Coloring squares of claw-free graphs

Lucas Pastor

November 15 2017

Joint-work with **Rémi de Joannis de Verclos** and **Ross J. Kang**

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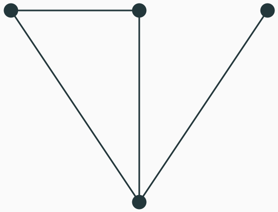

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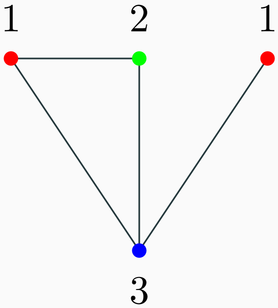

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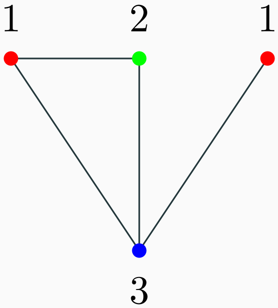

The **chromatic number**, $\chi(G)$, is the smallest k such that G is k-colorable.

A **(proper)** ***k*-edge-coloring** of G is an assignment of colors $\{1, \ldots, k\}$ to the edges of G such that any two adjacent edges (sharing a vertex) receive a different color.

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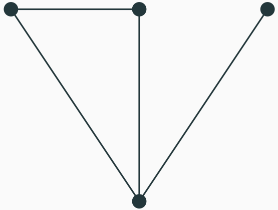

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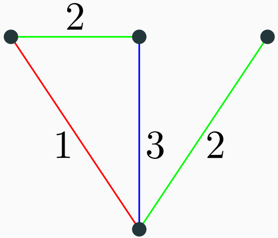

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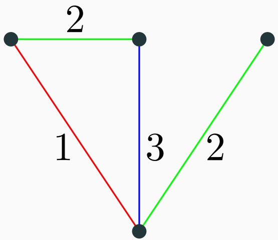

The **chromatic index**, $\chi'(G)$, is the smallest k such that G is k-edge-colorable.

Note that in an edge coloring, each color class is a **matching**.

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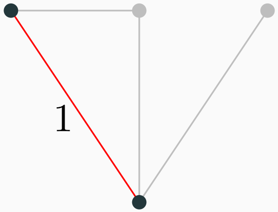

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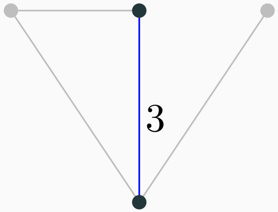

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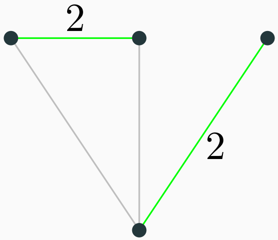

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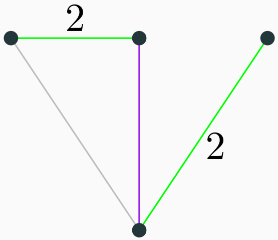

But not necessarily an **induced matching**!

A **strong k-edge-coloring** of G is a k-edge-coloring where each color class is an induced matching.

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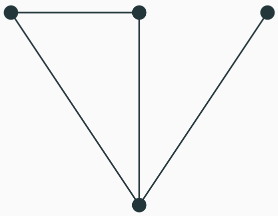

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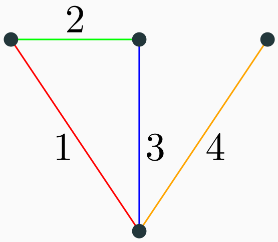

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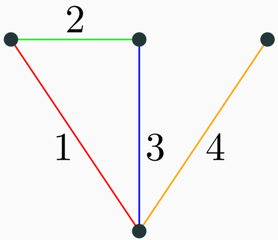

The **strong chromatic index**, $\chi'_s(G)$, is the smallest k such that G is strong k-edge-colorable.

Questions

Given a graph G with maximum degree $\Delta(G)$.

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$$\textbf{lower bound} \leq \chi'_s(G) \leq \textbf{upper bound}$$

Upper bound

Pick any edge e, and look at how large can be its neighborhood at distance 2.

