Mechanics: Surfaces, geodesics, connections, Riemannian curvature tensor

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In surfaces in $\mathbb{R}^n$, or more generally in varieties:

1- The branch of mathematics which adds to a geometry the connections (derivations), parallel transport, geodesics and curvature is called "affine geometry" (quantification of "variations").

2- The branch of mathematics which adds a metric to a geometry is called "Riemannian geometry".

3- The union of these two branches of mathematics gives "differential geometry".

Part I

Surfaces in $\mathbb{R}^n$

1 Surfaces and bases

1.1 Surface

Let $n = 1$, 2 or 3, let $\mathbb{R}^n$ be the affine geometric space, and let $\vec{\mathbb{R}^n}$ be the associated vector space. Let $(\vec{E}_i)_{i=1,...,n} =^{\text{written}} (\vec{E}_i)$ be a Euclidean basis in $\mathbb{R}^n$:

$$(\vec{E}_1, ..., \vec{E}_n) \text{ Euclidean basis.} \tag{1.1}$$

Let $m \in \mathbb{N}^*$, $m \le n$, and $\vec{\mathbb{R}^m} = \mathbb{R} \times ... \times \mathbb{R}$, (the Cartesian space of "parameters", e.g. $m = 2$ and $\vec{\mathbb{R}^2} = \{(r, \theta)\}$ the space of polar coordinates, see example 1.2). Let $(\vec{A}_i)_{i=1,...,m} =^{\text{written}} (\vec{A}_i)$ be the canonical basis in $\vec{\mathbb{R}^m}$, that is,

$$\vec{A}_1 = (1, 0, ..., 0), \ ..., \vec{A}_m = (0, ..., 0, 1). \tag{1.2}$$

Let U be an open set in $\vec{\mathbb{R}^m}$ (set of parameters).

Definition 1.1 A (parametric) regular surface in $\vec{\mathbb{R}^n}$ is a C^∞ map

$$\Phi : \begin{cases} U \subset \vec{\mathbb{R}^m} & \to S \subset \mathbb{R}^n, \\ \quad \vec{q} & \to p = \Phi(\vec{q}), \end{cases} \tag{1.3}$$

which is surjective on S and of rank m for all $\vec{q} \in U$ (the surface S is m-dimensional). And Φ is called a coordinate system on S. The geometric surface S is the range (or image) of Φ, that is,

$$S = \Phi(U) = \text{Im}\Phi = \bigcup_{\vec{q} \in U} \{p = \Phi(\vec{q})\}. \tag{1.4}$$

Example 1.2 (Polar coordinates in the affine geometric space $\mathbb{R}^2$.) m=n=2. E.g. $U = \{(r, \theta) \in \mathbb{R}^*_+ \times] - \pi, \pi[\} \subset \vec{\mathbb{R}}^2$. Let $(\vec{A}_1, \vec{A}_2)$ be the canonical basis in $\vec{\mathbb{R}^2}$, and $\vec{q} = r\vec{A}_1 + \theta\vec{A}_2 =^{\text{written}} (r, \theta) \in U$ (parameter). Let O be an origin in the affine geometric space $\mathbb{R}^2$ (the center of the disk in the following) and $(\vec{E}_1, \vec{E}_2)$ be a Euclidean basis in the geometric space $\vec{\mathbb{R}^2}$. And consider the polar coordinate system:

$$\Phi : \begin{cases} \quad U & \to \mathbb{R}^2 \\ \vec{q} = (r, \theta) & \to p = \Phi(\vec{q}) = O + r\cos\theta \vec{E}_1 + r\sin\theta \vec{E}_2 = O + \vec{x}, \end{cases} \tag{1.5}$$

so $\vec{x} = \overrightarrow{Op}$ and (column matrix representing $\vec{x}$ relative to $(\vec{E}_1, \vec{E}_2)$)

$$[\vec{x}]_{|\vec{E}} = [\overrightarrow{Op}]_{|\vec{E}} = [\overrightarrow{O\Phi(\vec{q})}]_{|\vec{E}} = \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}. \tag{1.6}$$

Interpretation or the parameters: $r = ||\vec{x}||$ is the length of $\vec{x}$ in the unit given by $(\vec{E}_i)$ and θ is the angle between $\vec{x}$ and $\vec{E}_1$. Here $S = \text{Im}\Phi$ is $\mathbb{R}^2$ without the left axis $\{(x, y) : x \le 0, \ y = 0\}$. ▚

Example 1.3 (Cercle.) n=2 and m=1. Let $R > 0$ and $U =] - \pi, \pi[$. With (1.5), consider

$$\Phi_R(\theta) = \Phi(R, \theta). \tag{1.7}$$

So:

$$\vec{x} = R\cos\theta \vec{E}_1 + R\sin\theta \vec{E}_2, \quad [\vec{x}]_{|\vec{E}} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \end{pmatrix}. \tag{1.8}$$

θ is the parameter. The geometric surface $S = \text{Im}\Phi_R$ is the circle $C(\vec{0}, R)$ with center $\vec{0}$ and radius R without the point $(-1, 0)$. Here, in $\mathbb{R}^2$, the surface Φ_R is a curve. ▚

Example 1.4 See § 3 for other usual examples. ▚

Definition 1.5 A material coordinate system is a coordinate system which depends on time:

$$\Phi : \begin{cases} \mathbb{R} \times U & \to \mathbb{R}^n \\ \ (t, \vec{q}) & \to p = \Phi(t, \vec{q}). \end{cases} \tag{1.9}$$

1.2 Coordinate lines, and the basis of a coordinate system

Consider a coordinate system Φ, cf. (1.3). Let $\vec{q_0} = \sum_{i=1}^m q_0^i \vec{A}_i \in U$, and $p_0 = \Phi(\vec{q_0})$. Let $j \in [1, m]_{\mathbb{N}}$. The j-th Cartesian line in $\vec{\mathbb{R}^m}$ at q_0 is $a_{q_0}^{(j)} : \begin{Bmatrix}] - \varepsilon, \varepsilon[& \longrightarrow \vec{\mathbb{R}^m} \\ u & \longrightarrow q = a_{q_0}^{(j)}(u) = \vec{q_0} + u\vec{A}_j \end{Bmatrix}$, and $\frac{da_{q_0}^{(j)}}{du}(u) = \vec{A}_j$ is the tangent vector at q.

Definition 1.6 The j-th coordinate line at $p_0 = \Phi(\vec{q}_0)$ is the curve

$$c_{p_0}^{(j)} := \Phi \circ a_{q_0}^{(j)} : \begin{cases}]-\varepsilon, \varepsilon[& \longrightarrow S \subset \mathbb{R}^n \\ u & \longrightarrow p = c_{p_0}^{(j)}(u) = \Phi(\vec{q}_0 + u\vec{A}_j) = \Phi(q_0^1, ..., q_0^{j-1}, q_0^j + u, q_0^{j+1}, ..., q_0^n). \end{cases} \tag{1.10}$$

In particular $c_{p_0}^{(j)}(0) = p_0$ (and ε has been chosen small enough for $\vec{q}_0 + u\vec{A}_j$ to be in U).

Definition 1.7 The j-th coordinate basis vector at $p_0 = \Phi(\vec{q}_0)$ is

$$\vec{e}_j(p_0) := \frac{dc_{p_0}^{(j)}}{du}(0) \quad (= \lim_{h \to 0} \frac{c_{p_0}^{(j)}(h) - c_{p_0}^{(j)}(0)}{h}), \tag{1.11}$$

tangent vector of the j-th coordinate curve at $p_0 = \Phi(\vec{q}_0)$. So, with $c_{p_0}^{(j)} := \Phi \circ a_{q_0}^{(j)}$, cf. (1.10), we have

$$\vec{e}_j(p_0) = d\Phi(\vec{q}).\vec{A}_j \overset{named}{=} \frac{\partial \Phi}{\partial q^j}(\vec{q}_0) \quad (= \lim_{h \to 0} \frac{\Phi(\vec{q}_0 + h\vec{A}_j) - \Phi(\vec{q}_0)}{h}). \tag{1.12}$$

And, Φ being regular, for all $p \in S$, the $\vec{e}_i(p)$ are independent ($i = 1, ..., m$): They form a basis at p_0 in $\text{Vect}\{\vec{e}_1(p_0), ..., \vec{e}_n(p_0)\}$ called the tangent space at p_0.

Example 1.8 Continuation of the example 1.2: $\overrightarrow{Op} = \overrightarrow{O\Phi(\vec{q})} = \overrightarrow{O\Phi(r, \theta)} = r\cos\theta \vec{E}_1 + r\sin\theta \vec{E}_2$, $c_p^{(1)} : r \to c_p^{(1)}(r) = \Phi_\theta(r)$ (radius), $c_p^{(2)} : \theta \to c_p^{(2)}(\theta) = \Phi_r(\theta)$ (circle), and (1.12) give

$$\begin{cases} \vec{e}_1(p) = d\Phi(p).\vec{A}_1 \overset{named}{=} \dfrac{\partial \Phi}{\partial r}(r, \theta) = \cos\theta \vec{E}_1 + \sin\theta \vec{E}_2, \quad [\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \\ \vec{e}_2(p) = d\Phi(p).\vec{A}_2 \overset{named}{=} \dfrac{\partial \Phi}{\partial \theta}(r, \theta) = -r\sin\theta \vec{E}_1 + r\cos\theta \vec{E}_2, \quad [\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}. \end{cases} \tag{1.13}$$

▪

Example 1.9 Continuation of the example 1.3. $m = 1$ and the canonical vector basis in $\mathbb{R}$ is written $A_1 = 1$. The coordinate line is $\theta \to p = \Phi_R(\theta)$ (circle), and the tangent vector to the coordinate line at p is $\vec{f}_1(p) = d\Phi_R(\theta).A_1 = \frac{d\Phi_R}{d\theta}(\theta) = \Phi_R'(\theta) = -R\sin\theta \vec{E}_1 + R\cos\theta \vec{E}_2$, cf. (1.12). Here, relatively to (1.13), $\vec{f}_1(p) = \vec{e}_2(p)$ at $p = \Phi(R, \theta)$. ▪

Exercise 1.10 Prove the mean value theorem in S surface in $\mathbb{R}^n$ (affine): If $T > 0$, if $c : t \in [-T, T] \to p_t = c(t) \in S$ is a regular curve in S and $\vec{v}(p_t) = c'(t)$ is the tangent vector at $p_t = c(t)$, then there exists $u \in]t, t+h[$ s.t. $p_u = c(u)$ satisifies

$$f(p_{t+h}) - f(p) = h\, df(p_u).\vec{v}(p_u). \tag{1.14}$$

Answer. Let $g(t) = f(c(t))$, thus $g'(t) = df(c(t)).c'(t)$, thus $\exists u \in]t, t+h[$ s.t. $g(t+h) - g(t) = h\, g'(u)$ (mean value theorem in $\mathbb{R}$), i.e. (1.14). ▪

1.3 The tangent space at p

Definition 1.11 With (1.12), the tangent space at p_0 of $S = \Phi(U)$ is

$$T_{p_0}S = \text{Vect}\{\vec{e}_1(p_0), ..., \vec{e}_m(p_0)\} \subset \vec{\mathbb{R}^n}. \tag{1.15}$$

(Φ being regular, $(\vec{e}_1(p_0), ..., \vec{e}_m(p_0))$ is a basis in $T_{p_0}S$, so $\dim T_{p_0}S = m$.)

(If $m = 2$ and $n = 3$ then T_pS is the tangent plane at p, and if $m = 1$ and $n = 2$ or 3 then T_pS is the tangent line at p.)

1.4 The fiber at p

Definition 1.12 Using (1.15), the fiber at $p \in S$ is the couple

$$(p, T_pS) \in S \times \vec{\mathbb{R}^n}. \tag{1.16}$$

(A vector in $\vec{\mathbb{R}^n}$ is "drawn anywhere". While an element $(p, \vec{w}_p)$ of the fiber (p, T_pS) is "the vector $\vec{w}_p$ drawn at p".)

1.5 The tangent bundle TS

Definition 1.13 The tangent bundle TS is the set of fibers:

$$TS = \bigcup_{p\in S}(p, T_pS) \ \subset\ S \times \vec{\mathbb{R}^n}. \tag{1.17}$$

(Subset of the cross product "affine space $\mathbb{R}^n$" $\times$ "vector space $\vec{\mathbb{R}^n}$".)

Example 1.14 In $\mathbb{R}^2$, let S be the circle $C(\vec{0},R)$, cf. (1.8). Then the fiber at $p\in S$ can be drawn as the tangent line at S at p. And the tangent bundle can be represented by the union of these fibers.

However, the cross-product $\mathbb{R}^n\times\vec{\mathbb{R}^n}$ makes us represent a fiber as a "vertical line" at p (a line on the cylinder), and the tangent bundle is then represented as the "vertical cylinder" through $C(\vec{0},R)$. ▪

A vector field is a map

$$\tilde{\vec{v}} : \begin{cases} S & \to TS = \bigcup_{p\in S}(p, T_pS) \\ p & \mapsto \tilde{\vec{v}}(p) = (p, \vec{v}(p)), \end{cases} \tag{1.18}$$

supposed to be C^∞. Then the range (image) $\mathrm{Im}(\tilde{\vec{v}}) = \bigcup_{p\in S}(p,\vec{v}(p)) = \mathrm{graph}(\vec{v})$ is the graph of $\vec{v}$. (So, in mechanics $\vec{v}$ is a Eulerian function.)

The set of vector fields is named $\Gamma(S)$:

$$\Gamma(S) = \{\text{the set of vector fields on } S\}. \tag{1.19}$$

If there is no ambiguity, we simply write

$$\tilde{\vec{v}}(p) = (p,\vec{v}(p)) \overset{\text{written}}{=} \vec{v}(p). \tag{1.20}$$

Example 1.15 in $\mathbb{R}^2$. Polar coordinates: $\overrightarrow{Op} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$, basis at p given by $\vec{e}_1(p) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\vec{e}_2(p) = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}$: And $\vec{e}_1$ and $\vec{e}_2$ are vector fields in S. Full notation: vector fields $\tilde{\vec{e}}_i$ given by $\tilde{\vec{e}}_i(p) = (p,\vec{e}_i(p))$, cf. (1.18). ▪

1.6 Jacobian matrix of the coordinate system

Let $\vec{q}\in U\subset\vec{\mathbb{R}^m}$, $p=\Phi(\vec{q})$. Thus

$$\overrightarrow{Op} = \sum_{i=1}^n \Phi^i(\vec{q})\vec{E}_i \quad \text{gives} \quad d\Phi(\vec{q})(.) = \sum_{i=1}^n (d\Phi^i(\vec{q})(.))\vec{E}_i, \tag{1.21}$$

which means $d\Phi(\vec{q}).\vec{u}_{\vec{q}} = \sum_{i=1}^n (d\Phi^i(\vec{q}).\vec{u}_{\vec{q}})\vec{E}_i$ for all $\vec{u}_{\vec{q}}$ vector at $\vec{q}\in U$. In particular

$$d\Phi^i(\vec{q}).\vec{A}_j = \sum_{i=1}^n (d\Phi^i(\vec{q}).\vec{A}_j)\vec{E}_i = \sum_{i=1}^n \frac{\partial\Phi^i}{\partial q^j}(\vec{q})\vec{E}_i, \quad \text{and} \quad [d\Phi(\vec{q})]_{|\vec{A},\vec{E}} = [\frac{\partial\Phi^i}{\partial q^j}(\vec{q})] \tag{1.22}$$

is the Jacobian matrix of Φ relative to $(\vec{A}_i)$ and $(\vec{E}_i)$. Thus

$$[(d\Phi(\vec{q}).\vec{u}_{\vec{q}}]_{\vec{E}} = [d\Phi(\vec{q})]_{|\vec{A},\vec{E}}.[\vec{u}_{\vec{q}}]_{|\vec{A}}. \tag{1.23}$$

1.7 Notation (dq^i) for the dual basis (A^i)

Let (A^i) be the dual basis of the canonical $(\vec{A}_i)$ of the space of parameters, cf. (1.2): The dual basis (A^i) is made of the linear forms $A^i\in\mathbb{R}^{m*} = \mathcal{L}(\mathbb{R}^m,\mathbb{R})$ defined by, for all $i,j=1,...,m$,

$$A^i\in\mathbb{R}^{m*}, \quad A^i(\vec{A}_j) = \delta^i_j, \quad \text{and} \quad A^i \overset{\text{named}}{=} dq^i. \tag{1.24}$$

Thus $d\Phi(\vec{q})$ has the tensorial expression (explicit reference to the bases in use)

$$d\Phi(\vec{q}) = \sum_{i=1}^n\sum_{j=1}^m \frac{\partial\Phi^i}{\partial q^j}(\vec{q})\vec{E}_i\otimes A^j = \sum_{i=1}^n\sum_{j=1}^m \frac{\partial\Phi^i}{\partial q^j}(\vec{q})\vec{E}_i\otimes dq^j. \tag{1.25}$$

And with the contraction rules we recover, with $\vec{u} \in \vec{\mathbb{R}^m}$ and $\vec{u} = \sum_{j=1}^m u^j \vec{A}_j$,

$$d\Phi(\vec{q}).\vec{u} = (\sum_{i,j} \frac{\partial \Phi^i}{\partial q^j}(\vec{q})\vec{E}_i \otimes dq^j).\vec{u} = \sum_{i,j} \frac{\partial \Phi^i}{\partial q^j}(\vec{q})\vec{E}_i(dq^j.\vec{u}) = \sum_{i,j} \frac{\partial \Phi^i}{\partial q^j}(\vec{q})u^j\,\vec{E}_i.$$

And $[d\Phi(\vec{q}).\vec{u}]_{|\vec{E}} = [d\Phi(\vec{q})]_{|\vec{A},\vec{E}}[\vec{u}]_{|\vec{A}}$.

1.8 Notation $(dq^i(p))$ for the dual basis $(e^i(p))$

One of the difficulty is notations... The basis $(\vec{e}_i(p))$ in T_pS, of the coordinate system has been defined at (1.12). Its dual basis $(e^i(p))_{i=1,...,m}$ is made of the linear forms $e^i(p) \in \mathcal{L}(T_pS;\mathbb{R})$ defined by

$$\forall i,j = 1,...,m,\ e^i(p).\vec{e}_j(p) = \delta^i_j, \quad \text{and} \quad e^i(p) \overset{\text{written}}{=} dq^i(p). \tag{1.26}$$

Why? With (1.5) and $S = \Phi(U)$, consider

$$\Phi^{-1} : \begin{cases} S & \to U \\ p & \to \vec{q} = \Phi^{-1}(p) \overset{\text{written}}{=} \vec{q}(p) = \sum_{i=1}^m q^i(p)\vec{A}_i. \end{cases} \tag{1.27}$$

So $d\Phi^{-1}(p) \in \mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^m})$ and $d\Phi^{-1}(p).\vec{w}_p = \sum_{i=1}^m (dq^i(p).\vec{w}_p)\vec{A}_i$ for all vector $\vec{w}_p$ at p. In particular

$$\vec{A}_j = d\Phi^{-1}(p).\vec{e}_j(p) = \sum_{i=1}^m (dq^i(p).\vec{e}_j(p))\vec{A}_i \quad \Longrightarrow \quad dq^i(p).\vec{e}_j(p) = \delta^i_j,\ \forall i,j, \tag{1.28}$$

hence $dq^i(p) = e^i(p)$, hence the notation $(1.26)_2$.

So: In $\vec{\mathbb{R}^m}$, the dual basis of $(\vec{A}_i)$ is (dq^i) since here the variable name is $\vec{q}$. And in T_pS the dual basis of $(\vec{e}_i(p))$ is $(dq^i(p))$ with dq^i the exact differential form = the differential of $q^i : S \to \mathbb{R}$. So beware of the context (either in the space of parameters or in the geometric space).

1.9 Bidual basis $(\partial_i) = (\frac{\partial}{\partial q^i})$

Let $p \in S$. Let $(\partial_i(p))$ be the dual basis of $(dq^i(p))$ (the bidual basis of $(\vec{e}_i(p))$): The $\partial_j(p) \in (T_pS^*)^* =^{named} T_pS^{**}$ (dual of T_pS^* and bidual of T_pS) satisfy, for all $i,j = 1,...,m$,

$$\partial_j(p).dq^i(p) = \delta^i_j \quad (\text{thus also} = dq^i(p).\vec{e}_j(p)). \tag{1.29}$$

With the natural canonical isomorphism (see Spivak [17])

$$J : \begin{cases} E & \to E^{**} \\ \vec{v} & \to J(\vec{v}) \quad \text{s.t.} \quad J(\vec{v}).\ell = \ell.\vec{v},\ \forall \vec{v} \in E, \end{cases} \tag{1.30}$$

we write

$$J(\vec{v}) \overset{\text{written}}{=} \vec{v}, \quad \text{so} \quad \partial_i(p) = \mathcal{J}(\vec{e}_i(p)) \overset{\text{written}}{=} \vec{e}_i(p), \tag{1.31}$$

and $\partial_i =^{\text{written}} \vec{e}_i$. Application:

1.10 The notation $\frac{\partial f_S}{\partial q^j}(p)$, and interpretation

Let $f_S : p \in S \to f_S(p) \in \mathbb{R}$ be a C^1 function (defined on the geometric surface S), and consider the function $f_U : \vec{q} \in U \to f_U(\vec{q}) \in \mathbb{R}$ defined (on the parametric open set U) by

$$f_U := f_S \circ \Phi, \quad \text{i.e.} \quad f_U(\vec{q}) = f_S(p) \quad \text{when} \quad p = \Phi(\vec{q}). \tag{1.32}$$

Remember the classical notation

$$\frac{\partial f_U}{\partial q^j}(\vec{q}) := df_U(\vec{q}).\vec{A}_j. \tag{1.33}$$

So $f_U(\vec{q}) = f_S(\Phi(\vec{q}))$, thus $df_U(\vec{q}).\vec{A}_j = df_S(\Phi(\vec{q})).d\Phi(\vec{q}).\vec{A}_j = df_S(p).\vec{e}_j(p)$, thus

$$df_S(p).\vec{e}_j(p) = \frac{\partial f_U}{\partial q^j}(\vec{q}) \quad \text{when} \quad p = \Phi(\vec{q}) : \tag{1.34}$$

Definition 1.16 (notation) Also f_S is a function acting on p, not on $\vec{q}$, we define

$$\frac{\partial f_S}{\partial q^j}(p) := \frac{\partial f_U}{\partial q^j}(\vec{q}), \quad \text{i.e.} \quad \frac{\partial f_S}{\partial q^j}(p) := df_S(p).\vec{e}_j(p) \quad (= \frac{\partial (f_S \circ \Phi)}{\partial q^j}(\vec{q}) = df_U(\vec{q}).\vec{A}_j). \tag{1.35}$$

In other words, $\frac{\partial f_S}{\partial q^i}(p)$ is the derivative of f_S along the i-th coordinate line, cf. (1.10):

$$\lim_{h\to 0} \frac{(f_S \circ c_p^{(i)})(h) - (f_S \circ c_p^{(i)})(0)}{h} = (f_S \circ c_p^{(i)})'(0) = df_S(p).\vec{e}_i(p) \overset{\text{written}}{=} \frac{\partial f_S}{\partial q^i}(p). \tag{1.36}$$

This is also the interpretation of $\partial_i(p)$, cf. (1.29): At p, $\partial_i(p)$ is the directional derivative along $\vec{e}_i(p)$: For $i = 1, ..., m$,

$$\partial_i(p) = \frac{\partial}{\partial q^i}(p) : \begin{cases} C^1(S;\mathbb{R}) & \to \mathbb{R}, \\ f_S & \to \partial_i(p)(f_S) = (\frac{\partial}{\partial q^i}(p))(f_S) := df_S(p).\vec{e}_i(p) \overset{named}{=} \frac{\partial f_S}{\partial q^i}(p). \end{cases} \tag{1.37}$$

Thus (1.35) and (1.26) give

$$df_S(p) = \sum_{i=1}^m \frac{\partial f_S}{\partial q^i}(p)\, dq^i(p) \quad (= \sum_{i=1}^m \frac{\partial f_S}{\partial q^i}(p)\, e^i(p)). \tag{1.38}$$

Indeed, with the contaction rules, for all $j = 1, ..., m$,

$$(\sum_{i=1}^m \frac{\partial f_S}{\partial q^i}(p)\, \underbrace{dq^i(p)).\vec{e}_j(p)} = \sum_{i=1}^m \frac{\partial f_S}{\partial q^i}(p)\, \underbrace{(dq^i(p).\vec{e}_j(p))}_{\delta_{ij}} \overset{(1.26)}{=} \frac{\partial f_S}{\partial q^j}(p) \overset{(1.35)}{=} df_S(p).\vec{e}_j(p). \tag{1.39}$$

Example 1.17 Polar coordinates. With (1.6): $x = r\cos\theta$, $y = r\sin\theta$, $\overrightarrow{Op} = (x, y)$. Let $f_S(p) =^{\text{written}} f_S(x, y)$. Let $f_U(r, \theta) = f_S(x, y)$. Thus $\frac{\partial f_S}{\partial r}(x, y) := \frac{\partial f_U}{\partial r}(r, \theta)$.

E.g. $f_S(p) = xy^2$ gives $f_S(p) = f_U(r, \theta) = r^3 \cos\theta \sin^2\theta$, thus $\frac{\partial f_S}{\partial r}(x, y) := \frac{\partial f_U}{\partial r}(r, \theta) = 3r^2 \cos\theta \sin^2\theta = 3\frac{xy^2}{\sqrt{x^2+y^2}} =^{\text{written}} \frac{\partial f_S}{\partial r}(x, y)$; And $\frac{\partial f_S}{\partial \theta}(x, y) := \frac{\partial f_U}{\partial \theta}(r, \theta) = r^3(-\sin^3\theta + 2\cos^2\theta \sin\theta) = -y^3 + 2x^2 y =^{\text{written}} \frac{\partial f_S}{\partial \theta}(x, y)$.

Check: $f_S(p) = xy^2$ gives $df_S(x, y) = y^2\, dx + 2xy\, dy$ (Cartesian coordinates); Thus, with $\vec{e}_1(p) = \cos\theta \vec{E}_1 + \sin\theta \vec{E}_2$, we get $\frac{\partial f_S}{\partial r}(p) := df_S(p).\vec{e}_1(\vec{x}) = y^2\cos\theta + 2xy\sin\theta = r^2 \sin^2\theta \cos\theta + 2r^2\cos\theta \sin\theta \sin\theta$; and, with $\vec{e}_2(p) = -r\sin\theta \vec{E}_1 + r\cos\theta \vec{E}_2$, we get $\frac{\partial f_S}{\partial \theta}(p) := df_S(p).\vec{e}_2(\vec{x}) = y^2(-r\sin\theta) + 2xy(r\cos\theta) = -r^3\sin^3\theta + 2r^3\cos\theta\sin\theta\cos\theta$. ▪

Remark 1.18 Classic issue when $f_S \in \mathcal{F}(S;\mathbb{R})$ and $f_S = 0$ outside S: The classic definition $\frac{\partial f_S}{\partial x^j}(p) = \lim_{h\to 0} \frac{f_S(p+h\vec{E}_j) - f_S(p)}{h} = df_S(p).\vec{E}_i$ is not defined in general, since $p + h\vec{E}_j \notin S$ give $f_S(p + h\vec{E}_j) = 0$ and then $\lim_{h\to 0} \frac{f_S(p+h\vec{E}_j) - f_S(p)}{h} = \pm\infty$ in general.

On the other hand, $\frac{\partial f_S}{\partial q^j}(p) = df_S(p).\vec{e}_j(p) := \lim_{h\to 0} \frac{(f_S\circ c_p^{(j)})(h) - (f_S\circ c_p^{(j)})(0)}{h}$ is well defined (finite value: The curve $c^{(j)}$ is in S). ▪

Exercise 1.19 Matrix calculations: Let $\Phi(\vec{q}) = \sum_{i=1}^n \Phi^i(\vec{q})\vec{E}_i$ and $[d\Phi(\vec{q})]_{|\vec{A},\vec{E}} = [\frac{\partial \Phi^i}{\partial q^j}(\vec{q})]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$. Let $f_U = f_S \circ \Phi$. Prove:

$$[df_U(\vec{q})]_{|\vec{A}} = [df_S(p)]_{|\vec{E}}.[d\Phi(\vec{q})]_{|\vec{A},\vec{E}} = [df_S(p)]_{|\vec{e}}. \tag{1.40}$$

Answer. We have $\frac{\partial f_U}{\partial q^j}(\vec{q}) = df_U(\vec{q}).\vec{A}_j$ for $j = 1, ..., m$ (usual Cartesian notation). And $\frac{\partial f_S}{\partial x^j}(p) = df_S(p).\vec{E}_j$ for $j = 1, ..., n$ (usual Cartesian notation). And $\frac{\partial f_S}{\partial q^j}(p) = df_S(p).\vec{e}_j(p)$ for $j = 1, ..., m$, cf. (1.35). So:

$$df_U(\vec{q}) = \sum_{j=1}^m \frac{\partial f_U}{\partial q^j}(\vec{q})\, dq^j, \quad \text{and} \quad df_S(p) = \sum_{i=1}^n \frac{\partial f_S}{\partial x^j}(p)\, dx^j = \sum_{j=1}^m \frac{\partial f_S}{\partial q^j}(p)\, dq^j(p). \tag{1.41}$$

Consider the matrices $[df_U(\vec{q})]_{|\vec{A}} = [\frac{\partial f_U}{\partial q^j}(\vec{q})]_{j=1,\dots,m}$, $[df_S(p)]_{|\vec{E}} = [\frac{\partial p}{\partial x^j}(p)]_{j=1,\dots,n}$, and $[df_S(p)]_{|\vec{e}} = [\frac{\partial f_S}{\partial q^j}(\vec{q})]_{j=1,\dots,m} = [df_U(\vec{q})]_{|\vec{A}}$, cf. (1.35) (line matrices for linear forms). $f_U = f_S \circ \Phi$ gives $df_U(\vec{q}).\vec{A}_j = df_S(p).d\Phi(\vec{q}).\vec{A}_j$, that is, $\frac{\partial f_U}{\partial q^j}(\vec{q}) = df_S(p).(\sum_{i=1}^n \frac{\partial \Phi^i}{\partial q^j}(\vec{q})\, \vec{E}_i) = \sum_{i=1}^n \frac{\partial \Phi^i}{\partial q^j}(\vec{q})\, df_S(p).\vec{E}_i = \sum_{i=1}^n \frac{\partial \Phi^i}{\partial q^j}(\vec{q})\, \frac{\partial f_S}{\partial x^i}(p)$, for all $j = 1, ..., m$, that is, $[df_U(\vec{q})]_{|\vec{A}} = [df_S(p)]_{|\vec{E}}.[d\Phi(\vec{q})]_{|\vec{A},\vec{E}} = [df_S(p)]_{|\vec{e}}$. ▪

1.11 Notation $\frac{\partial \vec{w}}{\partial q^j}(p)$

Let $\vec{w}_S : p \in S \to f_S(p) \in \vec{\mathbb{R}^n}$ be a C^1 function (defined in S), and consider the function $\vec{w}_U : \vec{q} \in U \to f_U(\vec{q}) \in \vec{\mathbb{R}^m}$ defined (in U) by

$$\vec{w}_U := \vec{w}_S \circ \Phi, \quad \text{i.e.} \quad \vec{w}_U(\vec{q}) = \vec{w}_S(p) \quad \text{when} \quad p = \Phi(\vec{q}). \tag{1.42}$$

So $\vec{w}_U(\vec{q}) = \vec{w}_S(\Phi(\vec{q}))$, thus $d\vec{w}_U(\vec{q}).\vec{A}_j = d\vec{w}_S(\Phi(\vec{q})).d\Phi(\vec{q}).\vec{A}_j$, that is,

$$d\vec{w}_S(p).\vec{e}_j(p) = d\vec{w}_U(\vec{q}).\vec{A}_j \quad \text{when} \quad p = \Phi(\vec{q}). \tag{1.43}$$

Definition (notation):

$$\frac{\partial \vec{w}_S}{\partial q^j}(p) := \frac{\partial \vec{w}_U}{\partial q^j}(\vec{q}), \quad \text{i.e.} \quad \frac{\partial \vec{w}_S}{\partial q^j}(p) := d\vec{w}_S(p).\vec{e}_j(p) \quad \Big(= \frac{\partial(\vec{w}_S \circ \Phi)}{\partial q^j}(p) = d\vec{w}_U(\vec{q}).\vec{A}_j\Big). \tag{1.44}$$

NB: $\vec{w}_S$ is a function of p, not a function of $\vec{q}$, thus the notation $\frac{\partial \vec{w}_S}{\partial q^j}$ would be meaningless without the definition (notation) (1.35).

Thus, with $p = \Phi(\vec{q})$ and $c_p^{(j)}(h) = \Phi(\vec{q} + h\vec{A}_j)$,

$$\begin{aligned}\frac{\partial \vec{w}_S}{\partial q^j}(p) := \frac{\partial \vec{w}_U}{\partial q^j}(\vec{q}) &= \lim_{h\to 0} \frac{(\vec{w}_U(\vec{q} + h\vec{A}_j) - \vec{w}_U(\vec{q})}{h} \\ &= \lim_{h\to 0} \frac{(\vec{w}_S \circ c_p^{(j)})(h) - (\vec{w}_S \circ c_p^{(j)})(0)}{h} = d\vec{w}_S(p).\vec{e}_j(p) = (\vec{w}_S \circ c_p^{(j)})'(0).\end{aligned} \tag{1.45}$$

Let $\vec{w}_U(p) = \sum_{i=1}^n w_U^i(\vec{q})\, \vec{A}_i$ and $\vec{w}_S(p) = \sum_{i=1}^n w_S^i(p)\, \vec{E}_i$. Then

$$d\vec{w}_S(p) = \sum_{i=1}^n \sum_{j=1}^m \frac{\partial w_S^i}{\partial q^j}(p) \vec{E}_i \otimes e^j(p), \quad [d\vec{w}_S(p)]_{|\vec{E},\vec{e}(p)} = [\frac{\partial w_S^i}{\partial q^j}(p)] = [\frac{\partial w_U^i}{\partial q^j}(\vec{q})]. \tag{1.46}$$

Indeed, $d\vec{w}_S(p) = \sum_{i=1}^n \vec{E}_i \otimes dw_S^i(p)$ gives $d\vec{w}_S(p).\vec{e}_j(p) = (\sum_{i=1}^n \vec{E}_i \otimes dw_S^i(p)).\vec{e}_j(p) = \sum_{i=1}^n \vec{E}_i(dw_S^i(p).\vec{e}_j(p)) = \sum_{i=1}^n \vec{E}_i \frac{\partial w_S^i}{\partial q^j}(p)$, which is also the result of $(\sum_{i=1}^n \sum_{j=1}^m \frac{\partial w_S^i}{\partial q^j}(p)\vec{E}_i \otimes e^j(p)).\vec{e}_j(p)$.

Exercise 1.20 Polar coordinate (1.6): $[\overrightarrow{Op}]_{|\vec{E}} = [\vec{x}]_{|\vec{E}} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$ and, cf. (1.13),

$$\vec{e}_1(p) = \cos\theta \vec{E}_1 + \sin\theta \vec{E}_2 \overset{\text{written}}{=} \vec{a}_1(r,\theta), \quad \text{and} \quad \vec{e}_2(p) = -r\sin\theta \vec{E}_1 + r\cos\theta \vec{E}_2 \overset{\text{written}}{=} \vec{a}_2(r,\theta). \tag{1.47}$$

With $\vec{q} = (r,\theta)$ and $p = \Phi(\vec{q}) = \Phi(r,\theta)$, prove:

$$[\frac{\partial \vec{e}_1}{\partial r}(p)]_{|\vec{E}} = \vec{0}, \quad [\frac{\partial \vec{e}_1}{\partial \theta}(p)]_{|\vec{E}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \quad [\frac{\partial \vec{e}_2}{\partial r}(p)]_{|\vec{E}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \quad [\frac{\partial \vec{e}_2}{\partial \theta}(p)]_{|\vec{E}} = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix}. \tag{1.48}$$

Deduce:

$$[d\vec{e}_1]_{|\vec{E},\vec{e}(p)} = \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{pmatrix} = [d\vec{a}_1]_{|\vec{A},\vec{E}}, \quad [d\vec{e}_2]_{|\vec{E},\vec{e}(p)} = \begin{pmatrix} -\sin\theta & -r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix} = [d\vec{a}_2]_{|\vec{A},\vec{E}}. \tag{1.49}$$

Answer. Let $\vec{a}_i(\vec{q}) := (\vec{e}_i \circ \Phi)(\vec{q}) = \vec{e}_i(p)$. Thus $d\vec{a}_i(\vec{q}).\vec{A}_j = d\vec{e}_i(p).\vec{e}_j(p)$.

$d\vec{a}_1(r,\theta).\vec{A}_1 = \frac{\partial \vec{a}_1}{\partial r}(r,\theta) = 0$ and $d\vec{a}_1(r,\theta).\vec{A}_2 = \frac{\partial \vec{a}_1}{\partial \theta}(r,\theta) = -\sin\theta \vec{E}_1 + \cos\theta \vec{E}_2$, thus $d\vec{a}_1(r,\theta) = -\sin\theta \vec{E}_1 \otimes \vec{A}_2 + \cos\theta \vec{E}_2 \otimes \vec{A}_2$.

$d\vec{a}_2(r,\theta).\vec{A}_1 = \frac{\partial \vec{a}_2}{\partial r}(r,\theta) = -\sin\theta \vec{E}_1 + \cos\theta \vec{E}_2$ and $d\vec{a}_2(r,\theta).\vec{A}_2 = \frac{\partial \vec{a}_2}{\partial \theta}(r,\theta) = -r\cos\theta \vec{E}_1 - r\sin\theta \vec{E}_2$, thus $d\vec{a}_2(r,\theta) = -\sin\theta \vec{E}_1 \otimes \vec{A}_1 + \cos\theta \vec{E}_2 \otimes \vec{A}_1 - r\cos\theta \vec{E}_1 \otimes \vec{A}_2 - r\sin\theta \vec{E}_2 \otimes \vec{A}_2$. ▪

2 Christoffel symbols γ_{ij}^k

2.1 Definition

2.1.1 A thickening S_+ of S

Let $\Phi : U \subset \vec{\mathbb{R}^m} \to \mathbb{R}^n$ be a parametric surface, cf. (1.3), and let $S = \Phi(U)$, cf. (1.4).

Hypothesis (makes the presentation easier): The surface S (dimension m) is considered as a part of a thickened surface S_+ (dimension n), that is, Φ is the restriction of a C^2-diffeomorphism

$$\Phi_+ : \left\{ \begin{array}{rl} U_+ \subset \vec{\mathbb{R}^n} = \vec{\mathbb{R}^m} \times \vec{\mathbb{R}^{n-m}} & \to S_+ \in \mathbb{R}^n, \\ \vec{q}_+ = (\vec{q}, \vec{z}) & \to p = \Phi(\vec{q}_+) \end{array} \right\} \quad \text{s.t.} \quad \Phi_+(\vec{q}, 0, ..., 0) = \Phi(\vec{q}),\ \forall \vec{q} \in U, \tag{2.1}$$

U_+ being an open set in $\vec{\mathbb{R}^n}$ and $U \subset U_+$; So $S = \{\Phi_+(\vec{q}, \vec{0}),\ \vec{q} \in U\} = \{\Phi(\vec{q}),\ \vec{q} \in U\}$.

Let $(\vec{A}_i)_{i=1,...,n}$ be the canonical basis in $\vec{\mathbb{R}^n} = \vec{\mathbb{R}^m} \times \vec{\mathbb{R}^{n-m}}$; Thus the basis $(\vec{e}_i(p))_{i=1,...,n}$ of the system Φ_+ at $p = \Phi_+(\vec{q}_+)$ is given by, cf. (1.12),

$$\forall j \in [1, n]_{\mathbb{N}},\ \vec{e}_j(p) = d\Phi_+(\vec{q}_+).\vec{A}_j = \frac{\partial \Phi_+}{\partial q^j}(\vec{q}_+), \tag{2.2}$$

and in particular for $p \in S$ and $j \in [1, m]_{\mathbb{N}}$ we have $\vec{e}_j(p) \in T_pS$.

2.1.2 Christoffel symbols in S_+

Definition 2.1 In S_+. Let $i, j \in [1, n]_{\mathbb{N}}$. The Christoffel symbols $(\gamma_{ij}^k(p))_{k=1,...,n}$ at $p \in S_+$ are the components of the vector $d\vec{e}_j(p).\vec{e}_i(p) \in \vec{\mathbb{R}^n}$ relative to the (local) basis $(\vec{e}_i(p))_{i\in[1,n]_{\mathbb{N}}}$, that is, for $i, j = 1, ..., n$,

$$d\vec{e}_j(p).\vec{e}_i(p) = \sum_{k=1}^{n} \gamma_{ij}^k(p)\vec{e}_k(p), \quad \text{i.e.} \quad \gamma_{ij}^k := e^k.(d\vec{e}_j.\vec{e}_i), \tag{2.3}$$

that is, for all $j, k \in [1, n]_{\mathbb{N}}$, $d\vec{e}_k(p).\vec{e}_j(p) = \sum_{i=1}^n \gamma_{jk}^i(p)\vec{e}_i(p)$, i.e. $\gamma_{jk}^i = e^i.(d\vec{e}_k.\vec{e}_j)$. So, for $k \in [1, n]_{\mathbb{N}}$,

$$d\vec{e}_k = \sum_{i,j=1}^{n} \gamma_{jk}^i \vec{e}_i \otimes e^j, \quad \text{and} \quad [d\vec{e}_k]_{|\vec{e}} = [\gamma_{jk}^i]_{\substack{i=1,...,n \\ j=1,...,n}}. \tag{2.4}$$

Example 2.2 Polar coordinates: See (3.12)-(3.11). ∴

2.1.3 Christoffel symbols in S surface in $\mathbb{R}^n$

Definition 2.3 In S. Let $i, j \in [1, m]_{\mathbb{N}}$. The Christoffel symbols $(\gamma_{ij}^k(p))_{k=1,...,n}$ at $p \in S$ are the components of the vector $d\vec{e}_j(p).\vec{e}_i(p) \in \vec{\mathbb{R}^n}$ relative to the basis $(\vec{e}_i(p))_{i\in[1,n]_{\mathbb{N}}}$, that is,

$$\forall i, j \in [1, m]_{\mathbb{N}}, \quad d\vec{e}_j(p).\vec{e}_i(p) = \sum_{k=1}^{n} \gamma_{ij}^k(p)\vec{e}_k(p), \quad \text{i.e.} \quad \gamma_{ij}^k = e^k.(d\vec{e}_j.\vec{e}_i),\ k \in [1, n]_{\mathbb{N}}. \tag{2.5}$$

NB: Although $\vec{e}_i$ and $\vec{e}_j$ are in T_pS in (2.5), the vector $d\vec{e}_j(p).\vec{e}_i(p)$ is not in T_pS in general: See e.g. (1.49) which gives $d\vec{e}_2(p).\vec{e}_2(p) = -R\vec{e}_1(p)$ which is not tangent to the circle whereas $\vec{e}_2(p)$ is. In other words, only considering the tangent vectors at p in S, we have

$$d\vec{e}_j = \sum_{k=1}^{n}\sum_{i=1}^{m} \gamma_{ij}^k \vec{e}_k \otimes e^i, \quad \text{i.e.} \quad d\vec{e}_k = \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_{jk}^i \vec{e}_i \otimes e^j \tag{2.6}$$

2.1.4 Usual Christoffel symbols in S on its own

Here we we cannot take height (we cannot gain altitude). Thus in (2.5) we can only see

$$\text{Proj}_{T_pS}(d\vec{e}_j(p).\vec{e}_i(p)) = \text{Proj}_{T_pS}(\sum_{k=1}^{n} \gamma_{ij}^k(p)\vec{e}_k(p)) = \sum_{k=1}^{m} \gamma_{ij}^k(p)\vec{e}_k(p). \tag{2.7}$$

Definition 2.4 The "usual Riemannian connection ∇" is S is characterized by, for all $p \in S$ and $i,j \in [1,m]_{\mathbb{N}}$,

$$\nabla_{\vec{e}_i}\vec{e}_j(p) = \sum_{k=1}^{m} \gamma_{ij}^k(p)\vec{e}_k(p) \quad (= \mathrm{Proj}_{T_pS}(d\vec{e}_j(p).\vec{e}_i(p))), \tag{2.8}$$

and the Christoffel symbols in S are the γ_{ij}^k for $i,j,k \in [1,m]_{\mathbb{N}}$. So, in S,

$$\gamma_{ij}^k = e^k.\nabla_{\vec{e}_i}\vec{e}_j, \tag{2.9}$$

$\nabla_{\vec{e}_i}\vec{e}_j$ being the covariant derivative of $\vec{e}_j$ along $\vec{e}_i$ restricted to T_pS.

2.2 Symmetry: $d\vec{e}_j.\vec{e}_i = d\vec{e}_i.\vec{e}_j$

We consider Φ_+ and S_+, Φ_+ being C^2, the results for Φ and S being obtained by restriction.

(2.2) gives $\vec{e}_j(\Phi_+(\vec{q})) = d\Phi_+(\vec{q}).\vec{A}_j$, thus $d\vec{e}_j(p).d\Phi_+(\vec{q}).\vec{A}_i = (d^2\Phi_+(\vec{q}).\vec{A}_i).\vec{A}_j$ for all i,j, thus,

$$d\vec{e}_j(p).\vec{e}_i(p) = \frac{\partial^2\Phi_+}{\partial q^i\partial q^j}(\vec{q}) = \frac{\partial^2\Phi_+}{\partial q^j\partial q^i}(\vec{q}) = d\vec{e}_i(p).\vec{e}_j(p) \quad (= \frac{\partial\vec{e}_j}{\partial q^i}(p) = \frac{\partial\vec{e}_i}{\partial q^j}(p)) \tag{2.10}$$

since Φ_+ is supposed C^2 (Schwarz theorem). That is, $\sum_{k=1}^m \gamma_{ij}^k\vec{e}_k = \sum_{k=1}^m \gamma_{ji}^k\vec{e}_k$ for all i,j, thus,

$$\forall i,j,k \in [1,n]_{\mathbb{N}}, \quad \gamma_{ij}^k = \gamma_{ji}^k, \tag{2.11}$$

and the Christoffel symbols are said to be (covariant) symmetric (symmetry for the bottom indices).

Corollary 2.5 *Let $f \in C^2(S_+;\mathbb{R})$ (thus d^2f is symmetric). Then, Φ_+ being C^2, for all $i,j = 1,...,n$,*

$$d(df.\vec{e}_j).\vec{e}_i = d^2f(\vec{e}_i,\vec{e}_j) + \sum_{k=1}^{n} \gamma_{ij}^k\, df.\vec{e}_k. \tag{2.12}$$

(NB: a first order derivative is still alive.) NB: with $f =^{\text{named}} f_S$ and f_U defined by $f_S(p) = f_U(\vec{q})$ when $p = \Phi(\vec{q})$, cf. (1.32), then (2.12) reads

$$\frac{\partial^2 f_U}{\partial q^i\partial q^j}(\vec{q}) = d^2f(p)(\vec{e}_i(p),\vec{e}_j(p)) + \sum_{k=1}^{n} \gamma_{ij}^k(p)\,\frac{\partial f}{\partial q^k}(p). \tag{2.13}$$

Proof. $f_U(\vec{q}) = f(\Phi(\vec{q})) = f(p)$ gives

$$\frac{\partial f_U}{\partial q^j}(\vec{q}) = df_U(\vec{q}).\vec{A}_j = df(\Phi(\vec{q})).d\Phi(\vec{q}).\vec{A}_j = df(\Phi(\vec{q})).\frac{\partial\Phi}{\partial q^j}(\vec{q}) \quad (= df(p).\vec{e}_j(p)).$$

Thus,

$$\begin{aligned}\frac{\partial\frac{\partial f_U}{\partial q^j}}{\partial q^i}(\vec{q}) &= \Big(d(df)(\Phi(\vec{q})).\frac{\partial\Phi}{\partial q^i}(\vec{q})\Big).\frac{\partial\Phi}{\partial q^j}(\vec{q}) + df(\Phi(\vec{q})).\Big(\frac{\partial\frac{\partial\Phi}{\partial q^j}}{\partial q^i}(\vec{q})\Big) \quad (= d(df(p).\vec{e}_j(p)).\vec{e}_i(p)) \\ &= \Big(d^2f(p).\vec{e}_i(p)\Big).\vec{e}_j(p) + df(p).(d\vec{e}_j(p).\vec{e}_i(p)),\end{aligned} \tag{2.14}$$

and $f \in C^2$, thus $(d^2f(p).\vec{e}_i(p)).\vec{e}_j(p) = (d^2f(p).\vec{e}_j(p)).\vec{e}_i(p) = d^2f(\vec{e}_i,\vec{e}_j)$. ∎

2.3 Geometric interpretation of $d\vec{e}_i.\vec{e}_j = d\vec{e}_j.\vec{e}_i$

It is the geometric interpretation of the Schwarz theorem: $\frac{\partial\frac{\partial\Phi_+}{\partial q^i}}{\partial q^j}(\vec{q}) = \frac{\partial\frac{\Phi_+}{\partial q^j}}{\partial q^i}(\vec{q})$ when Φ_+ is C^2: Let $p_0 = \Phi(\vec{q}_0) \in S$, and consider the coordinate lines $\vec{c}_{p_0}^{(i)} : t \to q = \vec{c}_{p_0}^{(i)}(t) = \Phi(\vec{q}_0 + t\vec{A}_i)$, cf. (1.10); Then, with $\vec{q} = \vec{q}_0 + t\vec{A}_i = \vec{c}^{(i)}_{p_0}(t)$ and $p = \Phi(\vec{q}) = \vec{c}^{(i)}_{p_0}(t)$, we get

$$\vec{c}^{(i)}_{p_0}{}'(t) = d\Phi(\vec{q}).\vec{A}_i = \vec{e}_i(p), \tag{2.15}$$

cf. (1.12). Then, see figure 2.1, let $i, j \in [1, m]_{\mathbb{N}}$, $i \neq j$, let $p_0 =^{\text{written}} P \in S$, let $h, k > 0$ ("small"), and let

$$P_i = \vec{c}^{(i)}_P(h), \quad P_j = \vec{c}^{(j)}_P(k). \tag{2.16}$$

Then let

$$P_{ij} = \vec{c}^{(j)}_{P_i}(k), \quad P_{ji} = \vec{c}^{(i)}_{P_j}(h). \tag{2.17}$$

Thus to get to P_{ij}, start from P, follow the trajectory $\vec{c}^{(i)}_P$ to the point P_i, then follow the trajectory $\vec{c}^{(j)}_{P_i}$ to the point P_{ij}. And to get to $P_{ji}(p)$, start from P, follow the trajectory $\vec{c}^{(j)}_P$ to the point P_j, then follow the trajectory $\vec{c}^{(i)}_{P_j}$ to the point $P_{ji}(p)$. See example 2.6.

