Disproving the normal graph conjecture

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October 12, 2016

Joint-work with **Ararat Harutyunyan** and **Stéphan Thomassé**

The normal graph conjecture

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Let G_1 and G_2 be two graphs. The co-normal product $G_1 * G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$, where (v_1, v_2) and (u_1, u_2) are adjacent if u_1 is adjacent to v_1 or u_2 is adjacent to v_2.

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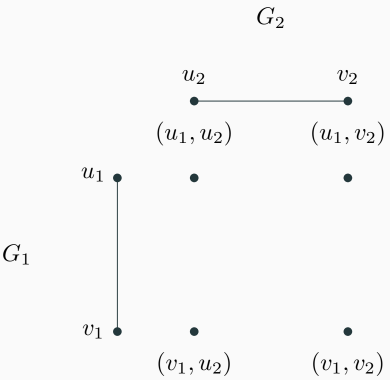

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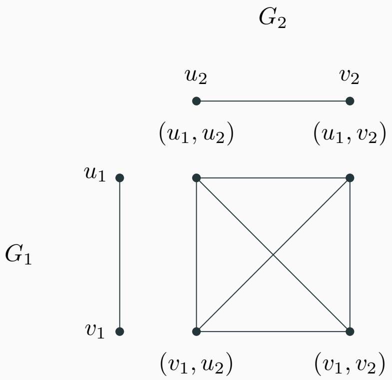

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Definition

A graph G is normal if there exists two coverings, $\mathbb{C}$ and $\mathbb{S}$ of its vertex set such that every member of $\mathbb{C}$ induces a clique in G, every member of $\mathbb{S}$ induces an independent set in G and $C \cap S \neq \emptyset$ for every $C \in \mathbb{C}$ and $S \in \mathbb{S}$.

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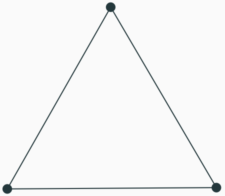

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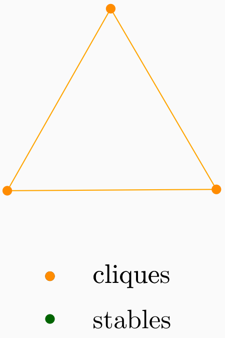

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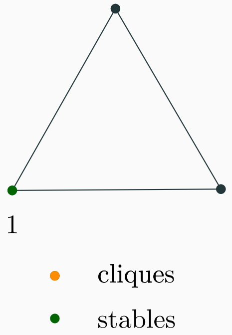

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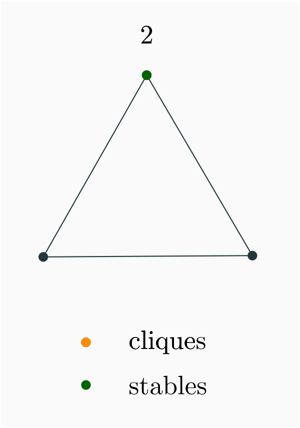

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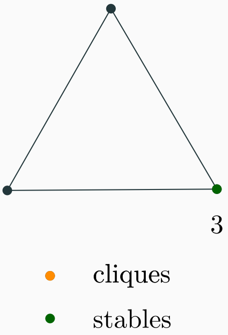

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Minimal non normal graphs

The only known minimally graphs which are not normal are C_5, C_7, $\overline{C_7}$.

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The entropy $H(G, P)$ of a graph G can be defined with respect to a probability distribution P on $V(G)$.

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Sub-additivity

The graph entropy is sub-additive with respect to complementary graphs:

$$H(P) \leq H(G, P) + H(\overline{G}, P)$$

Entropy of perfect graphs

Csiszár et. al showed that:

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Entropy of normal graphs

Körner and Marton showed that:

$$H(P) = H(G, P) + H(\overline{G}, P)$$

for at least one P if and only if G is normal.

The Normal Graph Conjecture [C. De Simone, J. Körner 1999]

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Tools

Random graphs!

