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Screw theory (torsor theory)
Vector and pseudo-vector representations, twist, wrench

Gilles Leborgne

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A screw (also called a torsor) is an affine antisymmetric vector field in a Euclidean setting. It is called a twist (or a kinematic screw, or a distributor) when it is the velocity field of a rigid body motion, and called a wrench when it is the moment of a force field.

To avoid confusions and misunderstandings, the first three paragraphs are devoted to the definitions of vectors, pseudo-vectors, vector products, pseudo-vector products, antisymmetric endomorphisms and their representations. The fourth fifth and sixth paragraphs define a screw, a twist and a wrench.

Contents

The notation $g:=f$ means: f being given, g is defined by $g=f$.
V is a dimension 3 vector space.

1 Dimension 3 vector spaces

1.1 The different $\overrightarrow{\mathbb{R}^3}$ in mechanics

1.1.1 Cartesian $\overrightarrow{\mathbb{R}^3}$

$\mathbb{R} := (\mathbb{R}, +, \times)$ is the usual field, with 0 the + identity element and 1 the $\times$ identity element; This 1 is theoretical: It is not linked to any "unit of measurement".

Then consider the Cartesian product $\mathbb{R}\times\mathbb{R}\times\mathbb{R} \ =^{\text{noted}}\ \overrightarrow{\mathbb{R}^3}$, and the usual operations $\vec u+\vec v = (u_1{+}v_1, u_2{+}v_2, u_3{+}v_3)$ and $\lambda.\vec u = (\lambda u_1, \lambda u_2, \lambda u_3) \ =^{\text{noted}}\ \lambda\vec u$ when $\vec u = (u_1,u_2,u_3)$, $\vec v = (v_1,v_2,v_3)$ and $\lambda\in\mathbb{R}$. It is a real vector space, and $(\vec E_1{:=}(1,0,0), \vec E_2{:=}(0,1,0), \vec E_3{:=}(0,0,1))$ is a basis called "the canonical basis".

1.1.2 $\mathcal{M}_{31}$ the space of real $3*1$ column matrices

$\mathcal{M}_{31} = \{[\vec v] = \begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix} : v_1,v_2,v_3\in\mathbb{R}\}$ is the usual set of real $3*1$ column matrices. It is a real vector space with its usual rules, $(\vec C_1{:=}\begin{pmatrix}1\\0\\0\end{pmatrix}, \vec C_2{:=}\begin{pmatrix}0\\1\\0\end{pmatrix}, \vec C_3{:=}\begin{pmatrix}0\\0\\1\end{pmatrix}) \ =^{\text{noted}}\ (\vec C_i)$ being its canonical basis (the identity element 1 is theoretical: It is not linked to any "unit of measurement"). So $[\vec v] = \begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}$ means $\vec v = \sum_i v_i\vec C_i$. And $\mathcal{M}_{31}$ is isomorphic to $\overrightarrow{\mathbb{R}^3}$ Cartesian.

Similarly with transposed matrices and $\mathcal{M}_{13} = \{[\vec v]^T : [\vec v]\in\mathcal{M}_{31}\}$ the set of row matrices.

Definition 1.1 A column matrix $[\vec v]\in\mathcal{M}_{31}$ is also called a pseudo-vector.

1.1.3 The many $V = \overrightarrow{\mathbb{R}^3}$ in mechanics

For a sum to be defined, we need "compatible dimensions" : You don't add bi-point vectors velocities with accelerations or forces or moments... Thus we define distinct real vector spaces corresponding to different dimensions: V_{bpv} for bi-point vectors, V_{vel} for the velocities, V_{acc} for accelerations, ... However, to simplify the notations, all these spaces are noted $\overrightarrow{\mathbb{R}^3}$. So pay attention to the context.

And, e.g. in $V_{bpv} \ =^{\text{noted}}\ \overrightarrow{\mathbb{R}^3}$, there is no canonical basis: a basis $(\vec a_1, \vec a_2, \vec a_3) = (\vec a_i)_{i=1,2,3} \ =^{\text{noted}}\ (\vec a_i)$ is chosen by some observer, e.g. with $\vec a_3$ giving the direction of the vertical at some point on Earth and with its length being 1 is some unit of measurement (e.g. 1 foot in aviation).

1.1.4 Quantification in V

V being a dimension 3 real vector space, let $\vec v\in V$.

Quantification. An observer chooses a basis $(\vec a_i)$ in V. Hence $\exists v_1,v_2,v_3\in\mathbb{R}$ s.t. $\vec v = \sum_{i=1}^3 v_i\vec a_i$, and the column matrix $[\vec v]_{|\vec a}{=}\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}\in\mathcal{M}_{31}$ is the usual matrix representation of $\vec v$ which quantifies $\vec v$ in the basis $(\vec a_i)$. (And, $[\vec v]_{|\vec a} = \begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix}$ means $\vec v = \sum_{i=1}^3 v_i\vec a_i$.)

Let $\mathcal{M}_{33}$ will be the space of $3*3$ real matrices.

Let $z(\cdot,\cdot) : V\times V\to\mathbb{R}$ be a bilinear form (e.g. a scalar dot product). Quantification: Let $[z]_{\vec a} := [z(\vec a_i,\vec a_j)]_{\substack{i=1,2,3\\ j=1,2,3}} \ =^{\text{noted}}\ [z(\vec a_i,\vec a_j)]\in\mathcal{M}_{33}$; This 3*3 matrix $[z]_{\vec a}$ is the usual matrix representation (quantification) of $z(\cdot,\cdot)$ relative to $(\vec a_i)$. So, for all $\vec v = \sum_{i=1}^3 v_i\vec a_i$ and $\vec w = \sum_{i=1}^3 w_i\vec a_i$ in V, the bilinearity of $z(\cdot,\cdot)$ gives $z(\vec v,\vec w) = \sum_{i,j=1}^3 v_iw_jz(\vec a_i,\vec a_j) = [\vec v]_{\vec a}^T.[z]_{\vec a}.[\vec w]_{\vec a}$ where $[\vec v]_{\vec a}^T := ([\vec v]_{|\vec a})^T$ (transposed matrix).

Let $L : V\to V$ be an endomorphism (linear map from a vector space to itself). Quantification: Let L_{ij} be the components of $L.\vec a_j$, i.e. $L.\vec a_j = \sum_{i=1}^3 L_{ij}\vec a_i$, for all j; The $3*3$ matrix $[L]_{\vec a} := [L_{ij}]\in\mathcal{M}_{33}$ is the usual representation of L relative to $(\vec a_i)$. So, with $\vec v = \sum_{i=1}^3 v_i\vec a_i$, the linearity of L gives $L.\vec v = \sum_{i,j=1}^3 L_{ij}v_j\vec a_i$, i.e. $[L.\vec v]_{|\vec a} = [L]_{|\vec a}.[\vec v]_{|\vec a}$.

1.1.5 Our usual affine space $\mathbb{R}^3$ and associated $\overrightarrow{\mathbb{R}^3}$

Affine setting: $\mathbb{R}^3$ is the usual affine space of points representing positions of particles in our classical 3-D world.

Associated setting vector space: $\overrightarrow{\mathbb{R}^3}$ is its associated vector space made of the bi-point vectors $\overrightarrow{AB} =^{\text{noted}} B-A$ for all $A, B \in \mathbb{R}^3$, and we write $B = A + \overrightarrow{AB}$.

1.1.6 Euclidean framework

Choose a unit of measure of length u in our affine space $\mathbb{R}^3$ (foot, metre...), then make a Euclidean associated basis $(\vec{e}_i)_{i=1,2,3} =^{\text{noted}} (\vec{e}_i)$ in $\overrightarrow{\mathbb{R}^3}$: The length of each $\vec{e}_i$ is 1 in the unit u, and the length of $3\vec{e}_i + 4\vec{e}_{i+1}$ is 5 (Pythagoras orthogonality) in the unit u, for all $i = 1, 2, 3$, where $\vec{e}_4 := \vec{e}_1$ and $\vec{e}_5 := \vec{e}_2$.

The associated Euclidean dot product $g_e(\cdot,\cdot) = (\cdot,\cdot)_{g_e} =^{\text{noted}} . \bullet_{g_e} . : \overrightarrow{\mathbb{R}^3} \times \overrightarrow{\mathbb{R}^3} \to \mathbb{R}$ (symmetric definite positive bilinear form) is defined by $(\vec{e}_i, \vec{e}_j)_{g_e} = \delta_{ij}$ for all i, j, i.e. $[g_e]_{|\vec{e}} = I$, so, for all $\vec{v} = \sum_{i=1}^3 v_i \vec{e}_i$ and $\vec{w} = \sum_{i=1}^3 w_i \vec{e}_i$,

$$\vec{v} \bullet_{g_e} \vec{w} := [\vec{v}]_{|\vec{e}}{}^T . [\vec{w}]_{|\vec{e}} = \sum_{i,j=1}^{3} v_i w_i \quad \text{(Euclidean case).} \tag{1.1}$$

The associated Euclidean norm $||.||_{g_e} : \overrightarrow{\mathbb{R}^3} \to \mathbb{R}_+$ is given by $||\vec{v}||_{g_e} := \sqrt{\vec{v} \bullet_{g_e} \vec{v}}$ $(= \sum_{i,j=1}^3 v_i^2)$.

Two vectors $\vec{v}, \vec{w} \in \overrightarrow{\mathbb{R}^3}$ are $(\cdot,\cdot)_{g_e}$-orthogonal iff $\vec{v} \bullet_{g_e} \vec{w} = 0$.

The algebraic (signed) volume of the parallelepiped limited by three vectors $\vec{u}, \vec{v}, \vec{w}$ is $\det_{\vec{e}}(\vec{u}, \vec{v}, \vec{w})$ (and the volume is the absolute value $|\det_{\vec{e}}(\vec{u}, \vec{v}, \vec{w})|$) where $\det_{\vec{e}} : (\overrightarrow{\mathbb{R}^3})^3 \to \mathbb{R}$ is the tri-linear alternated form defined by $\det_{\vec{e}}(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1$. That is, for all $\vec{u} = \sum_{i=1}^3 u_i \vec{e}_i$, $\vec{v} = \sum_{i=1}^3 v_i \vec{e}_i$, $\vec{w} = \sum_{i=1}^3 w_i \vec{e}_i$ in V,

$$\det_{\vec{e}}(\vec{u}, \vec{v}, \vec{w}) = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_2 w_1), \tag{1.2}$$

i.e. $\det_{\vec{e}}(\vec{u}, \vec{v}, \vec{w}) = \det\big([\vec{u}]_{|\vec{e}} \quad [\vec{v}]_{|\vec{e}} \quad [\vec{w}]_{|\vec{e}} \big)$ = the determinant of a $3*3$ matrix $\mathcal{M} = \big([\vec{u}]_{|\vec{e}} \quad [\vec{v}]_{|\vec{e}} \quad [\vec{w}]_{|\vec{e}} \big)$.