Example 2.6 Polar coordinates: $P = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$, $P_1 = \begin{pmatrix} (r+h)\cos\theta \\ (r+h)\sin\theta \end{pmatrix}$, $P_{12} = \begin{pmatrix} (r+h)\cos(\theta+k) \\ (r+h)\sin(\theta+k) \end{pmatrix}$, $P_2 = \begin{pmatrix} r\cos(\theta+k) \\ r\sin(\theta+k) \end{pmatrix}$ and $P_{21} = \begin{pmatrix} (r+h)\cos(\theta+k) \\ (r+h)\sin(\theta+k) \end{pmatrix} = P_{12}$. ▪

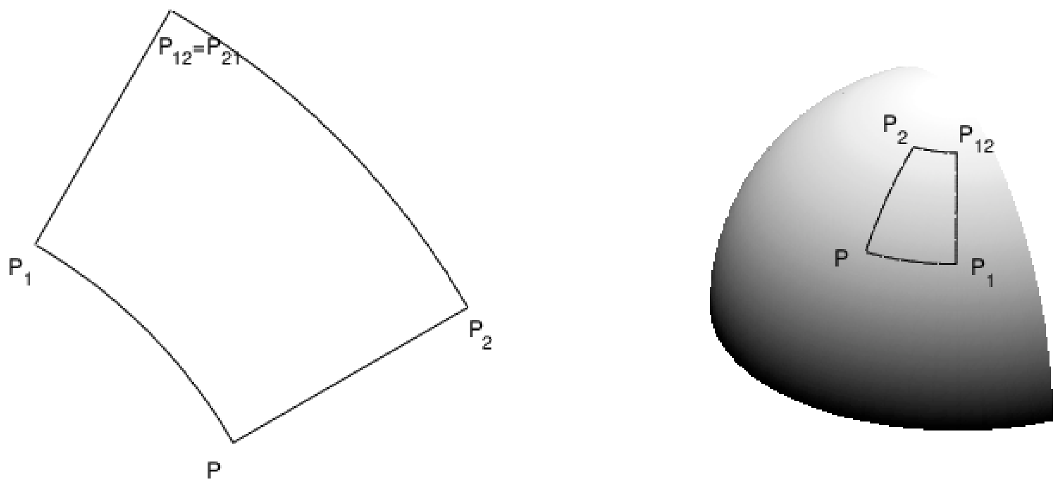

Figure 2.1: With coordinate lines, the curve is closed: $P_{12} = P_{21}$. On the left with polar coordinates, exemple 2.6, on the right with spherical coordinates along parallels and meridians.

Proposition 2.7

$$P_{ij} = P_{ji}. \tag{2.18}$$

And the interpretation of Schwarz theorem (for C^2 functions) is

$$\frac{\partial^2\Phi}{\partial q^j \partial q^i}(\vec{q}_0) = \frac{\partial^2\Phi}{\partial q^i \partial q^j}(\vec{q}_0) = \lim_{\substack{h\to 0 \\ k\to 0}} \frac{P_{ij} - P_j - P_i + P}{hk} = \lim_{\substack{h\to 0 \\ k\to 0}} \frac{P_{ji} - P_j - P_i + P}{hk}. \tag{2.19}$$

Proof. $P_i = \vec{c}^{(i)}_p(h) = \Phi(\vec{q}_0 + h\vec{A}_i)$ and $P_{ij} = \Phi((\vec{q}_0 + h\vec{A}_i) + k\vec{A}_j)$.

And $P_j = \vec{c}^{(j)}_p(k) = \Phi(\vec{q}_0 + k\vec{A}_j)$ and $P_{ji} = \Phi((\vec{q}_0 + k\vec{A}_j) + h\vec{A}_i)$. Thus $P_{ij} = P_{ji}$, i.e. (2.18).

And $\frac{\partial\Phi}{\partial q^i}(\vec{q}_0) = \lim_{h\to 0} \frac{\Phi(\vec{q}_0 + h\vec{A}_i) - \Phi(\vec{q}_0)}{h}$, thus,

$$\begin{aligned}
\frac{\partial \frac{\partial\Phi}{\partial q^i}}{\partial q^j}(\vec{q}_0) &= \lim_{k\to 0} \frac{\frac{\partial\Phi}{\partial q^i}(\vec{q}_0 + k\vec{A}_j) - \frac{\partial\Phi}{\partial q^i}(\vec{q}_0)}{k} \\
&= \lim_{\substack{h\to 0 \\ k\to 0}} \frac{\Phi(\vec{q}_0 + k\vec{A}_j + h\vec{A}_i) - \Phi(\vec{q}_0 + k\vec{A}_j) - \Phi(\vec{q}_0 + h\vec{A}_i) + \Phi(\vec{q}_0)}{hk} \\
&= \lim_{\substack{h\to 0 \\ k\to 0}} \frac{P_{ij} - P_j - P_i + P}{hk}.
\end{aligned}$$

Similarly $\frac{\partial \frac{\partial\Phi}{\partial q^j}}{\partial q^i}(\vec{q}_0) = \lim_{\substack{h\to 0 \\ k\to 0}} \frac{P_{ji} - P_i - P_j + P}{hk}$, therefore $P_{ij} = P_{ji}$ gives $\frac{\partial \frac{\partial\Phi}{\partial q^i}}{\partial q^j}(\vec{q}_0) = \frac{\partial \frac{\partial\Phi}{\partial q^j}}{\partial q^i}(\vec{q}_0)$. ▪

2.4 Identity $de^i.\vec{e}_j + e^i.d\vec{e}_j = 0$

We have seen that the dual basis $(e^i(p))$ is nothing but $(dq^i(p))$, cf. (1.26)-(1.28), where $\vec{q}(p) := \Phi^{-1}(p)$. And Φ being supposed to be at least a C^2 diffeomorphism, $de^i(p) = d^2(\Phi^{-1})(p)$ is well defined.

Proposition 2.8 *For all $i, j = 1, ..., m$, we have*

$$de^i.\vec{e}_j + e^i.d\vec{e}_j = 0, \quad \text{i.e.} \quad (de^i.\vec{v}).\vec{e}_j + e^i.(d\vec{e}_j.\vec{v}) = 0,\ \forall \vec{v} \in TS. \tag{2.20}$$

Thus $de^i = d(dq^i) = d^2q^i$ is symmetric, that is, $(de^i.\vec{u}).\vec{w} = (de^i.\vec{w}).\vec{u}$ for all $\vec{u}, \vec{w} \in \Gamma(S)$, and

$$de^i = d(dq^i) \overset{\text{written}}{=} d^2q^i. \tag{2.21}$$

Proof. $e^i(p).\vec{e}_j(p) = \delta^i_j$ gives $(de^i(p).\vec{v}_p).\vec{e}_j(p) + e^i(p).(d\vec{e}_j(p).\vec{v}_p) = 0$ for all $\vec{v}_p \in T_pS$, i.e. (2.20). Thus $(de^i.\vec{e}_k).\vec{e}_j = e^i.(d\vec{e}_j.\vec{e}_k) =^{(2.10)} e^i.(d\vec{e}_k.\vec{e}_j) =^{(2.20)} (de^i.\vec{e}_j).\vec{e}_k$, thus de^i is symmetric, or apply $de^i = d(dq^i) = d^2q^i$ with q^i C^2 since Φ is a C^2 diffeomorphism. ∎

2.5 Components of $de^i = d^2q^i$

(2.20) gives $(de^i.\vec{e}_j).\vec{e}_k + e^i.(d\vec{e}_k.\vec{e}_j) = 0$, thus $(de^i.\vec{e}_j).\vec{e}_k = -e^i.(\sum_{\ell=1}^n \gamma^\ell_{jk}\vec{e}_\ell) = -\gamma^i_{jk}$, hence

$$de^i.\vec{e}_j = -\sum_{k=1}^n \gamma^i_{jk} e^k. \tag{2.22}$$

(Einstein convention is satisfied.) So, the components of $de^i.\vec{e}_j$ are γ^i_{jk}.

$$d^2q^i = de^i = -\sum_{j,k=1}^n \gamma^i_{jk} e^j \otimes e^k. \tag{2.23}$$

Example 2.9 Cartesian basis: $\vec{e}_j$ and e^i are uniform, then γ^i_{jk} and $de^j = 0$. ∎

Exercise 2.10 Polar coordinates: compute $d^2r = de^1$ and $d^2\theta = de^2$.

Answer. (3.12) gives $d^2r = -\gamma^1_{11}e^1 \otimes e^1 - \gamma^1_{21}e^1 \otimes e^2 - \gamma^1_{12}e^2 \otimes e^1 - \gamma^1_{22}e^2 \otimes e^2 = re^2 \otimes e^2$.
And $d^2\theta = -\gamma^2_{11}e^1 \otimes e^1 - \gamma^2_{21}e^1 \otimes e^2 - \gamma^2_{12}e^2 \otimes e^1 - \gamma^2_{22}e^2 \otimes e^2 = -\frac{1}{r}(e^1 \otimes e^2 + e^2 \otimes e^1)$.
Thus

$$d^2r = r\,d\theta \otimes d\theta \quad \text{and} \quad d^2\theta = -\frac{1}{r}(dr \otimes d\theta + d\theta \otimes dr). \tag{2.24}$$

∎

2.6 Non holonomic basis

Consider m vector fields $\vec{b}_1, ..., \vec{b}_m$ in TS, so, such that $\vec{b}_i(p) \in T_pS$ for all $i \in [1, m]_{\mathbb{N}}$ and $p \in S$.

Definition 2.11 Let $p_0 \in S$ and $\vec{q}_0 = \Phi^{-1}(p_0)$. It there exists an open set V_{p_0} in S, an open set $U_{\vec{q}_0}$ in U, and a diffeomorphism $\Psi : \left\{\begin{array}{ll} U_{\vec{q}_0} & \to V_{p_0} \\ \vec{q} & \to p = \Psi(\vec{q}) \end{array}\right\}$ such that

$$\forall i \in [1, m]_{\mathbb{N}},\ \forall p \in V_{p_0},\ \vec{b}_i(p) = \frac{\partial \Psi}{\partial q^i}(\vec{q}) \quad \text{where} \quad \Psi(\vec{q}) = p, \tag{2.25}$$

then the basis $(\vec{b}_i(p))$ is said to be holonomic in V_{p_0}, associated to the local coordinate system Ψ.

Otherwise, $(\vec{b}_i)$ is not holonomic.

Example 2.12 In particular, the basis $(\vec{e}_i(p))$ of the coordinate system Φ is holonomic: Take $\Psi = \Phi$. E.g., the polar basis (3.3) is holonomic. ∎

Exercise 2.13 (Fundamental). Consider the (widely used) normalized polar basis $(\vec{b}_1, \vec{b}_2)$ given by

$$\vec{b}_1(p) = \frac{\vec{e}_1(p)}{||\vec{e}_1(p)||}, \quad \vec{b}_2(p) = \frac{\vec{e}_2(p)}{||\vec{e}_2(p)||}, \quad \text{i.e.} \quad [\vec{b}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad [\vec{b}_2(p)]_{|\vec{E}} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \tag{2.26}$$

Prove that $(\vec{b}_1, \vec{b}_2)$ is nowhere holonomic: Give two proofs, 1- the first one by proving

$$d\vec{b}_2.\vec{b}_1 - d\vec{b}_1.\vec{b}_2 = -\frac{1}{r}\vec{b}_2, \quad \text{so} \quad d\vec{b}_2.\vec{b}_1 - d\vec{b}_1.\vec{b}_2 \neq \vec{0}, \tag{2.27}$$

(and use the Schwarz equality), 2- the second one supposing that $(\vec{b}_1, \vec{b}_2)$ is holonomic.

Answer. 1) $\vec{b}_1(p) = \cos\theta\vec{E}_1 + \sin\theta\vec{E}_2$ and $\vec{b}_2(p) = -\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2$ give, with (1.26), $d\vec{b}_1(p) = -\sin\theta\vec{E}_1 \otimes d\theta + \cos\theta\vec{E}_2 \otimes d\theta$ and $d\vec{b}_2(p) = -\cos\theta\vec{E}_1 \otimes d\theta - \sin\theta\vec{E}_2 \otimes d\theta$. Thus, $d\theta.\vec{e}_1 = 0$ and $d\theta.\vec{e}_2 = 1$ give

$$\left\{ \begin{aligned} & d\vec{b}_1.\vec{b}_2 = d\vec{b}_1.\frac{\vec{e}_2}{r} = -\frac{1}{r}\sin\theta\vec{E}_1 + \frac{1}{r}\cos\theta\vec{E}_2 = -\frac{1}{r}\vec{b}_2, \\ & d\vec{b}_2.\vec{b}_1 = 0 + 0 = 0, \end{aligned} \right\}, \quad \text{and} \quad d\vec{b}_2.\vec{b}_1 - d\vec{b}_1.\vec{b}_2 = -\frac{1}{r}\vec{b}_2. \tag{2.28}$$

If $(\vec{b}_1, \vec{b}_2)$ is a basis of a coordinate system Ψ, then, at $p = \Psi(\vec{q})$, $\vec{b}_i(p) = \frac{\partial\psi}{\partial q^i}(\vec{q})$, and the Schwarz equality $\frac{\partial \frac{\partial\Psi}{\partial q^1}}{\partial q^2} = \frac{\partial \frac{\partial\Psi}{\partial q^2}}{\partial q^1}$ gives $d\vec{b}_2.\vec{b}_1 = d\vec{b}_1.\vec{b}_2$: But we have $d\vec{b}_2.\vec{b}_1 \neq d\vec{b}_1.\vec{b}_2$.

2) Suppose $\exists\Psi : \vec{q} = (q^1, q^2) \in Z \to p = \Psi(\vec{q})$ diffeomorphism, Z being a non empty open set, s.t. $\vec{b}_i(p) = \frac{\partial\Psi}{\partial q^i}(\vec{q})$. Let $\vec{Op} = \Psi^1(\vec{q})\vec{E}_1 + \Psi^2(\vec{q})\vec{E}_2$. Since $\vec{b}_1(p) = \vec{e}_1(p)$ and $\vec{b}_2(p) = \frac{\vec{e}_2(p)}{||\vec{e}_2(p)||}$ with $p = \Phi(r, \theta)$ and Φ being a diffeomorphism, then eventually replacing Ψ with $\Psi \circ \Phi$, and consider $\Psi(\vec{q}) = \psi^1(\vec{q})\vec{E}_1 + \psi^2(\vec{q})\vec{E}_2$ to be a function of $\vec{q} = (r, \theta)$. So $\cos\theta\vec{E}_1 + \sin\theta\vec{E}_2 = \vec{b}_1 = \frac{\partial\Psi}{\partial r}(r, \theta) = \frac{\partial\psi^1}{\partial r}(r, \theta)\vec{E}_1 + \frac{\partial\psi^2}{\partial r}(r, \theta)\vec{E}_2$ and $-\sin\theta\vec{E}_1 + \cos\theta\vec{E}_2 = \vec{b}_2 = \frac{\partial\Psi}{\partial\theta}(r, \theta) = \frac{\partial\psi^1}{\partial\theta}(r, \theta)\vec{E}_1 + \frac{\partial\psi^2}{\partial\theta}(r, \theta)\vec{E}_2$. Thus $\frac{\partial\psi^1}{\partial r}(r, \theta) = \cos\theta$ and $\frac{\partial\psi^1}{\partial\theta}(r, \theta) = -\sin\theta$. But $\frac{\partial\psi^1}{\partial\theta}(r, \theta) = -\sin\theta$ gives $\psi^1(r, \theta) = \cos\theta + g(r)$ thus $\frac{\partial\psi^1}{\partial r}(r, \theta) = g'(r)$ with $\frac{\partial\psi^1}{\partial r}(r, \theta) = \cos\theta$, hence $\cos\theta = g'(r)$ for all $(r, \theta) \in Z$: Absurd in any (non empty) open set Z. Thus ψ^1 does not exist, thus Ψ does not exist. ▪

Exercise 2.14 Consider the ellipse $p = \Phi(r, \theta) = \begin{pmatrix} ar\cos\theta \\ br\sin\theta \end{pmatrix}$, $0 < a < b$. Compute $(\vec{e}_1(p), \vec{e}_2(p))$ the basis of the system Φ. Then fix $r = R$, let $\varphi_R(\theta) = \varphi(R, \theta)$, and give a local coordinate system $\Psi(u, \theta)$ in $\mathbb{R}^2$ (a thickening of $S = \text{Im}(\Phi_R)$) such that: $\Psi(R, \theta) = \Phi(R, \theta)$, $\frac{\partial\Psi}{\partial\theta}(R, \theta) = \vec{e}_2(R, \theta)$ and $\frac{\partial\Psi}{\partial u}(R, \theta) \perp \vec{e}_2(R, \theta)$.

Answer. Let $\Psi(u, \theta) = \begin{pmatrix} \alpha(u)R\cos\theta \\ \beta(u)R\sin\theta \end{pmatrix}$ with $\alpha(0) = a$ and $\beta(0) = b$: We have $\Psi(R, \theta) = \Phi(R, \theta)$. Then $\frac{\partial\Psi}{\partial u}(u, \theta) = \begin{pmatrix} \alpha'(u)R\cos\theta \\ \beta'(u)R\sin\theta \end{pmatrix}$. We want $\frac{\partial\Psi}{\partial u}(R, \theta) \perp \vec{e}_2(R, \theta)$, i.e., $(\alpha'(0)R\cos\theta)(-aR\sin\theta) + (\beta'(0)R\sin\theta)(bR\cos\theta) = 0$, i.e. $a\alpha'(0) = b\beta'(0)$. Choose $\beta(u) = \frac{a}{b}\alpha(u) + c$, and $b = \beta(0) = \frac{a}{b}\alpha(0) + c = \frac{a^2}{b} + c$ gives $c = \frac{b^2 - a^2}{b}$. And, e.g., $\alpha(u) = a(1+u)$, thus $\beta(u) = \frac{a^2}{b}(1+u) + \frac{b^2 - a^2}{b}$. So $\Psi(u, \theta) = \begin{pmatrix} (au + a)R\cos\theta \\ (\frac{a^2}{b}u + b)R\sin\theta \end{pmatrix}$.
Check: $\Psi(0, \theta) = \begin{pmatrix} aR\cos\theta \\ bR\sin\theta \end{pmatrix}$, and $\frac{\partial\Psi}{\partial u}(0, \theta) = \frac{a}{b}\begin{pmatrix} bR\cos\theta \\ aR\sin\theta \end{pmatrix} \perp \begin{pmatrix} -aR\sin\theta \\ bR\cos\theta \end{pmatrix} = \vec{e}_2(R, \theta)$. The coordinate line at $p = \Phi_{\theta_0}(R) = \Psi_{\theta_0}(0)$ is $\Psi_{\theta_0}(u) = \begin{pmatrix} aR\cos\theta_0 \\ bR\sin\theta_0 \end{pmatrix} + u\begin{pmatrix} aR\cos\theta_0 \\ \frac{a^2}{b}R\sin\theta_0 \end{pmatrix}$, straight line. ▪

3 Examples

3.1 Cartesian coordinate system

Here $U = \mathbb{R}^m$ and $\Phi : U = \mathbb{R}^m \to S \subset \mathbb{R}^n$ is affine, that is, $\exists L \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ s.t.$L = d\Phi(\vec{q}) \in \mathcal{L}(\vec{\mathbb{R}^m}; \vec{\mathbb{R}^n})$ is independent of $\vec{q}$: for all $\vec{q}, \vec{q_0} \in \mathbb{R}^m$,

$$\Phi(\vec{q}) = \Phi(\vec{q_0}) + L.(\vec{q} - \vec{q_0}). \tag{3.1}$$

And $S = \Phi(U)$ is a affine sub-space in $\mathbb{R}^n$.

And with $(\vec{A}_i)$ the canonical basis in $\vec{\mathbb{R}^m}$, and with $p = \Phi(\vec{q})$, we get $d\Phi(\vec{q}).\vec{A}_i = \vec{e}_i(p) = L.\vec{A}_i =^{\text{written}} \vec{e}_i \in \vec{\mathbb{R}^n}$ independent of p, and $(\vec{e}_i)$ is the basis of the coordinate system. Then the Christoffel, cf. (2.5) vanish: $\gamma_{ij}^k = 0$ for all i, j, k, since $d\vec{e}_i(p) = 0$.

And the coordinate lines through p are the straight line $t \to c^i_{\vec{x}}(t) = p + t\vec{e}_i$ pour $i = 1, ..., m$.

E.g., $m = n = 2$, O origin and $(\vec{E}_1, \vec{E}_2)$ Cartesian basis in $\mathbb{R}^2$ (geometric), Φ given by $[d\Phi(\vec{q})]_{|\vec{A},\vec{E}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ invertible, thus $[\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} a \\ c \end{pmatrix}$, $[\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} b \\ d \end{pmatrix}$.

3.2 Polar coordinate system

$\mathbb{R}^n = \mathbb{R}^m = \mathbb{R}^2$. And $(\vec{A}_i)$ is the canonical basis in $\mathbb{R}^2$ (parameter space), and $(\vec{E}_i)$ is a Euclidean basis in $\mathbb{R}^2$ (geometric space).

3.2.1 The coordinate system

See (1.5): With $\vec{q} = (r, \theta)$,

$$\overrightarrow{Op} = \overrightarrow{O\Phi(\vec{q})} = r\cos\theta\vec{E}_1 + r\sin\theta\vec{E}_2 = \vec{x}, \quad [\vec{x}]_{|\vec{E}} = \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}. \tag{3.2}$$

The coordinate lines through p are the $c^{(i)}_p : h \in \mathbb{R} \to p = c^{(i)}_p(h) \in \mathbb{R}^2$ given by

$$\begin{cases} c^{(1)}_p(h) := \Phi(r{+}h, \theta), & \text{i.e.} \quad \overrightarrow{Oc^{(1)}_p(h)} = (r{+}h)\cos\theta\vec{E}_1 + (r{+}h)\sin\theta\vec{E}_2, \quad h > -r, \\ c^{(2)}_p(h) := \Phi(r, \theta{+}h), & \text{donc} \quad \overrightarrow{Oc^{(2)}_p(h)} = r\cos(\theta{+}h)\vec{E}_1 + r\sin(\theta{+}h)\vec{E}_2, \quad h \in \mathbb{R}. \end{cases}$$

(Straight line and circle.) That is,

$$[\overrightarrow{Oc^{(1)}_p(h)}]_{|\vec{E}} = \begin{pmatrix} (r{+}h)\cos\theta \\ (r{+}h)\sin\theta \end{pmatrix}, \quad \text{and} \quad [\overrightarrow{Oc^{(2)}_p(h)}]_{|\vec{E}} = \begin{pmatrix} r\cos(\theta{+}h) \\ r\sin(\theta{+}h) \end{pmatrix}.$$

The basis vectors at $p = \Phi(\vec{q})$ are

$$\begin{cases} \vec{e}_1(p) = d\Phi(\vec{q}).\vec{A}_1 = \dfrac{\partial\Phi}{\partial r}(\vec{q}) = \cos\theta\vec{E}_1 + \sin\theta\vec{E}_2, \quad [\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \\ \vec{e}_2(p) = d\Phi(\vec{q}).\vec{A}_2 = \dfrac{\partial\Phi}{\partial\theta}(\vec{q}) = -r\sin\theta\vec{E}_1 + r\cos\theta\vec{E}_2, \quad [\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}. \end{cases} \tag{3.3}$$

So, with (A^1, A^2) the dual basis of the canonical basis in $\mathbb{R}^{\vec{m}}$, we have

$$[d\Phi(\vec{q})]_{|\vec{A},\vec{E}} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} [\vec{e}_1(p)]_{|\vec{E}} & [\vec{e}_2(p)]_{|\vec{E}} \end{pmatrix}, \tag{3.4}$$

that is (tensorial expression to see the basis in use),

$$d\Phi(\vec{q}) = \cos\theta\vec{E}_1 \otimes A^1 - r\sin\theta\vec{E}_1 \otimes A^2 + \sin\theta\vec{E}_2 \otimes A^1 + r\cos\theta\vec{E}_2 \otimes A^2. \tag{3.5}$$

Thus, cf. (1.48) and (3.3),

$$(d\vec{e}_1.\vec{e}_1)(p) = \frac{\partial\vec{e}_1}{\partial r}(p) = \vec{0}, \tag{3.6}$$

$$(d\vec{e}_1.\vec{e}_2)(p) = \frac{\partial\vec{e}_1}{\partial\theta}(p) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \frac{1}{r}\vec{e}_2(p), \tag{3.7}$$

$$(d\vec{e}_2.\vec{e}_1)(p) = \frac{\partial\vec{e}_2}{\partial r}(p) = (d\vec{e}_1.\vec{e}_2)(p) = \frac{1}{r}\vec{e}_2(p), \tag{3.8}$$

$$(d\vec{e}_2.\vec{e}_2)(p) = \frac{\partial\vec{e}_2}{\partial\theta}(p) = \begin{pmatrix} -r\cos\theta \\ -r\sin\theta \end{pmatrix} = -r\vec{e}_1(p). \tag{3.9}$$

So (tensorial expression to see the basis in use),

$$d\vec{e}_1 = \frac{1}{r}\vec{e}_2 \otimes e^2 \quad \text{and} \quad d\vec{e}_2 = \frac{1}{r}\vec{e}_2 \otimes e^1 - r\vec{e}_1 \otimes e^2, \tag{3.10}$$

that is,

$$[d\vec{e}_1(p)]_{|\vec{e}(p),\vec{e}(p)} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix}, \quad [d\vec{e}_2(p)]_{|\vec{e}(p),\vec{e}(p)} = \begin{pmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{pmatrix}. \tag{3.11}$$

Thus, cf. (2.3)-(2.4), $\gamma^1_{11} = 0$, $\gamma^2_{11} = 0$, $\gamma^1_{12} = 0$, $\gamma^2_{12} = \frac{1}{r}$, and $\gamma^1_{21} = 0$, $\gamma^2_{21} = \frac{1}{r}$, $\gamma^1_{22} = -r$, $\gamma^2_{22} = 0$, and the non vanishing Christoffel symbols are

$$\gamma_{12}^2 = \frac{1}{r} = \gamma_{21}^2, \quad \text{and} \quad \gamma_{22}^1 = -r. \tag{3.12}$$

And (3.10) gives

$$\begin{aligned} d\vec{e}_1 &= -\sin\theta \vec{E}_1 \otimes e^2 + \cos\theta \vec{E}_2 \otimes e^2, \\ d\vec{e}_2(p) &= -\sin\theta \vec{E}_1 \otimes e^1(p) - r\cos\theta \vec{E}_1 \otimes e^2(p) + \cos\theta \vec{E}_2 \otimes e^1(p) - r\sin\theta \vec{E}_e^g \otimes e^2(p). \end{aligned} \tag{3.13}$$

Thus,

$$[d\vec{e}_1(p)]_{|\vec{e}(p),\vec{E}} = \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{pmatrix}, \quad [d\vec{e}_2(p)]_{|\vec{e},\vec{E}} = \begin{pmatrix} -\sin\theta & -r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix}. \tag{3.14}$$

Exercise 3.1 Prove:

$$[d\vec{e}_1(p)]_{|\vec{E},\vec{E}} = \frac{1}{r} \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}, \qquad [d\vec{e}_2(p)]_{|\vec{E},\vec{E}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.15}$$

(And $d\vec{e}_2(p)$ is a rotation with angle $\frac{\pi}{2}$.)

Answer. The transition matrix from $(\vec{E}_i)$ to $(\vec{e}_i(p))$ is $P = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$, cf. (3.3). Its inverse is $P^{-1} = \frac{1}{r} \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Thus, with (dx, dy) the dual basis of $(\vec{E}_1, \vec{E}_2)$,

$$\begin{aligned} e^1(p) &= \cos\theta dx + \sin\theta dy, \\ e^2(p) &= -\frac{\sin\theta}{r} dx + \frac{\cos\theta}{r} dy. \end{aligned}$$

Then (3.13) gives

$$\begin{aligned} d\vec{e}_1(p) &= -\sin\theta \vec{E}_1 \otimes (-\frac{\sin\theta}{r} dx + \frac{\cos\theta}{r} dy) + \cos\theta \vec{E}_2 \otimes (-\frac{\sin\theta}{r} dx + \frac{\cos\theta}{r} dy), \\ d\vec{e}_2(p) &= \ -\sin\theta \vec{E}_1 \otimes (\cos\theta dx + \sin\theta dy) - r\cos\theta \vec{E}_1 \otimes (-\frac{\sin\theta}{r} dx + \frac{\cos\theta}{r} dy) \\ &\quad + \cos\theta \vec{E}_2 \otimes (\cos\theta dx + \sin\theta dy) - r\sin\theta \vec{E}_e^g \otimes (-\frac{\sin\theta}{r} dx + \frac{\cos\theta}{r} dy) \end{aligned}$$

▪

Exercise 3.2 Ellipse coordinate system: let pour $a, b > 0$, $r \geq 0$, $\theta \in \mathbb{R}$, and

$$p = \Phi(r, \theta), \qquad [\vec{x}]_{|\vec{E}} = \begin{pmatrix} ar\cos\theta \\ br\sin\theta \end{pmatrix}, \tag{3.16}$$

Find $\vec{e}_1$ and $\vec{e}_2$, and give $d\vec{e}_1$ et $d\vec{e}_2$.

Answer. Let $\vec{q} = (r, \theta)$ and $p = \Phi(\vec{q}) = \Phi(r, \theta)$.

$$[\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix}, \quad [\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} -ar\sin\theta \\ br\cos\theta \end{pmatrix}, \quad [d\Phi(\vec{q})]_{|\vec{A},\vec{E}} = \begin{pmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{pmatrix}. \tag{3.17}$$

Hence,

$$\begin{aligned} [d\vec{e}_1(p).\vec{e}_1(p)] &= [\frac{\partial \vec{e}_1}{\partial r}(p)] = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}, \\ [d\vec{e}_1(p).\vec{e}_2(p)] &= [\frac{\partial \vec{e}_1}{\partial \theta}(p)] = \begin{pmatrix} -a\sin\theta \\ b\cos\theta \end{pmatrix} = [d\vec{e}_2(p).\vec{e}_1(p)] = \frac{1}{r}[\vec{e}_2(p)], \\ [d\vec{e}_2(p).\vec{e}_2(p)] &= [\frac{\partial \vec{e}_2}{\partial \theta}(p)] = \begin{pmatrix} -ar\cos\theta \\ -br\sin\theta \end{pmatrix} = -r[\vec{e}_1(p)]. \end{aligned} \tag{3.18}$$

So,

$$d\vec{e}_1 = \frac{1}{r}\vec{e}_2 \otimes e^2, \qquad d\vec{e}_2 = \frac{1}{r}\vec{e}_2 \otimes e^1 - r\,\vec{e}_1 \otimes e^2. \tag{3.19}$$

And the Christoffel symbols are those of the polar coordinate system. (NB: If $a \neq b$ then $\vec{e}_1(p) \not\perp \vec{e}_2(p)$, and the dual basis $(dr(p), d\theta(p))$ is not made of orthogonal projections. See next remark.) And the Jacobian matrices are

$$[d\vec{e}_1(p)]_{|\vec{e},\vec{e}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix}, \qquad [d\vec{e}_2(p)]_{|\vec{e},\vec{e}} = \begin{pmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{pmatrix}. \tag{3.20}$$

▪

Remark 3.3 If $m = n = 2$, then (3.19) tells that the Christoffel symbols γ_{ij}^k, with $i,j,k \in \{1,2\}$, of the elliptic coordinate system (3.16) are those of the polar coordinate system (3.2), cf. (3.10) and (3.19).

For the restriction to the circle and to the ellipse, that is with $\Phi_R(\theta) = \Phi(R,\theta)$, this will be false for the association Riemannian connection: The Riemannian connection in $\mathbb{R}^n$ on a surface (the usual connection) is the Euclidean orthogonal projection on the surface. Here $\vec{e}_1(p) \not\perp \vec{e}_2(p)$ when $a \neq b$, cf. (3.17). And, with E the ellipse curve $\Phi_R(\mathbb{R})$, cf. (3.16), and with $T_pE = \mathrm{Vect}\{\vec{e}_2(p)\}$ the tangent line to the ellipse at p, we have $(\nabla_{\vec{e}_2}\vec{e}_2)(p) = \mathrm{Proj}_{T_pE}(d\vec{e}_2(p).\vec{e}_2(p)) = \mathrm{Proj}_{T_pE}(-R\vec{e}_1(p))$. So, $(\vec{e}_1(p), \vec{e}_2(p))_{\mathbb{R}^2} = r(b^2-a^2)\cos\theta\sin\theta$, and $||\vec{e}_2(p)|| = r(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac12}$, give

$$(\nabla_{\vec{e}_2}\vec{e}_2)(p) = \gamma_{22}^2(p)\vec{e}_2(p) \quad \text{with} \quad \gamma_{22}^2(p) = -R\frac{(b^2-a^2)\cos\theta\sin\theta}{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac12}}\vec{e}_2(p).$$

(And $\gamma_{22}^2 = 0$ when $a = b$.) ▪

3.2.2 Inverse diffeomorphism

Let the domain of Φ be restricted to, e.g., $U = \mathbb{R}_+^* \times]-\pi,\pi[$. Then Φ is a diffeomorphism, and $\Phi^{-1} : \Omega \to \mathbb{R}_+^*\times]-\pi,\pi[$ with $\Omega = \mathbb{R}^2 - (\mathbb{R}_-\times\{0\})$.

Notation: $\overrightarrow{Op} = x\vec{E}_1 + y\vec{E}_2 = \vec{x}$. Then, if $x > 0$, i.e. if $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, then

$$[\Phi^{-1}(x,y)]_{\vec{A}} = \begin{pmatrix} r(x,y) = \sqrt{x^2+y^2} \\ \theta(x,y) = \arctan\dfrac{y}{x} \end{pmatrix} \overset{\text{written}}{=} [\vec{q}(x,y)]_{\vec{A}}. \tag{3.21}$$

General case, with $\theta \neq \pi$,

$$[\Phi^{-1}(x,y)]_{\vec{A}} = \begin{pmatrix} r(x,y) = \sqrt{x^2+y^2} \\ \theta(x,y) = 2\arctan\dfrac{y}{x+\sqrt{x^2+y^2}} \end{pmatrix} \overset{\text{written}}{=} [\vec{q}]_{\vec{A}}, \tag{3.22}$$

that is, $\Phi^{-1}(x,y) = \sqrt{x^2+y^2}\vec{A}_1 + 2\arctan\frac{y}{x+\sqrt{x^2+y^2}}\vec{A}_2 =^{\text{written}} \vec{q}(x,y)$.

Exercise 3.4 Prove:

$$d\Phi^{-1}(p) = \cos\theta\vec{A}_1\otimes dx + \sin\theta\vec{A}_1\otimes dy - \frac{1}{r}\sin\theta\vec{A}_2\otimes dx + \frac{1}{r}\cos\theta\vec{A}_2\otimes dy = d\vec{q}(p), \tag{3.23}$$

that is,

$$[d\Phi^{-1}(p)]_{|\vec{E},\vec{A}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \overset{\text{written}}{=} [d\vec{q}(p)]_{|\vec{E},\vec{A}}. \tag{3.24}$$

Answer. With (3.21) to simplify. $\Phi^{-1}(p) = \vec{q}(p) = (r(p),\theta(p)) = r(p)\vec{A}_1 + \theta(p)\vec{A}_2$, gives $d\vec{q}(p) = \vec{A}_1\otimes dr(p) + \vec{A}_2\otimes d\theta(p)$. And $\frac{\partial r}{\partial x}(x,y) = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$, $\frac{\partial r}{\partial y}(x,y) = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta$, $\frac{\partial\theta}{\partial x}(x,y) = -\frac{y}{x^2}\frac{1}{1+\frac{y^2}{x^2}} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}$, $\frac{\partial\theta}{\partial y}(x,y) = \frac{1}{x}\frac{1}{1+\frac{y^2}{x^2}} = \frac{x}{r^2} = \frac{\cos\theta}{r}$, which gives (3.24). ▪

3.3 Cylindrical coordinate system

The cylindrical coordinate system is

$$\Phi : \begin{cases} \mathbb{R}_+\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}^3 \\ \vec{q} = (r,\theta,z) \longmapsto p = \Phi(r,\theta,z), \quad [\overrightarrow{Op}]_{|\vec{E}} = \begin{pmatrix} x(r,\theta,z) = \Phi^1(r,\theta,z) = r\cos\theta \\ y(r,\theta,z) = \Phi^2(r,\theta,z) = r\sin\theta \\ z(r,\theta,z) = \Phi^3(r,\theta,z) = z \end{pmatrix}. \end{cases} \tag{3.25}$$

To get a diffeomorphism we can consider the restrition $\Phi : U \to \Omega$ where, e.g.,

$$U = \mathbb{R}_+^*\times]-\pi,\pi[\times\mathbb{R} \quad \text{and} \quad \Omega = \mathbb{R}^3 - (\mathbb{R}_-\times\{0\}\times\mathbb{R}). \tag{3.26}$$

Thus the coordinate curves are

$$c_p^{(1)}(h) = \begin{pmatrix} (r+h)\cos\theta \\ (r+h)\sin\theta \\ z \end{pmatrix}, \quad c_p^{(2)}(h) = \begin{pmatrix} r\cos(\theta+h) \\ r\sin(\theta+h) \\ z \end{pmatrix}, \quad c_p^{(3)}(h) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z+h \end{pmatrix}.$$

1- $c_p^{(1)}$ is the radius at altitude z at angle θ with $\vec{E}_1$,

2- $c_p^{(2)}$ is the cercle at altitude z centered at $(0,0,z)$ with radius r,
3- $c_p^{(3)}$ is the vertical line through p.

3.4 Spherical coordinate system

Let O be an origin and $(\vec{E}_i)$ be a Euclidean basis in $\mathbb{R}^3$. If $p \in \mathbb{R}^3$, let $\vec{x} = \overrightarrow{Op} = \sum_i x^i \vec{E}_i$. Let $\Phi : \vec{q} \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to p \in \mathbb{R}^3$. There are two usual spherical coordinate systems.

3.4.1 The GPS system

(GPS = Global Positioning System.) O is the center of the Earth, $\vec{E}_3$ gives the axis of rotation of the Earth, and $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ is a Euclidean basis. Then, with r the distance to O, θ the longitude, and φ the latitude:

$$\text{GPS:}\quad \overrightarrow{O\Phi(r,\theta,\varphi)} = [\vec{x}]_{|\vec{E}} = \begin{pmatrix} x = r\cos\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\sin\varphi \end{pmatrix}, \quad (r,\theta,\varphi) \in \mathbb{R}_+ \times [-\pi,\pi] \times [-\frac{\pi}{2},\frac{\pi}{2}]. \tag{3.27}$$

E.g., with r_0 the radius of the Earth and φ_0 a given latitude,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = r_0\cos\varphi_0 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \\ z = r_0 \sin\varphi_0 \end{pmatrix}, \qquad \theta \in [-\pi,\pi],$$

is the parallel at latitude φ_0; In particular $\varphi_0 = 0$ gives the equator.

And e.g., with r_0 the radius of the Earth and θ_0 a given longitude,

$$[\vec{x}]_{|\vec{E}} = \begin{pmatrix} x = r_0\cos\theta_0\cos\varphi \\ y = r_0\sin\theta_0\cos\varphi \\ z = r_0\sin\varphi \end{pmatrix}, \quad \varphi \in [-\frac{\pi}{2},\frac{\pi}{2}], \tag{3.28}$$

gives the meridian at longitude θ_0.

Example 3.5 GPS Coordinates at ISIMA: $\theta \simeq 3°07'E$ and $\varphi \simeq 45°45'N$ in degrees and minute. With radian: $\theta \simeq 0,054$ and $\varphi \simeq -0,80$. And $r \simeq 6370$ km gives the distance to the center of the Earth, and the altitude is given relative to the (mean) level of the oceans ($\simeq 400$ meter at ISIMA). ▪

The basis at $p = O + \vec{x}$ or the GPS system gives $(\vec{e}_i(p) = \frac{\partial\Phi}{\partial q^i}(\vec{q}) = c_p^{(i)\prime}(0))$

$$[\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos\theta\cos\varphi \\ \sin\theta\cos\varphi \\ \sin\varphi \end{pmatrix}, \quad [\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} -r\sin\theta\cos\varphi \\ r\cos\theta\cos\varphi \\ 0 \end{pmatrix}, \quad [\vec{e}_3(p)]_{|\vec{E}} = \begin{pmatrix} -r\cos\theta\sin\varphi \\ -r\sin\theta\sin\varphi \\ r\cos\varphi \end{pmatrix}. \tag{3.29}$$

Thus, omitting the writing of (p),

$$[d\vec{e}_1.\vec{e}_1]_{|\vec{E}} = [\frac{\partial\vec{e}_1}{\partial r}]_{|\vec{E}} = 0, \tag{3.30}$$

$$[d\vec{e}_1.\vec{e}_2]_{|\vec{E}} = [\frac{\partial\vec{e}_1}{\partial\theta}]_{|\vec{E}} = \begin{pmatrix} -\sin\theta\cos\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} = [\frac{1}{r}\,\vec{e}_2]_{|\vec{E}}, \tag{3.31}$$

$$[d\vec{e}_1.\vec{e}_3]_{|\vec{E}} = [\frac{\partial\vec{e}_1}{\partial\varphi}]_{|\vec{E}} = \begin{pmatrix} -\cos\theta\sin\varphi \\ -\sin\theta\sin\varphi \\ \cos\varphi \end{pmatrix} = [\frac{1}{r}\,\vec{e}_3]_{|\vec{E}}, \tag{3.32}$$

$$[d\vec{e}_2.\vec{e}_2]_{|\vec{E}} = [\frac{\partial\vec{e}_2}{\partial\theta}]_{|\vec{E}} = \begin{pmatrix} -r\cos\theta\cos\varphi \\ -r\sin\theta\cos\varphi \\ 0 \end{pmatrix} - r\cos^2\varphi\,[\vec{e}_1]_{|\vec{E}} + \cos\varphi\sin\varphi\,[\vec{e}_3]_{|\vec{E}}, \tag{3.33}$$

$$[d\vec{e}_2.\vec{e}_3] = [\frac{\partial\vec{e}_2}{\partial\varphi}]_{|\vec{E}} = \begin{pmatrix} r\sin\theta\sin\varphi \\ -r\cos\theta\sin\varphi \\ 0 \end{pmatrix} = [-\tan\varphi\,\vec{e}_2]_{|\vec{E}}, \tag{3.34}$$

$$[d\vec{e}_3.\vec{e}_3]_{|\vec{E}} = [\frac{\partial \vec{e}_3}{\partial \varphi}]_{|\vec{E}} = \begin{pmatrix} -r\cos\theta\cos\varphi \\ -r\sin\theta\cos\varphi \\ -r\sin\varphi \end{pmatrix} = [-r\,\vec{e}_1]_{|\vec{E}}. \tag{3.35}$$

Thus,

$$\gamma_{11}^1 = 0, \quad \gamma_{11}^2 = 0, \quad \gamma_{11}^3 = 0, \tag{3.36}$$

$$\gamma_{21}^1 = 0 = \gamma_{12}^1, \quad \gamma_{21}^2 = \frac{1}{r} = \gamma_{12}^2, \quad \gamma_{21}^3 = 0 = \gamma_{12}^3, \tag{3.37}$$

$$\gamma_{31}^1 = 0 = \gamma_{13}^1, \quad \gamma_{31}^2 = 0 = \gamma_{13}^2, \quad \gamma_{31}^3 = \frac{1}{r} = \gamma_{13}^3, \tag{3.38}$$

$$\gamma_{22}^1 = -r\cos^2\varphi, \quad \gamma_{22}^2 = 0, \quad \gamma_{22}^3 = \cos\varphi\sin\varphi, \tag{3.39}$$

$$\gamma_{32}^1 = 0 = \gamma_{23}^1, \quad \gamma_{32}^2 = -\tan\varphi = \gamma_{23}^2, \quad \gamma_{32}^3 = 0 = \gamma_{23}^3, \tag{3.40}$$

$$\gamma_{33}^1 = -r, \quad \gamma_{33}^2 = 0, \quad \gamma_{33}^3 = 0. \tag{3.41}$$

Therefore, the non vanishing Christoffel with the index 1 are

$$\gamma_{12}^2 = \gamma_{21}^2 = \frac{1}{r} = \gamma_{13}^3 = \gamma_{31}^3, \quad \gamma_{22}^1 = -r\cos^2\varphi, \quad \gamma_{33}^1 = -r, \tag{3.42}$$

and without the index 1, that is for Riemannian connection in the spherical surface with $r = R$ fixed,

$$\gamma_{22}^3 = \cos\varphi\sin\varphi, \quad \gamma_{23}^2 = \gamma_{32}^2 = -\tan\varphi. \tag{3.43}$$

Exercise 3.6 Express the Euclidean dot product $g(\cdot,\cdot) = (\cdot,\cdot)_{\mathbb{R}^3}$ in the GPS basis.

Answer. We have to compute $g_{ij} = g(\vec{e}_i(p), \vec{e}_j(p)) = (\vec{e}_i(p), \vec{e}_j(p))_{\mathbb{R}^n}$. The spherical basis being orthogonal, we get $g_{ij} = 0$ if $i \neq j$, and $g_{11}(p) = ||\vec{e}_1(p)||^2 = 1$, $g_{22}(p) = ||\vec{e}_2(p)||^2 = r^2\cos^2\varphi$, and $g_{33}(p) = ||\vec{e}_3(p)||^2 = r^2$. Thus $[g(p)]_{|\vec{e}} = [g_{ij}(p)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2\cos^2\varphi & 0 \\ 0 & 0 & r^2 \end{pmatrix}$. ▪

Exercise 3.7 Let $a,b,c > 0$. Ellipsoid $\vec{x} = \overrightarrow{Op} = \overrightarrow{O\Phi(r,\theta,\varphi)}$ with $[\vec{x}]_{|\vec{E}} = \begin{pmatrix} x = ar\cos\theta\cos\varphi \\ y = br\sin\theta\cos\varphi \\ z = cr\sin\varphi \end{pmatrix}$, $(r,\theta,\varphi) \in \mathbb{R}_+^* \times]-\pi,\pi[\times]-\frac{\pi}{2},\frac{\pi}{2}[$. Prove that the Christoffel symbols are still given by (3.42)-(3.43). Express the Euclidean dot product $g(\cdot,\cdot) = (\cdot,\cdot)_{\mathbb{R}^3}$ in the corresponding basis, in particular if $a = b$.

