Colouring squares of claw-free graphs

Lucas Pastor

April 17 2018

Joint-work with **Rémi de Joannis de Verclos** and **Ross J. Kang**

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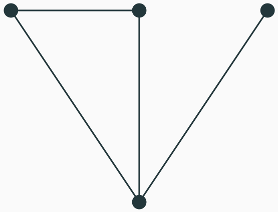

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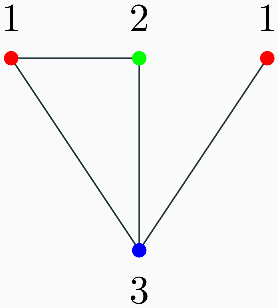

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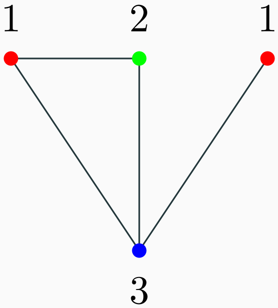

The **chromatic number**, $\chi(G)$, is the smallest k such that G is k-colorable.

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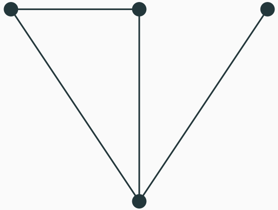

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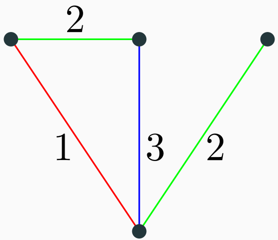

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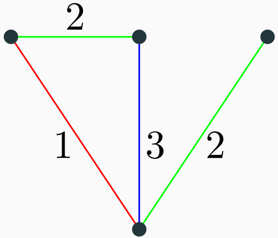

The **chromatic index**, $\chi'(G)$, is the smallest k such that G is k-edge-colorable.

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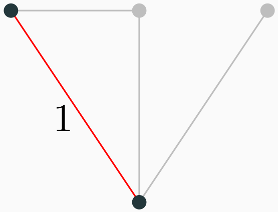

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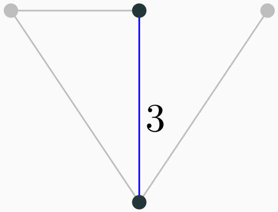

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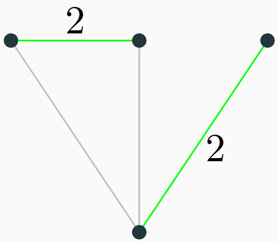

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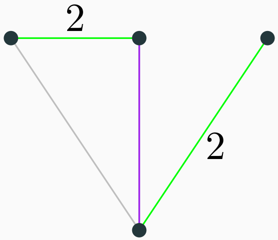

But not necessarily an **induced matching**!

A **strong k-edge-coloring** of G is a k-edge-coloring where each color class is an induced matching.

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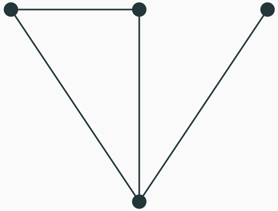

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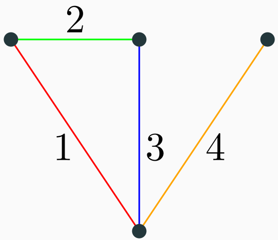

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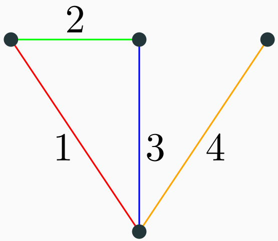

The **strong chromatic index**, $\chi'_s(G)$, is the smallest k such that G is strong k-edge-colorable.

Questions

Given a graph G with maximum degree $\Delta(G)$.

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$$\chi'_s(G)$$

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lower bound $\leq \chi'_s(G) \leq$ **upper bound**

Upper bound

Pick any edge e, and look at how large can be its neighborhood at distance 2.

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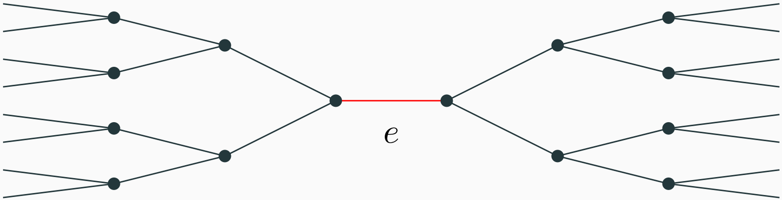

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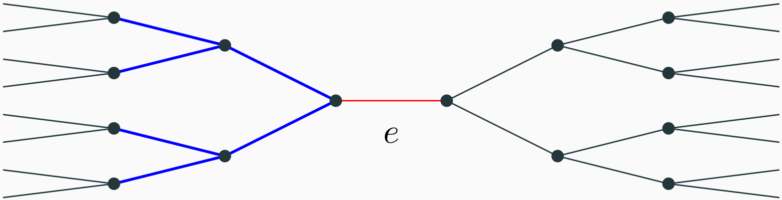

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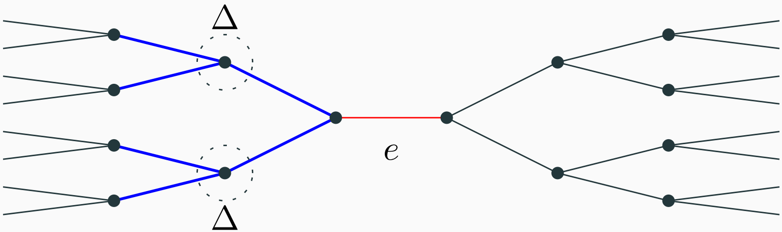

