)$ *is complete to* $Z_1 \setminus N(w^*)$.

Proof. Note that $v_i \in V_i$ for each $i \in \{1, \dots, 5\}$. It is easy to check from their definition that the sets $V_1, \dots, V_5, X, W, Z$ are pairwise disjoint.

(a) By Lemma 3.26 we may assume that $G[S]$ itself is not a magnet (and this can easily be checked in polynomial time). Then the second item of Lemma 3.28 implies the existence of a vertex that is complete to $\{v_1, v_4\}$ and anticomplete to $\{v_2, v_3, v_5\}$, so that vertex is in X. Thus item (a) holds.

(b) Consider any $x \in X$. Suppose that x has a neighbor v in $V_2 \cup V_3 \cup V_5$. If $v \in V_5$, then $\{v_1, v_2, v_3, v_4, v, x\}$ induces an F_2, a contradiction. So x is anticomplete to V_5; in particular x is not adjacent to v_5. If v in V_2, then $\{v_1, v, v_3, v_4, v_5, x\}$ induces an F_4. The same holds if $v \in V_3$. If x has a non-neighbor u in V_1, then $\{x, v_4, v_3, v_5, u\}$ induces a bull. The same holds if $u \in V_4$. Thus (b) holds.

By (a) we pick a vertex $x_0 \in X$. By (b) x_0 is not adjacent to v_5.

(c) Let u be any vertex in $V(G)$. First suppose that u is adjacent to both v_1, v_4. Then u has exactly one neighbor in $\{v_2, v_3\}$, for otherwise u is in V_5 or X. So assume that u is adjacent to v_2 and not to v_3. Then u is adjacent to v_5, for otherwise $\{u, v_1, \dots, v_5\}$ induces an F_4, and to x_0, for otherwise $\{x_0, v_1, u, v_5, v_3\}$ induces a bull; but then $\{u, v_1, \dots, v_5, x_0\}$ induces an F_6.
Now suppose that u is non-adjacent to both v_1, v_4. Then u has exactly one neighbor in $\{v_2, v_3\}$, for otherwise either u is in $W \cup Z$ or $\{v_1, v_2, u, v_3, v_4\}$ induces a bull. So assume that u is adjacent to v_2 and not to v_3; then u is adjacent to v_5, for otherwise $\{u, v_2, v_1, v_5, v_4\}$ induces a bull; and so $u \in V_1$.
Finally suppose, up to symmetry, that u is adjacent to v_1 and not to v_4. If u is not adjacent to v_2, then it is adjacent to v_5, for otherwise $\{u, v_1, v_2, v_5, v_4\}$ induces a bull, and to v_3, for otherwise $\{u, v_1, \dots, v_5\}$ induces an F_1; and so u is in V_2. So suppose that u is adjacent to v_2. If u is not adjacent to v_3, then it is adjacent to v_5, for otherwise $\{u, v_1, \dots, v_5\}$ induces an F_3; and so u is in V_1. So suppose that u is adjacent to v_3. If u is not adjacent to v_5, then it is adjacent to x_0, for otherwise $\{u, v_3, v_5, v_4, x_0\}$ induces a bull; but then $\{u, v_1, v_3, v_4, v_5, x_0\}$ induces an F_5. So u is adjacent to v_5, and so u is in V_2. Thus (c) holds.

(d) Consider any $v \in V_5$. Suppose that v has a non-neighbor $u \in V_1 \cup V_2$. Note that $v \neq v_5$ and $u \notin \{v_1, v_2\}$. By (b), v is not adjacent to x_0. If $u \in V_1$, then u is adjacent to v_1, for otherwise $\{u, v_2, v_1, v, v_4\}$ induces a bull; but then $\{u, v, v_1, v_2, v_3, v_4\}$ induces an F_3. If $u \in V_2$, then, by (b), u is not adjacent to x_0; but then $\{u, v_3, v, v_4, x_0\}$ induces a bull. Thus (d) holds.

(e) Consider any $w \in W$. By the definition of W, w has a neighbor v in V_5. Consider any $x \in X$. By (b), v is not adjacent to x. Then w is adjacent to x, for otherwise $\{w, v, v_3, v_4, x\}$ induces a bull. So w is complete to X. Now suppose for a contradiction and up to symmetry, that w has a neighbor $u \in V_1 \cup V_2$. We know that w is adjacent to x_0 as proved just above. By (b), x_0 is not adjacent to v. By (d) v is adjacent to u. If $u \in V_1$, then $\{u, v, w, v_2, v_3, v_4\}$ induces an F_1. If $u \in V_2$, then u is adjacent to v_2, for otherwise $\{v_2, v, u, w, x_0\}$ induces a bull; but then $\{w, u, v_2, v_3, v_4\}$ induces a bull, a contradiction. Thus (e) holds.

(f) Suppose, up to symmetry, that some vertex $z \in Z$ has a neighbor $u \in V_1 \cup V_2$. If $u \in V_1$, then $\{z, u, v_2, v_5, v_4\}$ induces a bull. If $u \in V_2$, then $\{z, u, v_1, v_5, v_4\}$ induces a bull. So (f) holds.

(g) Consider any $z \in Z_1$ and $x \in X$. By the definition of Z_1, there is a path z_0-$\cdots$-z_ℓ such that $z_0 \in W$, $z_1, \ldots, z_\ell \in Z_1$ and $z = z_\ell$. We take a shortest such path, so if $\ell \geq 2$ then $z_2, \ldots, z_\ell$ are not adjacent to z_0. By (e) x is adjacent to z_0. Then by induction on $i = 1, \ldots, \ell$, and by (b) and (f), we see that x is adjacent to z_i, for otherwise z_i-z_{i-1}-x-v_1-v_2-v_3 is an induced P_6. Thus (g) holds.

(h) Suppose that some component Y of X is not homogeneous; so there are adjacent vertices $x, y \in Y$ and a vertex $z \in V(G) \setminus Y$ such that z is adjacent to y and not to x. By (b), (e) and (g) we have $z \in Z_0$. Then, by (b), $\{z, y, x, v_1, v_2\}$ induces a bull. So Y is homogeneous, and consequently Y is a clique since G is quasi-prime. Thus (h) holds.

(i) Suppose that some component Y of Z_0 is not homogeneous; so there are adjacent vertices $y, z \in Y$ and a vertex $x \in V(G) \setminus Y$ such that x is adjacent to y and not to z. By (f) and the definition of Z_0 and Y, we have $x \in X$. Then, by (b), z-y-x-v_1-v_2-v_3 is an induced P_6. So Y is homogeneous, and consequently Y is a clique since G is quasi-prime. Thus (i) holds.

(j) By (b), (e) and (g), X is complete to $V_1 \cup V_4 \cup W \cup Z_1$ and anticomplete to $V_2 \cup V_3 \cup V_5$. So (j) holds.

(k) By the definition of Z_1, some vertex w^* in W has a neighbor in Z_1. Suppose that $Z_1 \cap N(w^*)$ is not complete to $Z_1 \setminus N(w^*)$. So there are non-adjacent vertices $y \in Z_1 \cap N(w^*)$ and $z \in Z_1 \setminus N(w^*)$. By (e) and (g), x_0 is complete to $\{w^*, y, z\}$. By the definition of W, w^* has a neighbor v in V_5. By (b) and the definition of Z, v is anticomplete to $\{x_0, y, z\}$. Then $\{v, w^*, y, x_0, z\}$ induces a bull. So (k) holds. This completes the proof of the lemma. □

3.3 Coloring (P_6, bull)-free graphs

The goal of the structure detailed in the previous sections is to give useful tools to determine if a (P_6, bull)-free graph is 4-colorable and if it is, give the coloring. The main theorem of this section is the following.

THEOREM 3.31

There is a polynomial-time algorithm that determines whether a (P_6, bull)-free graph G is 4-colorable, and if it is, produces a 4-coloring of G.

We split the proof of Theorem 3.31 into two parts. Firstly, we describe what happens when G is also gem-free and secondly when G has a gem. The reason behind this is that when G has the additional property of containing no gem, its clique-width is bounded by some constant and when it does contain a gem, the structure of G is well centered around this gem. Let us first prove the following theorem.

THEOREM 3.32

For any fixed k, there is a polynomial-time algorithm that determines whether a (P_6, bull, gem)-free graph is k-colorable, and if it is, produces a k-coloring of G.

Armed with Theorem 3.29 and with the useful results concerning the clique-width of graphs as depicted in the preliminaries at the beginning of the manuscript (more precisely when the clique-width of a graph is bounded), we are ready to prove that when there is no gem, the clique-width is bounded. In [10] it is proved that the clique-width c of (P_6, K_3)-free graphs is at most 40 and claimed that one can obtain $c \leq 36$. The following theorem refers to the same constant c.

THEOREM 3.33

Let G be a $(P_6, bull, gem)$-free graph that contains a C_5. Then G has bounded clique-width c, and a c-expression can be found in time $O(|V(G)|^2)$ for every graph G in this class.

Proof. We may assume that G is connected since the clique-width of a graph is the maximum of the clique-width of its components. Suppose that $\overline{G}$ is not connected. So $V(G)$ can be partitioned into two non-empty sets V_1 and V_2 that are complete to each other. Since G is gem-free, each of $G[V_1]$ and $G[V_2]$ is P_4-free, and consequently G itself is P_4-free; so G has clique-with at most 2 by Theorem 3.22. Therefore we may assume that G and $\overline{G}$ are connected. Let $M_1, \ldots, M_p$ be the maximal modules of G. Pick one vertex m_i from each M_i, and let $G' = G[\{m_1, \ldots, m_p\}]$. Since G and $\overline{G}$ are connected we know from the theory of modular decomposition (see Section 2.4) that $M_1, \ldots, M_p$ form a partition of $V(G)$, with $p \geq 4$, and that G' is a prime graph. Clearly G' is $(P_6, \text{bull}, \text{gem})$-free since it is an induced subgraph of G. We observe that:

$$G[M_i] \text{ is } P_4\text{-free, for each } i \in \{1, \ldots, p\}. \tag{3.16}$$

Proof: Since $p \geq 2$ and G is connected there is a module M_j such that $j \neq i$ and M_j is complete to M_i. If $G[M_i]$ contains a P_4, then m_j and the four vertices of this P_4 induce a gem, a contradiction. So (3.16) holds.

Consider any prime induced subgraph H of G. We claim that:

$$H \text{ contains at most one vertex from each maximal module } M_i. \tag{3.17}$$

Proof: Suppose that H contains two vertices from some M_i. By (3.16) the subgraph of G induced by $V(H) \cap M_i$ has a pair of twins; but this contradicts the fact that H is prime. So (3.17) holds.

By (3.17), H is isomorphic to an induced subgraph of G'. By Theorems 3.29 and 3.23, H has bounded clique-width. Hence and by Theorem 2.2, G has bounded clique-width. □

Proof of Theorem 3.32. Let G be a $(P_6, \text{bull}, \text{gem})$-free graph. Since G is P_6-free it contains no C_ℓ with $\ell \geq 7$, and since it is gem-free it contains no $\overline{C_\ell}$ with $\ell \geq 7$. So if G also contains no C_5, then it is a bull-free perfect graph. In that case we can use the algorithms from either [27] or [82] to find a $\chi(G)$-coloring of G in polynomial time, and we need only check whether $\chi(G) \leq k$. (When $k = 4$, we can do a little better: by Lemmas 3.26 and 3.28 we may assume that G is also F_5-free, so G contains no $\overline{C_\ell}$ for any $\ell \geq 6$. Then we can use the algorithm from [28], which is simpler than those in [27, 82].)

Now assume that G contains a C_5. Then Theorems 3.33 and 2.1 imply that the k-coloring problem can be solved in polynomial time. □

The last case to treat is when G contains a gem. Moreover, we know that G cannot contain any magnet, otherwise the 4-coloring problem is easily solved, and it cannot contains any K_5 nor double wheel otherwise it is not 4-colorable. Hence, we will prove the following theorem.

THEOREM 3.34

Let G be a (P_6, bull)-free graph. Assume that G is quasi-prime, contains no K_5, no double wheel and no F_0, F_1, …, F_6, and that G contains a gem. Then we can determine in polynomial time whether G is 4-colorable.

Proof. Let $v_1, \ldots, v_5$ be five vertices that induce a gem, with edges v_1v_2, v_2v_3, v_3v_4 and v_5v_i for each $i \in \{1, 2, 3, 4\}$, and let V_i $(i = 1, \ldots, 5)$, X, W, Z, Z_0 and Z_1 be the sets defined as in Subsection 3.2.4. In this proof all items (a)–(k) that we invoke refer to Theorem 3.30. First, we observe that:

G is 4-colorable if and only if $G \setminus Z_0$ is 4-colorable. Moreover, given any 4-coloring of $G \setminus Z_0$ we can make a 4-coloring of G in polynomial time. (3.1)

Proof: Clearly if G is 4-colorable then $G \setminus Z_0$ is 4-colorable. So let us prove the converse and the second sentence of the claim. Let c be a 4-coloring of $G \setminus Z_0$. Let t be the maximum size of a component of X. By item (h) we may assume, up to relabeling, that the colors used on such a component are $1, \ldots, t$. By items (b), (e) and (g), these colors are not used on $V_1 \cup V_4 \cup W \cup Z_1$. So for every component Y of X we can recolor the vertices of Y with colors $1, \ldots, |Y|$. Thus we obtain a 4-coloring c' of $G \setminus Z_0$. Now we can extend c' to Z_0 as follows. Let U be any component of Z_0. By the definition of Z and Z_0 and by item (f), we have $N(U) \subseteq X$. Let Y be the largest component of X that is adjacent to U. By items (h) and (i), U and Y are complete to each other and $U \cup Y$ is a clique. Since G contains no K_5, we have $|U| \leq 4 - |Y|$. Moreover, by the choice of Y, any component Y' of X that is adjacent to U is not larger than Y, so the colors used on Y' are also used on Y. So U can be colored with the colors from $\{1, 2, 3, 4\}$ that are not used on Y. We can proceed similarly for all U. This yields a 4-coloring of G. Thus (3.1) holds.

By (3.1) we may assume that $Z_0 = \emptyset$. By item (j) we may assume that X is a clique, with $|X| \leq 3$.

Now we can describe the coloring procedure. We "precolor" a set P of vertices (of size at most 8), that is, we try every 4-coloring f of P and check whether the precoloring f extends to a 4-coloring of G. Each vertex v in $V(G) \setminus P$ has a list $L(v)$ of available colors, which consists of the set $\{1, 2, 3, 4\}$ minus the colors given by f to the neighbors of v in P. Hence we want to solve the L-coloring problem on $G \setminus P$ or determine that it has no solution.

First suppose that $|X| \geq 2$. Let $P = \{v_1, v_2, v_3, v_4, v_5\} \cup X$. So $|P| \leq 8$. It follows from items (d), (e), and (g) that every vertex in $V(G) \setminus P$ has two adjacent neighbors in P. So every vertex v in $V(G) \setminus P$ satisfies $|L(v)| \leq 2$. Hence checking whether f extends to G is a 2-list-coloring problem on the vertices of $G \setminus P$, which can be solved in polynomial time. Therefore we may assume that $|X| = 1$, and so $X = \{x_0\}$.

Let $P = \{v_1, v_2, v_3, v_4, x_0\}$. Clearly $|f(\{v_1, v_2, v_3, v_4\})| \geq 2$; moreover we may assume that $|f(\{v_1, v_2, v_3, v_4\})| \leq 3$ for otherwise the precoloring cannot be extended to V_5 and we stop examining it. We distinguish two cases.

Case 1: $|f(\{v_1, v_2, v_3, v_4\})| = 3$.
We may assume up to relabeling that $f(\{v_1, v_2, v_3, v_4\}) = \{1, 2, 3\}$. Then $L(v) = \{4\}$ for all $v \in V_5$, so V_5 must be a stable set, for otherwise the precoloring cannot be extended to V_5 and we stop examining it. So let us assume that V_5 is a stable set, and let $f(v) = 4$ for all $v \in V_5$.
Suppose that $f(x_0) = 4$. In that case we have $L(v) = \{1, 2, 3\}$ for all $v \in W \cup Z$. We can check whether $G[W \cup Z]$ is 3-colorable with the known algorithms [83, 13]. On the other hand we have $|L(u)| \leq 2$ for all $u \in V_1 \cup V_2 \cup V_3 \cup V_4$, so checking whether f extends to $V_1 \cup V_2 \cup V_3 \cup V_4$ is a 2-list coloring problem. By items (e) and (f) the two sets $V_1 \cup V_2 \cup V_3 \cup V_4$ and $W \cup Z$ are anticomplete to each other, so extending the coloring to them can be done independently.
Now suppose that $f(x_0) \neq 4$. Then every vertex in W has a list of size 2 (the set $\{1, 2, 3, 4\} \setminus \{4, f(x_0)\}$). If $Z_1 \neq \emptyset$, we pick a vertex w^* from W as in item (k) and add w^* to P; moreover, if w^* is not complete to Z_1, we pick one vertex z^* from $N_{Z_1}(w^*)$

and add z^* to P. It follows from items (d), (e), (g) and (k) that every vertex in $G \setminus P$ has a list of size 2 (in particular every vertex in Z_1 is complete to either $\{x_0, w^*\}$ or $\{x_0, z^*\}$), so we can finish with a 2-list coloring problem.

Case 2: $|f(\{v_1, v_2, v_3, v_4\})| = 2$.
We may assume up to relabeling that $f(v_1) = f(v_3) = 1$ and $f(v_2) = f(v_4) = 2$. Suppose that V_1 contains two adjacent vertices a, b. Then $\{a, b, v_2\}$ is a clique of size 3. We add a, b to the set P. By item (d), in any possible 4-coloring of G the vertices of V_5 must all have the same color, say color 4. In that case we can argue as in Case 1 and conclude. The same argument can be applied if V_2 is not a stable set, and by symmetry if V_3 or V_4 is not a stable set. Therefore we may assume that each of V_1, V_2, V_3, V_4 is a stable set.
We have $L(v) = \{3, 4\}$ for all $v \in V_5$, and we may assume, up to symmetry, that $f(x_0) = 4$. So we have $L(v) = \{1, 2, 3\}$ for all $v \in W \cup Z_1$ by items (e), (f), and (g). We may assume that all vertices in $V_1 \cup V_3$ receive color 1 and all vertices in $V_2 \cup V_4$ receive color 2, because the only other vertices that may receive color 1 or 2 are in $W \cup Z_1$ and are anticomplete to $V_1 \cup V_2 \cup V_3 \cup V_4$. Therefore we must only extend the coloring to $V_5 \cup W \cup Z_1$.
Since $L(v) = \{3, 4\}$ for all $v \in V_5$, the set V_5 must be bipartite, for otherwise the precoloring cannot be extended to V_5 and we stop examining it. So assume that V_5 is bipartite. Let $D_1, \ldots, D_t$ be the components of V_5 of size at least 2 (which we call the *big* components of V_5), if any. For each D_i, let A_i, B_i be the two stable sets that form a partition of D_i; let $W_{A_i} = \{x \in W \mid x \text{ has a neighbor in } A_i \text{ and no neighbor in } B_i\}$, $W_{B_i} = \{x \in W \mid x \text{ has a neighbor in } B_i \text{ and no neighbor in } A_i\}$, and $W_i = \{x \in W \mid x \text{ has a neighbor in each of } A_i \text{ and } B_i\}$. We claim that:

For every big component D_i of V_5, each of A_i and B_i contains a vertex that is complete to W_i. (3.2)

Proof: Let d be a vertex in B_i (the proof is similar for A_i) that has the most neighbors in W_i, and suppose that there is still a vertex $u \in W_i$ that is not adjacent to d. By the definition of W_i vertex u has a neighbor a in A_i and a neighbor b in B_i. In D_i there is a shortest path Q from a to b, of odd length. It is easy to see that D_i is P_3-connected. If a, b are adjacent, then Lemma 3.27 is contradicted by D_i, u and x_0, because u is not adjacent to d. So a, b are not adjacent, and since G is P_6-free we have $Q = a\text{-}b'\text{-}a'\text{-}b$ for some $a' \in A_i$ and $b' \in B_i$, and u has no neighbor in $\{a', b'\}$. Then d is adjacent to a, for otherwise $\{u, a, b', v_1, d\}$ induces a bull, and d is adjacent to a', for otherwise $\{u, a, d, v_1, a'\}$ induces a bull. By the choice of d some vertex v in W_i is adjacent to d and not to b. Then v is adjacent to a, for otherwise $\{v, d, a, v_1, b\}$ induces a bull, and to a', for otherwise $\{x_0, v, a, d, a'\}$ induces a bull; but then $\{x_0, v, d, a', b\}$ induces a bull, a contradiction. Thus (3.2) holds.

For every big component D_i of V_5, one of W_{A_i} and W_{B_i} is empty. (3.3)

Proof: Suppose on the contrary that some vertex u in W has a neighbor a in A_i and no neighbor in B_i and some vertex v in W has a neighbor b in B_i and no neighbor in A_i. If

a, b are adjacent, then u, v are adjacent, for otherwise $\{u, a, v_1, b, v\}$ induces a bull; but then $\{v_1, a, b, u, v, x_0\}$ induces an F_5. Hence a, b are not adjacent. Since G is P_6-free, D_i contains a chordless path a-b'-a'-b of length 3. Then $\{u, a, b', v_1, b\}$ induces a bull, a contradiction. Thus (3.3) holds.

By (3.3) we may assume that $W_{B_i} = \emptyset$ for every big component D_i of V_5. For each big component D_i of V_5, take a vertex d_i that is complete to W_i, with $d_i \in B_i$, which is possible by (3.2); so d_i is anticomplete to W_{A_i}. Let $T = \{d_1, \ldots, d_t\}$. Note that T is a stable set. Let $H = G[Z_1 \cup W \cup T \cup \{v_1, v_2\}]$. We claim that:

$$f \text{ extends to a 4-coloring of } G \text{ if and only if } H \text{ is 3-colorable.} \tag{3.4}$$

Proof: Suppose that f extends to a 4-coloring c of G. Clearly every big component D_i of V_5 satisfies either $c(A_i) = 4$ and $c(B_i) = 3$ or vice-versa. If every big component D_i of V_5 satisfies $c(A_i) = 4$ and $c(B_i) = 3$, then the restriction of c to H is a 3-coloring, using colors $1, 2, 3$. So suppose that some component D_i satisfies $c(A_i) = 3$ and $c(B_i) = 4$. Then we swap colors 3 and 4 on that component, and we claim that the result is still a proper coloring. Indeed, vertices in $V_1 \cup V_2 \cup V_3 \cup V_4$ have color 1 or 2; vertices in W_i have a neighbor in each of A_i and B_i, so their color is 1 or 2; vertices in W_{A_i} do not have color 4 since they are adjacent to x_0; and all other vertices of G are anticomplete to D_i, by the definition of Z, D_i, W_{A_i}, W_i and because $W_{B_i} = \emptyset$. So the swap does not cause any two adjacent vertices to have the same color. We can repeat this operation for every such component D_i; thus we obtain a 4-coloring of G whose restriction to H is a 3-coloring.
Conversely, suppose that H admits a 3-coloring g, with colors $1, 2, 3$. Up to relabeling we may assume that $g(v_1) = 1$ and $g(v_2) = 2$. It follows that all vertices in T have color 3. Then we extend this coloring to G as follows. Assign color 1 to all vertices in $V_1 \cup V_3$ and color 2 to all vertices in $V_2 \cup V_4$. For every big component D_i of V_5, assign color 3 to all vertices in B_i and color 4 to all vertices in A_i. Also assign color 4 to all vertices in $V_5 \setminus (D_1 \cup \cdots \cup D_t)$ and to x_0. Thus we obtain a proper 4-coloring c of G, and clearly c is also an extension of f. So (3.4) holds.

By (3.4) we need only check whether the induced subgraph H is 3-colorable, which we can do with the known algorithms [83, 13]. This completes the proof of the theorem. □

The time complexity of the coloring algorithm given in Theorem 3.34 can be evaluated as follows. We test only a fixed number of precolorings, and for each of them we need to solve either a list-2-coloring problem, which takes time $O(n^2)$, or the problem of 3-coloring a certain P_6-free subgraph of G, which takes time $O(n^3)$ in [13]. So the complexity is $O(n^3)$.

Finally, Theorem 3.31 follows from Lemmas 3.24, 3.26 and 3.28 and Theorems 3.32 and 3.34.

The complexity of our general algorithm can be evaluated as follows. Assume that we are given a (P_6, bull)-free graph G on n vertices. We first apply the reduction steps described in Lemma 3.24; the complexity is $O(n^6)$ as discussed after the proof of this lemma. Then we test in time $O(n^5)$ whether G contains a gem. Suppose that G is gem-free. Then we test whether G is perfect, which in this case is equivalent to testing whether G is C_5-free and takes time $O(n^5)$. If G is perfect, we use the algorithm from [82] to compute the chromatic number of G in time $O(n^6)$. If G is not perfect, we use the algorithm from [10], based on the fact that the clique-width is bounded, which runs in $O(n^2)$. Finally, if the graph contains a gem, then we construct in time $O(n^2)$ the partition as in Theorem 3.30 and apply the method described in Theorem 3.34, which takes time $O(n^3)$. Hence the overall complexity is $O(n^6)$.

Chapter 4

List Coloring

4.1 Context and motivations

Suppose that in a graph G, a proper subset S of vertices is already precolored with k colors and let $H = G \setminus S$. Is it possible to extend this coloring to H? A natural way of formalizing this is to assign to every vertex v of H a list of colors equals to the set $\{1,\ldots,k\}$ minus the colors already appearing in the neighborhood of v. The question is then, is it possible to color H by assigning to each vertex a color from its list such that no adjacent vertices receive the same color?

More generally, the *list coloring problem*, introduced by Erdős, Rubin and Taylor [30] and by Vizing [52] is stated as follows. Let $L : V(G) \to \mathcal{P}(\mathbb{N})$ be a list assignment of colors on the vertices of a graph G. The question asked is, can we find a proper coloring c such that $c(v) \in L(v)$ for all v of $V(G)$. If such a coloring exists we say that G is *L-colorable* and that c is an *L-coloring*. If an integer k is given, a graph G is said to be *k-choosable* if it is L-colorable for any list assignement L such that $|L(v)| = k$ for every vertex v of G. In an analogous way as for the classical coloring problem, the *list-chromatic number* (also called the *choice number*), denoted by $ch(G)$, is the smallest k such that G is k-choosable. It is a straightforward observation that $\chi(G) \leq ch(G)$. To see this, put the list $\{1,\ldots,\chi(G)-1\}$ on every vertex and remark that this list assignment L is not L-colorable. Is it possible that $\chi(G) < ch(G)$ for some graph G? The answer is yes and in fact the gap can be as large as desired.

Lemma 4.1

For any integer $p \geq 1$, the list-chromatic number of the complete bipartite graph K_{p,p^p} is at least $p+1$.

Proof. Let (X,Y) be the bipartition of the complete bipartite graph K_{p,p^p} with $X = \{x_1,\ldots,x_p\}$ and $Y = \{y_1,\ldots,y_{p^p}\}$. Assign the lists on the vertices as follows. On the X side, for each x_i assign the list $L(x_i) = \{(i-1)p+1,\ldots,ip\}$ for $i \in \{1,\ldots,p\}$. On the Y side, assign all the possible p-tuples $(c_1,c_2,\ldots,c_p)$ with $c_i \in L(x_i)$ for

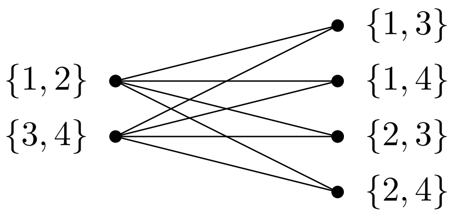


Figure 4.1: $K_{2,4}$ is not 2-choosable.

$i \in \{1, \ldots, p\}$ (there are p^p of them). See Figure 4.1 for a small example of the lists assignement. Assume now that K_{p,p^p} is L-colorable. It means that each vertex on the X side has been assigned a color from its list. But since on the Y side all p-tuples $(c_1, c_2, \ldots, c_p)$ with $c_i \in L(x_i)$ for $i \in \{1, \ldots, p\}$ are present, it means that there exist a $j \in \{1, \ldots, p^p\}$ for which $L(y_j) = \bigcup_{i=1}^{p} c(x_i)$. Hence the vertex y_j can not be colored, a contradiction. All color lists have size p, it follows that $ch(K_{p,p^p}) \geq p + 1$. $\square$

Since it is a bipartite graph we have $\chi(K_{p,p^p}) = 2$, and as shown above, $ch(K_{p,p^p}) \geq p + 1$.

A famous theorem in the history of list coloring takes root from a conjecture proposed by Jeffrey Dinitz (see [30] page 157). A latin square is $n \times n$ matrix with integral coefficients in $\{1, \ldots, n\}$ having the property that each coefficient appears exactly once on each row and each column. In a partial latin square, a cell takes its coefficients from a list of size n of possible coefficients (not necessarily the usual set $\{1, \ldots, n\}$). Dinitz asked if in a $n \times n$ matrix and given any assignment of n coefficients to the cells, is it always possible to construct a partial latin square? This problem is equivalent to a problem of list coloring the edges of a complete bipartite graph. The construction is as follows. Pick the complete bipartite graph $K_{n,n}$ and to each row of the matrix, associate a vertex on the left side, and to each column associate a vertex on the right side. Each edge in the graph is then a cell of the matrix. To every edge assign the list of colors of the corresponding cell. The question now is, is it possible to color properly the edges of this graph by choosing the color for each edge from its list? This has been answered positively by Galvin.

By analogy of the list coloring problem of the vertices to the edges of the graph, it is possible to define in an analogous way the *list-chromatic index*, denoted by $ch'(G)$, of a graph G as the minimum k such that G is L-colorable for any assignment L of colors to the edges of G with $|L(e)| = k$ for every $e \in E(G)$. Dinitz's problem can now be restated succinctly as follows. Is it true that $ch'(K_{n,n}) = n$? This was proved by Galvin who in fact proved the following more general theorem.

THEOREM 4.2 [35]

Every bipartite multigraph G satisfies $ch'(G) = \chi'(G)$.

Galvin's theorem is used in the proof of the result presented in this chapter, furthermore it is of self interest. Hence, we exhibit a simple proof of it.

$\{1,2,3\}$	$\{1,3,4\}$	$\{2,5,6\}$
$\{2,3,5\}$	$\{1,2,3\}$	$\{4,5,6\}$
$\{4,3,6\}$	$\{3,5,6\}$	$\{2,3,5\}$

1	3	2
2	1	5
6	5	3

Figure 4.2: A partial latin square and its solution.

A *kernel* in a directed graph G is a subset of vertices $K \subseteq V(G)$ such that K is a stable set and for every vertex $u \in V(G) \setminus K$, there exists a vertex $v \in K$ such that $\overrightarrow{uv} \in E(G)$. The following lemma is due to Bondy, Boppana and Siegel. However this lemma is not published but first appeared in the famous article of Alon and Tarsi [3] linking coloring and Eulerian subgraphs.

Lemma 4.3 Bondy, Boppana and Siegel

Let G be a directed graph such that every induced subgraph has a kernel. Then, for any list assignment L satisfying $|L(v)| \geq d^+(v) + 1$ for every vertex $v \in V(G)$, G is L-colorable.

Proof. Let c be a color present in one of the color lists of the graph and let H be the subgraph induced by all the vertices v such that $c \in L(v)$. The induced subgraph H has a kernel K. Color all the vertices in K with color c, delete those vertices from the graph G and delete the color c from all the vertices list in $H \setminus K$. Note that the vertices list in $G \setminus H$ remains unchanged, and every vertex in $H \setminus K$ loses a color in their list but also their out-degree is decreased by one. The lemma follows by induction on $|V(G)|$. □

The goal is to use this lemma, hence it suffices to find an orientation of our graph satisfying the conditions in the statement. An orientation of a multigraph orientation is *clique-acyclic* if no clique contains a directed cycle. A graph is *solvable* if every clique-acyclic orientation has a kernel. In 1992, Maffray [67] gave a characterization of solvable line-graphs with the following theorem.

Theorem 4.4

A line-graph (of a multigraph) is solvable if and only if it is perfect.

It is known that line-graphs of bipartite multigraphs are perfect, see Kőnig's Line Coloring Theorem [69, 56]. Thus, the previous theorem gives the following corollary.

Corollary 4.5

The line-graph of a bipartite multigraph is solvable.

Given a bipartite multigraph B, the only thing left to do to finish the proof of Galvin's theorem is to find a clique acyclic orientation of $G = \mathcal{L}(B)$ such that $d^+(v) \leq |L(v)| - 1$ for every $v \in G$. This is given by the neat idea of Galvin to use proper coloring of edges in bipartite multigraphs.

LEMMA 4.6

Let G be the line-graph of a bipartite multigraph B with bipartition (X, Y). Let f be a coloring of the vertices of G using $\omega(G)$ colors (it exists because G is perfect). Let D be the directed graph obtained from G by directing every edge uv as follows. Suppose that $f(u) < f(v)$. When the common end of edges u and v in B is in X, give the orientation $u \to v$, and when it is in Y give the orientation $u \leftarrow v$. Then every induced subgraph of D has a kernel and $d^+(v) \leq \omega(G) - 1$.