A $(\cdot,\cdot)_{g_e}$-orthonormal basis is a basis $(\vec{b}_i)$ s.t. $(\vec{b}_i, \vec{b}_j)_{g_e} = \delta_{ij}$, i.e. $\vec{b}_i \bullet_{g_e} \vec{b}_j = \delta_{ij}$ for all i, j, i.e. $[g_e]_{|\vec{b}} = I$.

A basis $(\vec{b}_i)$ as the same orientation as $(\vec{e}_i)$ iff $\det_{\vec{e}}(\vec{b}_1, \vec{b}_2, \vec{b}_3) > 0$. Otherwise it as the opposite orientation.

1.2 The vector product associated with a basis

Framework: $\overrightarrow{\mathbb{R}^3}$ Euclidean with $(\vec{e}_i)$ a chosen Euclidean basis, $(\cdot,\cdot)_{g_e}$ the associated Euclidean dot product and $\det_{\vec{e}}$ the associated algebraic volume.

Definition 1.2 The vector product $\times_e(\cdot,\cdot) : \left\{ \begin{array}{ll} V \times V & \to V \\ (\vec{u}, \vec{v}) & \to \times_e(\vec{u}, \vec{v}) \overset{\text{noted}}{=} \vec{u} \times_e \vec{v} \end{array} \right\}$ is the bilinear antisymmetric map defined by

$$(\vec{u} \times_e \vec{v}) \bullet_{g_e} \vec{w} = \det_{\vec{e}}(\vec{u}, \vec{v}, \vec{w}), \quad \forall \vec{w} \in \overrightarrow{\mathbb{R}^3}. \tag{1.3}$$

So the components of $\vec{u} \times_e \vec{v}$ in the basis $(\vec{e}_i)$ are the reals $(\vec{u} \times_e \vec{v}) \bullet_{g_e} \vec{e}_i = \det_{\vec{e}}(\vec{u}, \vec{v}, \vec{e}_i)$ for $i = 1, 2, 3$:

$$\begin{gathered}
[\vec{u} \times_e \vec{v}]_{|\vec{e}} := \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}, \quad \text{i.e.} \\
\vec{u} \times_e \vec{v} := (u_2 v_3 - u_3 v_2)\vec{e}_1 + (u_3 v_1 - u_1 v_3)\vec{e}_2 + (u_1 v_2 - u_2 v_1)\vec{e}_3 \overset{\text{noted}}{=} \det\left(\begin{pmatrix} \vec{e}_1 & u_1 & v_1 \\ \vec{e}_2 & u_2 & v_2 \\ \vec{e}_3 & u_3 & v_3 \end{pmatrix} \right),
\end{gathered} \tag{1.4}$$

the formal determinant being expanded along the first column. So $\times_e$ is indeed bilinear, easy check, and antisymmetric, i.e. $\vec{u} \times_e \vec{v} = -\vec{v} \times_e \vec{u}$, easy check.

In other words, $\times_e$ is the bilinear antisymmetric map defined by

$$\forall i = 1, 2, 3, \quad \vec{e}_i \times_e \vec{e}_{i+1} = \vec{e}_{i+2}, \tag{1.5}$$

where $\vec{e}_4 := \vec{e}_1$ and $\vec{e}_5 := \vec{e}_2$.

Proposition 1.3 *For all $\vec{u},\vec{v}\in V$.*
1- If $\vec{u}\times_e\vec{z}=\vec{v}\times_e\vec{z}$ for all $\vec{z}$, then $\vec{u}=\vec{v}$.
2- $\vec{u}\times_e\vec{v}$ is $(\cdot,\cdot)_{g_e}$-orthogonal to $\vec{u}$ and to $\vec{v}$.
3- If $\vec{u}$ is parallel to $\vec{v}$ then $\vec{u}\times_e\vec{v}=0$. 3'- If $\vec{u}\times_e\vec{v}\neq 0$ then $\vec{u}$ is not parallel to $\vec{v}$.
4- If $\vec{u}$ is not parallel to $\vec{v}$ then $\vec{u}\times_e\vec{v}\neq 0$. 4'- If $\vec{u}\times_e\vec{v}=0$ then $\vec{u}$ is parallel to $\vec{v}$.
5- If $\vec{u}$ is not parallel to $\vec{v}$ then the basis $(\vec{u},\vec{v},\vec{u}\times_e\vec{v})$ has the same orientation than $(\vec{e}_i)$.
6- $||\vec{u}\times_e\vec{v}||_{g_e}$ is the area or the parallelogram $(\vec{u},\vec{v})$ (in the unit chosen for $(\vec{e}_i)$).

Proof. 1- (1.4) give $[\vec{u}\times_e\vec{e}_1]_{|\vec{e}}=\begin{pmatrix}0\\u_3\\-u_2\end{pmatrix}$, similarly $[\vec{v}\times_e\vec{e}_1]_{|\vec{e}}=\begin{pmatrix}0\\v_3\\-v_2\end{pmatrix}$, thus $u_3=w_3$ and $u_2=v_2$. Similarly with $\vec{e}_2$ wich gives $u_1=v_1$.

2- $(\vec{u}\times_e\vec{v})\bullet_{g_e}\vec{u}=\det_{\vec{e}}(\vec{u},\vec{v},\vec{u})=0$ since $\det_{\vec{e}}$ is alternated, similarly $(\vec{u}\times_e\vec{v})\bullet_{g_e}\vec{v}=0$.

3- Trivial with (1.4). 3'- Contraposition.

4- If $\vec{u}$ is not parallel to $\vec{v}$ then let $\vec{z}\in V$ s.t. $(\vec{u},\vec{v},\vec{z})$ is a basis; Hence, $\det_{\vec{e}}(\vec{u},\vec{v},\vec{z})\neq 0$, thus $(\vec{u}\times_e\vec{v})\bullet_{g_e}\vec{z}\neq 0$, thus $\vec{u}\times_e\vec{v}\neq\vec{0}$. 4'- Contraposition.

5- $\det_{\vec{e}}(\vec{u},\vec{v},\vec{u}\times_e\vec{v})=(\vec{u}\times_e\vec{v})\bullet_{g_e}(\vec{u}\times_e\vec{v})=||\vec{u}\times_e\vec{v}||^2>0$ since $\vec{u}\not\parallel\vec{v}$.

6- If $\vec{u}$ is parallel to $\vec{v}$ then it is trivial (zero area). Otherwise $\vec{u}\times_e\vec{v}\neq\vec{0}$ thus $0\neq\det_{\vec{e}}(\vec{u},\vec{v},\frac{\vec{u}\times_e\vec{v}}{||\vec{u}\times_e\vec{v}||_{g_e}})=(\vec{u}\times_e\vec{v})\bullet_{g_e}\frac{\vec{u}\times_e\vec{v}}{||\vec{u}\times_e\vec{v}||_{g_e}}=||\vec{u}\times_e\vec{v}||_{g_e}=$ volume of the parallelepiped $(\vec{u},\vec{v},\frac{\vec{u}\times_e\vec{v}}{||\vec{u}\times_e\vec{v}||_{g_e}})$ (height 1). ∎

Exercise 1.4 $(\vec{a}_i)$ being a $(\cdot,\cdot)_{g_e}$-orthonormal basis, define the basis $(\vec{b}_i)$ by $\vec{b}_1=-\vec{a}_1$, $\vec{b}_2=\vec{a}_2$, $\vec{b}_3=\vec{a}_3$ (change of orientation). Prove:

$$\times_b=-\times_a \tag{1.6}$$

(the definition of a vector product is basis dependent), i.e. $\vec{v}\times_b\vec{w}=-\vec{v}\times_a\vec{w}$, for all $\vec{v},\vec{w}\in V$.

Answer. $\vec{b}_2\times_b\vec{b}_3=\vec{b}_1=-\vec{a}_1=-\vec{a}_2\times_a\vec{a}_3=-\vec{b}_2\times_a\vec{b}_3$, and $\vec{b}_3\times_b\vec{b}_1=\vec{b}_2=\vec{a}_2=\vec{a}_3\times_a\vec{a}_1=-\vec{b}_3\times_a\vec{b}_1$, and $\vec{b}_1\times_b\vec{b}_2=\vec{b}_3=\vec{a}_3=\vec{a}_1\times_a\vec{a}_2=-\vec{b}_1\times_a\vec{b}_2$; And $\times_a$ and $\times_b$ are bilinear antisymmetric, hence (1.6). ∎

Exercise 1.5 Check:

$$\vec{u}\times_e(\vec{v}\times_e\vec{w})=(\vec{u}\bullet_{g_e}\vec{w})\vec{v}-(\vec{u}\bullet_{g_e}\vec{v})\vec{w}. \tag{1.7}$$

Answer. $[\vec{u}\times_e(\vec{v}\times_e\vec{w})]_{|\vec{e}}=\begin{pmatrix}u_2(v_1w_2-v_2w_1)-u_3(v_3w_1-v_1w_3)\\u_3(v_2w_3-v_3w_2)-u_1(v_1w_2-v_2w_1)\\u_1(v_3w_1-v_1w_3)-u_2(v_2w_3-v_3w_2)\end{pmatrix}=\begin{pmatrix}(\sum_{i=1}^3u_iw_i)v_1-(\sum_{i=1}^3u_iv_i)w_1\\(\sum_{i=1}^3u_iw_i)v_2-(\sum_{i=1}^3u_iv_i)w_2\\(\sum_{i=1}^3u_iw_i)v_3-(\sum_{i=1}^3u_iv_i)w_3\end{pmatrix}$. ∎

Exercise 1.6 Let $\vec{v},\vec{w}\in V$. Prove: $\vec{z}:=\vec{v}\times_e\vec{w}$ is a "contravariant vector", i.e. satisfies the change of basis formula $[\vec{z}]_{|\vec{b}}=P^{-1}.[\vec{z}]_{|\vec{a}}$ where P is the transition matrix from a basis $(\vec{a}_i)$ to a basis $(\vec{b}_i)$.