Answer. $\vec{e}_1(p) = \begin{pmatrix} a\cos\theta\cos\varphi \\ b\sin\theta\cos\varphi \\ c\sin\varphi \end{pmatrix}$, $\vec{e}_2(p) = \begin{pmatrix} -ar\sin\theta\cos\varphi \\ br\cos\theta\cos\varphi \\ 0 \end{pmatrix}$, $\vec{e}_3(p) = \begin{pmatrix} -ar\cos\theta\sin\varphi \\ -br\sin\theta\sin\varphi \\ cr\cos\varphi \end{pmatrix}$. (The basis $(\vec{e}_i(p))$ isn't orthogonal in general). Then $\frac{\partial \vec{e}_1}{\partial r}(p) = \vec{0}$, $\frac{\partial \vec{e}_1}{\partial \theta}(p) = \begin{pmatrix} -a\sin\theta\cos\varphi \\ b\cos\theta\cos\varphi \\ 0 \end{pmatrix} = \frac{\vec{e}_2(p)}{r}$, $\frac{\partial \vec{e}_1}{\partial \varphi}(p) = \begin{pmatrix} -a\cos\theta\sin\varphi \\ -b\sin\theta\sin\varphi \\ c\cos\varphi \end{pmatrix} = \frac{\vec{e}_3(p)}{r}$, $\frac{\partial \vec{e}_2}{\partial r}(p) = \begin{pmatrix} -a\sin\theta\cos\varphi \\ b\cos\theta\cos\varphi \\ 0 \end{pmatrix} = \frac{\vec{e}_2(p)}{r}$, $\frac{\partial \vec{e}_2}{\partial \theta}(p) = \begin{pmatrix} -ar\cos\theta\cos\varphi \\ -br\sin\theta\cos\varphi \\ 0 \end{pmatrix} = -r\cos^2\varphi\,\vec{e}_1(p) + \cos\varphi\sin\varphi\,\vec{e}_3(p)$, $\frac{\partial \vec{e}_2}{\partial \varphi}(p) = \begin{pmatrix} ar\sin\theta\sin\varphi \\ -br\cos\theta\sin\varphi \\ 0 \end{pmatrix} = -\tan\varphi\vec{e}_2(p)$, $\frac{\vec{e}_3}{\partial r}(p) = \begin{pmatrix} -a\cos\theta\sin\varphi \\ -b\sin\theta\sin\varphi \\ c\cos\varphi \end{pmatrix} = \frac{\vec{e}_3(p)}{r}$, $\frac{\vec{e}_3}{\partial \theta}(p) = \begin{pmatrix} ar\sin\theta\sin\varphi \\ -br\cos\theta\sin\varphi \\ 0 \end{pmatrix} = -\tan\varphi\vec{e}_2(p)$, $\frac{\vec{e}_3}{\partial \varphi}(p) = \begin{pmatrix} -ar\cos\theta\cos\varphi \\ -br\sin\theta\cos\varphi \\ -cr\sin\varphi \end{pmatrix} = -r\vec{e}_1(p)$. Therefore the Christoffel symbols.

Then $g_{11}(p) = ||\vec{e}_1(p)|^2 = 1$, $g_{22}(p) = ||\vec{e}_2(p)|^2 = r^2\cos^2\varphi$, et $g_{33}(p) = ||\vec{e}_3(p)|^2 = r^2$.
$g_{12}(p) = (\vec{e}_1(p), \vec{e}_2(p))_{\mathbb{R}^n} = r\cos^2\varphi\cos\theta\sin\theta(-a^2+b^2)$
$g_{13}(p) = (\vec{e}_1(p), \vec{e}_3(p))_{\mathbb{R}^n} = r\cos\varphi\sin\varphi(c^2 - a^2\cos^2\theta - b^2\sin^2\theta)$
$g_{23}(p) = (\vec{e}_2(p), \vec{e}_3(p))_{\mathbb{R}^n} = r^2(a^2-b^2)\cos\theta\sin\theta\cos\varphi\sin\varphi$.

In particular, $a = b$ gives $[g(p)]_{|\vec{e}} = \begin{pmatrix} 1 & 0 & r\cos\varphi\sin\varphi(c^2-a^2) \\ 0 & r^2\cos^2\varphi & 0 \\ r\cos\varphi\sin\varphi(c^2-a^2) & 0 & r^2 \end{pmatrix}$. ▪

3.4.2 GPS system on the surface of the Earth

We have

$$\Phi_R : \begin{cases} U \subset \mathbb{R}^2 & \longrightarrow S \subset \mathbb{R}^3 \\ \vec{q} = (\theta, \varphi) & \longmapsto p = \Phi(\theta, \varphi) = \Phi_R(\theta, \varphi), \quad [\vec{x}] = \begin{pmatrix} \Phi^1(\theta, \varphi) = R\cos\theta\cos\varphi \\ \Phi^2(\theta, \varphi) = R\sin\theta\cos\varphi \\ \Phi^3(\theta, \varphi) = R\sin\varphi \end{pmatrix}. \end{cases} \tag{3.44}$$

And the non vanishing Christoffel symbols are given in (3.42)-(3.43):

$$\gamma_{22}^1 = -R\cos^2\varphi, \quad \gamma_{33}^1 = -R, \quad \gamma_{22}^3 = \cos\varphi\sin\varphi, \quad \gamma_{23}^2 = \gamma_{32}^2 = -\tan\varphi. \tag{3.45}$$

NB:

$$d\vec{e}_3(p).\vec{e}_3(p) = -R\,\vec{e}_1(p) = \gamma_{33}^1\,\vec{e}_1(p) \tag{3.46}$$

is not tangent to the surface, although $\vec{e}_3(p)$ is.

3.4.3 The "classical mechanic" spherical system

Here, instead of the latitude (φ above), we use the colatitude $\frac{\pi}{2} - \varphi$ (so $\psi = 0$ gives the North pole), and the colatitude is named φ, so, $\vec{x} = \overrightarrow{Op}$ and

$$\text{Mechanic:} \quad [\vec{x}]_{|\vec{E}} = [\Phi(r, \theta, \varphi)]_{|\vec{E}} = \begin{pmatrix} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{pmatrix}, \quad r \in \mathbb{R}_+, \quad \theta \in [-\pi, \pi], \quad \varphi \in [0, \pi]. \tag{3.47}$$

Thus

$$\begin{aligned} \vec{e}_1(p) &= \cos\theta\sin\varphi\vec{E}_1 + \sin\theta\sin\varphi\vec{E}_2 + \cos\varphi\vec{E}_3, \\ \vec{e}_2(p) &= -r\sin\theta\sin\varphi\vec{E}_1 + r\cos\theta\sin\varphi\vec{E}_2, \\ \vec{e}_3(p) &= r\cos\theta\cos\varphi\vec{E}_1 + r\sin\theta\cos\varphi\vec{E}_2 - r\sin\varphi\vec{E}_3, \end{aligned} \tag{3.48}$$

$$[\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{pmatrix}, \quad [\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} -r\sin\theta\sin\varphi \\ r\cos\theta\sin\varphi \\ 0 \end{pmatrix}, \quad [\vec{e}_3(p)]_{|\vec{E}} = \begin{pmatrix} r\cos\theta\cos\varphi \\ r\sin\theta\cos\varphi \\ -r\sin\varphi \end{pmatrix}. \tag{3.49}$$

Then

$$\begin{aligned} &d\vec{e}_1.\vec{e}_1 = \frac{\partial\vec{e}_1}{\partial r} = \vec{0}, \quad d\vec{e}_1.\vec{e}_2 = \frac{\partial\vec{e}_1}{\partial\theta} = \frac{\vec{e}_2}{r}, \quad d\vec{e}_1.\vec{e}_3 = \frac{\partial\vec{e}_1}{\partial\varphi} = \frac{\vec{e}_3}{r}, \\ &d\vec{e}_2.\vec{e}_1 = \frac{\vec{e}_2}{r}, \quad d\vec{e}_2.\vec{e}_2 = -r\sin^2\varphi\,\vec{e}_1 - \cos\varphi\sin\varphi\,\vec{e}_3, \quad d\vec{e}_2.\vec{e}_3 = \cot\varphi\,\vec{e}_2, \\ &d\vec{e}_3.\vec{e}_1 = \frac{\vec{e}_3}{r}, \quad d\vec{e}_3.\vec{e}_2 = \cot\varphi\,\vec{e}_2, \quad d\vec{e}_3.\vec{e}_3 = -r\vec{e}_1, \end{aligned} \tag{3.50}$$

and the non-vanishing christoffel symbols are

$$\gamma_{12}^2 = \frac{1}{r} = \gamma_{21}^2,\ \gamma_{13}^3 = \frac{1}{r} = \gamma_{31}^3,\ \gamma_{22}^1 = -r\sin^2\varphi,\ \gamma_{22}^3 = -\cos\varphi\sin\varphi,\ \gamma_{23}^2 = \cot\varphi = \gamma_{32}^2,\ ;\gamma_{33}^1 = -r. \tag{3.51}$$

Part II

Derivation operator on $\mathcal{F}(S)$

4 Operator $\tilde{\nabla}_{\vec{v}} f = \tilde{\mathcal{L}}_{\vec{v}} f = df.\vec{v}$

Let $\Phi : U \subset \mathbb{R}^m \to S \subset \mathbb{R}^n$ be a coordinate system in S, cf. (1.3), and $(\vec{e}_i(p))_{i=1,\dots,m}$ be the coordinate basis at $p \in S$, cf. (1.12).

4.1 Derivation operator at a point

Definition 4.1 Let $p_0 \in S$. A $\mathbb{R}$-linear form $\tilde{\mathcal{L}}_{p_0} \in L(\mathcal{F}(S);\mathbb{R})$ is a derivation in S at p_0 iff for all $f,g \in \mathcal{F}(S)$,

$$\tilde{\mathcal{L}}_{p_0}(fg) = \tilde{\mathcal{L}}_{p_0}(f)\,g(p_0) + f(p_0)\,\tilde{\mathcal{L}}_{p_0}(g). \tag{4.1}$$

(= Derivation formula $(fg)' = f'g + fg'$).

Let $1_S : p \in \mathbb{R}^n \to 1_S(p) \left\{ \begin{array}{l} = 1 \text{ if } p \in S \\ = 0 \text{ if } p \notin S \end{array} \right\}$ (indicator function).

Proposition 4.2 *If $c \in \mathbb{R}$, if $f = c1_S =^{\text{written}} c$ and if $\tilde{\mathcal{L}}_{p_0}$ is a derivation at $p_0 \in S$, then*

$$\tilde{\mathcal{L}}_{p_0}(c) = 0. \tag{4.2}$$

Proof. (4.1) gives $\tilde{\mathcal{L}}_{p_0}(1_S) = \tilde{\mathcal{L}}_{p_0}(1_S 1_S) = \tilde{\mathcal{L}}_{p_0}(1_S)1_S + 1_S\tilde{\mathcal{L}}_{p_0}(1_S) = 2\tilde{\mathcal{L}}_{p_0}(1_S)$, thus $\tilde{\mathcal{L}}_{p_0}(1_S) = 0$. And the linearity of $\tilde{\mathcal{L}}_{p_0}$ gives $\tilde{\mathcal{L}}_{p_0}(c1_S) = c\tilde{\mathcal{L}}_{p_0}(1_S) = 0$. ▪

4.2 Characterization

Proposition 4.3 *Let $p_0 \in S$. If $\vec{v}_{p_0} \in T_{p_0}S$, then the map $\tilde{\mathcal{L}}_{p_0} : \mathcal{F} \to \mathbb{R}$ defined by*

$$\tilde{\mathcal{L}}_{p_0} f := df(p_0).\vec{v}_{p_0} \overset{\text{written}}{=} \tilde{\nabla}_{\vec{v}} f(p_0) \overset{\text{written}}{=} \vec{v}[f](p_0) \tag{4.3}$$

is a derivation at p_0 on $\mathcal{F}(S)$ (we say that the vector field $\vec{v}$ acts on f as a derivation operator).

Conversely:

A derivation $\tilde{\mathcal{L}}_{p_0} \in L(\mathcal{F}(S);\mathbb{R})$ is a directional derivative,

that is, there exists $\vec{v}_{p_0} \in T_{p_0}S$ s.t. $\mathcal{L}_{p_0}$ is given by (4.3).

Thus, considering the coodinate basis $(\vec{e}_i(p_0))$ in $T_{p_0}S$, a derivation $\tilde{\mathcal{L}}_{p_0}$ is a linear combination of the elementary derivations $\partial_i(p_0) = \frac{\partial}{\partial q^i}(p_0)$ (bidual basis of $(\vec{e}_i(p_0))$): $\exists v^1_{p_0}, ..., v^m_{p_0} \in \mathbb{R}$ s.t.

$$\tilde{\mathcal{L}}_{p_0} f = \sum_{i=1}^m v^i_{p_0} \frac{\partial f}{\partial q^i}(p_0) \quad \Big(= \sum_{i=1}^m v^i_{p_0} df(p_0).\vec{e}_i(p_0) = df(p_0).\vec{v}_{p_0} = \partial_{\vec{v}} f(p_0)\Big). \tag{4.4}$$

Proof. $\Rightarrow$: The $\frac{\partial}{\partial q^i}(p_0)$ are derivations: $\frac{\partial (fg)}{\partial q^i}(p_0) = d(fg)(p_0).\vec{e}_i(p_0) = df(p_0).\vec{e}_i(p_0)g(p_0) + f(p_0)dg(p_0).\vec{e}_i(p_0) = \frac{\partial f}{\partial q^i}(p_0)g(p_0) + f(p_0)\frac{\partial g}{\partial q^i}(p_0)$. Thus $\tilde{\mathcal{L}}_{p_0} : f \to \tilde{\mathcal{L}}_{p_0} f = df(p_0).\vec{v}(p_0) = \sum_{i=1}^m \frac{\partial f}{\partial q^i}(p_0)v^i(p_0)$ is a derivation (trivial).

$\Leftarrow$: Converse: Let $f \in \mathcal{F}(S)$ and $f_U := f \circ \Phi \in \mathcal{F}(U)$, that is $f_U(\vec{q}) = f(p)$ when $p = \Phi(\vec{q})$. Let B be an open ball in S, let $p_0, p \in B$, let $\vec{q}_0 = \Phi^{-1}(p_0)$ and $\vec{q} = \Phi^{-1}(p)$. Consider $\alpha : t \to \alpha(t) = f_U(\vec{q}_0 + t(\vec{q}-\vec{q}_0))$, hence $\alpha'(t) = df_U(\vec{q}_0 + t(\vec{q}-\vec{q}_0)).(\vec{q}-\vec{q}_0)$. With $\vec{q} - \vec{q}_0 = \sum_{i=1}^m (q^i - q^i_0)\vec{A}_i$ and $df_U.\vec{A}_i = \frac{\partial f_U}{\partial q^i}$ we get

$$f_U(\vec{q}) - f_U(\vec{q}_0) = \alpha(1) - \alpha(0) = \int_0^1 \alpha'(t)\,dt = \sum_{i=1}^m (q^i - q^i_0) \int_0^1 \frac{\partial f_U}{\partial q^i}(\vec{q}_0 + t(\vec{q}-\vec{q}_0))\,dt.$$

Thus, with $p = \Phi(\vec{q})$ (and $\vec{q} = \Phi^{-1}(p) = \vec{q}(p)$),

$$f(p) - f(p_0) = \sum_{i=1}^m (q^i(p) - q^i_0)g_i(p), \quad \text{where} \quad g_i(p) = \int_0^1 \frac{\partial f_U}{\partial q^i}(\vec{q}_0 + t(\vec{q}(p)-\vec{q}_0))\,dt$$

Thus, $\tilde{\mathcal{L}}_p$ being $\mathbb{R}$-linear, and with (4.2) and (4.1), we get

$$\tilde{\mathcal{L}}_p f - 0 = \sum_{i=1}^m g_i(p)(\tilde{\mathcal{L}}_p q^i - 0) + (q^i(p) - q^i_0)\tilde{\mathcal{L}}_p g_i,$$

hence, at $p = p_0$,

$$\tilde{\mathcal{L}}_{p_0} f = \sum_{i=1}^m g_i(p_0)\tilde{\mathcal{L}}_{p_0} q^i + 0, \quad \text{with} \quad g_i(p_0) = \frac{\partial f_U}{\partial q^i}(\vec{q}_0).$$

Then $v^i_{p_0} = \tilde{\mathcal{L}}_{p_0} q^i = \tilde{\mathcal{L}}_{p_0}(\Phi^{-1})^i$ and $\vec{v}_{p_0} = \sum_{i=1}^m v^i_{p_0}\vec{e}_i(p_0)$ give $\tilde{\mathcal{L}}_{p_0} f = \sum_{i=1}^m v^i_{p_0} \frac{\partial (f \circ \Phi)}{\partial q^i}(\vec{q}_0)$, i.e., $\tilde{\mathcal{L}}_{p_0} f = \sum_{i=1}^m v^i_{p_0} df(p_0).d\Phi(p_0).\vec{e}_i = \sum_{i=1}^m df(p_0).\vec{v}_{p_0}$. ▪

Example 4.4 Counter-example. A second order derivative does not define a derivation: (4.1) is not satisfied; Indeed, $(fg)'' \neq f''g + fg''$ (in general) since $(fg)'' = ((fg)')' = f''g + 2f'g' + fg''$. ▪

4.3 Derivation operator in S

Definition 4.5 A linear map $\mathcal{L} \in L(\mathcal{F}(S);\mathcal{F}(S))$ is a derivation iff, for all $f, g \in \mathcal{F}(S)$,

$$\mathcal{L}(fg) = \mathcal{L}(f)\,g + f\,\mathcal{L}(g) \quad \in \mathcal{F}(S), \tag{4.5}$$

that is, for all $p \in S$, $\mathcal{L}(fg)(p) = \mathcal{L}(f)(p)\,g(p) + f(p)\,\mathcal{L}(g)(p) \in \mathbb{R}$.

Remark 4.6 With $\mathcal{F}(\mathbb{R})$, the first order derivation $f \to f'$ is a derivation : $(fg)' = f'g+fg'$. But the second order derivation $f \to f''$ is not: $(fg)'' \neq f''g+fg''$ (in general) since $(fg)'' = f''g+2fg+fg''$. ▪

Proposition 4.7 *If $\mathcal{L}$ si a derivation on $\mathcal{F}(S)$, then, for all $c \in \mathbb{R}$, $\mathcal{L}(c1_S) = 0$.*

Proof. (4.5) gives $\mathcal{L}(1_S) = \mathcal{L}(1_S 1_S) = \mathcal{L}(1_S)1 + 1\mathcal{L}(1_S) = 2\mathcal{L}(1_S)$, thus $\mathcal{L}(1_S) = 0$, thus $\mathcal{L}(c1_S) = c\mathcal{L}(1_S) = 0$ (linearity). ▪

4.4 Autonomous Lie derivative $\tilde{\mathcal{L}}^0_{\vec{v}} f = df.\vec{v} = \tilde{\nabla}_{\vec{v}} f$

Definition 4.8 Let $\vec{v} \in \Gamma(S)$, $f \in \mathcal{F}(S)$. The autonomous Lie derivative of f along $\vec{v}$ is the map $\tilde{\mathcal{L}}^0_{\vec{v}} : \mathcal{F}(S) \to \mathcal{F}(S)$ defined by, for all $p \in S$,

$$\tilde{\mathcal{L}}^0_{\vec{v}}(f)(p) = df(p).\vec{v}(p), \quad \text{and} \quad \tilde{\mathcal{L}}^0_{\vec{v}}(f) \overset{\text{written}}{=} \tilde{\nabla}_{\vec{v}}(f) \quad (= df.\vec{v} \overset{\text{written}}{=} \vec{v}(f)). \tag{4.6}$$

(So $\tilde{\mathcal{L}}^0_{\vec{v}}(f)(p)$ is the directional derivative of f at p along $\vec{v}(p)$.)

Proposition 4.9 *$\tilde{\mathcal{L}}^0_{\vec{v}}$ is a derivation on $\mathcal{F}(S)$.*

Conversely: If $\mathcal{L} : \mathcal{F}(S) \to \mathcal{F}(S)$ is a derivation, then $\mathcal{L}$ is an autonomous Lie derivative, that is, there exists $\vec{v} \in \Gamma(S)$ s.t.

$$\mathcal{L}(f) = \tilde{\mathcal{L}}^0_{\vec{v}} f \quad (= df.\vec{v} = \tilde{\nabla}_{\vec{v}} f = \vec{v}(f)), \tag{4.7}$$

Proof. $\tilde{\mathcal{L}}^0_{\vec{v}}(fg)(p) = d(fg)(p).\vec{v}(p) = (df(p).\vec{v}(p))g(p) + f(p)(dg(p).\vec{v}(p) = (\tilde{\mathcal{L}}^0_{\vec{v}}(f)g + f\tilde{\mathcal{L}}^0_{\vec{v}}(g))(p)$.

Converse. Let $\mathcal{L}$ be a derivation on S, and let $p \in S$. Then $\mathcal{L}^p : \mathcal{F}(S) \to \mathbb{R}$ defined by $\mathcal{L}^p(f) = \mathcal{L}(f)(p)$ is a derivation at p (trivial). Thus there exists $\vec{v}(p)$ s.t. $\mathcal{L}^p = \sum_{i=1}^n v^i(p)\,\frac{\partial}{\partial q^i}(p)$, cf. proposition 4.3. This defines the map $\vec{v} : S \to TS$, and (4.3) gives $\mathcal{L}(f)(p) = \sum_{i=1}^n v^i(p)\frac{\partial f}{\partial q^i}(p) = \sum_{i=1}^n v^i(p)df(p).\vec{e}_i(p) = \tilde{\mathcal{L}}^0_{\vec{v}}(f)(p)$. And $\mathcal{L}(f)$ and $\vec{e}_i$ C^∞, gives $\vec{v} = \sum_i v^i\vec{e}_i$ C^∞: $\vec{v} \in \Gamma(S)$. ▪

Part III
Usual Riemannian connection on vector fields

The usual connection $\nabla_{\vec{v}}$ along $\vec{v}$ in S, called the Riemannian connection, is given by $\nabla_{\vec{v}}\vec{w} := \mathrm{Proj}_{TS}(d\vec{w}.\vec{v})$ = the orthogonal projection of $d\vec{w}.\vec{v}$ on TS. This projection is the only component of $d\vec{w}.\vec{v}$ we have access to, when the "outside" of S is inaccessible.

A general connection is defined such that it "looks like" the usual Riemannian connection.

5 Connection $\tilde{\nabla}$ on $\mathcal{F}(S)$

5.1 The classical definition is problematic in S

Let S be a sphere in $\mathbb{R}^3$ and $f = 1_S$ (uniform density in S, zero outside S). Let $p \in S$ and $\vec{v}_p \in T_pS$. If we try the usual definition in $\mathbb{R}^n$, that is

$$df(p).\vec{v} := \lim_{\substack{h\to 0\\ h\neq 0}} \frac{f(p+h\vec{v}) - f(p)}{h}, \quad \text{we get} \quad df(p).\vec{v} = \lim_{\substack{h\to 0\\ h\neq 0}} \frac{0-1}{h} = \mp\infty \tag{5.1}$$

absurd result (not expected) in the sense: If we walk on the Earth surface and $f = 1_S$, then f does not vary, so we expect $df.\vec{v} = 0$ for all horizontal $\vec{v}$, not ∞. Thus definition in (5.1) is inadequate.

5.2 Covariant derivative, or autonomous Lie derivative, on $\mathcal{F}(S)$

Let

$$c : \begin{cases} [a,b] & \to S \subset \mathbb{R}^n \\ s & \to c(s) \end{cases} \tag{5.2}$$

be a regular curve in S. NB: the variable s is interpreted as a spatial coordinate in the following (the time variable t will be introduced for the "unsteady" case). Let $p = c(s) \in \text{Im}c \subset S$ and let

$$\vec{v}(p) := \vec{c}\,'(s) = \lim_{h\to 0} \frac{c(s+h)-c(s)}{h} \in \vec{\mathbb{R}^n} \quad \text{(the tangent vector at Im}c\text{ at } p = c(s)). \tag{5.3}$$

Definition 5.1 Let $f : S \to \mathbb{R}$. At $p = c(s)$, the covariant derivative $\nabla_{\vec{v}} f(p)$, also called the autonomous Lie derivative $\mathcal{L}^0{}_{\vec{v}} f(p)$, is the scalar defined by (if it exists)

$$\tilde{\nabla}_{\vec{v}} f(p) = \mathcal{L}^0{}_{\vec{v}} f(p) := \frac{d(f\circ c)}{ds}(s) = \lim_{h\to 0} \frac{f(c(s+h)) - f(c(s))}{h} \tag{5.4}$$

(To compare with (5.1).) And

$$\tilde{\nabla}_{\vec{v}} f(p) = \mathcal{L}^0{}_{\vec{v}} f(p) \overset{\text{written}}{=} \frac{Df}{ds}(p) = df(p).\vec{v}(p). \tag{5.5}$$

Remark: The exponent ^{0} in $\mathcal{L}^0{}_{\vec{v}} f$ corresponds to the steady case (time independent), while $\mathcal{L}_{\vec{v}} f$ refers to the general unsteady case, see previous manuscript.

Remark: If $\vec{v} \in \Gamma(S)$, then we dispose of its integral curves $c : [a,b] \to S$ in S, given by $c(s) = \int_{u=0}^{s} \vec{v}(c(u))\,du + c_0$ (with c_0 a constant, and $\vec{v}$ is independent of c_0), thus $\tilde{\nabla}_{\vec{v}} f(p)$ is well defined.

Thus we have defined the covariant derivative, or autonomous Lie derivative, along $\vec{v}$:

$$\tilde{\nabla}_{\vec{v}} = \mathcal{L}^0{}_{\vec{v}} : \begin{cases} \mathcal{F}(S) & \to \mathcal{F}(S), \\ f & \mapsto \tilde{\nabla}_{\vec{v}} f = \mathcal{L}^0{}_{\vec{v}} f := \tilde{\nabla}_{\vec{v}} f = \dfrac{Df}{ds} = df.\vec{v}. \end{cases} \tag{5.6}$$

Proposition 5.2 *If $\varphi \in \mathcal{F}(S)$ and $\vec{v}, \vec{w} \in \Gamma(S)$, then, for all $g \in \mathcal{F}(S)$ (algebraic formulas)*

$$\tilde{\nabla}_{\varphi\vec{v}+\vec{w}} g = \varphi\, \tilde{\nabla}_{\vec{v}} g + \tilde{\nabla}_{\vec{w}} g, \quad \text{i.e.} \quad \tilde{\nabla}_{\varphi\vec{v}+\vec{w}} g = \varphi \tilde{\nabla}_{\vec{v}} g + \tilde{\nabla}_{\vec{w}} g, \tag{5.7}$$

that is, $dg(\varphi\vec{v} + \vec{w}) = \varphi\, dg.\vec{v} + dg.\vec{w}$ (linearity). And, for all $f, g \in \mathcal{F}(S)$ (derivation formula),

$$\tilde{\nabla}_{\vec{v}}(fg) = f\, \tilde{\nabla}_{\vec{v}} g + (\tilde{\nabla}_{\vec{v}} f)\, g, \tag{5.8}$$

that is, $d(fg).\vec{v} = (df.\vec{v})\, g + f\,(dg.\vec{v})$ (written $d(fg) = (df)g + f(dg)$).

Proof. Adapt the classical proof, cf. (5.1), of the proposition in $\mathbb{R}^n$ to the definition given in (5.4). ∎

5.3 Differential $\tilde{\nabla} f$ in $\mathcal{F}(S)$

Definition 5.3 If $f \in C^1(S;\mathbb{R})$, then, with (5.4), the differential of f in S is the map, if it exists,

$$\tilde{\nabla} f : \begin{cases} \Gamma(S) & \to \mathcal{F}(S), \\ \vec{v} & \mapsto \tilde{\nabla} f.\vec{v} := \tilde{\nabla}_{\vec{v}} f \overset{\text{written}}{=} df.\vec{v}, \end{cases} \tag{5.9}$$

And $\tilde{\nabla} f =^{\text{written}} df$ (although S is not an open set in $\mathbb{R}^n$ if $m < n$).

5.4 Connection $\tilde{\nabla}$ in S

(5.9) enables to define the connection for scalar functions in S: It is the map

$$\tilde{\nabla} : \begin{cases} \Gamma(S) \times \mathcal{F}(S) & \to \mathcal{F}(S), \\ (\vec{v}, f) & \mapsto \tilde{\nabla}(\vec{v}, f) := \tilde{\nabla}_{\vec{v}} f = \tilde{\nabla} f.\vec{v} \quad (= \mathcal{L}^0{}_{\vec{v}} f = df.\vec{v}), \end{cases} \tag{5.10}$$

NB: Usually $\tilde{\nabla} =^{\text{written}} \nabla$, the context removing ambiguities. However, we will stick to $\tilde{\nabla}$ to avoid confusions with the connection ∇ on vectors fields.

Proposition 5.4 *$\tilde{\nabla}$ is $\mathcal{F}(S)$-linear in the first variable, and is derivation in the second variable.*

Proof. It is (5.7) and (5.8). ∎

5.5 Quantification of $\tilde\nabla_{\vec v} f$ and $\tilde\nabla f$

$(\vec A_i)$ being the canonical basis in $\vec{\mathbb{R}^m}$, the coordinate basis $(\vec e_i(p))$ at $p=\Phi(\vec q)$ is given by $\vec e_i(p) = d\Phi(\vec q).\vec A_i$, and $(e^i(p)) = (dq^i(p))$ is its dual basis, cf. (1.26). Then, if $f\in\mathcal{F}(S)$, the $\frac{\partial f}{\partial q^i}(p)$ are the components of $\tilde\nabla f(p)$ relative to the basis $(dq^i(p))$: If $\vec v = \sum_{i=1}^m v^i\vec e_i$ and $p=\Phi(\vec q)\in S$, then

$$(df(p)=)\quad \tilde\nabla f(p) = \sum_{i=1}^m \frac{\partial f}{\partial q^i}(p)dq^i(p), \quad\text{and}\quad \tilde\nabla_{\vec v} f = \sum_{i=1}^m \frac{\partial f}{\partial q^i} v^i. \tag{5.11}$$

Indeed, $\tilde\nabla f(p).\vec e_i(p) = \dfrac{d(f\circ c^{(i)})}{dt}(t) = \lim_{h\to 0}\dfrac{f(c^{(i)}(t+h)) - f(c^{(i)}(t))}{h} = df(p).\vec e_i(p) = \dfrac{\partial f}{\partial q^i}(p)$.

Proposition 5.5 *With a coordinate basis $(\vec e_i)$, we have, for all $f\in\mathcal{F}(S)$ and for all i,j,*

$$\tilde\nabla_{\vec e_i}(\tilde\nabla_{\vec e_j} f) = \tilde\nabla_{\vec e_j}(\tilde\nabla_{\vec e_i} f)\quad (=\frac{\partial^2 f}{\partial q^i\partial q^j}). \tag{5.12}$$

Proof. Let $p=\Phi(\vec q)$ and $f_U(\vec q) = f(p)$, that is, $f_U := f\circ\Phi$. Then $\frac{df_U}{dq^j}(s) = df_U(\vec q).\vec A_j = df(p).\vec e_j(p) = \tilde\nabla_{\vec e_j} f(p)$ $(=^{named}\frac{df}{dq^j}(s))$. And $\frac{\partial\frac{\partial f_U}{\partial q^i}}{\partial q^j} = \tilde\nabla_{\vec e_i}(\tilde\nabla_{\vec e_j} f)(p)$. And f and Φ being regular (at least C^2), f_U is regular (at least C^2), we have $\frac{\partial\frac{\partial f_U}{\partial q^i}}{\partial q^j} = \frac{\partial\frac{\partial f_U}{\partial q^j}}{\partial q^i}$, thus (5.12). ∎

6 Riemannian connection ∇ in S

6.1 The classical definition is problematic in S

As in paragraph 5.1, if $\vec w\in\Gamma(S)$ and is zero outside of S, then $\lim_{h\to 0}\frac{\vec w(p+h\vec v)-\vec w(p)}{h} = \pm\infty$, and the classical definition of the covariant derivative $d\vec w.\vec v = \lim_{h\to 0}\frac{\vec w(p+h\vec v)-\vec w(p)}{h}$ is inappropriate. Thus, as for (5.4), we consider a curve c, cf. (5.2), and $\vec v(c(s)) = \vec c\,'(s)$, cf. (5.3), and we define

$$d\vec w(p).\vec v(p) := \frac{d(\vec w\circ c)}{ds}(s). \tag{6.1}$$

But a new problem arises: Even if $\vec v,\vec w\in TS$ (tangent to S), the covariant derivative $d\vec w.\vec v\notin TS$ (not tangent to S) in general; E.g., (3.9) gives $d\vec e_2.\vec e_2 = -r\vec e_1$ (the centrifugal force), which is not tangent to the circle, whereas $\vec e_2$ is.

So one more step is required to get a derivation in S for vector fields in $\Gamma(S)$: To get read of the orthogonal component of $d\vec w.\vec v$ on S. That is, if $d\vec w(p).\vec v(p) = \vec u_{||}(p) + \vec u_\perp(p)\in T_pS\oplus T_pS^\perp = \vec{\mathbb{R}^n}$, we only consider $\vec u_{||}(p) = \mathrm{Proj}_{T_pS}(d\vec w(p).\vec v(p))$.

6.2 Projections Proj_{T_pS} and Proj_{TS}

Let $(\cdot,\cdot)_{\mathbb{R}^n}$ be a dot product in $\vec{\mathbb{R}^n}$ (often supposed to be Euclidean in classical mechanics). Let $p\in S$. The orthogonal projection in S at p is the map

$$\mathrm{Proj}_{T_pS}:\begin{cases}\vec{\mathbb{R}^n} &\to T_pS\subset\vec{\mathbb{R}^n}\\ \quad\vec u &\to \mathrm{Proj}_{T_pS}(\vec u),\end{cases} \tag{6.2}$$

where the projection $\mathrm{Proj}_{T_pS}(\vec u)$ is the unique vector in T_pS such that

$$(\mathrm{Proj}_{T_pS}(\vec u),\vec v_p)_{\mathbb{R}^n} = (\vec u,\vec v_p)_{\mathbb{R}^n},\quad \forall\vec v_p\in T_pS. \tag{6.3}$$

That is, $(\vec u - \mathrm{Proj}_{T_pS}(\vec u),\vec v_p)_{\mathbb{R}^n} = 0$ for all $\vec v_p\in T_pS$, i.e. $\vec u-\mathrm{Proj}_{T_pS}(\vec u)\perp T_pS$.

This gives the definition of the projection operator (the linear map) on vector fields in S:

$$\mathrm{Proj}_{TS}:\begin{cases}\Gamma(S) &\to\Gamma(S)\\ \quad\vec u &\to \mathrm{Proj}_{TS}(\vec u)\quad\text{where}\quad \mathrm{Proj}_{TS}(\vec u)(p) := \mathrm{Proj}_{T_pS}(\vec u(p)),\quad\forall p\in S.\end{cases} \tag{6.4}$$

E.g., S being the circle, (3.9) gives $d\vec e_2.\vec e_2 = -r\vec e_1$ (only a centrifugal force), thus $\mathrm{Proj}_{TS}(\vec e_2) = 0$ (there is no tangential force).

6.3 Riemannian covariant derivative $\nabla_{\vec{v}}\vec{w} = \frac{D\vec{w}}{dt}$

Let $c : [a,b] \to S$ be regular curve in $S \subset \mathbb{R}^n$, let $p = c(s)$ and $\vec{v}(p) = \vec{c}\,'(s)$, cf. (5.3).

Definition 6.1 If $\vec{w} \in \Gamma(S)$, cf.(2.1), then its Riemannian covariant derivative along $\vec{v}$ at p is

$$\nabla_{\vec{v}}\vec{w}(p) := \mathrm{Proj}_{T_pS}(\frac{d(\vec{w}\circ c)}{ds}(s)) = \mathrm{Proj}_{T_pS}(d\vec{w}(p).\vec{v}(p)) \overset{\text{written}}{=} \frac{D\vec{w}}{ds}(p), \tag{6.5}$$

And the Riemannian covariant derivative operator in $\Gamma(S)$ along $\vec{v}$ is the (linear) map

$$\nabla_{\vec{v}} : \begin{cases} \Gamma(S) & \to \Gamma(S), \\ \vec{w} & \mapsto \nabla_{\vec{v}}\vec{w} = \mathrm{Proj}_{TS}(d\vec{w}.\vec{v}) \overset{\text{written}}{=} \dfrac{D\vec{w}}{ds}. \end{cases} \tag{6.6}$$

6.4 Riemannian differential $\nabla\vec{w}$ on $\Gamma(S)$

If $\vec{v} \in \Gamma(S)$ then we consider its integral curves in S. And with (6.6):

Definition 6.2 Let $\vec{w} \in \Gamma(S)$. Its Riemannian differential at p is the map

$$\nabla\vec{w} : \begin{cases} \Gamma(S) & \to \Gamma(S), \\ \vec{v} & \mapsto \nabla\vec{w}.\vec{v} := \nabla_{\vec{v}}\vec{w} \quad (= \mathrm{Proj}_{TS}(d\vec{w}.\vec{v})). \end{cases} \tag{6.7}$$

6.5 Riemannian connection ∇ in S

Definition 6.3 With (6.6), the Riemannian connection ∇ in S is the map

$$\nabla : \begin{cases} \Gamma(S)\times\Gamma(S) & \to \Gamma(S), \\ (\vec{v},\vec{w}) & \mapsto \nabla(\vec{v},\vec{w}) := \nabla_{\vec{v}}\vec{w} \quad (= \mathrm{Proj}_{TS}(d\vec{w}.\vec{v})). \end{cases} \tag{6.8}$$

6.6 Properties

Proposition 6.4 *If $\varphi \in \mathcal{F}(S)$ and $\vec{u},\vec{v} \in \Gamma(S)$ then, for all $\vec{w} \in \Gamma(S)$, (algebraic formula)*

$$\nabla_{\varphi\vec{u}+\vec{v}}\vec{w} = \varphi\,\nabla_{\vec{u}}\vec{w} + \nabla_{\vec{v}}\vec{w}, \tag{6.9}$$

and (derivation formula)

$$\nabla_{\vec{v}}(\varphi\vec{w}) = \varphi\,\nabla_{\vec{v}}(\vec{w}) + \tilde{\nabla}_{\vec{v}}(\varphi)\,\vec{w}. \tag{6.10}$$

Proof. Apply the linear operator Proj_{T_pS} to $d\vec{w}.(\varphi\vec{u}+\vec{v}) = \varphi\,d\vec{w}.\vec{u}+d\vec{w}.\vec{v}$ and to $d(\varphi\vec{w}).\vec{v} = (d\varphi.\vec{v})\vec{w}+\varphi\,(d\vec{w}.\vec{v})$ at p. ∎

Corollary 6.5 *The derivation operator $\nabla_{\vec{v}} : \Gamma(S) \to \Gamma(S)$ is not tensorial.*

Proof. It would require $\nabla_{\vec{v}}(\varphi\vec{w}) = \varphi\nabla\vec{w}$, which is false if φ is not constant, cf. (6.10). ∎

On the other hand

Proposition 6.6 *If $\vec{w} \in \Gamma(S)$, the $\nabla\vec{w} : \left\{ \begin{array}{rl} \Gamma(S) & \to \Gamma(S) \\ \vec{v} & \to \nabla\vec{w}.\vec{v} \end{array} \right\}$ is a tensor.*

Proof. Corollary of $\nabla\vec{w}.(\varphi\vec{v}) = \varphi\,\nabla\vec{w}.\vec{v}$, cf. (6.9). ∎

6.7 Notation $\frac{D^2\vec{w}}{ds^2}$

With $\frac{D\vec{w}}{ds} = \nabla_{\vec{v}}\vec{w} \in \Gamma(S)$, cf. (6.13), then

$$\frac{D\frac{D\vec{w}}{ds}}{ds} \overset{\text{written}}{=} \frac{D^2\vec{w}}{ds^2} \quad (:= \nabla_{\vec{v}}(\nabla_{\vec{v}}\vec{w})) \tag{6.11}$$

gives the second order variations of $\vec{w}$ along an integral curve of $\vec{v}$ (will give the Riemann tensor).

6.8 Unsteady vector fields and $\frac{D\vec{w}}{Dt}$

Let $\vec{v} : [t_0, T] \times S \to \vec{v}(t,p) \in TS$ be an unsteady (Eulerian) velocity field in S (the velocity along an unsteady "real motion"). And let $\vec{w} : [t_0, T] \times S \to \vec{w}(t,p) \in TS$ be any unsteady vector field in S (usually interpreted as a "tangential force field"). And at any given t, let $\vec{v}_t(p) := \vec{v}(t,p)$ and $\vec{w}_t(p) := \vec{w}(t,p)$.

Definition 6.7 The covariant derivative $\nabla_{\vec{v}}\vec{w}(t,p)$ is defined by $\nabla_{\vec{v}}\vec{w}(t,p) := \nabla_{\vec{v}_t}\vec{w}_t(p)$, that is,

$$\nabla_{\vec{v}}\vec{w}(t,p) := \text{Proj}_{T_pS}(d\vec{w}_t(p).\vec{v}_t(p)) \overset{\text{written}}{=} \text{Proj}_{T_pS}(d\vec{w}(t,p).\vec{v}(t,p)) \quad \text{(space derivation at } t\text{)}. \tag{6.12}$$

The material derivative in S of $\vec{w}$ along $\vec{v}$ is, at t at $p \in S$,

$$\frac{D\vec{w}}{Dt}(t,p) := \frac{\partial\vec{w}}{\partial t}(t,p) + \nabla_{\vec{v}}\vec{w}(t,p) \quad \Big(= \frac{\partial\vec{w}}{\partial t}(t,p) + \text{Proj}_{T_pS}(d\vec{w}(t,p).\vec{v}(t,p))\Big). \tag{6.13}$$

(If S is open in $\mathbb{R}^n$ then $\frac{D\vec{w}}{Dt}(t,p) = \frac{\partial\vec{w}}{\partial t}(t,p) + d\vec{w}(t,p).\vec{v}(t,p) =$ the material derivative in $\mathbb{R}^n$.

6.9 Acceleration $\frac{D\vec{v}}{Dt}$

Let $\vec{v}$ be an unsteady vector field and let c be an integral curve of $\vec{v}$ at p. So $\vec{c}\,'(t) = \vec{v}(t,c(t)) = \vec{v}(t,p)$, when $p = c(t)$, is the velocity along c. Remember that the acceleration along c at $p = c(t)$ is $\vec{\gamma}(c(t)) = \vec{c}\,''(t)$.

Definition 6.8 The acceleration in S is at t at $p = c(t)$ is

$$\vec{\gamma}_S(p) := \text{Proj}_{T_pS}(\vec{\gamma}(p)) \quad (= \text{Proj}_{T_pS}(\vec{c}\,''(t))), \tag{6.14}$$

that is, with (6.13),

$$\vec{\gamma}_S(c(t)) := \frac{D\vec{v}}{Dt}(t,p) = \text{Proj}_{T_pS}(\frac{\partial\vec{v}}{\partial t} + d\vec{v}.\vec{v})(t,p)) = \frac{\partial\vec{v}}{\partial t}(t,p) + \nabla_{\vec{v}}\vec{v}(t,p). \tag{6.15}$$

7 Christoffel symbols

7.1 Definition

Definition 7.1 The Christoffel symbols $(\gamma_{ij}^k(p))_{k=1,\dots,n}$ at $p \in S \subset \mathbb{R}^n$ relative to the connection ∇, cf. (6.8), are the components of the vector $\nabla_{\vec{e}_i}\vec{e}_j(p) = \text{Proj}_{T_pS}(d\vec{e}_j(p).\vec{e}_i(p)) \in T_pS$ relative to $(\vec{e}_i(p))_{i\in[1,m]_\mathbb{N}}$, that is, for $i,j = 1,...,m$,

$$(\text{Proj}_{TS}(d\vec{e}_j.\vec{e}_i) =) \quad \nabla_{\vec{e}_i}\vec{e}_j = \sum_{k=1}^m \gamma_{ij}^k\vec{e}_k, \quad \text{and} \quad \gamma_{ij}^k = e^k.(d\vec{e}_j.\vec{e}_i) \quad \forall k = 1,...,m, \tag{7.1}$$

i.e., for all $p \in S$, $(\text{Proj}_{T_pS}(d\vec{e}_j(p).\vec{e}_i(p)) =)\ \nabla_{\vec{e}_i}\vec{e}_j(p) = \sum_{k=1}^m \gamma_{ij}^k(p)\vec{e}_k(p)$.

Comparison with definition 2.1: The sum is limited to $k = 1,...,m$. And remember, cf. (2.11):

$$\forall i,j,k \in [1,m]_\mathbb{N}, \ \ \gamma_{ij}^k = \gamma_{ji}^k. \tag{7.2}$$

Example 7.2 Polar coordinate system, cf. (3.10), $d\vec{e}_2.\vec{e}_2 = -r\vec{e}_1$. And restricted to the circle, $\nabla_{\vec{e}_2}\vec{e}_2 = \text{Proj}_{TS}(d\vec{e}_2.\vec{e}_2) = \text{Proj}_{TS}(-r\vec{e}_1) = \vec{0}$, thus $\gamma_{22}^2 = 0$ in the circle (the only symbol of Christoffel vanishes).

For spherical coordinates restricted to the sphere, the γ_{ij}^k are given at (3.43). ∴

7.2 Components $w^i_{|j}$ of $\nabla\vec{w}$

Let $\vec{w}\in\Gamma(S)$ and $\vec{w}=\sum_{i=1}^m w^i\vec{e}_i$, that is $\vec{w}(p)=\sum_{i=1}^m w^i(p)\vec{e}_i(p)$ for all $p\in S$; Then, for $j=1,...,m$,

$$\begin{aligned}\nabla_{\vec{e}_j}\vec{w}&=\text{Proj}_{T_pS}(d\vec{w}.\vec{e}_j)=\text{Proj}_{T_pS}(\sum_{i=1}^m(dw^i.\vec{e}_j)\vec{e}_i+\sum_{i=1}^m w^i d\vec{e}_i.\vec{e}_j)\\&=\text{Proj}_{T_pS}(\sum_{i=1}^m\frac{\partial w^i}{\partial q^j}\vec{e}_i+\sum_{i=1}^m\sum_{k=1}^n w^i\gamma^k_{ji}\vec{e}_k).\end{aligned}\tag{7.3}$$

Thus,

$$\nabla_{\vec{e}_j}\vec{w}=\sum_{i=1}^m w^i_{|j}\vec{e}_i\quad\text{with}\quad w^i_{|j}=\frac{\partial w^i}{\partial q^j}+\sum_{k=1}^m\gamma^i_{jk}w^k,\ \forall i=1,...,m.\tag{7.4}$$

So, the $w^i_{|j}=(\nabla\vec{w})^i_j$ are the components of the tensor $\nabla\vec{w}$ relative to $(\vec{e}_i)$:

$$\nabla\vec{w}=\sum_{i,j=1}^m w^i_{|j}\,\vec{e}_i\otimes e^j,\quad\text{and}\quad[\nabla\vec{w}]_{|\vec{e}}=[w^i_{|j}]_{\substack{i=1,...,m\\j=1,...,m}}.\tag{7.5}$$

Hence, if $\vec{v}\in\Gamma(S)$ and $\vec{v}=\sum_{j=1}^m v^j\vec{e}_j$, then $\nabla_{\vec{v}}\vec{w}=\sum_{i=1}^m v^j\nabla_{\vec{e}_j}\vec{w}$ gives

$$\nabla_{\vec{v}}\vec{w}=\sum_{i,j=1}^m w^i_{|j}v^j\,\vec{e}_i=\nabla\vec{w}.\vec{v}\quad(=\sum_{i,j=1}^m(\frac{\partial w^i}{\partial q^j}+\sum_{k=1}^m\gamma^i_{jk}w^k)v^j\vec{e}_i).\tag{7.6}$$

Exercise 7.3 Prove:

$$\nabla_{\vec{v}}\vec{w}-\nabla_{\vec{w}}\vec{v}=\sum_{i,j=1}^m(v^j\frac{\partial w^i}{\partial q^j}-w^j\frac{\partial v^i}{\partial q^j})\,\vec{e}_i.\tag{7.7}$$

(The Christoffel symbols "disappear", and $\mathcal{L}_{\vec{v}}\vec{w}:=\nabla_{\vec{v}}\vec{w}-\nabla_{\vec{w}}\vec{v}\in\Gamma(S)$.) ♣

7.3 Divergence in S : $\text{div}\vec{w}=\sum w^i_{|i}=\text{Tr}(\nabla\vec{w})$

Definition 7.4 Let $\vec{w}\in\Gamma(S)$. Its divergence is the trace of its differential $\nabla\vec{w}$, that is, with (7.5),

$$\text{div}\vec{w}=\text{Tr}(\nabla\vec{w})=\sum_{i=1}^m w^i_{|i}\quad(=\sum_{i=1}^m\frac{\partial w^i}{\partial q^i}+\sum_{i,k=1}^m\gamma^i_{ik}w^k).\tag{7.8}$$

NB: The divergence is independent of the coordinate basis, since $\nabla\vec{w}$ is a $\binom{1}{1}$ tensor (or apply (8.15)).