Proof. For any $v \in V(D)$, the out-degree of v is at most $\omega(G) - 1$ since f is an $\omega(G)$-coloring and is injective on the closed neighbourhood of v in D. What is left is to show is that every induced subdigraph of D has a kernel. Let v be any vertex in $V(D)$ (equivalently, this is an edge in B) and denote by $X(v)$ the set of vertices in D sharing a common end in $B[X]$ with v and similarly with $Y(v)$, the set of vertices in D sharing a common end in $B[Y]$ with v. Let $S \subseteq V(D)$ be any subset of vertices of D, we prove by induction on $|S|$ that S has a kernel. Let $T = \{v \in S \mid f(v) > f(u)$ for all $v \in (X(v) \cap S)$ different from $u\}$. If T is a stable set, then T is a kernel of S, thus we can assume that T has two elements, v_1, v_2, sharing a common end in $B[Y]$ with $f(v_1) > f(v_2)$ and let $Y(v_1) = Y(v_2) = Z$. Pick $v_0 \in Z \cap S$ such that $f(v_0) > f(u)$ for all $u \in C \cap S$ different of v_0. By the definition of T and the choice of v_0, we have $N[v_2] \cap S \subseteq C \cap S \subseteq N[v_0]$. By the induction hypothesis, $S \setminus \{v_0\}$ has a kernel K. Since K is a kernel, $N[v_2] \cap K \neq \emptyset$ and furthermore, $N[v_2] \cap K \subseteq N[v_0] \cap K$, it follows that either $v_0 \in K$ or v_0 has a neighbor in K, so K is a kernel of S. □

Moreover, Galvin, with his theorem, proved a subcase of what is probably the most famous conjecture in list coloring, stated independently by several authors including Vizing, Gupta, Albertson and Collins, and Bollobás and Harris (see [42]). The List Coloring Conjecture is stated as follows.

CONJECTURE 4.7

Every multigraph G satisfies $ch'(G) = \chi'(G)$.

The List Coloring Conjecture can then be restated in terms of line-graph.

CONJECTURE 4.8

Every multigraph G satisfies $ch(\mathcal{L}(G)) = \chi(\mathcal{L}(G))$.

Line-graphs are characterized by a list of nine forbidden induced subgraphs [6]. The smallest of these forbidden graphs is the claw graph and it is not hard to see that one cannot produce a claw in a line-graph. The only known examples where the chromatic number and list-chromatic number differ contain claws. This fact pushed

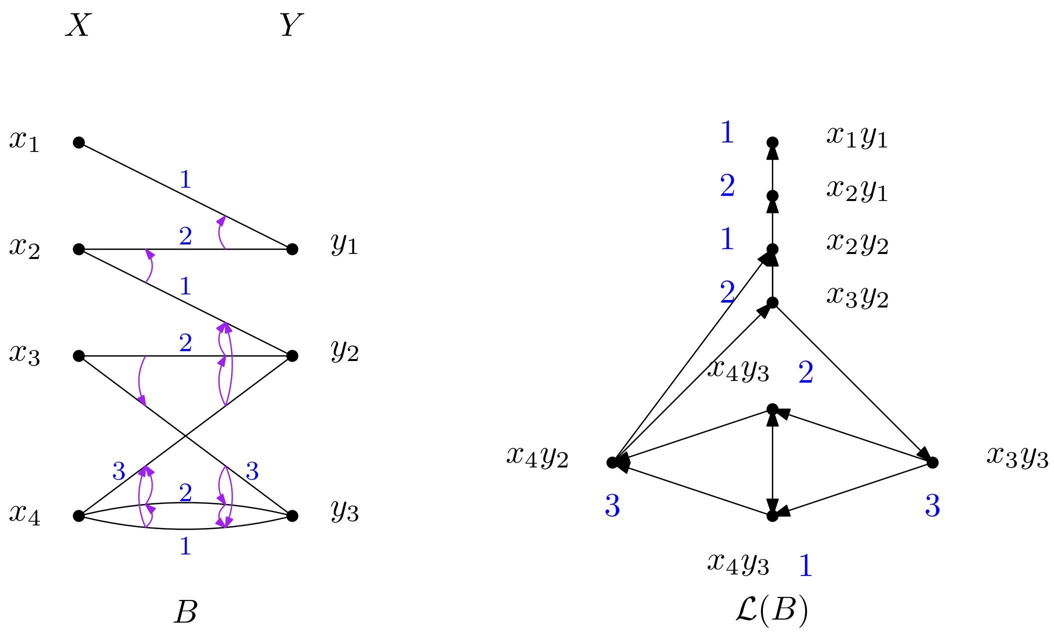


Figure 4.3: An example of Galvin's theorem with a bipartite multigraph B and its line-graph $\mathcal{L}(B)$.

Gravier and Maffray to generalize the List Coloring Conjecture to all claw-free graphs, more precisely, the following was conjectured in [37, 38].

CONJECTURE 4.9

Every claw-free graph G satisfies $ch(G) = \chi(G)$.

Even though this conjecture appears to be somehow too large, it is still widely open. An interesting subclass of claw-free graphs concerning this problem is the claw-free perfect graphs class.

4.2 Structure of claw-free perfect graphs

Claw-free perfect graphs are described by a decomposition theorem of Chvátal and Sbihi [20]. A *clique cutset* in a graph G is a clique C of G such that $G \setminus C$ is disconnected. A *minimal clique cutset* is a clique cutset that does not contain another clique cutset. Graph decomposition is a remarkable tool in many aspects of graph theory and allows to solve some of the most difficult problems by reducing the difficulties to easy (with the regard to the problem one wants to solve) classes of graphs and combining the solutions of multiple easy graphs to obtain a general solution for the initial graph. Given a graph G, a decomposition of G is a pair (G_1, G_2) where G_1 and G_2 are proper induced subgraphs of G. Many different types of decomposition

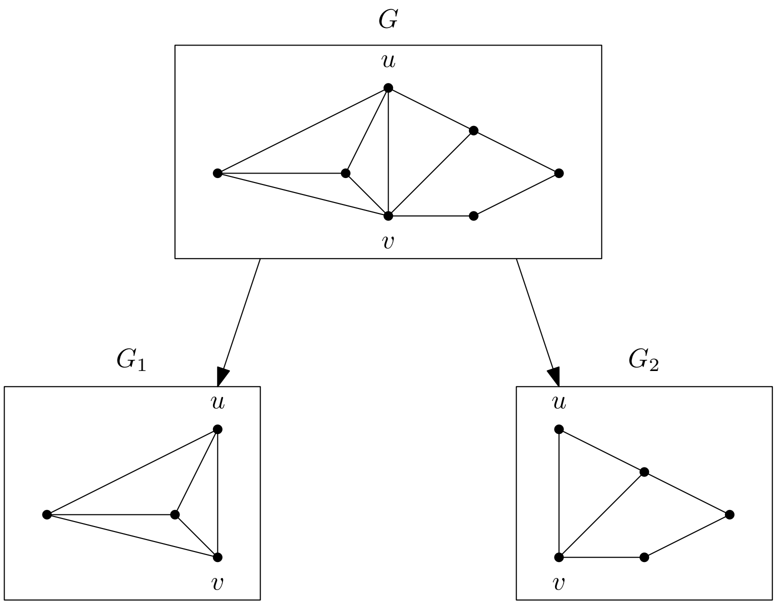


Figure 4.4: A graph G decomposed in graphs G_1 and G_2 by the clique cutset $\{u, v\}$.

exist, including component decomposition, anticomponent decomposition, module decomposition. One that will be of particular interest in this chapter is the *clique cutset decomposition*. If C is a minimal clique cutset in a graph G and $A_1, \dots, A_k$ are the vertex sets of the component of $G \setminus C$, we consider that G is decomposed into the collection of induced subgraphs $G[A_i \cup C]$ for $i \in \{1, \dots, k\}$. We can then recursively apply clique cutset decompositions on the newly obtained graphs, see Figure 4.4 for an example of a clique cutset decomposition.

Before going into the details of the structure of claw-free perfect graphs, we need to define what are the basic graphs of the decomposition. A graph is *elementary* if its edges can be colored with two colors (one color on each edge) in such a way that every induced two-edge path has its two edges colored differently. A graph G is *peculiar* if $V(G)$ can be partitioned into nine sets A_i, B_i, Q_i $(i = 1, 2, 3)$ that satisfy the following properties for each i, where subscripts are understood modulo 3:

- Each of the nine sets is non-empty and induces a clique.
- A_i is complete to $B_i \cup A_{i+1} \cup A_{i+2} \cup B_{i+2}$ and not complete to B_{i+1}.
- B_i is complete to $A_i \cup B_{i+1} \cup B_{i+2} \cup A_{i+1}$ and not complete to A_{i+2}.
- Q_i is complete to $A_{i+1} \cup B_{i+1} \cup A_{i+2} \cup B_{i+2}$ and anticomplete to $A_i \cup B_i \cup Q_{i+1} \cup Q_{i+2}$.

We say that $(A_1, B_1, A_2, B_2, A_3, B_3, Q_1, Q_2, Q_3)$ is a peculiar partition of G. See Figure 4.5.

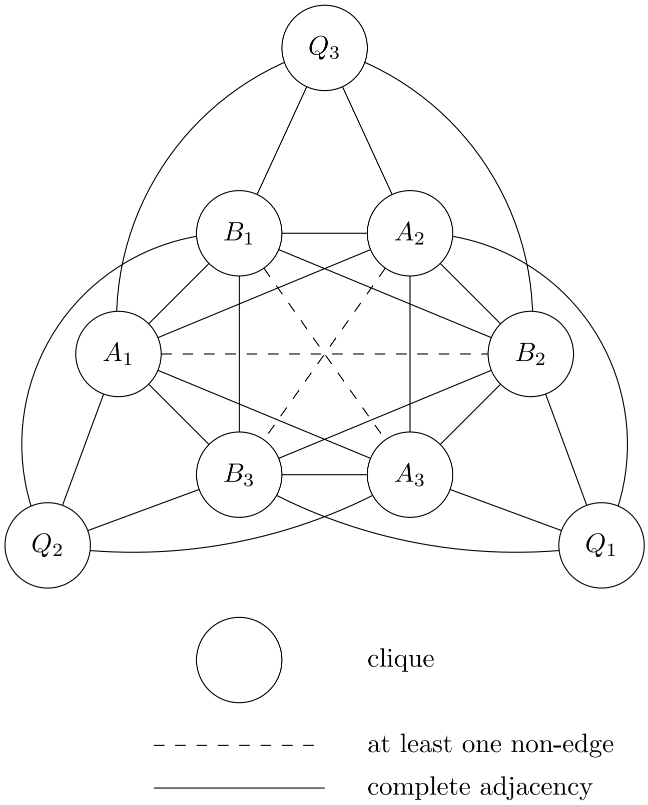


Figure 4.5: General structure of peculiar graphs.

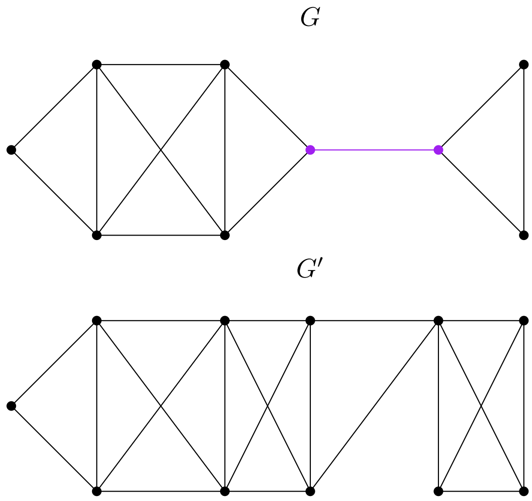


Figure 4.6: A flat edge of G in purple, and G' the graph obtained from its augmentation.

In 1988, Chvátal and Sbihi proved the following theorem, which will be useful for our problem.

THEOREM 4.10 Chvátal and Sbihi [20]

Every claw-free perfect graph either has a clique cutset or is a peculiar graph or an elementary graph.

The class of elementary graphs, to be fully described, requires the following definitions. A *flat edge* is an edge that is not contained in a triangle. A *flat edge augmentation* is the following process applied to a flat edge of G. Let xy be a flat edge in a graph G, and let A be a cobipartite graph such that $V(A)$ is disjoint from $V(G)$ and $V(A)$ can be partitioned into two cliques X, Y. We obtain a new graph G' by removing x and y from G and adding all edges between X and $N_G(x) \setminus \{y\}$ and all edges between Y and $N_G(y) \setminus \{x\}$. This operation is called *augmenting* the flat edge xy with the cobipartite graph A. In G' the pair (X, Y) is called the *augment*. When $x_1y_1, \ldots, x_ky_k$ are pairwise non-adjacent flat edges in a graph G, and $A_1, \ldots, A_k$ are pairwise vertex-disjoint cobipartite graphs, also vertex-disjoint from G, one can augment each edge x_iy_i with the graph A_i. Clearly the result is the same whatever the order in which the k operations are performed. We say that the resulting graph is an *augmentation* of G, see Figure 4.6 for an example.

The structure of peculiar graphs follows from their definition, but elementary remained not fully described until Maffray and Reed proved the following in 1999.

THEOREM 4.11 Maffray and Reed [69]

A graph G is elementary if and only if it is an augmentation of the line-graph H of a bipartite multigraph B. Moreover we may assume that each augment A_i satisfies the following:

- *There is at least one pair of non-adjacent vertices in A_i,*
- *The bipartite graph whose vertex-set is $X_i \cup Y_i$ and whose edges are the edges of A_i with one end in X_i and one in Y_i is connected (and consequently both $|X_i|, |Y_i| \geq 2$).*

The List Coloring Conjecture was proved in [39] for every claw-free perfect graph G with $\omega(G) \leq 3$. In the following section, we prove it for the case $\omega(G) \leq 4$ and whilst using similar techniques, we also provide new ones that can be of self interest.

For the sake of completeness we recall a classical theorem of Hall. Let $X_1, \ldots, X_k$ be a family of sets. A *system of distinct representatives* for the family is a subset $\{x_1, \ldots, x_k\}$ of k distinct elements of $X_1 \cup \cdots \cup X_k$ such that $x_i \in X_i$ for all $i = 1, \ldots, k$. Note that if G is a graph and L is a list assignment on $V(G)$, and the family $\{L(v) \mid v \in V(G)\}$ admits a system of distinct representatives, then this is an L-coloring of G.

THEOREM 4.12 Hall's theorem [43]

A family $\mathcal{F}$ of k sets has a system of distinct representatives if and only if, for all $\ell \in \{1, \ldots, k\}$, the union of any ℓ members of $\mathcal{F}$ has size at least ℓ.

4.3 List coloring claw-free perfect graphs

This section is dedicated to prove the following theorem.

THEOREM 4.13

Let G be a claw-free perfect graph with $\omega(G) \leq 4$. Then $ch(G) = \chi(G)$.

One tool that we will use is due to Galvin. The proof was given at the beginning of the chapter but we restate it here in a different form. An example in given in Figure 4.3.

THEOREM 4.14 Galvin [35]

Let G be the line-graph of a bipartite graph B, where $V(B)$ is partitioned into two stable sets X, Y. Let f be an $\omega(G)$-coloring of the vertices of G, with colors $1, 2, \ldots, \omega(G)$. Let D be the directed graph obtained from G by directing every edge uv as follows, assuming that $f(u) < f(v)$: when the common end of edges u, v in B is in X, then give the orientation $u \rightarrow v$, and when it is in Y give the orientation $u \leftarrow v$. Assume that L is a list assignment on $V(G)$ such that every vertex v of G satisfies $|L(v)| \geq d^+_D(v) + 1$. Then G is L-colorable.

As described by the theorem of Chvátal and Sbihi, first of all, we need to focus on the two types of basic graphs given by their decomposition. First, let us concentrate on peculiar graphs. Then on cobipartite graphs, followed by elementary graphs. Finally we will combine all those results to deal with claw-free perfect graphs.

4.3.1 Peculiar Graphs

The following useful lemma allows to treat peculiar graphs separately and to not bother with clique cutset decomposition since if a graph claw-free perfect graph contains a peculiar graph, the whole graph is peculiar.

LEMMA 4.15

Let G be a connected claw-free graph that contains a peculiar subgraph, and assume that G is also C_5-free. Then G is peculiar.

Proof. Let H be a peculiar subgraph of G that is maximal. If $H = G$ we are done. So let us assume that $H \neq G$. Since G is connected there is a vertex x of $V(G) \setminus V(H)$ that has a neighbor in H. Let $A_1, B_1, A_2, B_2, A_3, B_3, Q_1, Q_2, Q_3$ be nine cliques that form a partition of $V(H)$ as in the definition of a peculiar graph. For $i = 1,2,3$ we pick a pair of non-adjacent vertices $a_i \in A_i$ and $b_{i+1} \in B_{i+1}$, and we pick any $q_i \in Q_i$. (All subscripts are modulo 3.)

If x has no neighbor in $Q_1 \cup Q_2 \cup Q_3$, then it has a neighbor a in $A_i \cup B_i$ for some i; but then $\{a, x, q_{i+1}, q_{i+2}\}$ induces a claw. Therefore x has a neighbor in $Q_1 \cup Q_2 \cup Q_3$.

Suppose that x has a neighbor k in Q_1 and none in $Q_2 \cup Q_3$. Then x has no neighbor z in $A_1 \cup B_1$, for otherwise $\{z, x, q_2, q_3\}$ induces a claw. Also x is adjacent to one of a_2, b_3, for otherwise $\{x, k, a_2, b_3\}$ induces a claw; up to symmetry we assume that x is adjacent to a_2. Then x is adjacent to every vertex $a \in A_3$, for otherwise $\{a_2, q_3, a, x\}$ induces a claw; and to every vertex $y \in A_2 \cup B_2 \cup Q_1$, for otherwise $\{a_3, y, x, q_2\}$ induces a claw; and to every vertex $b \in B_3$, for otherwise $\{b_2, b, q_3, x\}$ induces a claw. Hence x is complete to $A_2 \cup B_2 \cup A_3 \cup B_3 \cup Q_1$ and anticomplete to $A_1 \cup B_1 \cup Q_2 \cup Q_3$. So $V(H) \cup \{x\}$ induces a peculiar subgraph of G, because x can be added to Q_1, a contradiction to the choice of H.

Therefore we may assume up to symmetry that x has a neighbor $k \in Q_1$ and a neighbor $k' \in Q_2$. Note that x has no neighbor $k'' \in Q_3$, for otherwise $\{x, k, k', k''\}$ induces a claw.

Suppose that x has a non-neighbor $a \in A_1$. Then x is adjacent to every vertex $u \in A_2$, for otherwise $\{x, k, u, a, k'\}$ induces a C_5; and then to every vertex $v \in B_2$, for otherwise either $\{a_2, a, x, v\}$ induces a claw (if $av \notin E(G)$) or $\{x, k, v, a, k'\}$ induces a C_5 (if $av \in E(G)$); and then to every vertex $w \in A_3 \cup B_3 \cup Q_1$, for otherwise $\{b_2, x, w, q_3\}$ induces a claw. Then a is adjacent to every vertex $b \in B_2$, for otherwise $\{x, k', a, q_3, b\}$ induces a C_5; and by the same argument the set $A_1 \setminus N(x)$ is complete to B_2. It follows that $a_1 \in N(x)$ since a_1 is not complete to B_2. Then x is adjacent to every vertex $q \in Q_2$, for otherwise $\{a_1, x, q_3, q\}$ induces a claw. But now we observe that $V(H) \cup \{x\}$

induces a larger peculiar subgraph of G, because x can be added to A_3 and the vertices of $A_1 \setminus N(x)$ can be moved to B_1.

Therefore we may assume that x is complete to A_1, and, similarly, to B_2. Then x is adjacent to every vertex u in $Q_2 \cup B_3$, for otherwise $\{a_1, x, u, q_3\}$ induces a claw, and similarly x is complete to $Q_1 \cup A_3$. It cannot be that x has both a non-neighbor $a' \in A_2$ and a non-neighbor $b' \in B_1$, for otherwise $\{x, k, a', b', k'\}$ induces a C_5. So, up to symmetry, x is complete to A_2. But now $V(H) \cup \{x\}$ induces a larger peculiar subgraph of G, because x can be added to A_3. This completes the proof of the lemma. □

We observe that (up to isomorphism) there is a unique peculiar graph G with $\omega(G) = 4$. Indeed if G is such a graph, with the same notation as in the definition of a peculiar graph, then for each i the set $Q_i \cup A_{i+1} \cup B_{i+1} \cup A_{i+2}$ is a clique, so, since G has no clique of size 5, the four sets $Q_i, A_{i+1}, B_{i+1}, A_{i+2}$ have size 1; and so the nine sets A_i, B_i, Q_i ($i = 1, 2, 3$) all have size 1. Hence G is the unique peculiar graph on nine vertices.

LEMMA 4.16

Let G be a peculiar graph with $\omega(G) = 4$. Then G is 4-choosable.

Proof. Let $(A_1, B_1, A_2, B_2, A_3, B_3, Q_1, Q_2, Q_3)$ be a peculiar partition of G. As observed above, we have $|A_i| = |B_i| = |Q_i| = 1$ for all $i = 1, 2, 3$. Hence let $A_i = \{a_i\}$, $B_i = \{b_i\}$ and $Q_i = \{q_i\}$, for all $i = 1, 2, 3$. Recall that a_i is not adjacent to b_{i+1}, for each i. Let $Q = \{q_1, q_2, q_3\}$.

Let L be a list assignment that satisfies $|L(v)| = 4$ for all $v \in V(G)$. Let us prove that G is L-colorable.

First suppose that for some $i \in \{1, 2, 3\}$ we have $L(a_i) \cap L(b_{i+1}) \neq \emptyset$, say for $i = 1$. Pick any $c \in L(a_1) \cap L(b_2)$. Let $G' = G \setminus \{a_1, b_2\}$ and let $L'(x) = L(x) \setminus \{c\}$ for all $x \in V(G')$. Clearly, G' is a claw-free perfect graph and $\omega(G') = 3$. Moreover, G' is elementary. To see this, define an edge coloring of G' by coloring blue the edges in $\{q_3b_1, q_3a_2, b_1a_2, b_3a_3, q_2a_3, b_3q_1\}$ and red the edges in $\{q_2b_1, q_2b_3, b_3b_1, q_1a_2, q_1a_3, a_2a_3\}$; it is a routine matter to check that this edge coloring is an elementary coloring. By [39], G' is 3-choosable, so it admits an L'-coloring. We can extend this coloring to a_1 and b_2 by assigning color c to them. Therefore we may assume that:

$$L(a_i) \cap L(b_{i+1}) = \emptyset \text{ for all } i = 1, 2, 3. \tag{4.1}$$

Now suppose that there are vertices $u, v \in Q$ such that $L(u) \cap L(v) \neq \emptyset$. Let w be the unique vertex in $Q \setminus \{u, v\}$. Pick any $c \in L(u) \cap L(v)$. Let $G' = G \setminus \{u, v\}$. Let $L'(x) = L(x) \setminus \{c\}$ for all $x \in V(G') \setminus \{w\}$, and let $L'(w) = L(w)$. We claim that the family $\{L'(x) \mid x \in V(G')\}$ admits a system of distinct representatives. Suppose the contrary. By Hall's theorem, there is a set $S \subseteq V(G')$ such that $|L'(S)| < |S|$. Since $|L'(x)| \geq 3$ for all $x \in V(G')$, we have $|L'(S)| \geq 3$, so $|S| \geq 4$; this implies that either (a) $S \supseteq \{a_i, b_{i+1}\}$ for some $i \in \{1, 2, 3\}$ or (b) S contains w. In case (a), (4.1) implies that c belongs to at most one of $L(a_i)$ and $L(b_{i+1})$, and so $|L'(S)| \geq |L'(a_i) \cup L'(b_{i+1})| \geq 7$, so $|S| \geq 8$, which is impossible because $|V(G')| = 7$. In case (b), since $|L'(w)| = 4$, we have $|L'(S)| \geq 4$, so $|S| \geq 5$, which implies that S satisfies (a) again, a contradiction. Thus the family $\{L'(x) \mid x \in V(G')\}$ admits a system of distinct representatives, which is an L'-coloring of G'. We can extend this coloring to u and v by assigning color c to them. Therefore we may assume that

$$L(u) \cap L(v) = \emptyset \text{ for all } u, v \in Q. \tag{4.2}$$

We claim that the family $\{L(x) \mid x \in V(G)\}$ admits a system of distinct representatives. Suppose the contrary. By Hall's theorem, there is a set $T \subseteq V(G)$ such that $|L(T)| < |T|$. Since $|L(x)| = 4$ for all $x \in V(G)$, we have $|L(T)| \geq 4$, so $|T| \geq 5$; this implies that either (a) $T \supseteq \{a_i, b_{i+1}\}$ for some $i \in \{1, 2, 3\}$ or (b) T contains two vertices from Q. In either case, (4.1) or (4.2) implies that $|L(T)| \geq 8$, so $|T| \geq 9$, that is, $T = V(G)$. But then $T \supset Q$, so (4.2) implies that $|L(T)| \geq 12$ and $|T| \geq 13$, which is impossible. Thus the family $\{L(x) \mid x \in V(G)\}$ admits a system of distinct representatives, which is an L-coloring of G. □

4.3.2 Cobipartite graphs

In this subsection we analyze the list colorability of certain cobipartite graphs with certain list assignments. The following lemmas will be useful for the final step of the proof.

LEMMA 4.17

Let H be a cobipartite graph, where $V(H)$ is partitioned into two cliques X and Y. Assume that $|X| \leq |Y|$ and that there are $|X|$ non-edges between X and Y and they form a matching in $\overline{H}$. Let L be a list assignment on $V(H)$ such that $|L(x)| \geq |X|$ for all $x \in X$ and $|L(y)| \geq |Y|$ for all $y \in Y$. Then H is L-colorable.

Proof. Let $X = \{x_1, \ldots, x_p\}$, and let $y_1, \ldots, y_p$ be vertices of Y such that $\{x_1, y_1\}, \ldots, \{x_p, y_p\}$ are the non-edges of H. The hypothesis implies that $y_1, \ldots, y_p$ are pairwise distinct. Since a clique in H can contain at most one of x_i, y_i for each $i = 1, \ldots, p$, we have $\omega(H) = |Y|$.

We proceed by induction on $|X|$. If $|X| = 0$, then H is a clique with $|L(v)| = |V(H)|$ for all $v \in V(H)$; so H is L-colorable by Hall's theorem. Now suppose that $|X| > 0$. If the family $\{L(v) \mid v \in V(H)\}$ admits a system of distinct representatives, then this is an L-coloring. So suppose the contrary. By Hall's theorem there is a set $T \subseteq V(H)$ such that $|L(T)| < |T|$. Then $|T| > |X|$, so T contains a vertex y from Y, and so $|T| > |L(y)| \geq |Y|$. Since $\omega(H) = |Y|$, it follows that T is not a clique. So T contains non-adjacent vertices x, y with $x \in X$ and $y \in Y$. We have $|L(x) \cup L(y)| \leq |L(T)| < |T| \leq |X| + |Y|$, which implies $L(x) \cap L(y) \neq \emptyset$. Pick a color $c \in L(x) \cap L(y)$. Set $L'(w) = L(w) \setminus \{c\}$ for all $w \in V(H) \setminus \{x, y\}$. Let $X' = X \setminus \{x\}$, $Y' = Y \setminus \{y\}$, and $H' = H \setminus \{x, y\}$. Clearly every vertex $x' \in X'$ satisfies $|L'(x')| \geq |X'|$ and every vertex $y' \in Y'$ satisfies $|L'(y')| \geq |Y'|$, and $|X'| \leq |Y'|$, and there are $|X'|$ non-edges between X' and Y', and they form a matching in $\overline{H'}$. By the

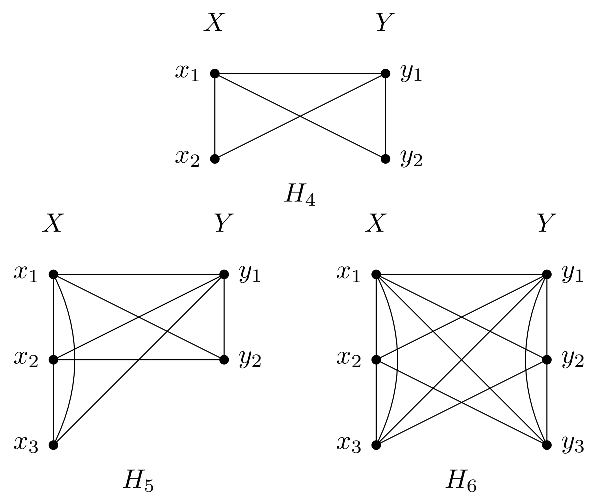


Figure 4.7: The graphs H_4, H_5 and H_6.

induction hypothesis, H' admits an L'-coloring. We can extend it to an L-coloring of H by assigning the color c to x and y. □

LEMMA 4.18

Let H be the cobipartite graph isomorphic to H_4 as depicted in Figure 4.7, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$, and $E(\overline{H}) = \{x_2y_2\}$. Let L be a list assignment on $V(H)$ such that $|L(u)| \geq 2$ for all $u \in V(H)$. Then H is L-colorable if and only if every clique Q of H satisfies $|L(Q)| \geq |Q|$.

Proof. This is a corollary of Claim 1 in [37]. For completeness, we restate the claim here: *The graph H is not L-colorable if and only if for some $v \in \{x_2, y_2\}$ we have $L(x_1) = L(y_1) = L(v)$ and these three lists are of size two.*

Clearly, if H is L-colorable, then every clique Q of H satisfies $|L(Q)| \geq |Q|$. Conversely, if every clique Q of H satisfies $|L(Q)| \geq |Q|$, then by the above claim, applied to the cliques $\{x_1, y_1, x_2\}$ and $\{x_1, y_1, y_2\}$, we obtain that H is L-colorable. □

LEMMA 4.19

Let H be the cobipartite graph isomorphic to H_5 as depicted in Figure 4.7, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, and $E(\overline{H}) = \{x_3y_2\}$. Let L be a list assignment on $V(H)$ such that $|L(x)| \geq 3$ for all $x \in X$ and $|L(y)| \geq 2$ for all $y \in Y$. Then H is L-colorable if and only if every clique Q of H satisfies $|L(Q)| \geq |Q|$.

Proof. If H is L-colorable then clearly every clique Q of H satisfies $|L(Q)| \geq |Q|$. Now let us prove the converse.

First suppose that $L(y_2) \subseteq L(x_3)$. Since $H \setminus \{x_3\}$ is a clique, every subset T of $V(H) \setminus \{x_3\}$ satisfies $|L(T)| \geq |T|$, and so, by Hall's theorem there is an L-coloring of

$H \setminus \{x_3\}$. Then we can extend any such coloring by assigning to x_3 the color assigned to y_2.

Now assume that $L(y_2) \not\subseteq L(x_3)$. This implies $|L(x_3) \cup L(y_2)| \geq 4$. Suppose that the family $\{L(x) \mid x \in V(H)\}$ does not have a system of distinct representatives. By Hall's theorem there is a set $T \subseteq V(H)$ such that $|L(T)| < |T|$. By the assumption, T is not a clique, so it contains x_3 and y_2. It follows that $|L(T)| \geq 4$. Hence $|T| = 5$, so $T = V(H)$, and $|L(T)| = 4$, and we may assume that $L(x_3) = \{1,2,3\}$ and $L(y_2) = \{3,4\}$ and $L(T) = \{1,2,3,4\}$. Assign color 3 to x_3 and y_2. Now assign a color c from $L(y_1) \setminus \{3\}$ to y_1 (there may be two choices for c). We may assume that this coloring fails to be extended to $\{x_1, x_2\}$; so it must be that $L(x_1) \setminus \{3, c\}$ and $L(x_2) \setminus \{3, c\}$ are equal and of size 1; so $L(x_1) = L(x_2) = \{b, c, 3\}$ for some $b \neq c$, with $b \in \{1,2,4\}$. Suppose that $3 \notin L(y_1)$. Then there is a second choice for c, and we may assume that this attempt fails similarly. Hence $L(y_1) = \{b, c\}$, with $b, c \in \{1,2,4\}$. If $\{b, c\} = \{1,2\}$, then the clique $Q_1 = \{x_1, x_2, x_3, y_1\}$ violates the assumption because $L(Q_1) = \{1,2,3\}$. If $\{b, c\} = \{1,4\}$ or $\{2,4\}$, then the clique $Q_2 = \{x_1, x_2, y_1, y_2\}$ violates the assumption because $L(Q_2) = \{b, c, 3\}$. So we may assume that $3 \in L(y_1)$, i.e., $L(y_1) = \{c, 3\}$. If $c = 4$, then Q_2 violates the assumption because $L(Q_2) = \{b, 3, 4\}$. So, up to symmetry, $c = 1$. If $b = 2$, then Q_1 violates the assumption because $L(Q_1) = \{1,2,3\}$. If $b = 4$, then Q_2 violates the assumption because $L(Q_2) = \{1,3,4\}$. Hence the family $\{L(x) \mid x \in V(H)\}$ admits a system of distinct representatives, which is an L-coloring of G. $\square$

LEMMA 4.20

Let H be the cobipartite graph isomorphic to H_6 as depicted in Figure 4.7, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, and $E(\overline{H}) = \{x_2y_2, x_3y_3\}$. Let L be a list assignment on $V(H)$ such that $|L(x)| \geq 3$ for all $x \in V(H)$. Then H is L-colorable if and only if every clique Q of H satisfies $|L(Q)| \geq |Q|$. In particular, if $|L(x_1) \cup L(y_1)| \geq 4$, then H is L-colorable.

