The coloring problem in graphs with structural restrictions

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Joint-work with Frédéric Maffray

G-SCOP

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k-coloring

For any integer k, a ***k*-coloring** of a graph G is a mapping $c : V(G) \rightarrow \{1, \ldots, k\}$ such that any two adjacent vertices u, v in G satisfies $c(u) \neq c(v)$.

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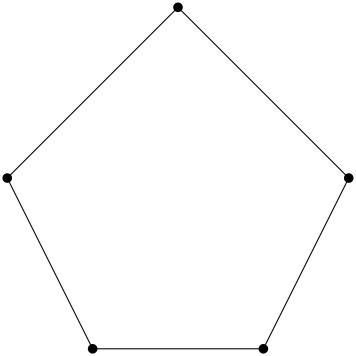
Chromatic number

The **chromatic number**, $\chi(G)$, of a graph G is the smallest integer k such that G is k-colorable.

Optimal coloring example



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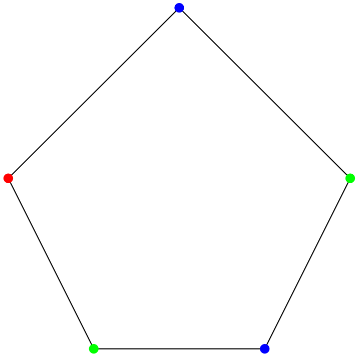

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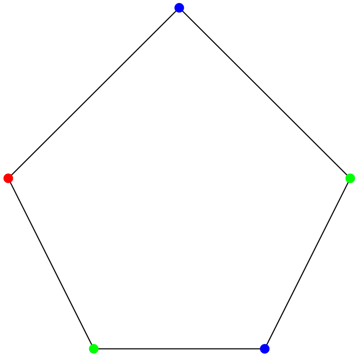

$$\chi(G) = 3$$

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- Deciding whether a graph G is k-colorable is ***NP*-complete** for each fixed $k \geq 3$ [*Stockmeyer 1973*].

Forbidden induced subgraph

A graph G is H-free if no induced subgraph of G is isomorphic to H.

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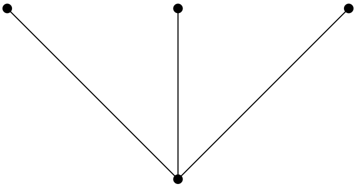
Without short cycles [*Kamiński, Lozin 2007*]

For any fixed $k, g \geq 3$, the k-coloring problem is NP-complete for the class of graphs with **girth at least g**.

girth: length of the shortest cycle.

Without claws [*Holyer 1981; Leven, Galil 1983*]

For any fixed $k \geq 3$, the k-coloring problem is NP-complete for the class of H-free graphs **where H contains a claw**.



claw

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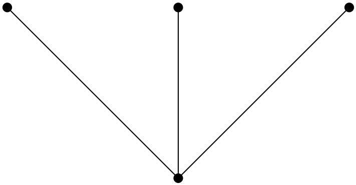

claw

Without cycles or claws

For any fixed $k \geq 3$ and H a forbidden induced subgraph that is **not a collection of paths**, deciding whether a graph is k-colorable is *NP*-complete

What is known?

k-coloring of P_ℓ-free graphs.

P_ℓ: induced path on ℓ vertices.

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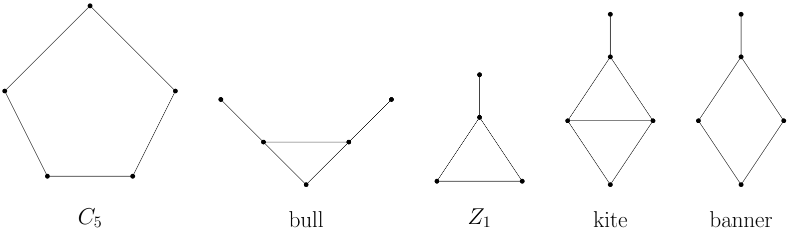
$\ell \backslash k$	1; 2	3	4	5	6	7	8	$\cdots$
≤ 4	*P*	*P*	*P*	*P*	*P*	*P*	*P*	*P*
5	*P*	*P*	*P*	*P*	*P*	*P*	*P*	*P*
6	*P*	*P*	**?**	*NPC*	*NPC*	*NPC*	*NPC*	*NPC*
7	*P*	*P*	*NPC*	*NPC*	*NPC*	*NPC*	*NPC*	*NPC*
8	*P*	**?**	*NPC*	*NPC*	*NPC*	*NPC*	*NPC*	*NPC*
$\cdots$	*P*	**?**	*NPC*	*NPC*	*NPC*	*NPC*	*NPC*	*NPC*

4-coloring polynomial-time algorithms in P_6-free graphs



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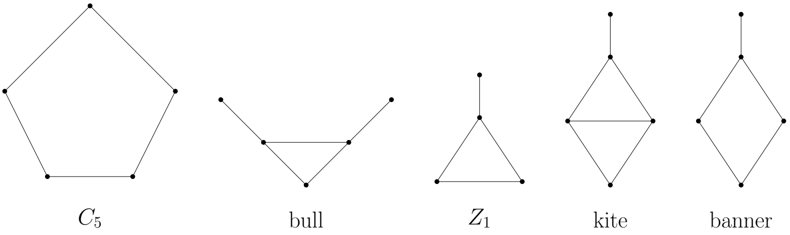
- (P_6, C_5)-free graphs [*Chudnovsky et al. 2014*].