Exercise 7.5 Prove:

$$\text{div}(f\vec{w})=f\,\text{div}\vec{w}+df.\vec{w}.\tag{7.9}$$

Answer. $f\vec{w}=\sum_i fw^i\vec{e}_i$, thus $(f\vec{w})^i=fw^i$, thus $\text{div}(f\vec{w})=\sum_{i=1}^m(fw^i)_{|i}=\sum_i f_{|i}w^i+f\sum_i w^i_{|i}$. Or, apply $d(f\vec{w})=\vec{w}\otimes df+f\,d\vec{w}$ (indeed, $d(f\vec{w}).\vec{v}=(df.\vec{v})\vec{w}+f\,(d\vec{w}.\vec{v})$). ♣

8 Change of coordinate system in S

8.1 Change of coordinate basis and transition matrix

8.1.1 The coordinate systems

Consider two coordinate systems describing (locally) S:

$$\Phi_a:\left\{\begin{array}{ll}U\subset\mathbb{R}^{\vec{m}}&\to S\\ \quad\vec{q}_a&\to p=\Phi_a(\vec{q}_a)\end{array}\right\}\quad\text{and}\quad\Phi_b:\left\{\begin{array}{ll}V\subset\mathbb{R}^{\vec{m}}&\to S\\ \quad\vec{q}_b&\to p=\Phi_b(\vec{q}_b)\end{array}\right\}.\tag{8.1}$$

E.g. in $\mathbb{R}^2$, Φ_a is a Cartesian system chosen by an observer A, and Φ_b is polar system chosen by an observer B, both systems being used to describe the same surface$S\subset\mathbb{R}^2$.

Consider $p \in S$, $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$ (the position p as described by the observers A and B), and $\vec{\psi}$ the change of parameter diffeomorphism (the translation from B to A)

$$\vec{\psi} := {\Phi_a}^{-1} \circ \Phi_b : \begin{cases} V \subset \mathbb{R}^m & \to U \subset \mathbb{R}^m, \\ \vec{q}_b & \to \vec{q}_a = \vec{\psi}(\vec{q}_b) := {\Phi_a}^{-1}(\Phi_b(\vec{q}_b)) \overset{\text{written}}{=} \vec{q}_a(\vec{q}_b). \end{cases} \tag{8.2}$$

Thus $\vec{q}_a(\vec{q}_b)$ is the name given to $\vec{\psi}(\vec{q}_b) = {\Phi_a}^{-1}(p)$, when $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$.

8.1.2 The coordinate bases

Let $(\vec{A}_i)$ be the canonical basis in $\vec{\mathbb{R}^m}$ (in the Cartesian space $\vec{\mathbb{R}^m} = \mathbb{R} \times ... \times \mathbb{R}$ there is only one Cartesian basis). Then the bases at p of the coordinate systems Φ_a and Φ_b are, with $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$, cf. (1.12),

$$\vec{e}_{i,a}(p) = d\Phi_a(\vec{q}_a).\vec{A}_i \quad \text{and} \quad \vec{e}_{i,b}(p) = d\Phi_b(\vec{q}_b).\vec{A}_i. \tag{8.3}$$

8.1.3 The change of coordinate basis

Consider the change of basis endomorphism $\mathcal{P}(p) : \vec{\mathbb{R}^m} \to \vec{\mathbb{R}^m}$ defined by, for all $j \in [1,m]_{\mathbb{N}}$,

$$\vec{e}_{j,b}(p) = \mathcal{P}(p).\vec{e}_{j,a}(p) = \sum_{i=1}^m P^i_j(p)\, \vec{e}_{i,a}(p), \quad [\mathcal{P}(p)]_{|\vec{e}_a} = [P^i_j(p)]. \tag{8.4}$$

Proposition 8.1 *With $\vec{\psi}(\vec{q}_b) = \sum_{i=1}^m \psi^i(\vec{q}_b)\vec{A}_i = \sum_{i=1}^m q_a^i(\vec{q}_b)\vec{A}_i$, then, for all $i,j \in [1,m]_{\mathbb{N}}$, at $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$,*

$$P^i_j(p) = \frac{\partial q_a^i}{\partial q_b^j}(\vec{q}_b), \quad [\mathcal{P}(p)]_{|\vec{e}_a} = [d\vec{q}_a(\vec{q}_b)]_{|\vec{A}} \overset{\text{written}}{=} [\frac{\partial \vec{q}_a}{\partial \vec{q}_b}(\vec{q}_b)]. \tag{8.5}$$

(The only difficulty is due to the understanding of the notations.)

Proof. $p = \Phi_b(\vec{q}_b) = \Phi_a(\vec{q}_a) = \Phi_a(\vec{\psi}(\vec{q}_b))$ gives $d\Phi_b(\vec{q}_b).\vec{A}_j = d\Phi_a(\vec{\psi}(\vec{q}_b)).d\vec{\psi}(\vec{q}_b).\vec{A}_j$. And $\vec{\psi}(\vec{q}_b) = \sum_{i=1}^m \psi^i(\vec{q}_b)\vec{A}_i$ gives $d\vec{\psi}(\vec{q}_b).\vec{A}_j = \sum_{i=1}^m (d\psi^i(\vec{q}_b).\vec{A}_j)(\vec{q}_b)\vec{A}_i = \sum_{i=1}^m \frac{\partial \psi^i}{\partial q_b^j}(\vec{q}_b)\vec{A}_i$. Thus

$$d\Phi_b(\vec{q}_b).\vec{A}_j = \sum_{i=1}^m \frac{\partial \psi^i}{\partial q_b^j}(\vec{q}_b)\, d\Phi_a(\vec{q}_a(\vec{q}_b)).\vec{A}_i, \quad \text{i.e.} \quad \vec{e}_{i,b}(p) = \sum_{i=1}^m \frac{\partial \psi^i}{\partial q_b^j}(\vec{q}_b)\vec{e}_{i,a}(p). \tag{8.6}$$

So, with (8.4), we get $P^i_j(p) = \frac{\partial \psi^i}{\partial q_b^j}(\vec{q}_b)$ for all i,j, and $q_a^i(\vec{q}_b) := \psi^i(\vec{q}_b)$ gives (8.5). ∎

Example 8.2 In $\mathbb{R}^2$ with a Euclidean basis $(\vec{E}_1, \vec{E}_2)$: Let Φ_a Cartesian: $\begin{Bmatrix} \vec{\mathbb{R}^2} & \to \mathbb{R}^2 \\ \vec{q}_a = (x,y) & \to p = \Phi_a(\vec{q}_a) = O + x\vec{E}_1 + y\vec{E}_2 \end{Bmatrix}$, and $\vec{e}_{i,a}(p) = \vec{E}_i$. And let Φ_b polar: $\begin{Bmatrix} \mathbb{R}_+ \times]-\pi,\pi] & \to \mathbb{R}^2 \\ \vec{q}_b = (r,\theta) & \to p = \Phi_b(\vec{q}_b) = O + r\cos\theta\vec{E}_1 + r\sin\theta\vec{E}_2 \end{Bmatrix}$, and $\vec{e}_{i,b}(p)$ is given by (3.3).

And $\vec{q}_a(\vec{q}_b) := \vec{\psi}(\vec{q}_b)$ is given by $[\vec{q}_a(\vec{q}_b)]_{\vec{e}_{i,a}} = \begin{pmatrix} q_a^1 = x = r\cos\theta = q_b^1 \cos q_b^2 \\ q_a^2 = y = r\sin\theta = q_b^1 \sin q_b^2 \end{pmatrix}$, thus $P = [\frac{\partial \vec{q}_a}{\partial \vec{q}_b}] = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = \begin{pmatrix} [\vec{e}_{1,b}(p)]_{|\vec{E}} & [\vec{e}_{2,b}(p)]_{|\vec{E}} \end{pmatrix} = [\mathcal{P}]_{|\vec{e}_a}$: The transition matrix P is the change of basis matrix from the Cartesian basis $(\vec{e}_{i,a}) = (\vec{E}_i)$ to the polar basis $(\vec{e}_{i,b}) = \vec{e}_i(p))$ as expected, cf. (3.3). ∎

8.2 Change of basis formula for vectors

Let $\mathcal{Q}(p) = \mathcal{P}(p)^{-1}$. So, cf. (8.5),

$$\vec{e}_{j,a}(p) = \mathcal{Q}(p).\vec{e}_{j,b}(p) = \sum_{i=1}^m Q^i_j(p)\, \vec{e}_{i,b}(p), \quad [\mathcal{Q}(p)]_{|\vec{e}_b} = [Q^i_j(p)] = Q(p) = P(p)^{-1}, \tag{8.7}$$

the last equality since $\vec{e}_{j,a} = \sum_{i=1}^m Q^i_j \sum_{k=1}^m P^k_i \vec{e}_{k,a} = \sum_{k=1}^m (\sum_{i=1}^m P^k_i Q^i_j)\vec{e}_{k,a}$ for all j, thus $(\sum_{i=1}^m P^k_i Q^i_j) = \delta^i_j$ for all j,k, i.e. $P.Q = I$. And, as for (8.5),

$$Q(p) = [d\vec{q}_b(\vec{q}_a)]_{|\vec{A}} = [Q^i_j(p)] \overset{\text{written}}{=} [\frac{\partial \vec{q}_b}{\partial \vec{q}_a}(\vec{q}_a)]. \tag{8.8}$$

Proposition 8.3 *Let $\vec{v} \in TS$, $\vec{v} = \sum_i v_a^i \vec{e}_{i,a} = \sum_i v_b^i \vec{e}_{i,b}$. Then (contravariance formula)*

$$[\vec{v}]_{|\vec{e}_b} = P^{-1}.[\vec{v}]_{|\vec{e}_a}, \quad \text{i.e.} \quad v_b^i = \sum_{j=1}^m Q_j^i v_a^j, \quad \forall i = 1, ..., m. \tag{8.9}$$

Proof. (8.5) gives $\sum_{i=1}^m v_a^i \vec{e}_{i,a} = \vec{v} = \sum_{j=1}^m v_b^j \vec{e}_{j,b} = \sum_{j=1}^m v_b^j (\sum_{i=1}^m P_j^i \vec{e}_{i,a}) = \sum_{i=1}^m (\sum_{j=1}^m P_j^i v_b^j) \vec{e}_{i,a}$, thus $v_a^i = \sum_{j=1}^m P_j^i v_b^j$ for all i, i.e., $[\vec{v}]_a = P.[\vec{v}]_b$, thus (8.9). ∎

8.3 Change of dual basis formula

Proposition 8.4 *Let $p \in S$. For all $j = 1, ..., m$, we have*

$$e_b^i = \sum_{j=1}^m Q_j^i \, e_a^j. \tag{8.10}$$

Proof. $(\sum_{\ell=1}^m Q_\ell^i \, e_a^\ell).\vec{e}_{j,b} = (\sum_{\ell=1}^m Q_\ell^i \, e_a^\ell).(\sum_{k=1}^m P_j^k \vec{e}_{k,a}) = \sum_{\ell,k=1}^m Q_\ell^i P_j^k \delta_k^\ell = \sum_{k=1}^m Q_k^i P_j^k = (P.Q)_j^i = \delta_j^i$, for all $i, j = 1, ..., m$. Thus $\sum_{\ell=1}^m Q_\ell^i \, e_a^\ell = e_b^i$ (by definition of the dual basis), i.e. (8.10). ∎

8.4 Change of component formula for linear forms

Let $p \in S$ and $\ell_p \in (T_pS)^* = \mathcal{L}(T_pS; \mathbb{R})$ a linear form at p. Let $\ell = \sum_{i=1}^m \ell_{i,a} e_a^i = \sum_{i=1}^m \ell_{i,b} e_b^i$, so $[\ell_a] = (\, \ell_{1,a} \;\; \dots \;\; \ell_{m,a} \,)$ and $[\ell_b] = (\, \ell_{1,b} \;\; \dots \;\; \ell_{m,b} \,)$ (line matrices since ℓ_p is a linear form).

Proposition 8.5 *(Covariance formula)*

$$[\ell_b] = [\ell_a].P, \quad \text{i.e.} \quad \ell_{j,b} = \sum_{i=1}^m \ell_{i,a} P_j^i, \quad j = 1, ..., m. \tag{8.11}$$

Proof. (8.10) gives $\sum_{j=1}^m \ell_{j,a} e_a^j = \sum_{i=1}^m \ell_{i,b} e_b^i = \sum_{i=1}^m \ell_{i,b} (\sum_{j=1}^m Q_j^i \, e_a^j) = \sum_{j=1}^m (\sum_{i=1}^m \ell_{i,b} Q_j^i) \, e_a^j$, for all i, j, thus $\ell_{j,a} = \sum_{i=1}^m \ell_{i,b} Q_j^i$, so $[\ell_a] = [\ell_b].Q$. ∎

8.5 Change of basis for the γ_{ij}^k

Consider the Riemannian connection $\nabla_{\vec{v}} \vec{w} = \text{Proj}_{TS}(d\vec{w}.\vec{v})$ for all $\vec{v}, \vec{w} \in \Gamma(S)$ (derivation along $\vec{v}$ in S). The Christoffel symbols relative to the coordinate systems Φ_a and Φ_b are, at $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$, cf. (2.3),

$$\nabla_{\vec{e}_{i,a}} \vec{e}_{j,a} = \sum_{k=1}^m \gamma_{ij,a}^k \vec{e}_{k,a} \quad \text{and} \quad \nabla_{\vec{e}_{i,b}} \vec{e}_{j,b} = \sum_{k=1}^m \gamma_{ij,b}^k \vec{e}_{k,b}. \tag{8.12}$$

Proposition 8.6

$$\gamma_{ij,b}^k = \sum_{\alpha,\beta,\lambda=1}^m P_i^\alpha P_j^\beta Q_\lambda^k \gamma_{\alpha\beta,a}^\lambda + \sum_{\alpha=1}^m \frac{\partial^2 q_a^\alpha}{\partial q_b^i \partial q_b^j} \frac{\partial q_b^k}{\partial q_a^\alpha}, \tag{8.13}$$

(Thus a connection is not a tensor because of the last term.)

Proof. (8.4) and(8.5) give $\vec{e}_{j,b}(p) = \sum_{\alpha=1}^m \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) \; \vec{e}_{\alpha,a}(p)$, thus, with $p = \Phi_b(\vec{q}_b)$,

$$d\vec{e}_{j,b}(p).d\Phi_b(\vec{q}_b).\vec{A}_i = \sum_{\alpha=1}^m (d(\frac{\partial q_a^\alpha}{\partial q_b^j})(\vec{q}_b).\vec{A}_i) \vec{e}_{\alpha,a}(p) + \sum_{\alpha=1}^m \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) d\vec{e}_{\alpha,a}(p).d\Phi_b(\vec{q}_b).\vec{A}_i,$$

i.e. $d\vec{e}_{j,b}(p).\vec{e}_{i,b}(p) = \sum_{\alpha=1}^m \frac{\partial^2 q_a^\alpha}{\partial q_b^j q_b^i}(\vec{q}_b) \vec{e}_{\alpha,a}(p) + \sum_{\alpha=1}^m \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) d\vec{e}_{\alpha,a}(p).\vec{e}_{i,b}(p)$, thus $d\vec{e}_{j,b}(p).\vec{e}_{i,b}(p) = \sum_{\alpha=1}^m \frac{\partial^2 q_a^\alpha}{\partial q_b^j q_b^i}(\vec{q}_b) \vec{e}_{\alpha,a}(p) + \sum_{\alpha,\beta=1}^m \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) \frac{\partial q_a^\beta}{\partial q_b^i}(\vec{q}_b) d\vec{e}_{\alpha,a}(p).\vec{e}_{\beta,a}(p)$. Hence $\vec{e}_{\alpha,a}(p) = \sum_{k=1}^m \frac{\partial q_b^k}{\partial q_b^\alpha}(\vec{q}_b) \; \vec{e}_{k,b}(p)$

gives

$$\sum_{k=1}^{n}\gamma_{ij,b}^{k}\vec{e}_{k,b} = \sum_{\alpha=1}^{m}\frac{\partial^2 q_a^\alpha}{\partial q_b^j q_b^i}(\vec{q}_b)\vec{e}_{\alpha,a}(p) + \sum_{\alpha,\beta=1}^{m}\sum_{\lambda=1}^{n}\frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b)\frac{\partial q_a^\beta}{\partial q_b^i}(\vec{q}_b)\gamma_{\alpha\beta,a}^{\lambda}\vec{e}_{\lambda,a}(p),$$

thus (with projection on TS)

$$\sum_{k=1}^{m}\gamma_{ij,b}^{k}\vec{e}_{k,b} = \sum_{\alpha,k=1}^{m}\frac{\partial^2 q_a^\alpha}{\partial q_b^j q_b^i}(\vec{q}_b)\frac{\partial q_b^k}{\partial q_b^\alpha}(\vec{q}_b)\ \vec{e}_{k,b}(p) + \sum_{\alpha,\beta,\lambda,k=1}^{m}\frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b)\frac{\partial q_a^\beta}{\partial q_b^i}(\vec{q}_b)\gamma_{\alpha\beta,a}^{\lambda}\frac{\partial q_b^k}{\partial q_b^\lambda}(\vec{q}_b)\ \vec{e}_{k,b}(p).$$

▪

8.6 Change of basis formula for $\nabla\vec{w}$

Let $\vec{w}\in\Gamma(S)$, $\vec{w}=\sum_{i=1}^{m}w_a^i\,\vec{e}_{i,a}=\sum_{i,j=1}^{m}w_b^i\,\vec{e}_{i,b}$, and

$$\nabla\vec{w} = \sum_{i,j=1}^{m} w_{a|j}^i\,\vec{e}_{i,a}\otimes e_a^j = \sum_{i,j=1}^{m} w_{b|j}^i\,\vec{e}_{i,b}\otimes e_b^j. \tag{8.14}$$

Then (8.4) and (8.10) give $\sum_{i,j=1}^{m}w_{b|j}^i\,\vec{e}_{i,b}\otimes e_b^j=\sum_{k,\ell=1}^{m}w_{a|\ell}^k\,\vec{e}_{k,a}\otimes e_a^\ell=\sum_{k,\ell,i,j=1}^{m}w_{a|\ell}^k\,Q_i^kP_\ell^j\,\vec{e}_{i,b}\otimes e_b^j$, thus $w_{b|j}^i=\sum_{k,\ell=1}^{m}Q_i^kw_{a|\ell}^kP_\ell^j$, i.e.

$$[\nabla\vec{w}]_{|\vec{e}_b} = P^{-1}.[\nabla\vec{w}]_{|\vec{e}_a}.P, \tag{8.15}$$

as expected for endomorphisms.

Part IV

Lie autonomous derivative and Lie bracket

For the mechanical interpretation, see manuscript "Objectivity".

9 Lie autonomous derivative

9.1 Second order derivation, and issues

Consider Φ_+, cf. (2.1). Let $f\in\mathcal{F}(S_+)$ and $\vec{v},\vec{w}\in\Gamma(S_+)$. Then we have (derivation along $\vec{w}$ then derivation along $\vec{v}$)

$$d(df.\vec{w}).\vec{v} = d^2f(\vec{v},\vec{w}) + df.(d\vec{w}.\vec{v}). \tag{9.1}$$

And the first order term $f.(d\vec{w}.\vec{v})$ remains on the right side (unless $\vec{w}$ is uniform).

Issue: If $f\in\mathcal{F}(S)$ is extended by $f=0$ outside S, then $df.(d\vec{w}.\vec{v})$ is meaningless since $d\vec{w}.\vec{v}\notin\Gamma(S)$ in general, even if $\vec{v}_{|S},\vec{w}_{|S}\in\Gamma(S)$:

Example 9.1 Polar coordinates (3.2), $R>0$, the circle $S=C(\vec{0},R)=\{(R\cos\theta,R\sin\theta),\ \theta\in\mathbb{R}\}$, and its thickening $S_+=\{(r\cos\theta,r\sin\theta),\ r\in[-\frac{R}{2},\infty[,\ \theta\in\mathbb{R}\}$.

If $\vec{w}=\vec{v}=\vec{e}_2$, then $d\vec{e}_2(p).\vec{e}_2(p)=-r\vec{e}_1(p)$ for all $p\in S_+$, cf. (3.9), and the right hand side of (9.1) gives

$$d^2f(\vec{e}_2,\vec{e}_2)+df.d\vec{e}_2.\vec{e}_2 = d^2f(\vec{e}_2,\vec{e}_2) - r\,df.\vec{e}_1. \tag{9.2}$$

Issue: If $f\in\mathcal{F}(S)$ is extended by $f=0$ outside S then $df.\vec{e}_1=\pm\infty$, cf. (5.1).

While the left-hand side $d(df.\vec{e}_2).\vec{e}_2$ of (9.1) is meaningful, since $\vec{e}_2\in\Gamma(S)$ gives $df.\vec{e}_2=\frac{\partial f}{\partial\theta}\in\mathcal{F}(S)$, cf. (5.4), and then $d(df.\vec{e}_2).\vec{e}_2=\frac{\partial^2 f}{\partial\theta^2}\in\mathcal{F}(S)$ (meaning $\frac{\partial^2 f_U}{\partial\theta^2})(r,\theta)$ when $f_U=f\circ\Phi$, cf. (1.35)).

So, in a surface S, the right hand side of (9.1) cannot be used to compute $d(df.\vec{e}_2).\vec{e}_2$. In fact, $d^2f(\vec{e}_2,\vec{e}_2)$ is not defined on the circle (would be equal to $d(df.\vec{e}_2).\vec{e}_2-df.(d\vec{e}_2.\vec{e}_2)=\text{finite}-\pm\infty$)..., that is the bilinear form d^2f does not exists here (case f zero outside $S=C(\vec{0},R)$). ▪

9.2 Lie autonomous derivative ${\mathcal{L}^0}_{\vec v}\vec w$ on $\mathcal{F}(S)$

Definition 9.2 Let $\vec v,\vec w\in\Gamma(S)$. The autonomous Lie derivative ${\mathcal{L}^0}_{\vec v}\vec w$ is the derivative operator

$$ {\mathcal{L}^0}_{\vec v}\vec w : \begin{cases} \mathcal{F}(S) & \to \mathcal{F}(S) \\ f & \to {\mathcal{L}^0}_{\vec v}\vec w(f) = d(df.\vec w).\vec v - d(df.\vec v).\vec w \overset{named}{=} [\tilde\nabla_{\vec v},\tilde\nabla_{\vec w}](f). \end{cases} \tag{9.3}$$

That is, cf. (5.10),

$$ {\mathcal{L}^0}_{\vec v}\vec w(f) = \tilde\nabla_{\vec v}(\tilde\nabla_{\vec w}f) - \tilde\nabla_{\vec w}(\tilde\nabla_{\vec v}f) \overset{named}{=} [\tilde\nabla_{\vec v},\tilde\nabla_{\vec w}](f). \tag{9.4}$$

Proposition 9.3 *${\mathcal{L}^0}_{\vec v}\vec w$ is a derivation on $\mathcal{F}(S)$, i.e., if $f,g\in\mathcal{F}(S)$, then ${\mathcal{L}^0}_{\vec v}\vec w(fg) = {\mathcal{L}^0}_{\vec v}\vec w(f)g + f{\mathcal{L}^0}_{\vec v}\vec w(g)$. Hence, there exists a vector field $\vec z\in\Gamma(S)$ such that, for all $f\in\mathcal{S}$,*

$$ {\mathcal{L}^0}_{\vec v}\vec w(f) = \mathcal{L}_{\vec z}(f) \quad (= df.\vec z = \tilde\nabla_{\vec z}f), \quad \text{and} \quad \vec z \overset{named}{=} [\vec v,\vec w]. \tag{9.5}$$

If $(\vec e_i)$ is a holonomic basis (is the basis of a coordinate system), then, for all i,j,

$$ {\mathcal{L}^0}_{\vec e_i}\vec e_j = {\mathcal{L}^0}_{\vec e_j}\vec e_i, \quad \text{i.e.} \quad [\vec e_i,\vec e_j]=0. \tag{9.6}$$

Proof. $d(fg).\vec w = (df.\vec w)g + f(dg.\vec w)$.

Thus $d(d(fg).\vec w).\vec v = d(df.\vec w).\vec v\, g + (df.\vec v)(dg.\vec w) + (df.\vec w)(dg.\vec v) + f d(dg.\vec w).\vec v$.

And $d(d(fg).\vec v).\vec w = d(df.\vec v).\vec w\, g + (df.\vec w)(dg.\vec v) + (df.\vec v)(dg.\vec w) + f d(dg.\vec v).\vec w$.

Thus ${\mathcal{L}^0}_{\vec v}\vec w(fg) = \Big(d(df.\vec w).\vec v - d(df.\vec v).\vec w\Big)g + f\Big(d(dg.\vec w).\vec v - d(dg.\vec v).\vec w\Big) = g{\mathcal{L}^0}_{\vec v}\vec w(f) + f{\mathcal{L}^0}_{\vec v}\vec w(g)$.

And (4.7) gives $\vec z$.

Then $\mathcal{L}_{\vec e_j}\vec e_i(f) = d(df.\vec e_j).\vec e_i - d(df.\vec e_i).\vec e_j = \frac{\partial^2 f}{\partial q^i\partial q^j} - \frac{\partial^2 f}{\partial q^j\partial q^i} = 0$. ∎

Exercise 9.4 Consider the normalized polar basis $(\vec b_1(p),\vec b_2(p))$ given by $\vec b_1(p)=\vec e_1(p)$ and $\vec b_2(p) = \frac{\vec e_2(p)}{r}$, cf. (3.3). Compute $d\vec b_2.\vec b_1$ and $d\vec b_1.\vec b_2$ and $\mathcal{L}_{\vec b_1}\vec b_2$.

Answer. (2.28) gives

$$ d\vec b_2.\vec b_1 - d\vec b_1.\vec b_2 = -\frac{1}{r}\vec b_2 = -\frac{1}{r^2}\vec e_2, \quad \text{and} \quad \mathcal{L}_{\vec b_1}\vec b_2(f) = -\frac{1}{r^2}df.\vec e_2 = -\frac{1}{r^2}\frac{\partial f}{\partial\theta}, \tag{9.7}$$

first order derivative $\frac{\partial}{\partial\theta} : f \to \frac{\partial f}{\partial\theta} = df.\vec e_2$. ∎

Proposition 9.5 *For all $\vec v,\vec w\in\Gamma(S)$ and $\varphi\in\mathcal{F}(S)$,*

$$ \begin{cases} {\mathcal{L}^0}_{\varphi\vec v}\vec w = \varphi\,{\mathcal{L}^0}_{\vec v}\vec w - {\mathcal{L}^0}_{\vec w}\varphi{\mathcal{L}^0}_{\vec v} \quad (= \varphi\,{\mathcal{L}^0}_{\vec v}\vec w - (d\varphi.\vec w){\mathcal{L}^0}_{\vec v}), \\ {\mathcal{L}^0}_{\vec v}(\varphi\vec w) = \varphi\,{\mathcal{L}^0}_{\vec v}\vec w + {\mathcal{L}^0}_{\vec v}\varphi{\mathcal{L}^0}_{\vec w} \quad (= \varphi\,{\mathcal{L}^0}_{\vec v}\vec w + (d\varphi.\vec v){\mathcal{L}^0}_{\vec w}), \end{cases} \tag{9.8}$$

that is, for all $f\in\mathcal{F}(S)$,

$$ \begin{cases} {\mathcal{L}^0}_{\varphi\vec v}\vec w(f) = \varphi\,{\mathcal{L}^0}_{\vec v}\vec w(f) - {\mathcal{L}^0}_{\vec w}\varphi{\mathcal{L}^0}_{\vec v}f \quad (= \varphi\, df.\vec z - (d\varphi.\vec w)(df.\vec v)), \\ {\mathcal{L}^0}_{\vec v}(\varphi\vec w)(f) = \varphi\,{\mathcal{L}^0}_{\vec v}\vec w(f) + {\mathcal{L}^0}_{\vec v}\varphi{\mathcal{L}^0}_{\vec w}f \quad (= \varphi\, df.\vec z + (d\varphi.\vec w)(df.\vec v)), \end{cases} \tag{9.9}$$

Proof.

${\mathcal{L}^0}_{\varphi\vec v}\vec w(f) = \tilde\nabla_{\varphi\vec v}(\tilde\nabla_{\vec w}f) - \tilde\nabla_{\vec w}(\tilde\nabla_{\varphi\vec v}f) = d(df.\vec w).(\varphi\vec v) - d(df.(\varphi\vec v)).\vec w = \varphi\, d(df.\vec w).\vec v - d(\varphi\,(df.\vec v)).\vec w = \varphi\, d(df.\vec w).\vec v - (d\varphi.\vec w)\,(df.\vec v)) - \varphi\, d(df.\vec v).\vec w = \varphi\,{\mathcal{L}^0}_{\vec v}\vec w(f) - {\mathcal{L}^0}_{\vec w}\varphi{\mathcal{L}^0}_{\vec v}f$.

${\mathcal{L}^0}_{\vec v}(\varphi\vec w)(f) = \tilde\nabla_{\vec v}(\tilde\nabla_{\varphi\vec w}(f)) - \tilde\nabla_{\varphi\vec w}(\tilde\nabla_{\vec v}f) = d(df.(\varphi\vec w)).\vec v - d(df.\vec v).(\varphi\vec w) = d(\varphi\,(df.\vec w)).\vec v - \varphi\, d(df.\vec v).\vec w = (d\varphi.\vec v)(df.\vec w) + \varphi\, d(df.\vec w).\vec v - \varphi\, d(df.\vec v).\vec w = (d\varphi.\vec v)(df.\vec w) + \varphi\,{\mathcal{L}^0}_{\vec v}\vec w(f)$. ∎

10 Lie bracket

10.1 Lie bracket $[\vec{v},\vec{w}]$

$\mathcal{L}^0{}_{\vec{v}}\vec{w} : \mathcal{F}(S) \to \mathcal{F}(S)$ being a derivation, cf. prop. 9.3, the vector field $\vec{z}$ in (9.5) is denoted

$$\vec{z} = [\vec{v},\vec{w}] \in \Gamma(S) \tag{10.1}$$

and $[\vec{v},\vec{w}] \in \Gamma(S)$ is called the Lie bracket of $\vec{v},\vec{w} \in \Gamma(S)$. So, for all $f \in \mathcal{F}(S)$,

$$\mathcal{L}^0{}_{\vec{v}}\vec{w}(f) = df.[\vec{v},\vec{w}] \overset{\text{written}}{=} [\vec{v},\vec{w}](f), \tag{10.2}$$

action of the vector field $[\vec{v},\vec{w}]$ on f, where the last notation $[\vec{v},\vec{w}](f)$ uses the natural canonical isomorphism $\mathcal{J}$, cf. (1.30) (vectors $\leftrightarrow$ directional derivation). See (4.7).

E.g., in S_+, with ∇ the usual connection ($\nabla_{\vec{v}}\vec{w} = d\vec{w}.\vec{v}$), (9.1) gives

$$[\vec{v},\vec{w}] = d\vec{w}.\vec{v} - d\vec{v}.\vec{w} = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}, \tag{10.3}$$

E.g., (9.7) gives $[\vec{b}_1,\vec{b}_2] = -\frac{1}{r}\vec{b}_2$, and $[\vec{b}_1,\vec{b}_2](f) = df.[\vec{b}_1,\vec{b}_2] = -\frac{1}{r^2}\frac{\partial f}{\partial\theta}$.

E.g., in S, with ∇ the usual connection ($\nabla_{\vec{v}}\vec{w} = \text{Proj}_{TS}(d\vec{w}.\vec{v})$), (9.1) gives

$$[\vec{v},\vec{w}] = \text{Proj}_{TS}(d\vec{w}.\vec{v} - d\vec{v}.\vec{w}) = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}. \tag{10.4}$$

Corollary 10.1 *(9.8) reads*

$$\begin{cases} [\varphi\vec{v},\vec{w}](f) = \varphi\,[\vec{v},\vec{w}](f) - (d\varphi.\vec{w})(df.\vec{v}) \quad (= \varphi\,df.[\vec{v},\vec{w}] - (d\varphi.\vec{w})(df.\vec{v})), \\ [\vec{v},\varphi\vec{w}](f) = \varphi\,[\vec{v},\vec{w}](f) + (d\varphi.\vec{v})(df.\vec{w}) \quad (= \varphi\,df.[\vec{v},\vec{w}] + (d\varphi.\vec{w})(df.\vec{v})). \end{cases} \tag{10.5}$$

10.2 $[\vec{v},\vec{w}]$ is tangential

Theorem 10.2 *If $\vec{v},\vec{w} \in \Gamma(S_+)$ are such that $\vec{v}_{|S},\vec{w}_{|S} \in \Gamma(S)$ (tangent to S), then*

$$[\vec{v},\vec{w}]_{|S} \in \Gamma(S) \quad \textit{(tangent to } S\textit{)}. \tag{10.6}$$

More precisely,

$$[\vec{v},\vec{w}]_{|S} = \text{Proj}_{TS}(d\vec{w}.\vec{v}) - \text{Proj}_{TS}(d\vec{v}.\vec{w}) = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}, \tag{10.7}$$

with ∇ the usual connection in S, cf. (6.6).

And $[\vec{v},\vec{w}]$ is antisymmetric.

Quantification: $(\vec{e}_i)_{i=1,\dots,m}$ being the coordinate basis of Φ, if $\vec{v} = \sum_{i=1}^m v^i\vec{e}_i$ and $\vec{w} = \sum_{j=1}^m w^j\vec{e}_j$ in S, then

$$[\vec{v},\vec{w}]_{|S} = (d\vec{w}.\vec{v} - d\vec{v}.\vec{w})_{|S} = \sum_{i,j=1}^m (\frac{\partial w^i}{\partial q^j}\,v^j - \frac{\partial v^i}{\partial q^j}\,w^j)\vec{e}_i \;\in \Gamma(S). \tag{10.8}$$

Proof. With the extended basis $(\vec{e}_1,\dots,\vec{e}_n)$ of S_+, we have in S:

$$\begin{aligned} d\vec{w}.\vec{v} &= \sum_{i=1}^m (dw^i.\vec{v})\vec{e}_i + \sum_{i=1}^m w^i(d\vec{e}_i.\vec{v}) = \sum_{i,j=1}^m v^j(dw^i.\vec{e}_j)\vec{e}_i + \sum_{i,j=1}^m w^iv^j d\vec{e}_i.\vec{e}_j \\ &= \sum_{i,j=1}^m \frac{\partial w^i}{\partial q^j}v^j\vec{e}_i + \sum_{i,j=1}^m w^iv^j(\sum_{k=1}^n \gamma_{ij}^k\vec{e}_k). \end{aligned} \tag{10.9}$$

And $\gamma_{ij}^k = \gamma_{ji}^k$ since $(\vec{e}_i)$ is a coordinate system, thus the γ_{ij}^k terms vanish in $d\vec{w}.\vec{v} - d\vec{v}.\vec{w}$, hence (10.8), thus (10.7), (10.6), and the antisymmetry (trivial). ∎

10.3 Jacobi identity

Proposition 10.3 (Jacobi identity) *The Lie bracket satisfies: For all $\vec{u},\vec{v},\vec{w}\in\Gamma(S)$,*

$$[\vec{u},[\vec{v},\vec{w}]]+[\vec{v},[\vec{w},\vec{u}]]+[\vec{w},[\vec{u},\vec{v}]]=0. \tag{10.10}$$

Proof. $[\vec{v},\vec{w}]=\tilde{\nabla}_{\vec{v}}\circ\tilde{\nabla}_{\vec{w}}-\tilde{\nabla}_{\vec{w}}\circ\tilde{\nabla}_{\vec{v}}$ gives

$$\begin{aligned}[\vec{u},[\vec{v},\vec{w}]]&=\tilde{\nabla}_{\vec{u}}\circ(\tilde{\nabla}_{\vec{v}}\circ\tilde{\nabla}_{\vec{w}}-\tilde{\nabla}_{\vec{w}}\circ\tilde{\nabla}_{\vec{v}})-(\tilde{\nabla}_{\vec{v}}\circ\tilde{\nabla}_{\vec{w}}-\tilde{\nabla}_{\vec{w}}\circ\tilde{\nabla}_{\vec{v}})\circ\tilde{\nabla}_{\vec{u}}\\&=\tilde{\nabla}_{\vec{u}}\circ\tilde{\nabla}_{\vec{v}}\circ\tilde{\nabla}_{\vec{w}}-\tilde{\nabla}_{\vec{u}}\circ\tilde{\nabla}_{\vec{w}}\circ\tilde{\nabla}_{\vec{v}}-\tilde{\nabla}_{\vec{v}}\circ\tilde{\nabla}_{\vec{w}}\circ\tilde{\nabla}_{\vec{u}}+\tilde{\nabla}_{\vec{w}}\circ\tilde{\nabla}_{\vec{v}}\circ\tilde{\nabla}_{\vec{u}}\end{aligned} \tag{10.11}$$

Idem for the two other terms (circular permutation). Thus (10.10). ∎

Definition 10.4 A Lie algebra is a quadruplet $(V,+,.,a)$ where $(V,+,.)$ is a vector space and $a: V\times V\to V$ is an antisymmetric bilinear map which satisfies Jacobi identity, that is,

$$a(a(\vec{u},\vec{v}),\vec{w})+a(a(\vec{v},\vec{w}),\vec{u})+a(a(\vec{w},\vec{u}),\vec{v})=\vec{0}. \tag{10.12}$$

Example 10.5 $\{n*n$ matrices$\}$ with $[A,B]=AB-BA$ is a Lie algebra. $(\mathbb{R}^n,+,.,[.,.])$ is a Lie algebra. ∎

10.4 Derivation formula $\mathcal{L}^0_{\vec{u}}[\vec{v},\vec{w}]=[\mathcal{L}^0_{\vec{u}}\vec{v},\vec{w}]+[\vec{v},\mathcal{L}^0_{\vec{u}}\vec{w}]$

With $\mathcal{L}^0_{\vec{u}}\vec{v}=[\vec{u},\vec{v}]$ and with $[\vec{u},\vec{v}]=-[\vec{v},\vec{u}]$, the Jacobi identity (10.10) gives the derivation rule

$$\mathcal{L}^0_{\vec{u}}[\vec{v},\vec{w}](f)=[\mathcal{L}^0_{\vec{u}}\vec{v},\vec{w}](f)+[\vec{v},\mathcal{L}^0_{\vec{u}}\vec{w}](f). \tag{10.13}$$

10.5 Geometric interpretation of $[\vec{v},\vec{w}]\neq\vec{0}$

Let $\vec{v},\vec{w}\in\Gamma(S)$, let $p=\Phi(\vec{q})\in S$, and let α_p and β_p be associated the integral curves at p, that is,

$$\frac{d\alpha_p}{ds}(s)=\vec{v}(\alpha_p(s)),\quad \alpha_p(0)=p,\quad\text{and}\quad \frac{d\beta_p}{dt}(t)=\vec{w}(\beta_p(t)),\quad \beta_p(0)=p, \tag{10.14}$$

for s close to 0. And consider the associated family of curves $\alpha(s,p)=\alpha_p(s)$ and $\beta(t,p):=\beta_p(t)$, so

$$\frac{\partial\alpha}{\partial s}(s,p):=\vec{v}(\alpha(s,p)),\quad\text{and}\quad \frac{\partial\beta}{\partial t}(t,p):=\vec{w}(\beta(t,p)). \tag{10.15}$$

Let $p=P_0=\alpha(0,P_0)=\beta(0,P_0)\in S$ and, see figure 10.1,

$$\begin{cases}P_1=\alpha_s(P_0),\\P_2=\beta_t(P_0),\\P_{12}=\beta_t(P_1)=(\beta_t\circ\alpha_s)(P_0)\overset{\text{written}}{=}\varphi(s,t),\\P_{21}=\alpha_s(P_2)=(\alpha_s\circ\beta_t)(P_0)\overset{\text{written}}{=}\psi(t,s).\end{cases} \tag{10.16}$$

Proposition 10.6 *With $\varphi(s,t)=\beta_t(\alpha_s(P_0))$ and $\psi(t,s)=\alpha_s(\beta_t(P_0))$, cf. (10.16), the second order Taylor expansion are we have*

$$\begin{cases}\varphi(s,t)=P_0+(s\,\vec{v}+t\,\vec{w})(P_0)+(s^2d\vec{v}.\vec{v}+st\,d\vec{v}.\vec{w}+t^2\,d\vec{w}.\vec{w})(P_0)+o(s^2+t^2),\\\psi(t,s)=P_0+(t\,\vec{w}+s\,\vec{v})(P_0)+(t^2d\vec{w}.\vec{w}+st\,d\vec{w}.\vec{v}+t^2\,d\vec{v}.\vec{v})(P_0)+o(s^2+t^2),\end{cases} \tag{10.17}$$

thus,

$$(\varphi(s,t)-\psi(t,s)=)\quad P_{12}-P_{21}=st[\vec{v},\vec{w}](p),+o(s^2+t^2). \tag{10.18}$$

Thus $[\vec{v},\vec{w}](P_0)$ is a measure of $\overrightarrow{P_{12}P_{21}}$ (the "aperture", see figure 10.1).

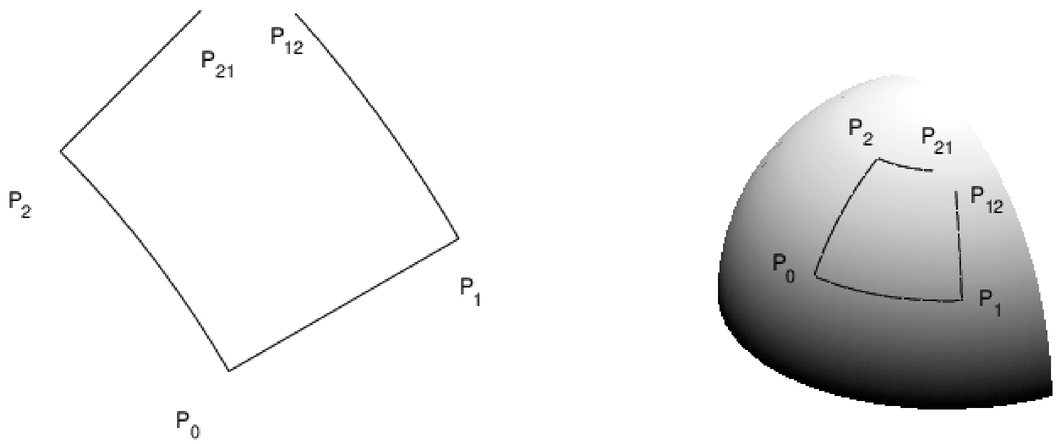

Figure 10.1: Along curves which are not curves of a coordinate system, the path joining P_{12} to P_{21} is not closed, and $\overrightarrow{P_{12}P_{21}} = st[\vec{v},\vec{w}](p) + o(s^2{+}t^2)$, thus $[\vec{v},\vec{w}]$ gives a measure of measure of the "aperture".

Proof. $\varphi(s,t) = \beta(t,\alpha(s,P_0))$, $\frac{\partial\alpha}{\partial s}(s,P_0) = \vec{v}(\alpha(s,P_0))$ and $\frac{\partial\beta}{\partial t}(t,\alpha(s,P_0)) = \vec{w}(\beta(t,\alpha(s,P_0)))$ give

$$\begin{cases} \dfrac{\partial\varphi}{\partial s}(s,t) = d\beta(t,\alpha(s,P_0)).\vec{v}(\alpha(s,P_0)), \\ \dfrac{\partial\varphi}{\partial t}(s,t) = \vec{w}(\beta(t,\alpha(s,P_0))). \end{cases}$$

Thus,

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial s^2}(s,t) = \Big(d^2\beta(t,\alpha(s,P_0)).\vec{v}(\alpha(s,P_0))\Big).\vec{v}(\alpha(s,P_0)) + d\beta(t,\alpha(s,P_0)).\Big(d\vec{v}(\alpha(s,P_0)).\vec{v}(\alpha(s,P_0))\Big), \\ \dfrac{\partial^2\varphi}{\partial t\partial s}(s,t) = d(\dfrac{\partial\beta}{\partial t})(t,\alpha(s,P_0)).\vec{v}(\alpha(s,P_0)) = d\vec{w}(\beta(t,\alpha(s,P_0))).\vec{v}(\alpha(s,P_0)), \\ \dfrac{\partial^2\varphi}{\partial t^2}(s,t) = d\vec{w}(\beta(t,\alpha(s,P_0)).\vec{w}(\beta(t,\alpha(s,P_0)). \end{cases}$$

In particular,

$$\begin{cases} \varphi(0,0) = P_0, \qquad \dfrac{\partial\varphi}{\partial s}(0,0) = \vec{v}(P_0), \qquad \dfrac{\partial\varphi}{\partial t}(0,0) = \vec{w}(P_0), \\ \dfrac{\partial^2\varphi}{\partial s^2}(0,0) = (d\vec{v}.\vec{v})(P_0), \qquad \dfrac{\partial^2\varphi}{\partial t\partial s}(0,0) = (d\vec{w}.\vec{v})(P_0) = \dfrac{\partial^2\varphi}{\partial s\partial t}(0,0), \qquad \dfrac{\partial^2\varphi}{\partial t^2}(0,0) = (d\vec{w}.\vec{w})(P_0). \end{cases}$$

Idem for ψ. Thus the second order Taylor expansion are:

$$\begin{cases} \varphi(s,t) = P_0 + s\,\vec{v}(P_0) + t\,\vec{w}(P_0) + \dfrac{s^2}{2}\,(d\vec{v}.\vec{v})(P_0) + st\,(d\vec{w}.\vec{v})(P_0) + \dfrac{t^2}{2}\,(d\vec{w}.\vec{w})(P_0) + o(s^2{+}t^2), \\ \psi(t,s) = P_0 + t\,\vec{w}(P_0) + s\,\vec{v}(P_0) + \dfrac{t^2}{2}\,(d\vec{w}.\vec{w})(P_0) + st\,(d\vec{v}.\vec{w})(P_0) + \dfrac{t^2}{2}\,(d\vec{v}.\vec{v})(P_0) + o(s^2{+}t^2). \end{cases}$$

Thus $\varphi(s,t) - \psi(t,s) = st((d\vec{w}.\vec{v})(P_0) - (d\vec{v}.\vec{w})(P_0)) + o(s^2{+}t^2) = st[\vec{v},\vec{w}](P_0) + o(s^2{+}t^2)$. ▪

Part V

Connections

11 General connection ∇ on $\Gamma(S)$

11.1 Definition

If $\vec{v} \in \Gamma(S)$ then we have defined $\tilde{\nabla} : \left\{ \begin{array}{ll} \Gamma(S) \times \mathcal{F}(S) & \to \mathcal{F}(S) \\ (\vec{v}, f) & \to \tilde{\nabla}(\vec{v}, f) = \tilde{\nabla}_{\vec{v}} f := df.\vec{v} = \tilde{\mathcal{L}}^0_{\vec{v}}(f) \end{array} \right\}$ the covariant derivation of scalar functions f along $\vec{v}$, cf. (5.4). For covariant derivation of vector fields:

Definition 11.1 An (affine) connection ∇ on $\Gamma(S)$ is a $\mathbb{R}$-bilinear map

$$\nabla : \left\{ \begin{array}{ll} \Gamma(S) \times \Gamma(S) & \to \Gamma(S), \\ (\vec{v}, \vec{w}) & \mapsto \nabla(\vec{v}, \vec{w}) \overset{\text{written}}{=} \nabla_{\vec{v}} \vec{w} \overset{\text{written}}{=} \nabla \vec{w}.\vec{v}, \end{array} \right. \tag{11.1}$$

such that

1. For all $\vec{w} \in \Gamma(S)$, the map $\nabla(.,\vec{w}) = \nabla \vec{w}.(.) = \nabla_{(.)}\vec{w} : \Gamma(S) \to \Gamma(S)$ is $\mathcal{F}(S)$-linear, i.e., for all $f \in \mathcal{F}(S)$ and all $\vec{u}, \vec{v} \in \Gamma(S)$ (algebraic formula),
$$\nabla_{f\vec{u}+\vec{v}} \vec{w} = f \, \nabla_{\vec{u}} \vec{w} + \nabla_{\vec{v}} \vec{w}, \tag{11.2}$$
i.e. $\nabla \vec{w}.(f\vec{u}+\vec{v}) = f \, \nabla \vec{w}.\vec{u} + \nabla \vec{w}.\vec{v}$, i.e. $\nabla(f\vec{u}+\vec{v}, \vec{w}) = f \, \nabla(\vec{u}, \vec{w}) + \nabla(\vec{v}, \vec{w})$ (linearity in the first variable), and
2. For all $\vec{v} \in \Gamma(S)$, the map $\nabla(\vec{v}, .) = \nabla_{\vec{v}}(.) : \Gamma(S) \to \Gamma(S)$ satisfies, for all $f \in \mathcal{F}(S)$ and all $\vec{w} \in \Gamma(S)$ (derivation formula),
$$\nabla_{\vec{v}}(f\vec{w}) = (\tilde{\nabla}_{\vec{v}} f)\vec{w} + f \, \nabla_{\vec{v}} \vec{w}, \tag{11.3}$$
i.e. $\nabla(f\vec{w}).\vec{v} = (df.\vec{v})\vec{w} + f \, \nabla \vec{w}.\vec{v}$, i.e. $\nabla(\vec{v}, f\vec{w}) = \tilde{\nabla}(\vec{v}, f)\vec{w} + f \, \nabla(\vec{v}, \vec{w})$, cf. (5.10)).