Answer. $g(\vec{u},\vec{v}\times_e\vec{w})_{g_e}=[\vec{u}]_{\vec{a}}^T.[g]_{\vec{a}}.[\vec{v}\times_e\vec{w}]_{\vec{a}}$ and $g(\vec{u},\vec{v}\times_e\vec{w})_{g_e}=[\vec{u}]_{\vec{b}}^T.[g]_{\vec{b}}.[\vec{v}\times_e\vec{w}]_{\vec{b}}$ with (change of basis formulas) $[\vec{v}]_{\vec{b}}=P^{-1}.[\vec{v}]_{\vec{a}}$ and $[g]_{\vec{b}}=P^T.[g]_{\vec{a}}.P$. So

$$g(\vec{u},\vec{v}\times_e\vec{w})_{g_e}=[\vec{u}]_{\vec{b}}^T.[g]_{\vec{b}}.[\vec{v}\times_e\vec{w}]_{\vec{b}}=([\vec{u}]_{\vec{a}}^T.P^{-T}).(P^T.[g]_{\vec{a}}.P).[\vec{v}\times_e\vec{w}]_{\vec{b}}=[\vec{u}]_{\vec{a}}^T.[g]_{\vec{a}}.P.[\vec{v}\times_e\vec{w}]_{\vec{b}},$$

for all $\vec{u},\vec{v},\vec{w}$, hence $[\vec{v}\times_e\vec{w}]_{\vec{a}}=P.[\vec{v}\times_e\vec{w}]_{\vec{b}}$, i.e. $[\vec{v}\times_e\vec{w}]_{\vec{b}}=P^{-1}.[\vec{v}\times_e\vec{w}]_{\vec{a}}$. ∎

2 Antisymmetric endomorphism and its representation vectors

2.1 Transpose of an endomorphism

V is a dimension n real vector space and $\mathcal{L}(V;V)$ is the set of endomorphisms $V\to V$.

Usual notation for a linear map: $L(\vec{v})\ =^{\text{noted}}\ L.\vec{v}$, hence $L.(\vec{v}+\lambda\vec{w})=L.\vec{v}+\lambda L.\vec{w}$ (distributivity notation = linearity notation).

Let $(\cdot,\cdot)_g:V\times V\to\mathbb{R}$ be a scalar dot product (required to define the transposed). (No basis required.)

Definition 2.1 The transposed of an endomorphism $L\in\mathcal{L}(V;V)$ relative to $(\cdot,\cdot)_g$ is the endomorphism $L_g^T\in\mathcal{L}(V;V)$ defined by, for all $\vec{v},\vec{w}\in V$,

$$(L_g^T.\vec{w},\vec{v})_g=(\vec{w},L.\vec{v})_g. \tag{2.1}$$

Quantification. Choose a basis $(\vec{e}_i)$ in V: (2.1) gives $[\vec{v}]_{|\vec{e}}^T.[g]_{|\vec{e}}.[L_g^T.\vec{w}]_{|\vec{e}}=[L.\vec{v}]_{|\vec{e}}^T.[g]_{|\vec{e}}[\vec{w}]_{|\vec{e}}$, thus $[\vec{v}]_{|\vec{e}}^T.[g]_{|\vec{e}}.[L_g^T]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}=[\vec{v}]_{|\vec{e}}^T.[L]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}$, for all $\vec{v},\vec{w}\in\overrightarrow{\mathbb{R}^3}$, thus $[L_g^T]_{|\vec{e}}=[g]_{|\vec{e}}^{-1}.[L]_{|\vec{e}}^T.[g]_{|\vec{e}}$.

Proposition 2.2 *If $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ are two Euclidean dot products (e.g. $(\cdot,\cdot)_a$ built with a foot and $(\cdot,\cdot)_b$ with a metre) then*

$$L_a^T = L_b^T \overset{\text{noted}}{=} L^T \quad \textit{(Euclidean setting)}: \tag{2.2}$$

The transposed of an endomorphism in $\overrightarrow{\mathbb{R}^3}$ Euclidean does not depend on the unit of measurement (foot, metre, ...) used to build Euclidean dot products.

Proof. $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ are both Euclidean thus $\exists\lambda > 0$ s.t. $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$, thus $(L_a^T.\vec{w},\vec{v})_a = (\vec{w},L.\vec{v})_a = \lambda^2(\vec{w},L.\vec{v})_b = \lambda^2(L_b^T.\vec{w},\vec{v})_b = (L_b^T.\vec{w},\vec{v})_a$ for all $\vec{v},\vec{w} \in \overrightarrow{\mathbb{R}^3}$, thus $L_a^T.\vec{w} = L_b^T.\vec{w}$ for all $\vec{w}$. ∎

Quantification, Euclidean setting. $(\cdot,\cdot)_{g_e}$-Euclidean basis $(\vec{e}_i)$, thus $[g_e]_{|\vec{e}} = I$, thus with (2.2),

$$L_g^T \overset{\text{noted}}{=} L^T, \quad [L^T]_{|\vec{e}} = [L]_{|\vec{e}}^T, \quad \text{i.e.} \quad (L^T)_{ij} = L_{ji} \ \forall i,j \quad \text{(Euclidean setting)}. \tag{2.3}$$

2.2 Symmetric and antisymmetric endomorphisms

Let $L \in \mathcal{L}(V;V)$ and let $(\cdot,\cdot)_g$ be a scalar dot product in V.

Definition 2.3

$$\begin{aligned} &\bullet\ L \text{ is } (\cdot,\cdot)_g\text{-symmetric iff } L_g^T = L, \quad \text{i.e.} \quad (L.\vec{w},\vec{v})_g = (\vec{w},L.\vec{v})_g,\ \forall\vec{v},\vec{w},\\ &\bullet\ L \text{ is } (\cdot,\cdot)_g\text{-antisymmetric iff } L_g^T = -L, \quad \text{i.e.} \quad (L.\vec{w},\vec{v})_g = -(\vec{w},L.\vec{v})_g,\ \forall\vec{v},\vec{w}. \end{aligned} \tag{2.4}$$

Proposition 2.4 *The space of $(\cdot,\cdot)_{g_e}$-symmetric endomorphisms is a vector space. The space of $(\cdot,\cdot)_{g_e}$-antisymmetric endomorphisms is a vector space.*

Proof. $(L+\lambda M)_g^T = L_g^T + \lambda M_g^T = (\pm L)+\lambda(\pm M) = \pm(L+\lambda M)$ with $+$ iff L and M are $(\cdot,\cdot)_{g_e}$-symmetric and $-$ iff L and M are antisymmetric. Thus, vector sub-spaces of $\mathcal{L}(V;V)$. ∎

Euclidean setting: Euclidean basis $(\vec{e}_i)$, associated Euclidean dot product $(\cdot,\cdot)_{g_e}$. With (2.2):

- L is Euclidean-symmetric iff $[L^T]_{|\vec{e}} = [L]_{|\vec{e}}$, (2.5)
- L is Euclidean-antisymmetric iff $[L^T]_{|\vec{e}} = -[L]_{|\vec{e}}$. (2.6)

2.3 Antisymmetric endomorphism and its representation vectors

Euclidean framework: $(\vec{e}_i)$ is a Euclidean basis and $(\cdot,\cdot)_{g_e}$ is the associated Euclidean dot product.

Let $L \in \mathcal{L}(\overrightarrow{\mathbb{R}^3};\overrightarrow{\mathbb{R}^3})$ be $(\cdot,\cdot)_{g_e}$-antisymmetric: (2.6) gives $L_{ii} = 0$ and $L_{ji} = -L_{ji}$ for all i,j, thus $\exists a,b,c \in \mathbb{R}$ s.t. $L.\vec{e}_1 = c\vec{e}_2 - b\vec{e}_3$, $L.\vec{e}_2 = -c\vec{e}_1 + a\vec{e}_3$ and $L.\vec{e}_3 = b\vec{e}_1 - a\vec{e}_2$. Then define the vector $\vec{\omega}_e \in \overrightarrow{\mathbb{R}^3}$ by $\vec{\omega}_e := a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$: We immediately have, for all $\vec{v} \in V$,

$$L.\vec{v} = \vec{\omega}_e \times_e \vec{v}. \tag{2.7}$$

In other words,

$$[L]_{|\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \quad \text{and} \quad [\vec{\omega}_e]_{|\vec{e}} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{give} \quad L.\vec{v} = \vec{\omega}_e \times_e \vec{v}, \quad \forall\vec{v} \in \overrightarrow{\mathbb{R}^3}. \tag{2.8}$$

Definition 2.5 The vector $\vec{\omega}_e$ is the $\times_e$-representation vector of the antisymmetric endomorphism L relative to the Euclidean basis $(\vec{e}_i)$.

Proposition 2.6 *The representation vector $\vec{\omega}_e$ (of L) is* ***not*** *intrinsic to L. In particular if $(\vec{b}_i)$ is another $(\cdot,\cdot)_{g_e}$-Euclidean basis which orientation is opposed to the orientation of $(\vec{e}_i)$ then*

$$\vec{\omega}_b = -\vec{\omega}_e. \tag{2.9}$$

Proof. $L.\vec{v} = \vec{\omega}_e \times_e \vec{v}$ and $L.\vec{v} = \vec{\omega}_b \times_b \vec{v}$ give $\vec{\omega}_e \times_e \vec{v} = \vec{\omega}_b \times_b \vec{v}$, thus $(\vec{\omega}_e \times_e \vec{v}) \bullet_{g_e} \vec{z} = (\vec{\omega}_b \times_b \vec{v}) \bullet_{g_e} \vec{z}$, thus (1.3) gives $\det_{\vec{e}}(\vec{\omega}_e,\vec{v},\vec{z}) = \det_{\vec{b}}(\vec{\omega}_b,\vec{v},\vec{z}) = -\det_{\vec{e}}(\vec{\omega}_b,\vec{v},\vec{z})$, for all $\vec{v},\vec{z}$, thus $\vec{\omega}_e = -\vec{\omega}_b$. ∎

2.4 Interpretation ($\pi/2$ rotation and dilation)

Consider (2.7)-(2.8), and let $\omega_e := ||\vec{\omega}_e||_{g_e} = \sqrt{a^2+b^2+c^2}$.