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- Circulant graphs are normal.
- A few classes of sparse graphs have been show to be normal.
- All subcubic triangle-free graphs are normal.
- Almost all d-regular graphs are normal when d is fixed.

Remark

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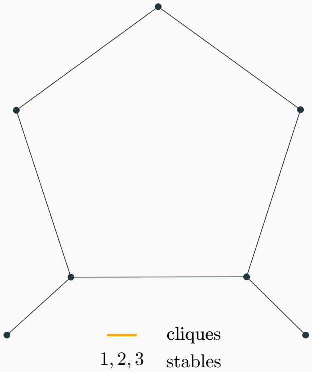

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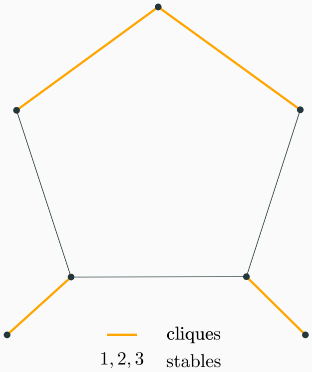

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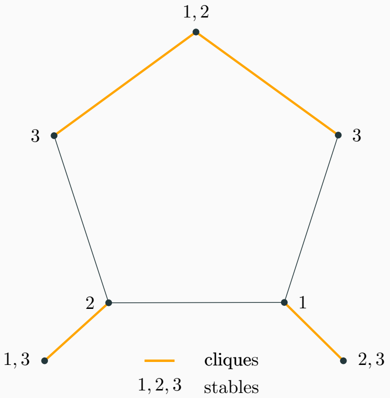

Probabilistic tools

The philosophy behind probabilistic arguments

In order to show that there exist an object O with some properties in a collection of objects $\mathbb{O}$, one can show that there is a **non-zero** probability to pick such an object if you choose at random in $\mathbb{O}$.

Union bound

For a countable set of events $A_1, \dots, A_n$, we have

$$\mathbb{P}(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \mathbb{P}(A_i).$$

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If X is any non-negative discrete random variable and $a > 0$, then

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Others inequalities

We also use other inequalities giving good concentration on $0/1$ valued random variables.

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3. We know that $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$. Together with $\alpha(G) = o(n)$, we could achieve high chromatic number.

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4. We want to generate a random graph that combine all these properties.

Cycles of length at most g

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- By the union bound, we have that the following probability is non-zero
$$\mathbb{P}[X < \frac{n}{2} \text{ and } \alpha(G_{n,p}) = o(n)] > 0.$$
- We just have found our graph. Now remove one vertex from each of the short cycles to get G' which have girth at least ℓ, then we can show that for some $\lambda > 0$
$$\chi(G') \geq \frac{n^{\lambda}}{6 \ln n}.$$

Sketch of the proof

Graphs of girth at least 8

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Note that in a graph of girth at least 8, there are no C_5, C_7 nor $\overline{C_7}$, hence we satisfy the required properties of the conjecture.

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- $X_7 \leq 4n^{0.7}$ with X_7 the number of cycles of length at most 7.
- $\alpha(G) < cn^{0.9} \log n$ with $c \geq 10$ a fixed constant.

Star covering

- Every member of $\mathbb{C}$ induces a clique K_2 or K_1 in G, where no K_1 is included in some K_2.