And $\nabla_{\vec{v}} \vec{w}$ is called the covariant derivative of $\vec{w}$ along $\vec{v}$.

Example 11.2 The Riemannian connection (6.6) satisfy (11.2) and (11.3), cf. (6.9)-(6.10): It is a connection. However the Riemannian connection (6.6) can only be defined if S is a surface in $\mathbb{R}^n$; It cannot be "naturally" defined if S lives on its own (e.g. S is the Earth surface and a "vertical" is not accessible, or e.g. S = our curved space-time set of general relativity). ∴

Remark 11.3 The notation $\nabla_{\vec{v}}(\vec{w}) = \nabla_{\vec{v}} \vec{w} = \nabla \vec{w}.\vec{v}$ seems to be universal, but the notation $\nabla(\vec{v}, \vec{w})$ depends on authors; Here we use Abraham and Marsden [1] notation, that is, $\nabla(\vec{v}, \vec{w}) = \nabla_{\vec{v}} \vec{w}$ (motivation: See § 11.8). (Misner–Thorne–Wheeler [15] use the notation $\nabla(\vec{w}, \vec{v}) = \nabla_{\vec{v}} \vec{w}$...) ∴

11.2 Covariant derivative $\nabla_{\vec{v}}$ on $\Gamma(S)$ and $\frac{D\vec{w}}{ds}$

(11.1) and (11.3) enable to define the covariant derivative along $\vec{v}$:

$$\nabla_{\vec{v}} : \left\{ \begin{array}{ll} \Gamma(S) & \to \Gamma(S), \\ \vec{w} & \mapsto \nabla_{\vec{v}}(\vec{w}) := \nabla(\vec{v}, \vec{w}) \overset{\text{written}}{=} \nabla_{\vec{v}} \vec{w} \overset{\text{written}}{=} \dfrac{D\vec{w}}{ds} \quad (\overset{\text{written}}{=} \nabla \vec{w}.\vec{v}). \end{array} \right. \tag{11.4}$$

In $\mathbb{R}^n$, $\nabla_{\vec{v}} \vec{w} := d\vec{w}.\vec{v}$ is the usual covariant derivative.
In $S \subset \mathbb{R}^n$, $\nabla_{\vec{v}} \vec{w} := \text{Proj}_{TS}(d\vec{w}.\vec{v})$ is the usual (Riemannian) covariant derivative.

11.3 Differential $\nabla \vec{w}$ on $\Gamma(S)$

(11.1) and (11.2) enable to define the ∇-differential of $\vec{w}$:

$$\nabla \vec{w} : \left\{ \begin{array}{ll} \Gamma(S) & \to \Gamma(S) \\ \vec{v} & \to \nabla \vec{w}.\vec{v} := \nabla_{\vec{v}} \vec{w} \quad (= \nabla(\vec{v}, \vec{w})). \end{array} \right. \tag{11.5}$$

Proposition 11.4 *Let $\vec{w} \in \Gamma(S)$. Then $\nabla \vec{w} : \Gamma(S) \to \Gamma(S)$ is a $\binom{1}{1}$ tensor in S.*

Proof. Thanks to the $\mathcal{F}(S)$-linearity (11.2), i.e., $\nabla \vec{w}.(f\vec{u}) = f \, \nabla \vec{w}(\vec{u})$. ∴

11.4 Naive connection ∇^0

Let $\vec{v} \in \Gamma(S)$.

Definition 11.5 Let $(\vec{a}_i)_{i=1,\dots m}$ be a basis in TS (not necessary holonomous), let $\vec{w} \in \Gamma(S)$. The naive connection ∇^0 relative to the basis $(\vec{a}_i)$ is defined by

$$\vec{w} = \sum_{j=1}^m w^j \vec{a}_j \quad \Longrightarrow \quad \nabla^0_{\vec{v}} \vec{w} := \sum_{j=1}^m (\tilde{\nabla}_{\vec{v}} w^j)\, \vec{a}_j \quad (= \sum_{j=1}^m (dw^j.\vec{v})\vec{a}_j = \nabla^0 \vec{w}.\vec{v} = \nabla^0(\vec{v}, \vec{w})). \tag{11.6}$$

In particular, for all $\vec{v} \in \Gamma(S)$ and all $i, j \in [1, m]_{\mathbb{N}}$,

$$\nabla^0_{\vec{a}_i} \vec{a}_j = \vec{0} = \nabla^0(\vec{a}_i, \vec{a}_j) = \nabla^0 \vec{a}_j.\vec{a}_i. \tag{11.7}$$

Proposition 11.6 *The naive connection is a connection.*

Proof. 1. $\nabla^0_{f\vec{u}+\vec{v}} \vec{w} = \sum_{i=1}^m (dw^i.(f\vec{u}+\vec{v}))\vec{a}_i = \sum_{i=1}^m f(dw^i.\vec{u})\vec{a}_i + \sum_{i=1}^m (dw^i.\vec{v})\vec{a}_i = f\nabla^0_{\vec{u}}\vec{w} + \nabla^0_{\vec{v}}\vec{w}$, i.e (11.2).
2. $\nabla^0_{\vec{v}}(f\vec{w}) = \sum_i (d(fw^i).\vec{v})\vec{a}_i = \sum_i (df.\vec{v})w^i\vec{a}_i + \sum_i f\,(dw^i.\vec{v})\vec{a}_i = (df.\vec{v})\vec{w} + f\,\nabla^0_{\vec{v}}\vec{w}$, i.e. (11.3). ▪

Proposition 11.7 *A naive connection is not a tensor.*

Proof. $\nabla^0_{\vec{v}}(f\vec{w}) = (df.\vec{v})\vec{w} + f\,\nabla^0_{\vec{v}}\vec{w} \neq f\nabla^0_{\vec{v}}(\vec{w})$ in general. ▪

11.5 Torsion of a connection

The Lie bracket $[\vec{v}, \vec{w}] \in \Gamma(S)$ of $\vec{v}, \vec{w} \in \Gamma(S)$ has been defined on $\mathcal{F}(S)$ by, cf. (10.2),

$$\begin{aligned} {\mathcal{L}^0}_{\vec{v}}\vec{w}(f) &= df.[\vec{v}, \vec{w}] = [\vec{v}, \vec{w}](f) = [\tilde{\nabla}_{\vec{v}}, \tilde{\nabla}_{\vec{w}}](f) \\ &= d(df.\vec{w}).\vec{v} - d(df.\vec{v}).\vec{w} = \tilde{\nabla}_{\vec{v}}(\tilde{\nabla}_{\vec{w}} f) - \tilde{\nabla}_{\vec{w}}(\tilde{\nabla}_{\vec{v}} f). \end{aligned} \tag{11.8}$$

(Remember that $\vec{v}$ is identified with $\tilde{\nabla}_{\vec{v}}$, cf. (1.30).)

Definition 11.8 Let ∇ be a connection, cf. (11.1)-(11.2)-(11.3), and let $\vec{v}, \vec{w} \in \Gamma(S)$. The torsion $T(\vec{v}, \vec{w}) \in \Gamma(S)$ due to $\vec{v}$ and $\vec{w}$ is the vector field (identified with the directional derivation operator) defined by, for all $f \in \mathcal{F}(S)$,

$$\begin{aligned} T(\vec{v}, \vec{w})(f) &= (\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v} - [\vec{v}, \vec{w}])(f) \\ &= df.(\nabla_{\vec{v}}\vec{w}) - df.(\nabla_{\vec{w}}\vec{v}) - d(df.\vec{w}).\vec{v} + d(df.\vec{v}).\vec{w} \end{aligned} \tag{11.9}$$

And the torsion T of ∇ is the map

$$T : \begin{cases} \Gamma(S) \times \Gamma(S) & \to \Gamma(S) \\ \qquad (\vec{v}, \vec{w}) & \to T(\vec{v}, \vec{w}) = (\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}) - [\vec{v}, \vec{w}]. \end{cases} \tag{11.10}$$

Example 11.9 With $\nabla = \nabla^0$ the naive connection relative to the the normalized polar basis $(\vec{b}_1, \vec{b}_2)$, we have $[\vec{b}_1, \vec{b}_2] = -\frac{1}{r}\,\vec{b}_2 \neq \vec{0}$, cf. (2.28), and we have $\nabla^0_{\vec{b}_1}\vec{b}_2 - \nabla^0_{\vec{b}_2}\vec{b}_1 = \vec{0}$, cf. (11.7). Thus $T(\vec{b}_1, \vec{b}_2) = \frac{1}{r}\,\vec{b}_2 \neq \vec{0}$: The torsion of the naive connection does not vanish. ▪

11.6 Torsion-free connection

Definition 11.10 A connection ∇ is torsion free iff $T = 0$, that is, iff ∇ and $\tilde{\nabla}$ satisfy, for all $\vec{v}, \vec{w} \in \Gamma(S)$,

$$\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v} = [\vec{v}, \vec{w}] \quad (= \tilde{\nabla}_{\vec{v}} \circ \tilde{\nabla}_{\vec{w}} - \tilde{\nabla}_{\vec{w}} \circ \tilde{\nabla}_{\vec{v}}), \tag{11.11}$$

that is, for all $\vec{v}, \vec{w} \in \Gamma(S)$ and all $f \in \mathcal{F}(S)$,

$$(\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v})(f) = [\vec{v}, \vec{w}](f). \tag{11.12}$$

(Also called a symmetric connection with reference to the Christoffel symbols: With the basis of a coordinate system, $\gamma^k_{ij} = \gamma^k_{ji}$ for all i, j, k, see (11.20).)

Example 11.11 (Fundamental.) The Riemannian connection in a surface S in $\mathbb{R}^n$ is torsion-free: Indeed, if $\vec{v},\vec{w}\in\Gamma(S)$ then $[\vec{v},\vec{w}]=d\vec{w}.\vec{v}-d\vec{v}.\vec{w}\in\Gamma(S)$, cf. (10.4)-(10.6), and, with (6.6) and (10.7), $\nabla_{\vec{v}}\vec{w}-\nabla_{\vec{w}}\vec{v}=\mathrm{Proj}_{TS}(d\vec{w}.\vec{v})-\mathrm{Proj}_{TS}(d\vec{v}.\vec{w})=\mathrm{Proj}_{TS}(d\vec{w}.\vec{v}-d\vec{v}.\vec{w})=[\vec{v},\vec{w}]$, thus, $T=[\vec{v},\vec{w}]-[\vec{v},\vec{w}]=0$. ▪

Exercise 11.12 We will only use torsion free connection in the following. However, prove that, in all cases, the torsion T is an antisymmetric $\binom{1}{2}$ tensor, i.e. $T(\vec{v},\vec{w})=-T(\vec{w},\vec{v})$ for all $\vec{v},\vec{w}\in\Gamma(S)$.

Answer. Antisymmetry is trivial since $[.,.]$ is. And T is a tensor iff the associated map $\tilde{T}:\Omega^1(S)\times\Gamma(S)\times\Gamma(S)\to\mathbb{R}$ defined by $\tilde{T}(\alpha,\vec{v},\vec{w})=\alpha.T(\vec{v},\vec{w})$ is $\mathcal{F}(S)$-multilinear. For the first component α it is trivial. And

$$
\begin{aligned}
T(\vec{v},f\vec{w}) &= (\nabla_{\vec{v}}(f\vec{w})-\nabla_{f\vec{w}}\vec{v})-([\vec{v},f\vec{w}])\\
&= \Big(f\,\nabla_{\vec{v}}\vec{w}+(df.\vec{v})\tilde{\nabla}\vec{w}-f\,\nabla_{\vec{w}}\vec{v}\Big)-\Big(f\,[\vec{v},\vec{w}]+(df.\vec{v})\tilde{\nabla}_{\vec{w}}\Big)\\
&= f(\nabla_{\vec{v}}\vec{w}-\nabla_{\vec{w}}\vec{v})-f[\vec{v},\vec{w}]=f\,T(\vec{v},\vec{w}).
\end{aligned}
$$

And T is antisymmetric, thus $T(f\vec{v},\vec{w})=-T(\vec{w},f\vec{v})=-fT(\vec{w},\vec{v})=fT(\vec{v},\vec{w})$. Thus T is a tensor. ▪

11.7 Tensor γ ...

Proposition 11.13 *If ∇ and N are two connections then*

$$\gamma=\nabla-N \text{ is a tensor.} \tag{11.13}$$

Proof. Consider the associated map $\tilde{\gamma}:\Omega^1(S)\times\Gamma(S)\times\Gamma(S)\to\mathbb{R}$ given by $\tilde{\gamma}(\alpha,\vec{v},\vec{w}):=\alpha.\gamma(\vec{v},\vec{w})$. The affirmation "$\gamma=\nabla-N$ is a tensor" means that "$\tilde{\gamma}$ is a $\binom{1}{2}$ tensor", that is, $\tilde{\gamma}$ is $\mathcal{F}(S)$-multilinear. For the first component α it is trivial. And $\gamma(\vec{v},f\vec{w})=\nabla_{\vec{v}}(f\vec{w})-N_{\vec{v}}(f\vec{w})=f\nabla_{\vec{v}}\vec{w}+(df.\vec{v}).\vec{w}-(fN_{\vec{v}}\vec{w}+(df.\vec{v})\vec{w})=f(\nabla_{\vec{v}}\vec{w}-N_{\vec{v}}\vec{w})=f\gamma(\vec{v},\vec{w})$: $\tilde{\gamma}$ is $\mathcal{F}(S)$-multilinear in its third component $\vec{w}$. And $\gamma(f\vec{v},\vec{w})=-\gamma(\vec{w},f\vec{v})=-f\gamma(\vec{w},\vec{v})=f\gamma(\vec{v},\vec{w})$: $\tilde{\gamma}$ is $\mathcal{F}(S)$-multilinear in its second component $\vec{v}$. ▪

Corollary 11.14 *If ∇ is a connection and $\nabla\neq0$, then ∇ is not a tensor.*

Proof. ∇ and ∇^0 (naive) being connections, $\gamma=\nabla-\nabla^0$ is a tensor, cf. (11.13). Thus, if ∇ was a tensor, then $\nabla^0=\nabla-\gamma$ would be a tensor (difference of two tensors). But ∇^0 is not a tensor. ▪

Exercise 11.15 Let ∇ be a connection, $\nabla\neq0$. Prove that $A=2\nabla$ is not a connection.

Answer. If A was, then $A-\nabla=\nabla$ is a tensor, cf. (11.13): But ∇ is not. cf.corollary 11.14. ▪

11.8 ... and its components γ^i_{jk} (Christoffel symbols)

Let ∇ be a connection, let $(\vec{a}_i)$ be a basis, let ∇^0 be the naive connection relative to $(\vec{a}_i)$, and consider the tensor $\gamma=\nabla-\nabla^0$, cf. prop. 11.13, and the associated tensor $\tilde{\gamma}\in T^1_2(S)$ given by $\tilde{\gamma}(\alpha,\vec{v},\vec{w}):=\alpha.\gamma(\vec{v},\vec{w})$ (cf. proof of prop. 11.13). Let C^i_{jk} be the components of $\tilde{\gamma}$ relative to the basis $(\vec{a}_i)$, that is,

$$\tilde{\gamma}=\sum_{i,j,k=1}^m C^i_{jk}\vec{a}_i\otimes a^j\otimes a^k,\quad \text{i.e.}\quad \tilde{\gamma}(a^i,\vec{a}_j,\vec{a}_k)=C^i_{jk}\quad (=a^i.\gamma(\vec{a}_j,\vec{a}_k)), \tag{11.14}$$

i.e., for all j,k,

$$\gamma_{\vec{a}_j}\vec{a}_k=\gamma(\vec{a}_j,\vec{a}_k)=\sum_{i=1}^m C^i_{jk}\vec{a}_i, \tag{11.15}$$

or, for all i,j, $\gamma_{\vec{a}_i}\vec{a}_j=\sum_{k=1}^m C^k_{ij}\vec{a}_k$. Then (11.7) and $\gamma=\nabla-\nabla^0$ give, for all i,j,

$$\gamma_{\vec{a}_i}\vec{a}_j=\nabla_{\vec{a}_i}\vec{a}_j-0=\sum_{k=1}^m C^k_{ij}\vec{a}_k\quad (=\gamma(\vec{a}_i,\vec{a}_j)). \tag{11.16}$$

Definition 11.16 If $(\vec{a}_i) = (\vec{e}_i)$ is an holonomic basis (= the basis of a coordinate system), then $C^k_{ij} =^{\text{written}} \gamma^k_{ij}$ are called the Christoffel symbols of the connection ∇ relative to $(\vec{e}_i)$. And (11.16) reads,

$$\nabla_{\vec{e}_i}\vec{e}_j = \sum_{k=1}^m \gamma^k_{ij}\vec{e}_k, \tag{11.17}$$

Then $\vec{w} = \sum_{j=1}^m w^j\vec{e}_j$ and (11.3) give $\nabla_{\vec{e}_i}\vec{w} = \sum_{j=1}^m (\tilde{\nabla}_{\vec{e}_i} w^j)\vec{e}_j + \sum_{i=1}^m w^j(\nabla_{\vec{e}_i}\vec{e}_j) = \sum_{j=1}^m \frac{\partial w^j}{\partial q^i}\vec{e}_j + \sum_{j,k=1}^m w^j\gamma^k_{ij}\vec{e}_k$, so

$$\nabla_{\vec{e}_i}\vec{w} = \sum_{i,j=1}^m w^j_{|i}\vec{e}_j \quad \text{with} \quad w^j_{|i} = \frac{\partial w^j}{\partial q^i} + \sum_{k=1}^m \gamma^j_{ik} w^k. \tag{11.18}$$

Then $\vec{v} = \sum_{i=1}^m v^i\vec{e}_i$ and (11.2) give $\nabla_{\vec{v}}\vec{w} = \nabla_{\sum_{i=1}^m v^i\vec{e}_i}\vec{w} = \sum_{i=1}^m v^i\nabla_{\vec{e}_i}\vec{w}$, so

$$\nabla_{\vec{v}}\vec{w} = \sum_{i,j=1}^m w^j_{|i} v^i\vec{e}_j = \sum_{i,j=1}^m (\frac{\partial w^j}{\partial q^i} v^i + \sum_{k=1}^m \gamma^j_{ik} w^k v^i)\vec{e}_j. \tag{11.19}$$

Proposition 11.17 *If ∇ is torsion-free, and if $(\vec{e}_i)$ is holonomic (is the basis of a coordinate system), then*

$$\forall i,j,k = 1,...,m, \quad \gamma^k_{ij} = \gamma^k_{ji}, \quad \text{and} \quad \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v} = \sum_{i,j=1}^m (\frac{\partial w^j}{\partial q^i} v^i - \frac{\partial v^j}{\partial q^i} w^i)\vec{e}_j \tag{11.20}$$

(the Christoffel symbols disappear in $[\vec{v},\vec{w}] := \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}$).

Proof. $\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v} = [\vec{v},\vec{w}]$, cf. (11.12), gives $(\nabla_{\vec{e}_i}\vec{e}_i - \nabla_{\vec{e}_i}\vec{e}_i)(f) = (\tilde{\nabla}_{\vec{e}_i} \circ \tilde{\nabla}_{\vec{e}_i} - \tilde{\nabla}_{\vec{e}_i} \circ \tilde{\nabla}_{\vec{e}_i})(f) = \frac{\partial \frac{\partial f}{\partial q^j}}{\partial q^i} - \frac{\partial \frac{\partial f}{\partial q^i}}{\partial q^j} = 0$ (Schwarz for $f_U = f \circ \Phi^{-1}$, cf. (1.35)). Hence $\sum_{k=1}^m \gamma^k_{ij}\vec{e}_k - \sum_{k=1}^m \gamma^k_{ji}\vec{e}_k = 0$. ∎

12 Connection $\tilde{\tilde{\nabla}}$ on $\Omega^1(S)$

12.1 Directional derivative $\tilde{\tilde{\nabla}}_{\vec{v}}\alpha = \frac{D\alpha}{ds}$

In $\mathbb{R}^n$, if $\alpha \in \Omega^1(\mathbb{R}^n)$ and $\vec{w} \in \Gamma(\mathbb{R}^n)$ then $\alpha.\vec{w} \in \mathcal{F}(\mathbb{R}^n)$, and

$$d(\alpha.\vec{w}).\vec{v} = (d\alpha.\vec{v}).\vec{w} + \alpha.(d\vec{w}.\vec{v}) \ \in \mathcal{F}(S). \tag{12.1}$$

In S, with $\tilde{\nabla}_{\vec{v}} f = df.\vec{v}$ and a connection ∇ on $\Gamma(S)$, we define $\tilde{\tilde{\nabla}}_{\vec{v}}$ on $\Omega^1(S)$ such that

$$\tilde{\nabla}_{\vec{v}}(\alpha.\vec{w}) = \tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w} + \alpha.\nabla_{\vec{v}}\vec{w} \tag{12.2}$$

(derivation of product):

Definition 12.1 The covariant derivation on $\Omega^1(S)$ along $\vec{v}$ is

$$\tilde{\tilde{\nabla}}_{\vec{v}} : \begin{cases} \Omega^1(S) \ \to \Omega^1(S), \\ \qquad \alpha \ \mapsto \tilde{\tilde{\nabla}}_{\vec{v}}\alpha, \quad \text{s.t.} \quad \tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w} := \tilde{\nabla}_{\vec{v}}(\alpha.\vec{w}) - \alpha.\nabla_{\vec{v}}\vec{w}, \quad \forall \vec{w} \in \Gamma(S). \end{cases} \tag{12.3}$$

E.g., for the Riemannian connection ∇,

$$\tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w} = d(\alpha.\vec{w}).\vec{v} - \alpha.\text{Proj}_{TS}(d\vec{w}.\vec{v}), \tag{12.4}$$

Notation:

$$\tilde{\tilde{\nabla}}_{\vec{v}}\alpha \overset{\text{written}}{=} \frac{D\alpha}{ds}, \quad \text{so} \quad \frac{D\alpha}{ds}.\vec{w} = \tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w}, \tag{12.5}$$

and

$$\frac{D\alpha}{ds}.\vec{w} = \frac{D(\alpha.\vec{w})}{ds} - \alpha.\frac{D\vec{w}}{ds}. \tag{12.6}$$

Proposition 12.2 *If $\varphi \in \mathcal{F}(S)$, $\vec{u},\vec{v} \in \Gamma(S)$ and $\alpha \in \Omega^1(S)$, then*

$$\tilde{\tilde{\nabla}}_{\varphi\vec{u}+\vec{v}}\alpha = \varphi\tilde{\tilde{\nabla}}_{\vec{u}}\alpha + \tilde{\tilde{\nabla}}_{\vec{v}}\alpha, \quad \text{and} \quad \tilde{\tilde{\nabla}}_{\vec{v}}(\varphi\alpha) = (\tilde{\nabla}_{\vec{v}}\varphi)\,\alpha + \varphi(\tilde{\tilde{\nabla}}_{\vec{v}}\alpha), \tag{12.7}$$

that is, for all $\vec{w} \in \Gamma(S)$,

$$\tilde{\tilde{\nabla}}_{\varphi\vec{u}+\vec{v}}\alpha.\vec{w} = \varphi\,\tilde{\tilde{\nabla}}_{\vec{u}}\alpha.\vec{w} + \tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w}, \quad \text{and} \quad \tilde{\tilde{\nabla}}_{\vec{v}}(\varphi\alpha).\vec{w} = (\tilde{\nabla}_{\vec{v}}\varphi)\alpha.\vec{w} + \varphi\tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w}. \tag{12.8}$$

Proof. (12.3) gives $\tilde{\tilde{\nabla}}_{\varphi\vec{u}}\alpha.\vec{w} = \tilde{\nabla}_{\varphi\vec{u}}(\alpha.\vec{w}) - \alpha.\nabla_{\varphi\vec{u}}\vec{w} = \varphi\tilde{\nabla}_{\vec{u}}(\alpha.\vec{w}) - \alpha.(\varphi\nabla_{\vec{u}}\vec{w}) = \varphi\tilde{\nabla}_{\vec{u}}(\alpha.\vec{w}) - \varphi\alpha.\nabla_{\vec{u}}\vec{w} = \varphi\tilde{\tilde{\nabla}}_{\vec{u}}\alpha$, and $\tilde{\tilde{\nabla}}_{\vec{v}}(\varphi\alpha).\vec{w} = \tilde{\nabla}_{\vec{v}}(\varphi\alpha.\vec{w}) - \varphi\alpha.\nabla_{\vec{v}}\vec{w} = (\tilde{\nabla}_{\vec{v}}\varphi)(\alpha.\vec{w}) + \varphi\tilde{\nabla}_{\vec{v}}(\alpha.\vec{w}) - \varphi\alpha.\nabla_{\vec{v}}\vec{w} = (\tilde{\nabla}_{\vec{v}}\varphi)(\alpha.\vec{w}) + \varphi(\tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w} + \alpha.\nabla_{\vec{v}}\vec{w}) - \varphi\alpha.\nabla_{\vec{v}}\vec{w} = (\tilde{\nabla}_{\vec{v}}\varphi)(\alpha.\vec{w}) + \varphi\tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w}$. ▪

12.2 Connection on $\Omega^1(S)$

Definition 12.3 The associated connection on $\Gamma(S) \times \Omega^1(S)$ is

$$\tilde{\tilde{\nabla}} : \begin{cases} \Gamma(S) \times \Omega^1(S) & \to \Omega^1(S), \\ (\vec{v}, \alpha) & \mapsto \tilde{\tilde{\nabla}}(\vec{v}, \alpha) := \tilde{\tilde{\nabla}}_{\vec{v}}\alpha. \end{cases} \tag{12.9}$$

12.3 Differential $\tilde{\tilde{\nabla}}\alpha$ on $\Omega^1(S)$

Definition 12.4 The associated differential is

$$\tilde{\tilde{\nabla}}\alpha : \begin{cases} \Gamma(S) & \to \Omega^1(S), \\ \vec{v} & \mapsto \tilde{\tilde{\nabla}}\alpha.\vec{v} := \tilde{\tilde{\nabla}}_{\vec{v}}\alpha \overset{\text{written}}{=} \dfrac{D\alpha}{dt}. \end{cases} \tag{12.10}$$

Proposition 12.5 *If $\alpha \in \Omega^1(S)$, then $\tilde{\tilde{\nabla}}\alpha$ is a tensor in $T^0_2(S)$.*

Proof. $\tilde{\tilde{\nabla}}\alpha.(f\vec{u} + \vec{v}) = f\,\tilde{\tilde{\nabla}}\alpha.\vec{u} + \tilde{\tilde{\nabla}}\alpha.\vec{v}$, for all $f \in \mathcal{F}(S)$ and $\vec{u},\vec{v} \in \Gamma(S)$, cf. (11.2). ▪

Example 12.6 In S_+ and ∇ the Riemannian connection, we get (the usual result)

$$\begin{aligned}(\tilde{\tilde{\nabla}}_{\vec{v}}\alpha).\vec{w} &= d(\alpha.\vec{w}).\vec{v} - \alpha.(d\vec{w}.\vec{v}) = (d\alpha.\vec{v}).\vec{w} + \alpha.(d\vec{w}.\vec{v}) - \alpha.(d\vec{w}.\vec{v}) \\ &= (d\alpha.\vec{v}).\vec{w} \qquad (= d\alpha(\vec{v},\vec{w})),\end{aligned} \tag{12.11}$$

In particular with $f \in \mathcal{F}(S)$ and $\alpha = df$, we get (the usual result,cf. (9.1))

$$(\tilde{\tilde{\nabla}}_{\vec{v}}df).\vec{w} = d(df.\vec{v}).\vec{w} = d^2f(\vec{w},\vec{v}) + df.(d\vec{v}.\vec{w}). \tag{12.12}$$

▪

13 Christoffel symbols for differential forms: $\nabla_{\vec{e}_j}e^i = -\sum_k \gamma^i_{jk}e^k$

Let ∇ be a connection in S. Let $(\vec{e}_i)$ be a holonomic basis and γ^k_{ij} be the associated Christoffel symbols, that is, $\nabla_{\vec{e}_j}\vec{e}_k = \sum_{i=1}^m \gamma^i_{jk}\vec{e}_i$. Then

Proposition 13.1

$$\begin{aligned} &\tilde{\tilde{\nabla}}_{\vec{e}_j}e^i = -\sum_{k=1}^m \gamma^i_{jk}e^k = \tilde{\tilde{\nabla}}e^i.\vec{e}_j, \quad \text{i.e.} \quad \gamma^i_{jk} = -(\tilde{\tilde{\nabla}}_{\vec{e}_j}e^i).\vec{e}_k, \\ \text{i.e.} \quad &\tilde{\tilde{\nabla}}e^i = \sum_{j=1}^m (\tilde{\tilde{\nabla}}_{\vec{e}_j}e^i) \otimes e^j = -\sum_{j,k=1}^m \gamma^i_{jk}e^k \otimes e^j. \end{aligned} \tag{13.1}$$

Proof. $e^i.\vec{e}_k = \delta^i_k$ gives $d(e^i.\vec{e}_k).\vec{e}_j = 0 = (\tilde{\tilde{\nabla}}_{\vec{e}_j}e^i).\vec{e}_k + e^i.(\nabla_{\vec{e}_j}\vec{e}_k)$, thus $(\tilde{\tilde{\nabla}}_{\vec{e}_j}e^i).\vec{e}_k = -\gamma^i_{jk}$. ▪

Example 13.2 Polar system and Riemannian connection: (3.6) gives

$$\begin{cases} d^2r.\vec{e}_1 = de^1.\vec{e}_1 = \nabla_{\vec{e}_1}e^1 = -\gamma^1_{11}e^1 - \gamma^1_{12}e^2 = 0, \\ d^2r.\vec{e}_2 = de^1.\vec{e}_2 = \nabla_{\vec{e}_2}e^1 = -\gamma^1_{21}e^1 - \gamma^1_{22}e^2 = re^2 = r\,d\theta, \\ d^2\theta.\vec{e}_1 = de^2.\vec{e}_1 = \nabla_{\vec{e}_1}e^2 = -\gamma^2_{11}e^1 - \gamma^2_{12}e^2 = -\frac{1}{r}e^2 = -\frac{1}{r}\,d\theta, \\ d^2\theta.\vec{e}_2 = de^2.\vec{e}_2 = \nabla_{\vec{e}_2}e^2 = -\gamma^2_{21}e^1 - \gamma^2_{22}e^2 = -\frac{1}{r}e^1 = -\frac{1}{r}dr, \end{cases} \tag{13.2}$$

so $d^2r = de^1 = \tilde{\tilde{\nabla}}e^1 = r\,e^2 \otimes e^2 = r\,d\theta \otimes d\theta$,
and $d^2\theta = de^2 = \tilde{\tilde{\nabla}}e^2 = -\frac{1}{r}e^2 \otimes e^1 - \frac{1}{r}e^1 \otimes e^2 = -\frac{1}{r}d\theta \otimes dr - \frac{1}{r}dr \otimes d\theta$. ∎

Example 13.3 Spherical GPS system and Riemannian connection: (3.30) and following give

$$\begin{cases} d^2r.\vec{e}_1 = de^1.\vec{e}_1 = \nabla_{\vec{e}_1}\vec{e}_1 = -\gamma^1_{11}e^1 - \gamma^1_{12}e^2 - \gamma^1_{13}e^3 = 0, \\ d^2r.\vec{e}_2 = de^1.\vec{e}_2 = -\gamma^1_{21}e^1 - \gamma^1_{22}e^2 - \gamma^1_{23}e^3 = +r\cos^2\varphi\, e^2 = r\cos^2\varphi\, d\theta, \\ d^2r.\vec{e}_3 = de^1.\vec{e}_3 = -\gamma^1_{31}e^1 - \gamma^1_{32}e^2 - \gamma^1_{33}e^3 = +r\, e^3 = r\, d\varphi, \\ d^2\theta.\vec{e}_1 = de^2.\vec{e}_1 = -\gamma^2_{11}e^1 - \gamma^2_{12}e^2 - \gamma^2_{13}e^3 = -\frac{1}{r}\, e^2 = -\frac{1}{r}\, d\theta, \\ d^2\theta.\vec{e}_2 = de^2.\vec{e}_2 = -\gamma^2_{21}e^1 - \gamma^2_{22}e^2 - \gamma^2_{23}e^3 = -\frac{1}{r}\, e^1 - \tan\varphi\, e^3 = -\frac{1}{r}\, dr - \tan\varphi\, d\varphi, \\ d^2\theta.\vec{e}_3 = de^2.\vec{e}_3 = -\gamma^2_{31}e^1 - \gamma^2_{32}e^2 - \gamma^2_{33}e^3 = +\tan\varphi\, e^2 = \tan\varphi\, d\theta, \\ d^2\varphi.\vec{e}_1 = de^3.\vec{e}_1 = -\gamma^3_{11}e^1 - \gamma^3_{12}e^2 - \gamma^3_{13}e^3 = -\frac{1}{r}\, e^3 = -\frac{1}{r}\, d\varphi, \\ d^2\varphi.\vec{e}_2 = de^3.\vec{e}_2 = -\gamma^3_{21}e^1 - \gamma^3_{22}e^2 - \gamma^3_{23}e^3 = -\cos\varphi\sin\varphi\, e^2 = -\cos\varphi\sin\varphi\, d\theta, \\ d^2\varphi.\vec{e}_3 = de^3.\vec{e}_3 = -\gamma^3_{31}e^1 - \gamma^3_{32}e^2 - \gamma^3_{33}e^3 = -\frac{1}{r}\, e^1 = -\frac{1}{r}\, dr, \end{cases} \tag{13.3}$$

∎

Let $\alpha \in \Omega^1(S)$ and $\alpha = \sum_i \alpha_i e^i \in \Omega^1(S)$. And let

$$\tilde{\tilde{\nabla}}\alpha = \sum_{i,j=1}^{m} \alpha_{i|j}\, e^i \otimes e^j, \tag{13.4}$$

that is, $[\nabla\alpha] = [\alpha_{i|j}]_{\substack{i=1,\dots,m \\ j=1,\dots,m}}$ is the Jacobian matrix of α relative to $(\vec{e}_i)$. Let $\vec{v} \in \Gamma(S)$ and $\vec{v} = \sum_j v^j \vec{e}_j \in \Gamma(S)$. Then,

$$\nabla_{\vec{v}}\alpha = \nabla\alpha.\vec{v} = \sum_{i,j=1}^{m} \alpha_{i|j} v^j\, e^i, \tag{13.5}$$

In particular,

$$\tilde{\tilde{\nabla}}\alpha.\vec{e}_j = \tilde{\tilde{\nabla}}_{\vec{e}_j}\alpha = \sum_{i=1}^{m} \alpha_{i|j}\, e^i, \quad \text{and} \quad \alpha_{i|j} = (\tilde{\tilde{\nabla}}_{\vec{e}_j}\alpha).\vec{e}_i. \tag{13.6}$$

Corollary 13.4

$$\alpha_{i|j} = \frac{\partial \alpha_i}{\partial q^j} - \sum_{k=1}^{m} \alpha_k \gamma^k_{ji}. \tag{13.7}$$

(In particular we find the previous result $\tilde{\tilde{\nabla}}_{\vec{e}_j} e^k = -\sum_i \gamma^k_{ji} e^i$, cf. (13.1).)

Proof. $\alpha_i = \alpha.\vec{e}_i$ gives $d\alpha_i.\vec{e}_j = \tilde{\nabla}_{\vec{e}_j}\alpha_i = \tilde{\nabla}_{\vec{e}_j}(\alpha.\vec{e}_i) = (\tilde{\tilde{\nabla}}_{\vec{e}_j}\alpha).\vec{e}_i + \alpha.(\nabla_{\vec{e}_j}\vec{e}_i)$, thus

$$\alpha_{i|j} = (\tilde{\tilde{\nabla}}_{\vec{e}_j}\alpha).\vec{e}_i = d\alpha_i.\vec{e}_j - \alpha.\nabla_{\vec{e}_j}\vec{e}_i = \frac{\partial \alpha_i}{\partial q^j} - (\sum_{\ell=1}^{m} \alpha_\ell e^\ell).(\sum_{k=1}^{m} \gamma^k_{ji}\vec{e}_k) = \frac{\partial \alpha_i}{\partial q^j} - \sum_{k=1}^{m} \alpha_k \gamma^k_{ji}. \tag{13.8}$$

∎

Example 13.5 If $\alpha = df = \sum_i \frac{\partial f}{\partial q^i}\, dq^i$, then $\alpha_i = \frac{\partial f}{\partial q^i}$, and we get

$$\nabla_{\vec{e}_j}(df) = \sum_{i=1}^m (\frac{\partial f}{\partial q^i})_{|j}\, dq^i \qquad \text{et} \qquad \nabla(df) = \sum_{i=1}^m \frac{\partial f}{\partial q^i}{}_{|j}\, dq^i \otimes dq^j, \tag{13.9}$$

with

$$(\frac{\partial f}{\partial q^i})_{|j} = \frac{\partial^2 f}{\partial q^j \partial q^i} - \sum_{k=1}^m \frac{\partial f}{\partial q^k}\gamma_{ji}^k \qquad (= (\nabla_{\vec{e}_j} df)_i = (\nabla_{\vec{e}_j} df).\vec{e}_i). \tag{13.10}$$

And $\vec{v} = \sum_j v^j \vec{e}_j$ gives

$$\nabla_{\vec{v}}(df) = \sum_{i,j=1}^m (\frac{\partial f}{\partial q^i})_{|j} v^j\, dq^i. \tag{13.11}$$

▪

14 Lie autonomous derivative of a differential form

14.1 Definition

Let $\vec{v}, \vec{w} \in \Gamma(S)$ and $\alpha \in \Omega^1(S)$. Thus $f = \alpha.\vec{w} \in \mathcal{F}(S)$, and (12.2) gives

$$\begin{aligned} \mathcal{L}^0{}_{\vec{v}}(\alpha.\vec{w}) &= \tilde{\nabla}_{\vec{v}}(\alpha.\vec{w}) = (\tilde{\tilde{\nabla}}_{\vec{v}}\alpha).\vec{w} + \alpha.(\nabla_{\vec{v}}\vec{w}) = (\tilde{\tilde{\nabla}}_{\vec{v}}\alpha).\vec{w} + \alpha.(\nabla_{\vec{w}}\vec{v} + \mathcal{L}^0{}_{\vec{v}}\vec{w}) \\ &= (\mathcal{L}^0{}_{\vec{v}}\alpha).\vec{w} + \alpha.(\mathcal{L}^0{}_{\vec{v}}\vec{w}) \end{aligned} \tag{14.1}$$

as soon as:

Definition 14.1 The autonomous Lie derivative of a differential form $\alpha \in \Omega^1(S)$ along $\vec{v} \in \Gamma(S)$ is the differential form $\mathcal{L}^0{}_{\vec{v}}\alpha \in \Omega^1(S)$ defined by, for all $\vec{w} \in \Gamma(S)$,

$$\begin{aligned} (\tilde{\tilde{\mathcal{L}}}^0{}_{\vec{v}}\alpha).\vec{w} &:= (\tilde{\tilde{\nabla}}_{\vec{v}}\alpha).\vec{w} + \alpha.(\nabla_{\vec{w}}\vec{v}) \\ &= (\tilde{\tilde{\nabla}}\alpha.\vec{v}).\vec{w} + \alpha.(\nabla\vec{v}.\vec{w}) \overset{\text{written}}{=} (\mathcal{L}^0{}_{\vec{v}}\alpha).\vec{w}, \end{aligned} \tag{14.2}$$

the last notation (without tilde) if there is no ambiguity.

Doing so, we have defined

$$\mathcal{L}^0{}_{\vec{v}} : \begin{cases} \Omega^1(S) & \to \Omega^1(S) \\ \alpha & \to \mathcal{L}^0{}_{\vec{v}}\alpha = \tilde{\tilde{\nabla}}\alpha.\vec{v} + \alpha.\nabla\vec{v} = \dfrac{D\alpha}{dt} + \alpha.\nabla\vec{v}. \end{cases} \tag{14.3}$$

14.2 Components of $\mathcal{L}^0{}_{\vec{v}}\alpha := \nabla_{\vec{v}}\alpha + \alpha\nabla\vec{v}$

Let $\vec{v} \in \Gamma(S)$ and $\vec{v} = \sum_{i=1}^m v^i \vec{e}_i$. Let $\nabla\vec{v} = \sum_{i,j=1}^m v^i_{|j}\vec{e}_i \otimes e^j$, where $v^i_{|j} = \frac{\partial v^i}{\partial q^j} + \sum_{k=1}^m \gamma_{jk}^i v^k$, cf. (7.4), and $[\nabla\vec{v}(p)]_{|\vec{e}} = [v^i_{|j}(p)]$ is the Jacobian matrix of $\vec{v}$ at p.

Let $\alpha \in \Omega^1(S)$ and $\alpha = \sum_{i=1}^m \alpha_i e^i$. Let $\tilde{\tilde{\nabla}}\alpha = \sum_{i,j=1}^m \alpha_{i|j} e^i \otimes e^j$, so $\alpha_{i|j} = \frac{\partial \alpha_i}{\partial q^j} - \sum_{k=1}^m \alpha_k \gamma_{ji}^k$, cf. (13.7), and $[\tilde{\tilde{\nabla}}\alpha(p)]_{|\vec{e}} = [\alpha_{i|j}(p)]$ is the Jacobian matrix of α at p.

Corollary 14.2

$$\mathcal{L}^0{}_{\vec{v}}\alpha = \sum_{i,j=1}^m (\alpha_i \frac{\partial v^i}{\partial q^j} + v^i \frac{\partial \alpha_j}{\partial q^i}) e^j. \tag{14.4}$$

Proof. $\tilde{\tilde{\nabla}}\alpha.\vec{v} = \sum_{i,j=1}^m \alpha_{i|j} v^j e^i = \sum_{i,j=1}^m \alpha_{j|i} v^i e^j$ and $\alpha.\nabla\vec{v} = \sum_{i,j=1}^m \alpha_i v^i_{|j} e^j$, thus $\mathcal{L}^0{}_{\vec{v}}\alpha = \sum_{i,j=1}^m (\alpha_{j|i} v^i + \alpha_i v^i_{|j}) e^j$; So $\mathcal{L}^0{}_{\vec{v}}\alpha = \sum_j (\mathcal{L}^0{}_{\vec{v}}\alpha)_j e^j$ gives

$$(\mathcal{L}^0{}_{\vec{v}}\alpha)_j = \sum_{i=1}^m \alpha_{j|i} v^i + \alpha_i v^i_{|j} = \sum_{i=1}^m \frac{\partial \alpha_j}{\partial q^i} v^j - \sum_{i,k=1}^m \alpha_k \gamma_{ij}^k v^j + \sum_{i=1}^m \alpha_i \frac{\partial v^i}{\partial q^j} + \sum_{i,k=1}^m \alpha_i \gamma_{jk}^i v^k. \tag{14.5}$$

And $\gamma_{ij}^k = \gamma_{ji}^k$ (coordinate system), hence (14.4). ▪

In particular

$$\mathcal{L}^0{}_{\vec{v}}e^i = \sum_{j=1}^m \frac{\partial v^i}{\partial q^j}e^j, \quad \text{i.e.} \quad (\mathcal{L}^0{}_{\vec{v}}e^i)_j = \frac{\partial v^i}{\partial q^j}, \quad \text{i.e.} \quad [\mathcal{L}^0{}_{\vec{v}}e^i] = \begin{pmatrix} \frac{\partial v^i}{\partial q^1} & \dots & \frac{\partial v^i}{\partial q^m} \end{pmatrix}, \tag{14.6}$$

and

$$\mathcal{L}^0{}_{\vec{e}_i}\alpha = \sum_{j=1}^m \frac{\partial \alpha_i}{\partial q^j}e^j, \quad \text{i.e.} \quad (\mathcal{L}^0{}_{\vec{e}_i}\alpha)_j = \frac{\partial \alpha_i}{\partial q^j}, \quad \text{i.e.} \quad [\mathcal{L}^0{}_{\vec{e}_i}\alpha] = \begin{pmatrix} \frac{\partial \alpha_i}{\partial q^1} & \dots & \frac{\partial \alpha_i}{\partial q^m} \end{pmatrix}, \tag{14.7}$$

and, for all i, j,

$$\mathcal{L}^0{}_{\vec{e}_i}e^j = 0. \tag{14.8}$$

14.3 "Universal" derivation property

Hence,

$$\mathcal{L}^0{}_{\vec{v}}(b.c) = (\mathcal{L}^0{}_{\vec{v}}b).c + b.(\mathcal{L}^0{}_{\vec{v}}c), \tag{14.9}$$

whenever $b.c$ is meaningful, that is, whenever
1- $b, c \in \mathcal{F}(S)$ (and $b.c = bc \in \mathcal{F}(S)$),
2- $b \in \mathcal{F}(S)$ and $c \in \Gamma(S)$ (and $b.c = bc \in \Gamma(S)$),
3- $b \in \mathcal{F}(S)$ and $c \in \Omega^1(S)$ (and $b.c = bc \in \Omega^1(S)$),
4- $b \in \Omega^1(S)$ and $c \in \Gamma(S)$ (and $b.c \in \mathcal{F}(S)$).

15 Connection ∇ on $T^r_s(S)$

15.1 Covariant derivative $\nabla_{\vec{v}}T$

Let ∇ be a connection in S. If $\vec{v} \in \Gamma(S)$, if $f \in \mathcal{F}(S)$, $\vec{w} \in \Gamma(S)$ and $\alpha \in \Omega^1(S)$, then the following covariant derivative along $\vec{v}$ have been defined: $\tilde{\nabla}_{\vec{v}}f \overset{\text{written}}{=} \nabla_{\vec{v}}f$, $\nabla_{\vec{v}}\vec{w}$, $\tilde{\tilde{\nabla}}_{\vec{v}}\alpha \overset{\text{written}}{=} \nabla_{\vec{v}}\alpha$.

Definition 15.1 Let $\vec{v} \in \Gamma(S)$. Let $T_1 \in T^{r_1}_{s_1}(S)$ and $T_2 \in T^{r_2}_{s_2}(S)$. The covariant derivative of $T = T_1 \otimes T_2$ along $\vec{v}$ is defined by

$$\nabla_{\vec{v}}T = \nabla_{\vec{v}}(T_1 \otimes T_2) := (\nabla_{\vec{v}}T_1) \otimes T_2 + T_1 \otimes (\nabla_{\vec{v}}T_2) \overset{\text{written}}{=} \frac{DT}{dt}. \tag{15.1}$$

This defines $\nabla_{\vec{v}} : T^r_s(S) \to T^r_s(S)$ the covariant derivative along $\vec{v}$ for tensors.

Example 15.2 Coordinate system Φ, and with $(\vec{e}_i(p))$ the coordinate basis at $p \in S$; If $T(p) = e^i(p) \otimes e^j(p)$, then

$$\begin{aligned} \nabla_{\vec{e}_k}(e^i \otimes e^j) &= (\nabla_{\vec{e}_k}e^i) \otimes e^j + e^i \otimes (\nabla_{\vec{e}_k}e^j) \\ &= -\sum_{\ell=1}^m \gamma^i_{k\ell}\, e^\ell \otimes e^j - \sum_{\ell=1}^m \gamma^j_{k\ell}\, e^i \otimes e^\ell. \end{aligned} \tag{15.2}$$

So, if $\vec{v} = \sum_k v^k \vec{e}_k$ then

$$\nabla_{\vec{v}}(e^i \otimes e^j) = -\sum_{k,\ell=1}^m \gamma^i_{k\ell}v^k\, e^\ell \otimes e^j - \sum_{\ell=1}^m \gamma^j_{k\ell}v^k\, e^i \otimes e^\ell. \tag{15.3}$$

∴

15.2 Differential ∇T

Definition 15.3 If $T \in T^r_s(S)$ then its differential is the tensor $\nabla T \in T^r_{s+1}(S)$ defined by, for all $\vec{v} \in \Gamma(S)$,

$$\nabla T.\vec{v} = \nabla_{\vec{v}}T, \tag{15.4}$$

That is, for all $\alpha_i \in \Omega^1(S)$ and all $\vec{w}_i \in \Gamma(S)$,

$$(\nabla T.\vec{v})(\alpha_1, ..., \alpha_r, \vec{w}_1, ...\vec{w}_s) = \nabla_{\vec{v}}T(\alpha_1, ..., \alpha_r, \vec{w}_1, ...\vec{w}_s). \tag{15.5}$$

Remark: we have implicitly used the expression "∇T is a tensor $T^r_{s+1}(S)$" to mean that the associated multilinear form $Z : (\Omega^1(S))^r \times (\Gamma(S))^{s+1} \to \mathbb{R}$ defined by

$$Z(\alpha_1,...,\alpha_r,\vec{w}_1,...\vec{w}_s,\vec{v}) := \nabla_{\vec{v}}T(\alpha_1,...,\alpha_r,\vec{w}_1,...\vec{w}_s) = \nabla T(\alpha_1,...,\alpha_r,\vec{w}_1,...\vec{w}_s).\vec{v}. \tag{15.6}$$

See next §: the components of $\nabla_{\vec{v}}T$ are indeed the components of Z.