Proposition 2.7 *Let* $[\vec{b}_3]_{|\vec{e}} = \frac{1}{\omega_e}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $[\vec{b}_1]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$, $\vec{b}_2 = \vec{b}_3 \times_e \vec{b}_1 = \frac{1}{\sqrt{a^2+b^2}}\frac{1}{\omega_e}\begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix}$. *Then* $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ *is a direct orthonormal basis, and*

$$[L]_{\vec{b}} = \omega_e \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \omega_e \begin{pmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) & 0 \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\vec{\omega}_e]_{\vec{b}} = \omega_e \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.10}$$

So $L.\vec{v}$ *rotates a vector* $\vec{v} = v_1\vec{b}_1 + v_2\vec{b}_2 \in \mathrm{Vect}\{\vec{b}_1, \vec{b}_2\}$ *through an angle* $\frac{\pi}{2}$ *radians in the plane* $\mathrm{Vect}\{\vec{b}_1, \vec{b}_2\}$ *and dilates by a factor* ω_e : $L.\vec{b}_1 = \omega_e\vec{b}_2$ *and* $L.\vec{b}_2 = -\omega_e\vec{b}_1$*; And it kills the third component :* $L.\vec{b}_3 = \vec{0}$.

Proof. $\det_{\vec{e}}(\vec{b}_1, \vec{b}_2, \vec{b}_3) > 0$: easy calculation. And $P = \begin{pmatrix} [\vec{b}_1]_{|\vec{e}} & [\vec{b}_2]_{|\vec{e}} & [\vec{b}_3]_{|\vec{e}} \end{pmatrix}$ (the transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$) gives $[L]_{|\vec{b}} = P^{-1}.[L]_{|\vec{e}}.P$ (change of basis formula for endomorphisms). And here $P^{-1} = P^T$ (change of orthonormal basis): We get (2.10). ∎

3 Antisymmetric matrix and its pseudo-vector representation

3.1 The pseudo-vector product

Here we are in the matrix world. Only the canonical basis in $\mathcal{M}_{31}$ is considered.

Definition 3.1 The pseudo-vector product is the map $\overset{\circlearrowleft}{\times}$: $\begin{cases} \mathcal{M}_{31} \times \mathcal{M}_{31} \to \mathcal{M}_{31} \\ ([\vec{u}], [\vec{v}]) \to \overset{\circlearrowleft}{\times}([\vec{u}], [\vec{v}]) = [\vec{u}] \overset{\circlearrowleft}{\times} [\vec{v}] \end{cases}$ defined by

$$[\vec{u}] \overset{\circlearrowleft}{\times} [\vec{v}] = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} \quad \text{when} \quad [\vec{u}] = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad [\vec{v}] = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \tag{3.1}$$

and the column matrix $[\vec{u}] \overset{\circlearrowleft}{\times} [\vec{v}]$ is called the pseudo-vector product of $[\vec{u}]$ and $[\vec{v}]$.

In other words $[\vec{u}] \overset{\circlearrowleft}{\times} [\vec{v}] := [\vec{u}]_{|\vec{C}} \times_C [\vec{v}]_{|\vec{C}}$ where $(\vec{C}_i)$ is the canonical basis in $\mathcal{M}_{31}$.

3.2 Antisymmetric matrix and its pseudo-vector representation

Let $M \in \mathcal{M}_{33}$ be an antisymmetric matrix, i.e. there exists $a, b, c \in \mathbb{R}$ s.t.

$$M = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}. \quad \text{Thus} \quad [\overset{\circlearrowleft}{\omega}] = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{gives} \quad M.[\vec{v}] = [\overset{\circlearrowleft}{\omega}] \overset{\circlearrowleft}{\times} [\vec{v}] \tag{3.2}$$

for all $[\vec{v}] \in \mathcal{M}_{31}$. The pseudo-vector (the column matrix) $[\overset{\circlearrowleft}{\omega}] \in \mathcal{M}_{31}$ is called the pseudo-vector representation (column matrix representation) of the matrix M.

3.3 Pseudo-vectors representation of an antisymmetric endomorphism

Euclidean framework: $(\vec{e}_i)$ is a Euclidean basis and $(\cdot,\cdot)_{g_e}$ is the associated Euclidean dot product.

Let $L \in \mathcal{L}(\overrightarrow{\mathbb{R}^3}; \overrightarrow{\mathbb{R}^3})$ be $(\cdot,\cdot)_{g_e}$-antisymmetric. Hence $[\vec{\omega}_e \times_e \vec{v}]_{|\vec{e}} \overset{(2.7)}{=} [L.\vec{v}]_{|\vec{e}} = [L]_{|\vec{e}}.[\vec{v}]_{|\vec{e}}$ gives, with (3.2) and $M = [L]_{|\vec{e}}$,

$$[\vec{\omega}_e \times_e \vec{v}]_{|\vec{e}} = [\overset{\circlearrowleft}{\omega}] \overset{\circlearrowleft}{\times} [\vec{v}]_{|\vec{e}} \quad \text{where} \quad [\overset{\circlearrowleft}{\omega}] := [\vec{\omega}_e]_{|\vec{e}}. \tag{3.3}$$

Definition 3.2 The matrix $[\overset{\circlearrowleft}{\omega}] := [\vec{\omega}_e]_{|\vec{e}} \in \mathcal{M}_{31}$ is the pseudo-vector representation of L relative to $(\vec{e}_i)$.

4 Screw (torsor)

4.0 Reminder

Let Ω be an open set in $\mathbb{R}^3$.

- A vector field in $\mathbb{R}^3$ is a function $\widetilde{\vec{u}} : \left\{ \begin{array}{ll} \Omega & \to \Omega \times \overrightarrow{\mathbb{R}^3} \\ A & \to \widetilde{\vec{u}}(A) := (A, \vec{u}(A)) \end{array} \right\}$, the couple $\widetilde{\vec{u}}(A) := (A, \vec{u}(A))$ being a "pointed vector at A", or "a vector at A". Drawing: $\vec{u}(A)$ has to be drawn at A, nowhere else. To compare with a vector $\vec{v} \in \overrightarrow{\mathbb{R}^3}$ which can be drawn anywhere (also called a free vector).

The sum of two vector fields $\widetilde{\vec{u}}, \widetilde{\vec{w}}$ and the multiplication by a real λ are defined by, at any $A \in \Omega$,

$$\widetilde{\vec{u}}(A) + \widetilde{\vec{w}}(A) = (A, \vec{u}(A) + \vec{w}(A)), \quad \text{and} \quad \lambda\widetilde{\vec{u}}(A) = (A, \lambda\vec{u}(A)) \tag{4.1}$$

(usual rules for "vectors at A"). To lighten the notations, $\widetilde{\vec{u}}(A) =^{\text{noted}} \vec{u}(A)$ (but don't forget it is a pointed vector).

The differential of a C^1 vector field $\widetilde{\vec{u}} : \Omega \to \Omega \times \overrightarrow{\mathbb{R}^3}$ at a point A is the "field of endomorphisms" $d\widetilde{\vec{u}} : \Omega \to \Omega \times \mathcal{L}(\overrightarrow{\mathbb{R}^3}; \overrightarrow{\mathbb{R}^3})$ defined by $d\widetilde{\vec{u}}(A) = (A, d\vec{u}(A))$ (an endomorphism at A) where $d\vec{u}(A)$ is the differential of $\vec{u}$ at A. So $\vec{u}(B) = \vec{u}(A) + d\vec{u}(A).\overrightarrow{AB} + o(||\overrightarrow{AB}||)$. And $d\widetilde{\vec{u}} =^{\text{noted}} d\vec{u}$.

- An affine vector field $\widetilde{\vec{u}} : \left\{ \begin{array}{ll} \Omega & \to \Omega \times \overrightarrow{\mathbb{R}^3} \\ A & \to \widetilde{\vec{u}}(A) := (A, \vec{u}(A)) \end{array} \right\}$ is a vector field s.t. $\vec{u} : \Omega \to \overrightarrow{\mathbb{R}^3}$ is affine, i.e. s.t. $d\vec{u}$ is uniform, i.e. s.t., for all A, B, $d\vec{u}(A) = d\vec{u}(B) =^{\text{noted}} d\vec{u}$, so s.t., for all $A, B \in \mathbb{R}^3$,

$$\vec{u}(B) = \vec{u}(A) + d\vec{u}.\overrightarrow{AB}. \tag{4.2}$$

4.1 Definition (Euclidean framework)

Euclidean framework required: $(\vec{e}_i)$ is a chosen Euclidean basis in $\overrightarrow{\mathbb{R}^3}$, $(\cdot,\cdot)_{g_e}$ is the associated Euclidean dot product, $\times_e$ is the associated vector product, and the transposed of an endomorphism L is L^T cf. (2.2).

Definition 4.1 A screw (a torsor) is the name given to an affine Euclidean antisymmetric vector field.

So a screw is a function $\widetilde{\vec{s}} : \left\{ \begin{array}{ll} \Omega & \to \Omega \times \overrightarrow{\mathbb{R}^3} \\ A & \to \widetilde{\vec{s}}(A) := (A, \vec{s}(A)) \end{array} \right\}$ s.t. $d\vec{s}$ is uniform and, with $\vec{\omega}_e$ the $\times_e$-representation vector of $d\vec{s}$ cf. (2.7), for all $A, B \in \Omega$,

$$\boxed{\vec{s}(B) = \vec{s}(A) + \vec{\omega}_e \times_e \overrightarrow{AB}}, \quad \text{so} \quad [\vec{s}(B)]_{|\vec{e}} = [\vec{s}(A)]_{|\vec{e}} + [\overset{\circlearrowleft}{\omega}] \overset{\circlearrowleft}{\times} [\overrightarrow{AB}]_{|\vec{e}}, \tag{4.3}$$

with $[\overset{\circlearrowleft}{\omega}] = [\vec{\omega}_e]_{|\vec{e}} := \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ when $[d\vec{s}]_{|\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$. Abusively written $\vec{s}(B) = \vec{s}(A) + \overset{\circlearrowleft}{\omega} \overset{\circlearrowleft}{\times} \overrightarrow{AB}$.

Definition 4.2 • The vector $\vec{\omega}_e \in \overrightarrow{\mathbb{R}^3}$ is the "resultant vector" of the screw $\vec{s}$ relative to $(\vec{e}_i)$.