Proof. If H is L-colorable then clearly every clique Q of H satisfies $|L(Q)| \geq |Q|$. Now let us prove the converse. We first claim that:

$$\text{We may assume that } |L(x_i) \cap L(y_i)| \leq 1 \text{ for each } i \in \{2,3\}. \tag{4.3}$$

Suppose on the contrary, and up to symmetry, that $|L(x_2) \cap L(y_2)| \geq 2$. Let $H' = H \setminus \{x_2\}$, and set $L'(y_2) = L(x_2) \cap L(y_2)$ and $L'(u) = L(u)$ for all $u \in \{x_1, x_3, y_1, y_3\}$. Thus H' and L' satisfy the hypothesis of Lemma 4.19. If every clique Q in H' satisfies $|L'(Q)| \geq |Q|$, then Lemma 4.19 implies that H' admits an L'-coloring, and we can extend it to an L-coloring of H by giving to x_2 the color assigned to y_2. Hence assume that some clique Q in H' satisfies $|L'(Q)| < |Q|$. We have $|L'(Q)| \geq 2$, so $|Q| \geq 3$, so $3 \leq |L'(Q)| < |Q| \leq 4$, and so $|L'(Q)| = 3$ and $|Q| = 4$. Since x_3 and y_3 play symmetric roles here, we may assume up to symmetry that $Q = \{x_1, y_1, y_2, y_3\}$, and $L'(Q) = \{a, b, c\}$, where a, b, c are three distinct colors. Hence $L(x_1) = L(y_1) = L(y_3) = \{a, b, c\}$. Since $|L(Q)| \geq 4$, there is a color $d \in L(y_2) \setminus \{a, b, c\}$. Since $|L(\{x_1, y_1, x_2, y_3\})| \geq 4$, there is a color $e \in L(y_2) \setminus \{a, b, c\}$. If $a \in L(x_3)$, then we can assign color a to x_3 and y_3, colors b and c to x_1 and y_1, color e to x_2 and color d to y_2. So assume that $a \notin L(x_3)$, and similarly that $b, c \notin L(x_3)$. Then we can assign colors a, b, c to x_1, y_1, y_3, color e to x_2, color d to y_2, and a color from $L(x_3) \setminus \{d, e\}$ to x_3. Thus (4.3) holds.

It follows from (4.3) that $|L(x_i) \cup L(y_i)| \geq 5$ for $i = 2, 3$. If the family $\{L(x) \mid x \in V(H)\}$ admits a system of distinct representatives, then this is an L-coloring. So suppose the contrary. By Hall's theorem there is a set $T \subseteq V(H)$ such that $|L(T)| < |T|$. By the assumption, T is not a clique, so it contains x_i and y_i for some $i \in \{2, 3\}$. By (4.3) we have $|L(T)| \geq 5$, so $|T| \geq 6$, hence $T = V(H)$, and $|L(T)| = 5$, and consequently $|L(x_i)| = |L(y_i)| = 3$ and $|L(x_i) \cap L(y_i)| = 1$ for each $i = 2, 3$. Let $L(x_i) \cap L(y_i) = \{c_i\}$ for $i = 2, 3$.

Suppose that $c_2 \neq c_3$. We assign color c_i to x_i and y_i for each $i = 2, 3$. If this coloring can be extended to $\{x_1, y_1\}$ we are done. So suppose the contrary. Then it must be that $L(x_1) = L(y_1) = \{b, c_2, c_3\}$ for some color $b \in L(H) \setminus \{c_2, c_3\}$. Then we can color H as follows. Assign colors c_2 and c_3 to x_1 and y_1. There are four ways to color x_2 and y_2 with one color from $L(x_2) \setminus \{c_2\}$ for x_2 and one color from $L(y_2) \setminus \{c_2\}$ for y_2; at most two of them use a pair of colors equal to $L(x_3) \setminus \{c_3\}$ or $L(y_3) \setminus \{c_3\}$, so we can choose another way, and there will remain a color for x_3 and a color for y_3.

Now suppose that $c_2 = c_3$; call this color c. Let $L'(v) = L(v) \setminus \{c\}$ for all $v \in V(H) \setminus \{x_3, y_3\}$. We may assume that the graph $H \setminus \{x_3, y_3\}$ does not admit an L'-coloring, for otherwise such a coloring can be extended to H by assigning color c to x_3 and y_3. Hence, by Lemma 4.18 there is a clique Q of size 3 in $H \setminus \{x_3, y_3\}$ such that $|L'(Q)| = 2$, say $L'(Q) = \{a, b\}$. So $L(u) = \{a, b, c\}$ for all $u \in Q$. Moreover Q consists of x_1, y_1 and one of x_2, y_2. We assign color a to x_1, color b to y_1, and color c to x_2 and y_2. Since $|L(Q \cup \{x_3\})| \geq 4$, there is a color $d \in L(x_3) \setminus \{a, b, c\}$, and similarly there is a color $e \in L(y_3) \setminus \{a, b, c\}$. We assign d to x_3 and e to y_3, and we obtain an L-coloring of H.

Finally we prove the last sentence of the lemma. Since x_1 and y_1 are in all cliques of size 4, the assumption that $|L(x_1) \cup L(y_1)| \geq 4$ implies that every clique Q of H satisfies $|L(Q)| \geq |Q|$. So H is L-colorable. □

LEMMA 4.21

Let H be a cobipartite graph with $\omega(H) \leq 4$. Let x, y be two adjacent vertices in H such that $N(x) \setminus \{y\}$ and $N(y) \setminus \{x\}$ are cliques and $V(H) = N(x) \cup N(y)$. Let L be a list assignment such that $|L(x)| \geq 2$, $|L(y)| \geq 2$, and $|L(v)| \geq 4$ for all $v \in V(H) \setminus \{x, y\}$. Then H is L-colorable.

Proof. Let $X = N(x) \setminus \{y\}$ and $Y = N(y) \setminus \{x\}$. Let $I = X \cap Y$. Since $\{x, y\} \cup I$ is a clique, we have $|I| \leq 2$.

First suppose that $|I| = 2$. Let $I = \{w, w'\}$. Since $\{x\} \cup X$ is a clique that contains I, we have $|X \setminus I| \leq 1$. Likewise $|Y \setminus I| \leq 1$. We may assume that we are in the situation where $X \setminus I$ and $Y \setminus I$ are non-empty and complete to each other, because any other situation can be reduced to that one by adding vertices or edges (which makes the coloring problem only harder). Let $X \setminus I = \{u\}$ and $Y \setminus I = \{v\}$. Suppose that $L(x) \cap L(v) \neq \emptyset$. Pick a color $a \in L(x) \cap L(v)$, assign it to x and v, and remove it from the lists of all other vertices. Pick a color b from $L(y) \setminus \{a\}$, assign it to y and remove it from the list of the vertices in I. Let L' be the reduced list assignment. Then $|L'(w)| \geq 2$, $|L'(w')| \geq 2$, and $|L'(u)| \geq 3$, so we can L'-color greedily w, w', u in this order. Hence assume that $L(x) \cap L(v) = \emptyset$, and similarly that $L(y) \cap L(u) = \emptyset$. Then $|L(x) \cup L(v)| \geq 6$ and $|L(y) \cup L(u)| \geq 6$. It follows that the family $\{L(z) \mid z \in V(H)\}$ satisfies Hall's condition, so H is L-colorable.

Now suppose that $|I| = 1$. Let $I = \{w\}$. Then $|X \setminus \{w\}| \leq 2$ and $|Y \setminus \{w\}| \leq 2$. We may assume that we are in the situation where $X \setminus I$ and $Y \setminus I$ have size 2 and there are three edges between them, because any other situation can be reduced to that one by adding vertices or edges. Let $X \setminus I = \{u, v\}$ and $Y \setminus I = \{s, t\}$, and let $us, ut, vs \in E(H)$ and $vt \notin E(H)$. Suppose that $L(x) \cap L(s) \neq \emptyset$. We pick a color $a \in L(x) \cap L(s)$, assign it to x and s, and remove it from the lists of all other vertices. Then it is easy to see that we can color y, t, w, u, v in this order, using colors from the reduced lists. Hence assume that $L(x) \cap L(s) = \emptyset$, and similarly that $L(y) \cap L(u) = \emptyset$. So $|L(x) \cup L(s)| \geq 6$ and $|L(y) \cup L(u)| \geq 6$.
Suppose that $L(x) \cap L(t) \neq \emptyset$. We pick a color $a \in L(x) \cap L(t)$, assign it to x and t, and remove it from the lists of all other vertices. Since $L(x) \cap L(s) = \emptyset$, the list $L(s)$ loses no color ($a \notin L(s)$). If $L(y) \setminus \{a\}$ and $L(v) \setminus \{a\}$ have a common element b, we assign it to y and v, and it is easy to see that w, u, s can be colored in this order with the reduced lists. On the other hand if $L(y) \setminus \{a\}$ and $L(v) \setminus \{a\}$ are disjoint, then it is easy to see that the family $\{L(z) \setminus \{a\} \mid z \in V(H) \setminus \{x, t\}\}$ satisfies Hall's condition, so H is L-colorable. Hence assume that $L(x) \cap L(t) = \emptyset$, and similarly that $L(y) \cap L(v) = \emptyset$. So $|L(x) \cup L(t)| \geq 6$ and $|L(y) \cup L(v)| \geq 6$.
Suppose that $L(t) \cap L(v) \neq \emptyset$. Pick a color $a \in L(t) \cap L(v)$ and assign it to t and v. Since $L(y) \cap L(v) = \emptyset$ and $L(x) \cap L(t) = \emptyset$ we have $L(y) = L(y) \setminus \{a\}$ and similarly $L(x) = L(x) \setminus \{a\}$. It follows that the family $\{L(z) \setminus \{a\} \mid z \in V(H) \setminus \{t, v\}\}$ satisfies Hall's condition. Finally assume that $L(t) \cap L(v) = \emptyset$. So $|L(t) \cup L(v)| \geq 8$. Then the family $\{L(z) \mid z \in V(H)\}$ satisfies Hall's condition, so H is L-colorable.

Finally suppose that $I = \emptyset$. We may assume that X and Y have size 3 and that the non-edges between them form a matching of size 2, because any other situation can be reduced to that one by adding vertices or edges. Let $X = \{u_1, u_2, u_3\}$, $Y = \{v_1, v_2, v_3\}$, and $E(\overline{H}) = \{u_2v_2, u_3v_3\}$. We can choose a color a from $L(x)$ and a color b from $L(y)$ such that $L(u_1) \setminus \{a\} \neq L(v_1) \setminus \{b\}$. Let $L'(u) = L(u) \setminus \{a\}$ for all $u \in X$ and $L'(v) = L(v) \setminus \{b\}$ for all $v \in Y$. By the last sentence of Lemma 4.20, $H \setminus \{x, y\}$ admits an L'-coloring, and we can extend it to an L-coloring of H by assigning color a to x and color b to y. $\square$

LEMMA 4.22

Let H be a cobipartite graph, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, and $E(\overline{H}) = \{x_1y_1, x_2y_2, x_3y_3, x_3y_1, x_1y_2\}$.

Let L be a list assignment on $V(H)$ such that $|L(x_3)| = 2$, $|L(y_2)| = 2$, and $|L(w)| = 3$ for every $w \in V(H) \setminus \{x_3, y_2\}$. Then H is L-colorable.

Proof. Suppose that $L(x_2) \cap L(y_2) \neq \emptyset$. Assign a color a from $L(x_2) \cap L(y_2)$ to x_2 and y_2. Let $L'(u) = L(u) \setminus \{a\}$ for all $u \in \{x_1, x_3, y_1, y_3\}$. Then we can L'-color x_3, x_1, y_3, y_1 greedily in this order, because x_3-x_1-y_3-y_1 is an induced path and the reduced lists' size pattern is $(\geq 1, \geq 2, \geq 2, \geq 2)$. The proof is similar when $L(x_3) \cap L(y_3) \neq \emptyset$. So we may assume that:

$$L(x_2) \cap L(y_2) = \emptyset \text{ and } L(x_3) \cup L(y_3) = \emptyset. \tag{4.4}$$

Suppose that $L(x_1) \cap L(y_2) \neq \emptyset$. Assign a color a from $L(x_1) \cap L(y_2)$ to x_1 and y_2. Let $L'(u) = L(u) \setminus \{a\}$ for all $u \in \{x_2, x_3, y_1, y_3\}$. By (4.4), we have $a \notin L(x_2)$, so $L'(x_2) = L(x_2)$, and a is in at most one of $L(x_3)$ and $L(y_3)$. If $a \in L(x_3)$, then we can L'-color greedily x_3, x_2, y_1, y_3 in this order. If $a \in L(y_3)$, then we can L'-color greedily y_3, y_1, x_2, x_3 in this order. The proof is similar when $L(x_3) \cap L(y_1) \neq \emptyset$. So we may assume that:

$$L(x_1) \cap L(y_2) = \emptyset \text{ and } L(x_3) \cap L(y_1) = \emptyset. \tag{4.5}$$

Suppose that $L(x_1) \cap L(y_1) \neq \emptyset$. Assign a color a from $L(x_1) \cap L(y_1)$ to x_1 and y_1. Let $L'(u) = L(u) \setminus \{a\}$ for all $u \in \{x_2, x_3, y_2, y_3\}$. By (4.5), we have $a \notin L(x_3)$ and $a \notin L(y_2)$. The graph $H \setminus \{x_1, y_1\}$ is an even cycle, and $|L'(u)| \geq 2$ for every vertex u in that graph, so it is L'-colorable. So we may assume that:

$$L(x_1) \cap L(y_1) = \emptyset. \tag{4.6}$$

By (4.4), (4.5) and (4.6), we have $|L(u) \cup L(v)| = 5$ whenever $\{u, v\}$ is any of $\{x_2, y_2\}$, $\{x_3, y_3\}$, $\{x_1, y_2\}$, $\{x_3, y_1\}$, and $|L(x_1) \cap L(y_1)| = 6$. It follows that the family $\{L(w) \mid w \in V(H)\}$ admits a system of distinct representatives, which is an L-coloring for H. $\square$

LEMMA 4.23

Let H be a cobipartite graph with $\omega(G) \leq 4$. Let $V(H)$ be partitioned into two cliques X, Y with $X = \{x_1, x_2, x_3\}$, such that x_1 is complete to Y. Let L be a list assignment such that $|L(x_1)| \geq 3$, $|L(x_2)| \geq 2$, $|L(x_3)| \geq 2$, and $|L(y)| \geq 4$ for all $y \in Y$. Then H is L-colorable.

Proof. Since $Y \cup \{x_1\}$ is a clique, we have $|Y| \leq 3$. If $|Y| \leq 2$, then Lemma 4.19 implies that H is L-colorable. So we may assume that $|Y| = 3$, say $Y = \{y_1, y_2, y_3\}$, and we may assume that $E(\overline{H}) = \{x_2y_2, x_3y_3\}$. If the family $\{L(w) \mid w \in V(H)\}$ admits a system of distinct representatives, then this is an L-coloring of H, so assume the contrary. So there is a set $T \subseteq V(H)$ such that $|L(T)| < |T|$. We have $|L(T)| \geq 2$, so $|T| \geq 3$, so $|L(T)| \geq 3$, so $|T| \geq 4$, so $T \cap Y \neq \emptyset$, so $|L(T)| \geq 4$, and so $|T| \geq 5$. It follows that T is not a clique. Hence assume that $x_2, y_2 \in T$. If $L(x_2) \cap L(y_2) = \emptyset$, then $|L(T)| \geq |L(x_2) \cup L(y_2)| = 6$, so $|T| \geq 7$, which is impossible. Hence $L(x_2) \cap L(y_2) \neq \emptyset$. Assign a color c_2 from $L(x_2) \cap L(y_2)$ to x_2 and y_2. Define $L'(u) = L(u) \setminus \{c_2\}$ for all $u \in V(H) \setminus \{x_2, y_2\}$. If $L'(x_3) \cap L'(y_3) \neq \emptyset$ assign a color c_3 from $L'(x_3) \cap L'(y_3)$ to x_3 and y_3. Then we have $|(L'(x_1) \cup L'(y_1)) \setminus \{c_2\}| \geq 2$, so we can extend the coloring to $\{x_1, y_1\}$. On the other hand, if $L'(x_3) \cap L'(y_3) = \emptyset$, the family $\{L'(w) \mid w \in V(H) \setminus \{x_2, y_2\}\}$ admits a system of distinct representatives. So H admits an L-coloring. $\square$

Lemma 4.24

Let H be a cobipartite graph, where $V(H)$ is partitioned into two cliques $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$, and $E(\overline{H}) = \{x_1y_1, x_1y_3, x_1y_4, x_2y_2, x_2y_3, x_2y_4, x_3y_3, x_4y_4\}$. Let L be a list assignment on $V(H)$ such that $|L(x_1)| = 2$, $|L(x_2)| = 2$ and $|L(w)| = 4$ for all $w \in V(H) \setminus \{x_1, x_2\}$. Then H is L-colorable.

Proof. We choose colors c_1, c_2 with $c_1 \in L(x_1)$, $c_2 \in L(x_2)$ and $c_1 \neq c_2$, such that if $|L(y_1) \cap L(y_2)| = 3$, then either $\{c_1\} \neq L(y_2) \setminus L(y_1)$ or $\{c_2\} \neq L(y_1) \setminus L(y_2)$. This is possible as follows: if $|L(y_1) \cap L(y_2)| = 3$, let α be the color in $L(y_1) \setminus L(y_2)$, then choose $c_2 \in L(x_2) \setminus \{\alpha\}$ and $c_1 \in L(x_1) \setminus \{c_2\}$. We assign color c_1 to x_1 and c_2 to x_2. Let $L'(y_1) = L(y_1) \setminus \{c_2\}$, $L'(y_2) = L(y_2) \setminus \{c_1\}$, $L'(x_3) = L(x_3) \setminus \{c_1, c_2\}$, $L'(x_4) = L(x_4) \setminus \{c_1, c_2\}$, $L'(y_3) = L(y_3)$ and $L'(y_4) = L(y_4)$. So $|L'(u)| \geq 2$ for $u \in \{x_3, x_4\}$, $|L'(v)| \geq 3$ for $v \in \{y_1, y_2\}$, and $|L'(w)| = 4$ for $w \in \{y_3, y_4\}$. Note that the choice of c_1 and c_2 implies that $|L'(y_1) \cup L'(y_2)| \geq 4$. Now we show that $H \setminus \{x_1, x_2\}$ is L'-colorable.

Suppose that $L'(x_3) \cap L'(y_3) \neq \emptyset$. Assign a color c_3 from $L'(x_3) \cap L'(y_3)$ to x_3 and y_3. Define $L''(u) = L'(u) \setminus \{c_3\}$ for all $u \in \{x_4, y_1, y_2, y_4\}$. Note that $|L''(x_4)| \geq 1$, $|L''(u)| \geq 2$ for $u \in \{y_1, y_2\}$, and $|L''(y_4)| \geq 3$. Assign a color c_4 from $L''(x_4)$ to x_4. Since $|L'(y_1) \cup L'(y_2)| \geq 4$, it follows that $|(L''(y_1) \cup L''(y_2)) \setminus \{c_4\}| \geq 2$. So we can L''-color greedily $\{y_1, y_2\}$ and then y_4. The proof is similar if $L'(x_4) \cap L'(y_4) \neq \emptyset$. Therefore we may assume that $L'(x_3) \cap L'(y_3) = \emptyset$ and $L'(x_4) \cap L'(y_4) = \emptyset$, and so $|L'(x_3) \cup L'(y_3)| = 6$ and $|L'(x_4) \cup L'(y_4)| = 6$. This and the choice of c_1, c_2 implies that the family $\{L'(w) \mid w \in V(H) \setminus \{x_1, x_2\}\}$ admits a system of distinct representatives. $\square$

Lemma 4.25

Let H be a cobipartite graph with $\omega(G) \leq 4$. Let C be a clique of size 3 in H such that for every $w \in C$, the set $N(w) \setminus C$ is a clique. Let L be a list assignment such that $|L(w)| = 3$ for all $w \in C$ and $|L(v)| = 4$ for all $v \in V(H) \setminus C$. Then H is L-colorable.

Proof. If H is not connected, it has two components H_1, H_2 and both are cliques of size at most 4. The hypothesis implies easily that for each $i \in \{1, 2\}$ the family $\{L(u) \mid u \in V(H_i)\}$ satisfies Hall's theorem, and consequently H is L-colorable. Hence we assume that H is connected. Let $n = |V(H)|$ and $V(H) = \{v_1, \ldots, v_n\}$. The hypothesis implies that $n \leq 8$. Let $\mu = n - 4$. Since $\omega(H) = 4$, Kőnig's theorem implies that $\overline{H}$ has a matching of size μ. We may assume that the pairs $\{v_i, v_{i+\mu}\}$ $(i = 1, \ldots, \mu)$ form such a matching. We may also assume that $E(H)$ is maximal under the hypothesis of the lemma, since adding edges can only make the problem harder.

First suppose that $n = 4$. The hypothesis implies that the family $\{L(u) \mid u \in V(H)\}$ satisfies Hall's theorem, and consequently H is L-colorable.

Now suppose that $n = 5$. So $\mu = 1$ and $v_1v_2 \in E(\overline{H})$. Up to symmetry, we have either $C = \{v_3, v_4, v_5\}$ or $C = \{v_1, v_3, v_4\}$. If $C = \{v_3, v_4, v_5\}$, then we can L-color greedily the vertices v_3, v_4, v_5, v_1, v_2 in this order. If $C = \{v_1, v_3, v_4\}$, then we can L-color greedily the vertices v_1, v_3, v_4, v_5, v_2 in this order.

Now suppose that $n = 6$. So $\mu = 2$ and $\{v_1v_3, v_2v_4\} \subseteq E(\overline{H})$. Up to symmetry, we have either $C = \{v_1, v_5, v_6\}$ or $C = \{v_1, v_2, v_5\}$. Suppose that $C = \{v_1, v_5, v_6\}$. Since $\{v_1, v_2, v_4\}$ is not a stable set of size 3 and $N(v_1) \setminus C$ is a clique, v_1 is adjacent to exactly one of v_2, v_4, say to v_4 and not to v_2. Then we can L-color greedily the vertices $v_1, v_5, v_6, v_4, v_3, v_2$ in this order. Suppose that $C = \{v_1, v_2, v_5\}$. By the maximality of $E(H)$ we may assume that $E(\overline{H}) = \{v_1v_2,\ v_3v_4\}$. Then Lemma 4.20 (with $X = C$, $Y = V(H) \setminus C$, $x_1 = v_5$ and $y_1 = v_6$) implies that H is L-colorable.

Now suppose that $n = 7$. So $\mu = 3$, and $\{v_1v_4, v_2v_5, v_3v_6\} \subseteq E(\overline{H})$. Up to symmetry, we have either $C = \{v_1, v_2, v_3\}$ or $C = \{v_1, v_2, v_7\}$. If $C = \{v_1, v_2, v_3\}$, then, by the maximality of $E(H)$ we may assume that $E(\overline{H}) = \{v_1v_4,\ v_2v_5,\ v_3v_6\}$, and by Lemma 4.17 (with $X = C$ and $Y = V(H) \setminus C$), H is L-colorable. So suppose that $C = \{v_1, v_2, v_7\}$. For each $i \in \{1, 2\}$, v_i has exactly one neighbor in $\{v_3, v_6\}$, for otherwise either $\{v_i, v_3, v_6\}$ is a stable set of size 3 or $N(v_i) \setminus C$ is not a clique. This leads to the following two cases (a) and (b):

(a) v_1 and v_2 have the same neighbor in $\{v_3, v_6\}$. We may assume that $v_1v_3, v_2v_3 \in E(H)$ and $v_1v_6, v_2v_6 \notin E(H)$. Since H is cobipartite, $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ are cliques, and by the maximality of $E(H)$ we may assume that $\{v_1v_5, v_2v_4, v_3v_4, v_3v_5\} \subseteq E(H)$ and that v_7 is complete to $\{v_1, \ldots, v_6\}$. Pick a color c from $L(v_7)$, assign it to v_7, and set $L'(u) = L(u) \setminus \{c\}$ for all $u \in V(H) \setminus \{v_7\}$. By Lemma 4.17 (with $X = \{v_1, v_2\}$ and $Y = \{v_3, v_4, v_5\}$), $H \setminus \{v_6, v_7\}$ admits an L'-coloring. This can be extended to v_6 since v_6 has only two neighbors in $H \setminus \{v_7\}$. So H is L-colorable.

(b) v_1 and v_2 do not have the same neighbor in $\{v_3, v_6\}$. We may assume that $v_1v_3, v_2v_6 \in E(H)$ and $v_1v_6, v_2v_3 \notin E(H)$. Since H is cobipartite, $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_6\}$ are cliques, and by the maximality of $E(H)$ we may assume that $v_4v_5, v_5v_6 \in E(H)$ and that v_7 is complete to $\{v_1, \ldots, v_6\}$. Pick a color c from $L(v_7)$, assign it to v_7, and set $L'(u) = L(u) \setminus \{c\}$ for all $u \in V(H) \setminus \{v_7\}$. By Lemma 4.22, $H \setminus \{v_7\}$ is L'-colorable. So H is L-colorable.

Now suppose that $n = 8$. So $\mu = 4$ and $\{v_1v_5, v_2v_6, v_3v_7, v_4v_8\} \subseteq E(\overline{H})$. Up to symmetry we have $C = \{v_1, v_2, v_3\}$. For each $i \in \{1, 2, 3\}$, v_i has exactly one neighbor in $\{v_4, v_8\}$, for otherwise either $\{v_i, v_4, v_8\}$ is a stable set of size 3 or $N(v_i) \setminus C$ is not a clique. This leads to two cases: (a) v_1, v_2, v_3 have the same neighbor in $\{v_4, v_8\}$; (b) only two of v_1, v_2, v_3 have a common neighbor in $\{v_4, v_8\}$.

Suppose that (a) holds. We may assume that v_1, v_2, v_3 are all adjacent to v_4 and not adjacent to v_8. Since H is cobipartite, $\{v_1, \ldots, v_4\}$ and $\{v_5, \ldots, v_8\}$ are cliques, and by the maximality of $E(H)$ we may assume that $E(\overline{H}) = \{v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_1v_8, v_2v_8, v_3v_8\}$. By Lemma 4.17 (with $X = \{v_1, v_2, v_3\}$ and $Y = \{v_4, v_5, v_6, v_7\}$), $H \setminus \{v_8\}$ admits an L'-coloring. This can be extended to v_8 since v_8 has only three neighbors in H. So H is L-colorable.

Therefore we may assume that (b) holds. We may assume that v_1v_4, v_2v_4, $v_3v_8 \in E(H)$ and $v_1v_8, v_2v_8, v_3v_4 \notin E(H)$. Since H is cobipartite, $\{v_1, v_2, v_4, v_7\}$ and $\{v_3, v_5, v_6, v_8\}$ are cliques, and by the maximality of $E(H)$ we may assume that $E(\overline{H}) = \{v_1v_5, v_2v_6, v_3v_7, v_4v_8, v_1v_8, v_2v_8, v_3v_4\}$.

Suppose that $L(v_3) \cap L(v_7) \neq \emptyset$. Assign a color c from $L(v_3) \cap L(v_7)$ to v_3 and v_7. Define $L'(w) = L(w) \setminus \{c\}$ for every $w \in V(H) \setminus \{v_3, v_7\}$. By Lemma 4.19, $H \setminus \{v_3, v_7, v_8\}$ admits an L'-coloring. This can be extended to v_8 since v_8 has only two neighbors in $H \setminus \{v_3, v_7\}$. So we may assume that:

$$L(v_3) \cap L(v_7) = \emptyset. \tag{4.7}$$

Suppose that $L(v_1) \cap L(v_5) \neq \emptyset$. Assign a color c from $L(v_1) \cap L(v_5)$ to v_1 and v_5. Define $L'(w) = L(w) \setminus \{c\}$ for every $w \in V(H) \setminus \{v_1, v_5\}$. By Lemma 4.22 the graph $H \setminus \{v_1, v_5\}$ is L'-colorable. The proof is similar if $L(v_2) \cap L(v_6) \neq \emptyset$. So we may assume that:

$$L(v_1) \cap L(v_5) = \emptyset \text{ and } L(v_2) \cup L(v_6) = \emptyset. \tag{4.8}$$

Suppose that $L(v_3) \cap L(v_4) \neq \emptyset$. Assign a color c from $L(v_3) \cap L(v_4)$ to v_3 and v_4. Define $L'(w) = L(w) \setminus \{c\}$ for every $w \in V(H) \setminus \{v_3, v_4\}$. By (4.7), we have $c \notin L(v_7)$, so $L'(v_7) = L(v_7)$. Hence and by (4.7) and (4.8), the family $\{L'(w) \mid w \in V(H) \setminus \{v_3, v_4\}\}$ admits a system of distinct representatives. So we may assume that:

$$L(v_3) \cup L(v_4) = \emptyset. \tag{4.9}$$

Suppose that $L(v_4) \cap L(v_8) \neq \emptyset$. Assign a color c from $L(v_4) \cap L(v_8)$ to v_4 and v_8. Define $L'(w) = L(w) \setminus \{c\}$ for every $w \in V(H) \setminus \{v_4, v_8\}$. By (4.9), we have $c \notin L(v_3)$, so $L'(v_3) = L(v_3)$. By (4.7), (4.8) and (4.9), the family $\{L'(w) \mid w \in V(H) \setminus \{v_4, v_8\}\}$ admits a system of distinct representatives. So we may assume that:

$$L(v_4) \cup L(v_8) = \emptyset. \tag{4.10}$$

By (4.7), (4.8), (4.9) and (4.10), we have $|L(v_i) \cup L(v_j)| = 7$ if the pair $\{i, j\}$ is any of $\{1, 5\}$, $\{2, 6\}$, $\{3, 7\}$ and $\{3, 4\}$, and $|L(v_4) \cup L(v_8)| = 8$. It follows easily that the family $\{L(w) \mid w \in V(H)\}$ admits a system of distinct representatives. □

4.3.3 Elementary graphs

Now we can consider the case of any elementary graph G with $\omega(G) \leq 4$.

THEOREM 4.26

Let G be an elementary graph with $\omega(G) \leq 4$. Then $ch(G) = \chi(G)$.

Proof. This theorem holds for every graph G with $\omega(G) \leq 3$ as proved in [39]. Hence we will assume that $\omega(G) = 4$. By Theorem 4.11, G is the augmentation of the line-graph $\mathcal{L}(H)$ of a bipartite multigraph H. Let $e_1, \ldots, e_h$ be the flat edges of $\mathcal{L}(H)$ that are augmented to obtain G. We prove the theorem by induction on h. If $h = 0$, then $G = \mathcal{L}(H)$; in that case the equality $ch(G) = \chi(G)$ follows from Galvin's theorem [35]. Now assume that $h > 0$ and that the theorem holds for elementary graphs obtained by at most $h - 1$ augmentations. Let (X, Y) be the augment in G that corresponds to the edge e_h of $\mathcal{L}(H)$. In $\mathcal{L}(H)$, let $e_h = xy$. So x, y are incident edges of H. In H, let $x = q_x q_{xy}$ and $y = q_y q_{xy}$; so their common vertex q_{xy} has degree 2 in H. Let G_{h-1} be the graph obtained from $\mathcal{L}(H)$ by augmenting only the $h - 1$ other edges $e_1, \ldots, e_{h-1}$. So G_{h-1} is an elementary graph.

Let L be a list assignment on $V(G)$ such that $|L(v)| = \omega(G)$ for all $v \in V(G)$. We will prove that G admits an L-coloring.

$$\text{We may assume that } |X \cup Y| > \omega(G). \tag{4.11}$$

Suppose that $|X \cup Y| \leq \omega(G)$. Let H' be the graph obtained from H by duplicating $|X| - 1$ times the edge x (so that there are exactly $|X|$ parallel edges between the two ends of x in H) and duplicating $|Y| - 1$ times the edge y. Let G'_{h-1} be the graph obtained from $\mathcal{L}(H')$ by augmenting the $h - 1$ edges $e_1, \ldots, e_{h-1}$ as in G. Then G'_{h-1} can also be obtained from G by adding all edges between non-adjacent vertices of $X \cup Y$. By the assumption, we have $\omega(G'_{h-1}) = \omega(G)$. By the induction hypothesis, G'_{h-1} admits an L-coloring. Then this is an L-coloring of G. Hence (4.11) holds.

Let $X = \{x_1, \ldots, x_{|X|}\}$ and $Y = \{y_1, \ldots, y_{|Y|}\}$. Let $N_X = \{v \in V(G) \setminus (X \cup Y) \mid v$ has a neighbor in $X\}$ and $N_Y = \{v \in V(G) \setminus (X \cup Y) \mid v$ has a neighbor in $Y\}$. By the definition of a line-graph and of an augment, the set N_X is a clique and is complete to X; hence $|N_X| \leq \omega(G) - |X|$. Likewise N_Y is a clique and is complete to Y, and $|N_Y| \leq \omega(G) - |Y|$. Let μ be the size of a maximum matching in the bipartite graph $\overline{G}[X \cup Y]$. By Kőnig's theorem we have $\mu + \omega(G) = |X| + |Y|$, so $\mu = |X| + |Y| - 4$. Moreover, we may assume that the edges of $\overline{G}[X \cup Y]$ form a matching of size μ (for otherwise we can add some edges to G, in $X \cup Y$, which makes the coloring problem only harder).