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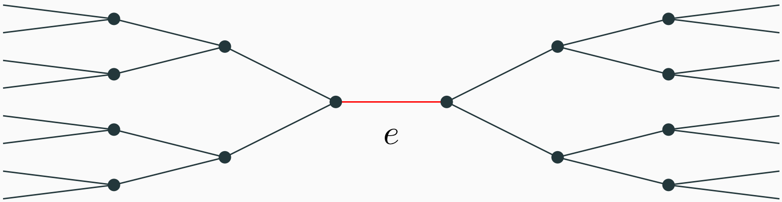

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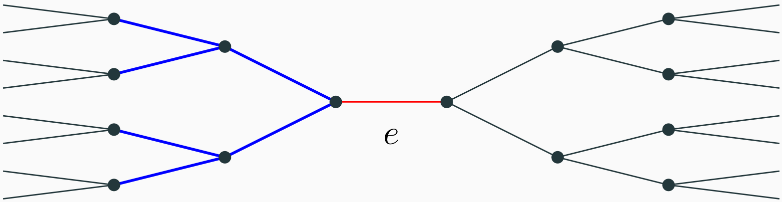

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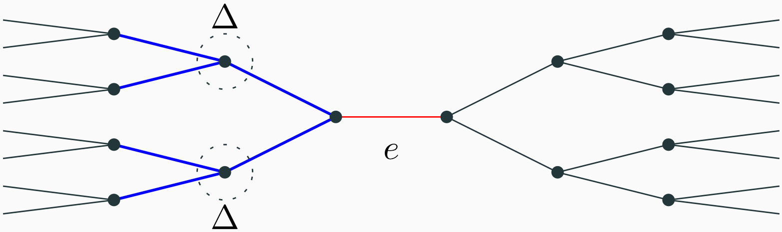

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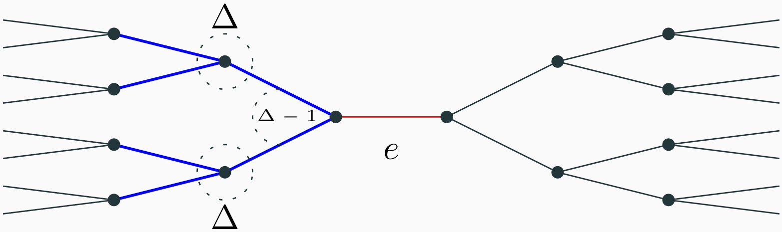

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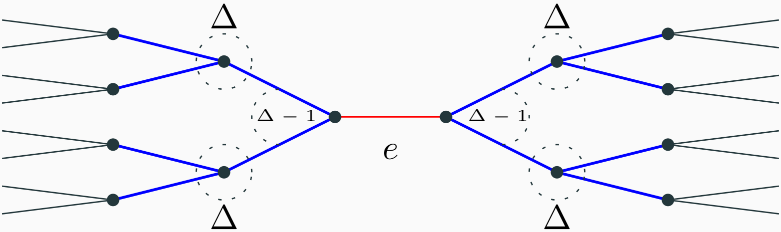

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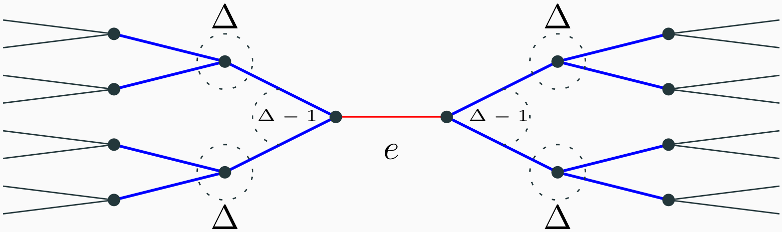

$$\chi'_s(G) \leq 2\Delta(\Delta - 1) + 1 = 2\Delta^2 - 2\Delta + 1.$$

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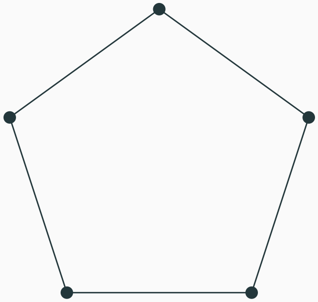

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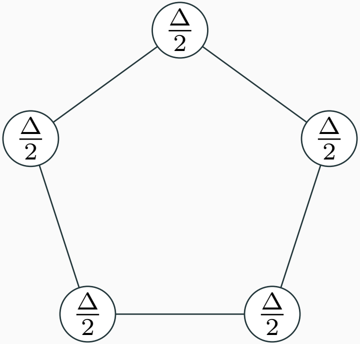