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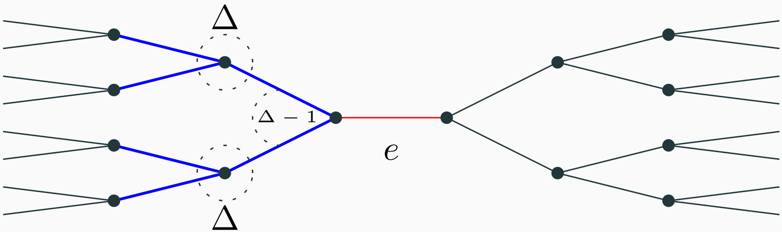

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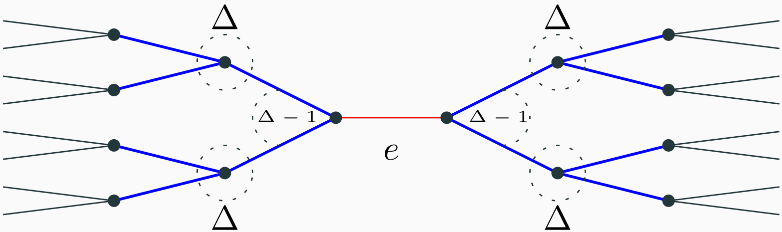

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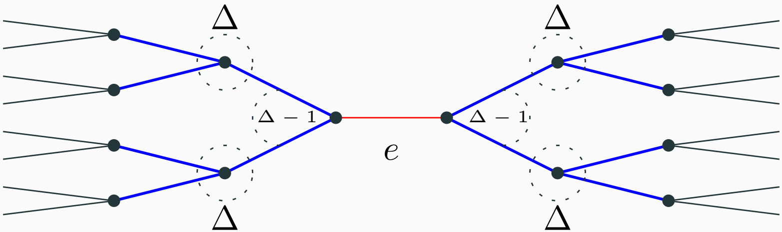

$$\chi'_s(G) \leq 2\Delta(\Delta - 1) + 1 = 2\Delta^2 - 2\Delta + 1.$$

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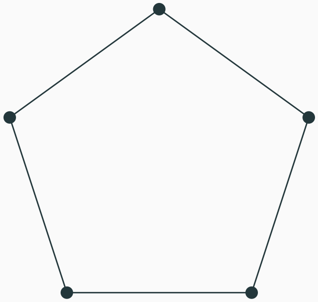

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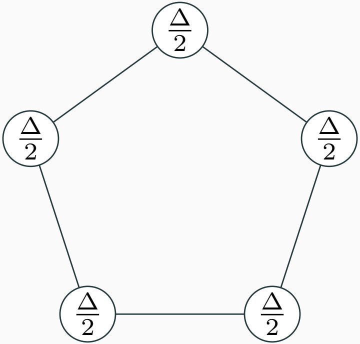

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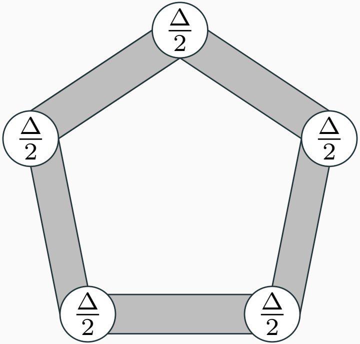

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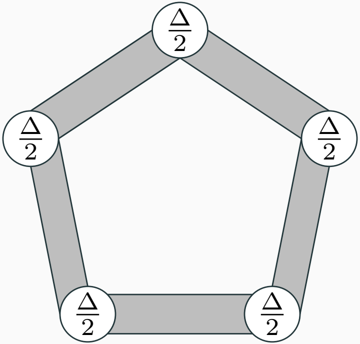

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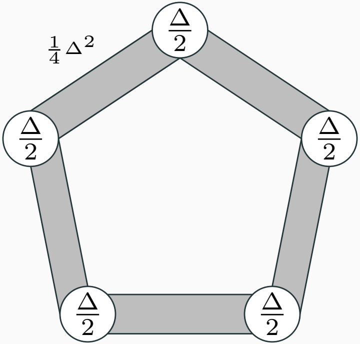

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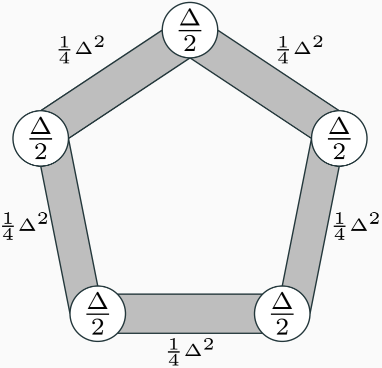

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Conjecture [Erdős, Nešetřil 1988]

The previous example is the worst you can get. In other words:

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The constant has been improved by Bruhn and Joos in 2015 to $\epsilon = 0.07$.

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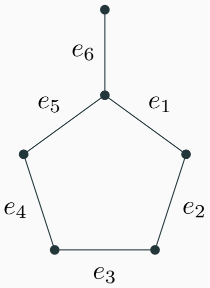

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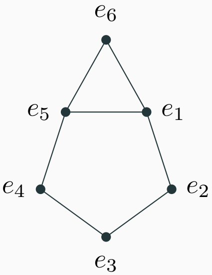

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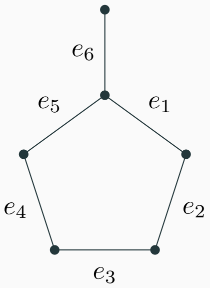

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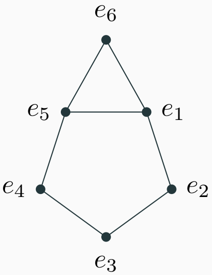

$\mathcal{L}(G)$

Note than if G is a simple graph, then $\omega(\mathcal{L}(G)) = \Delta(G)$ unless G is the disjoint union of a triangle, paths and cyles.