- The matrix (the pseudo-vector) $[\overset{\circlearrowleft}{\omega}] := [\vec{\omega}_e]_{|\vec{e}}$ is the "resultant" of the screw $\vec{s}$ relative to $(\vec{e}_i)$.
- $\vec{s}(A)$ is the moment of the screw $\vec{s}$ at $A \in \Omega$ (or moment of the torsor $\vec{s}$ at A).
- If $\vec{s} = \vec{0}$ then $\widetilde{\vec{s}}$ is a degenerate screw (a degenerate torsor).
- A constant screw $\vec{s}$ is non degenerate screw s.t. $\vec{s}(A) = \vec{s}(B)$ for all $A, B \in \Omega$ (i.e. s.t. $\vec{\omega}_e = \vec{0}$).
- The "reduction elements" at A are $[\overset{\circlearrowleft}{\omega}] := [\vec{\omega}_e]_{|\vec{e}}$ and $[\vec{s}(A)]_{|\vec{e}}$ (column matrices) relative to $(\vec{e}_i)$, written as the couple of matrices $([\overset{\circlearrowleft}{\omega}], [\vec{s}(A)]_{|\vec{e}})$ abusively written $(\overset{\circlearrowleft}{\omega}, \vec{s}(A))$.

Exercise 4.3 Let $\mathcal{S}$ be the set of the screws $\vec{s} : \Omega \to \overrightarrow{\mathbb{R}^3}$. Prove: $\mathcal{S}$ is a vector space.

Answer. If $\vec{s}_1, \vec{s}_2 \in \mathcal{S}$ and $\lambda \in \mathbb{R}$ then $\vec{s}_1+\lambda\vec{s}_2$ is affine antisymmetric: Indeed, at B, $(\vec{s}_1+\lambda\vec{s}_2)(B) = \vec{s}_1(B) + \lambda\vec{s}_2(B) = (\vec{s}_1(A) + d\vec{s}_1.\overrightarrow{AB}) + \lambda(\vec{s}_2(A) + d\vec{s}_2.\overrightarrow{AB}) = (\vec{s}_1+\lambda\vec{s}_2)(A) + (d\vec{s}_1+\lambda d\vec{s}_2).\overrightarrow{AB}$ with $d\vec{s}_1+\lambda d\vec{s}_2$ antisymmetric since $d\vec{s}_1$ and $d\vec{s}_2$ are; Thus $\vec{s}_1+\lambda\vec{s}_2 \in \mathcal{S}$ (affine with $L_{\vec{s}_1+\lambda\vec{s}_2} = d\vec{s}_1+\lambda d\vec{s}_2$ linear antisymmetric). ▪

Exercise 4.4 Let $\vec{s}$ be a screw and $\vec{\omega}_e$ its resultant vector. For all $\lambda \in \mathbb{R}$ and $A, B \in \mathbb{R}^3$, prove:

$$\vec{s}(A+\lambda\vec{\omega}_e) = \vec{s}(A), \quad \text{and} \quad \vec{s}(B) \bullet_{g_e} \vec{\omega}_e = \vec{s}(A) \bullet_{g_e} \vec{\omega}_e \ (\ = \text{constant}). \tag{4.4}$$

(Hence the definition: $s_{inv} := \vec{s}(A) \bullet_{g_e} \vec{\omega}_e$ is called the (scalar) invariant of the screw.) And prove:

$$\vec{s}(B) \bullet_{g_e} \overrightarrow{AB} = \vec{s}(A) \bullet_{g_e} \overrightarrow{AB}, \quad \text{called the equi-projectivity property.} \tag{4.5}$$

Answer. Let $B = A+\lambda\vec{\omega}_e$, so $\overrightarrow{AB} = \lambda\vec{\omega}_e$, thus $\vec{s}(B) =^{(4.3)} \vec{s}(A) + \vec{\omega}_e \times_e (\lambda\vec{\omega}_e) = \vec{s}(A) + \vec{0}$, i.e. $(4.4)_1$.
And $\vec{\omega}_e \times_e \overrightarrow{AB}$ orthogonal to both $\vec{\omega}_e$ and $\overrightarrow{AB}$, thus (4.3) gives $(4.4)_2$ and (4.5). ∎

Exercise 4.5 Fix a point $A \in \mathbb{R}^3$. Define $f_A : \left\{ \begin{array}{ll} \overrightarrow{\mathbb{R}^3} \times \overrightarrow{\mathbb{R}^3} & \to \mathcal{S} \\ (\vec{z}, \vec{w}) & \to \vec{s} = f_A(\vec{z}, \vec{w}) \end{array} \right\}$ by $\vec{s}(B) := \vec{z} + \vec{w} \times_e \overrightarrow{AB}$ for all $B \in \mathbb{R}^3$. Prove that f_A is linear and bijective (is one-to-one and onto).

Answer. Linearity: $f_A((\vec{z}_1, \vec{w}_1) + \lambda(\vec{z}_2, \vec{w}_2))(B) = f_A(\vec{z}_1{+}\lambda\vec{z}_2, \vec{w}_1{+}\lambda\vec{w}_2)(B) = \vec{z}_1{+}\lambda\vec{z}_2 + (\vec{w}_1{+}\lambda\vec{w}_2) \times_e \overrightarrow{AB} = \vec{z}_1{+}\vec{w}_1 \times_e \overrightarrow{AB} + \lambda(\vec{z}_2{+}\vec{w}_2 \times_e \overrightarrow{AB}) = (f_A(\vec{z}_1, \vec{w}_1) + \lambda f_A(\vec{z}_2, \vec{w}_2))(B)$.
One-to-one: $f_A(\vec{z}, \vec{w}) = 0$ iff $\vec{z} + \vec{w} \times_e \overrightarrow{AB} = \vec{0}$ for all B, in particular $B = A$ gives $\vec{z} = \vec{0}$ and then $\vec{w} = \vec{0}$.
Onto: Let $\vec{s} \in \mathcal{S}$, $\vec{s}(B) = \vec{s}(A) + \vec{\omega}_e \times_e \overrightarrow{AB}$, and take $\vec{z} = \vec{s}(A)$ and $\vec{w} = \vec{\omega}_e$. ∎

Exercise 4.6 Write $\times_e{=}\times$, $\bullet_{g_e}{=}\bullet$, $\vec{\omega}_e{=}\vec{\omega}$. Let $\vec{s}_1, \vec{s}_2 \in \mathcal{S}$, $\vec{s}_1(B) = \vec{s}_1(A) + \vec{\omega}_1 \times \overrightarrow{AB}$ and $\vec{s}_2(B) = \vec{s}_2(A) + \vec{\omega}_2 \times \overrightarrow{AB}$. Define the screw $\langle \vec{s}_1, \vec{s}_2 \rangle$ by $\langle \vec{s}_1, \vec{s}_2 \rangle(A) = \vec{\omega}_1 \bullet \vec{s}_2(A) + \vec{\omega}_2 \bullet \vec{s}_1(A)$. Prove $\langle \vec{s}_1, \vec{s}_2 \rangle$ is constant.

Answer. $\vec{\omega}_1 \bullet \vec{s}_2(B) + \vec{\omega}_2 \bullet \vec{s}_1(B) = \vec{\omega}_1 \bullet (\vec{s}_2(A) + \vec{\omega}_2 \times \overrightarrow{AB}) + \vec{\omega}_2 \bullet (\vec{s}_1(A) + \vec{\omega}_1 \times \overrightarrow{AB}) = \vec{\omega}_1 \bullet \vec{s}_2(A) + \vec{\omega}_2 \bullet \vec{s}_1(A) + \vec{\omega}_1 \bullet (\vec{\omega}_2 \times \overrightarrow{AB}) + \vec{\omega}_2 \bullet (\vec{\omega}_1 \times \overrightarrow{AB})$, with $\vec{\omega}_1 \bullet (\vec{\omega}_2 \times \overrightarrow{AB}) + \vec{\omega}_2 \bullet (\vec{\omega}_1 \times \overrightarrow{AB}) = \det_{\vec{e}}(\vec{\omega}_1, \vec{\omega}_2, \overrightarrow{AB}) + \det_{\vec{e}}(\vec{\omega}_2, \vec{\omega}_1, \overrightarrow{AB})$ hence $= 0$, thus $\vec{\omega}_1 \bullet \vec{s}_2(B) + \vec{\omega}_2 \bullet \vec{s}_1(B) = \vec{\omega}_1 \bullet \vec{s}_2(A) + \vec{\omega}_2 \bullet \vec{s}_1(A)$, for all A, B. ∎

4.2 Central axis

Let $\vec{s} : \Omega \to \overrightarrow{\mathbb{R}^3}$ be a screw, $\vec{s}(B) = \vec{s}(A) + \vec{\omega}_e \times_e \overrightarrow{AB}$, cf. (4.3).

Definition 4.7 The central axis (or instantaneous screw axis) of a non constant screw ($\vec{\omega}_e \neq \vec{0}$) is

$$\text{Ax}(\vec{s}) = \{C \in \mathbb{R}^3 : \vec{s}(C) \parallel \vec{\omega}_e\} = \{C \in \mathbb{R}^3 : \exists \lambda \in \mathbb{R},\ \vec{s}(C) = \lambda\vec{\omega}_e\} \tag{4.6}$$

called the set of central points. NB: Here $\vec{s}$ is affine thus Ω is implicitly extended to the whole $\mathbb{R}^3$, thus a point $C \in \text{Ax}(\vec{s})$ might be outside of Ω.