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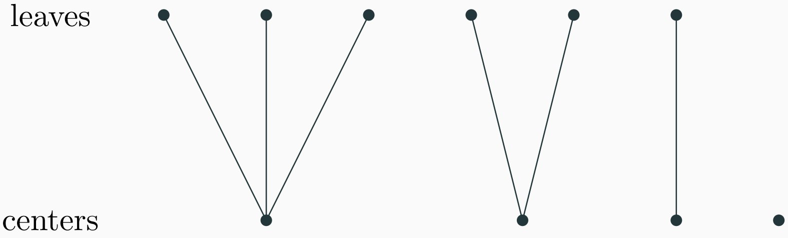
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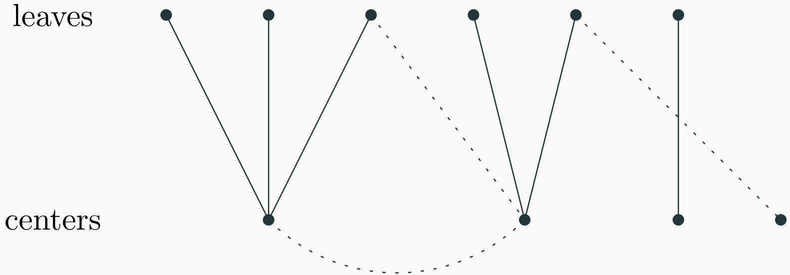
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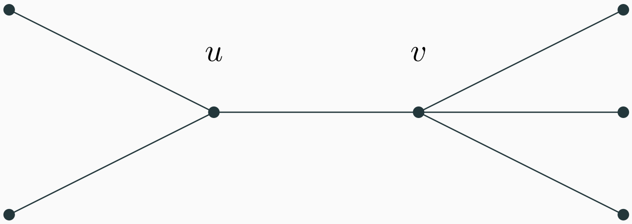
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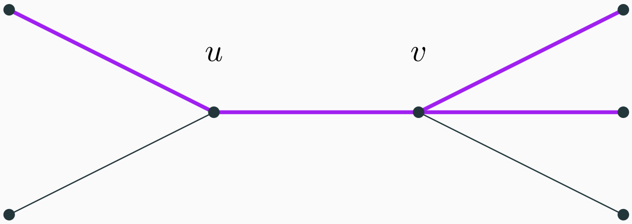
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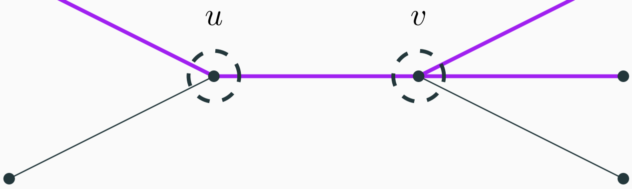
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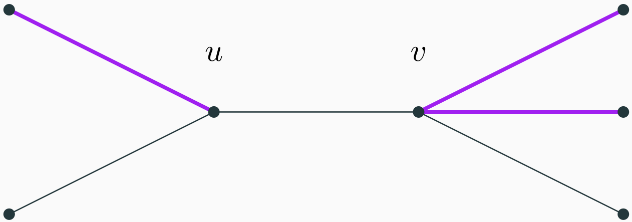

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Star system

A star system $(Q, \mathcal{S})$ is a spanning set of vertex disjoint stars where $\mathcal{S}$ is the set of stars and Q is the set of centers of the stars of $\mathcal{S}$.

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Directed star system

To every star system $(Q, \mathcal{S})$ we associate a directed graph Q^* on the vertex set Q and add a directed edge $x_i \rightarrow x_j$ whenever a leaf of S_i is adjacent to the center of S_j.

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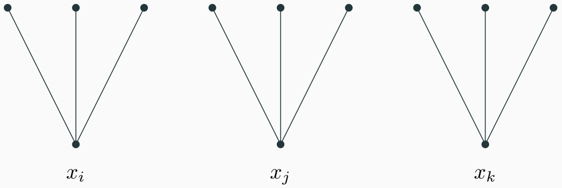

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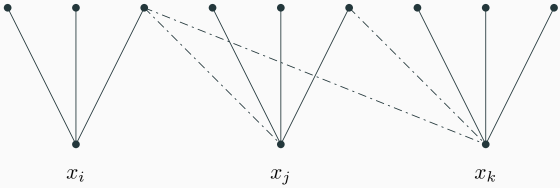

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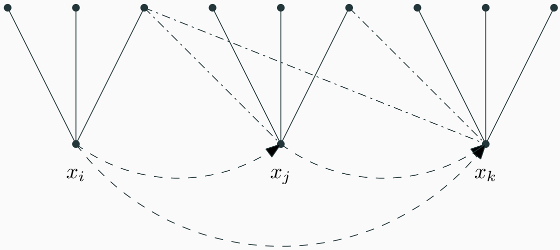

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A subset $X \subseteq Q$ is an out-section if there exists v in Q such that for each $x \in X$, there exists a directed path in Q^* from v to x.

