Disproving the normal graph conjecture

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Joint-work with **Ararat Harutyunyan** and **Stéphan Thomassé**

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This class of graphs forms a *weaker variant* of perfect graphs by means of a specific graph product.

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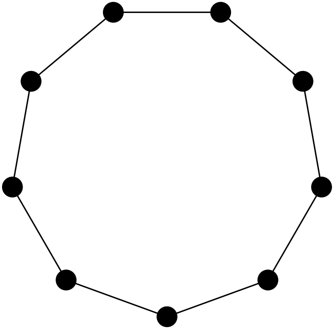

C_9, a **non** perfect graph

Co-normal product

The **co-normal product** $G_1 * G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$, where vertices (v_1, v_2) and (u_1, u_2) are adjacent if u_1 is adjacent to v_1 **or** u_2 is adjacent to v_2.

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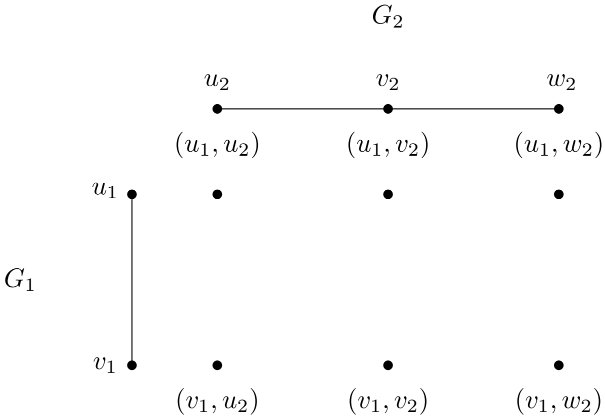

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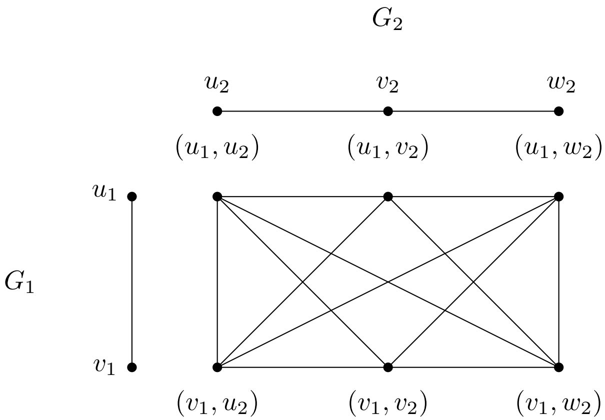

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Shannon capacity

Shannon capacity $C(G)$:

$$C(G) = \lim_{n\to\infty} \frac{1}{n} \log \omega(G^n).$$

Where G^n is the n^{th} co-normal power of G.

Shannon noticed that $\omega(G^n) = (\omega(G))^n$ whenever $\omega(G) = \chi(G)$.

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Körner and Longo 1973

Perfect graphs are **not** closed under co-normal products.

Definition

A graph G is **normal** if there exists a covering of $V(G)$, $\mathbb{C}$, of cliques and a covering of $V(G)$, $\mathbb{S}$, of stable sets such that $C \cap S \neq \emptyset$ for every $C \in \mathbb{C}$ and $S \in \mathbb{S}$.

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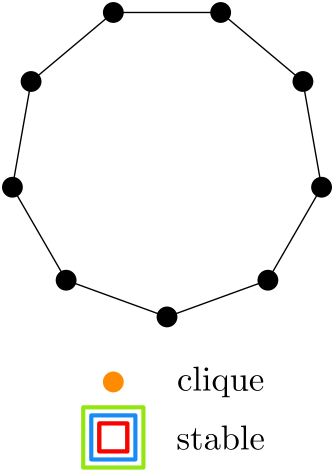

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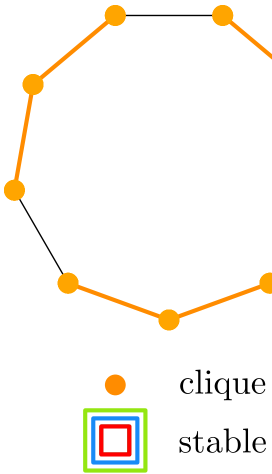

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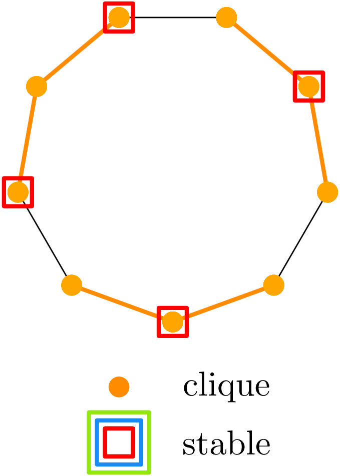