The graph $G_{h-1} \setminus \{x, y\}$ is elementary, and it has $h - 1$ augments, so, by the induction hypothesis, it admits an L-coloring f. We will try to extend f to G; if this fails, we will analyse why and then show that we can find another L-coloring of $G_{h-1} \setminus \{x, y\}$ that does extend to G. Let L' be the list assignment defined on $X \cup Y$ as follows: for all $u \in X$, let $L'(u) = L(u) \setminus f(N_X)$, and for all $v \in Y$, let $L'(v) = L(v) \setminus f(N_Y)$. Clearly, f extends to an L-coloring of G if and only if $G[X \cup Y]$ admits an L'-coloring. By (4.11) and up to symmetry, we may assume that either $|Y| = 4$ (and $|X| \leq 4$) or $(|X|, |Y|)$ is equal to $(3, 3)$ or $(2, 3)$. We deal with each case separately.

Case 1: $|Y| = 4$ and $|X| \leq 4$. We have $|N_X| \leq 4 - |X|$ and $|N_Y| = 0$, so $|L'(u)| \geq |X|$ for all $u \in X$ and $|L'(v)| = 4$ for all $v \in Y$. Since $\omega(G) = 4$, there are $|X|$ non-edges between X and Y that form a matching in $\overline{G}$. By Lemma 4.17, $G[X \cup Y]$ admits an L'-coloring.

Case 2: $|X| = |Y| = 3$. Here we have $\mu = 2$, and we may assume that the non-edges between X and Y are x_2y_2 and x_3y_3. We have $|N_X| \leq 1$ and $|N_Y| \leq 1$, so $|L'(u)| \geq 3$ for all $u \in X \cup Y$. If $G[X \cup Y]$ is L'-colorable we are done, so assume the contrary. By Lemma 4.20, there is a clique $Q \subset X \cup Y$ such that $|L'(Q)| < |Q|$. Thus $3 \leq |L'(Q)| < |Q| \leq 4$. This implies that $|Q| = 4$, and in particular Q contains x_1 and y_1. Moreover $|L'(Q)| = 3$, so $L'(x_1)$ and $L'(y_1)$ are equal and have size 3, so $|N_X| = 1$ and $|N_Y| = 1$. Let $N_X = \{u\}$ and $N_Y = \{v\}$. Thus there are colors a, b, c, d, d' such that $L(x_1) = \{a, b, c, d\}$, $L(y_1) = \{a, b, c, d'\}$, $f(u) = d$ and $f(v) = d'$ (possibly $d = d'$). In other words, f satisfies the following "bad" property:

$$\text{Either } L(x_1) = L(y_1) \text{ and } f(u) = f(v), \text{ or } |L(x_1) \cap L(y_1)| = 3 \text{ and } \{f(u)\} = L(x_1) \setminus L(y_1) \text{ and } \{f(v)\} = L(y_1) \setminus L(x_1). \tag{4.12}$$

Let G^* be the graph obtained from G by removing all edges between X and Y and adding two new vertices u^* and v^* with edges u^*v^*, u^*x_i ($i = 1, 2, 3$) and v^*y_i ($i = 1, 2, 3$). Let H^* be the graph obtained from H by removing the vertex q_{xy} and adding three vertices q_1, q_2, q_3, with edges q_1q_2 and q_2q_3, plus three parallel edges between q_x and q_1 and three parallel edges between q_3 and q_y. So H^* is bipartite, and it is easy to see that G^* is obtained from $\mathcal{L}(H^*)$ by augmenting $e_1, \ldots, e_{h-1}$ as in G. So G^* is elementary.

We define a list assignment L^* on G^* as follows. For all $v \in V(G \setminus (X \cup Y))$, let $L^*(v) = L(v)$. For all $v \in X \cup \{u^*, v^*\}$ let $L^*(v) = \{a, b, c, d\}$, and for all $v \in Y$ let $L^*(v) = \{a, b, c, d'\}$. By the induction hypothesis on h, the graph G^* admits an L^*-coloring f^*. In particular f^* is an L-coloring of $G \setminus (X \cup Y)$. We claim that if $d = d'$ then $f^*(u) \neq f^*(v)$, and if $d \neq d'$ then either $f^*(u) \neq d$ or $f^*(v) \neq d'$. Indeed we have $f^*(X) = \{a, b, c, d\} \setminus \{f^*(u)\}$ and $f^*(Y) = \{a, b, c, d'\} \setminus \{f^*(v)\}$, so if the claim fails then $f^*(X) = f^*(Y)$ and consequently $f^*(u^*) = f^*(v^*)$, a contradiction. So the claim holds. By the claim, we can use f^* instead of f above (as an L-coloring of $G \setminus (X \cup Y)$), because f^* does not satisfy (4.12); so we can extend it to an L-coloring of G.

Case 3: $|X| = 3$ and $|Y| = 2$. Here we have $\mu = 1$, and we may assume that the only non-edge between X and Y is x_3y_2. We have $|N_X| \leq 1$ and $|N_Y| \leq 2$, so $|L'(u)| \geq 3$ for all $u \in X$ and $|L'(v)| \geq 2$ for all $v \in Y$. If $G[X \cup Y]$ is L'-colorable we are done, so assume the contrary. By Lemma 4.19, there is a clique $Q \subset X \cup Y$ such that $|L'(Q)| < |Q|$. This inequality implies that $Q \not\subseteq Y$, so $Q \cap X \neq \emptyset$. Thus $3 \leq |L'(Q)| < |Q| \leq 4$. This implies that $|Q| = 4$, and in particular Q contains x_1, x_2 and y_1. Moreover $|L'(Q)| = 3$, so $L'(x_1)$ and $L'(x_2)$ are equal and have size 3, so $|N_X| = 1$, and $L'(y_1)$ has size at most 3, so $|N_Y| \geq 1$, and $L'(y_1) \subseteq L'(x_1)$. Let $N_X = \{u\}$. Thus $L(x_1) = L(x_2)$, and f satisfies the following "bad" property:

$$f(u) \in L(x_1) \text{ and } L(y_1) \setminus f(N_Y) \subseteq L(x_1) \setminus \{f(u)\}. \tag{4.13}$$

Let $G^* = G \setminus \{x_3\}$. Clearly G^* is elementary. Let H^* be the graph obtained from H by duplicating the edge q_xq_{xy} (so that there are two parallel edges between q_x and q_{xy}) and similarly duplicating q_yq_{xy}. It is easy to see that G^* is obtained from $\mathcal{L}(H^*)$ by augmenting $e_1, \ldots, e_{h-1}$ as in G. We define a list assignment L^* on G^* as follows.

For all $v \in V(G^*) \setminus \{y_2\}$, let $L^*(v) = L(v)$, and let $L^*(y_2) = L(y_1)$. By the induction hypothesis on h the graph G^* admits an L^*-coloring f^*. We claim that f^* does not satisfy the bad property (4.13). Indeed if it does, then $f^*(u) \in L^*(x_1)$ and $L^*(y_1) \setminus f^*(N_Y) \subseteq L^*(x_1) \setminus \{f^*(u)\}$. Since $L^*(y_2) = L^*(y_1)$, we also have $L^*(y_2) \setminus f^*(N_Y) \subseteq L^*(x_1) \setminus \{f^*(u)\}$, and this means that the four vertices x_1, x_2, y_1, y_2 (which induce a clique) are colored by f^* using colors from $L^*(x_1) \setminus \{f^*(u)\}$, which has size 3; but this is impossible. So the claim holds. By the claim, we can use f^* instead of f above (as an L-coloring of $G \setminus (X \cup Y)$) and we can extend it to an L-coloring of G. This completes the proof of the theorem. □

4.3.4 Claw-free perfect graphs

Now we can prove Theorem 4.13.

Proof. We may assume that G is connected. Let L be a list assignment on G such that $|L(v)| \geq 4$ for all $v \in V(G)$. Let us prove that G is L-colorable by induction on the number of vertices of G. If G is peculiar, then by Lemma 4.16 we know that the theorem holds. So assume that G is not peculiar. By Theorem 4.10 and Lemma 4.15, we know that G can be decomposed by clique cutsets into elementary graphs. We may assume that:

$$G \text{ has no simplicial vertex.} \tag{4.14}$$

Suppose that x is a simplicial vertex in G. By the induction hypothesis, $G \setminus \{x\}$ admits an L-coloring f. Since x is simplicial, it has at most three neighbors. So f can be extended to x by choosing in $L(x)$ a color not assigned by f to its neighbors. Thus (4.14) holds.

By the discussion after the definition of a clique cutset (Section 1), G admits an extremal cutset C, i.e., a minimal clique cutset such that for some component A of $G \setminus C$ the induced subgraph $G[A \cup C]$ is an atom (i.e., has no clique cutset). Since C is minimal, every vertex x of C has a neighbor in every component of $G \setminus C$ (for otherwise $C \setminus \{x\}$ would be a clique cutset), and it follows that $G \setminus C$ has only two components A_1, A_2 (for otherwise x would be the center of a claw). For $i = 1, 2$ let $G_i = G[C \cup A_i]$. Hence we may assume that G_2 is elementary.

By the induction hypothesis, the graph $G[C \cup A_1]$ is 4-choosable, so it admits an L-coloring f. We will show that we can extend this coloring to G.

By Theorem 4.11, G_2 is obtained by augmenting the line-graph $\mathcal{L}(H)$ of a bipartite graph H. For each augment (X, Y) of G_2, select a pair of adjacent vertices such that one is in X and the other is in Y. Also select all vertices of G_2 that are not in any augment. It is easy to see that $\mathcal{L}(H)$ is isomorphic to the subgraph of G_2 induced by the selected vertices. Without loss of generality it will be convenient to view $\mathcal{L}(H)$ as equal to that induced subgraph. We claim that:

If there is an augment (X, Y) in G_2 such that both $C \cap X$ and $C \cap Y$ are non-empty, then $V(G_2) = X \cup Y$. (4.15)

Suppose on the contrary, under the hypothesis of (4.15), that $V(G_2) \neq X \cup Y$. Let $Z = V(G_2) \setminus (X \cup Y)$. Let $Z_X = \{z \in Z \mid z \text{ has a neighbor in } X\}$ and $Z_Y = \{z \in Z \mid z \text{ has a neighbor in } Y\}$. By the definition of an augment, Z_X is complete to X and anticomplete to Y, and Z_Y is complete to Y and anticomplete to X, and $Z_X \cap Z_Y = \emptyset$. Since G_2 is connected, we may assume up to symmetry that $Z_X \neq \emptyset$. Pick any $z \in Z_X$. Since G_2 is an atom, X is not a cutset of G_2 (separating z from Y), so $Z_Y \neq \emptyset$, which restores the symmetry between X and Y. Since C is a clique and has a vertex in Y, C contains no vertex from Z_X; similarly, C contains no vertex from Z_Y; hence $C \subset X \cup Y$. Pick any $x \in C \cap X$. Since C is a minimal cutset, x has a neighbor a_1 in A_1. Then a_1 must be adjacent to every neighbor y of x in Y, for otherwise $\{x, a_1, z, y\}$ induces a claw; and it follows that $y \in C$. We can repeat this argument for every vertex in C; by the last item in Theorem 4.11 it follows that every vertex in $X \cup Y$ is adjacent to a_1 and, consequently, is in C. But this is a contradiction because C is a clique and $X \cup Y$ is not a clique. Thus (4.15) holds.

Now we distinguish two cases.

(I) First suppose that G_2 is not a cobipartite graph.

For every edge uv in the bipartite multigraph H, let C_{uv} be the subset of $V(G_2)$ defined as follows. If v has degree 2 in H, say $N_H(v) = \{u, u'\}$, and $\{vu, vu'\}$ is a flat edge in $\mathcal{L}(H)$ on which an augment (X, X') of G_2 is based (where X corresponds to vu and X' corresponds to vu'), then let $C_{uv} = X$. If uv is not such an edge, then let C_{uv} be the set of parallel edges in H whose ends are u and v. Now for every vertex u in H, let $C_u = \bigcup_{uv \in E(H)} C_{uv}$. Note that C_u is a clique in G_2. We claim that:

$$\text{There is a vertex } u \text{ in } H \text{ such that } C = C_u. \tag{4.16}$$

For every augment (X, Y) in G_2 we have $V(G_2) \neq X \cup Y$, because G_2 is not cobipartite, and so, by (4.15), either $C \cap X$ or $C \cap Y$ is empty. It follows that there is a vertex u in H such that $C \subseteq C_u$. Suppose that $C \neq C_u$. Then we can pick vertices $x \in C$ and $x' \in C_u \setminus C$ such that H has vertices v, v' with $x \in C_{uv}$ and $x' \in C_{uv'}$. Since C is a minimal cutset, x has a neighbor a_1 in A_1. Since G_2 is an atom, the set $C_u \setminus C_{uv}$ is not a cutset, so x has a neighbor z in $V(G_2) \setminus C_u$. Then $\{x, a_1, x', z\}$ induces a claw, a contradiction. So $C = C_u$ and (4.16) holds.

By (4.16), let u be a vertex in H such that $C = C_u$. Let $D = \{d \in A_1 \mid d \text{ has a neighbor in } C\}$. We claim that:

$$D \cup C \text{ is a clique.} \tag{4.17}$$

Pick any d in D. First suppose that d is not complete to C. Then we can find vertices $x \in C \cap N(d)$ and $x' \in C \setminus N(d)$ such that H has vertices v, v' with $x \in C_{uv}$ and $x' \in C_{uv'}$. Since G_2 is an atom, the set $C_u \setminus C_{uv}$ is not a cutset, so x has a neighbor z in $V(G_2) \setminus C_u$. Then $\{x, d, x', z\}$ induces a claw, a contradiction. It follows that D is complete to C. Now suppose that D contains non-adjacent vertices d, d'. Pick any $x \in C$. Then x has a neighbor z in $V(G_2) \setminus C_u$. Then $\{x, d, d', z\}$ induces a claw, a contradiction. So D is a clique. Thus (4.17) holds.

$$G[D \cup C \cup A_2] \text{ is an elementary graph.} \tag{4.18}$$

Let H^* be the bipartite graph obtained from H by adding $|D|$ vertices of degree 1 adjacent to vertex u. Then it is easy to see (by (4.16) and (4.17)) that $G[D \cup C \cup A_2]$ can be obtained from $\mathcal{L}(H^*)$ by augmenting the same flat edges as for G_2 and with the same augments. Thus (4.18) holds.

Let $D = \{d_1, \dots, d_p\}$. (Actually we have $|C| \geq 2$ by (4.16) and consequently $|D| \leq 2$ by (4.17), but we will not use this fact.) Recall that f is an L-coloring of G_1; so for $i = 1, \dots, p$ let $c_i = f(d_i)$.

The maximum degree in H^* is $\Delta(H^*) = \omega(\mathcal{L}(H^*)) \leq \omega(G_2) \leq \omega(G) \leq 4$. So we can color the edges of H^* with 4 colors in such a way that vertices $d_1, \dots, d_p$ receive colors $c_1, \dots, c_p$ respectively. Let L^* be a list assignment on $\mathcal{L}(H^*)$ defined as follows. If $v \in V(\mathcal{L}(H))$, let $L^*(v) = L(v)$. For $i = 1, \dots, p$, let $L^*(d_i) = \{c_1, \dots, c_i\}$. By Theorem 4.14, $\mathcal{L}(H^*)$ admits an L^*-coloring f^*. Now we can use the same technique as in the proof of Theorem 4.26 to extend f^* to an L-coloring of G_2. Moreover, we have $f^*(d_1) = c_1$ and consequently $f^*(d_i) = c_i = f(d_i)$ for all $i = 1, \dots, p$. Let f' be defined as follows. For all $v \in V(G_1) \setminus C$, let $f'(v) = f(v)$, and for all $v \in V(G_2)$, let $f'(v) = f^*(v)$. Then f' is an L-coloring of G. This completes the proof in case (I).

(II) We may now assume that G_2 is a cobipartite graph. Let D be the set of vertices of A_1 that have a neighbor in C. For all $x \in C$, let $N_1(x) = N(x) \cap A_1$, $N_2(x) = N(x) \cap A_2$ and $M_2(x) = A_2 \setminus N(x)$. We observe that:

$$N_1(x) \text{ and } N_2(x) \text{ are non-empty cliques, and } M_2(x) \text{ is a clique.} \tag{4.19}$$

We know that $N_1(x)$ and $N_2(x)$ are non-empty because C is a minimal cutset. For $i = 1, 2$ pick any $n_i \in N_i(x)$; then $N_i(x)$ is a clique, for otherwise x is the center of a claw with n_{3-i} and two non-adjacent vertices from $N_i(x)$. Also $M_2(x)$ is a clique, for otherwise G_2 contains a stable set of size 3. Thus (4.19) holds.

Suppose that $|C| = 1$. Let $C = \{x\}$. Then $M_2(x)$ is empty, for otherwise $N_2(x)$ is a clique cutset in G_2 (separating x from $M_2(x)$). So G_2 is a clique. Then every vertex in A_2 is simplicial, a contradiction to (4.14). So $|C| \geq 2$.

Suppose that two vertices x and y of C have inclusion-wise incomparable neighborhoods in A_1. So there is a vertex a in A_1 adjacent to x and not to y, and there is a vertex b in A_1 adjacent to y and not to x. If a vertex u in A_2 is adjacent to x, then it is adjacent to y, for otherwise $\{x, a, y, u\}$ induces a claw, and vice-versa. So $N_2(x) = N_2(y)$, and $|N_2(x)| \leq 2$ (because $N_2(x) \cup \{x, y\}$ is a clique), and $M_2(x) = M_2(y)$. Suppose that $M_2(x) \neq \emptyset$. Let $C' = \{u \in C \setminus \{x, y\} \mid u \text{ is complete to } N_2(x)\}$. Since $C' \cup N_2(x)$ is a clique, it cannot be a cutset of G_2, so some vertex z in $C \setminus (C' \cup \{x, y\})$ has a neighbor v in $M_2(x)$. Since $z \notin C'$, z has a non-neighbor u in $N_2(x)$. Then za is an edge, for otherwise $\{x, a, z, u\}$ induces a claw. But then $\{z, a, y, v\}$ induces a claw, a contradiction. So $M_2(x) = \emptyset$. Thus $A_2 = N_2(x) = N_2(y)$. If the vertices in A_2 have pairwise comparable neighborhoods in C, then it follows easily that the vertex in A_2 with the smallest degree is simplicial in G, a contradiction to (4.14). So there are two vertices u, v in A_2 and two vertices z, t in C such that tu, zv are edges and tv, zu are not edges. Clearly $z, t \notin \{x, y\}$, so $|C| = 4$. Then za is an edge, for otherwise $\{x, a, z, u\}$ induces a claw; and similarly, zb, ta, tb are edges. Then ab is an edge, for otherwise $\{z, a, b, v\}$ induces a claw. Recall that since G is perfect and claw-free, the neighborhood of every vertex can be partitioned into two cliques, and consequently (since $\omega(G) \leq 4$) every vertex has degree at most 6. Hence $N(x) = \{y, z, t, a, u, v\}$ (because we already know that x is adjacent to these six vertices), and similarly $N(y) = \{x, z, t, b, u, v\}$, $N(z) = \{x, y, t, a, b, v\}$, and $N(t) = \{x, y, z, a, b, u\}$. It follows that $A_2 = \{u, v\}$ and $D = \{a, b\}$. Here we view f as an L-coloring of $G_1 \setminus (C \cup \{a, b\})$ rather than of G_1, and we try to extend it to $\{a, b\} \cup C \cup A_2$. Let $S = \{s \in V(G_1) \setminus (C \cup \{a, b\}) \mid s$ has a neighbor in $\{a, b\}\}$. If a vertex $s \in S$ is adjacent to a and not to b, then $\{a, s, b, x\}$ induces a claw, a contradiction. By symmetry this implies that S is complete to $\{a, b\}$. Then S is a clique, for otherwise $\{a, s, s', x\}$ induces a claw from some non-adjacent $s, s' \in S$. So $S \cup \{a, b\}$ is a clique, and so $|S| \leq 2$. We remove the colors of $f(S)$ from the lists of a and b. By Lemma 4.24 we can color the vertices of $D \cup C \cup \{u, v\}$ with colors from the lists thus reduced. So G is L-colorable.

Therefore we may assume that any two vertices of C have inclusion-wise comparable neighborhoods in A_1. This implies that some vertex a_1 in A_1 is complete to C, and that some vertex x in C is complete to D. Since $\{a_1\} \cup C$ is a clique, we have $|C| \leq 3$. We have $D = N_1(x)$ and, by (4.19), D is a clique, so $|D| \leq 3$. Here we view f as an L-coloring of $G_1 \setminus C$ rather than of G_1, and we try to extend it to $C \cup A_2$. If $|D| = 1$ (i.e., $D = \{a_1\}$), we remove the color $f(a_1)$ from the list of the vertices in C. Then G_2 is a cobipartite graph which, with the reduced lists, satisfies the hypothesis of Lemma 4.21 or 4.25, so f can be extended to G_2. Hence assume that $|D| \geq 2$.

Suppose that D is complete to C. Then $D \cup C$ is a clique, so $|D| = 2$ and $|C| = 2$. Let $C = \{x, y\}$. Let $X = N_2(x)$, $Y = N_2(y)$, and $Z = A_2 \setminus (X \cup Y)$. Suppose that $Z \neq \emptyset$. By (4.19) $Z \cup (X \setminus Y)$ is a clique, since it is a subset of $M_2(y)$. Likewise, $Z \cup (Y \setminus X)$ is a clique. Moreover $X \setminus Y$ is complete to $Y \setminus X$, for otherwise $\{x, y, v, z, u\}$ induces a C_5 for some non-adjacent $u \in X \setminus Y$ and $v \in Y \setminus X$ and for any $z \in Z$. It follows that $X \cup Y$ is a clique cutset in G_2 (separating $\{x, y\}$ from Z), a contradiction. So $Z = \emptyset$, and $A_2 = X \cup Y$. Here we view f as an L-coloring of $G_1 \setminus C$ rather than of G_1, and we try to extend it to $C \cup A_2$. We remove the colors of $f(D)$ from the list of x and y. Since $|D| = 2$, each of these lists loses at most two colors. By Lemma 4.21 we can color the vertices of $C \cup A_2$ with colors from the lists thus reduced. So G is L-colorable.

Now assume that D is not complete to C. So some vertex d in D has a non-neighbor y in C. Then $N_2(x) \cup \{y\}$ is a clique, for otherwise $\{x, d, u, v\}$ induces a clique for any two non-adjacent vertices $u, v \in X \cup \{y\}$. Suppose that $M_2(x)$ is empty. So $A_2 = N_2(x)$. Then the vertices in A_2 have comparable neighborhoods in C (because they are complete to $\{x, y\}$ and $|C| \leq 3$), so the vertex in A_2 with the smallest degree is simplicial, a contradiction to (4.14). Therefore $M_2(x)$ is not empty. Since the clique $\{y\} \cup N_2(x)$ is not a cutset in G_2, some vertex z in $C \setminus \{x, y\}$ has a neighbor v in $M_2(x)$.

Hence $|C| = 3$. Then z has a non-neighbor u in $N_2(x)$, for otherwise $\{y,z\} \cup N_2(x)$ is a clique cutset in G_2 (separating x from v). Then zd is an edge, for otherwise $\{x,d,z,u\}$ induces a claw; and yv is an edge, for otherwise $\{z,d,y,v\}$ induces a claw; and uv is an edge since $N_2(y)$ is a clique. Moreover, if $N_2(x)$ contains a vertex u' adjacent to z, then vu' is an edge since $N_2(z)$ is a clique. Since this holds for every vertex in $M_2(x) \cap N(z)$, we deduce that $(M_2(x) \cap N(z)) \cup \{y\} \cup N_2(x)$ is a clique Q. If v' is any non-neighbor of z in $M_2(x)$, then Q is a clique cutset in G_2 (separating $\{x,z\}$ from v'), a contradiction. So $M_2(x) \subset N(z)$. Suppose that $|D| = 3$. Pick $a_3 \in D \setminus \{a_1, d\}$. Then a_3z is not an edge, for otherwise $D \cup \{x,z\}$ is a clique of size 5. So, by the same argument as for d, we deduce that a_3y is an edge. But this means that y and z have inclusion-wise incomparable neighborhoods in A_1 (because of d, a_3), a contradiction. So $|D| = 2$. We remove the color $f(a_1)$ from the lists of x, y, z and remove the color $f(d)$ from the list of x and z. By Lemma 4.23 we can color the vertices of $C \cup A_2$ with colors from the lists thus reduced. So G is L-colorable. This completes the proof of the theorem. □

Chapter 5

Maximum Weighted Stable Set

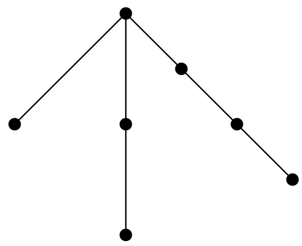
5.1 Context and motivations

The problem discussed in this chapter is an optimization problem that is often used in many different aspects of operational research and combinatorics. The *Maximum Stable Set Problem*, shortened *MSS*, is the problem of finding the stable set of maximum cardinality in a given graph. Let G be a graph, the weighted version of this problem is defined by the mean of a weight function on the vertices of G, $w : V(G) \rightarrow \mathbb{Q}$ that assign to each vertex v a weight $w(v)$. The *Maximum Weight Stable Set Problem*, shortened *MWSS*, is now to find the stable set of maximum weight, that we denote by $\alpha_w(G)$. It is well known that the MSS problem is NP-Hard in general, and so is the MWSS problem. Let $S_{i,j,k}$ be the graph obtained from the claw by subdividing its edges into i, j and k edges (see Figure 5.1 for an example). In 1983, Alekseev [1] proved that the MSS problem remains NP-Hard in the class of $\mathcal{F}$-free graphs whenever $\mathcal{F}$ is a finite set of graphs such that no member of $\mathcal{F}$ is a $S_{i,j,k}$ graph. In other words, to hope for a polynomial-time algorithm for those problems in a specific graphs class, a graph $S_{i,j,k}$ needs to be forbidden. Many results are known in specific classes of $S_{i,j,k}$-free graphs. For instance, a claw-free graph is nothing more than a $S_{1,1,1}$-free graph.

Several authors proved that the MWSS problem in claw-free graphs can be solved in polynomial time. The two first results were published in 1980 independently by Minty [70] and Sbihi [85] (non-weighted version). Later, in 2001, Nakamura et al. [76] improved Minty's algorithm as it failed in the weighted version for a few special cases. An implementation of this algorithm could be made to run in $\mathcal{O}(n^4 \log(n))$ where n is the number of vertices of the graph [77]. In 2011, Faenza et al. [31] improved the complexity to $\mathcal{O}(n^3)$. In 2015, Nobili et al. [78] lowered the complexity to $\mathcal{O}(n^2 \log(n))$.

Lozin and Milanič proved that the MWSS problem is polynomial solvable in fork-free graphs ($S_{1,1,2}$) [66].

Another important result concerning this problem is due to Lokshtanov, Vatshelle and Villanger who proved that the MWSS problem can be solved in polynomial time



$S_{1,2,3}$

Figure 5.1: The graph $S_{1,2,3}$.

for P_5-free graphs ($S_{0,2,2}$-free graphs) [64].

The union of the results cited above close the complexity of the MWSS problem in F-free graphs whenever F is any $S_{i,j,k}$ on at most five vertices. Hence, the graph classes worth taking a look at are the F-free graphs where F has at least six vertices. Several results on the existence of a polynomial-time algorithm for the MWSS problem in subclasses of P_6-free graphs have been published [54, 55, 68, 72, 73, 74]. Lokshtanov et al. [63] proved that the MWSS problem can be solved in quasi-polynomial time in the class of P_6-free graphs. Brandstädt and Mosca proved that there exists a polynomial-time algorithm for the MWSS problem in the class of (P_7, K_3)-free graphs [11].

One very useful theorem for tackling the MWSS problem is due to Lozin and Milanič. They proved that in a hereditary class of graphs $\mathcal{G}$, in order to prove that there exists a polynomial-time algorithm for the MWSS problem, it suffices to prove it for every prime graph of $\mathcal{G}$. This results gives strong structural properties. More formally, they proved the following.

Theorem 5.1 [66]

Let $\mathcal{G}$ be a hereditary class of graphs. Suppose that there is a constant $c \geq 1$ such that the MWSS problem can be solved in time $O(|V(G)|^c)$ for every prime graph G in $\mathcal{G}$. Then the MWSS problem can be solved in time $O(|V(G)|^c + |E(G)|)$ for every graph G in $\mathcal{G}$.

Because this theorem is central in our approach and of great importance for the MWSS problem, we include its proof. Lozin and Milanič produced an algorithm that, given a graph G, either reduce the problem to computing a maximum-weight stable set in a complete graph, an edgeless graph or a prime graph. Their algorithm is recursive but thanks to modular decomposition theory, they manage to show that the number of recursive calls is bounded by a linear function of m, the number of edges in G. Their algorithm is given in Algorithm 2.

Note that the graph G^* obtained at the end of the for-loop at step 15 is either a complete graph, an edgeless graph or a prime graph. Computing a maximum-weight independent set in a complete graph or an edgeless graph can be done in linear time,

Algorithm 2 ALPHA(G)

Input : A weighted graph G with weight function w
Output : An independent set of maximum weight in G

```
procedure ALPHA(G)
    if |V(G)| = 1 then
        return V(G)
    else if G is disconnected then
        partition G into components M_1, ..., M_k
    else if \overline{G} is disconnected then
        partition G into co-components M_1, ..., M_k
    else
        partition G into maximal modules M_1, ..., M_k
    end if
    for all i ∈ {1, ..., k} do
        I_i ← ALPHA(G[M_i])
    end for
    G* ← G
    for all i ∈ {1, ..., k} do
        in G*, contract M_i to a single vertex v_i
        assign to v_i the weight w(I_i)
    end for
    I* ← maximum-weight stable set of G*
    I ← ⋃_{v_i ∈ I*} I_i
    return I
end procedure
```

so the problem is reduced to prime induced subgraphs. Hence, to prove Theorem 5.1, it suffices to bound the number of recursive calls in Algorithm 2.

Proof of Theorem 5.1. Let G be a graph of $\mathcal{G}$ with n vertices and m edges. The recursive modular decomposition of G produced by Algorithm 2 can be implemented in $\mathcal{O}(n+m)$ [89]. The modular decomposition of G produces a decomposition tree $T(G)$ whose leaves correspond to vertices of G, internal nodes to modules of G and the root to $V(G)$. By doing a bottom-up approach on the tree $T(G)$, we can bound the time complexity of Algorithm 2 in the following way. We denote by $\mathcal{I}$ the set of all internal nodes of $T(G)$. Let U be any internal of $\mathcal{I}$ and let G_U be the subgraph of G induced by the vertices of U. The children of U in $T(G)$ correspond to the modular partition of G_U into modules $\{M_1, M_2, \dots, M_k\}$ for some integer k. Recall that G^*_U is the graph obtained from the induced subgraph G_U where each module M is contracted into a single vertex v, and the weight of v is the maximum-weight stable set of $G[M]$. If G_U or $\overline{G_U}$ is disconnected, then G^*_U is either an edgeless graph or a complete graph and the problem can be solved in linear time. If both G_U and $\overline{G_U}$ are connected, then G^*_U is a prime graph and by our hypothesis the problem can be solved in $\mathcal{O}(|V(G^*_U)|^c)$. With this approach, we can compute the maximum-weight stable set of G by starting at the leaves and going to the root. Hence, the total time complexity is bounded by the number of computational steps required to compute the maximum-weight stable set on every prime graph obtained with the contracting approach on the modular decomposition tree $T(G)$. This is bounded by $\mathcal{O}(\sum_{U\in\mathcal{I}} |V(G^*_U)|^c)$. Given any internal node U, remark that the number of vertices of G^*_U is the number of children of U in $T(G)$. Hence, we have that $\sum_{U\in\mathcal{I}} |V(G^*_U)|$ is exactly the number of edges of $T(G)$, which is $|V(T(G))| - 1$. The number of leaves of $T(G)$ is n and the number of internal nodes is at most $n-1$, so the number of edges in $T(G)$ is at most $2n-2$. We obtain the following:

$$\sum_{U\in\mathcal{I}} |V(G^*_U)|^c \leq \left(\sum_{U\in\mathcal{I}} |V(G^*_U)|\right)^c \leq (2n-2)^c = \mathcal{O}(n^c)$$

Adding the modular decomposition computed at the beginning of the algorithm, we obtain the complexity of $\mathcal{O}(n^c + m)$.

□

Clearly, the class of (P_6, bull)-free graphs and (P_7, bull)-free graphs are hereditary. By Theorem 5.1, in order to prove the existence of a polynomial algorithm computing the MWSS in those classes of graphs, it suffices to prove it for prime graphs. This is the object of the two following sections.

Let G be a graph and v any vertex of $V(G)$. Given v, one strategy to compute a maximum-weight stable set containing v is to look at the non-neighborhood of v, defined by $K = V(G) \setminus N[v]$, and to compute $\alpha_w(K)$. The maximum-weight stable set containing v has a total weight of $w(v) + \alpha_w(K)$. We repeat this for every vertex $v \in V(G)$ and keep the stable set of maximum weight among all weighted stable sets computed, which is the optimal solution.

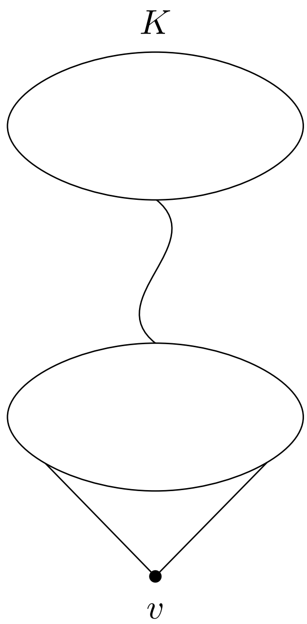


Figure 5.2: The non-neighborhood K of the vertex v.