Proposition 4.8 *Let $\vec{s}$ be a non constant screw. Let $O \in \mathbb{R}^3$. Define the point $C_0 \in \mathbb{R}^3$ by*

$$\overrightarrow{OC_0} = \frac{1}{||\vec{\omega}_e||^2} \vec{\omega}_e \times_e \vec{s}(O), \quad \text{i.e.} \quad C_0 := O + \frac{1}{||\vec{\omega}_e||^2} \vec{\omega}_e \times_e \vec{s}(O). \tag{4.7}$$

Then

1- $C_0 \in \text{Ax}(\vec{s})$, and

$$\text{Ax}(\vec{s}) = C_0 + \text{Vect}\{\vec{\omega}_e\} \quad \text{(affine straight line)}. \tag{4.8}$$

2- $\vec{s}$ is constant along $\text{Ax}(\vec{s})$: For all $C \in \text{Ax}(\vec{s})$, $\vec{s}(C) = \vec{s}(C_0)$.
3- $C \in \text{Ax}(\vec{s})$ iff $C = \arg\min_{A \in \mathbb{R}^3} ||\vec{s}(A)||_e$ (i.e. iff $||\vec{s}(C)||_e = \min_{A \in \mathbb{R}^3} ||\vec{s}(A)||_e$).
3'- $||\vec{s}(B)||_e > ||\vec{s}(C)||_e$ for all $C \in \text{Ax}(\vec{s})$ and all $B \notin \text{Ax}(\vec{s})$.
4- For all $B \in \Omega$ and $C \in \text{Ax}(\vec{s})$,

$$\vec{s}(B) = \vec{s}(C) + \vec{\omega}_e \times_e \overrightarrow{CB} \quad \in \text{Vect}\{\vec{\omega}_e\} \oplus^\perp \text{Vect}\{\vec{\omega}_e\}^\perp \quad \text{(orthogonal sum)}, \tag{4.9}$$

sum of the translation $\vec{s}(C)$ along the axis and of the rotation-dilation $\vec{\omega}_e \times_e \overrightarrow{CB}$ in $\text{Vect}\{\vec{\omega}_e\}^\perp$.

Proof. 1- $\vec{s}(C_0) = \vec{s}(O) + \vec{\omega}_e \times_e \overrightarrow{OC_0} = \vec{s}(O) + \vec{\omega}_e \times_e (\frac{1}{||\vec{\omega}_e||^2} \vec{\omega}_e \times_e \vec{s}(O)) = \vec{s}(O) + \frac{1}{||\vec{\omega}_e||^2} (\vec{\omega}_e \bullet_{g_e} \vec{s}(O))\vec{\omega}_e - \frac{1}{||\vec{\omega}_e||^2} ||\vec{\omega}_e||^2 \vec{s}(O) = \frac{1}{||\vec{\omega}_e||^2} (\vec{\omega}_e \bullet_{g_e} \vec{s}(O))\vec{\omega}_e$ is parallel to $\vec{\omega}_e$, thus $C_0 \in \text{Ax}(\vec{s})$.
Then $\vec{s}(C_0 + \lambda\vec{\omega}_e) = \vec{s}(C_0) + \vec{0}$ for all λ (because $\vec{\omega}_e \times_e \vec{\omega}_e = \vec{0}$), thus $\text{Ax}(\vec{s}) \supset C_0 + \text{Vect}\{\vec{\omega}_e\}$.
If $B \notin C_0{+}\text{Vect}\{\vec{\omega}_e\}$, then $\overrightarrow{C_0B} \nparallel \vec{\omega}_e$, i.e. $\vec{\omega}_e \times_e \overrightarrow{C_0B} \neq \vec{0}$, thus $\vec{s}(B) = \vec{s}(C_0) + \vec{\omega}_e \times_e \overrightarrow{C_0B} \in \text{Vect}\{\vec{\omega}_e\} \oplus^\perp \text{Vect}\{\vec{\omega}_e\}^\perp$ with $\vec{0} \neq \vec{\omega}_e \times_e \overrightarrow{C_0B}$, thus $\vec{s}(B) \nparallel \vec{\omega}_e$, hence $B \notin \text{Ax}(\vec{s})$. Thus $\text{Ax}(\vec{s}) = C_0 + \text{Vect}\{\vec{\omega}_e\}$.
2- $\vec{s}(C_0 + \lambda\vec{\omega}_e) = \vec{s}(C_0) + \vec{\omega}_e \times_e (\lambda\vec{\omega}_e) = \vec{s}(C_0) + \vec{0}$, thus $\vec{s}(C) = \vec{s}(C_0)$ for all $C \in C_0 + \text{Vect}\{\vec{\omega}_e\}$.
3- If $B \notin C_0 + \text{Vect}\{\vec{\omega}_e\}$ then $||\vec{s}(B)||_e^2 = ||\vec{s}(C_0) + \vec{\omega}_e \times_e \overrightarrow{C_0B}||_e^2 > ||\vec{s}(C_0)||_e^2$ (Pythagoras since $\vec{s}(C_0) \parallel \vec{\omega}_e$ is orthogonal to $\vec{\omega}_e \times_e \overrightarrow{C_0B}$).
4- $\vec{s}(B) =^{(4.3)} \vec{s}(C_0) + \vec{\omega}_e \times_e \overrightarrow{C_0B}$ with $\vec{s}(C_0) \parallel \vec{\omega}_e$ and $\vec{\omega}_e \times_e \overrightarrow{C_0B} \perp \vec{\omega}_e$. ∎

Exercise 4.9 How was the point C_0 in (4.7) found?

Answer. If $\vec{s}(O) \parallel \vec{\omega}_e$ then take $C_0 = O$. Else a drawing encourages to look for a $C_0 = O + \alpha\vec{\omega}_e \times_e \vec{s}(O)$ for some $\alpha \in \mathbb{R}$ because $\overrightarrow{OC_0}$ is then orthogonal to $\mathrm{Vect}\{\vec{\omega}_e\}$. Which gives $\vec{s}(C_0) = \vec{s}(O) + \vec{\omega}_e \times_e \overrightarrow{OC_0} = \vec{s}(O) + \vec{\omega}_e \times_e (\alpha\vec{\omega}_e \times_e \vec{s}(O)) = \vec{s}(O) + \alpha(\vec{\omega}_e \bullet_{\!g} \vec{s}(O))\vec{\omega}_e - \alpha||\vec{\omega}_e||^2\vec{s}(O)$. Hence we choose $\alpha = \frac{1}{||\vec{\omega}_e||^2}$: We get $\vec{s}(C_0) = \frac{1}{||\vec{\omega}_e||^2}(\vec{\omega}_e \bullet_{\!g} \vec{s}(O))\vec{\omega}_e$ parallel to $\vec{\omega}_e$, thus C_0 is in $\mathrm{Ax}(\vec{s})$: We have obtained (4.7). ▪

Exercise 4.10 Let $\vec{s}_1$ and $\vec{s}_2$ be two non constant screws s.t. $\vec{\omega}_{e1} + \vec{\omega}_{e2} \neq 0$. Find the axis of $\vec{s} := \vec{s}_1 + \vec{s}_2$.

Answer. $\vec{s}_1(B) = \vec{s}_1(O) + \vec{\omega}_{e1} \times_e \overrightarrow{OB}$ and $\vec{s}_2(B) = \vec{s}_2(O) + \vec{\omega}_{e2} \times_e \overrightarrow{OB}$ give $(\vec{s}_1+\vec{s}_2)(B) = (\vec{s}_1(O) + \vec{s}_2(O)) + (\vec{\omega}_1+\vec{\omega}_2) \times_e \overrightarrow{OB}$. Thus $\mathrm{Ax}(\vec{s}_1+\vec{s}_2) = C + \mathrm{Vect}\{\vec{\omega}_1+\vec{\omega}_2\}$ where $C : \overset{(4.7)}{=} O + \dfrac{1}{||\vec{\omega}_1+\vec{\omega}_2||^2}(\vec{\omega}_1+\vec{\omega}_2) \times_e \vec{s}(O)$. ▪

Exercise 4.11 Let $\vec{s}$ be a screw and $\vec{\omega}_e$ its resultant vector. Definition:

$$\vec{s}_{inv} := (\vec{s}(B) \bullet_{\!g} \frac{\vec{\omega}_e}{||\vec{\omega}_e||_e})\frac{\vec{\omega}_e}{||\vec{\omega}_e||_e} \text{ is called the vector invariant of the screw,} \tag{4.10}$$

i.e. $\vec{s}_{inv} := \frac{(\vec{s}(B) \bullet_{\!g} \vec{\omega}_e)\vec{\omega}_e}{\omega_e^2}$ where $\omega_e = ||\vec{\omega}_e||$. Prove: $\vec{s}(B)$ is independent of B and

$$\text{if } C \in \mathrm{Ax}(\vec{s}) \text{ then } \vec{s}(C) = \vec{s}_{inv}, \quad \text{thus} \quad \vec{s}(B) = \vec{s}_{inv} + \vec{\omega}_e \times_e \overrightarrow{CB}, \ \ \forall B \in \mathbb{R}^3. \tag{4.11}$$

Answer. $\vec{s}(B) \bullet_{\!g} \vec{\omega}_e = s_{inv}$, scalar invariant of the screw cf (4.4) independent of B). And $\vec{s}(B) = \vec{s}(C) + \vec{\omega}_e \times_e \overrightarrow{CB}$ with $\vec{s}(C) \parallel \vec{\omega}_e$ and $\vec{\omega}_e \times_e \overrightarrow{CB} \perp \vec{\omega}_e$, thus $\vec{s}_{inv} := (\vec{s}(C) \bullet_{\!g} \frac{\vec{\omega}_e}{||\vec{\omega}_e||_e})\frac{\vec{\omega}_e}{||\vec{\omega}_e||_e} = \vec{s}(C)$. ▪

5 Twist = kinematic torsor = distributor

5.1 Definition

Let $(\vec{e}_i)$ be a Euclidean basis and $\times_e \overset{\text{noted}}{=} \times$.

Definition 5.1 A twist[1] (or kinematic screw or distributor) is the name of the screw which is "the Eulerian velocity field of a rigid body".

So, let *Obj* be a rigid body, P_{Obj} its particles, $\widetilde{\Phi} : \left\{ \begin{array}{ll} [t_0,T] \times Obj & \to \mathbb{R}^3 \\ (t, P_{Obj}) & \to p(t) = \widetilde{\Phi}(t, P_{Obj}) \end{array} \right\}$ its motion (where $t_0, T \in \mathbb{R}$ and $t_0 < T$), and $\Omega_t := \widetilde{\Phi}(t, Obj) \subset \mathbb{R}^3$ its position in $\mathbb{R}^3$ at t.

Its Eulerian velocity field $\vec{v}$ is defined by $\vec{v}(t, p(t)) := \frac{\partial\widetilde{\Phi}}{\partial t}(t, P_{Obj})$ when $p(t) = \widetilde{\Phi}(t, P_{Obj})$.

Fix t and let $\vec{v}(t, p(t)) \overset{\text{noted}}{=} \vec{v}(p)$.

The body being rigid, $\vec{v}$ is affine and antisymmetric (is a screw called a twist): so, cf. (4.3) with $\vec{\omega} := \vec{\omega}_e$, for all $p, q \in \Omega_t$,

$$\vec{v}(q) = \vec{v}(p) + \vec{\omega} \times \vec{pq}. \tag{5.1}$$

Definition 5.2 $\vec{\omega}$ is the vector angular velocity, and $\omega := ||\vec{\omega}||$ is the angular velocity.