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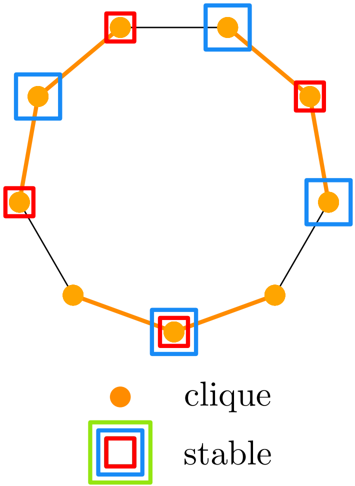

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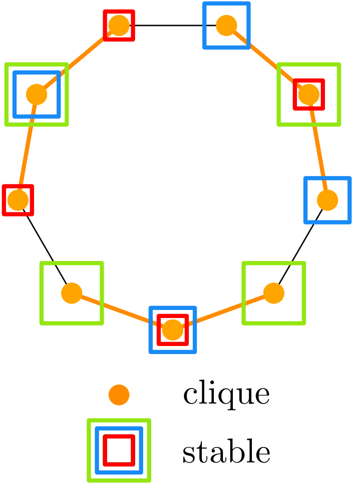

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Theorem [*Körner and Marton 1988*]

$H(P) = H(G,P) + H(\overline{G},P)$ **for at least one** $P \iff G$ is **normal**.

The Normal Graph Conjecture [*De Simone, Körner 1999*]

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- Almost all d-regular graphs are normal when d is fixed [*Hosseini, Mohar, Rezaei 2015*].

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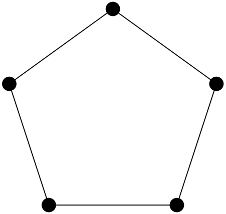

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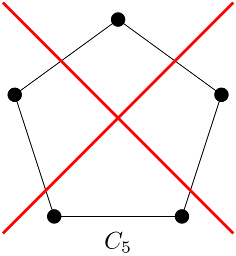

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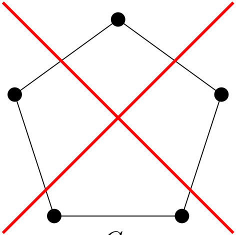

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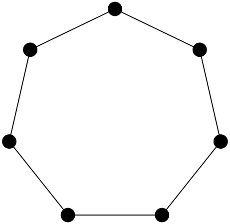

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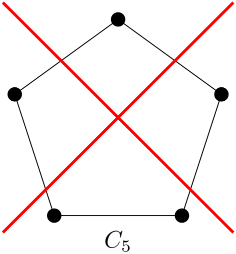

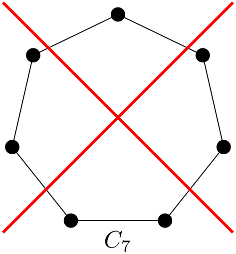

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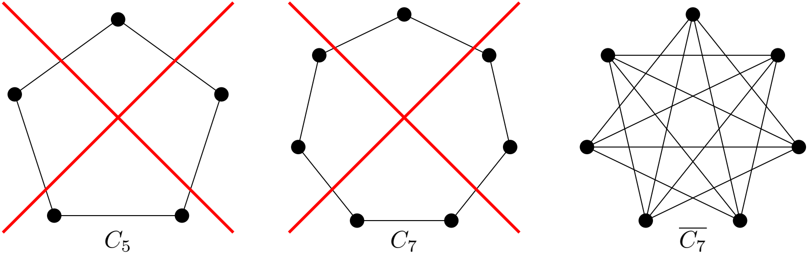

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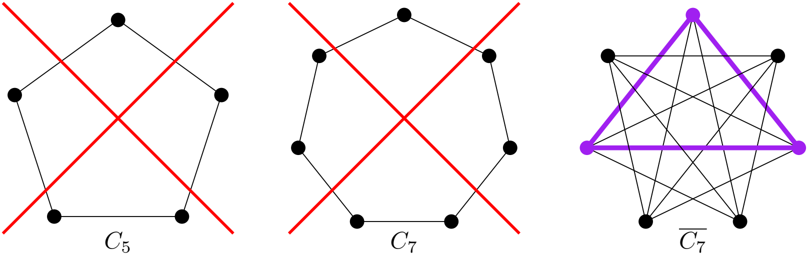

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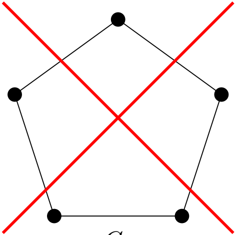