Example 15.4 Continuing example 15.2: (15.3) gives

$$\nabla(e^i \otimes e^j) = -\sum_{i,j,k=1}^{m} \gamma^i_{k\ell}\, e^\ell \otimes e^j \otimes e^k - \sum_{i,j,k=1}^{m} \gamma^j_{k\ell}\, e^i \otimes e^\ell \otimes e^k \quad \in T^0_3(S). \tag{15.7}$$

▪

15.3 Components $T^{i_1...i_r}_{j_1...j_s|k}$ of ∇T

Let $T \in T^r_s(S)$ and

$$T = \sum_{i_1,...,i_r,j_1,...,j_s=1}^{m} T^{i_1...i_r}_{j_1...j_s}\, \vec{e}_{i_1} \otimes ... \otimes \vec{e}_{i_r} \otimes e^{j_1} \otimes ... \otimes e^{j_s}. \tag{15.8}$$

Let $T^{i_1...i_r}_{j_1...j_s}{}_{|k}$ be the components of ∇T, that is,

$$\nabla T = \sum_{i_1,...,i_r,j_1,...,j_s,k=1}^{m} T^{i_1...i_r}_{j_1...j_s}{}_{|k}\, \vec{e}_{i_1} \otimes ... \otimes \vec{e}_{i_r} \otimes e^{j_1} \otimes ... \otimes e^{j_s} \otimes e^k. \tag{15.9}$$

So

$$\nabla_{\vec{e}_k}T = \nabla T.\vec{e}_k = \sum_{i_1,...,i_r,j_1,...,j_s=1}^{m} T^{i_1...i_r}_{j_1...j_s}{}_{|k}\, \vec{e}_{i_1} \otimes ...\vec{e}_{i_r} \otimes e^{j_1} \otimes ... \otimes e^{j_s}, \tag{15.10}$$

and

$$\nabla_{\vec{v}}T = \nabla T.\vec{v} = \sum_{i_1,...,i_r,j_1,...,j_s,k=1}^{m} T^{i_1...i_r}_{j_1...j_s}{}_{|k}\, v^k\, \vec{e}_{i_1} \otimes ...\vec{e}_{i_r} \otimes e^{j_1} \otimes ... \otimes e^{j_s}. \tag{15.11}$$

And (15.8) gives

$$\begin{aligned}\nabla T.\vec{e}_k = &\sum_{\substack{i_1,...,i_r\\ j_1,...,j_s=1}}^{m} (dT^{i_1,...,i_r}_{j_1,...,j_s}.\vec{e}_k)\, \vec{e}_{i_1} \otimes ... \otimes e^{j_s} \\ &+ T^{i_1,...,i_r}_{j_1,...,j_s}(\nabla_{\vec{e}_k}\vec{e}_{i_1}) \otimes \vec{e}_{i_2} \otimes ... \otimes e^{j_s} + \ldots + T^{i_1,...,i_r}_{j_1,...,j_s}\vec{e}_{i_1} \otimes ... \otimes e^{j_{s-1}} \otimes (\nabla_{\vec{e}_k}e^{j_s}).\end{aligned}$$

Thus, with $\nabla_{\vec{e}_k}\vec{e}_i = \sum_\ell \gamma^\ell_{ki}\vec{e}_\ell$ and $\nabla_{\vec{e}_k}\vec{e}^j = -\sum_\ell \gamma^k_{j\ell}e^\ell$ we get

$$\begin{aligned}T^{i_1...i_r}_{j_1...j_s}{}_{|k} = \frac{\partial T^{i_1...i_r}_{j_1...j_s}}{\partial q^k} &+ \sum_{\ell=1}^{m} T^{\ell\, i_2...i_r}_{j_1...j_s}\, \gamma^{i_1}_{k\ell} + \sum_{\ell=1}^{m} T^{i_1\ell\, i_3...i_r}_{j_1...j_s}\, \gamma^{i_2}_{k\ell} + ... \\ &- \sum_{\ell=1}^{m} T^{i_1...i_r}_{\ell\, j_2...j_{s-1}\ell}\, \gamma^\ell_{j_1k} - \sum_{\ell=1}^{m} T^{i_1...i_r}_{j_1\ell\, j_3...j_{s-1}\ell}\, \gamma^\ell_{j_2k} - ....\end{aligned} \tag{15.12}$$

16 Riemannian metric

Definition 16.1 A Riemannian metric g in S is a (regular) tensor $g \in T^0_2(S)$ such that, for all $p \in S$, $g_p := g(p)$ is a dot product in T_pS.

E.g., in $\mathbb{R}^n$ (affine space), a unit of measurement being chosen, an associated Euclidean basis $(\vec{E}_i)$ being chosen and the associated Euclidean dot product being named $(\cdot,\cdot)_{\mathbb{R}^n}$, the associated usual Riemannian metric is the (uniform) metric defined at any p by

$$g(p) = (\cdot,\cdot)_{\mathbb{R}^n} = \sum_{i=1}^{n} dx^i \otimes dx^i, \tag{16.1}$$

where $(dx^i) = (E^i)$ is the dual basis of $(\vec{E}_i)$ (same dot product $g_p = g(p)$ at all p). And if S is a surface in $\mathbb{R}^n$, the usual Riemannian metric is the restriction to S of a Euclidean dot product in $\mathbb{R}^n$, cf. (6.6): for all $\vec{v},\vec{w} \in \Gamma(S)$ and all $p \in S$, $g_p(\vec{v}_p,\vec{w}_p) = (\vec{v}_p,\vec{w}_p)_{\mathbb{R}^n}$.

Quantification: Let $\Phi : U \subset \mathbb{R}^m \to S$ be a coordinate system for S, let $(\vec{e}_i(p))_{i=1,...,m}$ be the basis of the system, cf. (1.12), and let $(e^i(p))_{i=1,...,m}$ be the dual basis at $p \in S$. Let $g \in T_2^0(S)$ be a metric and let g_{ij} be its components relative to $(\vec{e}_i)$, that is,

$$g = \sum_{i,j=1}^{m} g_{ij} e^i \otimes e^j, \quad \text{i.e.} \quad g_{ij} = g(\vec{e}_i, \vec{e}_j), \quad \text{and} \quad [g]_{|\vec{e}} = [g_{ij}]. \tag{16.2}$$

So, if $\vec{v}, \vec{w} \in \Gamma(S)$, $\vec{v} = \sum_{i=1}^m v^i \vec{e}_i$, $\vec{w} = \sum_{i=1}^m w^i \vec{e}_i$, then

$$g(\vec{v}, \vec{w}) = \sum_{i,j=1}^{m} g_{ij} v^i w^j = [\vec{v}]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}. \tag{16.3}$$

Example 16.2 $\mathbb{R}^2$, polar system, usual Riemannian metric:

$$(\cdot,\cdot)_{\mathbb{R}^n} = g_p = dr \otimes dr + r^2\, d\theta \otimes d\theta, \quad [g_p]_{pol} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \tag{16.4}$$

since $(\vec{e}_1(p), \vec{e}_2(p))_{\mathbb{R}^n} = 0 = g_{12} = g_{21}$, $||\vec{e}_1(p)||_{\mathbb{R}^n}^2 = 1 = g_{11}$ and $||\vec{e}_2(p)||_{\mathbb{R}^n}^2 = r^2 = g_{22}$.

Other calculation: $[g_p]_{pol} = P^T.[(\cdot,\cdot)_{\mathbb{R}^n}]_{\mathbb{R}^n}.P$ (change of basis formula for bilinear forms), thus $[g_p]_{pol} = P^T.I.P = P^T.P$, with $P = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ the transition matrix from $(\vec{E}_i)$ to $(\vec{e}_i(p))$. ∴

Example 16.3 $\mathbb{R}^2$, polar system, $S = C(\vec{0}, R)$, usual Riemannian metric:

$$g_p = R^2\, d\theta \otimes d\theta, \quad [g_p] = (\, R^2 \,),$$

since $(\vec{e}_2(p), \vec{e}_2(p))_{\mathbb{R}^n} = R^2$. ∴

Example 16.4 $\mathbb{R}^3$, GPS system $\Phi(r, \theta, \varphi) = \begin{pmatrix} x = r\cos\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\sin\varphi \end{pmatrix}$, usual Riemannian metric:

$$g_p = dr \otimes dr + r^2 \cos^2\varphi\, d\theta \otimes d\theta + r^2\, d\varphi \otimes d\varphi, \quad [g_{ij}(p)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2\cos^2\varphi & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \tag{16.5}$$

∴

Example 16.5 $\mathbb{R}^3$, GPS spherical system, $S = S(\vec{0}, R)$, usual Riemannian metric:

$$g_p = R^2 \cos^2\varphi\, d\theta \otimes d\theta + R^2\, d\varphi \otimes d\varphi, \qquad [g_{ij}(p)] = \begin{pmatrix} R^2\cos^2\varphi & 0 \\ 0 & R^2 \end{pmatrix}. \tag{16.6}$$

∴

Exercise 16.6 Check $\nabla g = 0$ for polar coordinate, directly from $g_p = dr \otimes dr + r^2\, d\theta \otimes d\theta$, cf. (16.4).

Answer.

$$dg_p.\vec{v} = (d^2r.\vec{v}) \otimes dr + dr \otimes (d^2r.\vec{v}) + (d(r^2).\vec{v}) d\theta \otimes d\theta + r^2 (d^2\theta.\vec{v}) \otimes d\theta + r^2 d\theta \otimes (d^2\theta.\vec{v}).$$

And $d(r^2) = 2r\, dr$ with $d^2r = r\, d\theta \otimes d\theta$ and $d^2\theta = -\frac{1}{r}(dr \otimes d\theta + d\theta \otimes dr)$, cf. (2.24), thus,

$$dg_p.\vec{e}_1 = 0 + 0 + 2r\, d\theta \otimes d\theta + r^2(-\frac{1}{r}) d\theta \otimes d\theta + r^2(-\frac{1}{r}) d\theta \otimes d\theta = 0,$$
$$dg_p.\vec{e}_2 = r\, d\theta \otimes dr + r\, dr \otimes d\theta + 0 - r^2 \frac{1}{r} dr \otimes d\theta - r^2 \frac{1}{r} d\theta \otimes dr = 0.$$

Thus $dg_p.\vec{v} = 0$ for all $\vec{v}$. ∴

Exercise 16.7 $\mathbb{R}^3$, GPS system, usual Riemannian metric: calculate dg with (16.5).

Answer. $g \in T^0_2(\mathbb{R}^3)$, thus $dg \in T^0_3(\mathbb{R}^3)$, $dg = \sum_{ijk} h_{ijk} e^i \otimes e^j \otimes e^k$, and $dg.\vec{v} \in T^0_2(\mathbb{R}^3)$ is given by

$$\begin{aligned} dg.\vec{v} &= (d^2r.\vec{v}) \otimes dr + dr \otimes (d^2r.\vec{v}) \\ &+ [2r\cos^2\varphi\,(dr.\vec{v}) - r^2\sin\varphi\cos\varphi\,(d\varphi.\vec{v})] d\theta\otimes d\theta + r^2\cos^2\varphi\,((d^2\theta.\vec{v})\otimes d\theta + d\theta\otimes(d^2\theta.\vec{v})) \\ &+ 2r(dr.\vec{v}) d\varphi\otimes d\varphi + r^2((d^2\varphi.\vec{v})\otimes d\varphi + d\varphi\otimes(d^2\varphi.\vec{v})). \end{aligned}$$

Thus with the Christoffel symbols, cf. (3.42)-(3.43),

$$\begin{aligned} dg.\vec{e}_1 &= 0 + 0 + [2r\cos^2\varphi + 0]\, d\theta\otimes d\theta + r^2\cos^2\varphi\,(-\frac{1}{r} d\theta\otimes d\theta - \frac{1}{r} d\theta\otimes d\theta) \\ &+ 2r\, d\varphi\otimes d\varphi + r^2(-\frac{1}{r}\, d\varphi\otimes d\varphi - \frac{1}{r}\, d\varphi\otimes d\varphi) = 0 \end{aligned}$$

Similarly $dg.\vec{e}_2 = dg.\vec{e}_3 = 0$. Thus $dg = 0$. (Trivial with $g = \sum_{i=1}^3 dx^i \otimes dx^i$.) ▪

17 Metric related to a connection

17.1 ∇g and $g_{ij|k}$

Consider a metric $g \in T^0_2(S)$, a holonomic basis $(\vec{e}_i)$, and $g_{ij} = g(\vec{e}_i, \vec{e}_j)$, so $g = \sum_{i,j=1}^m g_{ij} e^i \otimes e^j$. Then (15.12) (or direct calculation) gives

$$\nabla_{\vec{e}_k} g = \sum_{i,j=1}^m g_{ij|k} e^i \otimes e^j \quad \text{with} \quad g_{ij|k} = \frac{\partial g_{ij}}{\partial q^k} - \sum_\ell g_{\ell j}\gamma^\ell_{ki} - \sum_\ell g_{i\ell}\gamma^\ell_{kj}, \tag{17.1}$$

and

$$\nabla_{\vec{v}} g = \sum_{i,j,k=1}^m g_{ij|k} v^k\, e^i \otimes e^j \overset{\text{written}}{=} \frac{Dg}{ds}. \tag{17.2}$$

(The last equality refers to the derivation along an integral curve of $\vec{v}$, as in (5.4).)

Proposition 17.1 *Let $g \in T^0_2(S)$. For all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$:*

$$\tilde{\nabla}_{\vec{v}}(g(\vec{u}, \vec{w})) = (\nabla_{\vec{v}} g)(\vec{u}, \vec{w}) + g(\nabla_{\vec{v}}\vec{u}, \vec{w}) + g(\vec{u}, \nabla_{\vec{v}}\vec{w}), \tag{17.3}$$

written,

$$\frac{D(g(\vec{u}, \vec{w}))}{ds} = (\frac{Dg}{ds})(\vec{u}, \vec{w}) + g(\frac{D\vec{u}}{ds}, \vec{w}) + g(\vec{u}, \frac{D\vec{w}}{ds}). \tag{17.4}$$

(Derivation formula $(fgh)' = f'gh + fg'h + fgh'$.)

Proof. $g(\vec{u}, \vec{w}) = \sum_{ij} g_{ij} u^i w^j \in \mathcal{F}(S)$ gives

$$\begin{aligned} \tilde{\nabla}_{\vec{e}_k}(g(\vec{u}, \vec{w})) &= \sum_{ij} (\tilde{\nabla}_{\vec{e}_k} g_{ij}) u^i w^j + \sum_{ij} g_{ij} (\tilde{\nabla}_{\vec{e}_k} u^i) w^j + \sum_{ij} g_{ij} u^i (\tilde{\nabla}_{\vec{e}_k} w^j) \\ &= \sum_{ij} \frac{\partial g_{ij}}{\partial q^k} u^i w^j + \sum_{ij} g_{ij} \frac{\partial u^i}{\partial q^k} w^j + \sum_{ij} g_{ij} u^i \frac{\partial w^j}{\partial q^k}. \end{aligned} \tag{17.5}$$

Thus (17.1) and (11.18) give

$$\begin{aligned} \tilde{\nabla}_{\vec{e}_k}(g(\vec{u}, \vec{w})) &= \sum_{ij} \Big(g_{ij|k} + \sum_\ell g_{\ell j}\gamma^\ell_{ki} + g_{i\ell}\gamma^\ell_{kj} \Big) u^i w^j \\ &+ \sum_{ij} g_{ij} \Big(u^i_{|k} - \sum_\ell \gamma^i_{k\ell} u^\ell \Big) w^j + \sum_{ij} g_{ij} u^i \Big(w^j_{|k} - \sum_\ell \gamma^j_{k\ell} w^\ell \Big) \\ &= (\nabla_{\vec{e}_k} g)(\vec{u}, \vec{w}) + g(\nabla_{\vec{e}_k}\vec{u}, \vec{w}) + g(\vec{u}, \nabla_{\vec{e}_k}\vec{w}) \\ &+ \sum_{ij\ell} (g_{\ell j}\gamma^\ell_{ki} + g_{i\ell}\gamma^\ell_{kj}) u^i w^j - \sum_{ij\ell} g_{ij}\gamma^i_{k\ell} u^\ell w^j - \sum_{ij\ell} g_{ij}\gamma^j_{k\ell} u^i w^\ell \\ &= (\nabla_{\vec{e}_k} g)(\vec{u}, \vec{w}) + g(\nabla_{\vec{e}_k}\vec{u}, \vec{w}) + g(\vec{u}, \nabla_{\vec{e}_k}\vec{w}) \\ &+ \sum_{ij\ell} (g_{\ell j}\gamma^\ell_{ki} + g_{i\ell}\gamma^\ell_{kj} - g_{\ell j}\gamma^\ell_{ki} - g_{i\ell}\gamma^\ell_{kj}) u^i w^j, \end{aligned}$$

and the last sum vanishes. With $\tilde{\nabla}_{\vec{v}} = \sum_k v^k \tilde{\nabla}_{\vec{e}_k}$ we get (17.3). ▪

Exercise 17.2 Let $g(\cdot,\cdot)$ be a metric in $\mathbb{R}^n$. Let $\alpha \in T_1^0(\mathbb{R}^n)$. Let $\vec{\alpha}_g \in \Gamma(\mathbb{R}^n)$ be the $(\cdot,\cdot)_g$-Riesz representation vector of α, that is, for all $\vec{w} \in \mathbb{R}^n$:

$$\alpha.\vec{w} = g(\vec{\alpha}_g, \vec{w}). \tag{17.6}$$

Let $\vec{v} \in T_0^1(\mathbb{R}^n)$. Prove (with (12.3)):

$$\tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w} = g(\nabla_{\vec{v}}\vec{\alpha}_g, \vec{w}) + \nabla_{\vec{v}}g(\vec{\alpha}_g, \vec{w}). \tag{17.7}$$

In particular in $\mathbb{R}^n$ with a uniform metric $(\cdot,\cdot)_g$ (a dot product), $\tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w} = g(\nabla_{\vec{v}}\vec{\alpha}_g, \vec{w})$.

Answer. $\tilde{\tilde{\nabla}}_{\vec{v}}\alpha.\vec{w} = \tilde{\nabla}_{\vec{v}}(\alpha.\vec{w}) - \alpha.\nabla_{\vec{v}}\vec{w} = \tilde{\nabla}_{\vec{v}}(g(\vec{\alpha}_g, \vec{w})) - g(\vec{\alpha}_g, \nabla_{\vec{v}}\vec{w}) = \tilde{\nabla}_{\vec{v}}g(\vec{\alpha}_g, \vec{w}) + g(\tilde{\nabla}_{\vec{v}}\vec{\alpha}_g, \vec{w}) + g(\vec{\alpha}_g, \tilde{\nabla}_{\vec{v}}\vec{w}) - g(\vec{\alpha}_g, \nabla_{\vec{v}}\vec{w}) = \tilde{\nabla}_{\vec{v}}g(\vec{\alpha}_g, \vec{w}) + g(\tilde{\nabla}_{\vec{v}}\vec{\alpha}_g, \vec{w})$, thus (17.7). And if $g(\cdot,\cdot)$ is a uniform metric in $\mathbb{R}^n$ then $\nabla_{\vec{v}}g = 0$ for all $\vec{v}$ (indeed choose a Cartesian basis so that the g_{ij} are constants and use (17.1)). ▪

Proposition 17.3 *Let S be a surface in $\mathbb{R}^n$, $g(\cdot,\cdot) = (\cdot,\cdot)_g$ be the usual Riemannian metric in S and ∇ be the associated Riemannian connection ($\nabla_{\vec{v}}\vec{w} = \mathrm{Proj}_{TS}(d\vec{w}.\vec{v})$). Then*

$$\nabla g = 0. \tag{17.8}$$

Proof. $\nabla g = 0$ (take a Cartesian basis), thus $0 = \sum_{i=1}^n g_{ij|k} e^i \otimes e^j \otimes e^k$ in $\mathbb{R}^n$, thus $(\nabla g)_{|\Gamma(S)} = \sum_{i=1}^m g_{ij|k} e^i \otimes e^j \otimes e^k = 0$. ▪

17.2 Killing vectors and metrics (relative to a connection)

Definition 17.4 Let ∇ be a connection in S and $g \in T_2^0(S)$ be a metric in S. A Killing vector field $\vec{v} \in \Gamma(S)$, relative to $g(\cdot,\cdot)$ and ∇, is a vector field such that

$$\nabla_{\vec{v}}g = 0 \quad (= \frac{Dg}{ds}). \tag{17.9}$$

(The last equality refers to the derivation along an integral curve of $\vec{v}$.)

Definition 17.5 Let ∇ be a connection in S. A metric of Killing $g \in T_2^0(S)$ relative to ∇ is a metric such that

$$\nabla g = 0, \tag{17.10}$$

that is, such that $\nabla_{\vec{v}}g = 0$ for all $\vec{v} \in \Gamma(S)$.

(The German mathematician Wilhlem Killing, early 20th century, was a student of Weierstrass.)

Example 17.6 A Riemannian metric on a surface $S \subset \mathbb{R}^n$ is a metric of Killing relative to the usual Riemannian connection, cf. (17.8). ▪

In other words, a metric of Killing is uniform in S (equation $\nabla g = 0$). And with (17.3) we have, for all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$,

$$\tilde{\nabla}_{\vec{v}}(\vec{u}, \vec{w})_g = (\nabla_{\vec{v}}\vec{u}, \vec{w})_g + (\vec{u}, \nabla_{\vec{v}}\vec{w})_g, \quad \text{i.e.} \quad \frac{D}{ds}((\vec{u}, \vec{w})_g) = (\frac{D\vec{u}}{ds}, \vec{w})_g + (\vec{u}, \frac{D\vec{w}}{ds})_g. \tag{17.11}$$

Remark 17.7 If $(\cdot,\cdot)_g$ is a metric of Killing, then $\nabla_{\vec{v}}g = 0$ for all $\vec{v}$, that is, the first order derivatives vanish. But the second order derivatives don't vanish: They will give the curvature. ▪

17.3 Levi-Civita theorem 1

Theorem 17.8 *(Levi-Civita.) If $g(\cdot,\cdot)$ is a metric of Killing relative to a connection ∇, then, for all i, j, k,*

$$((\nabla_{\vec{e}_k}g)_{ij} =) \qquad g_{ij|k} = 0, \tag{17.12}$$

i.e.,

$$\frac{\partial g_{ij}}{\partial q^k} = \sum_{\ell=1}^m (g_{i\ell}\gamma_{jk}^\ell + g_{j\ell}\gamma_{ik}^\ell). \tag{17.13}$$

And:

$$2\sum_{\ell=1}^m g_{i\ell}\gamma_{jk}^\ell = \frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}, \tag{17.14}$$

that is, with $[g^{ij}] := [g_{ij}]^{-1}$,

$$\gamma_{jk}^i = \frac{1}{2}\sum_\ell g^{i\ell}(\frac{\partial g_{\ell j}}{\partial q^k} + \frac{\partial g_{\ell k}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell}). \tag{17.15}$$

Proof. $\nabla g = 0$ gives (17.12), thus (17.13), cf. (17.1). Thus (17.14) (circular permutation and sum). And, for all j,k, (17.14) reads $2[g].[\vec{a}] = [\vec{b}]$, thus $[\vec{a}] = \frac{1}{2}[g]^{-1}.[\vec{b}]$, thus (17.15). ∎

Corollary 17.9 *Let S be a surface of dimension $n{-}1$ in $\mathbb{R}^n$ and $(\vec{e}_1, ..., \vec{e}_{n-1})$ be the coordinate basis. Let $\vec{e}_n \in \Gamma(\mathbb{R}^n)$ be a vector field in $\mathbb{R}^n$ such that $||\vec{e}_n(p)||_g = 1$ and $g(\vec{e}_n(p), \vec{e}_j(p)) = \delta_{nj}$ for all $j = 1, ..., n{-}1$ and $p \in S$. Then let $g_+ = \sum_{i,j=1}^n g_{ij} e^i \otimes e^j$, hence $g_{nj} = \delta_{nj}$ for all $j \in [1,n]_\mathbb{N}$, and*

$$[g_+]_{|\vec{e}} = \begin{pmatrix} g_{11} & \dots & g_{1,n-1} & 0 \\ \vdots & & \vdots & \vdots \\ g_{n-1,1} & \dots & g_{n-1,n-1} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}. \tag{17.16}$$

In particular,

$$\frac{\partial g_{nj}}{\partial q^k} = 0, \quad \textit{and} \quad 2\gamma_{ij}^n = -\frac{\partial g_{ij}}{\partial q^n}. \tag{17.17}$$

Example 17.10 Sphere $S(\vec{0}, R) \subset \mathbb{R}^3$ and $\Phi : (r, \theta, \varphi) \to \Phi(r, \theta, \varphi)$ the GPS coordinate system, cf. (3.27). Let $\Psi(\theta, \varphi, r) := \Phi(r, \theta, \varphi)$, then $\vec{f}_1(p) = \Psi_{,1}(p) = \frac{\partial \Psi}{\partial \theta}(p) = \Phi_{,2}(p) = \vec{e}_2(p)$ (along a parallel), $\vec{f}_2(p) = \Psi_{,2}(p) = \frac{\partial \Psi}{\partial \varphi}(p) = \Phi_{,3}(p) = \vec{e}_3(p)$ (along a meridian, and $\vec{f}_3(p) = \Psi_{,3}(p) = \frac{\partial \Psi}{\partial r}(p) = \Phi_{,1}(p) = \vec{e}_1(p)$ (radial). Then

$$[g]_{|\vec{f}} = \begin{pmatrix} r^2 \cos^2 \varphi & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{\text{written}}{=} [g_{ij}], \tag{17.18}$$

cf. exercise 16.4 (here $1 \leftrightarrow \theta$, $2 \leftrightarrow \varphi$, $3 \leftrightarrow r$)). ∎

18 Application 2: Endomorphisms

Corollary 18.1 *Let $T \in T_1^1(S)$ and*

$$T = \sum_{i,j=1}^m T_j^i \vec{e}_i \otimes e^j, \quad \nabla T = \sum_{i,j,k=1}^m T_{j|k}^i \vec{e}_i \otimes e^j \otimes e^k. \tag{18.1}$$

Then

$$T_{j|k}^i = \frac{\partial T_j^i}{\partial q^k} - \sum_\beta T_\beta^i \gamma_{kj}^\beta + \sum_\alpha \gamma_{k\alpha}^i T_j^\alpha \quad (= \nabla T(e^i, \vec{e}_j, \vec{e}_k)). \tag{18.2}$$

Proof. Apply (15.2) or (15.11). ∎

Example 18.2 In $\mathbb{R}^n$, with a Euclidean basis and the usual Euclidean metric we get $T_{j|k}^i = \frac{\partial T_j^i}{\partial x^k}$. And for all $(\alpha, \vec{w}) \in \Omega^1(S) \times \Gamma^1(S)$,

$$\tilde{\nabla}_{\vec{v}}(T(\alpha, \vec{w})) = (\nabla_{\vec{v}} T)(\alpha, \vec{w}) + T(\nabla_{\vec{v}} \alpha, \vec{w}) + T(\alpha, \nabla_{\vec{v}} \vec{w}). \tag{18.3}$$

And

$$\frac{D}{ds}(T(\alpha, \vec{w})) = \frac{DT}{ds}(\alpha, \vec{w}) + T(\frac{D\alpha}{ds}, \vec{w}) + T(\vec{u}, \frac{D\vec{w}}{ds}). \tag{18.4}$$

∎

Part VI

Geodesics and parallel transport

19 Parallel transport in $\mathbb{R}^n$

Let $c : s \in [a,b] \to p = c(s) \in \mathbb{R}^n$ be a regular curve in $\mathbb{R}^n$ and $\vec{v}(p) = \vec{c}\,'(s)$ when $p = c(s) \in \text{Im}c$.

19.1 Parallel transport of a scalar function

Definition 19.1 $f : \mathbb{R}^n \to \mathbb{R}$ is parallel transported along c iff f is uniform on Imc, that is,

$$f \circ c \quad \text{is constant}, \quad \text{i.e.} \quad f(p) = f(\tilde{p}), \quad \forall p, \tilde{p} \in \text{Im}(c). \tag{19.1}$$

(So c is a level curve of f).

So $f \in \mathcal{F}(\mathbb{R}^n)$ is parallel transported along c iff $f(c(s)) = 0$ for all s, that is, iff

$$\forall p \in \text{Im}c, \quad df(p).\vec{v}(p) = 0, \quad \text{i.e.} \quad \tilde{\nabla}_{\vec{v}} f = 0 = \frac{Df}{ds}. \tag{19.2}$$

(See notation (5.4).) In the basis $(\vec{e}_i)$ of a coordinate system, with $\vec{v} = \sum_{i=1}^n v^i \vec{e}_i$, we get, with $\frac{\partial f}{\partial q^i} := df.\vec{e}_i$, cf. (1.35),

$$\sum_{i=1}^n \frac{\partial f}{\partial q^i} v^i = 0. \tag{19.3}$$

Example 19.2 If f is uniform in $\mathbb{R}^n$ then f is parallel transported along any curve. ▪

Example 19.3 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(p) = \sqrt{x^2 + y^2} = ||\vec{x}||_{\mathbb{R}^2}^2$ when $\vec{x} = \overrightarrow{Op}$. Let $c(s) = \begin{pmatrix} x(s) = R\cos s \\ y(s) = R \sin s \end{pmatrix}$ for $s \in [0, 2\pi]$. So $\vec{c}\,'(s) = \begin{pmatrix} -y(s) \\ x(s) \end{pmatrix}$, and $\vec{v}(p) = \begin{pmatrix} -y \\ x \end{pmatrix}$ along c. Thus $f \circ c$ is uniform along c (since $f(c(s)) = R$), so $\tilde{\nabla}_{\vec{v}} f(p) = \frac{Df}{ds}(p) = 0$ when $p \in \text{Im}c$, and f is parallel transported along c.

Or $p = c(s)$ gives $df(p).\vec{c}\,'(s) = (\,\frac{\partial f}{\partial x}(p) \quad \frac{\partial f}{\partial x}(p)\,) . \begin{pmatrix} c_1'(s) \\ c_2'(s) \end{pmatrix} = (\,\frac{x}{R} \quad \frac{y}{R}\,) . \begin{pmatrix} -y \\ x \end{pmatrix} = 0.$ ▪

19.2 Parallel transport of a vector field

Definition 19.4 A vector field $\vec{w} \in \Gamma(\mathbb{R}^n)$ is parallel transported along c iff $\vec{w}$ is uniform on Imc, that is,

$$\vec{w} \circ c \quad \text{is constant}, \quad \text{i.e.} \quad \vec{w}(p) = \vec{w}(\tilde{p}), \quad \forall p, \tilde{p} \in \text{Im}(c). \tag{19.4}$$

So, in $\vec{\mathbb{R}^n}$, $\vec{w}$ keeps its direction and its norm along c.

Hence $\vec{w} \in \mathcal{F}(\mathbb{R}^n)$ is parallel transported along c iff $\vec{w}(c(s)) = 0$ for all s, that is, iff

$$\forall p \in \text{Im}c, \quad d\vec{w}(p).\vec{v}(p) = 0, \quad \text{i.e.} \quad \nabla_{\vec{v}} \vec{w} = 0 = \frac{D\vec{w}}{ds}. \tag{19.5}$$

In the basis $(\vec{e}_i)$ of a coordinate system, with $\vec{v} = \sum_{i=1}^n v^i \vec{e}_i$ and $\vec{w} = \sum_{i=1}^n w^i \vec{e}_i$, we get

$$\sum_{k=1}^n w^i_{|k} v^k = 0, \quad \text{i.e.} \quad \sum_{k=1}^n \Big(\frac{\partial w^i}{\partial q^k} + \sum_{j=1}^n \gamma^i_{jk} w^j\Big) v^k = 0. \tag{19.6}$$

Example 19.5 If $\vec{w}$ is uniform in $\mathbb{R}^n$ then $\vec{w}$ is parallel transported along any curve. ▪

Example 19.6 $\mathbb{R}^2$; $p = c(\theta) = R\cos\theta \vec{E}_1 + R\sin\theta \vec{E}_2 = \Phi(R, \theta)$ with $\theta \in [0, 2\pi]$ (circle). Let $\vec{w}(p) = \vec{w}(c(\theta)) = \alpha(r,\theta)\vec{E}_1 + \beta(r,\theta)\vec{E}_2$. Then $(\vec{w} \circ c)'(\theta) = \frac{\partial \alpha}{\partial \theta}(r,\theta)\vec{E}_1 + \frac{\partial \beta}{\partial \theta}(r,\theta)\vec{E}_2$, and $\frac{D\vec{w}}{ds} = 0$ iff $\frac{\partial \alpha}{\partial \theta}(r,\theta) = \frac{\partial \beta}{\partial \theta}(r,\theta) = 0$, i.e. iff $\alpha(r,\theta) = \alpha(r)$ et $\beta(r,\theta) = \beta(r)$, i.e. iff $\vec{w}$ is independent of θ, which means that $\vec{W}_R(\theta) := \vec{w}(\Phi(\vec{q}))$ is independent of θ, i.e., $\vec{W}$ is uniform in Im(c). (And $\vec{0} = \frac{\partial \vec{W}}{\partial \theta}(\vec{q}) = d\vec{w}(\Phi(R,\theta)).\frac{\partial \Phi}{\partial \theta}(R,\theta) = d\vec{w}(p).\vec{v}(p) = 0$, cf. (19.5).) ▪

Exercise 19.7 Polar system and $\vec{w}(p) = \alpha(p)\vec{e}_1(p) + \beta(p)\vec{e}_2(p)$: Prove that

$$d\vec{w}.\vec{e}_2 = \Big(\frac{\partial \alpha}{\partial \theta} - r\beta\Big)\vec{e}_1 + \Big(\frac{\alpha}{r} + \frac{\partial \beta}{\partial \theta}\Big)\vec{e}_2, \tag{19.7}$$

And that $\frac{D\vec{w}}{dt} = 0$ along the circle $C(\vec{0}, R)$ iff $\vec{w}$ is uniform along $C(\vec{0}, R)$.

Answer. $d\vec{w}.\vec{e}_2 = (d\alpha.\vec{e}_2)\vec{e}_1 + \alpha(d\vec{e}_1.\vec{e}_2) + (d\beta.\vec{e}_2)\vec{e}_2 + \beta(d\vec{e}_2.\vec{e}_2)$ with $d\alpha.\vec{e}_2 = \frac{\partial \alpha}{\partial \theta}$, $d\beta.\vec{e}_2 = \frac{\partial \beta}{\partial \theta}$, $d\vec{e}_1.\vec{e}_2 = \frac{1}{r}\vec{e}_2$ and $d\vec{e}_2.\vec{e}_2 = -r\vec{e}_1$. Thus (19.7).

And $\frac{D\vec{w}}{dt} = 0$ along the circle $C(\vec{0}, R)$ iff $d\vec{w}.\vec{e}_2 = 0$. So $\frac{\partial\alpha}{\partial\theta} - R\beta = 0$ and $\frac{\alpha}{R} + \frac{\partial\beta}{\partial\theta} = 0$ at any $p \in C(\vec{0}, R)$. Thus $\frac{\partial^2\alpha}{\partial\theta^2} + \alpha = 0$ when $r = R$. Thus, with $\tilde{\alpha}_R(\theta) = \alpha(p)$ when $p = \Phi(R, \theta)$, we get $\tilde{\alpha}_R(\theta) = a_R \cos\theta + b_R \sin\theta$, with $a_R, b_R \in RR$. Thus, with $\tilde{\beta}_R(\theta) = \beta(p)$ when $p = \Phi(R, \theta)$, we get $\tilde{\beta}_R(\theta) = \frac{1}{R}(-a_R \sin\theta + b_R \cos\theta)$. Thus, with the Euclidean basis, at $p = \Phi(R, \theta)$ we get

$$\vec{w}(p) = (a_R \cos\theta + b_R \sin\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}_{|(\vec{E})} + (-a_R \sin\theta + b_R \cos\theta) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}_{|(\vec{E})} = \begin{pmatrix} a_R \\ b_R \end{pmatrix}_{|(\vec{E})},$$

so $\vec{w}$ is uniform along $C(\vec{0}, R)$. ▪

19.3 Geodesic in $\mathbb{R}^n$: A straight line

Definition 19.8 In $\mathbb{R}^n$, a geodesic is a curve $c : s \in [a, b] \to c(s) \in \mathbb{R}^n$ such that:
1- c is regular and s is an intrinsic paramater, that is, such that $||\vec{c}\,'(s)|| = 1$ for all $s \in [a, b]$, and
2- $\vec{v}(p) = \vec{c}\,'(s)$ at $p = c(s)$ is parallel transported along c, that is,

$$(\vec{c}\,'(s) =) \qquad (\vec{v} \circ c)(s) = (\vec{v} \circ c)(a), \quad \forall s \in [a, b], \tag{19.8}$$

i.e., for all $p \in \text{Im}c$,

$$d\vec{v}(p).\vec{v}(p) = 0 = \frac{D\vec{v}}{ds} \quad (\text{when } ||\vec{v}|| = 1). \tag{19.9}$$

And $\text{Im}c$ is also called a geodesic.

It is trivial that the straight lines are geodesic, since then $c(s) = c(s_0) + s\,\vec{v}_0$. Converse:

Proposition 19.9 *If c is a geodesic, then*

$$\vec{c}\,'' = 0, \tag{19.10}$$

and $\text{Im}c$ *is a straight line.*

Proof. $\vec{c}\,'(s) = \vec{c}\,'(a)$ gives $\vec{c}\,'' = 0$. Thus $c(s) = c(a) + s\,\vec{c}\,'(a)$, which is the equation of a straight line. ▪

Exercise 19.10 Let $\vec{v}_0 \in \mathbb{R}^n - \{\vec{0}\}$. Let $\alpha(t) = (t-a)^2\vec{v}_0 + \alpha_a$. Prove that $\text{Im}\alpha$ is a geodesic, and give the associated geodesic $c : s \in [a, b] \to c(s) \in \mathbb{R}^n$.

Answer. A trivial answer is $c(s) = s\frac{\vec{v}_0}{||\vec{v}_0||} + \alpha_a$.
Generic calculation: We look for an intrinsic parametrization of α. Let $s : t \to s(t)$ be a diffeomorphism. Let $c(s(t)) = \alpha(t)$. Thus $\vec{c}\,'(s(t))s'(t) = \alpha'(t)$, and we want $||\vec{c}\,'|| = 1$, thus $s'(t) = ||\vec{\alpha}\,'(t)|| = 2|t-a|\,||\vec{v}_0||$. Thus, $s(t) = (t-a)^2||\vec{v}_0||$ (up to a constant), hence $c(s) = s\frac{\vec{v}_0}{||\vec{v}_0||} + \alpha_a$, and $\vec{c}\,''(s) = 0$. ▪

20 Geodesic in a surface

20.1 Geodesic = a "short line" in a surface

Definition 20.1 Let S be a surface in $\mathbb{R}^n$. A geodesic in S is a regular curve $c : s \in [a, b] \to c(s) \in S$ such that:
1- s is an intrinsic paramater, i.e. such that $||\vec{c}\,'(s)|| = 1$ for all $s \in [a, b]$ (constant speed), and
2- The acceleration in S vanishes, that is,

$$\text{Proj}_{T_pS}(\vec{c}\,''(s)) = 0, \quad \forall s \in [a, b], \tag{20.1}$$

i.e., with $p = c(s)$ and $\vec{v}(p) = \vec{c}\,'(s)$, along $\text{Im}c$,

$$\nabla_{\vec{v}}\vec{v} = 0 \quad (= \frac{D\vec{v}}{ds} = \text{Proj}_{T_pS}(d\vec{v}.\vec{v}) = \text{ geodesic equation}), \tag{20.2}$$

i.e., $(\nabla_{\vec{v}}\vec{v})(p) = \vec{0}$ $(= \text{Proj}_{T_pS}(d\vec{v}(p).\vec{v}(p)))$ at all $p \in \text{Im}c$.
A geometric curve is a geodesic iff, parametrized with an intrinsic parameter, $\text{Proj}_{T_pS}(\vec{c}\,''(s)) = 0$.

Interpretation: On a geodesic, there is no "lateral acceleration" and no "longitudinal acceleration". So, the acceleration can only be orthogonal to S. In other words, a geodesic is obtained by "applying a straight line" on the surface.

Example 20.2 Thus on Earth, driving at constant speed, we are on a geodesic iff we don't feel any lateral forces or longitudinal forces (eventually we may feel vertical forces).

E.g., on the bridge over the Pontchartrain lake in Louisiana, we are on an Earth geodesic: In a car at constant speed, the bridge seems to be a straight line, but it is not, since the Earth is not flat. (And the length of the bridge makes it possible to see the roundness of the Earth.) ▪

Remark 20.3 In the definition we may replace $||\vec{c}\,'|| = 1$ by $||\vec{c}\,'|| = v_0$ (constant speed) for any $v_0 > 0$: It does not depend on the unit of measurement used to define $||.||$ ▪

20.2 Change of parameter

Let $\alpha : \begin{Bmatrix} [\tilde{a}, \tilde{b}] & \to S \\ t & \to p = \alpha(t) \end{Bmatrix}$ be a regular curve in S. To know if Imα is a geodesic, consider the increasing change of parameter $s : \begin{Bmatrix} [\tilde{a}, \tilde{b}] & \to [a, b] \\ t & \to s = s(t) \end{Bmatrix}$ (diffeomorphism with $s'(t) > 0$ for all t) such that the curve

$$c : \begin{Bmatrix} [a, b] & \to S \\ s & \to p = c(s) = \alpha(t) \quad \text{when} \quad s = s(t) \end{Bmatrix} \quad \text{satisfies} \quad ||\vec{c}\,'(s)||_{\mathbb{R}^n} = 1. \tag{20.3}$$

Thus $c \circ s = \alpha$, and

$$\vec{c}\,'(s(t))\, s'(t) = \vec{\alpha}\,'(t), \quad \text{thus} \quad s'(t) = ||\vec{\alpha}\,'(t)||_{\mathbb{R}^n}. \tag{20.4}$$

And then

$$\vec{\alpha}\,''(t) = \vec{c}\,''(s(t))(s'(t))^2 + \vec{c}\,'(s(t))s''(t). \tag{20.5}$$

And since $(\vec{c}\,'(s), \vec{c}\,'(s))_{\mathbb{R}^n} = 1$, we have $2(\vec{c}\,''(s), \vec{c}\,'(s))_{\mathbb{R}^n} = 0$, thus

$$\vec{c}\,''(s) \perp \vec{c}\,'(s), \quad \text{and} \quad s''(t) = (\vec{\alpha}\,''(t), \vec{c}\,'(s(t)))_{\mathbb{R}^n} = \frac{(\vec{\alpha}\,''(t), \vec{\alpha}\,'(t))_{\mathbb{R}^n}}{||\vec{\alpha}\,'(t)||}. \tag{20.6}$$

Hence, with (20.1), Imα is a geodesic (the parameter t is not necesseraly intrinsic) iff

$$\text{Proj}_{T_pS}(\alpha''(t)) = \frac{(\vec{\alpha}\,''(t), \vec{\alpha}\,'(t))_{\mathbb{R}^n}}{||\vec{\alpha}\,'(t)||^2}\, \vec{\alpha}\,'(t) \quad (= \vec{c}\,'(s(t))s''(t)). \tag{20.7}$$

(In particular, the acceleration in S is "purely longitudinal".)

In other words, with $p = \alpha(t)$ and $\vec{v}_\alpha(p) = \vec{\alpha}\,'(t) = \vec{v}_\alpha(\vec{\alpha}\,'(t))$, we have

$$\vec{\alpha}\,''(t) = d\vec{v}_\alpha(p).\vec{v}_\alpha(p), \tag{20.8}$$

thus with the usual Riemannian connection we have $\text{Proj}_{T_pS}(\alpha''(t)) = \nabla_{\vec{v}_\alpha}\vec{v}_\alpha = \frac{D\vec{v}_\alpha}{dt}$ and we get: Imα is a geodesic iff

$$(\nabla_{\vec{v}_\alpha}\vec{v}_\alpha)(p) = \frac{(d\vec{v}_\alpha(p).\vec{v}_\alpha(p), \vec{v}_\alpha(p))_{\mathbb{R}^n}}{||\vec{v}_\alpha(p)||^2}\, \vec{v}_\alpha(p) \quad (= \frac{(\vec{\alpha}\,''(t), \vec{\alpha}\,'(t))_{\mathbb{R}^n}}{||\vec{\alpha}\,'(t)||^2}\, \vec{\alpha}\,'(t)). \tag{20.9}$$

Exercise 20.4 Definition: A great circle on a sphere $S \subset \mathbb{R}^3$ is the intersection of S with a plane containing the center of S. And a parallel is the intersection of S with a plane parallel to the equator.

Prove that a parallel is a geodesic iff it is the equator.

Answer. Choose the origin O of $\mathbb{R}^n$ to be the center of S (we are interested in the derivatives, and the center won't be used, but to simplify the writings). So $S = S(\vec{0}, R)$. Choose a Euclidean basis and choose the GPS parametrization of S, so that parallel at latitude φ_0 is given by $p = c(\theta) = \begin{pmatrix} R\cos\theta\cos\varphi_0 \\ R\sin\theta\cos\varphi_0 \\ R\sin\varphi_0 \end{pmatrix}$. Thus $\vec{v}(p) = \vec{c}\,'(\theta) = \begin{pmatrix} -R\sin\theta\cos\varphi_0 \\ R\cos\theta\cos\varphi_0 \\ 0 \end{pmatrix}$ and $||\vec{v}(p)|| = R\cos\varphi_0$ is constant, and $\vec{c}\,''(\theta) = \begin{pmatrix} -R\cos\theta\cos\varphi_0 \\ -R\sin\theta\cos\varphi_0 \\ 0 \end{pmatrix} = d\vec{e}_2(p).\vec{e}_2(p) = -r\cos^2\varphi_0\,\vec{e}_1(p) + \cos\varphi_0\sin\varphi_0\,\vec{e}_3(p)$, cf. (3.33). Thus $\nabla_{\vec{v}}\vec{v}(p) = \text{Proj}_{TS}(\vec{c}\,''(t)) = \cos\varphi_0\sin\varphi_0\,\vec{e}_3(p)$. Therefore c is a geodesic iff $\cos\varphi_0\sin\varphi_0 = 0$, i.e., iff $\varphi_0 = 0$ or $\frac{\pi}{2}$; But at $\frac{\pi}{2}$ the curve is reduced to a point (the North Pole): Not a regular curve. Thus only the equator is a geodesic. ▪

Exercise 20.5 1- Give the equation of a great circle.

2- Give the equation of a great circle which makes an angle $\alpha \in]0,\pi[$ with a meridian.

Answer. Choose the origin at the center of S, so $S = S(\vec{0}, R)$.

1- A great circle is the intersection of S with a plane containing S, so is the set of $p = (x, y, z)$ s.t.

$$\begin{cases} x^2 + y^2 + b^2 = R^2, \\ ax + by + cz = 0. \end{cases} \tag{20.10}$$

(The unknowns are a, b, c.) Choose the GPS coordinates $p = c(\theta, \varphi) = \begin{pmatrix} x = R\cos\theta\cos\varphi \\ y = R\sin\theta\cos\varphi \\ z = R\sin\varphi \end{pmatrix}$, thus $x^2+y^2+b^2 = R^2$ is satisfied, and the great circles satisfy

$$a\cos\theta\cos\varphi + b\sin\theta\cos\varphi + c\sin\varphi = 0, \quad \theta \in [0, 2\pi], \quad \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{20.11}$$

In particular, if $c = 0$ and $\theta = \theta_0$ then $a\cos\theta_0\cos\varphi + b\sin\theta_0\cos\varphi = 0$, and we can choose, e.g., $a = -\sin\theta_0$ and $b = \cos\theta_0$, so a meridian is a great circle.