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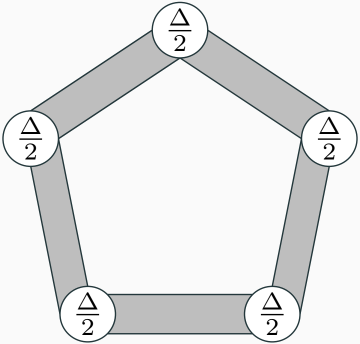

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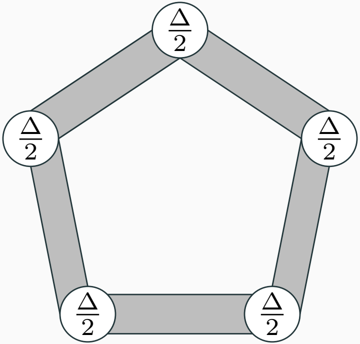

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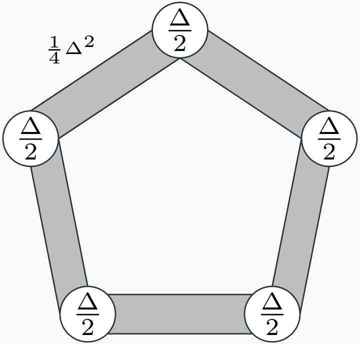

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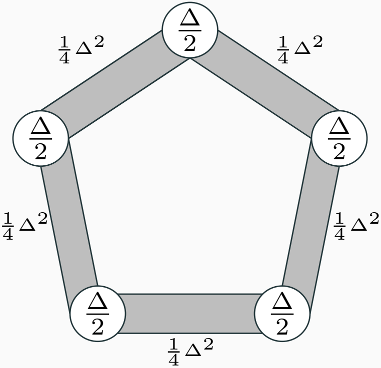

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Conjecture [Erdős, Nešetřil 1988]

The previous example is the worst you can get. In other words:

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The constant has been improved by Bruhn and Joos in 2015 to $\epsilon = 0.07$.

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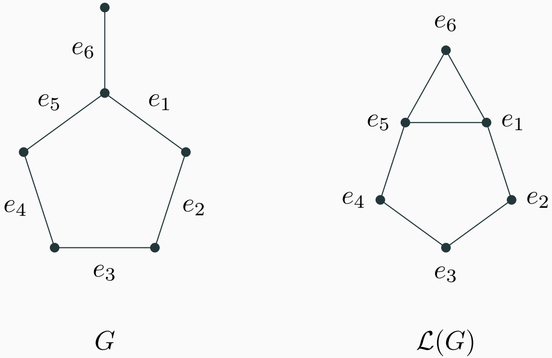

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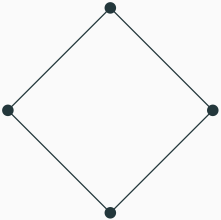

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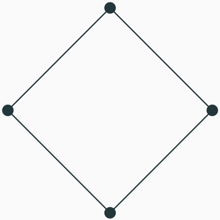

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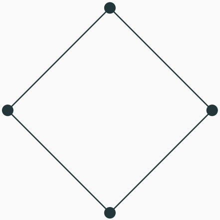

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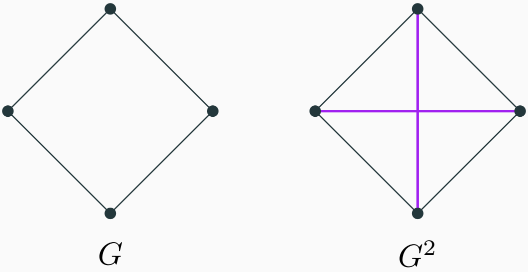

Strong coloring

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Molloy and Reed's theorem

Let G be the line-graph of any simple graph, then:

$$\chi(G^2) \leq (2-\epsilon)\omega(G)^2.$$

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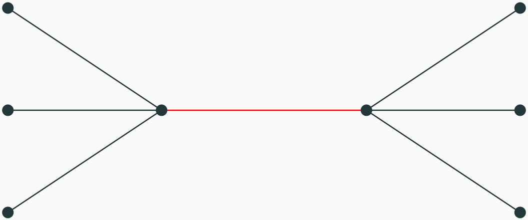

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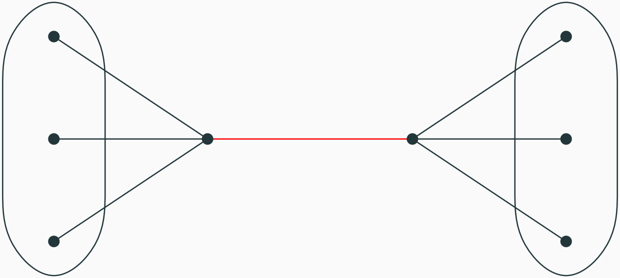