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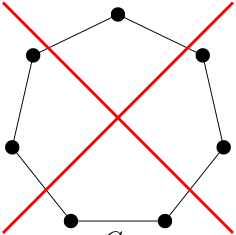

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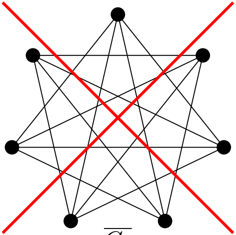

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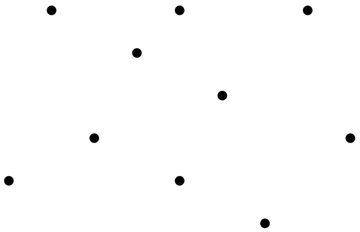

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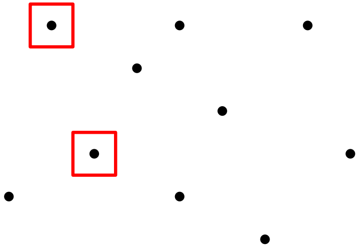

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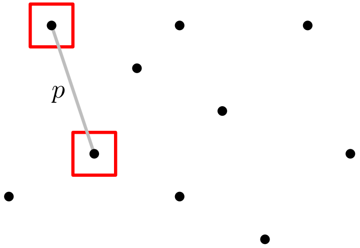

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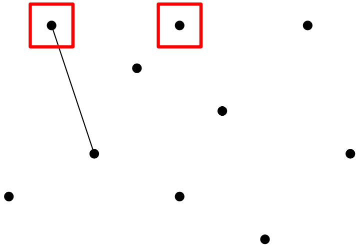

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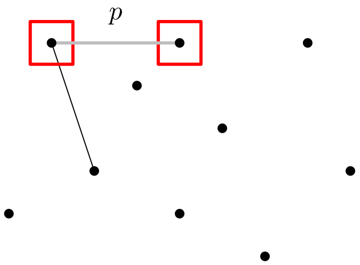

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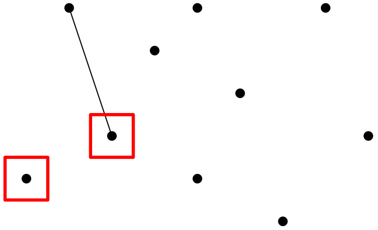

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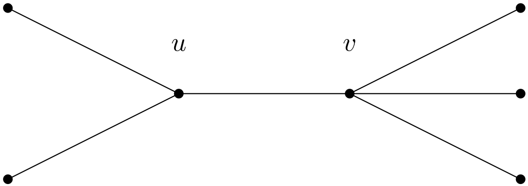

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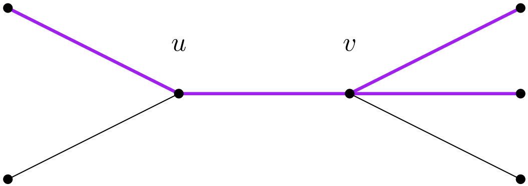

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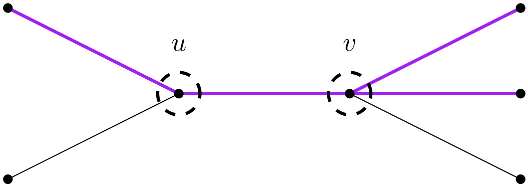

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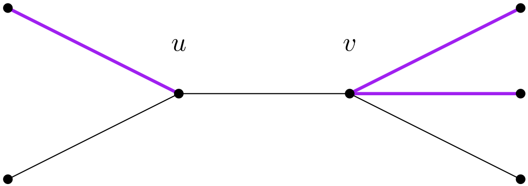

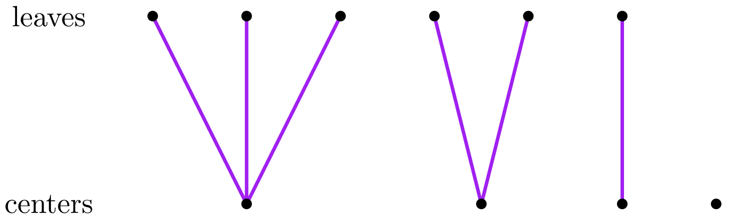
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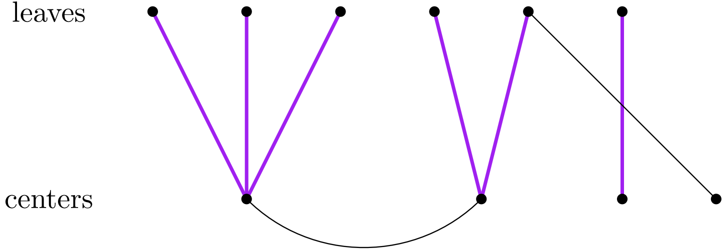
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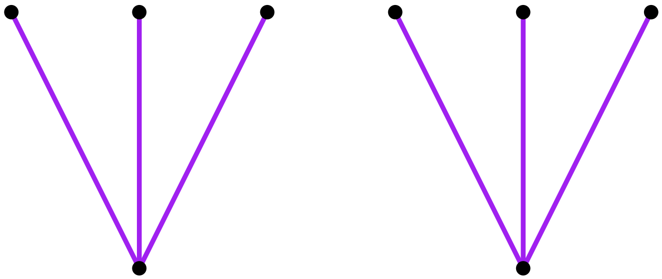