5.2 Structure of bull-free graphs

The goal of this section is to give a general structure of bull-free graphs, and more precisely of prime bull-free graphs. The main reason behind the fact that we are mainly focused on prime graphs is Theorem 5.1 and the fact that prime graphs are more structured and is of course a smaller class, hence easier to look at, than all bull-free graphs.

A *k-wheel* is a graph that consists of a cycle on k vertices plus a vertex (called the center) adjacent to all vertices of the cycle (see Figure 5.3). The following lemma was proved for $k \geq 7$ in [84]; actually the same proof holds for all $k \geq 6$ as observed in [28].

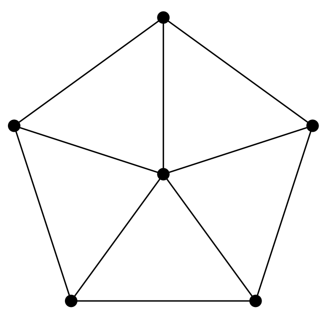


Figure 5.3: The 5-wheel graph.

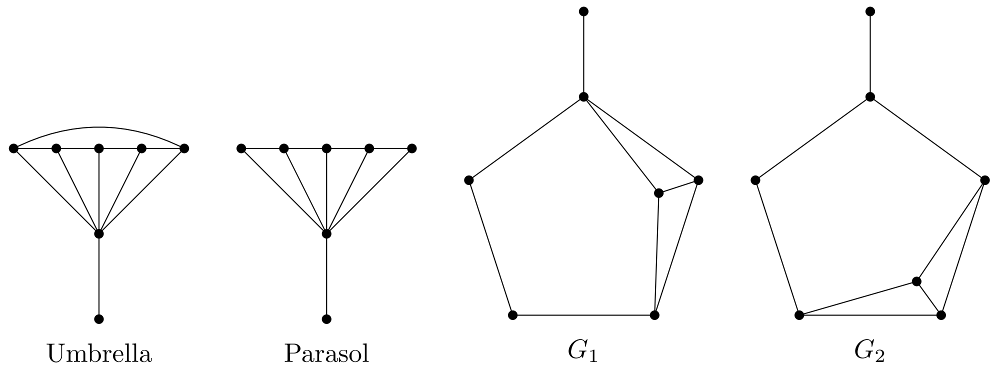


Figure 5.4: From left to right: Umbrella, parasol, G_1, G_2.

Lemma 5.2 [84, 28]

Let G be a bull-free graph. If G contains a k-wheel for any $k \geq 6$, then G has a proper homogeneous set.

Note that the bull is a self-complementary graph, so the preceding lemma also says that if G is prime then it does not contain the complementary graph of a k-wheel with $k \geq 6$.

An *umbrella* is a graph that consists of a 5-wheel plus a vertex adjacent to the center of the 5-wheel only.

Lemma 5.3

A prime bull-free graph contains no umbrella.

Proof. Let C be the 5-cycle of the umbrella, with vertices $c_1, \ldots, c_5$ and edges $c_i c_{i+1}$ for all i modulo 5. Let A be the set of vertices that are complete to C, and let Z be the set of vertices that are anticomplete to C. Let:

$$\begin{aligned} A' &= \{a \in A \mid a \text{ has a neighbor in } Z\}. \\ A'' &= \{a \in A \setminus A' \mid a \text{ has a non-neighbor in } A'\}. \end{aligned}$$

By the hypothesis that C is part of an umbrella, we have $A' \neq \emptyset$. Let H be the component of $G \setminus (A' \cup A'')$ that contains $V(C)$. We claim that:

$$A' \cup A'' \text{ is complete to } V(H). \tag{5.1}$$

Proof: Pick any $b \in A' \cup A''$ and $u \in V(H)$, and let us prove that b is adjacent to u. We use the following notation. If $b \in A'$, then b has a neighbor $z \in Z$. If $b \in A''$, then b has a non-neighbor $a' \in A'$, and a' has a neighbor $z \in Z$, and b is not adjacent to z, for otherwise we would have $b \in A'$.

By the definition of H, there is a shortest path u_0-$\cdots$-u_p in H with $u_0 \in V(C)$ and $u_p = u$, and $p \geq 0$. We know that b is adjacent to u_0 by the definition of A. First, we show that b is adjacent to u_1 and finally by induction on $j = 2, \dots p$, we show that b is adjacent to u_j.

Now suppose that $p \geq 1$. The vertex u_1 is a k-neighbor of C for some $k \geq 1$. If $k \in \{1, 2\}$, then b is adjacent to u_1 by Lemma 5.6 (iii). Suppose that $k \in \{3, 4\}$. Then there is an integer i such that u_1 is adjacent to c_i and not to c_{i+1}. By Lemma 5.6 (v), z is not adjacent to u_1. If $b \in A'$, then b is adjacent to u_1, for otherwise $\{z, b, c_{i+1}, c_i, u_1\}$ induces a bull. If $b \in A''$, then, by the preceding sentence we know that a' is adjacent to u_1; and then b is adjacent to u_1, for otherwise $\{z, a', u_1, u_0, b\}$ induces a bull. Suppose that $k = 5$. So $u_1 \in A$. Then u_1 is not adjacent to z, for otherwise we would have $u_1 \in A'$. If $b \in A'$, then b is adjacent to u_1 for otherwise we would have $u_1 \in A''$. If $b \in A''$, then, by the preceding sentence we know that a' is adjacent to u_1; and then b is adjacent to u_1, for otherwise $\{z, a', u_1, u_0, b\}$ induces a bull.

Finally suppose that $p \geq 2$. So $u_2, \dots, u_p$ are non-neighbors of C. Since $u_2 \in Z$, we have $k \neq 5$, for otherwise we would have $u_1 \in A'$. So there is an integer h such that u_1 is adjacent to c_h and not to c_{h+2}. We may assume up to relabeling that $u_0 = c_h$. It follows that c_{h+2} has no neighbor in $\{u_0, \dots, u_p\}$. Then, by induction on $j = 2, \dots, p$, the vertex b is adjacent to u_j, for otherwise $\{c_{h+2}, b, u_{j-2}, u_{j-1}, u_j\}$ induces a bull. So b is adjacent to u. Thus (5.1) holds.

Let $R = V(G) \setminus (A' \cup A'' \cup V(H))$. By the definition of H, there is no edge between $V(H)$ and R. By (5.1), $V(H)$ is complete to $A' \cup A''$. Hence $V(H)$ is a homogeneous set that contains $V(C)$, and it is proper since $A' \neq \emptyset$. $\square$

LEMMA 5.4

A prime bull-free graph contains no parasol.

Proof. Let G be a prime bull-free graph, and suppose that it contains a parasol, with vertices $p_1, \dots, p_5, x, y$ and edges $p_i p_{i+1}$ for $i = 1, 2, 3, 4$, and xp_j for $j = 1, \dots, 5$ and xy. Let $P = \{p_1, \dots, p_5\}$. Let A be the set of vertices that are complete to P, and let Z be the set of vertices that are anticomplete to P. Let:

$$\begin{aligned} A' &= \{a \in A \mid a \text{ has a neighbor in } Z\}. \\ A'' &= \{a \in A \setminus A' \mid a \text{ has a non-neighbor in } A'\}. \end{aligned}$$

Note that $y \in Z$ and $x \in A'$, so $A' \neq \emptyset$, and that A'' is anticomplete to Z, by the definition of A'. Let H be the component of $G \setminus (A' \cup A'')$ that contains P. We claim that:

$$A' \cup A'' \text{ is complete to } V(H). \tag{5.2}$$

Proof: Suppose on the contrary that there exist non-adjacent vertices a, u with $a \in A' \cup A''$ and $u \in V(H)$. We use the following notation. If $a \in A'$, let z be a neighbor of a in Z. If $a \in A''$, let b be a non-neighbor of a in A', and let z be a neighbor of b in Z; in that case we know that a is not adjacent to z, since $a \notin A'$. By the definition of H, there is a path u_0-$\cdots$-u_ℓ in H with $u_0 \in P$ and $u_\ell = u$, and $\ell \geq 0$. We know that a is adjacent to u_0 by the definition of A, so $\ell \geq 1$. We choose u that minimizes ℓ, so the path u_0-$\cdots$-u_ℓ is chordless, and a is complete to $\{u_0, \dots, u_{\ell-1}\}$, and if $\ell \geq 2$ then $u_2, \dots, u_\ell \in Z$.
Suppose that $\ell = 1$. Suppose that $u_1 \in A$. By the definition of H we have $u_1 \in A \setminus (A' \cup A'')$, so u_1 is not adjacent to z and is complete to A', and so $a \notin A'$, hence $a \in A''$, and u_1 is adjacent to b. Then $\{z, b, u_1, u_0, a\}$ induces a bull, a contradiction. Hence $u_1 \notin A$. So there is an integer $i \in \{1, 2, 3, 4\}$ such that u_1 has a neighbor and a non-neighbor in $\{p_i, p_{i+1}\}$. Suppose that u_1 is not adjacent to z. If $a \in A'$, then $\{z, a, p_i, p_{i+1}, u_1\}$ induces a bull. If $a \in A''$, then u_1 is adjacent to b, for otherwise $\{z, b, p_i, p_{i+1}, u_1\}$ induces a bull; but then $\{z, b, u_1, p, a\}$ induces a bull (for $p \in \{p_i, p_{i+1}\} \cap N(u_1)$). Hence u_1 is adjacent to z. It follows that there is no integer j such that $\{u_1, p_j, p_{j+1}\}$ induces a triangle, for otherwise there is an integer k such that $\{z, u_1, p_k, p_{k+1}, p_{k+2}\}$ induces a bull. If we can take $i = 1$, then u_1 is adjacent to p_4, for otherwise $\{u_1, p_1, p_2, a, p_4\}$ induces a bull; and similarly u_1 is adjacent to p_5; but then $\{u_1, p_4, p_5\}$ induces a triangle, a contradiction. Hence u_1 is either complete or anticomplete to $\{p_1, p_2\}$, and actually it is anticomplete to that set since $\{u_1, p_1, p_2\}$ does not induce a triangle. Likewise u_1 is anticomplete to $\{p_4, p_5\}$. Hence u_1 is adjacent to p_3. But then $\{u_1, p_3, p_2, a, p_5\}$ induces a bull, a contradiction.
Therefore $\ell \geq 2$. We have $u_1 \notin A$, for otherwise we would have $u_1 \in A'$ because $u_2 \in Z$. Since $u_1 \notin A$ and the graph $\overline{P_5}$ is connected, there are non-adjacent vertices $p, q \in P$ such that u_1 is adjacent to p and not to q. We may assume up to relabeling that $u_0 = p$. Then $\{u_\ell, u_{\ell-1}, u_{\ell-2}, a, q\}$ induces a bull, a contradiction. Thus (5.2) holds.

Let $R = V(G) \setminus (A' \cup A'' \cup V(H))$. By the definition of H, there is no edge between $V(H)$ and R. By (5.2), $V(H)$ is complete to $A' \cup A''$. Hence $V(H)$ is a homogeneous set, and it is proper because $P \subseteq V(H)$ and $A' \neq \emptyset$. $\square$

Let G_1 be the graph with vertices $p_1, \ldots, p_5, d, a$ such that p_1-p_2-p_3-p_4-p_5-p_1 is a C_5, d is adjacent to p_5, a is adjacent to p_5, p_1, p_2, and there is no other edge. Let G_2 be the graph with vertices $p_1, \ldots, p_5, d, a$ such that p_1-p_2-p_3-p_4-p_5-p_1 is a C_5, d is adjacent to p_5, a is adjacent to p_1, p_2, p_3, and there is no other edge. See Figure 5.4.

Lemma 5.5

A prime bull-free graph G contains no G_1 and no G_2.

Proof. First suppose that G contains a G_1, with the same notation as above. Let $X = \{x \in V(G) \mid xp_5, xp_2 \in E(G) \text{ and } xd, xp_3, xp_4 \notin E(G)\}$ (so $a, p_1 \in X$), and let Y be the vertex-set of the component of $G[X]$ that contains a and p_1. Since G is prime, Y is not a homogeneous set, so there are adjacent vertices $y, z \in Y$ and a vertex $b \in V(G) \setminus Y$ such that $by \in E(G)$ and $bz \notin E(G)$. Suppose that $bp_5 \notin E(G)$. Then $bd \in E(G)$, for otherwise $\{b, y, z, p_5, d\}$ induces a bull; and similarly $bp_4 \in E(G)$. If $bp_2 \notin E(G)$, then $bp_3 \in E(G)$, for otherwise $\{b, y, z, p_2, p_3\}$ induces a bull; but then $\{p_2, p_3, b, p_4, p_5\}$ induces a bull; so $bp_2 \in E(G)$. Then $bp_3 \in E(G)$, for otherwise $\{d, b, y, p_2, p_3\}$ induces a bull; but then $\{d, b, p_3, p_2, z\}$ induces a bull. Hence $bp_5 \in E(G)$. Suppose that $bp_2 \notin E(G)$. Then $bd \in E(G)$, for otherwise $\{p_2, y, b, p_5, d\}$ induces a bull; and $bp_4 \in E(G)$, for otherwise $\{p_2, y, b, p_5, p_4\}$ induces a bull; and $bp_3 \in E(G)$, for otherwise $\{z, p_5, b, p_4, p_3\}$ induces a bull; but then $\{d, b, p_4, p_3, p_2\}$ induces a bull. Hence $bp_2 \in E(G)$. If $bp_3 \in E(G)$, then $bp_4 \in E(G)$, for otherwise $\{z, p_2, b, p_3, p_4\}$ induces a bull, and $bd \in E(G)$, for otherwise $\{d, p_5, p_4, b, p_2\}$ induces a bull; but then $\{z, p_5, d, b, p_3\}$ induces a bull. Hence $bp_3 \notin E(G)$. Then $bp_4 \notin E(G)$, for otherwise $\{p_3, p_4, b, p_5, z\}$ induces a bull, and $bd \notin E(G)$, for otherwise $\{d, b, y, p_2, p_3\}$ induces a bull. But now we see that $b \in Y$, a contradiction.

Now suppose that G contains a G_2, with the same notation as above. Let $X = \{x \in V(G) \mid xp_1, xp_3 \in E(G) \text{ and } xd, xp_5, xp_4 \notin E(G)\}$ (so $a, p_2 \in X$), and let Y be the vertex-set of the component of $G[X]$ that contains a and p_2. Since Y is not a homogeneous set, there is a vertex $b \in V(G) \setminus Y$ and two adjacent vertices $x, y \in Y$ such that b is adjacent to x and not adjacent to y. If $bp_4 \notin E(G)$, then $bp_3 \in E(G)$, for otherwise $\{b, x, y, p_3, p_4\}$ induces a bull, and $bp_1 \in E(G)$, for otherwise $\{p_1, x, b, p_3, p_4\}$ induces a bull, and $bp_5 \notin E(G)$, for otherwise $\{y, p_1, b, p_5, p_4\}$ induces a bull, and $bd \notin E(G)$, for otherwise $\{d, b, x, p_3, p_4\}$ induces a bull; but then we see that $b \in Y$, a contradiction. Hence $bp_4 \in E(G)$. If $bp_5 \in E(G)$, then $bd \in E(G)$, for otherwise $\{x, b, p_4, p_5, d\}$ induces a bull, and $bp_3 \notin E(G)$, for otherwise $\{y, p_3, p_4, b, d\}$ induces a bull, and $bp_1 \in E(G)$, for otherwise $\{p_3, p_4, b, p_5, p_1\}$ induces a bull; but then $\{d, b, p_1, x, p_3\}$ induces a bull. Hence $bp_5 \notin E(G)$. Then $bp_3 \notin E(G)$, for otherwise $\{y, p_3, b, p_4, p_5\}$ induces a bull, and $bp_1 \in E(G)$, for otherwise $\{b, x, y, p_1, p_5\}$ induces a bull; but then $\{p_5, p_1, b, x, p_3\}$ induces a bull, a contradiction. $\square$

In a graph G, let H be a subgraph of G. For each $k > 0$, a *k-neighbor* of H is any vertex in $V(G) \setminus V(H)$ that has exactly k neighbors in H.

LEMMA 5.6

Let G be a bull-free graph. Let C be an induced 5-cycle in G, with vertices $c_1, \ldots, c_5$ and edges $c_i c_{i+1}$ for each i modulo 5. Then:

(i) Every 2-neighbor of C is adjacent to c_i and c_{i+2} for some i.

(ii) Every 3-neighbor of C is adjacent to c_i, c_{i+1} and c_{i+2} for some i.

(iii) Every 5-neighbor of C is adjacent to every k-neighbor with $k \in \{1, 2\}$.

(iv) If C has a 4-neighbor non-adjacent to c_i for some i, then every 1-neighbor of C is adjacent to c_i.

(v) If a non-neighbor of C is adjacent to a k-neighbor of C, then $k \in \{1, 2, 5\}$.

Proof. If either (i) or (ii) fails, there is a vertex x that is either a 2-neighbor adjacent to c_i and c_{i+1} or a 3-neighbor adjacent to c_i, c_{i+1} and c_{i+3} for some i, and then $\{c_{i-1}, c_i, x, c_{i+1}, c_{i+2}\}$ induces a bull.

(iii) Let u be a 5-neighbor of C and x be a k-neighbor of C with $k \in \{1, 2\}$. So for some i the vertex x is adjacent to c_i and maybe to c_{i+2}. Then u is adjacent to x, for otherwise $\{x, c_i, c_{i+1}, u, c_{i+3}\}$ induces a bull.

(iv) Let f be a 4-neighbor of C non-adjacent to c_i. Suppose that there is a 1-neighbor x not adjacent to c_i. So, up to symmetry, x is adjacent to c_{i+1} or c_{i+2}. Then x is adjacent to f, for otherwise $\{x, c_{i+1}, c_{i+2}, f, c_{i-1}\}$ induces a bull; but then $\{x, f, c_{i-2}, c_{i-1}, c_i\}$ induces a bull.

(v) Let z be a non-neighbor of C that is adjacent to a k-neighbor x with $k \in \{3, 4\}$. So there is an integer i such that x is adjacent to c_i and c_{i+1} and not adjacent to c_{i+2}. Then $\{z, x, c_i, c_{i+1}, c_{i+2}\}$ induces a bull. $\square$

The following lemma is straightforward and we omit its proof. However, to ease the reader understanding we would like to emphasize that any configuration not matching with what is described below induces a bull in G.

LEMMA 5.7

Let G be a bull-free graph. Let C be an induced C_7 in G, with vertices $c_1, \ldots, c_7$ and edges $c_i c_{i+1}$ for each i modulo 7. Then:

- *Any 2-neighbor of C is adjacent to c_i and either c_{i+2} or c_{i+3} for some i.*
- *Any 3-neighbor of C is adjacent to either to c_i, c_{i+1} and c_{i+2} or to c_i, c_{i+2} and c_{i+4} for some i.*
- *C has no k-neighbor for any $k \in \{4, 5, 6\}$.*

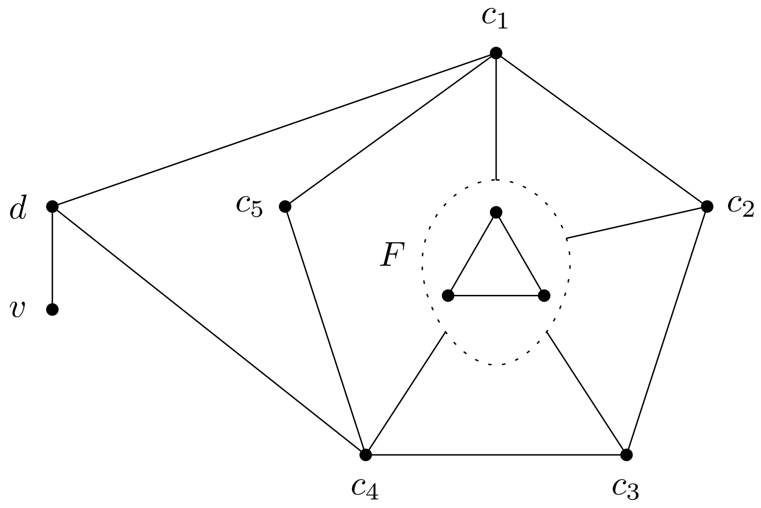


Figure 5.5: The structure of a component in the non-neighborhood of v.

5.3 MWSS in (P_6, bull)-free graphs

The main goal of this section is to prove that there exist a polynomial algorithm that solves the MWSS problem in the class of (P_6, bull)-free graphs. We were able to provide the following theorem.

THEOREM 5.8

MWSS can be solved in time $O(n^7)$ for every graph on n vertices in the class of $(P_6, bull)$-free graphs.

In order to prove Theorem 5.8, we refine the structure of prime (P_6, bull)-free graphs. The structure we use relies on the fact that in a prime (P_6, bull)-free graph, the non-neighborhood of a vertex either contains a C_5 or is perfect. In the latter case we can solve the problem by using already known algorithms. Hence, we can assume that the non-neighborhood contains a C_5. The following is the key theorem that helps us solve the MWSS problem in (P_6, bull)-free graphs, see Figure 5.5 for an illustration of what is proved.

THEOREM 5.9

Let G be a prime $(P_6, bull)$-free graph, and let x be any vertex in G. Suppose that there is a 5-cycle induced by non-neighbors of x. Then there is a (possibly empty) clique F in G such that the induced subgraph $G \setminus F$ is triangle-free, and such a set F can be found in time $O(n^2)$.

The proof of Theorem 5.9 is given in the next section. We close this section by showing how to obtain a proof of Theorem 5.8 on the basis of Theorem 5.9.

Our algorithm relies on results concerning graphs of bounded clique-width (we refer the reader to the preliminaries of this manuscript to learn more about the notion of clique-width). We restate the following theorem of Brandstädt et al. proving that the class of (P_6, triangle)-free graphs has bounded clique-width.

THEOREM 3.23[10]

The class of $(P_6, triangle)$-free graphs has bounded clique-width c, and a c-expression can be found in time $O(|V(G)|^2)$ for every graph G in this class.

Hence, as observed in [10], Theorems 2.1 and 3.23 imply the following.

COROLLARY 5.10 [10]

For any $(P_6, triangle)$-free graph G on n vertices one can find a maximum-weight stable set of G in time $O(n^2)$.

Our proof relies heavily on Corollary 5.10 since our strategy is to fall back on a (P_6, triangle)-free graph and call the algorithm of Corollary 5.10. By doing so, we were able to prove the following theorem.

THEOREM 5.11

Let G be a prime $(P_6, bull)$-free graph on n vertices. Then a maximum-weight stable set of G can be found in time $O(n^7)$.

Proof. Let G be a prime (P_6, bull)-free graph. Let $w : V(G) \to \mathbb{N}$ be a weight function on the vertex set of G. To find the maximum weight stable set in G it is sufficient to compute, for every vertex x of G, a maximum-weight stable set containing x. So let x be any vertex in G. We want to compute the weight of a maximum stable set containing x. Clearly it suffices to compute the maximum-weight stable set in each component of the induced subgraph $G \setminus (\{x\} \cup N(x))$ and make the sum over all components. Let K be any component of $G \setminus (\{x\} \cup N(x))$. We claim that:

$$\text{Either } K \text{ is perfect or it contains a 5-cycle.} \tag{5.3}$$

Proof of (5.3): Suppose that K is not perfect. Note that K contains no odd hole of length at least 7 since G is P_6-free. By the Strong Perfect Graph Theorem K contains an odd antihole C. If C has length at least 7 then $V(C) \cup \{x\}$ induces a wheel in $\overline{G}$, so G has a proper homogeneous set by Lemma 5.2, a contradiction because G is prime. So C has length 5, i.e., C is a 5-cycle. So (5.3) holds.

We can test in time $O(n^5)$ if K contains a 5-cycle. This leads to the following two cases.

Suppose that K contains no 5-cycle. Then (5.3) implies that K is perfect. In that case we can use the algorithms from either [27] or [82], which compute a maximum-weight stable set in a bull-free perfect graph in polynomial time. The algorithm from [82] has time complexity $O(n^6)$.

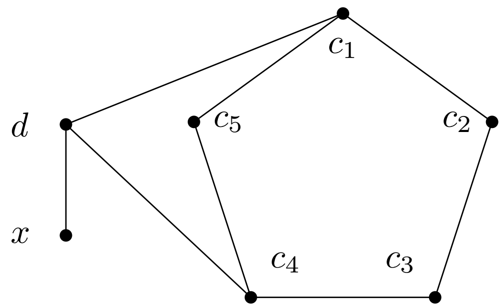


Figure 5.6: The graph G_7.

Now suppose that K contains a 5-cycle. Then by Theorem 5.9 we can find in time $O(n^2)$ a clique F such that $G \setminus F$ is triangle-free. Consider any stable set S in K. If S contains no vertex from F, then S is in the subgraph $G \setminus F$, which is triangle-free. By Corollary 5.10 we can find a maximum-weight stable set S_F in $G \setminus F$ in time $O(n^2)$. If S contains a vertex f from F, then $S \setminus f$ is in the subgraph $G \setminus (\{f\} \cup N(f))$, which, since F is a clique, is a subgraph of $G \setminus F$ and consequently is also triangle-free. By Corollary 5.10 we can find a maximum-weight stable set S'_f in $G \setminus (\{f\} \cup N(f))$ in time $O(n^2)$. Then we set $S_f = S'_f \cup \{f\}$. We do this for every vertex $f \in F$. Now we need only compare the set S_F and the sets S_f (for all $f \in F$) and select the one with the largest weight. This takes time $O(n^3)$ for each component K that contains a 5-cycle.

Repeating the above for each component takes time $O(n^6)$ as the components are disjoint. Repeating this for every vertex x, the total complexity is $O(n^7)$. □

Now Theorem 5.8 follows directly from Theorems 5.11 and 5.1.

5.3.1 Structure of the non-neighborhood

This section is dedicated to describe the non-neighborhood of a fixed vertex in the graph G. We start by proving the following lemma.

LEMMA 5.12

Let G be a prime (P_6,bull)-free graph. Let C be an induced 5-cycle in G. If a non-neighbor of C is adjacent to a k-neighbor of C, then $k = 2$.

Proof. Let C have vertices $c_1, \ldots, c_5$ and edges c_ic_{i+1} for each i modulo 5. Suppose that a non-neighbor z of C is adjacent to a k-neighbor x of C. By Lemma 5.6 (v), we have $k \in \{1, 2, 5\}$. If $k = 1$, say x is adjacent to c_i, then z-x-c_i-c_{i+1}-c_{i+2}-c_{i+3} is an induced P_6 in G. If $k = 5$, then $V(H) \cup \{x, y\}$ induces an umbrella, so, by Lemma 5.3, G has a proper homogeneous set, a contradiction. So $k = 2$. □

Let G_7 be the graph with vertex-set $\{c_1, \ldots, c_5, d, x\}$ and edge-set $\{c_ic_{i+1} \mid \text{for all } i \bmod 5\} \cup \{dc_1, dc_4, dx\}$. See Figure 5.6.

Lemma 5.13

Let G be a prime $(P_6, bull)$-free graph. Assume that G contains a 5-cycle C, with vertices $c_1, \dots, c_5$ and edges $c_i c_{i+1}$ for all i mod 5. Moreover assume that C has a non-neighbor x in G. Then:

(i) There is a neighbor d of x that is a 2-neighbor of C. And consequently, $V(C) \cup \{d, x\}$ induces a G_7.

(ii) C has no 3-neighbor and no 5-neighbor.

(iii) If the vertex d from (i) is (up to symmetry) adjacent to c_1 and c_4, then every 4-neighbor of C is non-adjacent to c_5.

Proof. Since G is prime it is connected, so there is a shortest path from C to x in G. Let x_0-$\cdots$-x_p be such a path, where $x_0 \in V(C)$ and $x_p = x$, and $p \geq 2$. By Lemma 5.12, x_1 is a 2-neighbor of C, so up to relabeling we may assume that x_1 is adjacent to c_1 and c_4. Then $p = 2$ for otherwise x_3-x_2-x_1-c_1-c_2-c_3 is an induced P_6. So (i) holds with $d = x_1$. Clearly, $\{c_1, \dots, c_5, x_1, x\}$ induces a G_7.

Therefore we may assume, up to symmetry, that the vertex d from (i) is adjacent to c_1 and c_4.

Suppose that there is a vertex u that is either a 5-neighbor of C or a 4-neighbor adjacent to c_5. In either case we may assume, up to symmetry, that u is adjacent to c_1, c_3 and c_5. Then u is adjacent to d, for otherwise $\{d, c_1, c_5, u, c_3\}$ induces a bull, and u is adjacent to x, for otherwise $\{x, d, c_1, u, c_3\}$ induces a bull. But then u and x contradict Lemma 5.12. This proves item (iii) and that C has no 5-neighbor.

Finally suppose that C has a 3-neighbor u, adjacent to c_{i-1}, c_i, c_{i+1}; we may assume up to symmetry that $i \in \{5, 1, 2\}$. Let X be the set of vertices that are complete to $\{c_{i-1}, c_{i+1}\}$ and anticomplete to $\{c_{i-2}, c_{i+2}\}$, and let Y be the vertex-set of the component of $G[X]$ that contains c_i and u. Since G is prime, Y is not a homogeneous set, so there is a vertex t in $V(G) \setminus Y$ and vertices y, z in Y such that t is adjacent to y and not to z, and since Y is connected we may choose y and z adjacent. We claim that:

$$t \text{ is adjacent to } c_{i-2} \text{ and } c_{i+2} \text{ and to at least one of } c_{i-1} \text{ and } c_{i+1}. \tag{5.4}$$

Proof: If t has no neighbor in $\{c_{i-1}, c_{i+1}\}$, then t is adjacent to c_{i-2}, for otherwise $\{t, y, z, c_{i-1}, c_{i-2}\}$ induces a bull, and similarly t is adjacent to c_{i+2}; but then $\{c_{i-1}, c_{i-2}, t, c_{i+2}, c_{i+1}\}$ induces a bull. Hence t has a neighbor in $\{c_{i-1}, c_{i+1}\}$. Suppose that t is adjacent to both c_{i-1} and c_{i+1}. Since t is not in Y it must have a neighbor in $\{c_{i-2}, c_{i+2}\}$, and actually t is complete to $\{c_{i-2}, c_{i+2}\}$, for otherwise t is a 3-neighbor of the 5-cycle induced by $\{z, c_{i-1}, c_{i-2}, c_{i+2}, c_{i+1}\}$ that violates Lemma 5.6 (ii). Now suppose that t is adjacent to exactly one of c_{i-1}, c_{i+1}, say up to symmetry to c_{i-1}. Then t is adjacent to c_{i-2}, for otherwise $\{c_{i-2}, c_{i-1}, t, y, c_{i+1}\}$ induces a bull, and t is adjacent to c_{i+2}, for otherwise $\{c_{i+2}, c_{i-2}, t, c_{i-1}, z\}$ induces a bull. Thus (5.4) holds.

Now we claim that:

$$x \text{ has no neighbor in } Y \cup \{t\}. \tag{5.5}$$

Proof: Suppose that x has a neighbor in Y. Since x also has a non-neighbor c_i in Y, and Y is connected, there are adjacent vertices v, v' in Y such that x is adjacent to v and not to v', and then $\{x, v, v', c_{i-1}, c_{i-2}\}$ induces a bull, a contradiction. So x has no neighbor in Y. In particular x is not adjacent to y, so x has no neighbor in the 5-cycle C_y induced by $\{y, c_{i-1}, c_{i-2}, c_{i+2}, c_{i+1}\}$. By (5.4), t is a 3- or 4-neighbor of C_y. By Lemma 5.12, x is not adjacent to t. Thus (5.5) holds.

Suppose that $i = 5$. By (5.4), t is adjacent to c_2 and c_3 and, up to symmetry, to c_1. Then d is not adjacent to y, for otherwise $\{x, d, y, c_1, c_2\}$ induces a bull, and d is not adjacent to t, for otherwise $\{x, d, c_1, t, c_3\}$ induces a bull; but then $\{d, c_1, y, t, c_3\}$ induces a bull, a contradiction.

Suppose that $i = 1$. By (5.4), t is adjacent to c_3 and c_4. Then d is adjacent to y, for otherwise x-d-c_4-c_3-c_2-y is an induced P_6, and similarly d is adjacent to z. Then t is adjacent to d, for otherwise $\{x, d, z, y, t\}$ induces a bull, and t is adjacent to c_2, for otherwise $\{x, d, t, y, c_2\}$ induces a bull; but then $\{x, d, c_4, t, c_2\}$ induces a bull.

Finally suppose that $i = 2$. By (5.4), t is adjacent to c_4 and c_5. Then d is not adjacent to y, for otherwise $\{x, d, c_1, y, c_3\}$ induces a bull, and d is adjacent to t, for otherwise $\{d, c_4, c_5, t, y\}$ induces a bull; but then $\{x, d, c_4, t, y\}$ induces a bull, a contradiction. □

THEOREM 5.14

Let G be a prime $(P_6, bull)$-free graph. Suppose that G contains a G_7, with vertex-set $\{c_1, \ldots, c_5, d, x\}$ and edge-set $\{c_i c_{i+1} \mid \text{for all } i \bmod 5\} \cup \{dc_1, dc_4, dx\}$. Let:

- *C be the 5-cycle induced by $\{c_1, \ldots, c_5\}$;*
- *F be the set of 4-neighbors of C;*
- *T be the set of 2-neighbors of C;*
- *W be the set of 1-neighbors and non-neighbors of C.*

Then the following properties hold:

(i) $V(G) = \{c_1, \ldots, c_5\} \cup F \cup T \cup W$.