Thus if $c \in \mathrm{Ax}(\vec{v})$ (so $\vec{v}(c)$ is the velocity along $\mathrm{Ax}(\vec{v})$) then (orthogonal decomposition of $\vec{v}(q)$)

$$\forall q \in \Omega_t, \quad \vec{v}(q) = \vec{v}(c) + \vec{\omega} \times \vec{cq} \in \mathrm{Vect}\{\vec{\omega}\} \oplus^{\perp} \mathrm{Vect}\{\vec{\omega}\}^{\perp}. \tag{5.2}$$

5.2 Pitch

Definition 5.3 For a non constant twist ($\omega \neq 0$), the pitch is, for $c \in \mathrm{Ax}(\vec{v})$,

$$p := 2\pi\frac{||\vec{v}(c)||}{\omega} \overset{\text{noted}}{=} 2\pi\frac{\text{linear speed}}{\text{angular speed}}. \tag{5.3}$$

In other words, $\vec{v}(c) \parallel \vec{\omega}$ gives $\vec{v}(c) = h\vec{\omega}$ and $p = 2\pi h$.

It is the "thread pitch" or a nut (or of a screw), i.e. the distance from the crest of one thread to the next, or from one groove to the next. (The pitch vanishes for a pure rotation defined by $\vec{v}(c) = 0$.)

[1] Definition of a twist by R.S. Ball [1]: "A body is said to receive a twist about a screw when it is rotated about the screw, while it is at the same time translated parallel to the screw, through a distance equal to the product of the pitch and the circular measure of the angle of rotation; hence, *the canonical form to which the displacement of a rigid body can be reduced is a twist about a screw.*"

Exercise 5.4 Recall the definition of the angular speed (ω here), and explain the pitch.

Answer. 1- Plane motion immersed in $\mathbb{R}^3$: $\vec{r}(t) = \begin{pmatrix} R\cos(\omega_0 t) \\ R\sin(\omega_0 t) \\ 0 \end{pmatrix}$ where $\omega \in \mathbb{R}^*$ (with prop. 2.7); Eulerian velocity $\vec{v}(t,\vec{r}(t)) = \vec{r}\,'(t) = R\omega_0 \begin{pmatrix} -\sin(\omega_0 t) \\ \cos(\omega_0 t) \\ 0 \end{pmatrix} = R\omega_0 \vec{u}(t)$ where $\vec{u}(t) = \frac{\vec{r}\,'(t)}{||\vec{r}\,'(t)||}$ (unit tangent vector). Definitions: ω_0 is the angular speed and $\vec{\omega}_0 = \begin{pmatrix} 0 \\ 0 \\ \omega_0 \end{pmatrix}$ the angular velocity, so $\vec{v}(t,\vec{r}(t)) = \vec{\omega}_0 \times \vec{r}(t)$; It gives (5.2) when $\vec{v}(c) = \vec{0}$ and $\overrightarrow{cq} = \vec{r}(t)$.

2- The pitch is given by the helix $\vec{r}(t) = \begin{pmatrix} x(t) = R\cos(\omega_0 t) \\ y(t) = R\sin(\omega_0 t) \\ z(t) = at \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ at \end{pmatrix} + \begin{pmatrix} R\cos(\omega_0 t) \\ R\sin(\omega_0 t) \\ 0 \end{pmatrix}$, sum of a translation along the vertical axis and of a plane rotation in the horizontal plane. Its projection on the horizontal plane (cf. 1-) is periodic with period $\frac{2\pi}{\omega_0}$ (because $\omega_0(t+\frac{2\pi}{\omega_0}) = \omega_0 t + 2\pi$), and the pitch is $p = z(t+\frac{2\pi}{\omega_0}) - z(t) = a\frac{2\pi}{\omega_0}$ = the distance "between two grooves of a screw". This corresponds in (5.2) to $\vec{v}(c) = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}$, so $||\vec{v}(c)|| = a$ = linear speed (speed along the axis), so $p = 2\pi\frac{a}{\omega_0} = 2\pi\frac{||\vec{v}(c)||}{\omega_0} = 2\pi\frac{\text{linear speed}}{\text{angular speed}}$. ∎

Exercise 5.5 (5.1) gives the "equiprojectivity property": $\vec{v}(p).\overrightarrow{pq} = \vec{v}(q).\overrightarrow{pq}$. Prove it starting from $||\overrightarrow{p(t)q(t)}||_e$ = constant (rigid body) for all particles $P_{Obj}, Q_{Obj} \in Obj$ where $p(t) = \widetilde{\Phi}(t, P_{Obj})$ and $q(t) = \widetilde{\Phi}(t, Q_{Obj})$.

Answer. Choose a $O \in \mathbb{R}^3$. let $p(t) = \widetilde{\Phi}(t, P_{Obj})$ and $q(t) = \widetilde{\Phi}(t, Q_{Obj})$. Thus $\frac{d}{dt}\overrightarrow{p(t)q(t)} = \frac{d}{dt}\overrightarrow{Oq(t)} - \frac{d}{dt}\overrightarrow{Op(t)} = \vec{v}(t,q(t)) - \vec{v}(t,p(t))$. And $||\overrightarrow{p(t)q(t)}||_e^2 = (\overrightarrow{p(t)q(t)}, \overrightarrow{p(t)q(t)})_g$ = constant, thus $\frac{d}{dt}(\overrightarrow{p(t)q(t)}, \overrightarrow{p(t)q(t)})_g = 0 = 2(\frac{d}{dt}\overrightarrow{p(t)q(t)}, \overrightarrow{p(t)q(t)})_g$, thus $0 = (\vec{v}(t,q(t)) - \vec{v}(t,p(t))), \overrightarrow{p(t)q(t)})_g$ (equiprojectivity property). ∎

5.3 Pure rotation

Definition 5.6 A pure rotation is a non constant twist $\vec{v}$ s.t. $\exists c_0 \in \mathbb{R}^3$, $\vec{v}(c_0) = \vec{0}$.

Hence such a c_0 is $\in \text{Ax}(\vec{v})$, cf prop. 4.8-3, so, for all $q \in \mathbb{R}^3$,

$$\vec{v}(q) = \vec{\omega}_e \times_e \overrightarrow{c_0 q} \quad \text{with} \quad \vec{\omega}_e \neq \vec{0}. \tag{5.4}$$

(So here $\vec{v}(q) \perp \vec{\omega}_e$ for all $q \in \mathbb{R}^3$ and $\text{Ax}(\vec{v}) = c_0 + \text{Vect}\{\vec{\omega}_e\}$).

Exercise 5.7 Prove: A twist $\vec{v}$ is the sum of a pure rotation and a translation.

Answer. With $\vec{v}(p) = \vec{v}(O) + \vec{\omega}_e \times_e \overrightarrow{Op}$: Call $\vec{v}_r$ the pure rotation defined by $\vec{v}_r(p) = \vec{\omega}_e \times_e \overrightarrow{Op}$ and call $\vec{v}_t$ the translation defined by $\vec{v}_t(p) = \vec{v}(O)$. We have $(\vec{v}_t + \vec{v}_r)(p) = \vec{v}(p)$, for all p, hence $\vec{v} = \vec{v}_r + \vec{v}_t$. ∎

Exercise 5.8 Fix $(\vec{e}_i)$, write $\times_e = \times$ and $\vec{\omega}_e = \vec{\omega}$, let $\vec{v}_1(q) = \vec{\omega}_1 \times \overrightarrow{c_1 q}$ and $\vec{v}_2(q) = \vec{\omega}_2 \times \overrightarrow{c_2 q}$.

1- Suppose $\text{Ax}(\vec{v}_1) \parallel \text{Ax}(\vec{v}_2)$, axes disjoint, and $\vec{\omega}_1 + \vec{\omega}_2 \neq \vec{0}$. Find $\text{Ax}(\vec{v}_1+\vec{v}_2)$ and prove that $\vec{v}_1+\vec{v}_2$ is a pure rotation.

1'- Suppose $\text{Ax}(\vec{v}_1) \parallel \text{Ax}(\vec{v}_2)$, axes disjoint, and $\vec{\omega}_1 + \vec{\omega}_2 = \vec{0}$. Prove that $\vec{v}_1+\vec{v}_2$ is a translation.

2- Suppose $\text{Ax}(\vec{v}_1) \nparallel \text{Ax}(\vec{v}_2)$ and the axes intersect at only one point O. Find $\text{Ax}(\vec{v}_1+\vec{v}_2)$, and prove that $\vec{v}_1+\vec{v}_2$ is a pure rotation.

3- Suppose $\text{Ax}(\vec{v}_1) \nparallel \text{Ax}(\vec{v}_2)$ and the axes don't intersect. Find $\text{Ax}(\vec{v}_1+\vec{v}_2)$, and prove that $\vec{v}_1+\vec{v}_2$ is not a pure rotation. Give a "simple" particular $c_0 \in \text{Ax}(\vec{v}_1+\vec{v}_2)$.

Answer. The notations tells: $c_1 \in \text{Ax}(\vec{v}_1)$, $c_2 \in \text{Ax}(\vec{v}_2)$, $(\vec{v}_1+\vec{v}_2)(q) = \vec{\omega}_1 \times \overrightarrow{c_1 q} + \vec{\omega}_2 \times \overrightarrow{c_2 q}$ for all q.

1- Here $\vec{\omega}_2 = \lambda\vec{\omega}_1$ with $\lambda \neq -1$, thus $(\vec{v}_1+\vec{v}_2)(q) = \vec{\omega}_1 \times (\overrightarrow{c_1 q} + \lambda\overrightarrow{c_2 q}) = (\lambda+1)\vec{\omega}_1 \times (\frac{1}{\lambda+1}\overrightarrow{c_1 q} + \frac{\lambda}{\lambda+1}\overrightarrow{c_2 q})$. Hence choose $c_0 \in \mathbb{R}^3$ s.t. $\frac{1}{\lambda+1}\overrightarrow{c_1 c_0} + \frac{\lambda}{\lambda+1}\overrightarrow{c_2 c_0} = \vec{0}$ (barycentric point on the straight line containing c_1 and c_2): We get $\vec{v}(c_0) = \vec{0}$ and $\text{Ax}(\vec{v}_1+\vec{v}_2) = c_0 + \text{Vect}\{\vec{\omega}_1+\vec{\omega}_2\}$. Remark (on barycentric points): We have $\overrightarrow{c_1 c_0} = \frac{1}{\lambda+1}\overrightarrow{c_1 c_2}$, thus c_0 in between c_1 and c_2 iff $0 < \frac{1}{\lambda+1} < 1$, i.e. iff $\lambda > 0$, i.e. iff $\vec{\omega}_1$ and $\vec{\omega}_2$ have the same orientation.