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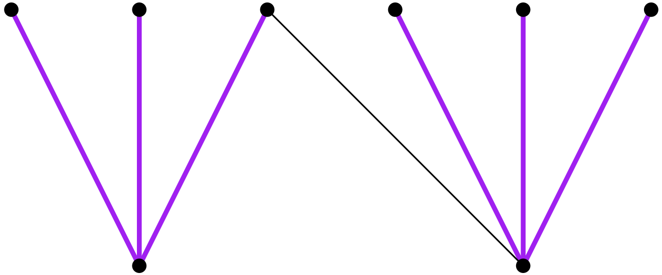

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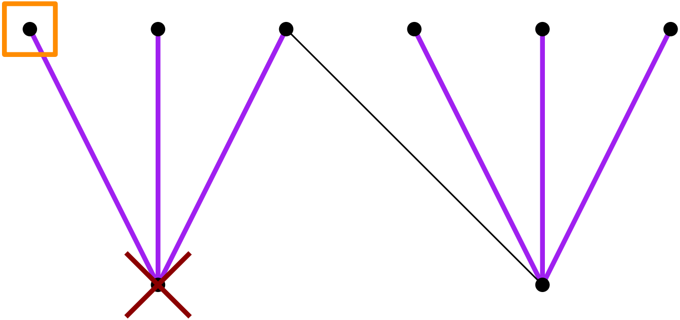

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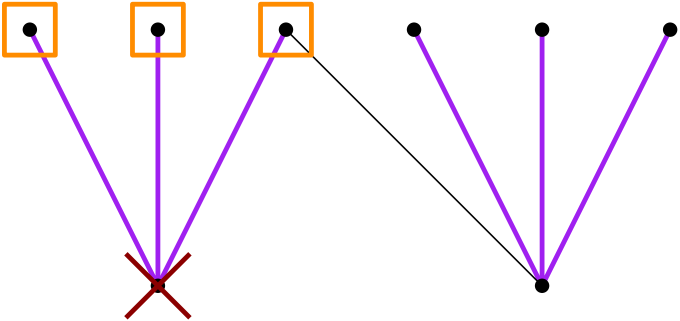

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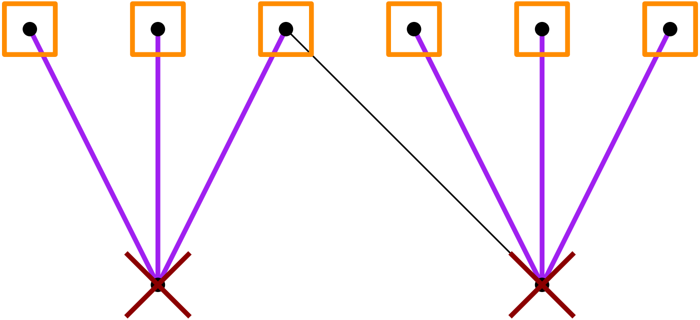

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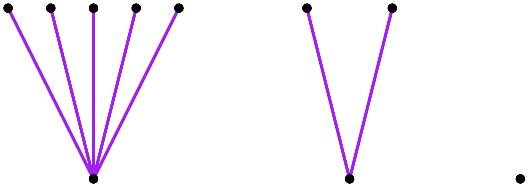

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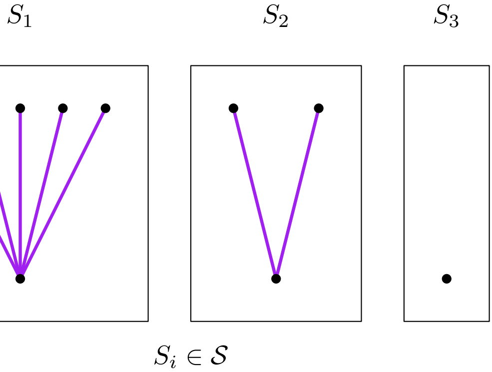