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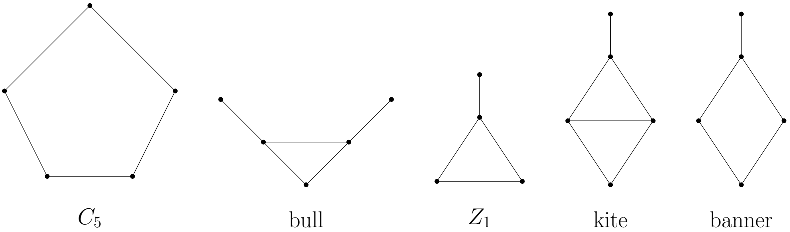

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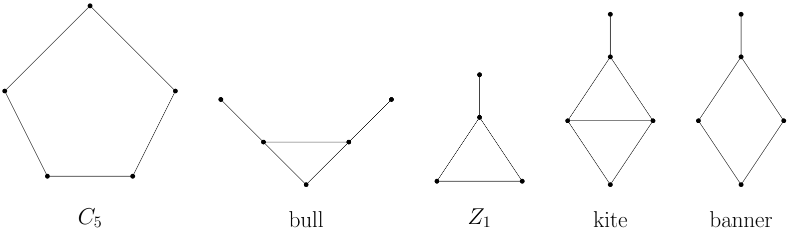
- (P_6, banner)-free graphs [*Huang 2016*].



Homogeneous set

A homogeneous set is a set $S \subseteq V(G)$ such that every vertex in $V(G) \setminus S$ is either complete to S or anticomplete to S.

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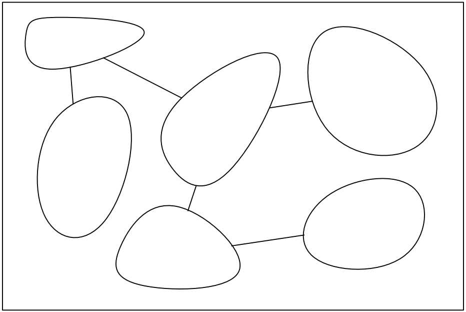

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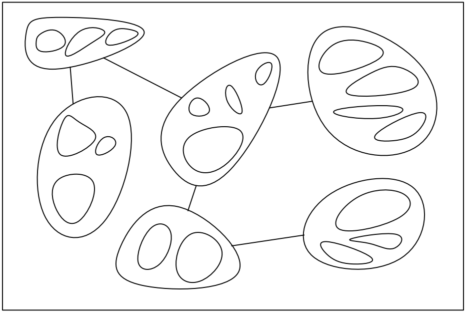

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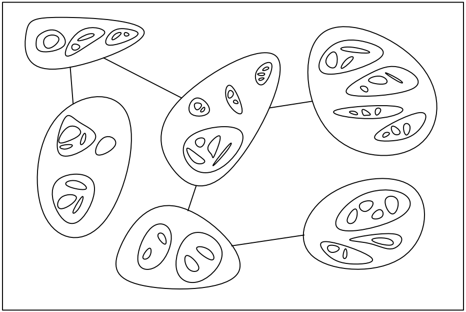

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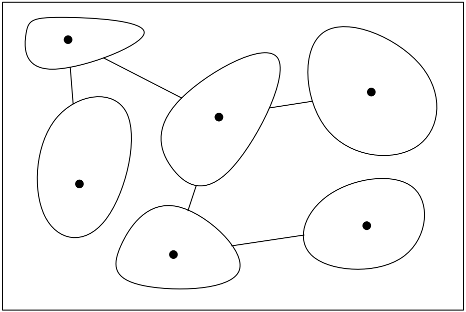

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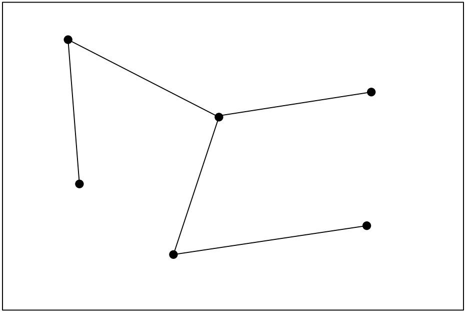

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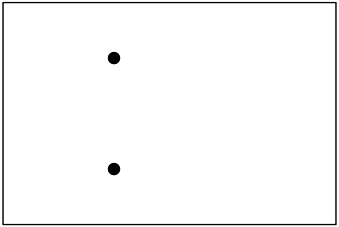

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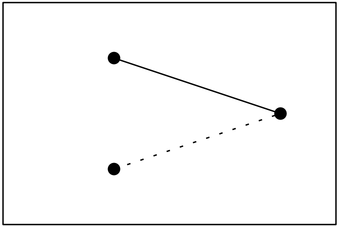

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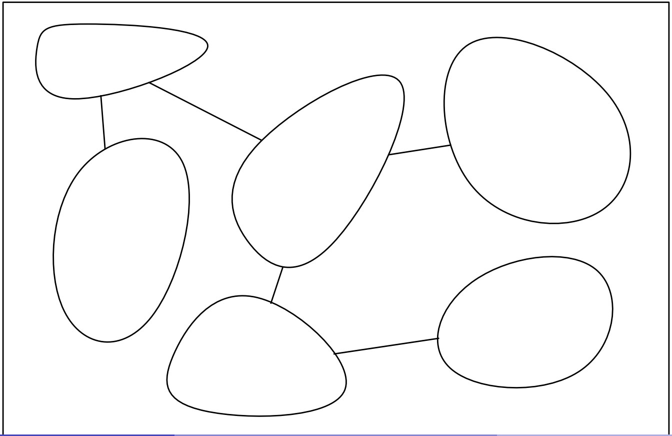

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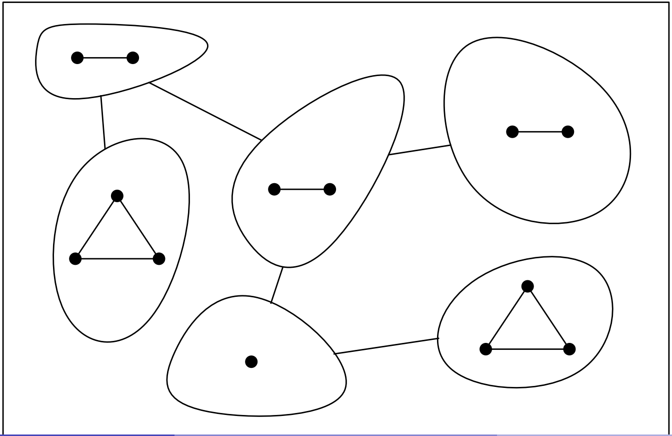

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Warning

We do not claim that K_5 and the double-wheel are the only 5-vertex-critical graphs!