(ii) *F is complete to $\{c_1, \ldots, c_4\}$ and anticomplete to $\{c_5, x, d\}$.*

(iii) *F is a clique.*

(iv) *$G \setminus F$ is triangle-free.*

Proof. Note that $d \in T$ and $x \in W$. Clearly the sets $\{c_1, \ldots, c_5\}$, F, T, and W are pairwise disjoint subsets of $V(G)$. We observe that item (i) follows directly from the definition of the sets F, T, W and Lemma 5.13 (ii).

Now we prove item (ii). Consider any $f \in F$. By Lemma 5.13 (iii), f is non-adjacent to c_5, and consequently f is complete to $\{c_1, \ldots, c_4\}$. Then f is not adjacent to

x, for otherwise $\{x, f, c_3, c_4, c_5\}$ induces a bull; and f is not adjacent to d, for otherwise $\{x, d, c_1, f, c_3\}$ induces a bull. Thus (ii) holds.

Now we prove item (iii). Suppose on the contrary that F is not a clique. So $G[F]$ has an anticomponent whose vertex-set F' satisfies $|F'| \geq 2$. Since G is prime, F' is not a homogeneous set, so there are vertices $y, z \in F'$ and a vertex $t \in V(G) \setminus F'$ that is adjacent to y and not to z, and since F' is anticonnected we may choose y and z non-adjacent. By the definition of F', we have $t \notin F$. By (ii), we have $t \notin V(C)$. Therefore, By (i), we have $t \in T \cup W$.
Suppose that $t \in T$, so t is adjacent to c_{i-1} and c_{i+1} for some i in (up to symmetry) $\{1, 2, 5\}$. If $i = 1$, then $\{z, c_2, y, t, c_5\}$ induces a bull. If $i = 2$, then $\{t, c_3, z, c_4, c_5\}$ induces a bull. So $i = 5$. Then t is not adjacent to x, for otherwise $\{x, t, c_1, y, c_3\}$ induces a bull. Then x is a non-neighbor of the 5-cycle induced by $\{c_1, c_2, c_3, c_4, t\}$, and y is a 5-neighbor of that cycle, which contradicts Lemma 5.13.
Hence $t \in W$. By Lemma 5.6 (iv), t is anticomplete to $\{c_1, c_2, c_3, c_4\}$. Then t is adjacent to each $u \in \{c_5, d\}$, for otherwise $\{t, y, c_3, c_4, u\}$ induces a bull. So t is a 1-neighbor of C, and by Lemma 5.12, t is not adjacent to x. But then x-d-t-y-c_3-z is an induced P_6. Thus (iii) holds.

There remains to prove item (iv). Suppose on the contrary that $G \setminus F$ contains a triangle, with vertex-set $R = \{u, v, w\}$. Clearly C and R have at most two common vertices. Moreover:

$$C \text{ and } R \text{ have at most one common vertex.} \tag{5.6}$$

Proof: Suppose on the contrary that $u, v \in V(C)$, and consequently $w \notin V(C)$. By Lemma 5.6 (i), w is a k-neighbor of C for some $k \geq 3$. Since $w \notin F$, we have $k \neq 4$, so $k \in \{3, 5\}$; but this contradicts Lemma 5.13 (ii). So (5.6) holds.

Suppose that $G \setminus F$ is not connected. Consider the component K of $G \setminus F$ that contains C; then K also contains T. Pick any vertex z in another component. By Lemma 5.13 (i), the vertex z must have a neighbor in T, a contradiction. Hence $G \setminus F$ is connected. It follows that there is a path from C to R in $G \setminus F$. Let $P = p_0\text{-}\cdots\text{-}p_\ell$ be a shortest such path, with $p_0 \in V(C)$, $p_\ell = u$, and $\ell \geq 0$. Note that if $\ell \geq 1$, the vertices $p_1, \ldots, p_\ell$ are not in C. We choose R so as to minimize ℓ. Let H be the component of $G[N(u)]$ that contains v and w. Since G is prime, $V(H)$ is not a homogeneous set, so there are two vertices $y, z \in V(H)$ and a vertex $a \in V(G) \setminus V(H)$ such that a is adjacent to y and not to z, and since H is connected we may choose y and z adjacent. By the definition of H, the vertex a is not adjacent to u.

Suppose that $\ell = 0$. So $u = p_0 = c_i$ for some $i \in \{1, \ldots, 5\}$. By (5.6) the vertices y, z are not in C and are anticomplete to $\{c_{i-1}, c_{i+1}\}$. So, by Lemma 5.6 (ii), each of y and z is a 1- or 2-neighbor of C. The vertex a is adjacent to c_{i-1}, for otherwise $\{a, y, z, c_i, c_{i-1}\}$ induces a bull; and similarly a is adjacent to c_{i+1}. Note that this implies $a \notin V(C)$. Suppose that a has no neighbor in $\{c_{i-2}, c_{i+2}\}$. Then one of y, z has a neighbor in $\{c_{i-2}, c_{i+2}\}$, for otherwise z-y-a-c_{i+1}-c_{i+2}-c_{i-2} is an induced P_6. So assume up to symmetry that one of y, z is adjacent to c_{i+2}. Then both y, z are adjacent to c_{i+2}, for otherwise $\{c_{i+2}, y, z, c_i, c_{i-1}\}$ induces a bull. So y and z are 2-neighbors of C, and they are not adjacent to c_{i-2}. But then $\{a, y, z, c_{i+2}, c_{i-2}\}$ induces a bull, a contradiction. Hence a has a neighbor in $\{c_{i-2}, c_{i+2}\}$. By Lemma 5.6 (ii) and Lemma 5.13 (ii), a must be adjacent to both c_{i-2}, c_{i+2}, so a is a 4-neighbor of C. Hence $a \in F$, and $i = 5$, and by (iii) a has no neighbor in $\{d, x\}$. The vertex z is not adjacent to c_2, for otherwise $\{z, c_2, c_1, a, c_4\}$ induces a bull; and similarly z is not adjacent to c_3. Then y is not adjacent to c_2, for otherwise $\{c_4, c_5, z, y, c_2\}$ induces a bull; and similarly y is not adjacent to c_3. So y and z are 1-neighbors of C, and by Lemma 5.12 they are not adjacent to x. Then d is adjacent to y, for otherwise $\{d, c_1, c_2, a, y\}$ induces a bull, and d is not adjacent to z, for otherwise $\{x, d, z, y, a\}$ induces a bull; but then z-y-d-c_1-c_2-c_3 is an induced P_6, a contradiction. Therefore $\ell \geq 1$.

We deduce that:

$$\text{Every vertex } c_i \text{ in } C \text{ has at most one neighbor in } \{u, y, z\}. \tag{5.7}$$

For otherwise, c_i and two of its neighbors in $\{u, y, z\}$ form a triangle that contradicts the choice of R (the minimality of ℓ). Thus (5.7) holds.

Suppose that $\ell \geq 2$. By Lemma 5.12 (applied to p_1 and p_2), p_1 is a 2-neighbor of C, adjacent to c_{i-1} and c_{i+1} for some i. The vertex y has no neighbor c_j in C, for otherwise the path c_j-y contradicts the choice of P. The vertex p_2 has no neighbor c_j in C, for otherwise the path c_j-p_2-$\cdots$-p_ℓ contradicts the choice of P. Put $p' = p_3$ if $\ell \geq 3$ and $p' = y$ if $\ell = 2$. Then p'-p_2-p_1-c_{i+1}-c_{i+2}-c_{i-2} is an induced P_6, a contradiction.

Therefore $\ell = 1$, so $u = p_1$. By (i), and since $u \notin F$, u is either a 1-neighbor or a 2-neighbor of C.

Suppose that u is a 1-neighbor of C, adjacent to c_i for some i. By (5.7), y and z are not adjacent to c_i. Then a is adjacent to c_i, for otherwise $\{a, y, z, u, c_i\}$ induces a bull. If a has a neighbor in $\{c_{i-1}, c_{i+1}\}$, then, by Lemma 5.6 (ii) and Lemma 5.13 (ii), a is a 4-neighbor of C; but then a and u violate Lemma 5.6 (iv). So a has no neighbor in $\{c_{i-1}, c_{i+1}\}$. Then z is not adjacent to c_{i+1}, for otherwise, by (5.7), $\{a, y, u, z, c_{i+1}\}$ induces a bull; and z has no neighbor c in $\{c_{i-2}, c_{i+2}\}$, for otherwise, by (5.7), $\{c_i, u, y, z, c\}$ induces a bull. But then z-u-c_i-c_{i+1}-c_{i+2}-c_{i-2} is an induced P_6, a contradiction.

Therefore u is a 2-neighbor of C, adjacent to c_{i-1} and c_{i+1} for some i. By (5.7), y and z are anticomplete to $\{c_{i-1}, c_{i+1}\}$. The vertex c_{i+2} has no neighbor in $\{y, z\}$, for otherwise, by (5.7), $\{c_{i+2}, y, z, u, c_{i-1}\}$ induces a bull. Likewise, c_{i-2} has no neighbor in $\{y, z\}$. The vertex a is adjacent to c_{i-1}, for otherwise $\{a, y, z, u, c_{i-1}\}$ induces a bull, and similarly a is adjacent to c_{i+1}. Then a has a neighbor in $\{c_{i-2}, c_{i+2}\}$, for otherwise z-y-a-c_{i+1}-c_{i+2}-c_{i-2} is an induced P_6. By Lemma 5.6 (ii) and Lemma 5.13 (ii), a is a 4-neighbor of C, so $i = 5$, and a has no neighbor in $\{c_5, d, x\}$. Then y is adjacent to c_5, for otherwise $\{y, a, c_3, c_4, c_5\}$ induces a bull; and by (5.7), z is not adjacent to c_5. But then z-y-c_5-c_4-c_3-c_2 is an induced P_6, a contradiction. This completes the proof of the theorem. □

Finally, Theorem 5.9 follows as a direct consequence of Lemma 5.13 and Theorem 5.14.

5.4 MWSS in (P_7, bull)-free graphs

We prove the following theorem. In the same flavour as for the previous section, we first describe structural properties of (P_7, bull)-free graphs and use these to compute the MWSS.

THEOREM 5.15

The Maximum Weight Stable Set problem can be solved in time $\mathcal{O}(n^9)$ in the class of (P_7, bull)-free graphs.

Before giving the proof of Theorem 5.15 we need another lemma.

LEMMA 5.16

Let G be a connected $(P_7, bull)$-free graph. Assume that G contains a C_7 but no C_5 and no 7-wheel. Then $V(G)$ can be partitioned into seven non-empty sets $A_1, \dots, A_7$ such that for each $i \in \{1, \dots, 7\} \pmod 7$ the set A_i is complete to $A_{i-1} \cup A_{i+1}$ and anticomplete to $A_{i-3} \cup A_{i-2} \cup A_{i+2} \cup A_{i+3}$.

Proof. Since G contains a C_7, there exist seven pairwise disjoint and non-empty sets $A_1, \dots, A_7 \subset V(G)$ such that for each $i \in \{1, \dots, 7\} \pmod 7$ the set A_i is complete to $A_{i-1} \cup A_{i+1}$ and anticomplete to $A_{i-3} \cup A_{i-2} \cup A_{i+2} \cup A_{i+3}$. We choose these sets so as to maximize their union $U = A_1 \cup \dots \cup A_7$. Hence we need only prove that $V(G) = U$, so suppose the contrary. Since G is connected, there is a vertex x in $V(G) \setminus U$ that has a neighbor in U. For each $i \in \{1, \dots, 7\}$ pick a vertex $c_i \in A_i$ so that x has a neighbor in the cycle C induced by $\{c_1, \dots, c_7\}$. So x is a k-neighbor of C for some $k > 0$. Since G contains no 7-wheel, and by Lemma 5.7, we have $k \in \{1, 2, 3\}$. If $k = 1$, say x is adjacent to c_1, then x-c_1-c_2-c_3-c_4-c_5-c_6 is an induced P_7. If $k = 2$ and x is adjacent to c_i and c_{i+3} for some i, then $\{x, c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ induces a C_5. If $k = 3$ and x is adjacent to c_i, c_{i+2} and c_{i+4} for some i, then $\{x, c_i, c_{i-1}, c_{i-2}, c_{i-3}\}$ induces a C_5. Therefore, by Lemma 5.7, it must be that $N_C(x)$ is equal to either $\{c_{i-1}, c_{i+1}\}$ or $\{c_{i-1}, c_i, c_{i+1}\}$ for some i, say $i = 7$. Pick any $c' \in A_1 \setminus \{c_1\}$ and let C' be the cycle induced by $(V(C) \setminus \{c_1\}) \cup \{c'\}$. Then by the same arguments applied to C' and x, we deduce that x is adjacent to c'. So x is complete to A_1, and similarly x is complete to A_6. Likewise, Lemma 5.7 and the fact that G is C_5-free implies that x has no neighbor in $A_2 \cup A_3 \cup A_4 \cup A_5$. But now the sets $A_1, \dots, A_6, A_7 \cup \{x\}$ contradict the maximality of U. So $V(G) = U$ and the lemma holds. $\square$

Now we can prove the main result of this section.

Proof of Theorem 5.15. Let G be a (P_7, bull)-free graph, and let w be a weight function on the vertex set of G. By Theorem 5.1, we may assume that G is prime. By Lemmas 5.2—5.5, G contains no k-wheel and no k-antiwheel for any $k \geq 6$, no umbrella, no parasol, no G_1 and no G_2. To find the maximum-weight stable set in G it is sufficient to compute, for every vertex c of G, a maximum-weight stable set containing c, and to choose the best set over all c. So let c be any vertex in G. The maximum weight of a stable set that contains c is equal to $w(c) + \sum_K \alpha_w(K)$, where the sum is over all components K of $G \setminus (\{c\} \cup N(c))$ (the non-neighborhood of c) and $\alpha_w(K)$ is the maximum weight of any stable set in K. So let K be an arbitrary component of $G \setminus (\{c\} \cup N(c))$. If K is perfect, we can use the algorithm from [82] to find a maximum-weight stable set in K. Therefore let us assume that K is not perfect. We note that K contains no antihole of length at least 6, for otherwise the union of such a subgraph with c forms an antiwheel. Hence, by the Strong Perfect Graph Theorem [18], and since G is P_7-free, K contains a C_5 or a C_7.

Since G is prime it is connected, so there is a neighbor d of c that has a neighbor in K. Let $H = N_K(d)$ and $Z = V(K) \setminus H$. We claim that every C_5 in K contains at most two vertices from H, and if it contains two they are non-adjacent. Indeed, in the opposite case, there is a C_5 in K with vertices $v_1, \ldots, v_5$ and edges v_iv_{i+1} (mod 5) such that $v_1, v_2 \in H$. Then $v_3 \in H$, for otherwise $\{c, d, v_1, v_2, v_3\}$ induces a bull; and similarly $v_4, v_5 \in H$; but then $\{v_1, \ldots, v_5, d, c\}$ induces an umbrella, which contradicts Lemma 5.3. So the claim is established. Henceforth, for $q \in \{0, 1, 2\}$ we say that a C_5 in K is of type q if it contains exactly q vertices from H. So every C_5 in K is of type 0, 1 or 2, and if it is of type 2 its two vertices from H are non-adjacent. Our proof follows the pattern from [11], but in some parts we will use different arguments.

Case 1: K **contains a** C_7 **and no** C_5**.**

Since K is connected and contains no 7-wheel, Lemma 5.16 implies that $V(K)$ can be partitioned into seven non-empty sets $A_1, \ldots, A_7$ such that for each $i \in \{1, \ldots, 7\}$ (mod 7) the set A_i is complete to $A_{i-1} \cup A_{i+1}$ and anticomplete to $A_{i-3} \cup A_{i-2} \cup A_{i+2} \cup A_{i+3}$.
Clearly we have $\alpha_w(K) = \max_{i \in \{1,\ldots,7\}} \{\alpha_w(G[A_i]) + \alpha_w(G[A_{i+2}]) + \alpha_w(G[A_{i+4}])\}$, so we need only compute $\alpha_w(G[A_i])$ for each $i \in \{1, \ldots, 7\}$. For each i pick a vertex $a_i \in A_i$. The graph $G[A_i]$ contains no C_5, no P_5 and no $\overline{P_5}$, for otherwise adding a_{i+1} and either a_{i+2} or a_{i+3} to such a subgraph we obtain an umbrella or a parasol in G or $\overline{G}$, which contradicts Lemmas 5.3 and 5.4. By results from [19] and [47], MWSS can be solved in time $O(n^3)$ in graphs with no C_5, P_5 and $\overline{P_5}$. Hence, since the A_i's are pairwise disjoint, MWSS can be solved in time $O(|V(K)|^3)$ in K.

Case 2: K **contains a** C_5 **of type** 2 **and no** C_5 **of type** 1 **or** 0**.**

For adjacent vertices u, v in Z we say that the edge uv is *red* if there exists a P_4 h'-u-v-h'' for some $h', h'' \in H$. For every vertex h in H we define its *score*, $sc(h)$, as the number of red edges that contain a neighbor of h. Let h be a vertex of maximum score in H.

> Suppose that $K \setminus N(h)$ contains a C_5 of type 2 t-h_1-a-b-h_2-t, with $h_1, h_2 \in H$ and $a, b, t \in Z$. Then $hh_1, hh_2 \notin E(G)$, and Z contains vertices y_1, z_1, y_2, z_2 such that $y_1z_1, y_2z_2, hy_1, hy_2 \in E(G)$, $hz_1, hz_2, h_1y_1, h_1z_1, h_2y_2, h_2z_2 \notin E(G)$, and, up to symmetry, $\{y_1, y_2\}$ is complete to a and anticomplete to b, and $\{z_1, z_2\}$ is anticomplete to a, and $bz_2 \in E(G)$. (5.8)

Proof: Clearly $h \notin \{h_1, h_2\}$. Note that ab is a red edge. There must be a red edge y_1z_1 (with $y_1, z_1 \in Z$) that is counted in $sc(h)$ and not in $sc(h_1)$, for otherwise we have

$sc(h_1) \geq sc(h) + 1$ (because of ab), which contradicts the choice of h. So h_1 has no neighbor in $\{y_1, z_1\}$. We may assume that $hy_1 \in E(G)$. Let h'-y_1-z_1-h'' be a P_4 with $h', h'' \in H$. If $hz_1 \in E(G)$, then $hh' \notin E(G)$, for otherwise $\{c, d, h', h, z_1\}$ induces a bull; and similarly $hh'' \notin E(G)$; but then $\{h', y_1, h, z_1, h''\}$ induces a bull. Hence $hz_1 \notin E(G)$. Clearly $a \notin \{y_1, z_1\}$. If a has no neighbor in $\{y_1, z_1\}$, then b has a neighbor in $\{y_1, z_1\}$, for otherwise b-a-h_1-d-h-y_1-z_1 is an induced P_7; and b is adjacent to both y_1, z_1, for otherwise $\{c, d, h_1, a, b, y_1, z_1\}$ induces a P_7; but then $\{h, y_1, z_1, b, a\}$ induces a bull, a contradiction. So a has a neighbor in $\{y_1, z_1\}$. If a is adjacent to both y_1, z_1, then $\{h, y_1, z_1, a, h_1\}$ induces a bull. So a has exactly one neighbor in $\{y_1, z_1\}$, which leads to the following two cases:
— (i) $ay_1 \in E(G)$ and $az_1 \notin E(G)$. Then also $y_1 b \notin E(G)$, for otherwise $\{h, y_1, b, a, h_1\}$ induces a bull.
— (ii) $az_1 \in E(G)$ and $ay_1 \notin E(G)$. Then also $z_1 b \notin E(G)$, for otherwise either $\{h_1, a, z_1, b, h_2\}$ induces a bull (if $z_1 h_2 \notin E(G)$), or $\{c, d, h_1, a, z_1, b, h_2\}$ induces a G_2 (if $z_1 h_2 \in E(G)$), which contradicts Lemma 5.5. Moreover, $y_1 b \in E(G)$, for otherwise c-d-h-y_1-z_1-a-b is an induced P_7.
Similarly, there is a red edge $y_2 z_2$ (with $y_2, z_2 \in Z$) that is counted in $sc(h)$ and not in $sc(h_2)$, so h_2 has no neighbor in $\{y_2, z_2\}$. We may assume that $hy_2 \in E(G)$, and by the same argument as above we have $hz_2 \notin E(G)$ and either:
— (iii) $by_2 \in E(G)$, $bz_2 \notin E(G)$, and $y_2 a \notin E(G)$, or
— (iv) $bz_2 \in E(G)$, $by_2 \notin E(G)$, $z_2 a \notin E(G)$, and $y_2 a \in E(G)$.
Now if either (i) and (iii) occur, or (ii) and (iv) occur, then either $\{d, h, y_1, y_2, a\}$ induces a bull (if $y_1 y_2 \in E(G)$) or $\{h, y_1, y_2, a, b\}$ induces a C_5 of type 1 (if $y_1 y_2 \notin E(G)$), a contradiction. Therefore we may assume, up to symmetry, that (i) and (iv) occur. Thus (5.8) holds.

Now we claim that:

If $K \setminus N(h)$ contains a C_5 of type 2, with the same notation as in (5.8), then $K \setminus (N(h) \cup N(a))$ contains no C_5 of type 2. (5.9)

Proof: Let y_1, z_1, y_2, z_2 be vertices of Z as in (5.8). Suppose that $K \setminus (N(h) \cup N(a))$ contains a C_5 of type 2 t'-h_3-a'-b'-h_4-t', with $h_3, h_4 \in H$ and $t', a', b' \in Z$. By the analogue of (5.8) there exist vertices y_4, z_4 in Z such that $y_4 z_4, hy_4 \in E(G)$, $hz_4, h_4 y_4, h_4 z_4 \notin E(G)$, and, up to symmetry, $y_4 a', z_4 b' \in E(G)$ and $y_4 b', z_4 a' \notin E(G)$. We have $y_4 a \notin E(G)$, for otherwise c-d-h_4-b'-a'-y_4-a is an induced P_7; and $y_4 y_1 \notin E(G)$, for otherwise $\{d, h, y_4, y_1, a\}$ induces a bull; and $y_4 b \notin E(G)$, for otherwise $\{h, y_1, a, b, y_4\}$ induces a C_5 of type 1. Then $ba' \notin E(G)$, for otherwise c-d-h-y_4-a'-b-a is an induced P_7. If $y_1 b' \in E(G)$, then $y_1 z_4 \in E(G)$, for otherwise $\{h, y_1, b', z_4, y_4\}$ induces a C_5 of type 1, and $y_1 h_4 \in E(G)$, for otherwise $\{h, y_1, z_4, b', h_4\}$ induces a bull; but then $\{d, h_4, b', y_1, a\}$ induces a bull. So $y_1 b' \notin E(G)$. Then $bb' \notin E(G)$, for otherwise c-d-h-y_1-a-b-b' is an induced P_7. Then $a' y_1 \in E(G)$, for otherwise b-a-y_1-h-y_4-a'-b' is an induced P_7. Then $h_3 y_1 \notin E(G)$, for otherwise $\{d, h_3, a', y_1, a\}$ induces a bull, and $h_3 b \notin E(G)$, for otherwise $\{h_3, a', y_1, a, b\}$ induces a C_5 of type 1. But then c-d-h_3-a'-y_1-a-b is an induced P_7, a contradiction. Thus (5.9) holds.

Case 3: K **contains a** C_5 **of type** 0 **or** 1.

We will prove that:

There is a vertex $x \in V(K)$ such that $K \setminus N(x)$ contains no C_5 of type 0 or 1. (5.10)

We first make some remarks about the C_5's of type 1 and make a few more claims. Let $H_1 = \{h \in H \mid h \text{ lies in a } C_5 \text{ of type } 1\}$.

Let $h \in H_1$, and let $C = h\text{-}p_1\text{-}p_2\text{-}p_3\text{-}p_4\text{-}h$ be any C_5 of type 1 that contains h. Let a be any vertex in Z. Then either $N_C(a)$ is a stable set, or $N_C(a) = \{p_1, p_2, p_3, p_4\}$. (5.11)

Proof: Suppose that $N_C(a)$ is not a stable set. If a is adjacent to h and one of p_1, p_4, say $ap_1 \in E(G)$, then $ap_2 \in E(G)$, for otherwise $\{d, h, a, p_1, p_2\}$ induces a bull, and $ap_3 \notin E(G)$, for otherwise $\{d, h, p_1, a, p_3\}$ induces a bull, and $ap_4 \notin E(G)$, for otherwise $\{d, h, p_4, a, p_2\}$ induces a bull. But then $\{p_1, p_2, p_3, p_4, h, d, a\}$ induces a G_1, which contradicts Lemma 5.5. Now suppose that $ah \notin E(G)$. If a is adjacent to p_2 and p_3, then a also has a neighbor in $\{p_1, p_4\}$, for otherwise $\{p_1, p_2, a, p_3, p_4\}$ induces a bull. So in any case, up to symmetry, we may assume that a is adjacent to p_1 and p_2. Then $ap_3 \in E(G)$, for otherwise $\{h, p_1, a, p_2, p_3\}$ induces a bull, and $ap_4 \in E(G)$, for otherwise $\{p_1, p_2, p_3, p_4, h, d, a\}$ induces a G_2, which contradicts Lemma 5.5. Thus (5.11) holds.

Let $h \in H_1$, and let $C = h\text{-}t\text{-}u\text{-}v\text{-}w\text{-}h$ be any C_5 of type 1 that contains h. Suppose that $C' = h'\text{-}t'\text{-}u'\text{-}v'\text{-}w'\text{-}h'$ is a C_5 of type 1 in which h has no neighbor, with $h' \in H$. Then either $N_{C'}(t) = \{u', w'\}$ and $N_{C'}(w) = \{t', v'\}$, or vice-versa. (5.12)

Proof: Clearly $h \neq h'$. Let $Y = \{t, u, v, w\}$ and $Y' = \{t', u', v', w'\}$. Suppose that $\{t, w\}$ is anticomplete to Y'. Then $h'w \in E(G)$, for otherwise $w\text{-}h\text{-}d\text{-}h'\text{-}w'\text{-}v'\text{-}u'$ is an induced P_7, and similarly $h't \in E(G)$. If $h'u \in E(G)$, then $ut' \notin E(G)$ (by (5.11) applied to C' and u), but then $\{h, t, u, h', t'\}$ induces a bull. So $h'u \notin E(G)$, and similarly $h'v \notin E(G)$. Then one of u, v, say u, has a neighbor in Y', for otherwise $u\text{-}v\text{-}w\text{-}h'\text{-}w'\text{-}v'\text{-}u'$ is an induced P_7; moreover u is complete to Y', for otherwise c, d, h, t, u plus two vertices from Y' induce a P_7. Then v has no neighbor $y' \in Y'$, for otherwise $\{t, u, y', v, w\}$ induces a bull; but then $\{h', t', u', u, v\}$ induces a bull. So $\{t, w\}$ is not anticomplete to Y', and we may assume up to symmetry that w has a neighbor in Y'.
We have $|N_{Y'}(w)| \geq 2$ and $N_{Y'}(w) \neq \{t', w'\}$, for otherwise c, d, h, w plus three vertices from Y' induce a P_7; and w is not complete to Y', for otherwise, by (5.11), $\{h, w, v', w', h'\}$ induces a bull. Hence, by (5.11) and up to symmetry, we have $N_{C'}(w) = \{t', v'\}$. Since $t'h \notin E(G)$, we have $t'v \notin E(G)$, for otherwise, by (5.11), $\{h, w, v, t', h'\}$ induces a bull. If also t has a neighbor in Y', then by the same argument as with w we have either (i) $N_{C'}(t) = \{u', w'\}$ or (ii) $N_{C'}(t) = \{t', v'\}$. In case (i) we obtain the desired result, so assume that (ii) holds. By (5.11), $t'u \notin E(G)$. Then h' has a

neighbor in $\{u,v\}$, for otherwise c-d-h'-t'-w-v-u is an induced P_7; say $h'u \in E(G)$. Then $h'v \notin E(G)$, for otherwise $\{t,u,h',v,w\}$ induces a bull. Then v' has neighbor in $\{u,v\}$, for otherwise c-d-h'-u-v-w-v' is an induced P_7; and by (5.11) we have $N_C(v') = Y$. But then $\{h,t,v',u,h'\}$ induces a bull, a contradiction. So we may assume that t has no neighbor in Y'. Then $th' \in E(G)$, for otherwise t-h-d-h'-t'-u'-v' is an induced P_7; and $uh' \notin E(G)$, for otherwise by (5.11), $N_C(h') = Y$, which would imply $N_{C'}(w) \neq \{v',t'\}$; and $vh' \in E(G)$, for otherwise c-d-h'-t-u-v-w is an induced P_7. By (5.11) we have $|N_{Y'}(v)| \leq 1$ and $N_{Y'}(v) \subset \{u',v'\}$. We have $vv' \notin E(G)$, for otherwise $\{h,w,v,v',u'\}$ induces a bull, so we have $vu' \in E(G)$, for otherwise c-d-h'-v-w-v'-u' is an induced P_7. Then $uu' \notin E(G)$ by (5.11) (since $wu' \notin E(G)$). But then c-d-h-t-u-v-u' is an induced P_7. Thus (5.12) holds.

Now we deal with C_5's of type 0. Clearly any such C_5 lies in a component of $G[Z]$, and any such component has a neighbor in H since G is connected.

Let T be any component of $G[Z]$ that contains a C_5, let C be any C_5 in T, and let h be any vertex in H that has a neighbor in T. Then h is a 2-neighbor of C. (5.13)

Proof: There is a shortest path p_0-p_1-p_2-$\cdots$-p_r such that $p_0 = c$, $p_1 = d$, $p_2 = h$ and p_r has a neighbor in C, and $r \geq 2$. By Lemma 5.6, p_r is either a 1-neighbor, a 2-neighbor or a 5-neighbor of C. If p_r is a 5-neighbor, then $V(C) \cup \{p_r, p_{r-1}\}$ induces an umbrella, which contradicts Lemma 5.3. If p_r is a 1-neighbor of C, then p_{r-2}, p_{r-1}, p_r and four vertices of C induce a P_7. So p_r is a 2-neighbor of C. Now if $r \geq 3$, then $p_{r-3}, p_{r-2}, p_{r-1}, p_r$ and three vertices of C induce a P_7. So $r = 2$, and (5.13) holds.

At most one component of $G[Z]$ contains a C_5. (5.14)

Proof: Suppose that two components T and T' of $G[Z]$ contain a C_5. Let C a C_5 in T, with vertices $c_1, \ldots, c_5$ and edges c_ic_{i+1} (mod 5), and let C' a C_5 in T', with vertices $c'_1, \ldots, c'_5$ and edges $c'_ic'_{i+1}$ (mod 5). Pick any $h \in H$ that has a neighbor in T, and pick any h' in H that has a neighbor in T'. By (5.13) and Lemma 5.6 we may assume that $N_C(h) = \{c_1, c_4\}$ and $N_{C'}(h') = \{c'_1, c'_4\}$. If h has a neighbor in T', then, by (5.13) and Lemma 5.6, we have $N_{C'}(h) = \{c'_j, c'_{j+2}\}$ for some j. But then c_3-c_2-c_1-h-c'_j-c'_{j-1}-c'_{j-2} is an induced P_7. So h has no neighbor in T', and similarly h' has no neighbor in T. Then either c_3-c_2-c_1-h-d-h'-c'_1 or c_3-c_2-c_1-h-h'-c'_1-c'_2 is an induced P_7. So (5.14) holds.

If a component T of $G[Z]$ contains a C_5, and h is any vertex in H that has a neighbor in T, then $K \setminus N(h)$ has no C_5 of type 0 or 1. (5.15)

Proof: By (5.13) and (5.14), $K \setminus N(h)$ has no C_5 of type 0. So suppose that there is a C_5 of type 1 $C' = h'$-t'-u'-v'-w'-h' (with $h' \in H$) in which h has no neighbor. Let C be a C_5 in T, with vertices $c_1, \ldots, c_5$ and edges c_ic_{i+1} (mod 5). By (5.13) and Lemma 5.6, we may assume that $N_C(h) = \{c_1, c_4\}$. Let $C_h = h$-c_1-c_2-c_3-c_4-h; so C_h is a C_5 of type 1. By (5.12) and up to symmetry, we have $N_{C'}(c_1) = \{t', v'\}$ and $N_{C'}(c_4) = \{u', w'\}$, and $t', u', v', w' \in T$. Then c_5 has a neighbor in $\{u', v'\}$, for otherwise $\{c_1, v', u', c_4, c_5\}$ induces a C_5 of type 0 in which h' has at most one neighbor, contradicting (5.13). If $c_5u' \in E(G)$, then $c_5v' \in E(G)$, for otherwise $\{h, c_4, c_5, u', v'\}$ induces a bull. If $c_5v' \in E(G)$, then $c_5u' \in E(G)$, for otherwise $\{h, c_1, c_5, v', u'\}$ induces a bull. In both cases, by (5.11), c_5 is complete to $\{t', u', v', w'\}$. But then $\{h, c_1, t', c_5, w'\}$ induces a bull. Thus (5.15) holds.