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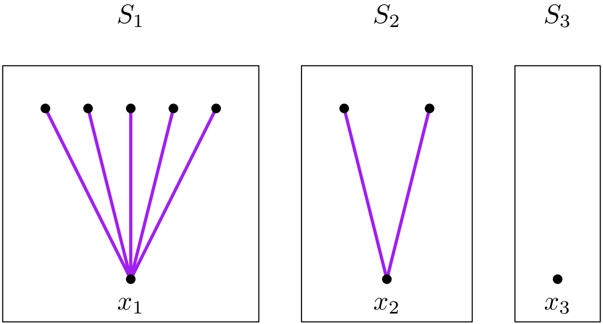

$$S_i \in \mathcal{S}$$

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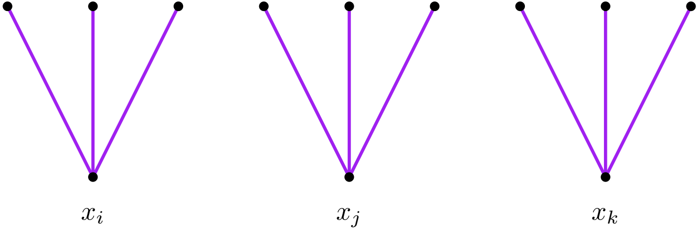

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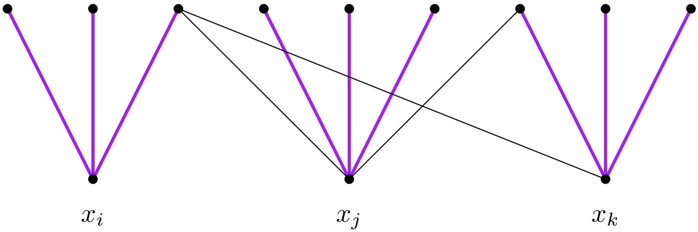

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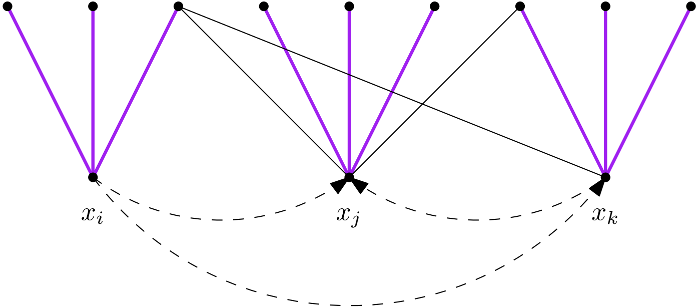

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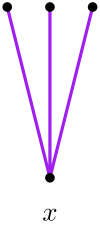

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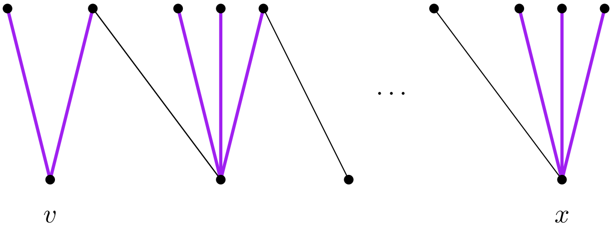

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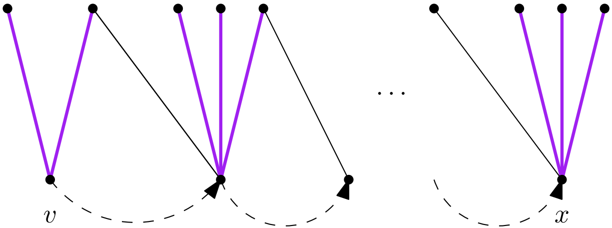

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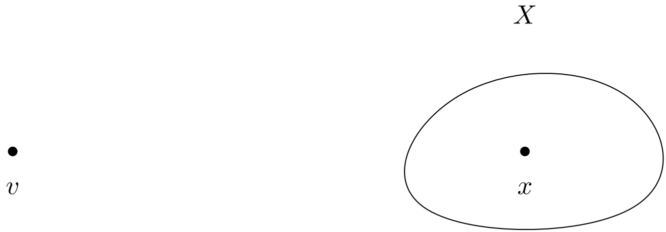

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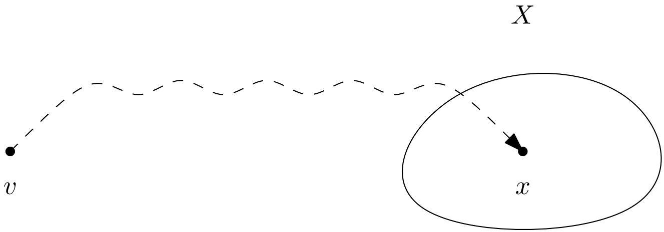