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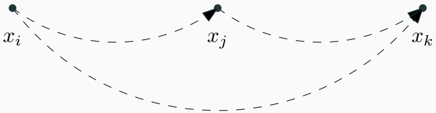

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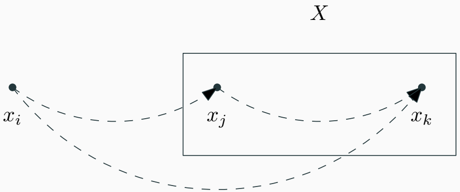

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Proof

Let $x_1, \ldots, x_k$ be the centers of the stars and let $S \in \mathbb{S}$ be some independent set. Then, for each x_i, either x_i or a leaf of x_i belongs to S. Which gives

$$k \leq |S| \leq \alpha(G).$$

Key Lemma

Let G be a normal triangle-free graph with a star covering $(\mathbb{C}, \mathbb{S})$ and denote by X the outsection of the associated star-system. Then the set of leaves of the stars with centers in X form an independent set of G.

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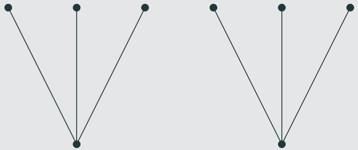

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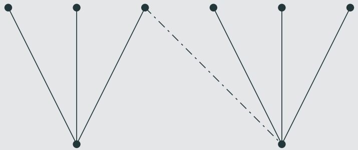

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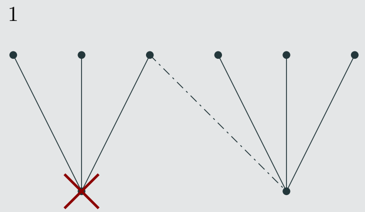

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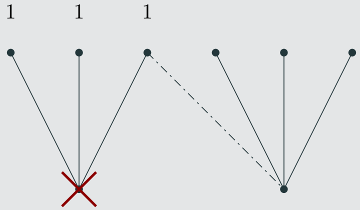

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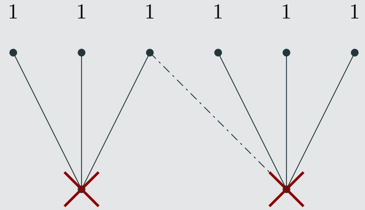

Private neighbors

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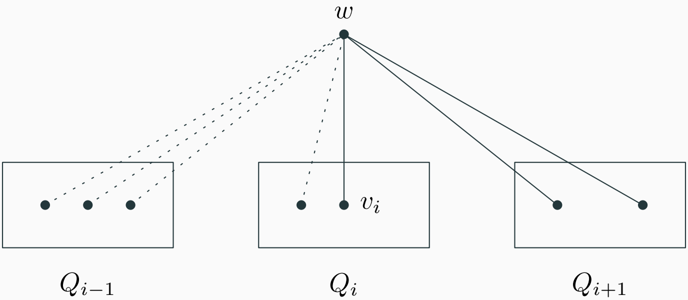

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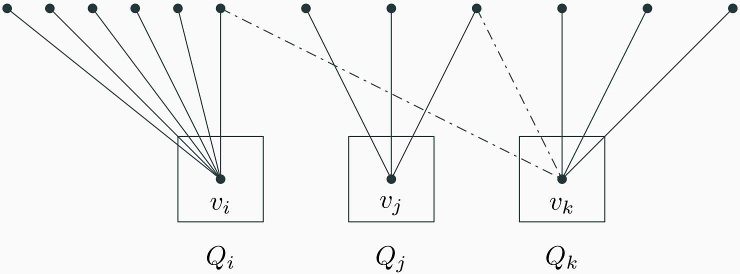