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    - We need $G[V_1]$ and $G[V_2]$ to be 3-colorable. We use the known algorithms for that.
    - If they are 3-colorable, we can precisely determine their chromatic number by testing if they are bipartite or edgeless.

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Assume that G is not quasi-prime. So it has a homogeneous set X that is not a clique.

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Assume that G is not quasi-prime. So it has a homogeneous set X that is not a clique.

- A necessary condition for G to be 4-colorable is that $\chi(G[X]) \leq 3$.
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- Contract X into a clique of size $\chi(G[X])$.

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- Repeat this until we obtain a quasi-prime graph G'.

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- Contract X into a clique of size $\chi(G[X])$.
- Repeat this until we obtain a quasi-prime graph G'.
- We can show that G is 4-colorable if and only if G' is 4-colorable.

Quasi-prime

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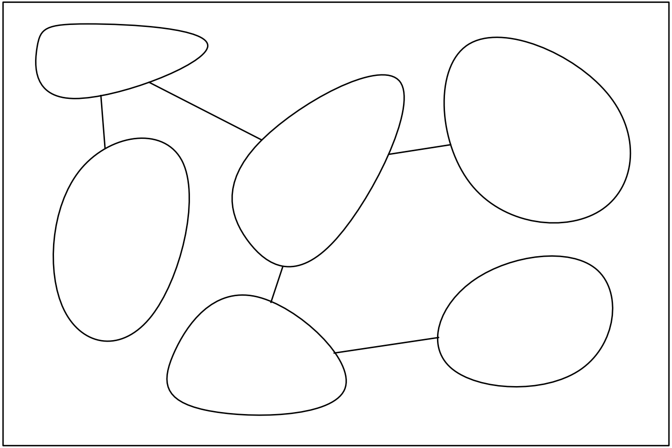

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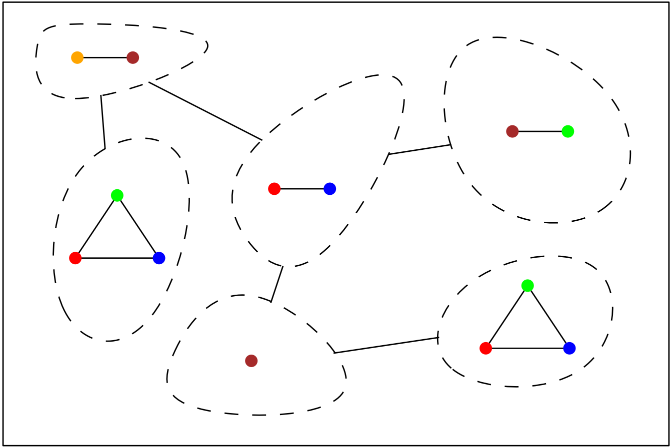
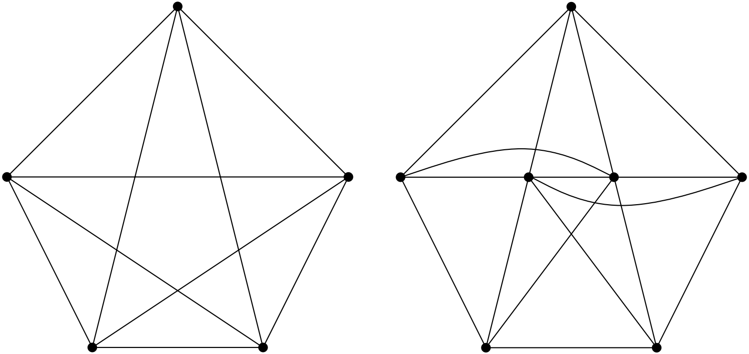


Proof of 3

It is easy to check that K_5 and the double-wheel are not 4-colorable.
Hence, if G contains a K_5 or the double-wheel it is not 4-colorable.



K_5 double-wheel

Structural Lemma

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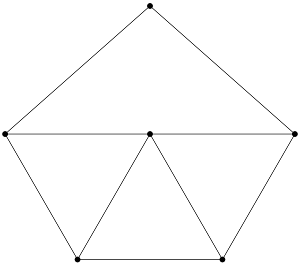
- G contains a magnet.



special graph

Structural Lemma

Let G be a quasi-prime bull-free graph that contains no K_5 and no double-wheel. Then at least one of the following holds:

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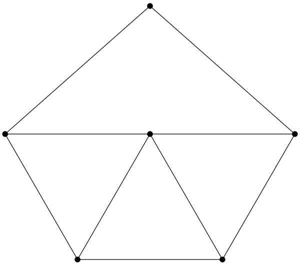
- G contains a gem and a special graph.



special graph

Structural Lemma

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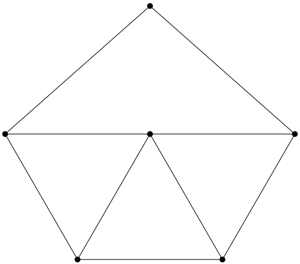
- G contains a magnet.
- G contains a gem and a special graph.
- G is gem-free.



special graph

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How to use it?