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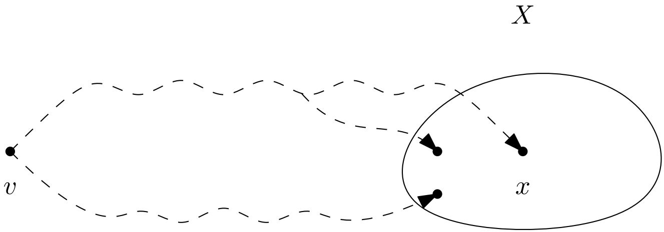

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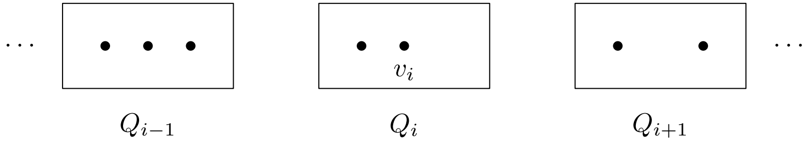

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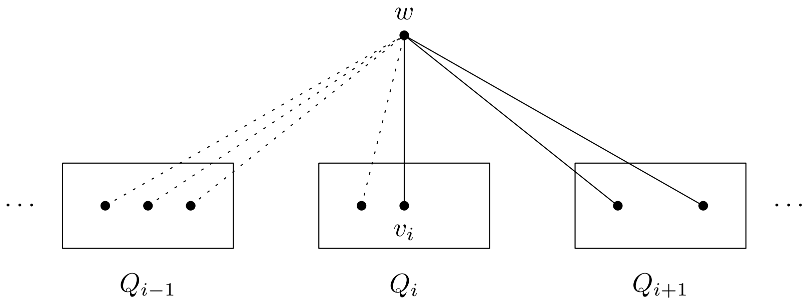

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Probabilistic arguments on $G_{n,p}$ with $p = n^{-9/10}$:

- Union bound.
- Markov's bound.
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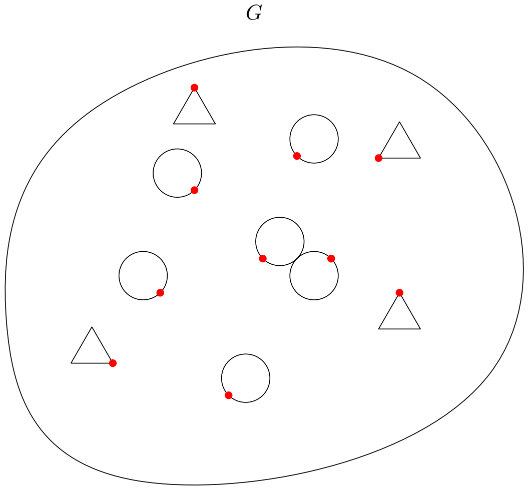

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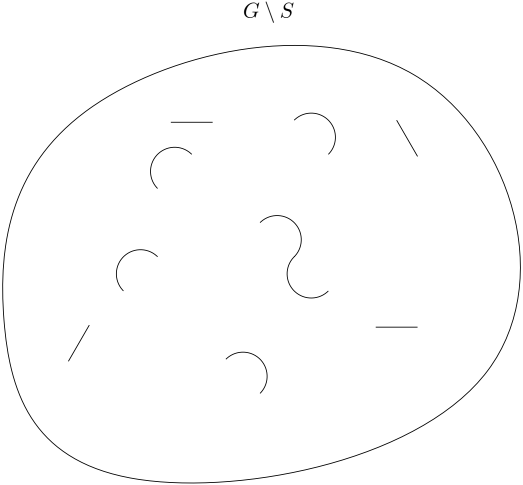

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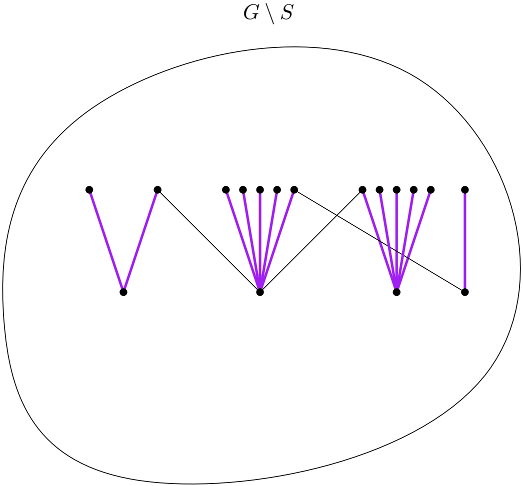