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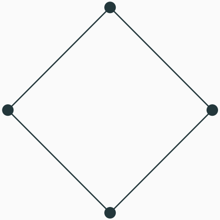

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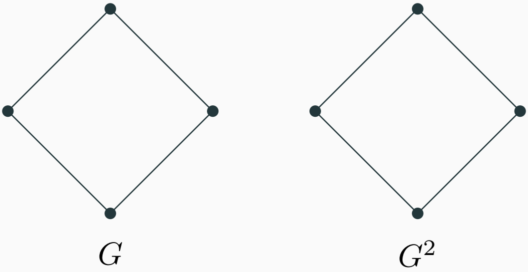

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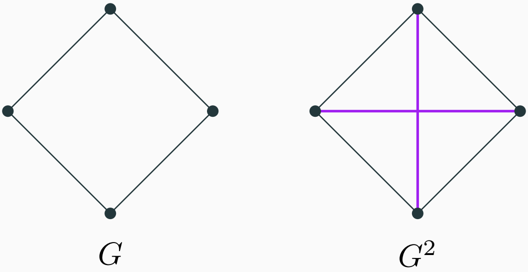

Strong coloring

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Molloy and Reed's theorem

Let G be the line-graph of any simple graph, then:

$$\chi(G^2) \leq (2-\epsilon)\omega(G)^2.$$

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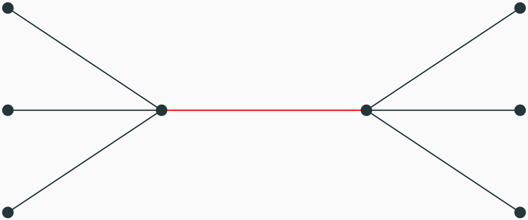

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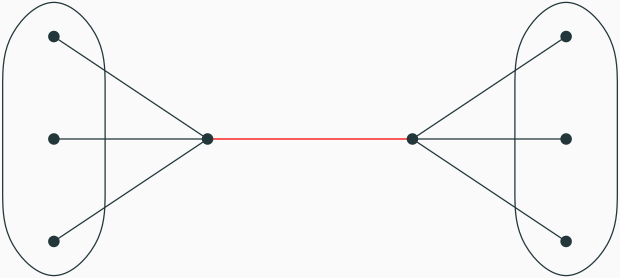

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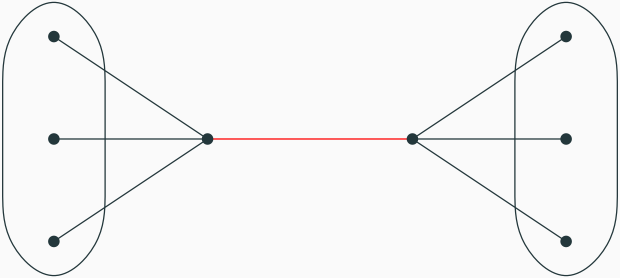

The class of graphs having this property is the class of **quasi-line** graphs.

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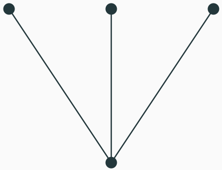

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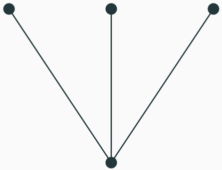

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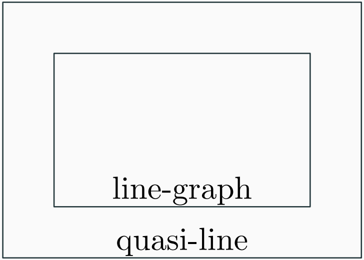
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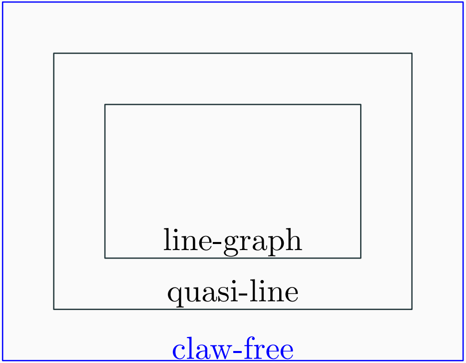
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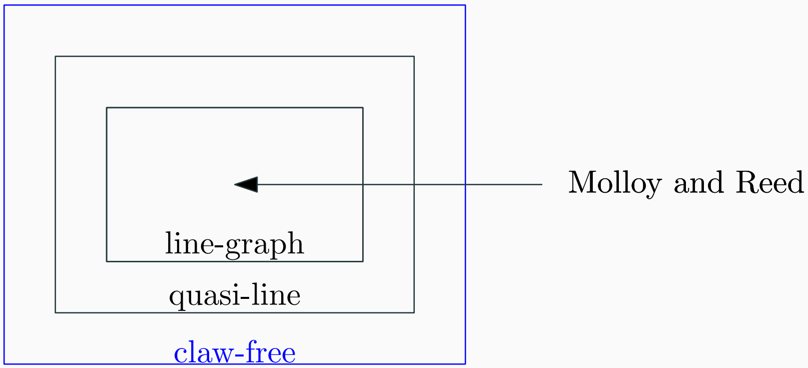
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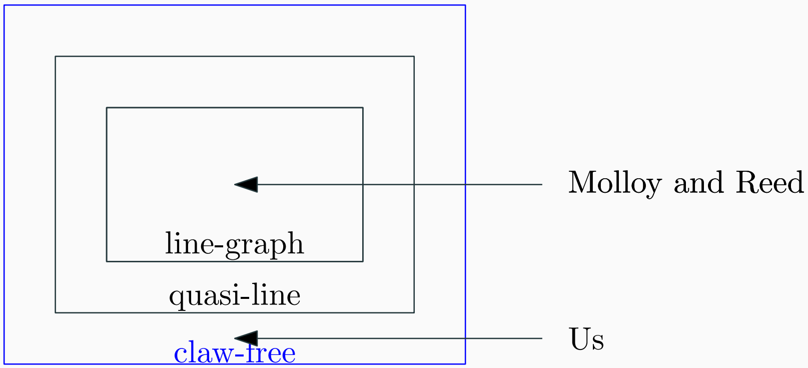
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Roadmap