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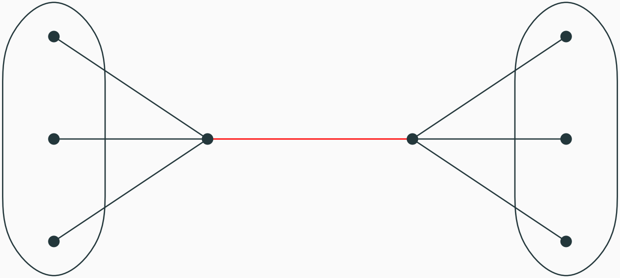

The class of graphs having this property is the class of **quasi-line** graphs.

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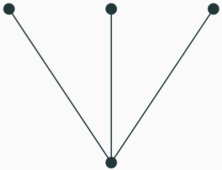

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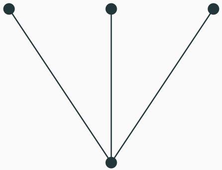

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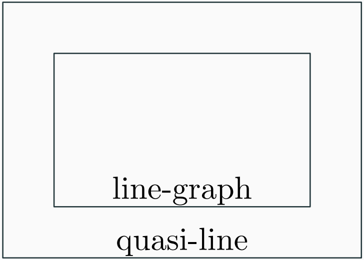
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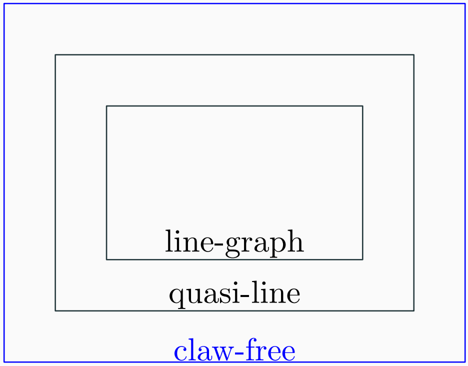
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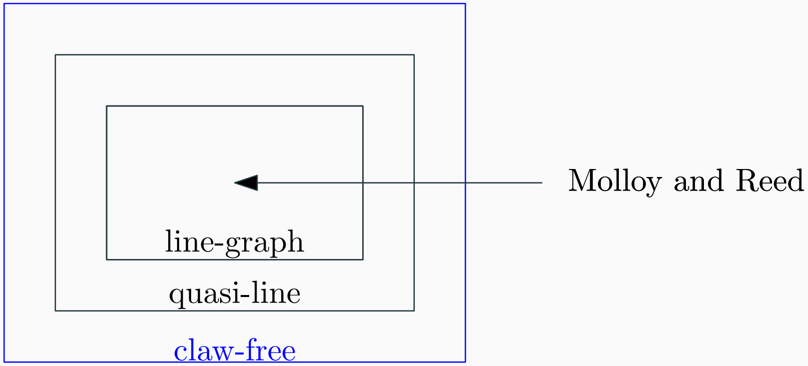
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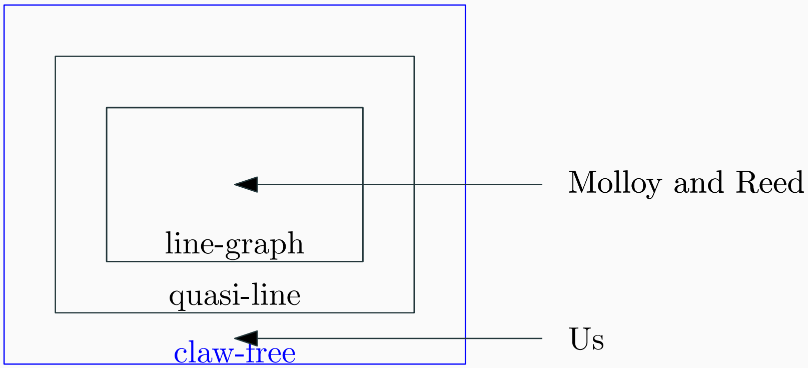
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Theorem [de Joannis de Verclos, Kang, P.]

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Roadmap

1. From claw-free to quasi-line graphs.

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1. From claw-free to quasi-line graphs.
2. From quasi-line graphs to line-graphs of multigraphs.

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1. From claw-free to quasi-line graphs.
2. From quasi-line graphs to line-graphs of multigraphs.
3. Prove the theorem for line-graphs of multigraphs.