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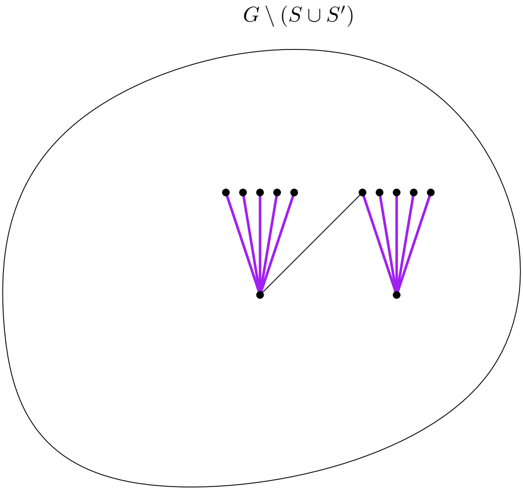

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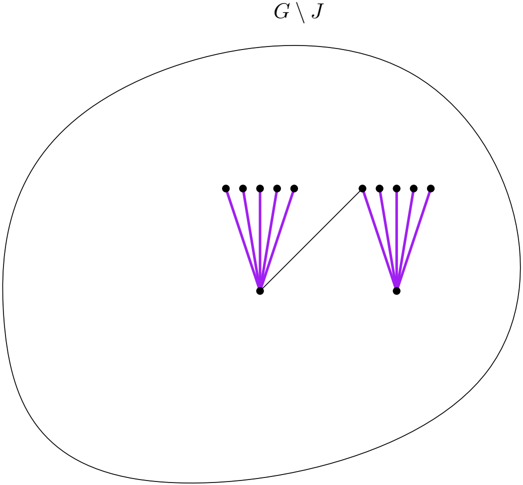

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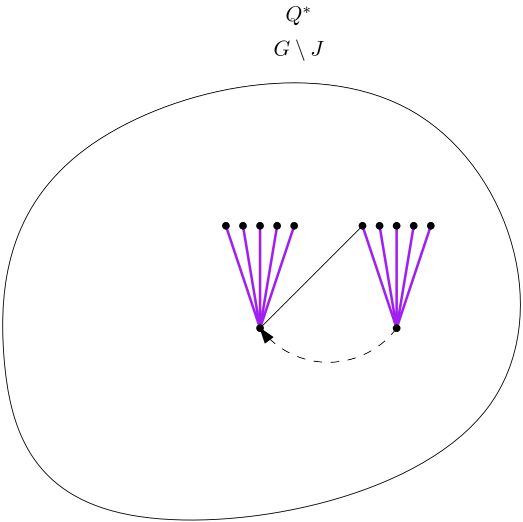

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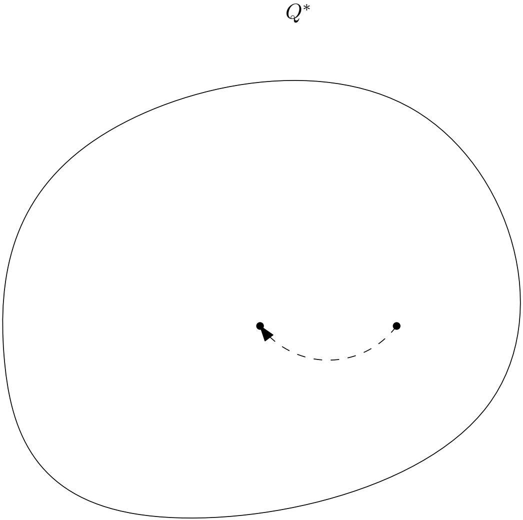

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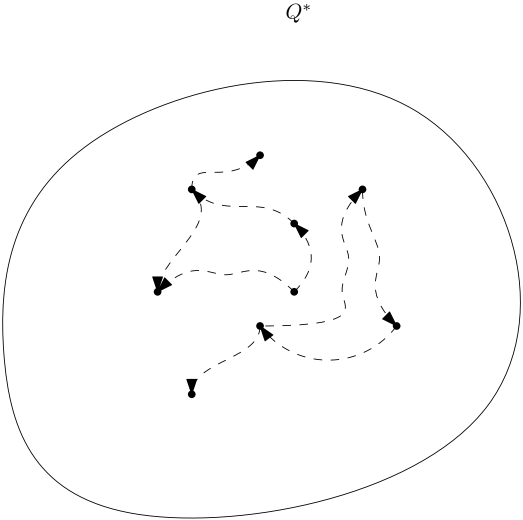

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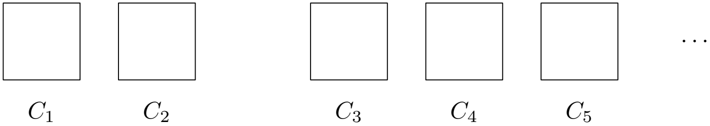