1'- $(\vec{v}_1+\vec{v}_2)(q) = (\vec{v}_1+\vec{v}_2)(p) + (\vec{\omega}_1+\vec{\omega}_2) \times \overrightarrow{pq} = (\vec{v}_1+\vec{v}_2)(p) + \vec{0}$ for all p, q, so $\vec{v}_1+\vec{v}_2$ is constant; Suppose $\exists q \in \mathbb{R}^3$ s.t. $(\vec{v}_1+\vec{v}_2)(q) = \vec{0}$: Hence $\vec{\omega}_1 \times \overrightarrow{c_1 q} + (-\vec{\omega}_1) \times \overrightarrow{c_2 q} = \vec{0}$, thus $\vec{\omega}_1 \times \overrightarrow{c_1 c_2} = \vec{0}$, thus $\vec{\omega}_1 \parallel \overrightarrow{c_1 c_2}$, absurd because the axes are parallel and disjoint. Thus $\vec{v}_1+\vec{v}_2 \neq \vec{0}$.

2- Take c_1=c_2=0, thus $(\vec{v}_1+\vec{v}_2)(q) = (\vec{\omega}_1+\vec{\omega}_2) \times \overrightarrow{Oq}$, thus $(\vec{v}_1+\vec{v}_2)(O) = \vec{0}$ and $\text{Ax}(\vec{v}_1+\vec{v}_2) = O + \text{Vect}\{\vec{\omega}_1+\vec{\omega}_2\}$.

3- Here $\vec\omega := \vec\omega_1+\vec\omega_2 \neq \vec 0$ and (4.7) tells that c_0 defined by $\overrightarrow{c_1c_0} = \frac{1}{||\vec\omega||^2}\vec\omega \times (\vec v_1+\vec v_2)(c_1) = \frac{1}{||\vec\omega||^2}\vec\omega \times \vec v_2(c_1) = \frac{1}{||\vec\omega||^2}\vec\omega \times (\vec\omega_2 \times \overrightarrow{c_2c_1})$, i.e.

$$\overrightarrow{c_1c_0} = \frac{1}{||\vec\omega||^2}\Big((\vec\omega \bullet_{ge} \overrightarrow{c_2c_1})\vec\omega_2 - (\vec\omega \bullet_{ge} \vec\omega_2)\overrightarrow{c_2c_1}\Big) \tag{5.5}$$

is in $\text{Ax}(\vec v_1+\vec v_2)$, so $\text{Ax}(\vec v_1+\vec v_2) = c_0 + \text{Vect}\{\vec\omega_1+\vec\omega_2\}$.

In particular, choose c_1 and c_2 s.t. $\overrightarrow{c_1c_2} \perp \vec\omega_1$ and $\perp \vec\omega_2$, i.e. the segment $[c_1, c_2]$ is the shortest segment joining $\text{Ax}(\vec v_1)$ and $\text{Ax}(\vec v_2)$. Thus $\overrightarrow{c_1c_2} \in \text{Vect}\{\vec\omega_1, \vec\omega_2\}^\perp$ and $\overrightarrow{c_1c_2} \perp \vec\omega_1+\vec\omega_2$. Thus

$$\overrightarrow{c_1c_0} = -\frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2}\overrightarrow{c_2c_1}, \quad \text{and} \quad \overrightarrow{c_2c_0} = \overrightarrow{c_2c_1} + \overrightarrow{c_1c_0} = (1 - \frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2})\overrightarrow{c_2c_1}. \tag{5.6}$$

In particular c_0 is in the straight line containing c_1, c_2. Thus $\vec v_1(c_0) = \vec\omega_1 \times \overrightarrow{c_1c_0} = -\frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2}\vec\omega_1 \times \overrightarrow{c_2c_1}$, and $\vec v_2(c_0) = \vec\omega_2 \times \overrightarrow{c_2c_0} = (1 - \frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2})\vec\omega_2 \times \overrightarrow{c_2c_1}$. Thus $(\vec v_1+\vec v_2)(c_0) = (-\frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2}\vec\omega_1 + (1 - \frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2})\vec\omega_2) \times \overrightarrow{c_2c_1}$. And $\vec\omega_1$ and $\vec\omega_2$ are independent, thus $\vec\omega$ and $\vec\omega_2$ are independent, thus $\vec\omega \bullet_{ge} \vec\omega_2 \neq 0$ and $(-\frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2}\vec\omega_1 + (1 - \frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2})\vec\omega_2) \neq \vec 0$, together with $(-\frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2}\vec\omega_1 + (1 - \frac{\vec\omega \bullet_{ge} \vec\omega_2}{||\vec\omega||^2})\vec\omega_2) \perp \overrightarrow{c_2c_1} \neq \vec 0$; Thus $(\vec v_1+\vec v_2)(c_0) \neq \vec 0$, thus $\vec v_1+\vec v_2$ isn't a pure rotation. ▪

6 Wrench = static torsor

6.1 Definition

Let $(\vec e_i)$ be a Euclidean basis and $\times_e \overset{\text{noted}}{=} \times$.

Definition 6.1 Let $P_0 \in \mathbb{R}^3$ (e.g. the position of a bolt). Let $P_{\vec f} \in \mathbb{R}^3$ and let $\vec f(P_{\vec f})$ be a vector at $P_{\vec f}$ interpreted as a force at $P_{\vec f}$. The moment $\vec M_{\vec f}(P_0)$ called the torque at P_0 applied by the force $\vec f(P_{\vec f})$ is

$$\vec M_{\vec f}(P_0) := \vec f(P_{\vec f}) \times \overrightarrow{P_{\vec f}P_0} \quad (\in \text{Vect}\{\vec f(P_{\vec f}), \overrightarrow{P_{\vec f}P_0}\}^\perp). \tag{6.1}$$

The "moment arm" at P_0 is the distance between the straight line $P_{\vec f} + \text{Vect}\{\vec f(P_{\vec f})\}$ and P_0, i.e. the distance between P_0 and its orthogonal projection on $P_{\vec f} + \text{Vect}\{\vec f(P_{\vec f})\}$.

Definition 6.2 If Ω is a set in $\mathbb{R}^3$ then the wrench due to $\vec f(P_{\vec f})$ is the screw $\vec M_{\vec f} : \Omega \to \overrightarrow{\mathbb{R}^3}$ defined by: For all $P \in \Omega$,

$$\vec M_{\vec f}(P) = \vec f(P_{\vec f}) \times \overrightarrow{P_{\vec f}P} \quad (= \overrightarrow{PP_{\vec f}} \times \vec f(P_{\vec f})). \tag{6.2}$$

$\vec f(P_{\vec f})$ is the resultant vector of the wrench, and $\vec M_{\vec f}(P)$ is the moment at P. (So $\vec M_{\vec f}(P_{\vec f}) = \vec 0$ and $\text{Ax}(\vec M_{\vec f}) = P_{\vec f} + \text{Vect}\{\vec f(P_{\vec f})\}$).

Remark 6.3 So: A torque $\vec M_{\vec f}(P_0)$ is used to screw a nut which is at P_0. A wrench $\vec M_{\vec f}$ gives the torque $\vec M_{\vec f}(P)$ on any point P in $\mathbb{R}^3$ due to $\vec f(P_{\vec f})$ at $P_{\vec f}$. ▪

6.2 Couple of forces and resulting wrench

Consider two vectors (forces) $\vec f_1(P_{f_1})$ and $\vec f_2(P_{f_2})$ at two distinct points P_{f_1} and P_{f_2}.

Let $P_0 = P_{f_1} + \frac12\overrightarrow{P_{f_1}P_{f_2}}$ (the midpoint, e.g. P_0 is the position of a nut holding a car wheel and P_{f_1} and P_{f_2} are the ends of a lug wrench used to unscrew the nut, drawing)). So $\overrightarrow{P_{f_2}P_0} = -\overrightarrow{P_{f_1}P_0}$. And suppose that $\vec f_2(P_{f_2}) = -\vec f_1(P_{f_1})$ and $\vec f_1(P_{f_1}) \perp \overrightarrow{P_{f_1}P_0}$ (drawing). We get: The sum of the torques at P_0 is

$$\vec M(P_0) := \vec M_{\vec f_1}(P_0) + \vec M_{\vec f_2}(P_0) = \vec f_1(P_{f_1}) \times \overrightarrow{P_{f_1}P_0} + \vec f_2(P_{f_2}) \times \overrightarrow{P_0P_{f_2}} = 2\,\vec f_1(P_{f_1}) \times \overrightarrow{P_{f_1}P_0} \tag{6.3}$$

(expected result).

More generally, let Ω be the segment $[P_{f_1}, P_{f_2}]$ and $P \in [P_{f_1}, P_{f_2}]$ (so $P = P_{f_1} + \lambda\overrightarrow{P_{f_1}P_{f_2}}$). We get the wrenches $\vec M_{\vec f_1}$ and $\vec M_{\vec f_2}$ defined in $[P_{f_1}, P_{f_2}]$ and their sum:

$$\vec M(P) := (\vec M_{\vec f_1}+\vec M_{\vec f_2})(P) = \vec f_1(P_{f_1}) \times \overrightarrow{P_{f_1}P} + \vec f_2(P_{f_2}) \times \overrightarrow{PP_{f_2}} = \vec f_1(P_{f_1}) \times \overrightarrow{P_{f_1}P_{f_2}} = \overrightarrow{\text{constant}} \tag{6.4}$$

(independent of P); In fact, the "moment arms" $d(P, P_{f_1})$ and $d(P, P_{f_2})$ ("one short and one long") give (6.4). This wrench $\vec M$ is a constant screw along $[P_{f_1}, P_{f_2}]$.

More generally Ω is extended to $\mathbb{R}^3$: we also get (6.4): The wrench $\vec M$ is a constant screw in $\mathbb{R}^3$.

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