Second neighborhood

The **second neighborhood** of v, denoted by $N_G^2(v)$, is the set of vertices at distance exactly two from v, i.e.
$N_G^2(v) = N_{G^2}(v) \setminus N_G(v)$.

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The **square degree** of v, denoted by $\deg_{G^2}(v)$, is equal to $\deg_G(v) + |N_G^2(v)|$.

Lemma

For G claw-free, either G is a quasi-line graph or there exist $v \in V(G)$ with $\deg_{G^2}(v) \leq \omega(G)^2 + (\omega(G)+1)/2$ **whose neighborhood is a clique of** $(G \setminus v)^2$.

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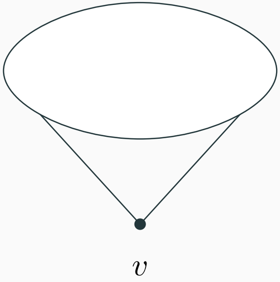
1. The proof is by induction on $|V(G)|$.
2. Note that $(G \setminus v)^2 \neq G^2 \setminus v$.


v



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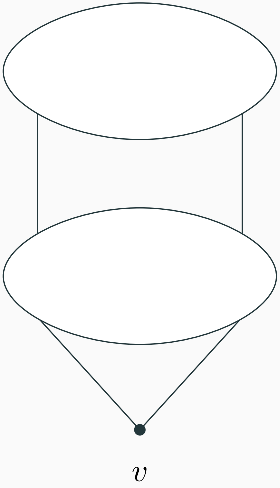
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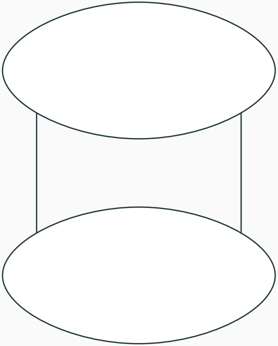

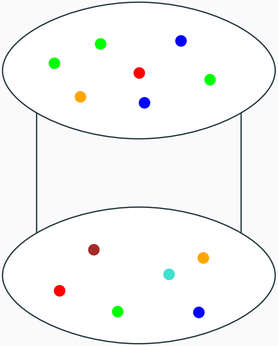

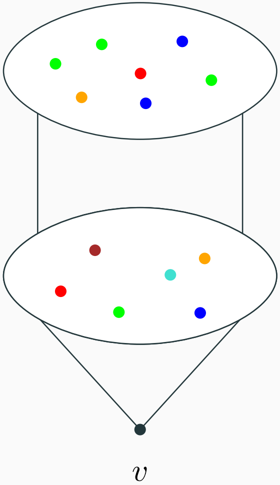
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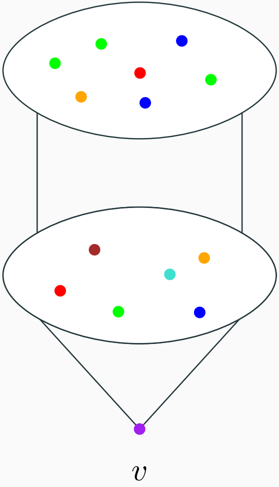
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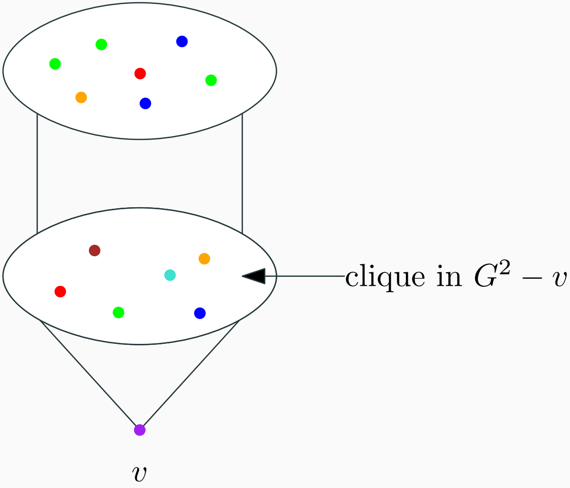
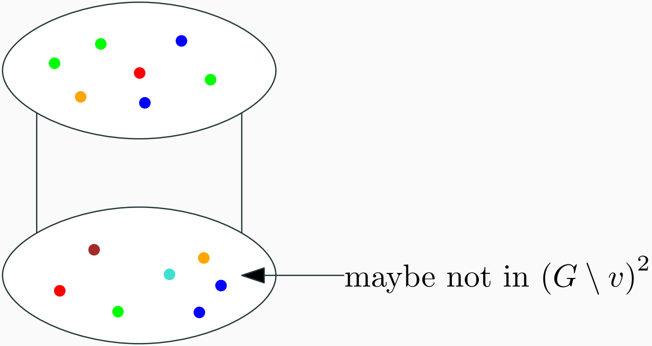