We can fix a coloring on F and use the 2-list-coloring algorithms to try extend it to the rest of the graph in polynomial time.

List-coloring problem

Every vertex v of the graph G has a list $L(v)$ of admissible colors. We want to find a proper coloring such that $c(v) \in L(v)$, for every vertex $v \in V(G)$.

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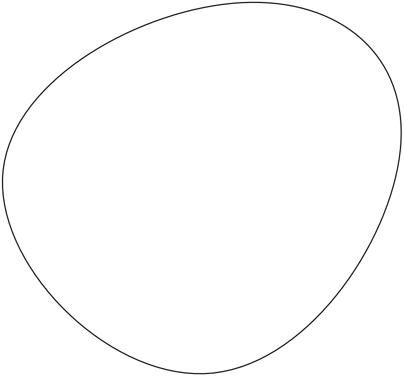
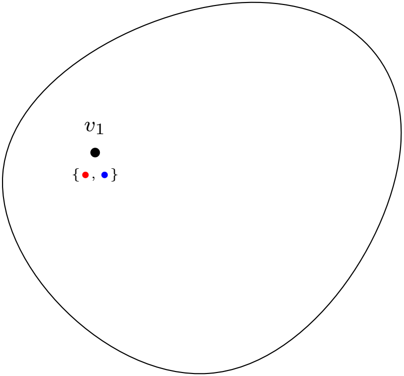
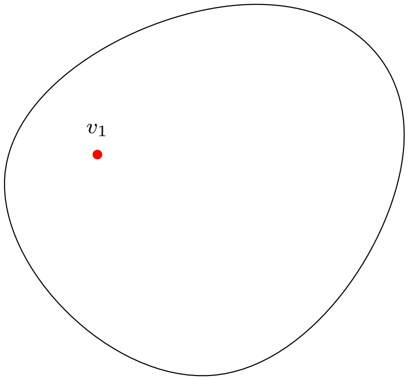
2-list coloring

List-coloring problem where all lists are of size 2. The 2-list coloring problem can be solved in polynomial time.




v_1
{•, •}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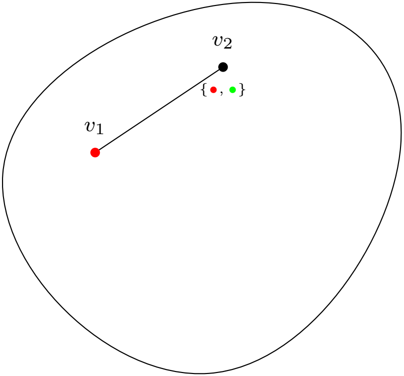
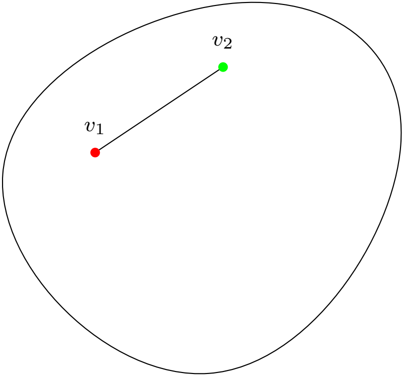
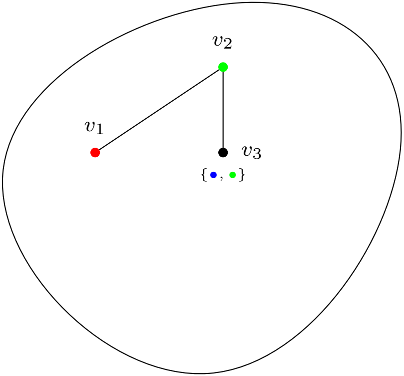
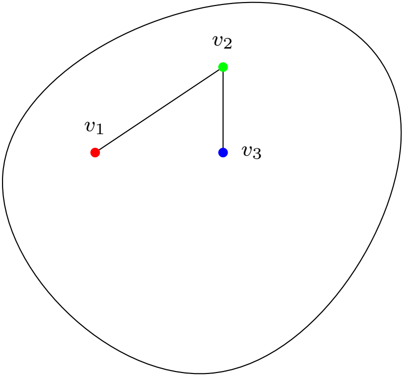


v_2
$\{\bullet, \bullet\}$
v_1


v_2
v_1



v_2
v_1
v_3
{ , }


v_2
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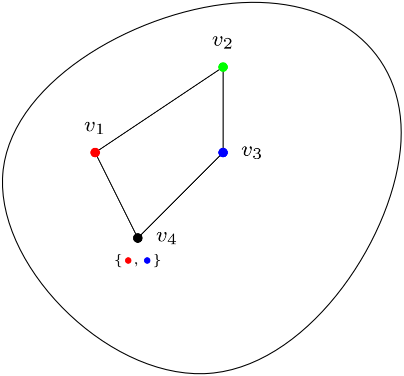
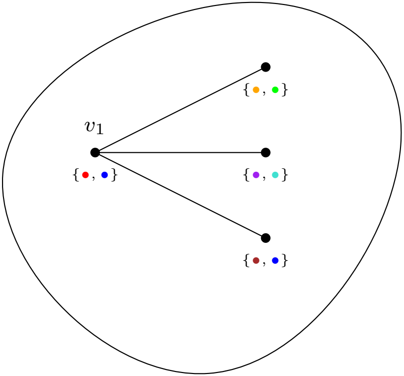