1. From claw-free to quasi-line graphs.

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Roadmap

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2. From quasi-line graphs to line-graphs of multigraphs.

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1. From claw-free to quasi-line graphs.
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3. Prove the theorem for line-graphs of multigraphs.

Second neighborhood

The **second neighborhood** of v, denoted by $N_G^2(v)$, is the set of vertices at distance exactly two from v, i.e.
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The **square degree** of v, denoted by $\deg_{G^2}(v)$, is equal to $\deg_G(v) + |N_G^2(v)|$.

Lemma

For G claw-free, either G is a quasi-line graph or there exist $v \in V(G)$ with $\deg_{G^2}(v) \leq \omega(G)^2 + (\omega(G)+1)/2$ **whose neighborhood is a clique of** $(G \setminus v)^2$.

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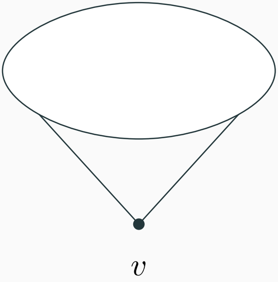
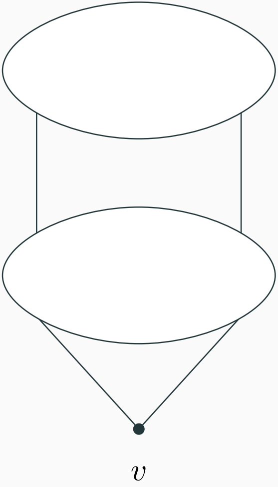
1. The proof is by induction on $|V(G)|$.
2. Note that $(G \setminus v)^2 \neq G^2 \setminus v$.


v


v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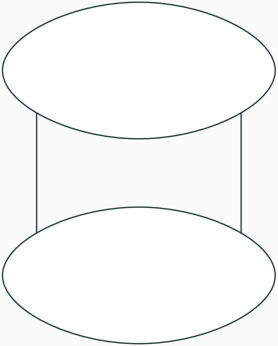
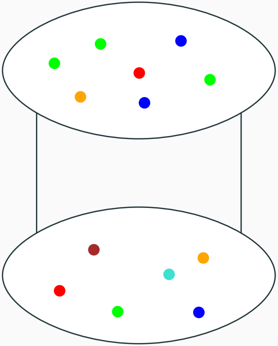
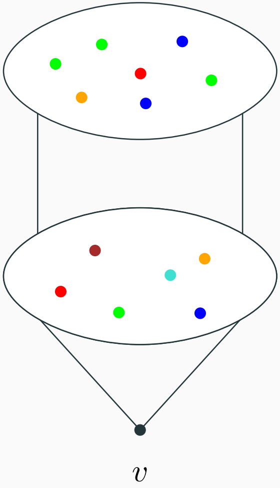
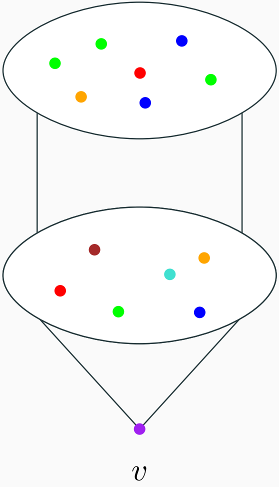
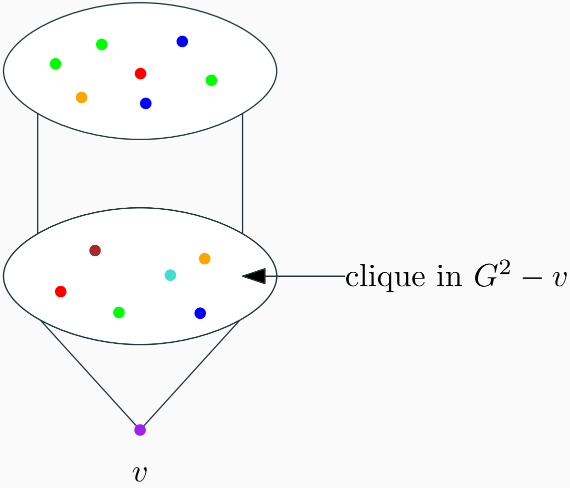




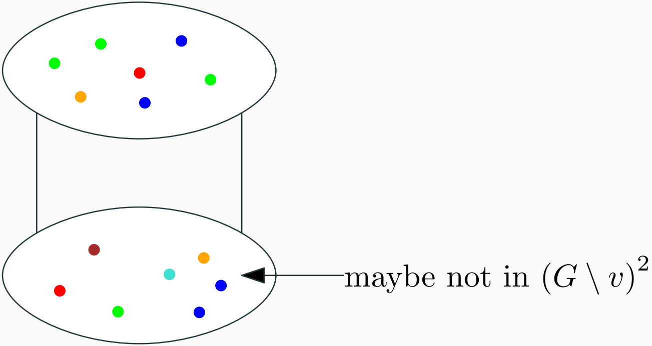




v



clique in $G^2 - v$
v


maybe not in $(G \setminus v)^2$

Lemma

For G claw-free, either G is a quasi-line graph or there exist $v \in V(G)$ with $\deg_{G^2}(v) \leq \omega(G)^2 + (\omega(G)+1)/2$ **whose neighborhood is a clique of** $(G \setminus v)^2$.