Suppose $c \neq 0$; And eventually dividing $(20.10)_2$ by c, suppose $c = -1$. Thus

$$a\cos\theta + b\sin\theta = \tan\varphi, \quad \text{i.e.} \quad \varphi = \tan^{-1}(a\cos\theta + b\sin\theta) = \varphi(\theta), \tag{20.12}$$

and a great circle other than a meridian is a curve $p(\theta) = c(\theta, \varphi(\theta))$.

2- Let $p_0 = c(\theta_0, \varphi_0)$ be a point in S. Consider the meridian which passes through p_0, that is, the curve $\varphi \to c(\theta_0, \varphi)$ which is normal to $\vec{e}_2(p_0) \parallel \vec{n}_m = \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \\ 0 \end{pmatrix}$. A plane through p_0 other than the meridian, equation $ax + by - z = 0$, is normal to $\vec{n}_p = \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$. Thus

$$\cos\alpha = (\frac{\vec{n}_m}{||\vec{n}_m||}, \frac{\vec{n}_p}{||\vec{n}_p||})_{\mathbb{R}^3} = \frac{-a\sin\theta_0 + b\cos\theta_0}{\sqrt{a^2 + b^2 + 1}}. \tag{20.13}$$

So, (20.12) and (20.13) give: a and b satisfy

$$\begin{cases} a\cos\theta_0 + b\sin\theta_0 = \tan\varphi_0, \\ -a\sin\theta_0 + b\cos\theta_0 = \sqrt{a^2 + b^2 + 1}\,\cos\alpha. \end{cases} \tag{20.14}$$

Thus,

$$\begin{cases} a^2\cos^2\theta_0 + b^2\sin^2\theta_0 + 2ab\cos\theta_0\sin\theta_0 = \tan^2\varphi_0, \\ a^2\sin^2\theta_0 + b^2\cos\theta_0^2 - 2ab\cos\theta_0\sin\theta_0 = (a^2 + b^2)\cos^2\alpha + \cos^2\alpha. \end{cases}$$

so (summation),

$$(a^2 + b^2)(1 - \cos^2\alpha) = \tan^2\varphi_0 + \cos^2\alpha$$

Let $\begin{cases} a = \rho\cos\gamma, \\ b = \rho\sin\gamma, \end{cases}$: Thus,

$$\rho^2 = \frac{\tan^2\varphi_0 + \cos^2\alpha}{1 - \cos^2\alpha}, \tag{20.15}$$

and (20.14) gives

$$\begin{cases} \rho(\cos\gamma\cos\theta_0 + \sin\gamma\sin\theta_0) = \tan\varphi_0 & = \rho\cos(\gamma - \theta_0), \\ \rho(-\cos\gamma\sin\theta_0 + \sin\gamma\cos\theta_0) = \sqrt{\rho^2 + 1}\,\cos\alpha = \rho\sin(\gamma - \theta_0). \end{cases} \tag{20.16}$$

If p is not on the equator ($\alpha \neq \frac{\pi}{2}$) then $\rho^2 + 1 = \frac{\tan^2\varphi_0 + 1}{1 - \cos^2\alpha}$ and

$$\tan(\gamma - \theta_0) = -\frac{\cos\alpha}{\tan\varphi_0}\sqrt{\frac{\tan^2\varphi_0 + 1}{1 - \cos^2\alpha}} \tag{20.17}$$

▪

Exercise 20.6 With (20.12) prove that the great circles are geodesics.

Answer. A meridian $c(\theta_0,\varphi)$ is a geodesic (easy). Otherwise, $c=-1$ gives

$$\varphi=\varphi(\theta)=\tan^{-1}(a\cos\theta+b\sin\theta),\quad c(\theta)=R\begin{pmatrix}\cos\theta\cos(\varphi(\theta))\\ \sin\theta\cos(\varphi(\theta))\\ \sin(\varphi(\theta))\end{pmatrix}.$$

Thus $\vec{c}\,'(\theta)=R\begin{pmatrix}-\sin\theta\cos(\varphi(\theta))\\ \cos\theta\cos(\varphi(\theta))\\ 0\end{pmatrix}+R\varphi'(\theta)\begin{pmatrix}-\cos\theta\sin(\varphi(\theta))\\ -\sin\theta\sin(\varphi(\theta))\\ \cos(\varphi(\theta))\end{pmatrix}$, i.e., with $p=c(\theta)$,

$$\vec{c}\,'(\theta)=\vec{e}_2(p)+\varphi'(\theta)\vec{e}_3(p).$$

Thus, with lighten notations,

$$\begin{aligned}\vec{c}\,'' &= d\vec{e}_2.\vec{c}\,'+\varphi''\vec{e}_3+\varphi' d\vec{e}_3.\vec{c}\,'\\ &= d\vec{e}_2.\vec{e}_2+\varphi' d\vec{e}_2.\vec{e}_3+\varphi''\vec{e}_3+\varphi' d\vec{e}_3.\vec{e}_2+\varphi'^2 d\vec{e}_3.\vec{e}_3,\\ &= -R\cos^2\varphi\vec{e}_1+\sin\varphi\cos\varphi\vec{e}_3-2\varphi'\tan\varphi\vec{e}_2+\varphi''\vec{e}_3-\varphi'^2R\vec{e}_1.\end{aligned}$$

Thus,

$$\mathrm{Proj}_{TS}(\vec{c}\,''(\theta))=-2\varphi'\tan\varphi\vec{e}_2+(\varphi''+\sin\varphi\cos\varphi)\vec{e}_3.$$

We will use (20.7). $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are orthogonal, $||\vec{e}_2||^2=R^2\cos^2\varphi$, $||\vec{e}_3||^2=R^2$, thus, $||\vec{c}\,'||^2=R^2(\cos^2\varphi+(\varphi')^2)$ and

$$\frac{(\vec{c}\,'',\vec{c}\,')_{\mathbb{R}^n}}{||\vec{c}\,'||^2}=\frac{-2\varphi'\tan\varphi\cos^2\varphi+\varphi'(\varphi''+\sin\varphi\cos\varphi)}{\cos^2\varphi+(\varphi')^2}=\frac{-\varphi'\sin\varphi\cos\varphi+\varphi'\varphi''}{\cos^2\varphi+(\varphi')^2}.$$

Thus, with (20.7) we have to check that

$$\begin{cases}-2\varphi'\tan\varphi(\cos^2\varphi+(\varphi')^2)=-\varphi'\sin\varphi\cos\varphi+\varphi'\varphi'',\\ (\varphi''+\sin\varphi\cos\varphi)(\cos^2\varphi+(\varphi')^2)=\varphi'(-\varphi'\sin\varphi\cos\varphi+\varphi'\varphi'')\end{cases}$$

that is,

$$\begin{cases}-\sin\varphi\cos\varphi-2(\varphi')^2\tan\varphi=\varphi'',\\ \varphi''\cos^2\varphi+\sin\varphi\cos^3\varphi+2(\varphi')^2\sin\varphi\cos\varphi=0\end{cases}$$

We have $\varphi(\theta)=\tan^{-1}(a\cos\theta+b\sin\theta)$ and $\tan^{-1}{}'(x)=\frac{1}{1+x^2}$.

Thus $\varphi'=\frac{-a\sin\theta+b\cos\theta}{1+\tan^2(\varphi)}=\cos^2\varphi(-a\sin\theta+b\cos\theta)$.

Thus $\varphi''=-2\sin\varphi\cos\varphi\varphi'(-a\sin\theta+b\cos\theta)-\cos^2\varphi(a\cos\theta+b\sin\theta)=-2\varphi'^2\tan\varphi-\cos\varphi\sin\varphi$ (the first equation).

Thus $\varphi''+\sin\varphi\cos\varphi=-2(\varphi')^2\tan\varphi$ (the second equation). ▪

20.3 Curve in S and coordinate systems

With $\Phi:\vec{q}\in U\subset\vec{\mathbb{R}^m}\to p=\Phi(\vec{q})\in S\subset\mathbb{R}^n$ a parametrization of S, consider a regular curve

$$\alpha:\begin{cases}]a,b[&\to S\\ t&\to p=\alpha(t)=\Phi(\vec{q}(t)),\end{cases}\tag{20.18}$$

where $\vec{q}=\Phi^{-1}\circ\alpha:\left\{\begin{array}{ll}]a,b[&\longrightarrow U\\ t&\to\vec{q}(t)\end{array}\right\}$ is a curve in $U\subset\vec{\mathbb{R}^m}$ (space of parameters).

With $\vec{q}(t)=\sum_{i=1}^m q^i(t)\vec{A}_i$ we have

$$\frac{d\vec{q}}{dt}(t)=\sum_{i=1}^m\frac{dq^i}{dt}(t)\vec{A}_i,\quad\text{written}\quad\vec{q}\,'(t)=\sum_{i=1}^m(q^i)'(t)\vec{A}_i.$$

Thus, at $p=\alpha(t)$,

$$\vec{v}(p)=\vec{\alpha}\,'(t)=\sum_{i=1}^m(q^i)'(t)\,\vec{e}_i(p),\quad\text{written}\quad\vec{v}=\sum_{i=1}^m(q^i)'\,\vec{e}_i,\tag{20.19}$$

is the velocity along α at $p=\alpha(t)$. And the acceleration is

$$\vec{\alpha}''(t)=\sum_{i=1}^{m}(q^i)''(t)\vec{e}_i(\alpha(t))+\sum_{j=1}^{m}(q^j)'(t)\,d\vec{e}_j(\alpha(t)).\vec{\alpha}'(t)=\sum_{i=1}^{m}(q^i)''(t)\vec{e}_i(p)+\sum_{j,k=1}^{m}(q^j)'(t)(q^k)'(t)\,d\vec{e}_j(p).\vec{e}_k(p),$$

that is, with and $d\vec{e}_j(p).\vec{e}_k(p)=\sum_{i=1}^{n}\gamma_{jk}^i\vec{e}_i$ and $\nabla_{\vec{e}_i}\vec{e}_j(p)=\sum_{i=1}^{m}\gamma_{jk}^i\vec{e}_i$,

$$\vec{\alpha}''(t)=\nabla_{\vec{v}}\vec{v}(p)=\frac{D\vec{v}}{dt}(p)=\sum_{i=1}^{m}\Big((q^i)''(t)+\sum_{j,k=1}^{m}\gamma_{jk}^i(p)(q^j)'(t)(q^k)'(t)\Big)\,\vec{e}_i(p). \tag{20.20}$$

20.4 Geodesic in a coordinate system

Corollary 20.7 *The following proposisitions are equivalent:*

(i) Imc *is a geodesic,*

(ii) c being travelled at constant speed and, for all $i=1,...,m$,

$$(q^i)''+\sum_{j,k=1}^{m}\gamma_{jk}^i\,(q^j)'\,(q^k)'=0, \tag{20.21}$$

meaning $(q^i)''(s)+\sum_{j,k=1}^{m}\gamma_{jk}^i(p)\,(q^j)'(s)\,(q^k)'(s)=0$ for all s and all $i=1,...,m$, with $p=c(s)$.

Proof. (i)⇒(ii). If $\mathrm{Proj}_{TS}(\vec{c}''(s))=0$, then (20.20) gives (20.21).

(ii)⇒(i). If (20.21), then (20.20) implies $\mathrm{Proj}_{TS}(\vec{c}''(s))=0$, so, the speed being constant, c is a geodesic. ∎

Exercise 20.8 Converse of exercise 20.6. Prove with (20.21) that the geodesics on $S=S(0,R)\subset\mathbb{R}^3$ are great circles.

Answer. 1- ODE satisfied by a great circle: Consider a regular curve $t\in]-\varepsilon,\varepsilon[\to c(\theta(t),\varphi(t))\in S$, let $c(0)=p_0$, and choose a basis such that in (20.11) we can take $c=-1$. Thus

$$a\cos\theta(t)+b\sin\theta(t)=\tan\varphi(t).$$

Thus

$$\theta'(-a\sin\theta+b\cos\theta)=\varphi'(1+\tan^2\varphi),$$

i.e., $\theta''(-a\sin\theta+b\cos\theta)+\theta'^2(-a\cos\theta-b\sin\theta)=\varphi''(1+\tan^2\varphi)+2\varphi'^2\tan\varphi(1+\tan^2\varphi)$,
i.e., $\theta''(-a\sin\theta+b\cos\theta)-\theta'\tan\varphi=\varphi''(1+\tan^2\varphi)+2\varphi'\theta'\tan\varphi(-a\sin\theta+b\cos\theta)$,
i.e.θ and φ satisfy the ODE

$$(\theta''-2\varphi'\theta'\tan\varphi)(-a\sin\theta+b\cos\theta)=\frac{1}{\cos^2\varphi}(\varphi''+\theta'\cos\varphi\sin\varphi). \tag{20.22}$$

2- And consider a geodesic $t\in]-\varepsilon,\varepsilon[\to c(\theta(t),\varphi(t))\in S$ with $||\vec{c}'(t)||$ =cste. Then (20.21) gives

$$\begin{cases}\theta''-2\theta'\varphi'\tan\varphi=0,\\ \varphi''+\theta'^2\sin\varphi\cos\varphi=0.\end{cases}$$

And these equations trivially satisfy (20.22). ∎

20.5 Geodesic: The shortest curve

Consider $\mathbb{R}^n$ with a Euclidean dot product that defines the usual metric $(\cdot,\cdot)_g$, and $||\vec{v}||=\sqrt{g(\vec{v},\vec{v})}$. Let $c:[a,b]\to S$ be a regular curve in S. Its length is

$$L(c)=\int_a^b||\vec{c}'(t)||\,dt\overset{\text{written}}{=}\tilde{L}(\mathrm{Im}c), \tag{20.23}$$

the length being independent of the parametrization.

Proposition 20.9 *Let A and B be two close points in S, and let $\mathcal{C}$ be the set of regular curves in S from A to B. The curve c realizing the $\min_{c\in\mathcal{C}}(L(c))$ is a geodesic.*

Proof. Consider a family of curves $\alpha_u : t \in [a,b] \to \alpha_u(t) \in S$ for $u \in [-1,1]$ in $\mathcal{C}$ s.t. $\alpha_u(a) = A$ and $\alpha_u(b) = B$ for all $u \in [0,1]$. Let $\alpha(u,t) := \alpha_u(t)$ (defined on $[-1,1] \times [a,b]$). The length of α_u is

$$\ell(u) = L(\alpha_u) = \int_a^b ||\vec{\alpha}_u{}'(t)||\,dt = \int_a^b ||\frac{\partial\alpha}{\partial t}(u,t)||\,dt. \tag{20.24}$$

The curves being regular and $[a,b]$ being compact, we get $\ell'(u) = \int_a^b \frac{\partial}{\partial u}(||\frac{\partial\alpha}{\partial t}(u,t)||)\,dt$. And

$$\frac{\partial}{\partial u}(||\frac{\partial\alpha}{\partial t}(u,t)||) = \frac{\partial}{\partial u}((\frac{\partial\alpha}{\partial t}(u,t), \frac{\partial\alpha}{\partial t}(u,t))_{\mathbb{R}^n})^{\frac{1}{2}}) = \frac{(\frac{\partial^2\alpha}{\partial u\partial t}(u,t), \frac{\partial\alpha}{\partial t}(u,t))_{\mathbb{R}^n}}{||\frac{\partial\alpha}{\partial t}(u,t)||}.$$

Thus,

$$\begin{aligned}\ell'(u = & \int_a^b (\frac{\partial^2\alpha}{\partial u\partial t}(u,t), \vec{w}_u(t))_{\mathbb{R}^n}\,dt, \quad \text{where} \quad \vec{w}_u(t) = \frac{\alpha_u{}'(t)}{||\alpha_u{}'(t)||}, \\ = & -\int_a^b (\frac{\partial\alpha}{\partial u}(u,t), \vec{w}_u{}'(t))_{\mathbb{R}^n}\,dt + (\frac{\partial\alpha}{\partial u}(u,b), \vec{w}_u(b))_{\mathbb{R}^n} - (\frac{\partial\alpha}{\partial u}(u,a), \vec{w}_u(a))_{\mathbb{R}^n}.\end{aligned} \tag{20.25}$$

And $\alpha(u,a) = B$ constant for all u, thus $\frac{\partial\alpha}{\partial u}(u,a) = 0$, idem $\frac{\partial\alpha}{\partial u}(u,b) = 0$. Thus a curve α_{u_0} realizing the minimum satisfies

$$\ell'(u_0) = 0 = \int_a^b (\frac{\partial\alpha}{\partial u}(u_0,t), \vec{w}_{u_0}{}'(t))_{\mathbb{R}^n}\,dt = 0.$$

And this is true for all $\int_{\tilde{a}}^{\tilde{b}}$ with $[\tilde{a},\tilde{b}] \subset [a,b]$. Thus $\frac{\partial\alpha}{\partial u}(u_0,t) \perp \vec{w}_{u_0}{}'(t)$ for all t. Considering all the family of curves we get $\vec{w}_{u_0}{}'(t) \perp T_pS$ with $p = c_{u_0}(t)$, that is $\text{Proj}_{T_pS}\vec{w}_{u_0}{}'(t) = 0$ for all t.

With $\vec{w}_u(t) = z(t)\alpha_u{}'(t)$ where $z(t) = ||\alpha_u{}'(t)||^{-1} = (\alpha_u{}'(t), \alpha_u{}'(t))_g^{-\frac{1}{2}}$, thus $z'(t) = (-\frac{1}{2})2(\alpha_u{}''(t), \alpha_u{}'(t))_g(\alpha_u{}'(t), \alpha_u{}'(t))_g^{-\frac{3}{2}} = -||\alpha_u{}'(t)||^{-3}(\alpha_u{}''(t), \alpha_u{}'(t))_g$. Thus $\vec{w}_u{}' = z'\alpha_u{}' + z\alpha_u{}''$ gives

$$\vec{w}_u{}' = -||\alpha_u{}'||^{-3}(\alpha_u{}'', \alpha_u{}')_g\alpha_u{}' + ||\alpha_u{}'||^{-1}\alpha_u{}'')$$

Thus $\text{Proj}_{TS}\vec{w}_{u_0}{}' = 0 = \text{Proj}_{TS}(||\alpha_{u_0}{}'||^3\vec{w}_{u_0}{}')$ gives $\text{Proj}_{TS}(||\alpha_{u_0}{}'||^2\alpha_{u_0}{}'') = (\alpha_{u_0}{}'', \alpha_{u_0}{}')_g\alpha_{u_0}{}'$, that is (20.7): α_{u_0} is a geodesic. ∎

20.6 Geodesic: The minimum energy curve

Consider the (kinematic) energy along c:

$$E(c) = \frac{1}{2}\int_a^b ||\vec{c}'(t)||^2\,dt. \tag{20.26}$$

NB: The energy E depends on the parametrization of Imc. Indeed, if $t : u \in [c,d] \to t(u) \in [a,b]$ is a diffeormophism, if $\alpha(u) = c(t(u))$, then $\vec{\alpha}'(u) = \vec{c}'(t(u))\,t'(u)$, and

$$\int_{t\in[a,b]} ||\vec{c}'(t)||^2\,dt = \int_{u\in[c,d]} ||\vec{\alpha}'(u)||^2 \frac{1}{t'(u)^2}\,|t'(u)|\,du \neq \int_{u\in[c,d]} ||\vec{\alpha}'(u)||^2\,du, \tag{20.27}$$

unless $|t'(u)| = 1$.

Proposition 20.10 *Let A and B be two close points in S. Let $\mathcal{C}$ be the set of regular curves c connecting A and B such that $||c'(t)|| = 1$ (intrinsic parametrization). If a curve realizes $\min_{c\in\mathcal{C}} E(c)$ then this curve is a geodesic.*

Proof. Consider a family of curves $(c_u : [a,b] \to S)_{u\in[-1,1]}$ in $\mathcal{C}$ (so $c_u(a) = A$ and $c_u(b) = B$ for all $u \in [0,1]$ and $||\vec{c}_u{}'(t)|| = 1$ for all t). Let $c(u,t) := c_u(t)$ (defined on $[-1,1] \times [a,b]$). Thus

$$E(c_u) = \frac{1}{2}\int_a^b ||\frac{\partial c}{\partial t}(u,t)||^2\,dt. \tag{20.28}$$

And $\frac{\partial}{\partial u}(||\frac{\partial c}{\partial t}(u,t)||^2) = \frac{\partial}{\partial u}(\frac{\partial c}{\partial t}(u,t), \frac{\partial c}{\partial t}(u,t))_{\mathbb{R}^n} = 2(\frac{\partial^2 c}{\partial u\partial t}(u,t), \frac{\partial c}{\partial t}(u,t))_{\mathbb{R}^n}$ gives

$$\begin{aligned}\frac{d}{du}(E(c_u)) = & \int_a^b (\frac{\partial^2 c}{\partial u\partial t}(u,t), \frac{\partial c}{\partial t}(u,t))_{\mathbb{R}^n}\,dt \\ = & -\int_a^b (\frac{\partial c}{\partial u}(u,t), \frac{\partial^2 c}{\partial t^2}(u,t))_{\mathbb{R}^n}\,dt + [(\frac{\partial c}{\partial u}(u,t), \frac{\partial c}{\partial t}(u,t))_{\mathbb{R}^n}]_{t=a}^b.\end{aligned}$$

And $c(u,a) = B$ constant for all u, thus $\frac{\partial c}{\partial u}(u,a) = 0$, idem $\frac{\partial c}{\partial u}(u,0) = 0$. And this is true for all $\int_c^d$

with $[c,d]\subset[a,b]$. Thus $\frac{\partial c}{\partial u}(u_0,t)\perp\frac{\partial^2 c}{\partial t^2}(u_0,t)$, for all $t\in[a,b]$. Considering all the family of curves we get $\mathrm{Proj}_{TS}\frac{\partial^2 c}{\partial t^2}(u_0,t)=0$ for all t: c_{u_0} is a geodesic. ▪

Remark 20.11

$$L^2(c)\le\frac{1}{2(b-a)}E(c). \tag{20.29}$$

Indeed, Cauchy–Schwarz theorem gives

$$L(c)=\int_a^b||\vec c\,'(t)||\,dt=(1_{[a,b]},||\vec c\,'||)_{L^2(]a,b[)}\le||1||_{L^2(]a,b[)}||\vec c\,'||_{L^2(]a,b[)},$$

with $||1||^2_{L^2(]a,b[)}=\int_a^b 1^2\,dt=b-a$ and $||\vec c\,'||^2_{L^2(]a,b[)}=\int_a^b||\vec c\,'(t)||^2\,dt=2E$. ▪

20.7 Geodesic and normal curvatures in $\mathbb{R}^3$

Let $\Phi:U\subset\vec{\mathbb{R}^2}\to S\subset\mathbb{R}^3$ be a coordinate system of a dimension 2 regular surface S, let $(\vec A_1,\vec A_2)$ be the canonical basis in $\vec{\mathbb{R}^2}$, let $(\vec e_i(p))=(d\Phi(\vec q).\vec A_i)$ he the coordinate system basis at $p=\Phi(\vec q)$.

Consider $\mathbb{R}^3$ with a Euclidean dot product. Let $\vec n(p)$ be the unit normal vector at T_pS such that $(\vec e_1(p),\vec e_2(p),\vec n(p))$ is a direct basis, that is,

$$\vec n(p)\perp T_pS,\quad ||\vec n_p||=1,\quad \det(\vec e_1(p),\vec e_2(p),\vec n(p))>0. \tag{20.30}$$

Consider a curve $c:\left\{\begin{array}{ll}[0,L]&\to S\\ s&\to p=c(s)\end{array}\right\}$ with s an intrinsic parameter: At any $p=c(s)$,

$$\vec v(p)=\vec c\,'(s)\overset{\text{written}}{=}\vec t(p),\quad\text{and}\quad ||\vec t(p)||=||\vec v(p)||=1, \tag{20.31}$$

$\vec t(p)$ being the tangent unit vector at p along c.

Definition 20.12 For c, the normal geodesic vector at $p=c(s)$ is the vector $\vec n_g(p)\in T_pS$ (tangent to S) defined by

$$\vec n_g(p)=\vec n(p)\wedge\vec t(p)\ \in T_pS. \tag{20.32}$$

Thus $(\vec n(p),\vec t(p),\vec n_g(p))$ is a direct orthogonal basis ins $\mathbb{R}^3$ (Serret–Frénet basis $(\vec t(p),\vec n_g(p),\vec n(p))$).
Since $||\vec c\,'(s)||^2_{\mathbb{R}^n}=1$ gives $(\vec c\,'(s),\vec c\,''(s))_{\mathbb{R}^n}=0$, the acceleration $\vec c\,''(s)$ satisfies

$$\vec c\,''(s)\perp\vec t(p), \tag{20.33}$$

thus

$$\vec c\,''(s)=\kappa_n(p)\vec n(p)+\kappa_g(p)\vec n_g(p)\quad\in\mathrm{Vect}\{\vec n(p)\}\oplus^\perp\mathrm{Vect}\{\vec n_g(p)\} \tag{20.34}$$

where

$$\left\{\begin{array}{l}\kappa_n(p)=(\vec c\,''(s),\vec n(p))_{\mathbb{R}^3},\\ \kappa_g(p)=(\vec c\,''(s),\vec n_g(p))_{\mathbb{R}^3},\end{array}\right\}\quad\text{and}\quad\kappa_g(p)\vec n_g(p)=\mathrm{Proj}_{T_pS}(\vec c\,''(s)). \tag{20.35}$$

Definition 20.13
The normal curvature at p is $\kappa_n(p)$ (normal acceleration when $||\vec v(p)||=1$),
The geodesic curvature at p is $\kappa_g(p)$ (tangential acceleration in S when $||\vec v(p)||=1$).

(If c is a geodesic, then $\kappa_g=0$.)
(if c is not a geodesic then $|\kappa_g(p)|=||\vec c\,''(s)\wedge\vec n(p)||$ gives a measure of the "rotation about $\vec n(p)$").

Example 20.14 Sphere $S=S(\vec 0,R)$, and GPS coordinates. A parallel in intrinsic curvilinear coordinate is a curve

$$c(s)=\begin{pmatrix}R\cos(\frac{s}{R\cos\varphi})\cos\varphi\\ R\sin(\frac{s}{R\cos\varphi})\cos\varphi\\ R\sin\varphi\end{pmatrix}.$$

(Intrinsic since $||\vec c\,'(s)||=||\begin{pmatrix}-\sin(\frac{s}{R\cos\varphi})\\ \cos(\frac{s}{R\cos\varphi})\\ 0\end{pmatrix}||=1$.) The acceleration is $\vec c\,''(s)=-\frac{1}{R\cos\varphi}\begin{pmatrix}\cos\frac{s}{R\cos\varphi}\\ \sin\frac{s}{R\cos\varphi}\\ 0\end{pmatrix}$. The unit normal vector s.t. $(\vec e_1(p),\vec e_2(p),\vec n_p)$ is direct (here $\vec e_1(p)$ is along a parallel and $\vec{e}_2(p)$ is along a meridian) is $\vec{n}(p) = \begin{pmatrix} \cos\frac{s}{R\cos\varphi}\cos\varphi \\ \sin\frac{s}{R\cos\varphi}\cos\varphi \\ \sin\varphi \end{pmatrix}$. Thus

$$(\vec{c}''(s), \vec{n}(p))_{\mathbb{R}^3} = \frac{-1}{R} = \kappa_n(p). \tag{20.36}$$

And

$$\vec{c}''(s) + \frac{1}{R}\vec{n}(p) = \frac{1}{R}\tan\varphi \begin{pmatrix} \cos\frac{s}{R\cos\varphi}\sin\varphi \\ \sin\frac{s}{R\cos\varphi}\sin\varphi \\ \cos\varphi \end{pmatrix} = \frac{1}{R}\tan\varphi\,\vec{n}_g. \tag{20.37}$$

Thus

$$\kappa_g = \frac{1}{R}\tan\varphi. \tag{20.38}$$

In particular on the equator, $\varphi = 0$ and $\kappa_g = 0$ (the equator is a geodesic). And κ_g increases with φ. ▪

21 Parallel transport in $S \subset \mathbb{R}^n$

21.1 Definition

In the 2-sphere $S = S(\vec{0}, R)$ in $\mathbb{R}^3$, a vector field in S cannot be uniform: The sphere is not flat, so the direction of a vector in TS changes when moving along a curve, from the point of view of $\mathbb{R}^3$.

Definition 21.1 Let S be a m differential manifold. Let ∇ be a connection in S. A vector field $\vec{w} \in \Gamma(S)$ is parallel transported in S (relative to ∇) along a curve $c : t \in [a,b] \to c(t) \in S$ iff, for all $p = c(t) \in c([a,b])$,

$$(\nabla_{\vec{v}}\vec{w})(p) = 0, \quad \text{i.e.} \quad \frac{D\vec{w}}{dt}(p) = 0. \tag{21.1}$$

E.g., with S a surface in $\mathbb{R}^n$ and ∇ is the Riemannian usual connection, that is given by $\nabla_{\vec{v}}\vec{w} = \text{Proj}_{TS}(d\vec{w}.\vec{v})$, then along a curve c, and with $\vec{v}(c(t)) = \vec{c}'(t)$, a vector field $\vec{w} \in TS$ is parallel transported along c iff

$$\text{Proj}_{TS}(d\vec{w}.\vec{v}) = 0. \tag{21.2}$$

Example 21.2 Sphere, see figure 21.1, A = the north pole, B = the south pole. Consider two meridians $c_i : [0,\pi] \to S$ connecting A and B, $i = 1,2$, the meridians making an angle of 90 degrees in the plane tangent at the north pole. Let $\vec{v}_i(c(t)) = {c_i}'(t)$ be the velocities. The parallel transport of $\vec{v}_1(A)$ along $\vec{c}_1$ (geodesic) gives $\vec{v}_1(B)$ (tangent vector to $\vec{c}_1$ at B). Whereas the parallel transport of $\vec{v}_1(A)$ (orthogonal to $\vec{v}_2(A)$) along $\vec{c}_2$ (geodesic) gives $\vec{w}_2(B)$ (orthogonal to $\vec{v}_2(B)$): And $\vec{v}_2(A) = -\vec{w}_2(B)$; Thus the result of a parallel transportation depends on the curve chosen to go from A to B. ▪

Remark 21.3 Beware of vocabulary: a parallel on Earth is "parallel to the equator", but along such a parallel a vector is not "parallel tranported" since a parallel is not a geodesic. See proposition 22.15 and example 22.17. ▪

Proposition 21.4 *Let $c : [a,b] \to S$ be a curve in S surface in $\mathbb{R}^n$, and consider the usual Riemannian connection. Then the parallel transport along c in S does not depend on the parametrization of c.*

Proof. Let $\alpha : s \in [\tilde{a}, \tilde{b}] \to p = \alpha(s) \in S$ be another parametrization of Imc, and let $t : s \in [\tilde{a}, \tilde{b}] \to t(s) \in [a,b]$ be the change of parameter (diffeomorphism); So $\alpha = c \circ t$, that is $\alpha(s) = c(t(s)) = p$. And $\vec{w}(p) = (\vec{w} \circ \alpha)(s) = (\vec{w} \circ c)(t(s))$ gives

$$(\vec{w} \circ \alpha)'(s) = (\vec{w} \circ c)'(t(s))\, t'(s).$$

Thus $\text{Proj}_{T_pS}(\vec{w} \circ \alpha)'(s) = \text{Proj}_{T_pS}(\vec{w} \circ c)'(t(s))\, t'(s)$, that is, $\frac{D\vec{w}}{ds}(p) = \frac{D\vec{w}}{dt}(p)\, t'(s)$; Thus $\frac{D\vec{w}}{ds}(p) = 0$ iff $\frac{D\vec{w}}{dt}(p) = 0$. ▪

With the basis of a coordinate system, (7.6) gives $\frac{D\vec{w}}{dt} = \nabla_{\vec{v}}\vec{w} = \sum_{i,j=1}^m v^j w^i_{|j}\,\vec{e}_i = \sum_{i,j=1}^m (\frac{\partial w^i}{\partial q^j} + \sum_{k=1}^m w^k \gamma^i_{jk}) v^j \vec{e}_i$. Thus $\vec{w}$ is parallel transported along a curve c in S iff, for all i,

$$\sum_{j=1}^m w^i_{|j} v^j = 0, \qquad \text{i.e.} \qquad \sum_{j=1}^m (\frac{\partial w^i}{\partial q^j} + \sum_{k=1}^m \gamma^i_{jk} w^k) v^j = 0. \tag{21.3}$$

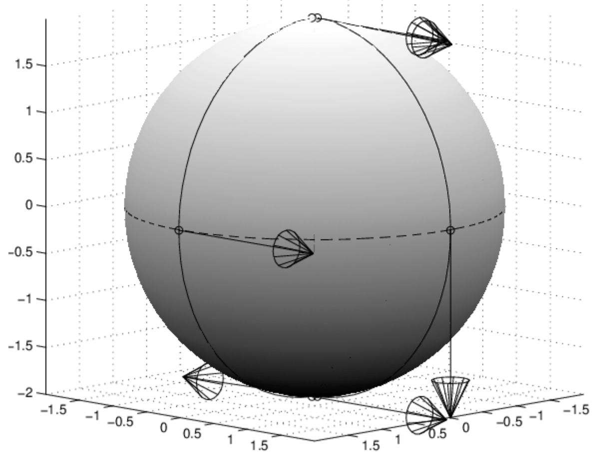

Figure 21.1: Parallel transport in the sphere along two meridians (at 0 and 90 degrees on the figure): The parallel transport along the two meridians of a vector at the north pole gives two different vectors at the south pole.

Remark 21.5 We cannot confuse

• The parallel transport equation $\nabla_{\vec{v}}\vec{w} = \frac{D\vec{w}}{dt} = 0$ which is independent of a parametrization of the curve, cf. proposition 21.4, with

• The geodesic equation $\nabla_{\vec{v}}\vec{v} = 0$ which requires $||\vec{v}|| = 1$ (intrinsic parametrization): Cf. (20.9) if the parameter is not intrinsic. ▪

Remark 21.6 The Levi–Civita theorem 22.23 will states that: If the connection parallel transports the metric, then proposition 21.4 will be valid (case of a usual Riemannian metric and connection in $\mathbb{R}^n$). ▪

22 The parallel transport operator

22.1 The shifter $J_t^{t_0}$ in $\mathbb{R}^n$

Let $a < b$, let $c : t \in [a, b] \to c(t) \in \mathbb{R}^n$ be a regular curve in $\mathbb{R}^n$, let $t_0, t \in]a, b[$, $p_{t_0} = c(t_0)$ and $p_t = c(t)$.

Definition 22.1 The shifter between t_0 and t along c is the translation

$$J_{c,t}^{t_0} : \begin{cases} \{p_{t_0}\} \times T_{p_{t_0}}\mathbb{R}^n & \to \{p_t\} \times T_{p_t}\mathbb{R}^n \\ (p_{t_0}, \vec{w}_{p_{t_0}}) & \to J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) := (p_t, \vec{w}_{p_t}), \quad \text{where} \quad \vec{w}_{p_t} = \vec{w}_{p_{t_0}}, \end{cases} \tag{22.1}$$

that is, $J_{c,t}^{t_0}$ translates the vector $\vec{w}_{p_{t_0}}$ at p_{t_0} at t_0 toward the vector $\vec{w}_{p_t} = \vec{w}_{p_{t_0}}$ at p_t at t.

Thus, if $A = p_{t_0} = c(t_0) = \tilde{c}(t_0)$ and $B = p_t = c(t) = \tilde{c}(t)$ are two points connected by two curves c and $\tilde{c}$, then $J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) = J_{\tilde{c},t}^{t_0}(\vec{w}_{p_{t_0}})$: The result is independent of the curves that connect A and B:

$$J_{c,t}^{t_0} \overset{\text{written}}{=} J_t^{t_0}. \tag{22.2}$$

Hence, with (22.1), we define $J^{t_0} : \left\{ \begin{array}{ll}]a, b[& \to \mathcal{F}(TS_{t_0}, T\mathbb{R}^n) \\ t & \to J^{t_0}(t) := J_t^{t_0} \end{array} \right\}$, and $J^{t_0}(t)(\vec{w}_{p_{t_0}}) =^{\text{written}} J^{t_0}(t, \vec{w}_{p_{t_0}})$.

If there is no ambiguity, then (shorten notation)

$$J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) \overset{\text{written}}{=} \vec{w}_{p_t}. \tag{22.3}$$

Example 22.2 In $\mathbb{R}^2$, with $c : t \to O + \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix}$, if $t_0 = 0$ and $\vec{w}_{p_{t_0}} = \vec{E}_2$ at $p_{t_0} = O + \vec{E}_2$, then at $t = \frac{\pi}{2}$ we have $p_t = O + \vec{E}_2$ and $J_t^{t_0}(\vec{w}_{p_{t_0}}) = \vec{E}_2$ (parallel transport in $\mathbb{R}^2$); And, relative to the tangent vector $\vec{v}(p_t) = \vec{c}\,'(t)$, the vector $\vec{w}_{p_t}$ "turns to the right as t increases". ▪

22.2 Parallel transport $J^{t_0}_{c,t}$ in S

Setting (obvious but has to be given): Let S be a surface in $\mathbb{R}^n$, let $t_0,T \in \mathbb{R}$ s.t. $t_0 < T$, let S_{t_0} be an open set in S, let $\Phi^{t_0} : [t_0,T] \times S_{t_0} \to S$ be a regular map (a motion). Let $t \in [t_0,T]$, let $\Phi^{t_0}(t,p_{t_0}) =^{\text{written}} \Phi^{t_0}_t(p_t)$, let $S_t = \Phi^{t_0}_t(S_{t_0})$, and $\Phi^{t_0}_t : S_{t_0} \to S_t$ is supposed to be a diffeomorphism. Let $c : [t_0,T] \to S$ be a regular curve in S such that $c(t) \in S_t$ for all t. And let ∇ be the usual Riemannian connection ∇ in S.

Proposition 22.3 *Let $p_{t_0} = c(t_0)$, $\vec{w}_{p_{t_0}} \in T_{p_{t_0}}S$, and $p_t = c(t)$. There exists a unique vector field $\vec{w}_c$ defined at all points p_t ($\vec{w}_c \in \Gamma(\text{Im}c)$) such that*

$$(\nabla_{\vec{v}}\vec{w}_c(p_t)) =) \quad \frac{D\vec{w}_c}{dt}(p_t) = \vec{0} \quad \text{and} \quad \vec{w}_c(p_{t_0}) = \vec{w}_{p_{t_0}}. \tag{22.4}$$

(E.g., see figure 22.1.)

Proof. (22.4) is an ODE with initial condition: Apply Cauchy–Lipschitz theorem. ∎

Definition 22.4 $\vec{w}_c$ being the solution of (22.4), the vector $\vec{w}_c(p_t)$ at $p_t = c(t)$ at t, is called the parallel transported vector from $\vec{w}_{p_{t_0}}$ at $p_{t_0} = c(t_0)$ at t_0 in S along c. And $\vec{w}_c(p_t) =^{\text{written}} J^{t_0}_{c,t}(\vec{w}_{p_{t_0}})$. This defines the shifter (the "parallel transport operator") $J^{t_0}_{c,t}$ along c from $T_{p_{t_0}}S_{t_0}$ to $T_{p_t}S_t$:

$$J^{t_0}_{c,t} : \begin{cases} \{p_{t_0}\} \times T_{p_{t_0}}S_{t_0} & \to \{p_t\} \times T_{p_t}S_t \\ (p_{t_0}, \vec{w}_{p_{t_0}}) & \to J^{t_0}_{c,t}(\vec{w}_{p_{t_0}}) := (p_t, \vec{w}_{p_t}), \quad \text{where} \quad \vec{w}_{p_t} = \vec{w}_c, \end{cases} \tag{22.5}$$

and the shifter (the "parallel transport operator") $J^{t_0}_c$ along c.

$$J^{t_0}_c : \begin{cases}]a,b[\times\{p_{t_0}\} \times T_{p_{t_0}}S & \to TS \\ (t, p_{t_0}, \vec{w}_{p_{t_0}}) & \to J^{t_0}_c(t, \vec{w}_{p_{t_0}}) := (p_t, \vec{w}_c(p_t)) \stackrel{\text{written}}{=} J^{t_0}_{c,t}(\vec{w}_{p_{t_0}}), \end{cases} \tag{22.6}$$

Shorten notation:

$$J^{t_0}_{c,t}(\vec{w}_{p_{t_0}}) = J^{t_0}_c(t, \vec{w}_{p_{t_0}}) = \vec{w}_{p_t} = \vec{w}_c. \tag{22.7}$$

Example 22.5 In $\mathbb{R}^2$, with $c : t \to p_t = c(t) = O + \begin{pmatrix} x = R\cos t \\ y = R\sin t \end{pmatrix}$; We have $[\vec{v}(p_t)]_{|\vec{E}} = \begin{pmatrix} -R\sin t = -y \\ R\cos t = x \end{pmatrix}$, and $T_{p_t}S = \text{Vect}\{\vec{v}(p_t)\}$; Let $t_0 = 0$, $p_{t_0} = c(0) = O + \vec{E}_1$, and $\vec{w}_{p_{t_0}} = \vec{E}_2 \in T_{p_{t_0}}S$; We look for $\vec{w}_c \in TS$, so $\vec{w}_c(p_t) = w(p_t)\vec{v}(p_t)$, s.t. $\text{Proj}_{T_{p_t}S}(d\vec{w}(p_t).\vec{v}(p_t)) = \vec{0}$ and $\vec{w}_c(p_{t_0}) = \vec{E}_2$.

We have $d\vec{w}(p_t).\vec{v}(p_t) = (dw(p_t).\vec{v}(p_t))\vec{v}(p_t) + w(p_t)d\vec{v}(p_t).\vec{v}(p_t)$, with $d\vec{v} = \sum_{i,j} \frac{\partial v^i}{\partial x^j}\vec{E}_i \otimes E^j = -\vec{E}_1 \otimes E^2 + \vec{E}_2 \otimes E^1$, thus $d\vec{v}.\vec{v} = -x\vec{E}_1 - y\vec{E}_2 \perp \vec{v}$, thus $\text{Proj}_{T_pS}(d\vec{w}(p_t).\vec{v}(p_t)) = dw(p_t).\vec{v}(p_t)\vec{v}(p_t)$ vanishes iff $dw(p_t).\vec{v}(p_t) = 0 = \frac{\partial w}{\partial x}(-y) + \frac{\partial w}{\partial y}(x)$, i.e. $y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0$: A solution is $w = k$ for any $k \in \mathbb{R}$. Thus $\vec{w}_c = k\vec{v}$, and $\vec{w}_{p_{t_0}} = \vec{E}_2$ gives $k = \frac{1}{R}$, thus $\vec{w}_c = \frac{1}{R}\vec{v}$: The parallel transported vector is always tangent to c (it is in TS) and keeps its length, that is, $\vec{w}_c$ "turns with the circle". ∎

Example 22.6 See § 22.6.4. ∎

22.3 The shifter is $\mathbb{R}$-linear

Proposition 22.7 $J^{t_0}_{c,t}$ *is $\mathbb{R}$-linear. Thus $dJ^{t_0}_{c,t}(\vec{w}_{p_{t_0}}) = J^{t_0}_{c,t}$ for all $\vec{w}_{t_0} \in T_{p_{t_0}}S$.*

Proof. Let $\vec{w}_{p_{t_0}}, \vec{z}_{p_{t_0}} \in T_{p_{t_0}}S$ and $h \in \mathbb{R}$. The ODE (22.4) being linear (the right hand side is linear since it vanishes), we get $J^{t_0}_{c,t}(\vec{w}_{p_{t_0}} + h\vec{z}_{p_{t_0}}) = J^{t_0}_{c,t}(\vec{w}_{p_{t_0}}) + hJ_{t,t_0}(\vec{z}_{p_{t_0}})$. ∎

$J^{t_0}_{c,t}$ being linear, $J^{t_0}_{c,t}(\vec{w}_{p_{t_0}}) =^{\text{written}} J^{t_0}_{c,t}.\vec{w}_{p_{t_0}}$.
Quantification with a coordinate basis:

$$J^{t_0}_{c,t} = \sum_{i,j=1}^{m} (J^{t_0}_{c,t})^i_j \, \vec{e}_i(p_t) \otimes e^j(p_{t_0}), \tag{22.8}$$

so, with $\vec{w}_{p_{t_0}} = \sum_{j=1}^m w^j_{p_{t_0}} \, \vec{e}_j(p_{t_0})$, $\vec{w}_c(p_t) = J^{t_0}_{c,t}.\vec{w}_{p_{t_0}} = \sum_{i,j=1}^m (J^{t_0}_{c,t})^i_j w^j_{p_{t_0}}(p_{t_0}) \, \vec{e}_i(p_t)$, i.e.,

$$[\vec{w}_c(p_t)]_{|\vec{e}_i(p_t)} = [J^{t_0}_{c,t}]_{|\vec{e}_i(p_{t_0}),\vec{e}_i(p_t)}.[\vec{w}_{p_{t_0}}]_{|\vec{e}_i(p_{t_0})}. \tag{22.9}$$

22.4 Composition of shifters S

Let $J_c(t,t_0) := J^{t_0}_{c,t}$.

Proposition 22.8 *c being a regular curve in S, we have, for all $t,t_1 \in]a,b[$,*

$$J^t_{c,t_1} \circ J^{t_0}_{c,t} = J^{t_0}_{c,t_1}, \quad \text{i.e.} \quad J_c(t_1,t)\circ J_c(t,t_0) = J_c(t_1,t). \tag{22.10}$$

(Linearity notation: $J^t_{c,t_1}.J^{t_0}_{c,t} = J^{t_0}_{c,t_1}$, or $J_c(t_1,t).J_c(t,t_0) = J_c(t_1,t)$.) In particular, $(J^{t_0}_{c,t})^{-1} = J^t_{c,t_0}$.

Proof. The existence and uniqueness (Cauchy–Lipschitz theorem) gives (22.10). ∎

22.5 Interpretation

Reminder: In $\mathbb{R}^n$, with $p_t = c(t)$ and $\vec{v}(p_t) = \vec{c}\,'(t)$, cf. (6.1),

$$d\vec{w}(p_{t_0}).\vec{v}(p_{t_0}) = \frac{d(\vec{w}\circ c)}{dt}(t_0) = \lim_{t\to t_0}\frac{\vec{w}(c(t)) - \vec{w}(c(t_0))}{t-t_0} = \lim_{t\to t_0}\frac{\vec{w}(p_t)-\vec{w}(p_{t_0})}{t-t_0}. \tag{22.11}$$

But the rate $\frac{\vec{w}(p_t)-\vec{w}((p_{t_0})}{t-t_0}$ is meaningless with the mathematical definition of a vector field, cf. (1.18), since the difference $\vec{w}(p_t)-\vec{w}(p_{t_0})$ is a nonsense: Not only $T_{p_{t_0}}S \neq T_{p_t}S$ in general, but with the full notation of a vector field, cf. (1.18), (22.11) reads

$$(p_{t_0}, \frac{D\vec{w}}{dt}(p_{t_0})) = \lim_{t\to t_0}\frac{(p_t,\vec{w}(p_t)) - (p_{t_0},\vec{w}(p_{t_0}))}{t-t_0}, \tag{22.12}$$

and the difference in the limit is meaningless since the base points p_t and p_{t_0} are different. In $\mathbb{R}^n$, (22.12) is made meaningful thanks to the shifter $J^{t_0}_t$, cf. (22.2):

$$\begin{aligned}(p_{t_0}, \frac{D\vec{w}}{dt}(p_{t_0})) &= \lim_{t\to t_0}\frac{(p_{t_0}, {J^{t_0}_t}^{-1}(\vec{w}(p_t)) - (p_{t_0},\vec{w}(p_{t_0}))}{t-t_0}\\ &= \lim_{t\to t_0}\frac{(p_t,\vec{w}(p_t) - (p_t, J^{t_0}_t(\vec{w}(p_{t_0})))}{t-t_0},\end{aligned} \tag{22.13}$$

simply written as (22.11).