v_2
v_1
v_3
v_4
$\{ \bullet , \bullet \}$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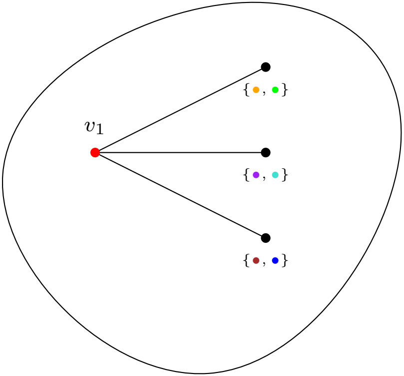
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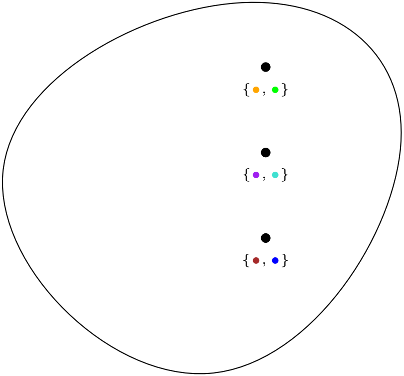
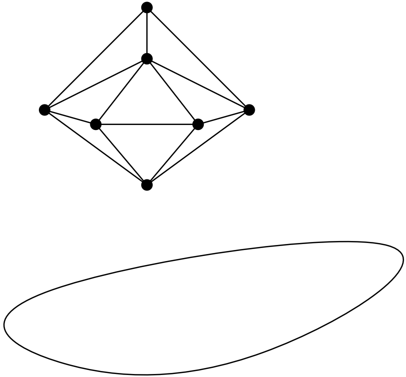


Magnet coloring



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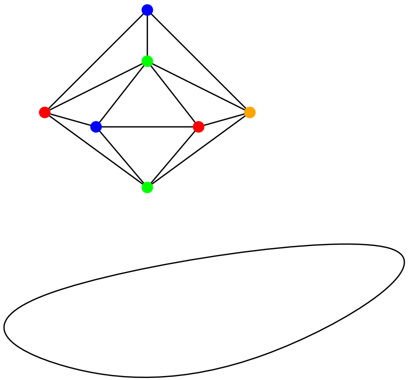

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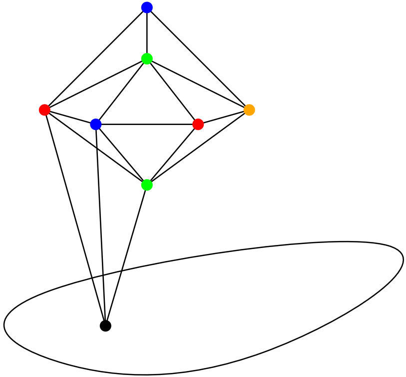

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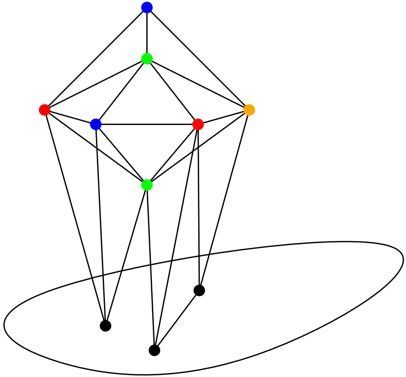

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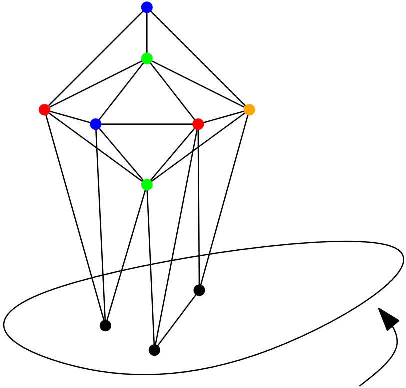
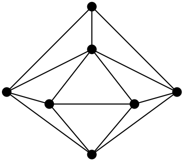
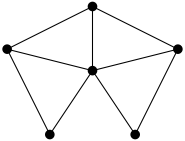


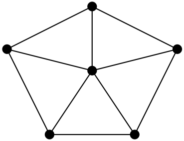
2-list-coloring problem



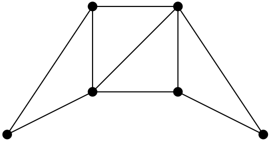
F_0



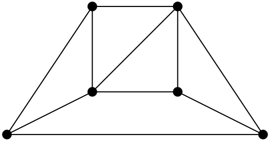
F_1



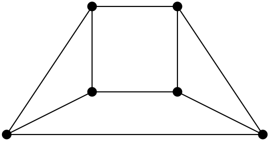
F_2



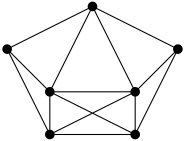
F_3



F_4



F_5



F_6

Theorem [*Maffray, P.*]

For any fixed k, there is a polynomial algorithm that determines if a (P_6, bull, gem)-free graph is k-colorable and if it is, produces a k-coloring.

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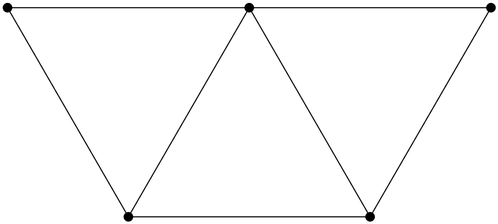

Perfect graph

A graph G is *perfect* if every induced subgraph H of G satisfies $\chi(H) = \omega(H)$.

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Strong Perfect Graph Theorem [*Chudnovsky, Robertson, Seymour and Thomas 2006*]

A graph is perfect if and only if it contains no C_ℓ and no $\overline{C_\ell}$ for any odd $\ell \geq 5$.

Coloring (P_6, bull, gem)-free graphs

Since G is P_6-free, it contains no C_ℓ with $\ell \geq 7$, and since it is gem-free, it contains no $\overline{C_\ell}$ with $\ell \geq 7$.

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- If G contains no C_5, then it is bull-free perfect.