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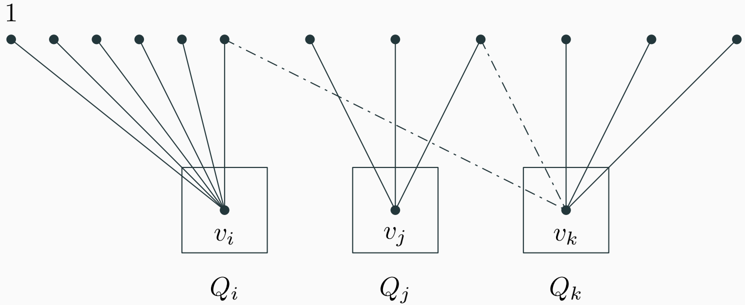

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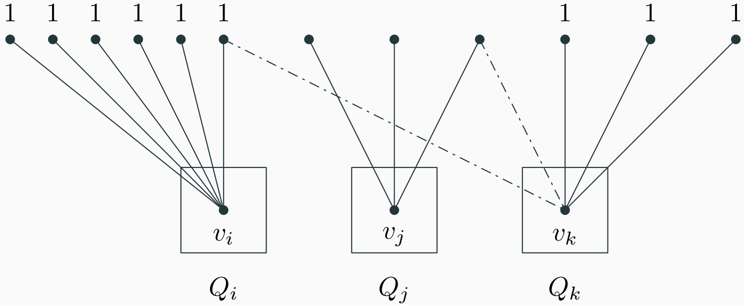

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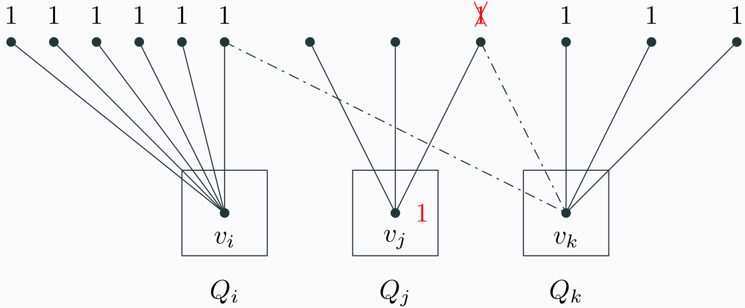

Property JQ

We say that G satisfies property JQ if for every choice of pairwise disjoint subsets of vertices $J, Q_1, \ldots, Q_{10}$, with $|J|$ and $|Q_i|$ of *good* sizes, the private directed graph Q^* defined on $G \setminus J$ has an out-section whose set of private neighbors have total size at least $n^{0.95}$.

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Lemma

A random graph $G_{n,p}$ with $p = n^{-0.9}$ will almost surely have property JQ.

Small sketch of why $G_{n,p}$ has property JQ

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- We first show that with good probability, every vertex in Q_i has many neighbors in Q_i.
- Then, that almost every vertex of Q_i has a good number of private neighbors.
- Finally, we show that with such properties, the probability of not having a big out-section is bounded by above by $o(1)$.

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By previous Lemmas and Claims and thanks to the union bound, for n sufficiently large, there exists a n-vertex graph G satisfying the followings:

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- G has property JQ.

Keeping the good stars and girth high

Let S be the set of vertices formed by picking one vertex from each short cycles. Assume now for contradiction that $G[V \setminus S]$ is a normal graph (note that $G[V \setminus S]$ has girth at least 8).

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- Contradiction to the fact that $\alpha(G) < 10n^{0.9} \log n$.