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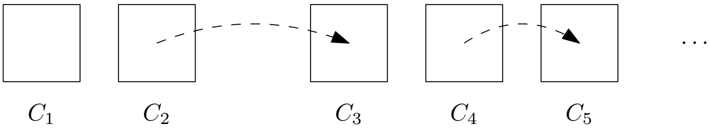

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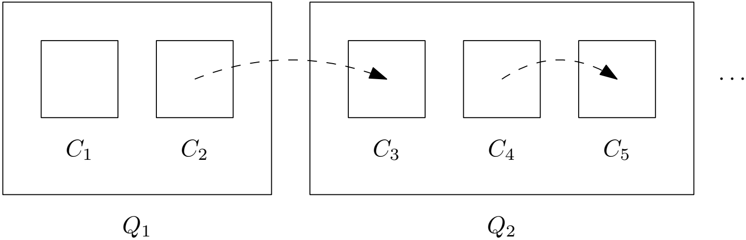

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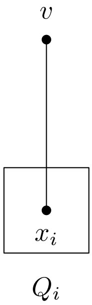

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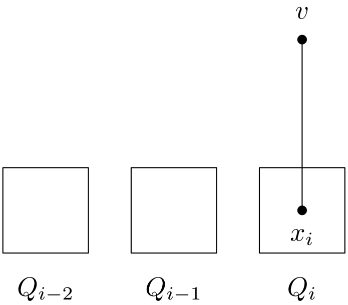

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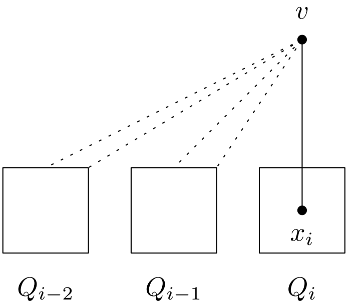

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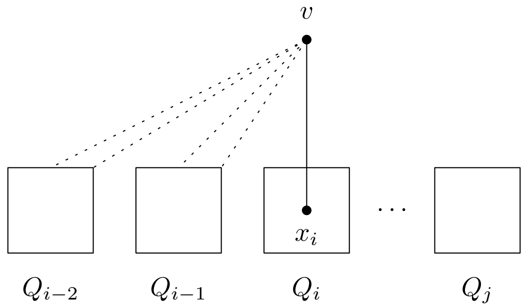

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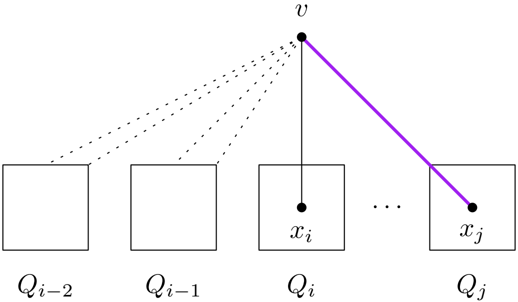

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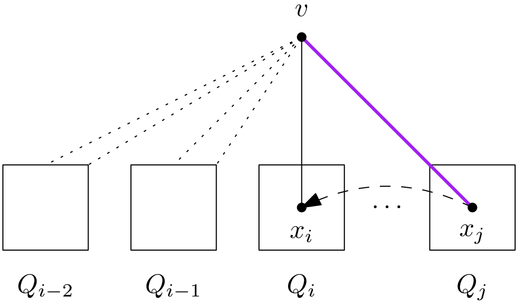

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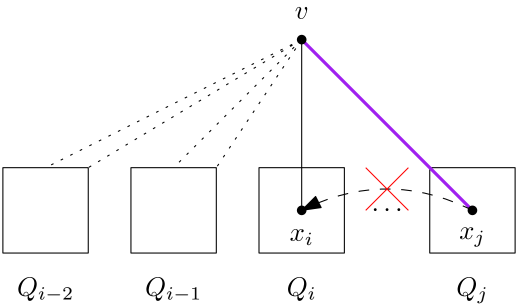

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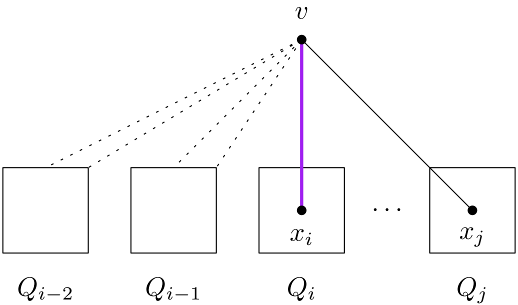

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Theorem [*Harutyunyan, Pastor, Thomassé*]

There exists a graph G of girth at least 8 that is not normal.

Counter-example to the Normal Graph Conjecture!

Conclusion

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Thank you!