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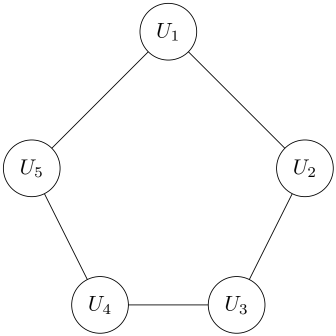
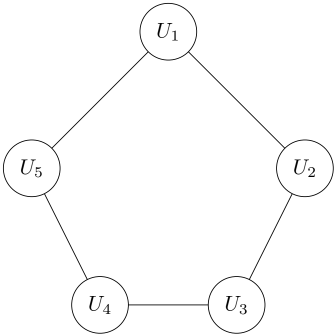
- If G contains no C_5, then it is bull-free perfect.
- If G contains a C_5, we prove that it is triangle-free.

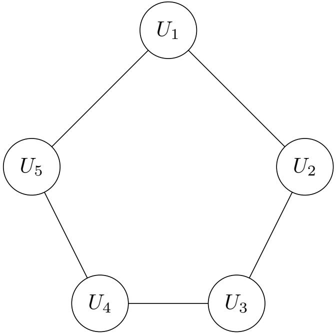

U_1
U_5
U_2
U_4
U_3

U_i is anticomplete to $U_{i-2} \cup U_{i+2}$



U_i is anticomplete to $U_{i-2} \cup U_{i+2}$

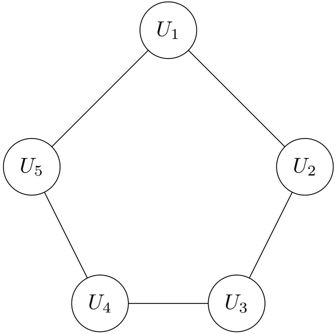
U_i contains a vertex that is complete to $U_{i-1} \cup U_{i+1}$



U_i is anticomplete to $U_{i-2} \cup U_{i+2}$

U_i contains a vertex that is complete to $U_{i-1} \cup U_{i+1}$

Each of $U_1, \ldots, U_5$ is a stable set



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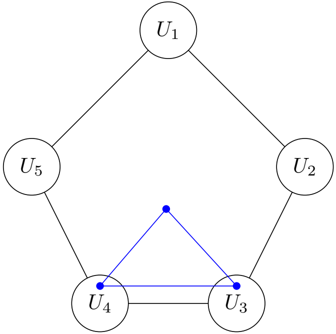
There is no blue triangle



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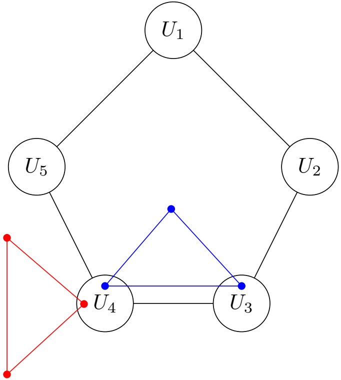

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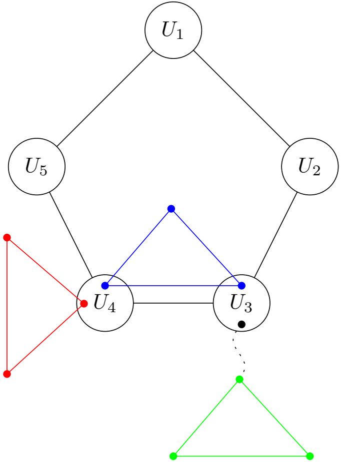

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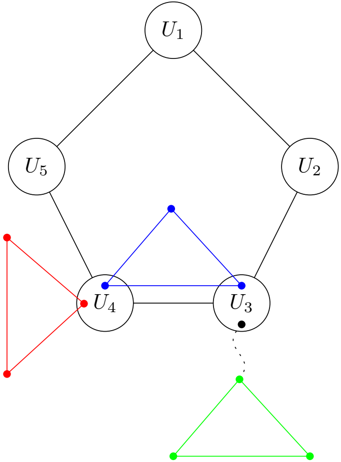
Hence, there is no triangle



Theorem [*Brandstädt et al. 2006*]

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For any fixed k, k-coloring a (P_6, K_3)-free graph is polynomial solvable.

Lemma

Let G be a prime (P_6, bull, gem)-free graph that contains a 5-hole. Then G is triangle-free.

[*Maffray, P.*]

There is a polynomial time algorithm that determines whether a (P_6, bull)-free graphs is 4-colorable, and if it is, produces a 4-coloring.

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Structural Lemma

Let G be a quasi-prime bull-free graph that contains no K_5 and no double-wheel. Then at least one of the following holds:

- G contains a magnet.
- G contains a gem and a special graph.
- G is gem-free.

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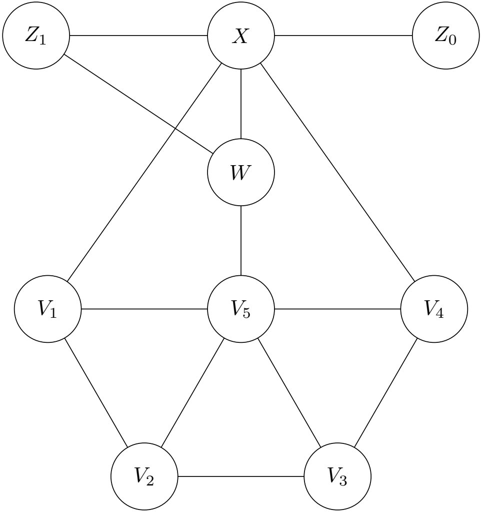
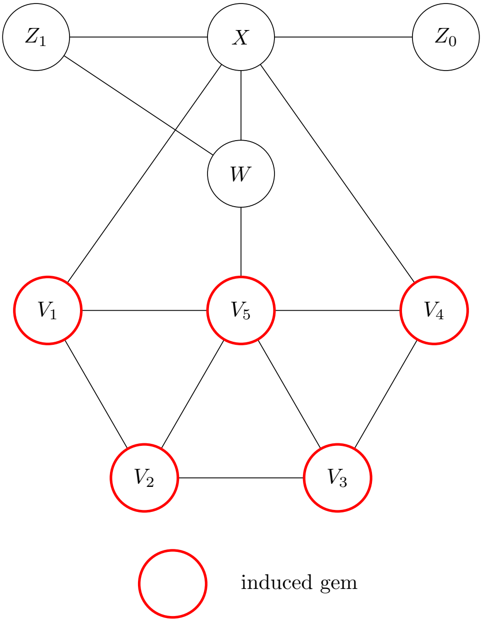
Structure