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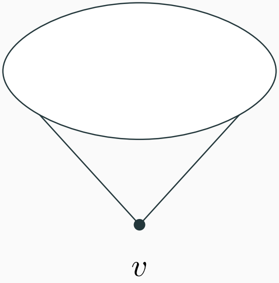

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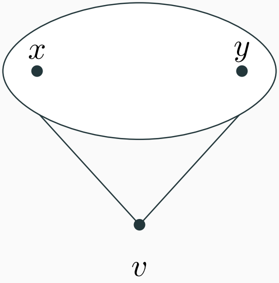

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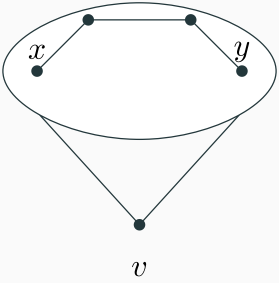

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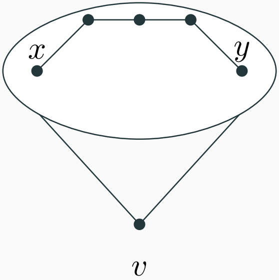

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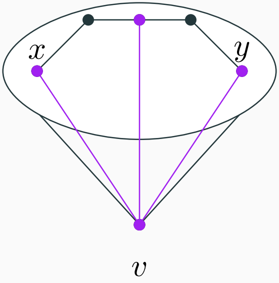

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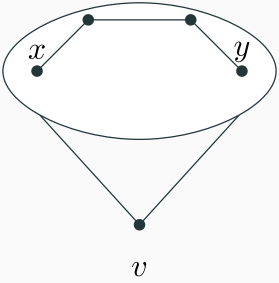

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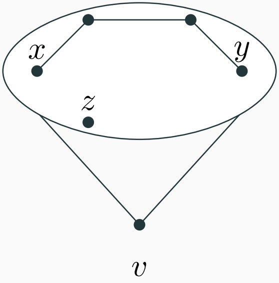

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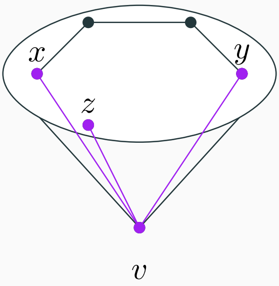

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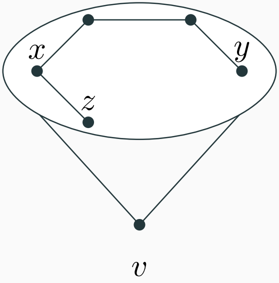

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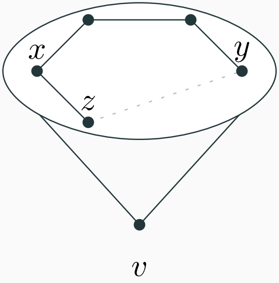

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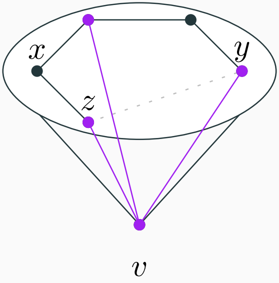

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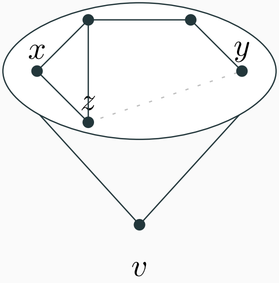

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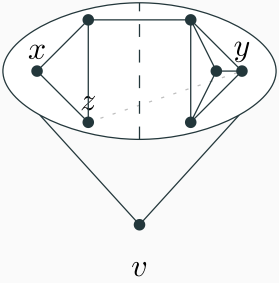

Upper bound on $\deg_{G^2}(v)$.

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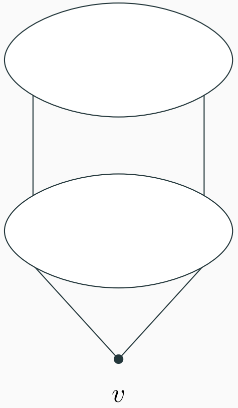

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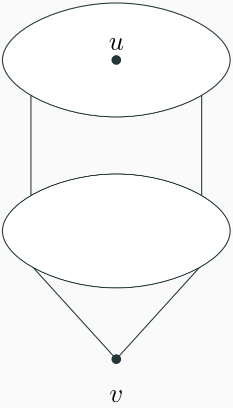
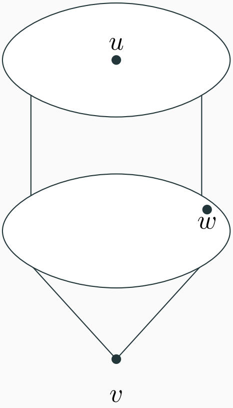


Let u with minimum $|N(u) \cap N(v)| = k$

Upper bound on $\deg_{G^2}(v)$.