Suppose that there is no C_5 of type 0. Pick any $h \in H_1$, and suppose that there is a C_5 of type 1 $C' = h'$-b_2-u-v-a_2-h' in which h has no neighbor. Then $K \setminus N(u)$ has no C_5 of type 1. (5.16)

Proof: Let h-a_1-v'-u'-b_1-h be any C_5 of type 1 that contains h. By (5.12), we may assume that $N_{C'}(a_1) = \{b_2, v\}$ and $N_{C'}(b_1) = \{a_2, u\}$. Let $C = h$-a_1-v-u-b_1-h; then C is a C_5 of type 1 in which h' has no neighbor, so h and h' play symmetric roles. Let $C_{a_1} = h$-a_1-b_2-u-b_1-h and $C_{a_2} = h'$-a_2-b_1-u-b_2-h'. Suppose that there is a C_5 of type 1 $C'' = h''$-t''-u''-v''-w''-h'' in which u has no neighbor. Let $X = \{a_1, b_1, a_2, b_2, u, v\}$ and $Y'' = \{t'', u'', v'', w''\}$.
We observe that $G[X \cup Y'']$ is bipartite: indeed in the opposite case, and since K contains no C_5 of type 0 and no C_7, there is a triangle in $G[X \cup Y'']$, and so there is either (i) a vertex $y'' \in Y''$ with two adjacent neighbors in X, or (ii) a vertex $x \in X$ with two adjacent neighbors in Y''. In case (i), by (5.11) applied to y'' and the cycles C, C', C_{a_1}, C_{a_2}, we see that y'' is complete to X, which is not possible since $uy'' \notin E(G)$. So suppose we have case (ii). By (5.11) we have $N_{C''}(x) = Y''$. Clearly $x \neq u$. Moreover, $x \notin \{b_1, b_2, v\}$, for otherwise $\{u, x, v'', w'', h''\}$ induces a bull. So, up to symmetry, $x = a_1$. By case (i) we have $v''b_2, w''b_2 \notin E(G)$; but then $\{h'', w'', v'', a_1, b_2\}$ induces a bull. So $G[X \cup Y'']$ is bipartite. Let A, B be a bipartition of $X \cup Y''$ in two stable sets. Up to symmetry we may assume that $A = \{a_1, a_2, u, u'', w''\}$ and $B = \{b_1, b_2, v, t'', v''\}$.
Note that h'' has a neighbor in C, for otherwise (5.12) is contradicted (since u has no neighbor in $\{t'', w''\}$), and similarly h'' has a neighbor in C', in C_{a_1} and in C_{a_2}. Suppose that $h''a_1 \in E(G)$. Then $h''b_2 \notin E(G)$, for otherwise $\{d, h'', a_1, b_2, u\}$ induces a bull, and $h''b_1 \in E(G)$, for otherwise c-d-h''-a_1-b_2-u-b_1 is an induced P_7, and $h''a_2 \notin E(G)$, for otherwise $\{d, h'', a_2, b_1, u\}$ induces a bull, and $h''h' \notin E(G)$, for otherwise $\{c, d, h'', h', a_2\}$ induces a bull. By (5.11), h'' is not adjacent to v. But then h'' has no neighbor in C', a contradiction. So $h''a_1 \notin E(G)$, and similarly $h''a_2 \notin E(G)$. So $h'' \notin \{h, h'\}$; moreover $h''h \notin E(G)$, for otherwise $\{c, d, h'', h, a_1\}$ induces a bull, and similarly $h''h' \notin E(G)$. Then h'' has a neighbor in $\{b_1, b_2\}$, say $h''b_1 \in E(G)$, for otherwise h'' has no neighbor in C_{a_1}; and then $h''b_2 \in E(G)$, for otherwise c-d-h''-b_1-u-b_2-a_1 is an induced P_7, and $h''v \in E(G)$, for otherwise c-d-h''-b_1-a_2-v-a_1 is an induced P_7. So $N_X(h'') = \{b_1, b_2, v\}$. By (5.11), b_1, b_2 and v have no neighbor in $\{t'', w''\}$; and since B is a stable set they are not adjacent to v''.
Suppose that $b_1u'' \in E(G)$. Then $a_1v'' \notin E(G)$, for otherwise c-d-h''-b_1-u''-v''-a_1 is an induced P_7, and $hu'' \notin E(G)$, for otherwise $\{d, h, u'', b_1, u\}$ induces a bull. Then h has exactly one neighbor in $\{v'', w''\}$, for otherwise either c-d-h-b_1-u''-v''-w'' is an induced P_7 or $\{d, h, w'', v'', u''\}$ induces a bull. However, if $hw'' \in E(G)$, then $b_2u'' \in E(G)$,

for otherwise u''-v''-w''-h-a_1-b_2-u is an induced P_7, and then c-d-h-w''-v''-u''-b_2 is an induced P_7; while if $hv'' \in E(G)$, then $u''v \notin E(G)$, for otherwise c-d-h-v''-u''-v-u is an induced P_7, and then u''-v''-h-d-h''-v-u is an induced P_7, a contradiction. Hence $b_1u'' \notin E(G)$ and, by symmetry, b_1 and b_2 have no neighbor in Y''.
If $vu'' \in E(G)$, then $hu'' \in E(G)$, for otherwise c-d-h-b_1-u-v-u'' is an induced P_7, but then c-d-h-u''-v-u-b_2 is an induced P_7. So $vu'' \notin E(G)$, and so v has no neighbor in Y''. Then $a_1v'' \notin E(G)$, for otherwise c-d-h''-v-a_1-v''-u'' is an induced P_7; and $a_1t'' \notin E(G)$, for otherwise b_1-u-v-a_1-t''-u''-v'' is an induced P_7. Hence, by symmetry, a_1 and a_2 have no neighbor in Y''. Now h has a neighbor in $\{t'', u'', v'', w''\}$, for otherwise a_1-h-d-h''-t''-u''-v'' is an induced P_7. If h has two adjacent neighbors in Y'', then h is complete to Y'', for otherwise d, h plus three consecutive vertices of Y'' induce a bull; but then $\{h'', t'', u'', h, a_1\}$ induces a bull. So we may assume that $N_{C''}(h) = \{t'', v''\}$, for otherwise u, v, a_1, h and three consecutive vertices in Y'' induce a P_7. But then u''-v''-h-d-h''-v-u is an induced P_7, a contradiction. Thus (5.16) holds.

Now, (5.10) follows from (5.15) and (5.16). This completes the proof in Case 3.

To conclude, we give the general outline of the algorithm to solve MWSS in K. For each type $q \in \{0, 1, 2\}$, we find a vertex x such that $K \setminus N(x)$ contains no C_5 of type q. We then solve the MWSS in $K \setminus N(x)$ and in $K \setminus \{x\}$. Since every maximum-weight stable set of K either contains x or not, the best of these two solutions is a solution for the MWSS in K. We repeat this until there are no more C_5's of this type. More formally:

(I) Suppose that K contains no C_5. If K also contains no C_7, then K is perfect, so we can solve the MWSS in K by using the algorithm from [82]. If K contains a C_7, then MWSS can be solved in time $O(|K|^3)$ as explained in Case 1 of the proof.

(II) Suppose that K contains a C_5 of type 2 and no C_5 of type 0 or 1. Let h be a vertex of maximum score as in Case 2 of the proof. Then MWSS in K can be solved by successively solving the MWSS in (a) $G[K \setminus N(h)]$ and in (b) $G[K \setminus \{h\}]$.

Step (a) can be done as follows: If $G[K \setminus N(h)]$ contains no C_5, then we are in (I). If $G[K \setminus N(h)]$ contains a C_5 (of type 2), then by (5.9) there is a vertex a in this C_5 such that $G[K \setminus (N(h) \cup N(a))]$ contains no C_5. Hence we solve MWSS in (a1) $G[K \setminus (N(h) \cup N(a))]$ and in (a2) $G[K \setminus (N(h) \cup \{a\})]$. Step (a1) can be done in polynomial time by referring to (I). Step (a2) can be computed by recursively calling Step (a). The number of recursive calls is bounded by $|Z|$.

Step (b) can be computed by recursively calling (II). After a number of calls there is no longer any C_5 of type 2, so we are in (I). The number of recursive calls is bounded by $|H|$.

(III) Suppose that K contains a C_5 of type 1 and no C_5 of type 0. Let u be a vertex such that $K \setminus N(u)$ has no C_5 of type 1, as in Claim (5.16). Then MWSS in K can be solved by successively solving the MWSS in (a) $G[K \setminus N(u)]$ and in (b) $G[K \setminus \{u\}]$. Step (a) can be done in polynomial time by referring to (II) or (I). Step (b) can be computed by recursively calling (III). After a number of calls there is no longer any C_5 of type 1, so we are in (II) or (I). The number of recursive calls is bounded by $|K|$.

(IV) Suppose that K contains a C_5 of type 0. Let T be the component of $G[Z]$ (unique by Claim (5.14)) that contains a C_5. Let $H_0 = \{h \in H \mid h \text{ has a neighbor in } T\}$. Let h be any vertex in H_0. By (5.15) we know that $G[K \setminus N(h)]$ contains no C_5 of type 0 or 1. Then the MWSS in K can be solved by successively solving the MWSS in (a) $G[K \setminus N(h)]$ and in (b) $G[K \setminus \{h\}]$. Step (a) can be computed in polynomial time by calling (II) or (I). Step (b) can be computed by recursively calling (IV). The number of recursive calls is equal to $|H_0|$. At the end of this step, the component T becomes isolated because we have removed all vertices of H_0, but we still need to solve MWSS in T. This can be done as follows. Consider any vertex $h \in H_0$. By Claim (5.13) every C_5 in T contains exactly two vertices from $N(h_0) \cap V(T)$, and these two vertices are not adjacent. Hence MWSS can be solved in T using the same technique as in (II) and the analogue of Claim (5.9).

The total number of recursive calls is in $O(n)$ since there are three different cycle types. For each computation of MWSS in K, we end up calling the algorithm in [82] which runs in $O(n^6)$. Furthermore, at each step we need to compute the list of all the cycles of length 5, which takes $O(n^5)$, but this is additive. We need to run all the previous steps on every connected component K of the non-neighborhood of a fixed vertex of $V(G)$, there are at most n such components. Finally, we repeat this for every vertex in $V(G)$, so the overall complexity of our algorithm is $O(n^9)$. This completes the proof of Theorem 5.15.

One may wonder whether Claims (5.9) and (5.10) could be subsumed by the following single claim: There is a vertex x in K such that $K \setminus N(x)$ contains no C_5 of any type. Here is an example showing that such a claim does not hold. Let Z have six vertices $c_1, \dots, c_5$ and z, such that $c_1, \dots, c_5$ induce a C_5 with edges $c_i c_{i+1}$ (i mod 5), and z has no neighbor in this C_5. Let H have five vertices $h_1, \dots, h_5$ such that for each i we have $N_Z(h_i) = \{c_{i-1}, c_{i+1}, z\}$. Let $V(G) = \{c, d, h_1, \dots, h_5, c_1, \dots, c_5, z\}$. It is a routine matter to check that G is (P_7, K_3)-free and that $K \setminus N(x)$ contains a C_5 for every vertex $x \in K$.

Chapter 6

Normal Graphs

6.1 Context and motivations

The study of normal graphs takes root from perfect graphs. Recall that a graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G. The *co-normal product* (also called the *strong product*) of two graphs G_1, G_2, denoted by $G_1 * G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$, where vertices (u_1, u_2) and (v_1, v_2) are adjacent if u_1 is adjacent to v_1 or u_2 is adjacent to v_2. See Figure 6.1 for an example. The *co-normal power* denoted by G^k for some positive integer k of a graph G is defined by:

$$G^k = \underbrace{G * \cdots * G}_{k}.$$

Claude Berge's motivation to study perfect graphs came in part from determining the zero-error capacity of a discrete memoryless channel. This can be measured by the following quantity, called *Shannon capacity* and is defined by:

$$C(G) = \lim_{n \to \infty} \frac{1}{n} \log(\omega(G^n))$$

For a longer introduction to Shannon capacity, we refer the reader to the survey of Gábor Simonyi [88] on graph entropy.

Shannon noticed that $\omega(G^n) = (\omega(G))^n$ whenever $\omega(G) = \chi(G)$. From this observation, one might have expected that perfect graphs are closed under co-normal products. However, Körner et al. [59] proved this to be false. This motivated Körner in [58] to study the class of graphs closed under co-normal products.

A *normal graph* is a graph for which there exist two coverings, $\mathbb{C}$ and $\mathbb{S}$ of its vertex set such that every member of $\mathbb{C}$ induces a clique in G, every member of $\mathbb{S}$ induces a stable set in G and $C \cap S \neq \emptyset$ for every $C \in \mathbb{C}$ and $S \in \mathbb{S}$. Such a covering is called a *normal covering*. Since C and S are respectively a clique and a stable set, their intersection contains at most one vertex. Figure 6.2 illustrates a normal covering in the complete graph on three vertices.

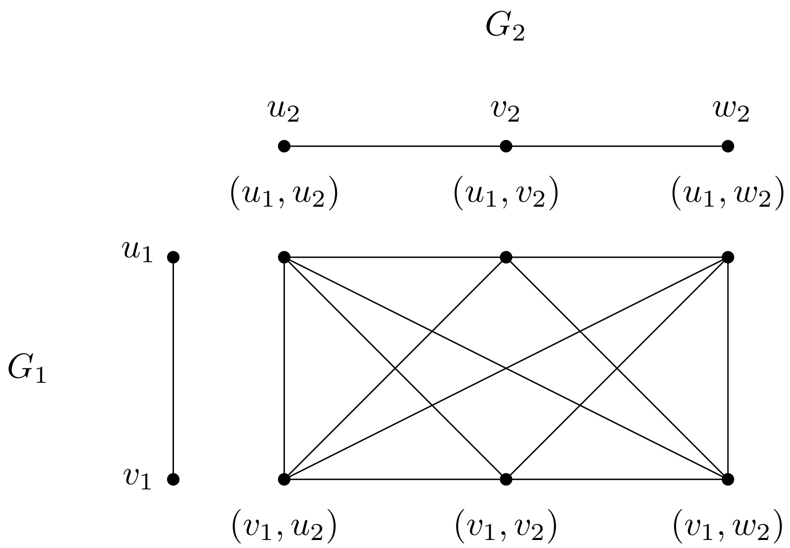


Figure 6.1: The co-normal product of a graph G_1 with another graph G_2.

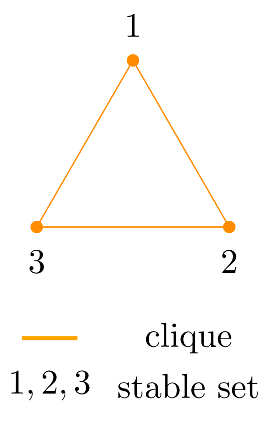


Figure 6.2: A normal covering of K_3. The clique is in orange (the whole graph) and the three stables are numbered from 1 to 3.

Körner [58] showed that all co-normal products of normal graphs are normal. He also showed that all perfect graphs are normal.

Given a graph G the *graph entropy* of G with respect to a probability distribution P on its vertex set $V(G)$ is defined as:

$$H(G,P) = \lim_{t\to\infty} \min_{U\subseteq V(G^t), P^t(U)>1-\epsilon} \frac{1}{t} \log \chi(G^t[U])$$

where $P^t(U) = \sum_{\mathbf{x}\in U} \prod_{i=1}^{t} P(x_i)$ and $\epsilon \in (0,1)$. Note that the limit is independent of ϵ as shown by Körner [57]. Normal graphs and perfect graphs have a close relationship with graph entropy. The graph entropy is sub-additive with respect to complementary graphs [59]:

$$H(P) \leq H(G,P) + H(\overline{G},P)$$

for all graph G and all probability distribution P on $V(G)$, where the entropy of P is given by $H(P) = \sum_{i=1}^{n} p_i \log(\frac{1}{p_i})$. Csiszár et al. showed that equality holds if and only if G is perfect. They proved the following theorem:

THEOREM 6.1 [26]

$H(P) = H(G,P) + H(\overline{G},P)$ for all P if and only if G is perfect.

Körner et al. proved a relaxed version for normal graphs, which is another way of seeing that every perfect graph is also normal:

THEOREM 6.2 [60]

$H(P) = H(G,P) + H(\overline{G},P)$ for at least one (nowhere vanishing) P if and only if G is normal.

The only minimally known graphs which are not normal are C_5, C_7 and $\overline{C_7}$. The previous observation and the fact that normal graphs and perfect graphs have many properties in common, as seen above, motivated De Simone and Körner to conjecture the following:

CONJECTURE 6.3 Normal Graph Conjecture [87]

A graph with no C_5, C_7 and $\overline{C_7}$ as an induced subgraph is normal.

Note that this conjecture is not stated in a "if and only if" way as it is for the Strong Perfect Graph Theorem. In fact, there exist graphs containing a C_5, C_7 or $\overline{C_7}$ that are normal. See Figure 6.3 for an example. Moreover, Wagler [95] proved that given any graph G, there exists a normal graph G^* containing G as an induced subgraph.

THEOREM 6.4 [95]

For any graph G, there is a normal graph G^* containing G as an induced subgraph.

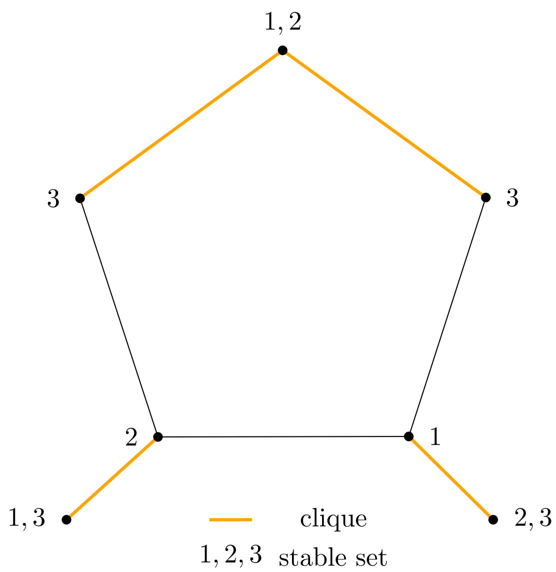


Figure 6.3: A normal graph containing an induced C_5.

This result strikes out the hope to have a characterization of normal graphs by forbidden induced subgraph.

A *cubic graph* is a graph in which all vertices have degree exactly three. For example, K_4 is a cubic graph. Patakfalvi proved that every line-graph of cubic graphs are normal.

Theorem 6.5 [81]

The line-graph of every cubic graph is normal.

A *circulant graph*, denoted by C_n^k, is a graph on n vertices $v_1, v_2, \ldots, v_n$ in which the vertex v_i is adjacent to the $2k$ vertices $v_{i\pm 1}, v_{i\pm 2}, \ldots, v_{i\pm k}$. Another definition that is more intuitive is as follows. The graph C_n^k is the graph obtain from the cycle graph C_n with the addition of edges between those pairs of vertices at distance at most k. Wagler verified the Normal Graph Conjecture for circulant graphs.

Theorem 6.6 [94]

The Normal Graph Conjecture is true for circulants and their complements.

Berry et al. also verified the Normal Graph Conjecture for a few well-known graph classes [8]. Normal graphs have also been studied within the context of random graphs by Hosseini et al. [49].

A graph that has a girth of at least 8 does not contain any triangle, C_5, C_7 and $\overline{C_7}$ since a $\overline{C_7}$ contains triangle. With a probabilistic proof, we managed to show that their exists a graph of girth at least 8 that does not admit a normal covering. Since our graph cannot contain any of the forbidden induced subgraphs and is not normal, it disproves Conjecture 6.3. We proved the following theorem:

THEOREM 6.7

There exists a graph G of girth at least 8 that is not normal.

The next section is dedicated to make a small overview of the tools used in the proof of Theorem 6.7.

6.2 The Probabilistic Method

To prove that an object exists you can exhibit it by construction or give a precise example. Another way is also to show its existence without providing a constructive proof neither an example. This is the way of a probabilistic proof. The idea behind the probabilistic method is to pick at random an object from a bag of objects of the same type and show that there is a positive probability of choosing the special object you are looking for, which shows its existence. First we will give basic definitions and give a complete proof of a famous theorem using probabilistic arguments. The most elementary notions are given informally and we refer the reader to [71] for a more formal introduction to discrete probability theory.

Given an event A, we denote by $\mathbb{P}[A]$, the *probability* that A is realized. Let X be a discrete random variable X that can take value x_1 with probability p_1, value x_2 with probability p_2 and so on up to x_k for some integer k, the *expectation* of X, denoted by $\mathbb{E}[X]$, is defined as follows:

$$\mathbb{E}[X] = \sum_{i=1}^{k} x_i p_i.$$

Intuitively, this is the average value of a random variable on a *long* series of repetitions of the experiment it represents. A *Bernoulli random variable* is a random variable that can take value 0 or 1. Given a random variable X, it is often very useful to be able to tell how X can deviate from some value. One very famous bound regarding this is Markov's inequality.

THEOREM 6.8 Markov's Inequality

If X is any non-negative discrete random variable and $a > 0$, then:

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}$$

More importantly, how far a random variable can deviate from its expected value? The Chernoff's inequality gives a narrow enclosing of how this deviation behaves for a Bernoulli random variable.

THEOREM 6.9 Chernoff's Inequality

Let $X_1, \dots, X_n$ be independent Bernoulli random variables where $\mathbb{P}[X_i = 1] = p_i$. Let $X = \sum_{i=1}^n X_i$ and let $\mu = \sum_{i=1}^n p_i$ be the expectation of X. Then, for all $0 < \delta < 1$, we have:

$$\mathbb{P}[X \leq (1-\delta)\mu] \leq e^{-\mu\delta^2/2}$$
$$\mathbb{P}[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3}$$

Given any finite or countable sets of events, Boole's inequality, also called the Union Bound, gives an upper bound on the probability that at least one of the events happens.

THEOREM 6.10 Union Bound

For any finite or countably infinite sequence of events $E_1, E_2, \dots$ we have:

$$\mathbb{P}\left[\bigcup_{i\geq 1} E_i\right] \leq \sum_{i\geq 1} \mathbb{P}(E_i)$$

Now that we have stated some of the basic tools used in probability theory, we can continue on an example of how to use some of these results to prove a non trivial result. One classical example of the power of the probabilistic method is the so called *High Girth and High Chromatic Number* theorem due to Erdős in 1959.

THEOREM 6.11 [29]

For all k, ℓ there exists a graph G with $\mathrm{girth}(G) > \ell$ and $\chi(G) > k$.

The proof can be found in the famous book called *The Probabilistic Method* of Alon and Spencer [2]. This proof is a canonical example of how the probabilistic method works. Also, this is a perfect introduction to the techniques used in the proof of Theorem 6.7 for the reader not familiar with random graphs. For those reasons, we include a pedagogical proof of it.

Proof of Theorem 6.11. The main idea is to generate and modify a *random graph*, denoted by $G_{n,p}$, on n vertices where each edge appears independently with probability p. An intuitive way to see this is to start with a stable set of n vertices, and for each pair of vertices, draw a magic coin that gives 1 as an outcome with probability p and 0 with probability $1-p$. If the outcome is 1 there is an edge, otherwise there is no edge. Our goal now is to show that by generating a graph at random with this method, there is a strictly positive probability that we obtain a graph with some special properties.

First, we want to aim for a random graph $G_{n,p}$ with girth greater than ℓ for some fixed $\ell \geq 3$. Let $\lambda \in (0, \frac{1}{\ell})$ and $p = n^{\lambda-1}$. Let us generate the random graph $G_{n,p}$ with the edge probability p given above and see what we can say about it.

We want to see how the girth behaves in $G_{n,p}$. Let us compute the number of cycles of length at most ℓ. Let X be this number and X_j the number of cycles of length at most j. One upper bound on X_j is obtained by seeing a cycle of length j as a word of length j on an alphabet of size n, hence we have this rough upper bound:

$$X_j \leq n^j.$$

Each of those cycles appears with probability p^j (there are j edges in a cycle of length j and each appears with probability p). Hence we have:

$$\begin{aligned}\mathbb{E}[X] &\leq \sum_{j=3}^{\ell} n^j p^j \\ &= \sum_{j=3}^{\ell} n^{\lambda j}.\end{aligned}$$

This is a sum of a geometric series starting at 3 and ending at ℓ. It gives the following:

$$\begin{aligned}\mathbb{E}[X] &\leq \sum_{j=3}^{\ell} n^{\lambda j} \\ &= \frac{n^{3\lambda} - n^{\lambda\ell+\lambda}}{1 - n^{\lambda}} = \frac{n^{\lambda\ell+\lambda} - n^{3\lambda}}{n^{\lambda} - 1} \\ &= \frac{n^{\lambda\ell+\lambda} - n^{3\lambda}}{n^{\lambda}(1 - n^{-\lambda})} = \frac{n^{\lambda\ell} - n^{2\lambda}}{1 - n^{-\lambda}} \\ &\leq \frac{n^{\lambda\ell}}{1 - n^{-\lambda}}.\end{aligned}$$

Since $\lambda\ell < 1$, we have that $\frac{n^{\lambda\ell}}{1-n^{-\lambda}}$ is smaller than $\frac{n}{c}$ for any $c > 1$ and n sufficiently large. By choosing $c = 4$ and for n sufficiently large, we have the following upper bound on the expectation of X:

$$\mathbb{E}[X] < \frac{n}{4}.$$

Hence, by Markov's inequality, we have:

$$\mathbb{P}[X \geq \frac{n}{2}] < \frac{n}{4} \times \frac{2}{n} = \frac{1}{2}.$$

To sum up, we know that the probability that $G_{n,p}$ contains at least $\frac{n}{2}$ short cycles is strictly less than $\frac{1}{2}$. We will keep this fact for later and will now deal with the stability number of our graph.

As explained in the first Chapter of this manuscript, for any graph G the chromatic number is lower bounded by the following:

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

Rather than directly looking at the chromatic number, we will deal with the stability number. Let $a = \left\lceil \frac{3}{p} \ln n \right\rceil$ and consider the event *there is a stable set of size a*. The probability of this event is given by:

$$\begin{aligned}
\mathbb{P}[\alpha(G) \geq a] &\leq \binom{n}{a}(1-p)^{\binom{a}{2}} \\
&\leq n^a e^{\frac{-p(a(a-1))}{2}} \quad \text{where } \binom{n}{a} \leq n^a \text{ and } (1+r)^x \leq e^{rx} \text{ for any real } x \text{ and } r > 0 \\
&= n^a e^{-\frac{3 \ln n(a-1)}{2}} \\
&= n^a e^{-\frac{3(a-1)}{2}}
\end{aligned}$$

It follows that $n^a n^{-\frac{3(a-1)}{2}}$ tends to 0 as n grows large. Hence we have that for n sufficiently large:

$$\mathbb{P}[\alpha(G) \geq a] < \frac{1}{2}$$

Now, the union bound gives the following probability:

$$\mathbb{P}[X \geq \frac{n}{2} \text{ or } \alpha(G) \geq a] < 1$$

By looking at the probability of the complementary of this event, we have that:

$$\mathbb{P}[X < \frac{n}{2} \text{ and } \alpha(G) < a] = 1 - \mathbb{P}[X \geq \frac{n}{2} \text{ or } \alpha(G) \geq a] > 0$$

This means that with strictly positive probability, there exists a graph G such that the number of *short* cycles is less than $\frac{n}{2}$ **and** the stability number is less than a.

There is one final step that needs to be done. We know that there are not too many short cycles, but we still need to produce a graph of girth at least ℓ. Let S be the vertex set obtained from picking exactly one vertex from each of these short cycles and let $G' = G[V \setminus S]$. Note that G' has girth at least ℓ, has at least $\frac{n}{2}$ vertices (since $|S| < \frac{n}{2}$) and that $\alpha(G') < a$ since taking an induced subgraph cannot increase the stability number. Now we can get the following lower bound on the chromatic number of G':

$$\begin{aligned}
\chi(G') \geq \frac{|V(G')|}{\alpha(G')} &\geq \frac{n}{2} \times \frac{p}{3 \ln n} \\
&= \frac{n}{2} \times \frac{n^{\lambda-1}}{3 \ln n} \\
&= \frac{n^{\lambda}}{6 \ln n}
\end{aligned}$$

Since $\lambda > 0$, as n grows large so does the chromatic number. Hence, for any girth ℓ, we can show that there exists a graph of girth at least ℓ and arbitrarily large chromatic number. $\square$

6.3 The random graph

Let $G_{n,p}$ denote the random graph on n vertices where each edge is drawn randomly and independently with probability p. We now consider the random graph $G = G_{n,p}$ with $p = n^{-9/10}$ and denote by $d = np = n^{1/10}$. First we will show that some properties hold in G and then prove our main theorem.

6.3.1 Properties

In this section we prove several properties satisfied by G. Let X_7 be the number of cycles of G of length at most 7. We always assume that n is sufficiently large whenever we refer to a property of G that holds asymptotically on the number of vertices of G.

Lemma 6.12

The following properties hold for the graph G.

(a) $\mathbb{P}[X_7 > 4n^{7/10}] < 1/2$.

(b) Let $c \geq 10$ be a fixed constant. Then $\mathbb{P}[\alpha(G) \geq cn^{9/10}\log n] \leq n^{-\frac{c^2 n^{0.9}\log n}{3}}$.

(c) Let D be the event that G has a vertex of degree greater than $2d$. Then $\mathbb{P}[D] \leq e^{-n^{0.1}/10}$.

Proof. (a) Note that by linearity of expectation,

$$\mathbb{E}[X_7] \leq \sum_{l=3}^{7} \binom{n}{l}(l-1)!p^l \leq \sum_{l=3}^{7}(np)^l \leq 2n^{7/10}.$$

The result now follows by Markov's inequality.

(b) is well-known and can be deduced from, for example, Frieze [33]. We include the proof for completeness. By the Union Bound, we have

$$\begin{aligned}\mathbb{P}[\alpha(G(n,p)) \geq x] &\leq \binom{n}{x}(1-p)^{\binom{x}{2}} \\ &\leq n^x(e^{-p(x-1)/2})^x \leq (ne^{-n^{-0.9}(x-1)/2})^x\end{aligned}$$

Now, setting $x := cn^{0.9}\log n$ yields the result.

(c) Clearly, $\mathbb{P}[D] \leq n\mathbb{P}[\deg(v) > 2d]$, where v is some fixed vertex. By Chernoff's inequality $\mathbb{P}[\deg(v) > 2d] \leq e^{-n^{0.1}/3}$. The claim now follows. □

Let G be a bipartite graph with m edges on vertex bipartition (A, B). We denote by d its average degree in A, that is $d = m/|A|$ and by $e(X,Y)$ the number of edges between the set X and Y for any $X \subseteq A$, $Y \subseteq B$. A *partial cover* of G is a set of pairs

(x_i, Y_i) where the x_i's are distinct vertices of A, the Y_i's are disjoint sets of B, x_i is a neighbor of all vertices of Y_i, the size of each Y_i is $\lceil d/3 \rceil$ and finally the union of Y_i's has size at least $|B|/3$.

LEMMA 6.13

Let G be a random bipartite graph on vertex bipartition (A, B), where each possible edge appears with some probability p, independently. If $|B| \geq |A| > 10^{100}p^{-1}$ then G has $e(A, B) \in [0.99p|A||B|, 1.01p|A||B|]$ and a partial cover with probability at least $1 - e^{-cp|A||B|}$, where $c > 0$ is an absolute constant.

Proof. Let A' be the set of vertices of A with degree in $[0.99p|B|, 1.01p|B|]$ in B and B' be the set of vertices of B with degree in $[0.99p|A|, 1.01p|A|]$ in A. By Chernoff's inequality, there exists a constant $c > 0$, such that the probability that (i) $|A'| < 0.99|A|$ or, (ii) $|B'| < 0.99|B|$, or (iii) $m := e(A, B) \notin [0.99p|A||B|, 1.01p|A||B|]$ is at most $e^{-cp|A||B|}$. Indeed, note that probability of (i) is at most

$$\binom{|A|}{0.01|A|}(2e^{-(0.01)^2 p|B|/3})^{0.01|A|} < 2^{|A|}e^{-(0.01)^4 p|A||B|} < e^{-c_1 p|A||B|}$$

for some constant $c_1 > 0$ (here we used the fact that $10^{100}p^{-1} < |A| \leq |B|$). Similarly the probability of (ii) is at most $2^{|B|}e^{-(0.01)^4 p|A||B|} < e^{-c_2 p|A||B|}$ for some constant $c_2 > 0$ (here again we used the fact that $|B| \geq |A| > 10^{100}p^{-1}$). The probability of (iii) is clear.

Now, we claim that if G satisfies $|A'| \geq 0.99|A|$, $|B'| \geq 0.99|B|$ and m is in the interval $[0.99p|A||B|, 1.01p|A||B|]$, then it has a partial cover. Observe first that at least $3m/4$ edges of G must be between A' and B' (call these *good edges*). Now greedily pick pairs (x_i, Y_i) where $x_i \in A'$ and $Y_i \subseteq B' \cap N(x_i)$ has size exactly $\lceil m/3|A| \rceil$ in order to construct a partial cover. If the process stops with $Y := Y_1 \cup \cdots \cup Y_k$ of size at least $|B|/3$, we have our partial cover. If not, denote by X the set $\{x_1, \ldots, x_k\}$, and note that this implies that every vertex in $A' \setminus X$ has degree less than $\lceil m/(3|A|) \rceil$ in $B' \setminus Y$. Note that the size of X is negligible compared to the size of A'. Indeed, $|X| < |B|/\lceil m/(3|A|) \rceil < 4p^{-1} < |A'|/10^{10}$. Hence the number of good edges incident to X is negligible compared to the number of good edges. In particular, at least $2.99m/4$ good edges are incident to $A' \setminus X$. However, since every vertex in $A' \setminus X$ has degree at most $\lceil m/(3|A|) \rceil$ in $B' \setminus Y$, $e(A' \setminus X, Y) > 2.99m/4 - \lceil m/(3|A|) \rceil(|A'| - |X|) > 2.99m/4 - m/3$. Now, since $|Y| < |B|/3$, and every vertex in Y has degree at most $1.01p|A|$, it follows that $e(A' \setminus X, Y) < 1.01p|A||B|/3 < 1.01m/(3 \cdot 0.99)$. This implies that $2.99m/4 - m/3 < 1.01m/(3 \cdot 0.99)$, a contradiction.

