List-coloring in claw-free perfect graphs

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Joint-work with Sylvain Gravier and Frédéric Maffray

G-SCOP

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List-coloring

- Let G be a graph. Every vertex $v \in V(G)$ has a list $L(v)$ of prescribed colors, we want to find a proper vertex-coloring c such that $c(v) \in L(v)$.
- When such a coloring exists, G is L-colorable.

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Choice number

The smallest k such that for every list assignment L of size k, the graph G is L-colorable.

Vizing's conjecture

For every graph G, $\chi(\mathcal{L}(G)) = ch(\mathcal{L}(G))$. In other words, $\chi'(G) = ch'(G)$ with $ch'(G)$ the list chromatic index of G.

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Special case

We are interested in the case where G is perfect.

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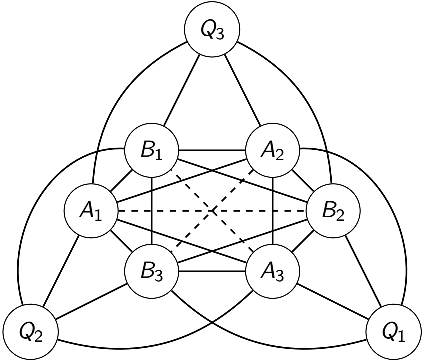
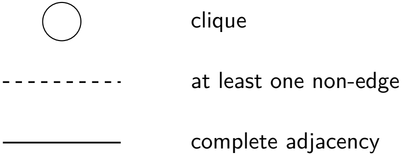
Claw-free graph

The claw is the graph $K_{1,3}$. A graph is said to be claw-free if it has no induced subgraph isomorphic to $K_{1,3}$.

Theorem [Chvátal and Sbihi, 1988]

Every claw-free perfect graph either has a clique-cutset, or is a peculiar graph, or is an elementary graph.


Q_3
B_1
A_2
A_1
B_2
B_3
A_3
Q_2
Q_1


clique
at least one non-edge
complete adjacency

Theorem [Maffray and Reed, 1999]

A graph G is elementary if and only if it is an augmentation of the line-graph H (called the **skeleton** of G) of a bipartite multigraph B (called the **root** graph of G).

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- Let G' be a graph obtained from G after removing x and y.

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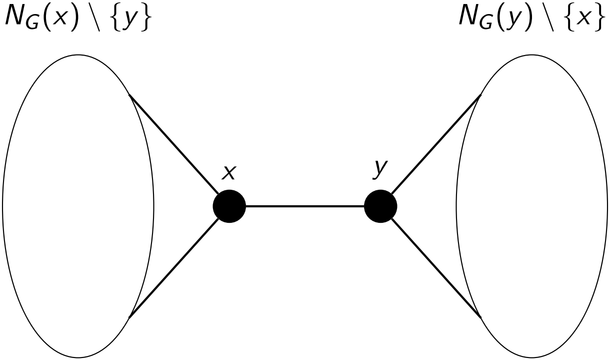
- Add all edges between Y and $N_G(y) \setminus \{x\}$ in G'.


$N_G(x) \setminus \{y\}$
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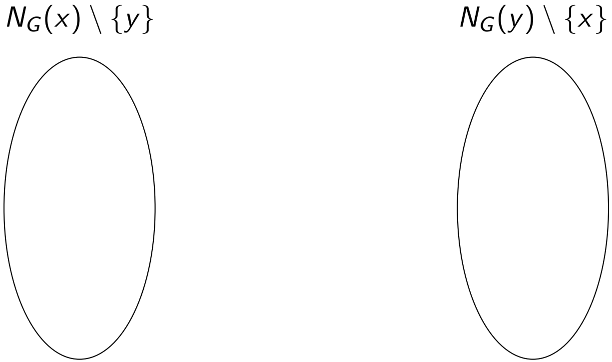
x
y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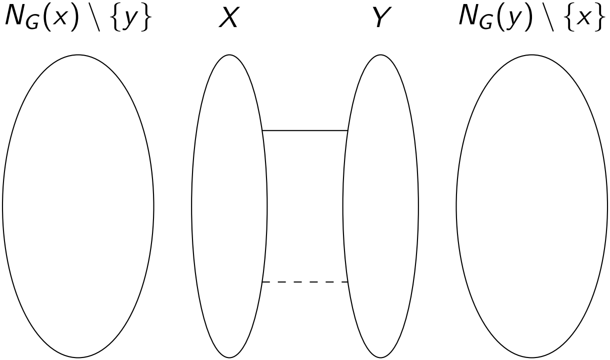
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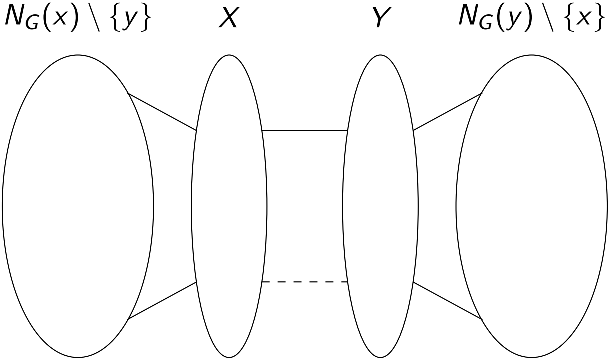
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Theorem [Gravier, Maffray, P.]

Let G be a claw-free perfect graph with $\omega(G) \leq 4$. Then $\chi(G) = ch(G)$.

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- Let H be a peculiar proper subgraph of G that is maximal.
- Since G is connected there is a vertex x of $V(G) \setminus V(H)$ having a neighbour in H.
- In order to avoid claws, odd holes and odd anti holes, x has many neighbours in H from several sets of the peculiar partition. In fact, x is in one of those sets, hence $H \cup \{x\}$ is a peculiar graph.

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- If some pairs of non-adjacent vertices share a color, we can color G.
- If no such pair exists, we can find a coloring by Hall's theorem.

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- If the coloring of G' can be extended to (X, Y) we are done.
- If not, we can show thanks to a gadget that there exists a coloring of G' that can be extended to G.

Proof of the main theorem

Let G be a claw-free perfect graph and C a clique cutset. The graph $G \setminus C$ has two components A_1 and A_2. Let $G_1 = G[C \cup A_1]$ and $G_2 = G[C \cup A_2]$. We may assume that G_1 is colored and we want to extend it to G_2. Let us assume that G_2 is elementary. There are two cases:

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Proof of 1

We manually prove that the coloring of C can be extended to G_2.

Proof of 2

We use a Galvin argument to show that the graph G_2 is colorable with forced colors on the clique C.

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A word on our method

- Proving that elementary graphs are chromatic-choosable by induction on the number of augmented flat edges gives us interesting tools for the extension of a coloring to an elementary graph.
- It is still not clear whether the gadget trick is a good option for the generalization or not.

Thank you for listening.
Do you have any questions?