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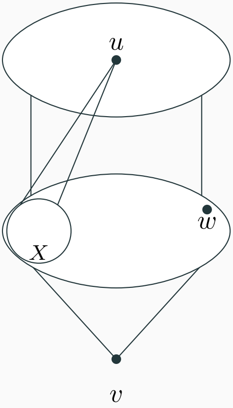

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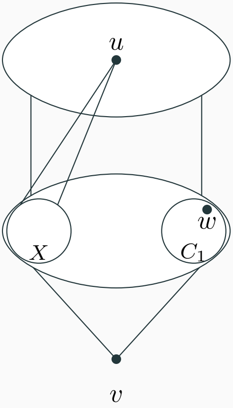

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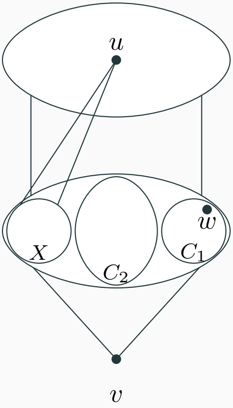

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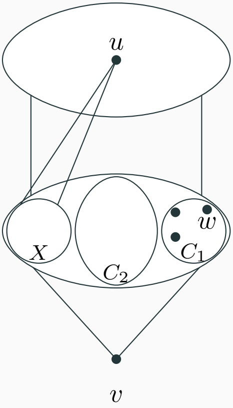

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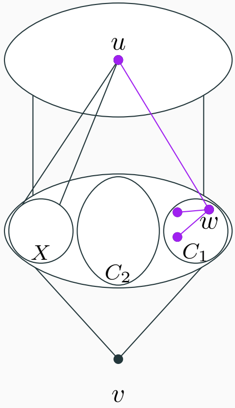

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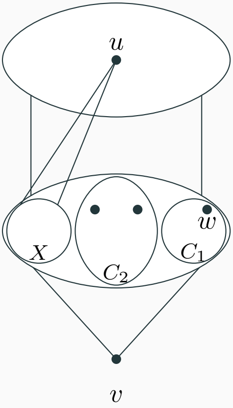

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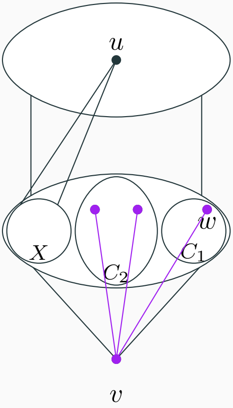

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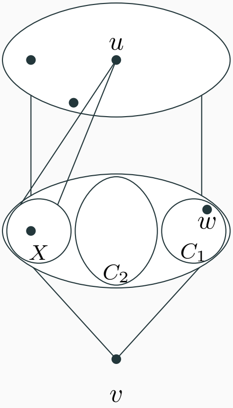

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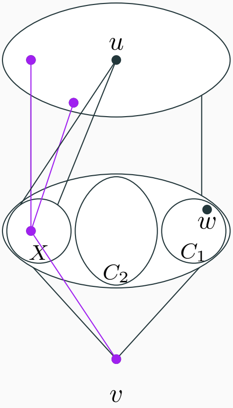

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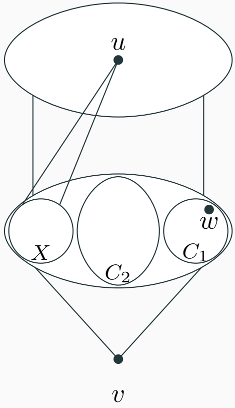

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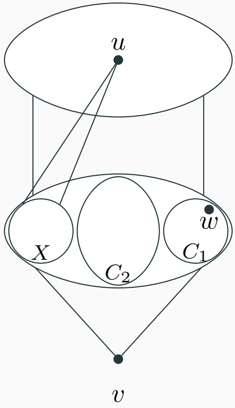

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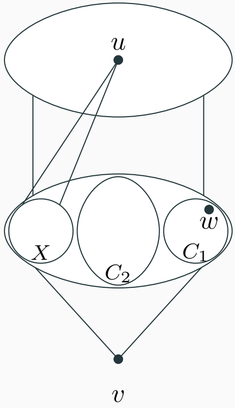

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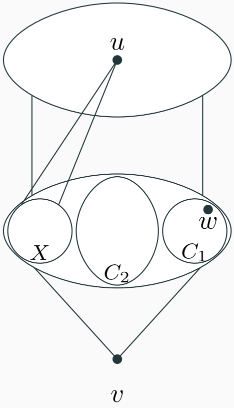

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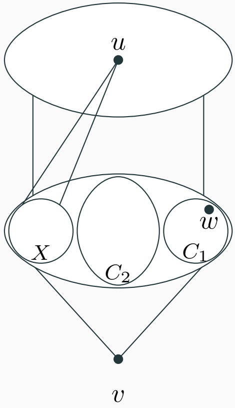

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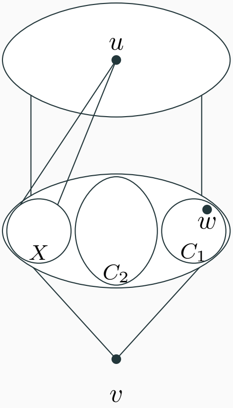

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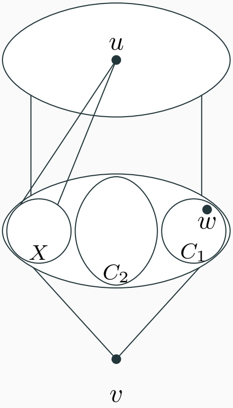

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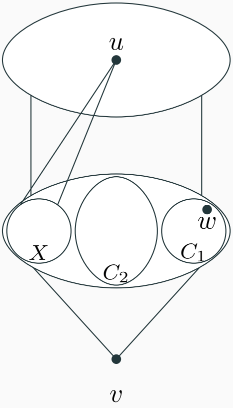

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if $k \geq 2$, $\deg_{G^2}(v) \leq \omega^2 + O(\omega)$

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From quasi-line graphs to line-graphs of multigraphs.