□

6.3.2 The proof

In this section we prove our main result. We say that a graph G admits a *star covering* if there exist two coverings, $\mathbb{C}$ and $\mathbb{S}$, of $V(G)$ such that:

(a) every member of $\mathbb{C}$ induces a clique K_2 or K_1 in G, where no K_1 is included in some K_2.

(b) the graph on $V(G)$ consisting of the edges of $\mathbb{C}$, denoted by $E[\mathbb{C}]$, is a vertex-disjoint union of stars (the isolated K_1 being stars just consisting of an isolated center).

(c) every member of $\mathbb{S}$ induces a stable set in G.

(d) $C \cap S \neq \emptyset$ for every $C \in \mathbb{C}$ and $S \in \mathbb{S}$.

Every graph G admitting a star covering is normal, and the converse holds for triangle-free graphs:

CLAIM 6.14

If G is a normal triangle-free graph, then G admits a star covering $(\mathbb{C}, \mathbb{S})$ where $E[\mathbb{C}]$ contains at most $\alpha(G)$ stars.

Proof. Let $(\mathbb{C}', \mathbb{S}')$ be a normal covering of G. Since G is triangle-free, all cliques in $\mathbb{C}'$ are K_2's or K_1's. The cliques K_1 included in some K_2 can be deleted from $\mathbb{C}'$. All that remains to show is that we can reduce to cliques inducing vertex-disjoint stars. Indeed, suppose that $E[\mathbb{C}']$ contains two adjacent vertices u, v with $d_{E[\mathbb{C}']}(u) \geq 2$ and $d_{E[\mathbb{C}']}(v) \geq 2$. Deleting the edge uv from $\mathbb{C}'$ gives another covering (since u and v are also covered by other edges) that is still intersecting with $\mathbb{S}'$. Repeating this, we obtain a star covering $(\mathbb{C}, \mathbb{S})$ of G.

Now, we show that the number of stars in $E[\mathbb{C}]$ is at most $\alpha(G)$. Indeed, let $x_1, \ldots, x_k$ be the centers of the stars (some centers x_i may be trivial stars) in $E[\mathbb{C}]$, and let $S \in \mathbb{S}$ be any stable set. Then for each x_i, S must contain either x_i or a neighbor of x_i in $\mathbb{C}$. Since the stars are disjoint, it follows that $k \leq |S| \leq \alpha(G)$. □

Let $G = (V, E)$ be a graph. A *star system* $(Q, \mathcal{S})$ of G is a spanning set of vertex disjoint stars where $\mathcal{S}$ is the set of stars, and Q is the set of centers of the stars of $\mathcal{S}$. Therefore every $x_i \in Q$ is the center of some star S_i of $\mathcal{S}$. Moreover, the union of vertices of the S_i's is equal to V. Note that some stars can be trivial, i.e. simply consisting of their center. To every star system $(Q, \mathcal{S})$, we associate a directed graph Q^* on vertex set Q by letting $x_i \to x_j$ whenever a leaf of S_i is adjacent to x_j. Of particular interest here is the following notion of *out-section*: A subset X of Q is an out-section if there exists v in Q such that for each $x \in X$, there exists a directed path in Q^* from v to x.

Observe that to every star-covering we can associate the star-system $E[\mathbb{C}]$.

LEMMA 6.15

Let G be a normal triangle-free graph with a star covering $(\mathbb{C}, \mathbb{S})$. We denote by $(Q, \mathcal{S})$ its associated star-system. Assume that X is an out-section of Q^*. Then the set of leaves of the stars with centers in X form a stable set of G.

Proof. To see this, consider a vertex v in Q which can reach every vertex x of X in Q^* by a directed path $v = x_0 \to x_1 \to \cdots \to x_k = x$. For all i, we denote by S_i the star with center x_i (observe that they all have leaves, except possibly S_k). Consider an stable set I of $\mathbb{S}$ which contains any leaf of S_0. Since I is a stable set, it does not contain x_0, and hence by definition of normal cover I must contain all the leaves of S_0. Now since $x_0 \to x_1$, there is a leaf of S_0 adjacent to x_1. In particular, x_1 is not in I, implying that every leaf of S_1 belongs to I. Applying the same argument, all leaves of S_i belong to I, for each i. Since this argument can be done for every directed path starting at v, any star S_j whose center is reachable from v in Q^* by a directed path has all its leaves contained in I. In particular, all the leaves with centers in X form a stable set. $\square$

This lemma provides a roadmap to a disproof of the normal graph conjecture. Namely, a normal, high girth, dense enough random graph will have a star covering with large out-sections, in particular, large stable sets. By tuning the density we can contradict the typical stability of such graphs. To achieve this, we need to introduce the following definitions:

Given a graph G and a subset Q of its vertices partitioned into $Q_1, \ldots, Q_{10}$, we say that $w \in V \setminus Q$ is a *private neighbor* of a vertex $v_i \in Q_i$ if w is adjacent to v_i but not to any other vertex in $Q_1, \ldots, Q_i$. Hence, every vertex $v_i \in Q_i$ is the center of some (possibly trivial) star S_i whose leaves are the private neighbors of v_i. We define as previously our directed graph Q^* based on the induced star system consisting of Q and the set of stars S_i. Observe that by definition of private neighbors, any arc $u \to v$ of Q^* with $u \in Q_i$ and $v \in Q_j$ satisfies $i < j$. Given $Q_1, \ldots, Q_{10}$ in some graph G, we refer to this star system as the *private* star system over $Q_1, \ldots, Q_{10}$. The directed graph Q^* is called *private* directed graph over $Q_1, \ldots, Q_{10}$.

Let us now turn to our fundamental property:

Property JQ:

We say that G satisfies property JQ if for every choice of pairwise disjoint subsets of vertices $J, Q_1, ..., Q_{10}$, with $|J| \leq n^{0.91}$ and $\frac{n^{0.9}}{1000} \leq |Q_i| \leq \frac{n^{0.9}}{500}$ for all $i = 1, \ldots, 10$, the private directed graph Q^* defined on the induced subgraph $G \setminus J$ has an out-section whose set of private neighbors has total size at least $n^{0.95}$.

The crucial point is that a random graph $G := G_{n,p}$ with $p = n^{-9/10}$ will almost surely have property JQ, as claimed by the lemma below.

LEMMA 6.16

$$\mathbb{P}[G \in JQ] = 1 - o(1).$$

We postpone the proof of this lemma to Section 6.4. Now, we show that Lemmas 6.12, 6.16 and Claim 6.14 are sufficient to prove our main theorem.

Proof of Theorem 6.7. We consider a random graph $G' := G'_{n,p}$ with $p = n^{-9/10}$. Using Lemma 6.12 and Lemma 6.16 and the Union Bound, for n sufficiently large, there exists an n-vertex graph G satisfying: (a) G has less than $4n^{0.7}$ cycles of length at most seven, (b) $\alpha(G) < 10n^{0.9}\log n$, (c) G has maximum degree at most $2n^{0.1}$, (d) G has property JQ.

Consider a set S of at most $4n^{0.7}$ vertices in G intersecting all cycles of length at most 7. Note that $G[V \setminus S]$ has girth at least 8. Remark that this type of alteration is inspired from the original proof of Erdős of Theorem 6.11. Assume now for contradiction that $G[V \setminus S]$ is a normal graph. By Claim 6.14, there is a star covering $(\mathbb{C}, \mathbb{S})$ of $G[V \setminus S]$ with the number of stars at most $10n^{0.9}\log n$. Let S' be the set of those stars which have size at most $10^{10}\log n$. Let $J = S \cup S'$. Observe that $|J| \leq 10^{10}\log n \cdot 10n^{0.9}\log n + 4n^{0.7} < n^{0.91}$. Now, consider $G[V \setminus J]$ and call Q the set of centers of the remaining stars. Observe that the set of stars centered at Q still form a star covering of $G[V \setminus J]$. Indeed, $\mathbb{C}$ and $\mathbb{S}$ restricted to $G[V \setminus J]$ is a star covering.

Note that since $|Q| < 10n^{0.9}\log n$, it follows that $|V \setminus (J \cup Q)| > n - n^{0.91} - 10n^{0.9}\log n$. Now, since Q is a dominating set in $G[V \setminus J]$, and the degree of every vertex in $G[V \setminus J]$ is at most $2n^{0.1}$, it follows that $|Q| > \frac{n^{0.9}}{3}$.

We now define the directed graph Q^* on Q based on the star covering of $G[V \setminus J]$.

CLAIM 6.17

Every strongly connected component C of Q^* has size at most $n^{0.9}/1000$.

Proof. Observe that C is an out-section of any of its vertices, hence by Lemma 6.15 the set of leaves of stars with centers in C is a stable set. Since each star in the star covering of $G[V \setminus J]$ has size at least $10^{10}\log n$, it follows that $G[V \setminus J]$ has a stable set of size $10^{10}\log n \cdot |C|$. The result follows now from the fact that $\alpha(G) < 10n^{0.9}\log n$. □

Let $C_1, \ldots, C_k$ be the strongly connected components of Q^*, enumerated in such a way that all arcs xx' of Q^* with $x \in C_i$ and $x' \in C_j$ satisfy $i \leq j$.

We concatenate subsets of the components $C_1, \ldots, C_k$ into blocks $Q_1, Q_2, ..., Q_{10}$ with $Q_1 = C_1C_2...C_{i_1}$, $Q_2 = C_{i_1+1}...C_{i_2}$,..., $Q_{10} = C_{i_9+1}...C_{i_{10}}$ for some $i_1, ..., i_{10}$ such that for each Q_i, $1 \leq i \leq 10$, $n^{0.9}/1000 \leq |Q_i| \leq n^{0.9}/500$. This is clearly possible since for each $i \leq k$, $|C_i| < n^{0.9}/1000$ and $|Q| > n^{0.9}/3$.

The crucial remark now is that if a vertex v of $G \setminus (J \cup Q)$ is a private neighbor of a vertex x_i in Q_i, then the edge x_iv must be an edge of the star covering. Indeed, v has a unique neighbor in $Q_1 \cup \cdots \cup Q_i$ by definition, and any edge vx_j where x_j is in $Q \setminus (Q_1 \cup \cdots \cup Q_i)$ cannot belong to $\mathbb{C}$ since this would imply $x_j \to x_i$. Now, by property JQ, we know that the private directed graph Q'^* defined on the stars formed by the private neighbors of the Q_i's has an out-section O of size at least $n^{0.95}$. Since Q'^* is a subdigraph of Q^*, the set O is also an out-section of Q^*. Hence the set of leaves with centers in O forms an stable set of size $n^{0.95}$ by Lemma 6.15, contradicting the fact that $\alpha(G) < 10n^{0.9}\log n$.

□

6.4 Proof of Lemma 6.16

In this section, we prove Lemma 6.16 to conclude the proof of Theorem 6.7.

Proof of Lemma 6.16. We prove that $\mathbb{P}[JQ^c] = o(1)$. We first fix the sets $J, Q_1, ..., Q_{10}$. There are at most $\sum_{i=1}^{n^{0.91}} \binom{n}{i} \leq 2n^{n^{0.91}}$ possible sets for J and at most $(\sum_{i=n/1000d}^{n/500d} \binom{n}{i})^{10} \leq 2^{10} n^{n^{0.9}/50}$ sets for the $Q_1, ..., Q_{10}$. Thus, there are at most $2^{11} n^{2n^{0.91}}$ ways to fix the sets $J, Q_1, ..., Q_{10}$. Let M_1 be the event that for some *fixed* sets $J, Q_1, ..., Q_{10}$ the property JQ^c holds. Clearly, $\mathbb{P}[JQ^c] \leq 2^{11} n^{2n^{0.91}} \mathbb{P}[M_1]$. Now, we bound $\mathbb{P}[M_1]$.

Denote by $B := G \setminus \{\cup_{i=1}^{10} Q_i \cup \{J\}\}$. For a vertex $v \in Q_1$, let D_v be the number of neighbors of v in B. Let D_{Q_1} be the event that at least $0.01|Q_1|$ vertices v in Q_1 have $D_v \notin (0.99d, 1.01d)$.

Note that,

$$\begin{aligned}
\mathbb{P}[D_{Q_1}] &\leq \binom{|Q_1|}{0.01|Q_1|} (\mathbb{P}[D_v \notin (0.99d, 1.01d)])^{0.01|Q_1|} \\
&\leq \binom{n/500d}{n/50000d} (\mathbb{P}[D_v \notin (0.99d, 1.01d)])^{n/100000d} \\
&\leq (n/500d)^{n/50000d} (e^{-d/10})^{n/10^5 d} \\
&< e^{-n/10^7}.
\end{aligned}$$

where we used the fact that D_v is a binomial random variable with mean $p|B| \in (0.999d, d)$ and thus Chernoff's inequality applies.

For a vertex $v \in B$, let X_v be the random variable counting the number of vertices in Q_1 adjacent to v, and X be the number of vertices in B that have degree equal to 1 in Q_1. Then X is a binomial random variable. Now,

$$\begin{aligned}
\mathbb{E}[X] &= |B| \times \mathbb{P}[X_v = 1] \\
&\geq 0.99n\mathbb{P}[X_v = 1] \\
&\geq 0.99n|Q_1|\tfrac{d}{n}(1 - d/n)^{|Q_1|-1} \\
&\geq 0.99|Q_1|de^{-1/250} \\
&\geq 0.985|Q_1|d.
\end{aligned}$$

By Chernoff's inequality, since $\mathbb{E}[X] \geq 0.985n/1000$,

$$\mathbb{P}[\{X < 0.98|Q_1|d\}] \leq e^{-n/10^7}.$$

Next, let Z_E be the number of edges from Q_1 to B. Note that Z_E is a binomial random variable with mean $\mu = |Q_1||B|\frac{d}{n}$. Note that $\mu \in (0.99|Q_1|d, |Q_1|d)$. Then

$$\begin{aligned}
\mathbb{P}[\{Z_E \notin (0.98|Q_1|d, 1.01|Q_1|d)\}] &\leq \mathbb{P}[\{Z_E \notin (0.98|Q_1|d, 1.01|Q_1|d)\}] \\
&\leq e^{-n/10^7}.
\end{aligned}$$

by Chernoff's inequality. Now, let M_2 be the event

$$M_2 := M_1 \cap \{Z_E \in (0.98|Q_1|d, 1.01|Q_1|d)\} \cap D^c_{Q_1} \cap \{X > 0.98|Q_1|d\}.$$

$$\begin{aligned} \mathbb{P}[JQ^c] &\leq 2^{11} n^{2n^{0.91}} \mathbb{P}[M_1] \\ &\leq 2^{11} n^{2n^{0.91}} (\mathbb{P}[M_2] + 3e^{-n/10^7}) \\ &\leq 2^{11} n^{2n^{0.91}} \mathbb{P}[M_2] + o(1). \end{aligned}$$

Thus, it suffices to bound $\mathbb{P}[M_2]$.

Let N_{Q_1} be the event that at least $0.8|Q_1|$ vertices in Q_1 have at least $d/2$ private neighbors. We claim that if M_2 holds then so does N_{Q_1}.

Assume that M_2 holds. Let us call an edge e a *good* edge if its endpoint in Q_1 has degree in the interval $(0.99d, 1.01d)$ in B and its endpoint in B has degree exactly 1 in Q_1. We compute the number of non-good edges. First, let us count the number of edges whose endpoint in B has degree greater than 1.

Note that the number of vertices in B that have degree 1 in Q_1 is at least $0.98|Q_1|d$. These vertices contribute at least $0.98|Q_1|d$ edges. Thus, the number of edges between Q_1 and B whose endpoint in B is not of degree 1 is at most $1.01|Q_1|d - 0.98|Q_1|d \leq 0.03|Q_1|d$.

Next, we count the number of edges between Q_1 and B whose endpoint in Q_1 is not of degree in the interval $(.99d, 1.01d)$. Since at least $0.99|Q_1|$ vertices in Q_1 have degree in the interval $(.99d, 1.01d)$, they contribute at least $.99^2|Q_1|d$ edges. The remaining number of edges is at most $1.01|Q_1|d - 0.99^2|Q_1|d \leq 0.05|Q_1|d$.

Thus, the number of edges which are not good is at most $0.08|Q_1|d$ which implies that the number of good edges is at least $0.98|Q_1|d - 0.08|Q_1|d \geq 0.9|Q_1|d$.

Now, we prove our claim that if M_2 holds then N_{Q_1} holds as well. We know that at least $0.99|Q_1|$ vertices in Q_1 have degree at least $0.99d$ in B. Let us compute the number of vertices (called *bad* vertices) which do not have at least $d/2$ private neighbors. Such a vertex is adjacent to at least $0.49d$ non-good edges since its degree is at least $0.99d$. Since the total number of non-good edges is at most $0.08|Q_1|d$ it follows that the number of bad vertices is easily at most $0.2|Q_1|$. Therefore, at least $0.8|Q_1|$ vertices in Q_1 have at least $d/2$ private neighbors, proving the claim. Summarizing,

$$\begin{aligned} \mathbb{P}[M_2] &= \mathbb{P}[N_{Q_1} \cap M_2] + \mathbb{P}[N^c_{Q_1} \cap M_2] \\ &= \mathbb{P}[N_{Q_1} \cap M_2]. \end{aligned}$$

So it is sufficient to bound $\mathbb{P}[N_{Q_1} \cap M_2]$.

Now, define $B_2 = B \setminus \Gamma(Q_1)$, where $\Gamma(Q_1)$ is the set of neighbors of Q_1 in B. Note that $\mathbb{P}[N_{Q_1} \cap M_2 \cap \{|B_2| < 0.99n\}]$ is at most $O(e^{-n/10^{10}})$ since Z_E is at most of size $n/400$ with this probability and $|B| \geq n - 11n^{0.9} \log n$. Thus, it is sufficient to bound $\mathbb{P}[N_{Q_1} \cap M_2 \cap \{|B_2| > 0.99n\}]$. Define N_{Q_2} to be the event that at least $0.8|Q_2|$ vertices in Q_2 have at least $d/2$ private neighbors in B_2. By an identical argument as before, we know the probability of the complement of this event is $O(e^{-n/10^7})$ as $|B_2| \geq 0.99n$. For each i, $2 \leq i \leq 10$, by defining the sets B_i by $B_{i+1} = B_i \setminus \Gamma(Q_i)$ and N_{Q_i} as the event at least $0.8|Q_i|$ vertices in Q_i have at least $d/2$ private neighbors in B_i we obtain that with probability at least $1 - O(e^{-n/10^{10}})$ the event $\mathbb{P}[M_2 \cap_{i=1}^{10} N_{Q_i}]$ holds. It follows that

$$\begin{aligned}\mathbb{P}[JQ^c] &\leq 2^{11} n^{2n^{0.91}} \mathbb{P}[M_2] + o(1) \\ &\leq 2^{11} n^{2n^{0.91}} (\mathbb{P}[M_2 \cap_{i=1}^{10} N_{Q_i}] + O(e^{-n/10^{10}})) + o(1) \\ &\leq 2^{11} n^{2n^{0.91}} \mathbb{P}[M_2 \cap_{i=1}^{10} N_{Q_i}] + o(1).\end{aligned}$$

So it remains to bound $\mathbb{P}[M_2 \cap_{i=1}^{10} N_{Q_i}]$.

We use the fact that at least $0.8|Q_i| \geq \frac{0.8n}{1000d} > \frac{n}{2000d}$ vertices of each Q_i have at least $d/2$ private neighbors. By deleting vertices if necessary, we may assume that each Q_i has size $|Q_i| = \lfloor \frac{n}{2000d} \rfloor$ and each vertex in each Q_i has exactly $\left\lfloor \frac{d}{2} \right\rfloor$ private neighbors.

In what follows we will assume that $M_2 \cap_{i=1}^{10} N_{Q_i}$ holds and by applying Lemma 6.13 we will conclude that in fact there is an out-section in Q_{10} whose corresponding private neighbors have size at least $n^{0.95}$.

We inductively prove the following claim (*):

(*) there exist positive constants ϵ_i, ϵ_i' and C_i such that with probability at least $1 - e^{-\epsilon_i' n}$, in each Q_i, $2 \leq i \leq 10$, there exist at least $\frac{\epsilon_i n}{d^i}$ disjoint out-sections (with respect to only the private neighbors of the vertices in Q_i's) each of size at least $\frac{d^{i-1}}{C_i}$.

Let J_i be the i^{th} event in the above statement. We first show that $\mathbb{P}[J_2] \geq 1 - e^{-\epsilon_2' n}$ for some values of ϵ_2, C_2 and ϵ_2'. We use Lemma 6.13.

Consider the bipartite graph $H_1 = (Q_1, Q_2)$ with bipartition Q_1 and Q_2 where there is an edge between $v_1 \in Q_1$ and $v_2 \in Q_2$ if at least one of the $\lfloor \frac{d}{2} \rfloor$ private neighbors of v_1 is adjacent to v_2. Thus, H_1 is a random bipartite graph where the probability of any edge is $p_1 = 1 - (1-p)^{\lfloor d/2 \rfloor}$. It is easily seen that $\frac{d^2}{4n} \leq p_1 \leq \frac{d^2}{n}$.

We apply Lemma 6.13. Indeed, $10^{100} p_1^{-1} < 10^{100} n/d^2 < n/2000d = |Q_1|$, if n is sufficiently large. Thus, H_1 has a partial cover and $e(Q_1, Q_2) \in [0.99 p_1 |Q_1||Q_2|, 1.01 p_1 |Q_1||Q_2|]$ with probability at least $1 - e^{-cp_1|Q_1||Q_2|} > 1 - e^{-c_1 n}$, for some constant $c_1 > 0$. Let $(x_1, Y_1), ..., (x_k, Y_k)$ be the set of pairs in the partial cover. It follows that the size of each Y_i is $\lceil e(Q_1, Q_2)/3|Q_1| \rceil > d/C_2$ for some $C_2 > 0$ and at least $|Q_2|/3$ of the vertices of Q_2 are covered by the Y_i's. Since $e(Q_1, Q_2) < 1.01 p_1 |Q_1||Q_2|$, it follows that $k > \frac{\epsilon_2 n}{d^2}$ for some $\epsilon_2 > 0$. This establishes the claim for J_2.

Now, suppose that we know that $\mathbb{P}[J_i] \geq 1 - e^{-\epsilon_i' n}$ with the corresponding constants C_i and ϵ_i.

Then $\mathbb{P}[J_{i+1}] \geq \mathbb{P}[J_{i+1} \mid J_i](1 - e^{-\epsilon_i' n})$. Therefore, it suffices to lower bound $\mathbb{P}[J_{i+1} \mid J_i]$.

We argue similarly as for the case $i = 1$. In the set Q_i we will have disjoint out-sections each of which has size at least d^{i-1}/C_i for some constant $C_i > 0$ such that the number of out-sections will be at least $\epsilon_i n/d^i$ for some $\epsilon_i > 0$. By truncating, we may assume that the number of out-sections in Q_i is exactly $\lceil \epsilon_i n/d^i \rceil$ and each out-section has size exactly $\lceil d^{i-1}/C_i \rceil$. Now, contract each out-section of Q_i into a single vertex and denote the resulting set of vertices by Q'_i.

Consider the bipartite graph $H_i = (Q'_i, Q_{i+1})$ with bipartition Q'_i and Q_{i+1} where there is an edge between $v_i \in Q'_i$ and $v_{i+1} \in Q_{i+1}$ if at least one of the $\lfloor \frac{d}{2} \rfloor$ private neighbors of at least one of the vertices in the out-section of Q_i corresponding to v_i is adjacent to v_{i+1}. Thus, H_i is a random bipartite graph where the probability of any edge is $p_i = 1 - (1-p_1)^{\lceil d^{i-1}/C_i \rceil}$. It is easily seen that $\frac{d^{i+1}}{4C_i n} < p_i < \frac{2d^{i+1}}{C_i n}$.

We again apply Lemma 6.13. Indeed, $10^{100} p_i^{-1} < 10^{100} \frac{4C_i n}{d^{i+1}} < \lceil \epsilon_i n/d^i \rceil = |Q'_i|$, if n is sufficiently large. Thus, H_i has a partial cover and $e(Q'_i, Q_{i+1}) \in [0.99 p_i |Q'_i||Q_{i+1}|, 1.01 p_i |Q'_i||Q_{i+1}|]$ with probability at least $1 - e^{-cp_i|Q'_i||Q_{i+1}|} > 1 - e^{-c_1 n}$, for some constant $c_1 > 0$. Let $(x_1, Y_1), ..., (x_k, Y_k)$ be the set of pairs in the partial cover. It follows that the size of each Y_j is $\lceil e(Q'_i, Q_{i+1})/3|Q'_i| \rceil > d^i/C_{i+1}$ for some $C_{i+1} > 0$ and at least $|Q_{i+1}|/3$ of the vertices of Q_{i+1} are covered by the Y_i's. Since $e(Q'_i, Q_{i+1}) < 1.01 p_i |Q'_i||Q_{i+1}|$, it follows that $k > \frac{\epsilon_{i+1} n}{d^{i+1}}$ for some $\epsilon_{i+1} > 0$. Thus, the size of each out-section and the number of out-sections is as required.

Thus, we have

$$\begin{aligned} \mathbb{P}[J_{i+1}] &\geq \mathbb{P}[J_{i+1} \mid J_i](1 - e^{-\epsilon'_i n}) \\ &> (1 - e^{-c_1 n})(1 - e^{-\epsilon'_i n}) \\ &> 1 - e^{-\epsilon'_{i+1} n} \end{aligned}$$

for some constant ϵ'_{i+1}, as required. This proves the claim (*).

Now, considering J_{10} we have that there exist at least $\epsilon_{10} n/d^{10} = \epsilon_{10} > 0$ out-sections of size at least d^9/C_{10}. Therefore, there is at least one out-section of size at least $\frac{n}{C_{10} d}$ with probability at least $1 - e^{-\epsilon'_{10} n}$. Now, if $M_2 \cap_{i=1}^{10} N_{Q_i}$ holds then every vertex in each Q_i has $d/2$ private neighbors, yielding a set of at least $n/2C_{10} > n^{0.95}$ total private neighbors corresponding to the out-section. Thus,

$$\begin{aligned} \mathbb{P}[JQ^c] &\leq 2^{11} n^{2n^{0.91}} \mathbb{P}[M_2 \cap_{i=1}^{10} N_{Q_i}] \\ &= 2^{11} n^{2n^{0.91}} \mathbb{P}[M_2 \cap_{i=1}^{10} N_{Q_i} \cap J_{10}^c] \\ &\leq 2^{11} n^{2n^{0.91}} e^{-\epsilon'_{10} n} = o(1). \end{aligned}$$

This completes the proof of the lemma. □

Note that by setting $p = n^{-1+1/10g}$ and reproducing the same arguments, one can show that for every g, there exists a graph of girth g which is not normal.

Conclusion

Several results of different types were presented in this manuscript. We present below a summary of all the results and directions for further research.

In Chapter 3 we provided two different polynomial-time algorithms for the coloring problem. The first one solves the 4-coloring problem in the class of (P_6, bull)-free graphs and the second one solves the k-coloring problem in the class of (P_6, bull, gem)-free graphs for any positive integer k. Both use structural properties of (P_6, bull)-free graphs and the latter uses the fact that the clique-width is bounded by a constant on this class of graphs. The big question is to determine the complexity of the 4-coloring problem of P_6-free graphs. In fact, Huang [51] conjectures that it is polynomial to decide the 4-colorability of any P_6-free graph. Using the same techniques we used is unlikely to work as P_6-free graphs have obviously less structure than (P_6, bull)-free graphs. One first step would be to try for another type of class. For example, as stated by the authors in [16], it might be a good bet to try to solve the 4-coloring problem of (P_6, W_5)-free graphs where W_5 is the wheel graph on five vertices. Also, it would be interesting to know if there is a finite number of 5-critical (P_6, bull)-free graphs and if the answer is yes, would it be possible to produce the list of all 5-critical (P_6, bull)-free graphs? Goedgebeur and Schaudt [36] provided an algorithm generating all k-critical $\mathcal{H}$-free graphs. It would be very interesting to try to implement their algorithm and see what it can outputs for the class of (P_6, bull)-free graphs.

Chapter 4 was dedicated to proving that for any claw-free perfect graph G where $\omega(G) \leq 4$, we have the following chromatic equality, $\chi(G) = ch(G)$. If one would try to prove a better result while using the same techniques, the first thing to try would be the elementary graphs. In fact, even though we were able to provide working gadgets for every example we tried for graphs with a higher clique number, we did not manage to make it work in the general case. However, it is worth trying since some specific cases are working. On the other hand, there are other techniques that we did not use and that might be worth a try. For instance, the structural description of claw-free perfect graphs provided by Chudnovsky and Plumettaz [17] would be a good start. We also feel that trying to prove the general case of all peculiar graphs (in case one would want to stick with decomposition used in our proof) is worth a try. The complete structure is given and we were able to prove small cases of higher clique number, however we did not manage to prove it for the general case. There is certainly still room for improvement.

We elaborated three different algorithms in Chapter 5. And there is still a lot to do. Recall the result of Alekseev [1] proving that the MWSS problem remains NP-Hard in the class of $\mathcal{F}$-free graphs whenever $\mathcal{F}$ does not contain any subdivision of $S_{i,j,k}$. Lokshtanov et al. [64] proved that the MWSS can be solved in polynomial time in the class of P_5-free graphs and Lokshtanov et al. [63] proved that it can be solved in quasi-polynomial time in the class of P_6-free graphs. Natural questions arise from these results. Does there exist an integer $k \geq 6$ for which the MWSS problem in the class of P_k-free graphs is NP-Hard? Or the other way around, how far can we push k to find a polynomial-time algorithm for the MWSS problem in the class of P_k-free graphs. One could ask the same question for quasi-polynomiality for $k \geq 7$. It is known that the MWSS is polynomial-time solvable in the class of claw-free graphs. There are a few results of polynomiality in $S_{i,j,k}$-free graphs for specific integers i, j, k. One could aim at trying for other values of $S_{i,j,k}$ or in a more general way as stated above (but also harder), does there exist an i, j, k for which the MWSS is NP-hard in the class of $S_{i,j,k}$-free graphs? A powerful theorem comes to mind while dealing with P_k-free graphs: Camby and Schaudt [14] proved that any connected P_k-free graph G with $k \geq 4$ admits a connected dominating set that induces either a P_{k-2}-free graph or a graph isomorphic to P_{k-2}. This theorem might a good tool to try for new results in P_k-free graphs.

In Chapter 6 we disproved a conjecture of De Simone and Körner. Even though our result disproves a 17 year-old conjecture, there is still a lot of work to do. First, our counter-example uses the probabilistic method and is not constructive. The first task that comes to mind is the following. Try to provide a constructive counter-example to the Normal Graph Conjecture? On the other hand, several authors proved the conjecture to be true in specific graph classes. For which other classes does the conjecture hold?

Contributions

Results presented in this manuscript can be found in the following research articles, which are either published or accepted in specialized journals:

1. **4-coloring (P_6, bull)-free graphs**, with F. Maffray.
   Discrete Applied Mathematics, 231 (2017), 198–210.

2. **On the choosability of claw-free perfect graphs**, with S. Gravier and F. Maffray.
   Graphs and Combinatorics, 32:6 (2016), 2393–2413.

3. **The maximum weight stable set problem in (P_6, bull)-free graphs**,
   with F. Maffray. *Proceedings of WG 2016*, pages 85–96, Springer, Berlin, Heidelberg, 2016.

4. **Maximum Weight Stable Set in (P_7, bull)-free graphs and ($S_{1,2,3}$, bull)-free graphs**, with F. Maffray. ArXiv preprint arXiv:1611.09663.
   Accepted in *Discrete Mathematics*.

5. **Disproving the normal graph conjecture**, with A. Harutyunyan and
   S. Thomassé. ArXiv preprint arXiv:1508.05487.
   Accepted in *Journal of Combinatorial Theory, Series B*.

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Summary

This thesis deals with graph coloring, list coloring, maximum weight stable set (shortened as MWSS) and structural graph theory.

First, we provide polynomial-time algorithms for the 4-coloring problem in subclasses of P_6-free graphs. These algorithms rely on a precise understanding of the structure of these classes of graphs for which we give a full description.

Secondly, we study the list coloring conjecture and prove that for any claw-free perfect graph with clique number bounded by 4, the chromatic number and the choice number are equal. This result is obtained by using a decomposition theorem for claw-free perfect graphs, a structural description of the basic graphs of this decomposition and by using Galvin's famous theorem.

Next by using the structural description given in the first chapter and strengthening other aspects of this structure, we provide polynomial-time algorithms for the MWSS problem in subclasses of P_6-free and P_7-free graphs.

In the last chapter of the manuscript, we disprove a conjecture of De Simone and Körner made in 1999 related to normal graphs. Our proof is probabilistic and is obtained by the use of random graphs.