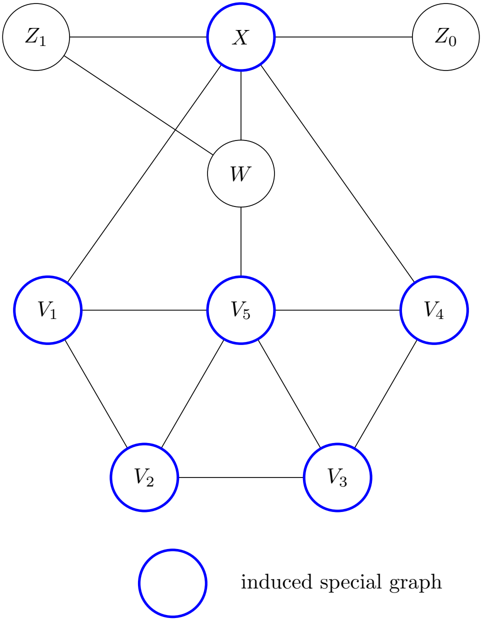
By the structural Lemma and further examining the structure we can partition the graph into sets as in the next drawing.

How to use this structure?

We will see that we can use this structure to color all the graph. In fact we color a fixed number of vertices and try to extend it.


Z_1
X
Z_0
W
V_1
V_5
V_4
V_2
V_3


Z_1
X
Z_0
W
V_1
V_5
V_4
V_2
V_3
induced gem


Z_1
X
Z_0
W
V_1
V_5
V_4
V_2
V_3
induced special graph

How to color

The main idea is to precolor a set P of vertices of bounded size (at most 8), more precisely:

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- After precoloring P, every vertex v in $V(G) \setminus P$ has a list of available colors $L(v)$ which consist of the set $\{1, 2, 3, 4\}$ minus the colors given by f to the neighbors of v in P.

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- We try every 4-coloring f of P and check if we can extend it to $G \setminus P$.
- After precoloring P, every vertex v in $V(G) \setminus P$ has a list of available colors $L(v)$ which consist of the set $\{1, 2, 3, 4\}$ minus the colors given by f to the neighbors of v in P.
- We solve the associated L-coloring problem on $G \setminus P$ or determine that is has no solution.

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- Since we choose a set P of fixed size we can try all the possible 4-colorings of it.

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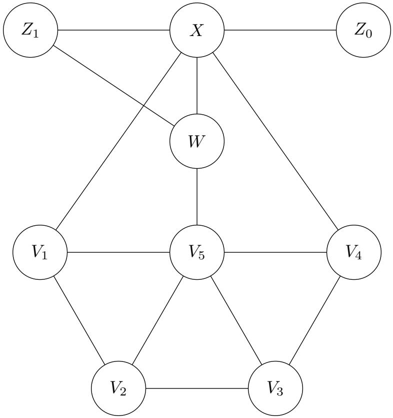
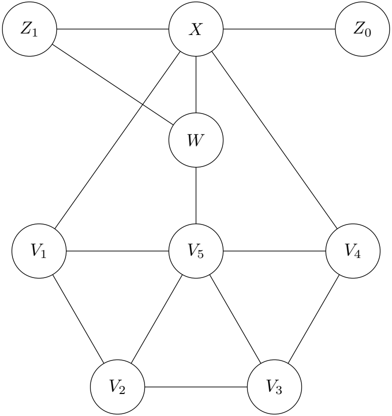
- Since we choose a set P of fixed size we can try all the possible 4-colorings of it.
- Thanks to the structure, we can use polynomial time algorithms to color the rest of the graph.

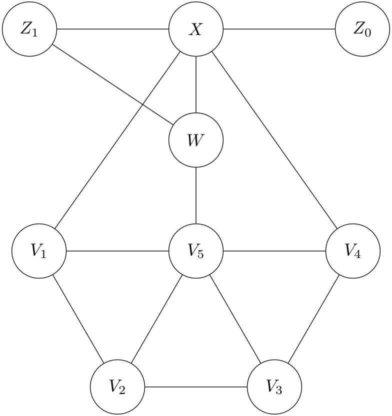

Z_1
X
Z_0
W
V_1
V_5
V_4
V_2
V_3

X is not empty



X is not empty

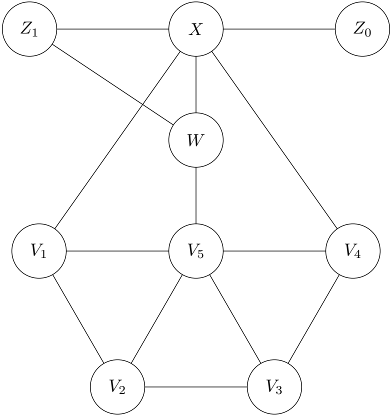
V_5 is complete to $V_1 \cup \ldots \cup V_4$