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- Structure theorem of claw-free graphs due to Chudnovsky and Seymour.

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From quasi-line graphs to line-graphs of multigraphs.

- Structure theorem of claw-free graphs due to Chudnovsky and Seymour.
- Either there is a good vertex, or G is the line-graph of a multigraph.

Lemma

For G line-graph of multigraph, there is an absolute constant $\epsilon > 0$ such that $\chi(G^2) \leq (2 - \epsilon)\omega(G)^2$.

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The idea is to generalize the proof of Molloy and Reed to line graphs of multigraphs.

Molloy and Reed

For any $\epsilon > 0$, there exist $\delta > 0$ and Δ_0 such that the following holds. For all $\Delta \geq \Delta_0$, if G is a graph with $\Delta(G) \leq \Delta$ and with at most $(1-\epsilon)\binom{\Delta}{2}$ edges in each neighbourhood, then $\chi(G) \leq (1-\delta)\Delta$.

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If the neighborhood is not too dense, then the chromatic number is not too big.

Theorem

There are some absolute constants $\epsilon > 0$ and Δ_0 such that $\chi'_s(F) \leq (2 - \epsilon)\Delta(F)^2$ for any multigraph F with $\Delta(F) \geq \Delta_0$.

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Lemma

There are absolute constants $\epsilon > 0$ and Δ_0 such that the following holds. For all $\Delta \geq \Delta_0$, if $F = (V, E)$ is a multigraph with $\Delta(F) \leq \Delta$, then $N_{\mathcal{L}(F)^2}(e)$ induces a subgraph of $\mathcal{L}(F)^2$ with at most $(1-\epsilon)\binom{2\Delta(\Delta-1)}{2}$ edges for any $e \in E$.

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Since $\Delta(\mathcal{L}(F)^2) \leq 2\Delta(F)(\Delta(F)-1)$, we apply the theorem of Molloy and Reed to $\mathcal{L}(F)^2$.

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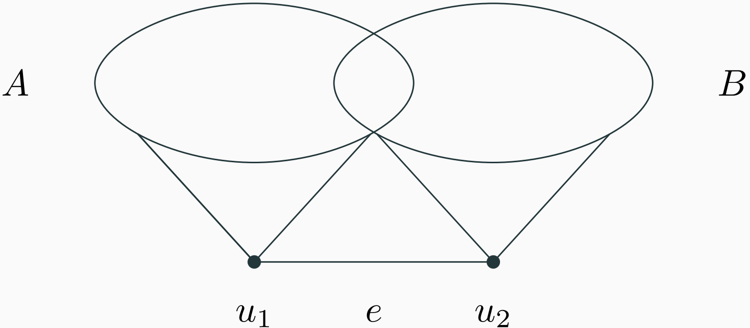

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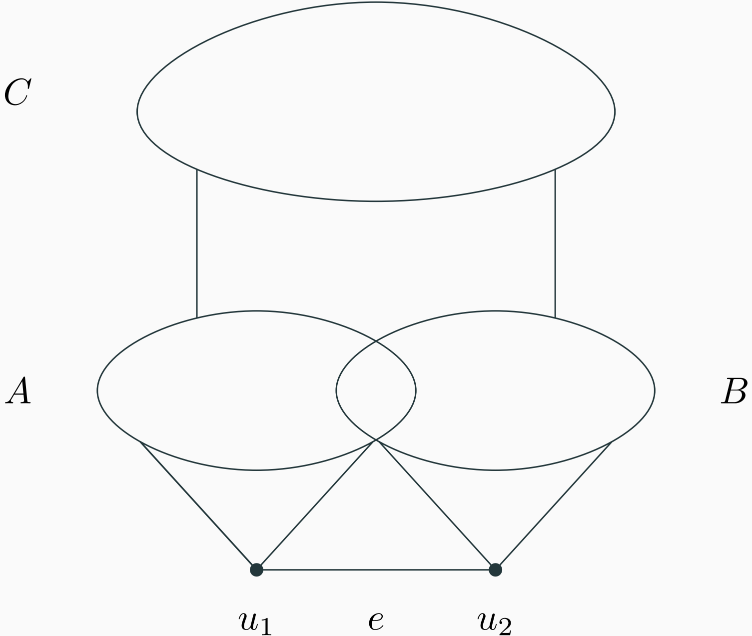

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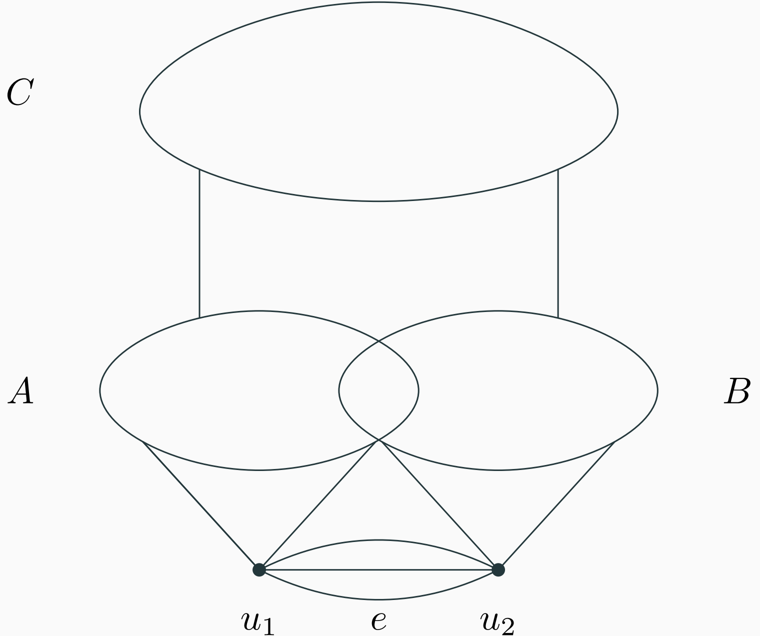