clique in $G^2 - v$
v


maybe not in $(G \setminus v)^2$

Lemma

For G claw-free, either G is a quasi-line graph or there exist $v \in V(G)$ with $\deg_{G^2}(v) \leq \omega(G)^2 + (\omega(G)+1)/2$ **whose neighborhood is a clique of** $(G \setminus v)^2$.

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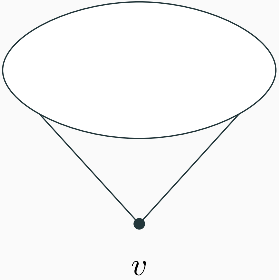

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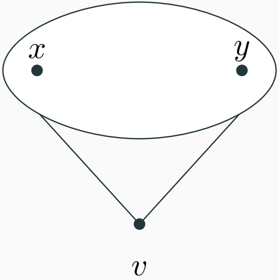

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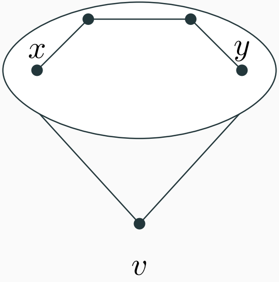

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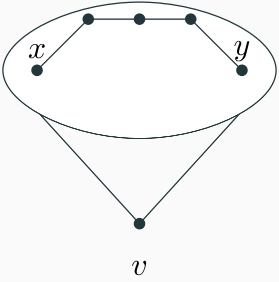

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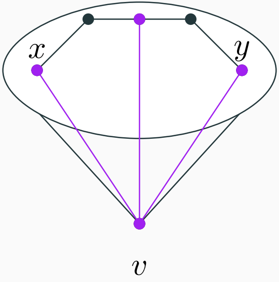

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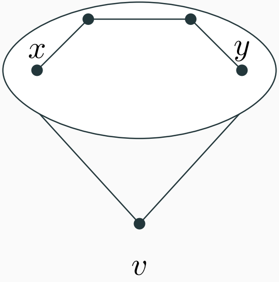

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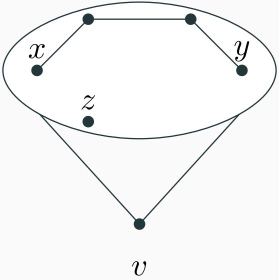

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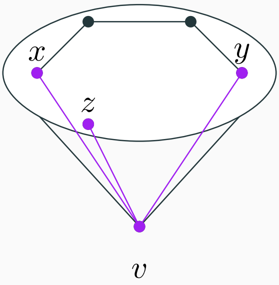

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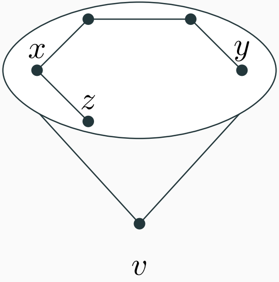

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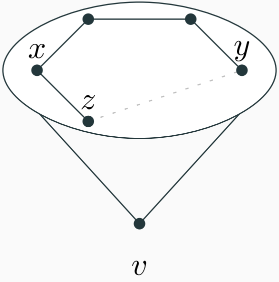

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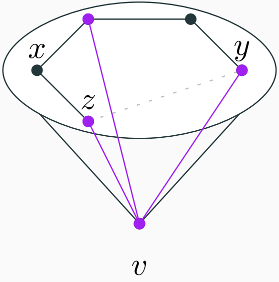

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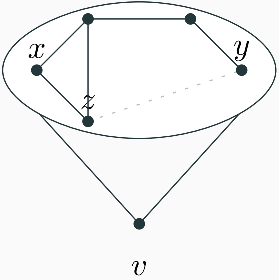

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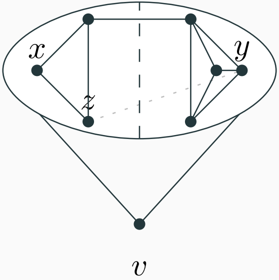

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- The conjecture for bipartite graphs is $\chi'_s(G) \leq \Delta(A)\Delta(B)$.

Thank you for your attention.