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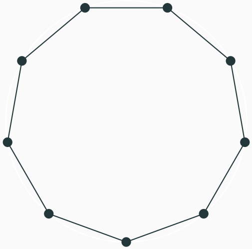
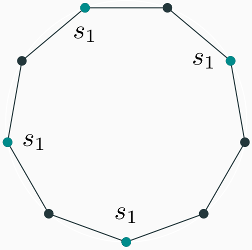
- Maybe there are precise classes for which it fails?
- What if the forbidden family is finite, what are the bad graphs?

Thank you for your attention.

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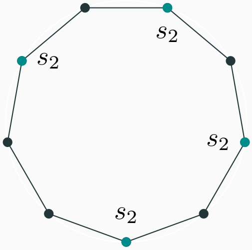
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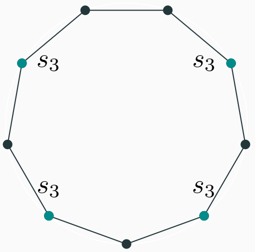
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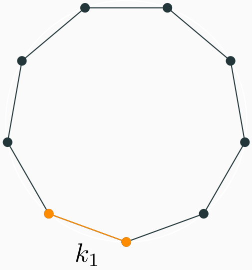
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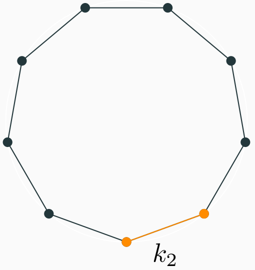
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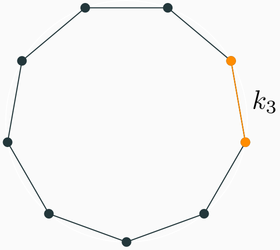
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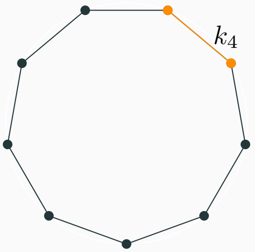
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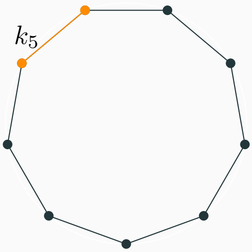
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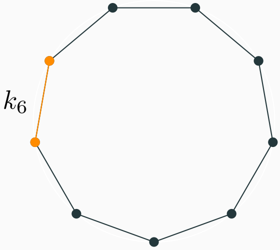
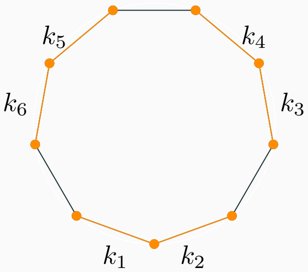

k_6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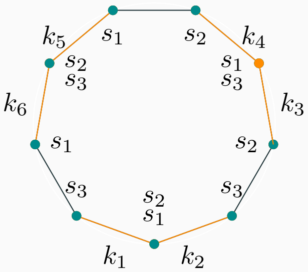

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We need to generate the random graph $G_{n,p}$, where each edge appears independently with probability p. Let $\lambda \in (0, \frac{1}{\ell})$ and $p = n^{\lambda - 1}$.

Let's compute the number of cycles of length at most ℓ in $G_{n,p}$. Let X be this number and X_j the number of cycles of length at most j. If you see a cycle of length j as a word of length j on an alphabet of size n, we have this large upper bound $X_j \leq n^j$. Each of those cycles appears with probability p^j (there are j edges in a cycle of length j, and each appears with probability p). Hence we have

$$
\begin{aligned}
\mathbb{E}[X] &\leq \sum_{j=3}^{\ell} n^j p^j \quad \text{(replace } p^j \text{ by its value)} \\
&= \sum_{j=3}^{\ell} n^{\lambda j}
\end{aligned}
$$