X is not empty

V_5 is complete to $V_1 \cup \ldots \cup V_4$

W is complete to X and anticomplete to $V_1 \cup \ldots \cup V_4$



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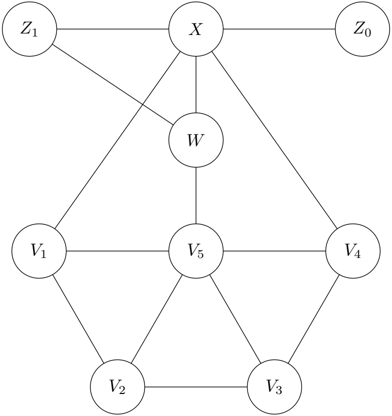
Z_1 is complete to X



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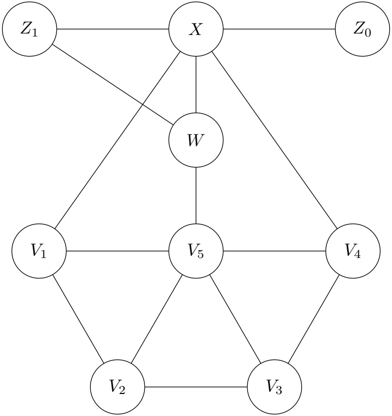
Any 4-coloring of $G \setminus Z_0$ extend to a 4-coloring of G



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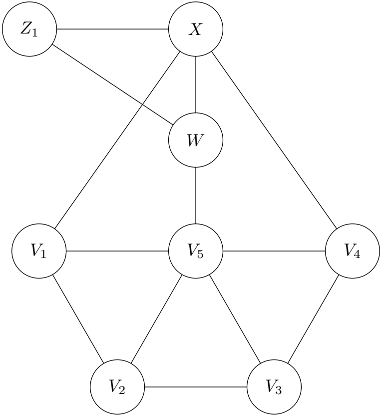

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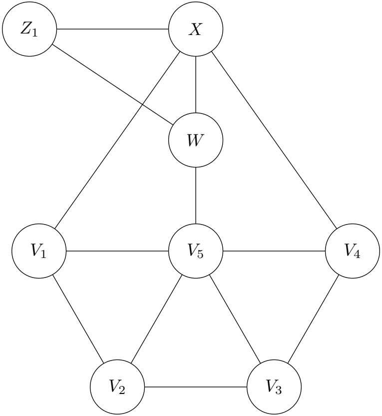

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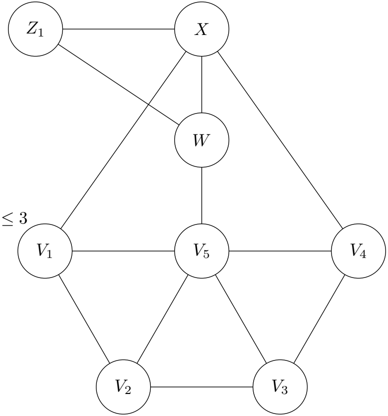

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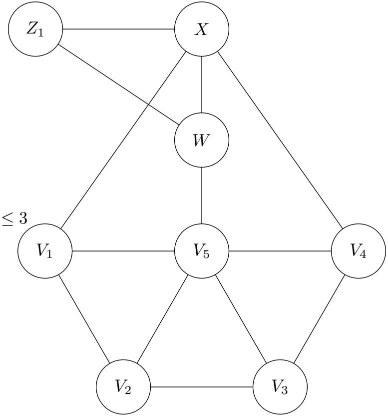
If $|X| \geq 2$, let $P = \{v_1, v_2, v_3, v_4, v_5\} \cup X$



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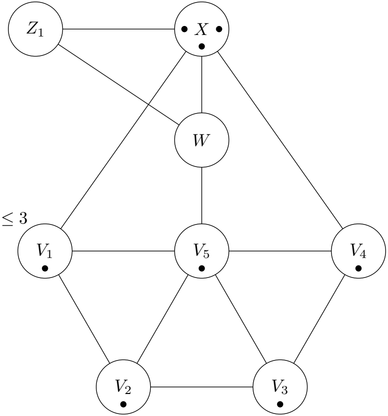

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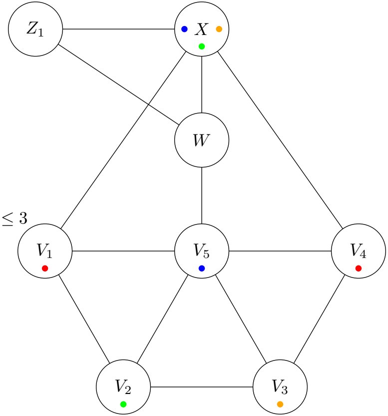

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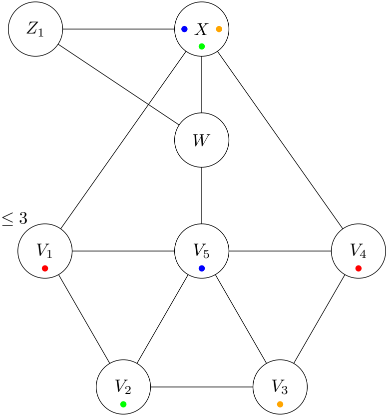
Every vertex v in $V(G) \setminus P$ satisfies $|L(v)| \leq 2$



Explore all the cases

There a few more cases to treat, but the idea is the same. In the most complicated ones you need to further examine the structure of some sets and use the 3-coloring algorithm on a subgraph of G.

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Output a 4-coloring

The algorithm gives a 4-coloring in polynomial time or stops if no such coloring exists.

An interesting question

- Is there a finite family of 5-vertex-critical (P_6, bull)-free graphs?

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Conjecture

The 4-coloring problem can be solved in polynomial time for P_6-free graphs [*Huang 2013*].

Thank you for listening.