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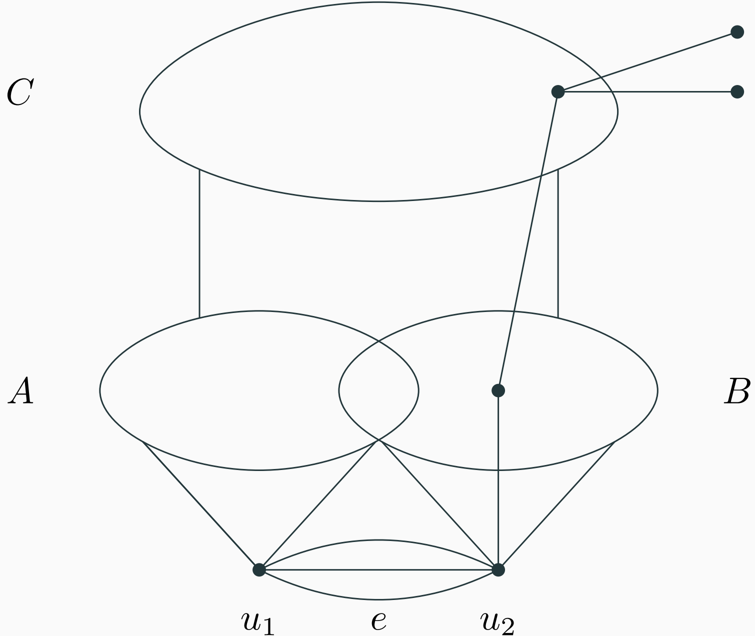

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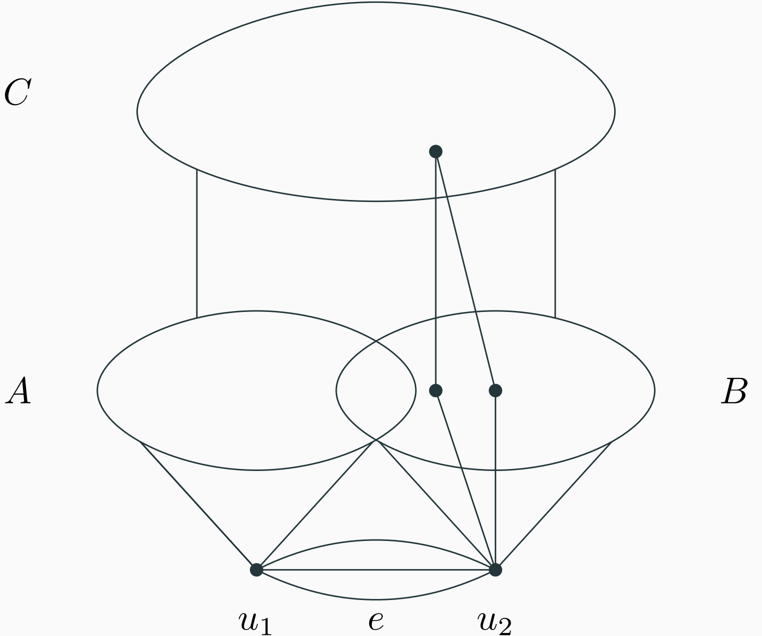

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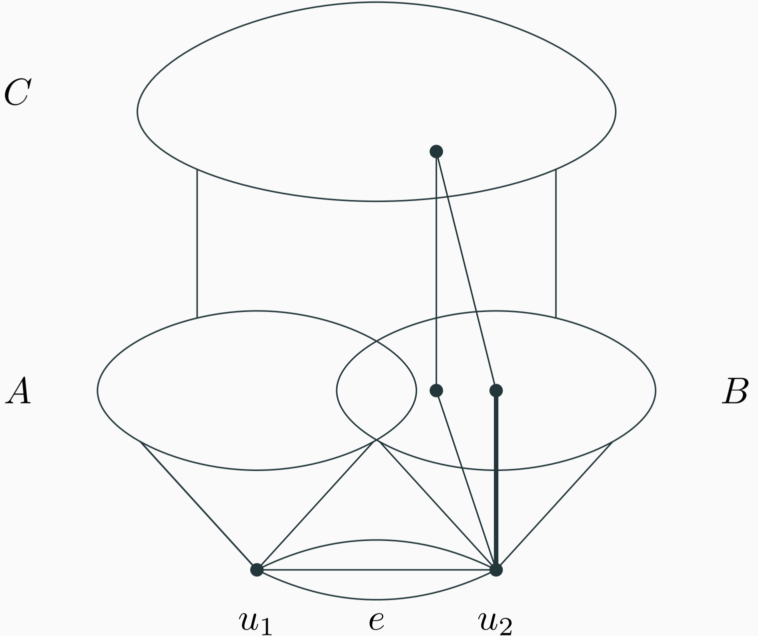

Conclusion

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- The conjecture for bipartite graphs is $\chi'_s(G) \leq \Delta(A)\Delta(B)$.

Thank you for your attention.