Recall that the sum of a geometric series starting at $j = a$ and ending at ℓ for $r \neq 1$ is

$$\sum_{j=a}^{\ell} r^j = \frac{r^a - r^{l+1}}{1 - r}$$

which gives us

$$\begin{aligned}
\mathbb{E}[X] &\leq \sum_{j=3}^{\ell} n^{\lambda j} \\
&= \frac{n^{3\lambda} - n^{\lambda\ell+\lambda}}{1 - n^{\lambda}} \text{ (multiply by -1 the denominator and the numerator)} \\
&= \frac{n^{\lambda\ell+\lambda} - n^{3\lambda}}{n^{\lambda} - 1} = \frac{n^{\lambda\ell+\lambda} - n^{3\lambda}}{n^{\lambda}(1 - n^{-\lambda})} \\
&= \frac{n^{\lambda\ell} - n^{2\lambda}}{1 - n^{-\lambda}} \quad \leq \frac{n^{\lambda\ell}}{1 - n^{-\lambda}}
\end{aligned}$$

In fact, $\frac{n^{\lambda\ell}}{1-n^{-\lambda}}$ is smaller than $\frac{n}{c}$, for any $c > 1$ and n sufficiently large. To see this, set the following inequation

$$
\begin{aligned}
& \frac{n^{\lambda\ell}}{1-n^{-\lambda}} < \frac{n}{c} \\
\iff \qquad & n^{\lambda\ell} < \frac{n}{c} - \frac{n^{1-\lambda}}{c} \\
\iff n^{\lambda\ell} + & \frac{n^{1-\lambda}}{c} < \frac{n}{c}
\end{aligned}
$$

which holds for n sufficiently large because $\lambda\ell < 1$ and $1-\lambda < 1$. By setting $c = 4$ we have the following upper bound on the expectation of X, $\mathbb{E}[X] \leq \frac{n}{4}$. Hence, by Markov's inequality, we have

$$P[X \geq \frac{n}{2}] < \frac{n}{4} \times \frac{2}{n} = \frac{1}{2}$$

Note that $\chi(G) \geq \frac{n}{\alpha(G)}$. So to deal with the chromatic number, we'll look at the independence number. Let $a = \left\lceil \frac{3}{p} \ln n \right\rceil$ and consider the event *there is an independent set of size a*. The probability of this event is given by

$$\begin{aligned}
\mathbb{P}[\alpha(G) \geq a] &\leq \binom{n}{a}(1-p)^{\binom{a}{2}} \text{ on all sets of size } a\text{, we want no edges} \\
&\leq n^a e^{\frac{-p(a(a-1))}{2}} \text{ where } \binom{n}{a} \leq n^a \text{ and } (1+r)^x \leq e^{rx} \\
&= n^a e^{-\frac{3\ln n(a-1)}{2}} \\
&= n^a n^{-\frac{3(a-1)}{2}}
\end{aligned}$$

Note that $a = 3(\ln n)n^{1-\lambda}$ with $\lambda < 1$, so this implies that $n^a n^{-\frac{3(a-1)}{2}}$ tends to 0 as n growth large. Hence we have that for n sufficiently large

$$\mathbb{P}[\alpha(G) \geq a] \leq \frac{1}{2}$$

Now, the union bound gives the following

$$\mathbb{P}[X \geq \frac{n}{2} \text{ or } \alpha(G) \geq a] < 1$$

So for n large enough, there is a graph such that none of these properties are satisfied. Which is equivalent to saying that

$$\mathbb{P}[X < \frac{n}{2} \text{ and } \alpha(G) < a] > 0$$

so there **exists** such a graph!

Let G be this graph and delete **one** vertex from each of these short cycles and let G' be this **induced** subgraph. Note that G' has girth at least ℓ and has at least $\frac{n}{2}$ vertices (we deleted strictly less than $\frac{n}{2}$). Now we can get the following lower bound on the chromatic number

$$\begin{aligned}
\chi(G') \geq \frac{|V(G')|}{\alpha(G')} &\geq \frac{n}{2} \times \frac{p}{3 \ln n} \\
&= \frac{n}{2} \times \frac{n^{\lambda-1}}{3 \ln n} \\
&= \frac{n^{\lambda}}{6 \ln n}
\end{aligned}$$

We just got a graph G' with high girth and high chromatic number.

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