Proposition 22.9 *In a regular surface $S\subset\mathbb{R}^n$, we have*

$$\begin{aligned}(p_{t_0},\nabla_{\vec{v}}\vec{w}(p_{t_0})) = (p_{t_0}, \frac{D\vec{w}}{dt}(p_{t_0})) &= \mathrm{Proj}_{T_{p_{t_0}}S}(\lim_{t\to t_0}\frac{(p_{t_0}, {J^{t_0}_{c,t}}^{-1}.\vec{w}(p_t)) - (p_{t_0},\vec{w}(p_{t_0}))}{t-t_0})\\ &= \mathrm{Proj}_{T_{p_{t_0}}S}(\lim_{t\to t_0}\frac{(p_t,\vec{w}(p_t)) - (p_t, J^{t_0}_{c,t}.\vec{w}(p_{t_0}))}{t-t_0}).\end{aligned} \tag{22.14}$$

(The last equality is not used in a surface since $\vec{w}(p_t) - J^{t_0}_{c,t}.\vec{w}(p_{t_0}) \in T_{p_t}S$ and $T_{p_t}S$ varies as $t\to t_0$.) Simply written:

$$\nabla_{\vec{v}}\vec{w}(p_{t_0}) = \frac{D\vec{w}}{dt}(p_{t_0}) = \mathrm{Proj}_{T_{p_{t_0}}S}(\lim_{t\to t_0}\frac{{J^{t_0}_{c,t}}^{-1}.\vec{w}(p_t) - \vec{w}(p_{t_0})}{t-t_0}). \tag{22.15}$$

Proof. $\nabla_{\vec{v}}\vec{w}(p_{t_0}) = \mathrm{Proj}_{T_{p_{t_0}}S}(d\vec{w}(p_{t_0}).\vec{v}(p_{t_0})) = \mathrm{Proj}_{T_{p_{t_0}}S}(\lim_{t\to t_0}\frac{\vec{w}(\vec{c}(t))-\vec{w}(c(t_0))}{t-t_0})$. Let $\vec{w}_c(p_t) = J^{t_0}_{c,t}(\vec{w}(p_{t_0}))$. Then $\frac{\vec{w}(\vec{c}(t))-\vec{w}(c(t_0))}{t-t_0} = \frac{\vec{w}(\vec{c}(t))-\vec{w}_c(\vec{c}(t))}{t-t_0} + \frac{\vec{w}_c(\vec{c}(t))-\vec{w}(c(t_0))}{t-t_0}$, with $\lim_{t\to t_0}\frac{\vec{w}_c(\vec{c}(t))-\vec{w}(c(t_0))}{t-t_0} = \vec{0}$, cf. (22.4). Thus $\nabla_{\vec{v}}\vec{w}(p_{t_0}) = \frac{D\vec{w}}{dt}(p_{t_0}) = \mathrm{Proj}_{T_{p_{t_0}}S}(\lim_{t\to t_0}\frac{\vec{w}(p_t)-\vec{w}_c(p_{t_0})}{t-t_0})$. And $J^{t_0}_{c,t} = I$ gives $\lim_{t\to t_0}\frac{\vec{w}(p_t)-\vec{w}_c(p_t)}{t-t_0}) = \lim_{t\to t_0}\frac{(J^{t_0}_{c,t})^{-1}.\vec{w}(p_t)-(J^{t_0}_{c,t})^{-1}.\vec{w}_c(p_t)}{t-t_0})$ ∎

22.6 Examples

22.6.1 Parallel transport in a curve in $\mathbb{R}^2$

Let S be a one dimensional regular manifold in $\mathbb{R}^2$ parametrized as a regular curve $c : t \in]a,b[\to c(t)$, $\mathrm{Im}c = S$.

Proposition 22.10 *Let $\vec{w}$ be a vector field in S, that is, $\vec{w}(p) = \alpha(t)\vec{c}\,'(t)$ at $p = c(t)$ with $\alpha \in \mathcal{F}(]a,b[;\mathbb{R})$. If $||\vec{c}\,'(t)|| = 1$ for all t, then $\vec{w}$ is parallel transported along* $\mathrm{Im}(c)$ *iff α is constant. (In particular, the velocity field $\vec{v}(c(t)) := \vec{c}\,'(t)$ is parallel transported along c, and a geodesic in a curve is a part of this curve).*

Proof. $||\vec{c}\,'(t)||^2_{\mathbb{R}^2} = 1$ gives $\frac{d}{dt}(\vec{c}\,'(t),\vec{c}\,'(t))_{\mathbb{R}^2} = 0 = 2(\vec{c}\,''(t),\vec{c}\,'(t))$, thus $\vec{c}\,''(t) \perp \vec{c}\,'(t)$ for all t. And $(\vec{w}\circ c)'(t) = \alpha'(t)\vec{c}\,'(t) + \alpha(t)\vec{c}\,''(t)$ gives $\frac{D\vec{w}}{dt}(t) = \mathrm{Proj}_{T_pS}((\vec{w}\circ c)'(t)) = \alpha'(t)\vec{c}\,'(t)$. Since $\vec{c}\,'(t) \neq \vec{0}$ (its norm equals 1), $\frac{D\vec{w}}{dt}(t) = 0$ iff $\alpha' = 0$. ∎

Example 22.11 (Calculations.) Polar coordinates, $S = C(\vec{0},R) = \mathrm{Im}c$ with $p = c(\theta) = \begin{pmatrix} R\cos\theta \\ R\sin\theta \end{pmatrix}$. So $||\vec{v}(p)|| = ||\vec{c}\,'(t)|| = R = ||\vec{e}_2(p)|| =$ constant. Let $\vec{w} \in \Gamma(S)$ given by, with $p = c(\theta)$,

$$\vec{w}(p) = \beta(p)\vec{e}_2(p), \quad \text{so} \quad d\vec{w} = d\beta\,\vec{e}_2 + \beta\,d\vec{e}_2.$$

Thus

$$d\vec{w}.\vec{e}_2 = (d\beta.\vec{e}_2)\vec{e}_2 + \beta(d\vec{e}_2.\vec{e}_2) = \frac{\partial\beta}{\partial\theta}\vec{e}_2 + \beta(\gamma^1_{22}\vec{e}_1 + \gamma^2_{22}\vec{e}_2).$$

Thus

$$\nabla_{\vec{e}_2}\vec{w} = \mathrm{Proj}_{TS}(d\vec{w}.\vec{e}_2) = (\frac{\partial\beta}{\partial\theta} + \beta\gamma^2_{22})\vec{e}_2. \tag{22.16}$$

With $\gamma^2_{22} = 0$, cf. (3.9), thus $\frac{D\vec{w}}{dt} = 0$ iff $\frac{\partial\beta}{\partial\theta} = 0$, i.e., iff $\beta = \beta_0$ is constant. In particular, $\vec{v} = \vec{e}_2$ is parallel transported along itself. ∎

22.6.2 Parallel transport in a cylinder in $\mathbb{R}^3$

Surface $\Phi(\theta,z) = R\cos\theta\vec{E}_1 + R\sin\theta\vec{E}_2 + z\vec{E}_3 =^{\text{written}} \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ z \end{pmatrix}$, and $S = \mathrm{Im}\Phi$ (cylinder). Basis at $p = \Phi(\theta,z)$ in T_pS: $\vec{e}_1(p) = \frac{\partial\Phi}{\partial\theta}(\theta,z) = \begin{pmatrix} -R\sin\theta \\ R\cos\theta \\ 0 \end{pmatrix}$ and $\vec{e}_2(p) = \vec{E}_3 =^{\text{written}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Let $a \neq 0$ and consider $c(t) = \Phi(t,at) = \begin{pmatrix} R\cos t \\ R\sin t \\ at \end{pmatrix} \in S$ (spiral). At $p = c(t)$, $\vec{e}_1(p) = \begin{pmatrix} -R\sin t \\ R\cos t \\ 0 \end{pmatrix}$ and $\vec{e}_2(p) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Consider the vector fields $\vec{e}_1$ and $\vec{e}_2$ in S; $\frac{D\vec{e}_1}{dt}(p) = \mathrm{Proj}_{T_pS}(\frac{d(\vec{e}_1\circ c]}{dt}(t)) = \mathrm{Proj}_{T_pS}(\begin{pmatrix} -R\cos t \\ -R\sin t \\ 0 \end{pmatrix}) = \vec{0}$ et $\frac{D\vec{e}_2}{dt}(p) = \mathrm{Proj}_{T_pS}(\vec{0}) = \vec{0}$. Thus any vector field $\vec{w} = \alpha\vec{e}_1(p) + \beta\vec{e}_2(p)$ with α and β constants is parallel transported along $\mathrm{Im}c$.

And a vector field in S reads $\vec{w}(p) = \alpha(t)\vec{e}_1(p) + \beta(t)\vec{e}_2(p)$, thus $\frac{D\vec{w}}{dt}(t) = \alpha'(t)\vec{e}_1(p) + \beta'(t)\vec{e}_2(p)$, since $\frac{D\vec{e}_1}{dt}(t) = \frac{D\vec{e}_2}{dt}(t) = 0$. Thus $\vec{w}$ is parallel transported along $\mathrm{Im}c$ iff α and β are constant.

Exercise 22.12 Compute $\nabla_{\vec{v}}\vec{e}_1$ and $\nabla_{\vec{v}}\vec{e}_2$ from $d\vec{e}_1$ and $d\vec{e}_2$.

Answer. With (3.14) and the shift of index $2 \to 1$ and $3 \to 2$ we get $d\vec{e}_1(p) = -R\cos\theta\vec{E}_1\otimes d\theta - R\sin\theta\vec{E}_2\otimes d\theta$, and $d\vec{e}_2(p) = 0$.

And $\vec{v}(p) = \vec{c}\,'(t) = -R\sin t\vec{E}_1 + R\cos t\vec{E}_2 + a\vec{E}_3 = \vec{e}_1(p) + a\,\vec{e}_2(p) \in T_PS$.

Thus $d\vec{e}_1(p).\vec{v}(p) = (-R\cos t\vec{E}_1\otimes d\theta - R\sin t\vec{E}_2\otimes d\theta).(\vec{e}_1(p) + a\,\vec{e}_2(p)) = a(-R\cos t\vec{E}_1 - R\sin t\vec{E}_2) \perp \vec{e}_1(p)$ and $\perp \vec{e}_2(p)$, thus $\frac{D\vec{e}_1}{dt}(p) = \mathrm{Proj}_{T_pS}(d\vec{e}_1(p).\vec{v}(p)) = \vec{0}$. And $\frac{D\vec{e}_2}{dt}(p) = \mathrm{Proj}_{T_pS}(\vec{0}) = \vec{0}$. ∎

22.6.3 Parallel transport in a sphere in $\mathbb{R}^3$

Example 22.13 c is a meridian $\varphi \to c(\varphi) = \begin{pmatrix} R\cos\theta_0\cos\varphi \\ R\sin\theta_0\cos\varphi \\ R\sin\varphi \end{pmatrix}$. Consider the vector field $\vec{w}(p) = \frac{\vec{e}_2(p)}{||\vec{e}_2(p)||} = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$, so if $p = c(\varphi)$ then $\vec{w}(p)$ is a unit vector orthogonal to the meridian. And $\frac{d(\vec{w}\circ c)}{d\varphi}(\varphi) = \vec{0}$, thus $\mathrm{Proj}_{TS}(\frac{d\vec{w}\circ c}{d\varphi}(\varphi)) = \vec{0}$, thus $\vec{w}$ is parallel transported along the meridian.

And any vector field $\vec{w} = \alpha\vec{e}_3(p) + \beta\frac{\vec{e}_2(p)}{||\vec{e}_2(p)||}$ with α and β constant is parallel transported along the meridian. Here $||\vec{w}|| = \sqrt{R^2\alpha^2 + \beta^2} =)$ constant, and "turns with the meridian (parallel transport). E.g.: If a plane flies along a meridian, then its wings are parallel transported along this meridian.

Consider $\vec{w}(p) = \vec{e}_2(p)$: parallel to a meridian, but its length $||\vec{e}_2(p)|| = R\cos\varphi$ varies with φ. So $\vec{w}(p) = (\vec{w}\circ c)(\varphi) = R\begin{pmatrix} -\sin\theta_0\cos\varphi \\ \cos\theta_0\cos\varphi \\ 0 \end{pmatrix}$, and $\frac{d(\vec{w}\circ c)}{d\varphi}(\varphi) = R\begin{pmatrix} \sin\theta_0\sin\varphi \\ -\cos\theta_0\sin\varphi \\ 0 \end{pmatrix} = -\frac{\cos\varphi}{\sin\varphi}\vec{e}_2(p)$ (every where but on the equator), thus $\mathrm{Proj}_{TS}(\frac{d\vec{w}\circ c}{d\varphi}(\varphi)) = -\frac{\cos\varphi}{\sin\varphi}\vec{e}_2(p) \neq \vec{0}$ (every where but on the equator and at the poles). Thus $\vec{e}_2$ is not parallel transported along the meridian. ▪

22.6.4 Parallel transport along a parallel

Continuing proposition 20.4 and remark 21.3. $\Phi(\theta,\varphi) = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\sin\theta\cos\varphi \\ R\sin\varphi \end{pmatrix}$, $\vec{e}_2(p) = \frac{\partial\Phi}{\partial\theta}(\theta,\varphi) = \begin{pmatrix} -R\sin\theta\cos\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix}$ and $\vec{e}_3(p) = \frac{\partial\Phi}{\partial\varphi}(\theta,\varphi) = \begin{pmatrix} -R\cos\theta\sin\varphi \\ -R\sin\theta\sin\varphi \\ R\cos\varphi \end{pmatrix}$, cf. (3.29). And parallel $c_{\varphi_0}(\theta) = \begin{pmatrix} R\cos\theta\cos\varphi_0 \\ R\sin\theta\cos\varphi_0 \\ R\sin\varphi_0 \end{pmatrix}$. Let $p_{t_0} = c_{\varphi_0}(\theta_0)$, and let $\vec{u}_{p_{t_0}} \in T_{p_{t_0}}S$ be given, $\vec{u}_{p_{t_0}} \neq \vec{0}$. Consider $p = c_{\varphi_0}(\theta)$ and the parallel transported vector

$$\vec{u}_c(p) = (J^{t_0}_{c,t}.\vec{u}_{t_0})(p) = \alpha(\theta)\vec{e}_2(p) + \beta(\theta)\vec{e}_3(p), \tag{22.17}$$

where $\alpha(\theta) = \alpha_{\varphi_0}(\theta)$ and $\beta(\theta) = \beta_{\varphi_0}(\theta)$ depends on φ_0 (the curve=parallel).

Remark 22.14 1- If $\vec{u}_c(p) = \alpha(\theta)\vec{e}_2(p)$ (tangent to the parallel), then, cf. (3.33) :

$$\frac{D\vec{u}_c}{d\theta}(p) = \alpha'(\theta)\vec{e}_2(p) + \alpha(\theta)\nabla_{\vec{e}_2}\vec{e}_2(p) = \alpha'(\theta)\vec{e}_2(p) + \alpha(\theta)\cos\varphi_0\sin\varphi_0\,\vec{e}_3(p).$$

So $\frac{D\vec{u}_c}{d\theta}(p) = 0$ along c_{φ_0} iff $\alpha' = 0$ and $\alpha(\theta)\cos\varphi_0\sin\varphi_0 = 0$, i.e., iff α is constant and $\varphi_0 = 0$: A parallel transport is only possible along the equator. In particular, $\vec{e}_2(p)$ is not parallel transported along a parallel which is not the equator.

2- If $\vec{u}_c(p) = \beta(\theta)\vec{e}_3(p)$ (tangent to a meridian), then, cf. (3.34) :

$$\frac{D\vec{u}_c}{d\theta}(p) = \beta'(\theta)\vec{e}_3(p) + \beta(\theta)\nabla_{\vec{e}_2}\vec{e}_3(p) = \beta'(\theta)\vec{e}_3(p) - \beta(\theta)\tan\varphi_0\,\vec{e}_2(p).$$

Thus $\frac{D\vec{u}_c}{d\theta}(p) = 0$ along c_{φ_0} iff $\beta' = 0$ and $\beta(\theta)\tan\varphi_0 = 0$, i.e., iff β is constant and $\varphi_0 = 0$: A parallel transport is only possible along the equator. In particular, $\vec{e}_3(p)$ is not parallel transported along a parallel which is not the equator. ▪

Proposition 22.15 *Let $s_0 = |\sin\varphi_0|$. The parallel transported vector $\vec{u}_c(p)$, cf. (22.17), is of the type*

$$\begin{cases} \alpha(\theta) = c_1\cos(s_0\theta) + c_2\sin(s_0\theta), \\ \beta(\theta) = (-c_1\sin(s_0\theta) + c_2\cos(s_0\theta))\cos\varphi_0, \end{cases} \tag{22.18}$$

where c_1 and c_2 are constants depending on $\vec{u}_{p_{t_0}} = U^2\vec{e}_2 + U^3\vec{e}_3$ given by

$$\begin{cases} c_1 = \cos(s_0\theta_0)U^2 - \sin(s_0\theta_0)\dfrac{U^3}{\cos\varphi_0}, \\ c_2 = \sin(s_0\theta_0)U^2 + \cos(s_0\theta_0)\dfrac{U^3}{\cos\varphi_0}. \end{cases} \tag{22.19}$$

Proof. Let $p = c_{\varphi_0}(\theta)$. (22.17) gives, with (3.43),

$$\frac{D\vec{u}_c}{d\theta}(p) = (\alpha'(\theta) - \beta(\theta)\tan\varphi_0)\,\vec{e}_2(p) + (\beta'(\theta) + \alpha(\theta)\cos\varphi_0\sin\varphi_0)\,\vec{e}_3(p). \tag{22.20}$$

Thus $\frac{D\vec{u}_c}{d\theta}(p) = 0$ gives

$$\begin{cases} \alpha'(\theta) - \beta(\theta)\tan\varphi_0 = 0, \\ \beta'(\theta) + \alpha(\theta)\cos\varphi_0\sin\varphi_0 = 0. \end{cases} \tag{22.21}$$

($\alpha = \beta = 0$ is not possible since $\vec{u}_0 \neq \vec{0}$, and we get the results in remark (22.14).)

Thus $\alpha'' = \beta'\tan\varphi_0$ and $\beta'' = -\alpha'\cos\varphi_0\sin\varphi_0$, thus

$$\begin{cases} \alpha'' + \alpha\tan\varphi_0\cos\varphi_0\sin\varphi_0 = \alpha'' + \alpha\sin^2\varphi_0 = 0, \\ \beta'' + \beta\tan\varphi_0\cos\varphi_0\sin\varphi_0 = \beta'' + \beta\sin^2\varphi_0 = 0. \end{cases}$$

With $s_0 = |\sin\varphi_0|$ we get: there exists $c_1, c_2, c_3, c_4 \in \mathbb{R}$ s.t.

$$\begin{cases} \alpha(\theta) = c_1\cos(s_0\theta) + c_2\sin(s_0\theta), \\ \beta(\theta) = c_3\cos(s_0\theta) + c_4\sin(s_0\theta). \end{cases}$$

Thus $\left\{ \begin{array}{l} \alpha'(\theta) = -c_1 s_0\sin(s_0\theta) + c_2 s_0\cos(s_0\theta), \\ \beta'(\theta) = -c_3 s_0\cos(s_0\theta) + c_4 s_0\cos(s_0\theta). \end{array} \right\}$ Thus, with (22.21) (the sin and cos functions are independent),

$$-c_1 s_0 = c_4\tan\varphi_0, \quad \text{and} \quad c_2 s_0 = c_3\tan\varphi_0,$$

that is, $-c_1\cos\varphi_0 = c_4$ and $c_2\cos\varphi_0 = c_3$, thus (22.18). Thus

$$\beta(\theta) = c_2\cos\varphi_0\cos(s_0\theta) - c_1\cos\varphi_0\sin(s_0\theta).$$

Thus:

$$\vec{u}_c(p) = (c_1\cos(s_0\theta) + c_2\sin(s_0\theta))\vec{e}_2(p) + (c_2\cos(s_0\theta) - c_1\sin(s_0\theta))(\cos\varphi_0\vec{e}_3(p)).$$

With $||\vec{e}_2(p)|| = R\cos\varphi_0$, $||\vec{e}_3(p)|| = R$ and $\vec{e}_2(p) \perp \vec{e}_3(p)$, we have $||\vec{u}_c|| = (c_1^2 + c_2^2)^{\frac{1}{2}}$ = constant (the parallel transport keeps the metric). And the initial condition $\vec{u}_c(p_{t_0}) = \vec{U} = U^2\vec{e}_2 + U^3\vec{e}_3$ gives

$$\begin{pmatrix} \alpha(\theta_0) \\ \frac{\beta(\theta_0)}{\cos\varphi_0} \end{pmatrix} = \begin{pmatrix} \cos(s_0\theta_0) & \sin(s_0\theta_0) \\ -\sin(s_0\theta_0) & \cos(s_0\theta_0) \end{pmatrix} . \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} U^2 \\ \frac{U^3}{\cos\varphi_0} \end{pmatrix}.$$

With $\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}^{-1} = \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix}$, we get (22.19). ∎

Example 22.16 On the equator: $\varphi_0 = 0$, $s_0 = 0$, thus $c_1 = U^2$, $c_2 = U^3$, so $\alpha(\theta) = U^2$ and $\beta(\theta) = U^3$, thus $\vec{u}_c(p) = U^2\vec{e}_2(p) + U^3\vec{e}_3(p)$, and the equator is a geodesic: Already known. ∎

Example 22.17 See figure 22.1. Parallel $\varphi_0 = \frac{\pi}{4}$ (radian, that is 45 degrees) north. So $s_0 = |\sin\varphi_0| = \frac{\sqrt{2}}{2}$. Choose $\theta_0 = 0$, $p_{t_0} = c_{\varphi_0}(0) = \begin{pmatrix} R\cos\varphi_0 \\ 0 \\ R\sin\varphi_0 \end{pmatrix}$ (on the Greenwich meridian), and travel along the 45-th parallel.

1- Initial vector $\vec{U} = \vec{e}_2(p_{t_0})$ (parallel to the paralllel toward East). See figure 22.1. Thus (22.19) and (22.18) give

$$\begin{cases} c_1 = U^2 = 1, \\ c_2 = \dfrac{U^3}{\cos\varphi_0} = 0, \end{cases} \quad \text{and} \quad \begin{cases} \alpha(\theta) = \cos(s_0\theta), \\ \beta(\theta) = -\sin(s_0\theta)\cos\varphi_0, \end{cases}$$

and

$$\vec{u}_c(\theta) = \cos(s_0\theta)\vec{e}_2(p) - \sin(s_0\theta)\cos\varphi_0\vec{e}_3(p). \tag{22.22}$$

Since $||\vec{e}_2(p)|| = ||\cos\varphi_0\vec{e}_3(p)|| = R\cos\varphi_0$, $\vec{u}_c(\theta)$ keeps its length and"turns", see figure 22.1.

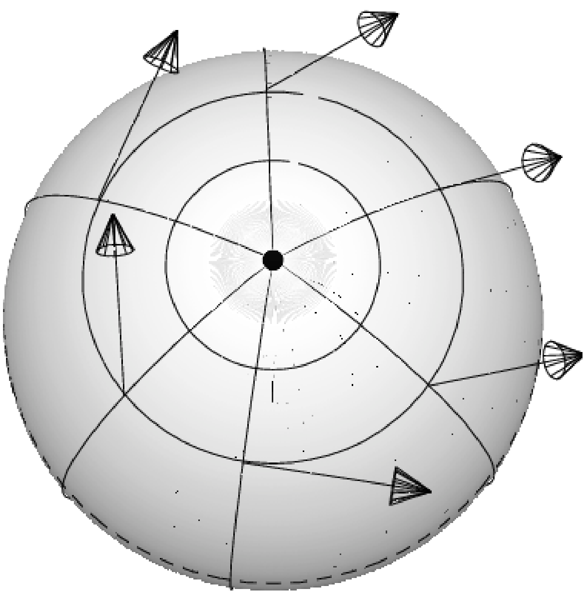

Figure 22.1: Parallel transport along the 45-th parallel, with, at $\theta=0$, $\vec{w}$ oriented to the East: The parallel transported vector turned south at $\theta\simeq 127$ degrés, see example 22.17. (Parallel transport along the 30-th parallel, with, at $\theta=0$, $\vec{w}$ oriented to the East: The parallel transported vector turned south at $\theta\simeq 180$ degrés. On the equator (geodesic) $\vec{w}$ stays parallel to the equator.)

• If θ is s.t. $s_0\theta=\frac{\pi}{2}$, i.e. $\theta=\frac{\pi}{2s_0}=\frac{\pi}{\sqrt{2}}$ (toward East and $\frac{\pi}{\sqrt{2}}\frac{180}{\pi}\simeq 127$ degrees longitude), then $\vec{u}_c(\theta)$ is now oriented South. The orientation toward the South was expected: Indeed, the geodesic (great circle) through p_{t_0} which is parallel to $\vec{e}_2(p_{t_0})$ "goes toward the equator" (a geodesic looks like a straight line in S); Thus $\vec{u}_c(\theta)$ goes toward the South.

• If θ is s.t. $s_0\theta=\pi$, i.e. $\theta=\frac{\pi}{s_0}=\frac{2\pi}{\sqrt{2}}$ (toward East and $\frac{2\pi}{\sqrt{2}}\frac{180}{\pi}\simeq 154$ degrees longitude), then $\vec{u}_c(\theta)=-\vec{e}_2(p)$.

• If θ is s.t. $s_0\theta=2\pi$, i.e.$\theta=\frac{\pi}{s_0}=\frac{4\pi}{\sqrt{2}}$ (toward East and $\frac{2\pi}{\sqrt{2}}\frac{180}{\pi}\simeq 308$ degrees longitude) then $\vec{u}_c(\theta)=\vec{e}_2(p)$.

2- Initial vector $\vec{U}=\vec{e}_3(p_{t_0})$ (parallel to the meridian toward North). Then

$$c_1=U^2=0,\qquad c_2=U^3=1,\qquad \alpha(\theta)=\sin(s_0\theta),\qquad \beta(\theta)=\frac{\sqrt{2}}{2}\cos(s_0\theta),$$

and

$$\vec{u}_c(\theta)=\sin(s_0\theta)\vec{e}_2(p)+\cos(s_0\theta)(\cos\varphi_0\vec{e}_3(p)),\tag{22.23}$$

and $\vec{u}_c(\theta)$ turns, see figure 22.1 with intial vector $-\vec{e}_3(p)$. ∴

22.7 The shifter preserves the Riemannian metric

Proposition 22.18 *With a usual Euclidean metric $(\cdot,\cdot)_g=(\cdot,\cdot)_{\mathbb{R}^n}$ and the associated Riemannian connection (given by $\nabla_{\vec{v}}\vec{w}=\mathrm{Proj}_{TS}(d\vec{w}.\vec{v})\in\Gamma(S)$), the shifter $J^{t_0}_{c,t}$ in S along a curve c satisfies*

$$(J^{t_0}_{c,t}.\vec{u}_{p_{t_0}},J^{t_0}_{c,t}.\vec{w}_{p_{t_0}})_{\mathbb{R}^n}=(\vec{u}_{p_{t_0}},\vec{w}_{p_{t_0}})_{\mathbb{R}^n}.\tag{22.24}$$

(The shifter is an isometry: keeps lengths and angles). I.e.,

$$\frac{d}{dt}\Big(J^{t_0}_{c,t}.\vec{u}_{p_{t_0}},J^{t_0}_{c,t}.\vec{w}_{p_{t_0}})\Big)_{\mathbb{R}^n}=0.\tag{22.25}$$

Proof. Let $J^{t_0}_{c,t}(\vec{w}_{p_{t_0}})=\vec{w}_c(c(t))$, cf. (22.6), so $\frac{D\vec{w}_c}{dt}(p_t)=0$, idem with $\vec{u}_{p_{t_0}}$. Thus

$$\begin{aligned}\frac{d}{dt}(J^{t_0}_c(t,\vec{u}_{p_{t_0}}),J^{t_0}_c(t,\vec{w}_{p_{t_0}}))_{\mathbb{R}^n}&=(\frac{\partial J^{t_0}_c}{\partial t}(t,\vec{u}_{p_{t_0}}),J^{t_0}_c(t,\vec{w}_{p_{t_0}}))_{\mathbb{R}^n}+(J^{t_0}_c(t,\vec{u}_{p_{t_0}}),\frac{\partial J^{t_0}_c}{\partial t}(t,\vec{w}_{p_{t_0}}))_{\mathbb{R}^n}\\&=(\mathrm{Proj}_{T_pS}\frac{\partial J^{t_0}_c}{\partial t}(t,\vec{u}_{p_{t_0}}),J^{t_0}_c(t,\vec{w}_{p_{t_0}}))_{\mathbb{R}^n}+(J^{t_0}_c(t,\vec{u}_{p_{t_0}}),\mathrm{Proj}_{T_pS}\frac{\partial J^{t_0}_c}{\partial t}(t,\vec{w}_{p_{t_0}}))_{\mathbb{R}^n},\end{aligned}$$

since $J_c^{t_0}(t,\vec{u}_{p_{t_0}}), J_c^{t_0}(t,\vec{w}_{p_{t_0}}) \in T_pS$. That is,

$$\frac{d}{dt}(J_c^{t_0}(t,\vec{u}_{p_{t_0}}), J_c^{t_0}(t,\vec{w}_{p_{t_0}}))_{\mathbb{R}^n} = (\frac{D\vec{u}_c}{dt}(c(t)), \vec{u}_c(c(t)))_{\mathbb{R}^n} + (\vec{u}_c(c(t)), \frac{D\vec{w}_c}{dt}(c(t)))_{\mathbb{R}^n} = 0+0=0,$$

since $\frac{D\vec{u}_c}{dt} = 0 = \frac{D\vec{w}_c}{dt}$. ∎

Proposition 22.19 *Let ∂_i be the derivation relative to the i-th variable. For all $t_0, t, t_1 \in]a,b[$, we have*

$$\partial_2 J_c(t_1,t).J_c(t,t_0) + J_c(t_1,t).\partial_1 J_c(t,t_0) = 0, \tag{22.26}$$

written

$$\frac{dJ^t_{c,t_1}}{dt}.J^{t_0}_{c,t} + J^t_{c,t_1}.\frac{dJ^{t_0}_{c,t}}{dt} = 0, \tag{22.27}$$

or, $\frac{dJ_c(t_1,t)}{dt}.J_c(t,t_0) + J_c(t_1,t).\frac{dJ_c(t,t_0)}{dt} = 0$. *In particular,*

$$\partial_2 J_c(t_0,t_0) + \partial_1 J_c(t_0,t_0) = 0, \quad i.e. \quad \frac{dJ^t_{c,t_0}}{dt}_{|t=t_0} = -\frac{dJ^{t_0}_{c,t}}{dt}_{|t=t_0}, \tag{22.28}$$

or, $\frac{dJ_{c,t,t_0}}{dt}_{|t=t_0} = -\frac{dJ_{c,t_0,t}}{dt}_{|t=t_0}$.

Proof. (22.10) gives $J(t_1,t).J(t,t_0).\vec{w}_{p_{t_0}} = J(t_1,t_0).\vec{w}_{p_{t_0}}$, thus

$$\partial_2 J(t_1,t).J(t,t_0).\vec{w}_{p_{t_0}} + J(t_1,t).\partial_1 J(t,t_0).\vec{w}_{p_{t_0}} = \vec{0}, \tag{22.29}$$

for all $\vec{w}_{p_{t_0}} \in T_{p_{t_0}}S$, i.e. (22.26). In particular, $\partial_2 J(t_0,t_0).J(t_0,t_0).\vec{w}_{p_{t_0}} + J(t_0,t_0).\partial_1 J(t_0,t_0).\vec{w}_{p_{t_0}} = \vec{0}$, thus $\partial_2 J(t_0,t_0) + \partial_1 J(t_0,t_0) = \vec{0}$, since $J_c(t_0,t_0) = I$. ∎

Corollary 22.20 *Let $p_t = c(t)$, $\vec{c}\,'(t) = \vec{v}(p_t) = \sum_{k=1}^m v^k(p_t)\vec{e}_k(p_t)$, and $J^{t_0}_{c,t} = \sum_{i,j=1}^m (J^{t_0}_{c,t})^i_j\, \vec{e}_i(p_t) \otimes e^j(p_{t_0})$, cf. (22.8), Then, for all i,j,*

$$\frac{d(J^{t_0}_{c,t})^i_j}{dt} = -\sum_{k,\ell=1}^m (J^{t_0}_{c,t})^\ell_j v^k \gamma^i_{k\ell}, \quad thus \quad \frac{d(J^{t_0}_{c,t})^i_j}{dt}_{|t=t_0} = -\sum_{k=1}^m \gamma^i_{kj} v^k. \tag{22.30}$$

Proof. Let $\vec{w}_{p_{t_0}} = \vec{e}_j(p_{t_0})$ and $\vec{w}_c(p_t) = J_c(t,t_0).\vec{e}_j(p_t) = \sum_{i=1}^m J_c(t,t_0)^i_j \vec{e}_i(c(t))$, cf. (22.8). Thus $\nabla_{\vec{v}}\vec{w}_c = \frac{D\vec{w}_c}{dt} = \vec{0}$, cf. (22.4), that is $\mathrm{Proj}_{T_{p_t}S}\frac{d(\vec{w}_c o c)}{dt} = \vec{0}$, gives

$$\sum_{i=1}^m \frac{d(J^{t_0}_{c,t})^i_j}{dt}\vec{e}_i(p_t) + \sum_{i=1}^m (J^{t_0}_{c,t})^i_j \mathrm{Proj}_{T_{p_t}S}(d\vec{e}_i(p_t).\vec{v}(p_t)) = \vec{0}.$$

With $\mathrm{Proj}_{TS}(d\vec{e}_i.\vec{v}) = \nabla_{\vec{v}}\vec{e}_i = \sum_{k=1}^m v^k \nabla_{\vec{e}_k}\vec{e}_i = \sum_{k,\ell=1}^m v^k \gamma^\ell_{ki}\vec{e}_\ell$, thus

$$\sum_{i=1}^m \frac{d(J^{t_0}_{c,t})^i_j}{dt}\vec{e}_i(p_t) + \sum_{i,k,\ell=1}^m (J^{t_0}_{c,t})^i_j v^k \gamma^\ell_{ki}\vec{e}_\ell = \vec{0}.$$

And $\sum_{i,k,\ell=1}^m (J^{t_0}_{c,t})^i_j v^k \gamma^\ell_{ki}\vec{e}_\ell = \sum_{i,k,\ell=1}^m (J^{t_0}_{c,t})^\ell_j v^k \gamma^i_{k\ell}\vec{e}_i$ gives (22.30). ∎

22.8 Theorem 2 of Levi-Civita

Manifold S with a connection ∇.

Definition 22.21 A metric $g(\cdot,\cdot)$ in S is parallel transported in S relative to the connection ∇ iff (22.25) is satisfied for all curve c in S, all $p_{t_0} \in S$ and all $\vec{u}_{t_0}, \vec{w}_{t_0} \in T_{p_{t_0}}S$, that is,

$$((J^{t_0}_{c,t}.\vec{u}_{t_0})(p_t), (J^{t_0}_{c,t}.\vec{w}_{t_0})(p_t))_{g_{p_t}} = (\vec{u}_{t_0}(p_{t_0}), \vec{u}_{t_0}(p_{t_0}))_{g_{p_{t_0}}}, \tag{22.31}$$

when $p_{t_0} = c(t_0)$, $p_t = c(t)$, that is,

$$(\vec{u}_c(p_t), \vec{w}_c(p_t))_{g_{p_t}} = (\vec{u}_{t_0}(p_{t_0}), \vec{u}_{t_0}(p_{t_0}))_{g_{p_{t_0}}}. \tag{22.32}$$

That is, iff

$$\frac{d}{dt}\Big(((J^{t_0}_{c,t}.\vec{u}_{t_0})(c(t)), (J^{t_0}_{c,t}.\vec{w}_{t_0})(c(t)))_{g_{c(t)}}\Big) = 0, \tag{22.33}$$

i.e., $\frac{d}{dt}\Big((\vec{u}_c(c(t)), \vec{w}_c(c(t)))_{g_{c(t)}}\Big) = 0$.

Proposition 22.22 *If $g(\cdot,\cdot)$ is parallel transported relative to ∇, then $g(\cdot,\cdot)$ is a metric of Killing relative to ∇.*

Proof. $\frac{Dg}{dt} = 0 = \nabla_{\vec{v}} g$ along any curve tells that $\vec{v}$ is a vector of Killing for all $\vec{v} \in \Gamma(S)$, thus $g(\cdot,\cdot)$ is a metric of Killing. ∴

Theorem 22.23 (**Levi–Civita, and definition**.) *Let $(S, g(\cdot,\cdot))$ be a Riemannian manifold. Then there exists a unique connection ∇ in S such that:*
1- ∇ is torsion free, cf. (11.11), and
2- ∇ parallel transport $g(\cdot,\cdot)$, cf. (22.31).
This connection is called the "metric connection" or the "Levi–Civita connection".

(In other words: there exists a unique torsion free connection ∇ in S such that the shifter preserves the Riemannian metric in S.)

Using a coordinate system, the connection is given by, for all i, j, k,

$$\gamma^i_{jk} = \frac{1}{2}\sum_\ell g^{i\ell}(\frac{\partial g_{j\ell}}{\partial q^k} - \frac{\partial g_{kj}}{\partial q^\ell} + \frac{\partial g_{\ell k}}{\partial q^j}), \tag{22.34}$$

where $[g^{ij}] := [g_{ij}]^{-1}$.

Proof. Apply the previous proposition and theorem 17.8. ∴

(If S is a surface in $\mathbb{R}^n$, and $g(\cdot,\cdot)$ a usual Euclidean metric, then the Levi-Civita connection is the usual Riemannian connection.)

Part VII

Normal and second fundamental form

23 Metric and volume in $\mathbb{R}^n$

Let $(\vec{E}_i)$ be a Euclidean basis in $\mathbb{R}^n$, (dx^i) the dual basis, and $g(\cdot,\cdot)$ the associated Euclidean dot product, that is,

$$g(\cdot,\cdot) = \sum_i dx^i \otimes dx^i \overset{\text{written}}{=} (\cdot,\cdot)_{\mathbb{R}^n}, \quad \text{i.e.} \quad [g(\cdot,\cdot)]_{|\vec{E}} = I. \tag{23.1}$$

And the Euclidean algebraic volume element and the Euclidean volume element (non negative) are

$$dx^1 \wedge ... \wedge dx^n, \quad \text{and} \quad d\Omega = |dx^1 \wedge ... \wedge dx^n| \overset{\text{written}}{=} dx^1 ... dx^n. \tag{23.2}$$

So the algebraic volume of the parallelepiped limited by n vectors $\vec{w}_1$, ..., $\vec{w}_n$ is

$$\det_{|\vec{E}}(\vec{w}_1, ..., \vec{w}_n) = \det[w^i_j] \quad \text{when} \quad \vec{w}_j = \sum_i w^i_j \vec{E}_i, \tag{23.3}$$

the volume being $|\det_{|\vec{E}}(\vec{w}_1, ..., \vec{w}_n)| = |\det[w^i_j]|$.

23.1 Expression in a coordinate system basis in $\mathbb{R}^n$

Let S be an open set in $\mathbb{R}^n$, and $\Phi_+ : \vec{q} \in U \subset \vec{\mathbb{R}^n} \to p = \Phi(\vec{q}) \in S \subset \mathbb{R}^n$ be a coordinate system in S, cf. (2.1). Let $(\vec{e}_i(p)) = (d\Phi(\vec{q}).\vec{A}_i)_{i=1,\dots,n}$ be the basis of the system at $p = \Phi_+(\vec{q})$, and $(e^i(p) = dq^i(p))_{i=1,\dots,n}$ be the dual basis, and let $g_{ij}(p) := g(\vec{e}_i(p), \vec{e}_j(p)) = (\vec{e}_i(p), \vec{e}_j(p))_{\mathbb{R}^n}$, so the Euclidean metric $g(\cdot,\cdot) = (\cdot,\cdot)_{\mathbb{R}^n}$, cf. (23.1), also reads, at any $p \in S$,

$$g(p) = \sum_{i,j=1}^{n} g_{ij}(p)\, dq^i(p) \otimes dq^j(p), \quad [g(p)]_{|\vec{e}} = [g_{ij}(p)]. \tag{23.4}$$

Proposition 23.1 *Let $p = \Phi_+(\vec{q})$ and $J(\vec{q}) = \det_{|\vec{E}}(\vec{e}_1(p), \dots, \vec{e}_n(p))$ = the algebraic volume of the parallelepiped limited by $\vec{e}_1(p)$, ..., $\vec{e}_n(p)$. Then*

$$J(\vec{q}) = \pm\sqrt{\det([g(p)]_{|\vec{e}})}, \tag{23.5}$$

with a + sign if the basis $(\vec{e}_i(p))$ is direct, that is if $J(\vec{q}) > 0$, and with a − sign if not. And the algebraic Euclidean volume element, cf. (23.2), also reads, at any p,

$$\pm\sqrt{\det([g(p)]_{|\vec{e}})}\, dq^1(p) \wedge \dots \wedge dq^n(p), \tag{23.6}$$

and the Euclidean volume element is $d\Omega(p) = \sqrt{\det([g(p)]_{|\vec{e}})}\, dq^1(p)...dq^n(p)$.

So, the algebraic volume of the parallelepiped limited by n vectors $\vec{w}_1(p)$, ..., $\vec{w}_n(p)$, cf. (23.3), is

$$\det_{|\vec{E}}(\vec{w}_1(p), \dots, \vec{w}_n(p)) = \sqrt{\det([g(p)]_{|\vec{e}})}\, \det[w^i_j(p)] \quad \textit{when} \quad \vec{w}_j(p) = \sum_i w^i_j(p)\vec{e}_i(p). \tag{23.7}$$

Proof. Let P be the transition matrix from $(\vec{E}_i)$ to $(\vec{e}_i)$. Then $J(\vec{q})^2 = \det_{|\vec{E}}(\vec{e}_1(p), \dots, \vec{e}_n(p))^2 = \det([\vec{e}_1(p)]_{|\vec{E}}, \dots, [\vec{e}_n(p)]_{|\vec{E}})^2 = \det(P)^2 = \det(P)\det(P) = \det(P^T)\det(P) = \det(P^T.P) = \det\left(\left[[\vec{e}_i(p)]^T_{|\vec{E}}.[\vec{e}_j(p)]_{|\vec{E}}\right]\right) = \det([(\vec{e}_i(p), \vec{e}_j(p))_{\mathbb{R}^n}]) = \det([g_{ij}(p)]) = \det([g(p)]_{|\vec{e}})$. Thus (23.5) and (23.7). ∎

Example 23.2 Polar coordinates, $r = r(p) = \sqrt{x^2 + y^2}$; $\det_{\vec{E}}(p)(\vec{e}_1(p), \vec{e}_2(p)) = \sqrt{\det[g_p]_{|\vec{e}}} = r$. And $[g(p)]_{|\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, $J_\Phi(\vec{q}) = \sqrt{\det[g(p)]_{|\vec{e}}} = r$, $\det_{\vec{E}}(p) = r\, dr(p) \wedge d\theta(p)$ (algebraic volume at p in the polar system), and the volume element at p is $|\det_{\vec{E}}(p)| = d\Omega(p) =^{\text{written}} r\, dr d\theta$ (positive volume at p). ∎

23.2 Volume in a surface

Let $\Phi : \vec{q} \in \vec{\mathbb{R}^m} \to p = \Phi(\vec{q}) \in S \subset \mathbb{R}^n$ a coordinate system in S, cf. (1.3), $(\vec{e}_i(p))_{i=1,\dots,m}$ the system basis at p, cf. (1.12), and $(dq^i(p))_{i=1,\dots,m}$ the dual basis at p, cf. (1.26). Consider a metric $g(\cdot,\cdot)$ in S written as, at $p \in S$,

$$g(p)(\cdot,\cdot) = \sum_{i,j=1}^{m} g_{ij}(p)\, dq^i(p) \otimes dq^j(p), \quad \text{so} \quad g_{ij} = g(\vec{e}_i, \vec{e}_j), \quad \text{and} \quad [g(p)]_{|\vec{e}} = [g_{ij}(p)]_{\substack{i=1,\dots,m \\ j=1,\dots,m}}. \tag{23.8}$$

Definition 23.3 The algebraic volume in S (the algebraic area element) at p is

$$\det_{\vec{E}}(p) := \sqrt{\det([g(p)]_{|\vec{e}})}\, dq^1(p) \wedge \dots \wedge dq^n(p). \tag{23.9}$$

And the algebraic volume in T_pS limited by m vectors $\vec{w}_1(p), ...\vec{w}_m(p) \in T_pS$ is

$$\det_{\vec{E}}(p)(\vec{w}_1(p), \dots, \vec{w}_m(p)) = \sqrt{\det([g(p)]_{|\vec{e}})}\, \det([w^i_j(p)]), \quad \text{when} \quad \vec{w}_j = \sum_{i=1}^{m} w^i_j(p)\vec{e}_i(p), \tag{23.10}$$

and the non negative volume is $|\det_{\vec{E}}(p)(\vec{w}_1(p), \dots, \vec{w}_m(p))| = \sqrt{\det([g(p)]_{|\vec{e}})}\, |\det([w^i_j(p)])|$.

23.3 Unit normal form

Let $m = n-1$, so S is a regular hyper-surface in $\mathbb{R}^n$. Let $\vec{n}(p) =^{\text{written}} \vec{e}_n(p)$ be one (of the two) unit normal vector at S at p, so

$$\forall i = 1, ..., n-1, \ (\vec{e}_i(p), \vec{n}(p))_{\mathbb{R}^n} = 0, \quad \text{and} \quad ||\vec{n}(p)||_{\mathbb{R}^n} = 1. \tag{23.11}$$

Thus $(\vec{e}_i(p))_{i=1,...,n}$ is a basis at p in $\mathbb{R}^n$.

Let $(e^i(p))_{i=1,...,n}$ be the dual basis, and let

$$e^n(p) \overset{\text{written}}{=} n^\flat(p). \tag{23.12}$$

Definition 23.4 The linear form $n^\flat(p)$ is called the unit normal form at S at p.

With $\vec{w}(p) = \vec{w}_{||}(p) + \vec{w}_\perp(p) \in T_pS \oplus^\perp \text{Vect}\{\vec{n}(p)\}$, we get, for all $\vec{w} \in \Gamma(S)$,

$$n^\flat.\vec{w} = (\vec{n}, \vec{w})_g, \tag{23.13}$$

that is, $\vec{n}$ is the $(\cdot,\cdot)_g$-Riesz representation vector of $n^\flat$. And $n^\flat(p).\vec{w}_p$ is the normal component at S at p of $\vec{w}_p$.

Example 23.5 Polar coordinate:

- $p = \Phi_R(\theta) = \begin{pmatrix} R\cos\theta \\ R\sin\theta \end{pmatrix}$, then $\vec{n}(p) = \pm\vec{e}_1(p)$ and $n^\flat(p) = \pm dr(p)$.
- $p = \Phi_{\theta_0}(r) = \begin{pmatrix} r\cos\theta_0 \\ r\sin\theta_0 \end{pmatrix}$, then $\vec{n}(p) = \pm\frac{\vec{e}_2(p)}{r}$ and $n^\flat(p) = \pm r\, d\theta(p)$. ▪

Example 23.6 Spherical coordinates, $p = \Phi_R(\theta,\varphi) = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\sin\theta\cos\varphi \\ R\sin\varphi \end{pmatrix}$, tangent vectors $\vec{f}_1(p) = \frac{\partial\Phi_R}{\partial\theta}(\theta,\varphi) = \vec{e}_2(p)$ and $\vec{f}_2(p) = \frac{\partial\Phi_R}{\partial\varphi}(\theta,\varphi) = \vec{e}_3(p)$, thus

$$\vec{n}(p) = \pm\frac{\vec{f}_1 \wedge \vec{f}_2}{||\vec{f}_1 \wedge \vec{f}_2||}(p) = \pm\vec{e}_1(p) \quad \text{and} \quad n^\flat(p) = \pm dr(p). \tag{23.14}$$

And the endomorphism $\vec{n}_p \otimes n^\flat_p : \vec{\mathbb{R}^3} \to \vec{\mathbb{R}^3}$ is the projection on T_pS: With $\vec{v} = \vec{v}_{||} + \vec{v}_\perp = \vec{v}_{||} + v_\perp\vec{n}_p \in T_pS \oplus^\perp \text{Vect}\{\vec{n}_p\}$ we get

$$(\vec{n}_p \otimes n^\flat_p).\vec{v} = \vec{v}_\perp, \tag{23.15}$$

since $(\vec{n}_p \otimes n^\flat_p).\vec{v} = \vec{n}_p\,(n^\flat_p.\vec{v}) = v_\perp\vec{n}_p = \vec{v}_\perp$. ▪

23.4 * Notation $*\ell$ de Hodge

23.4.1 Definition

The Hodge operator could be defined on vectors: E.g. in $\vec{\mathbb{R}^n}$, it would be the operator $* : \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by, for all orthonormal direct basis $(\vec{v}_1, ..., \vec{v}_n)$,

$$*(\vec{v}_1, ..., \vec{v}_{n-1}) = \vec{v}_n.$$

But it is defined on linear forms, in particular for Maxwell equations to exchange the roles of $\vec{E}$ and $\vec{B}$.

So let Ω^k be the set of k-alternate multilinear forms in $\mathbb{R}^n$, $1 \le k \le n-1$. Since $\dim\Omega^k = \dim\Omega^{n-k} = \binom{n}{k}$ there is an isomorphism between Ω^k and Ω^{n-k}. Let $(\vec{E}_i)$ be the canonical basis in $\vec{\mathbb{R}^n}$ and $(\cdot,\cdot)_{\mathbb{R}^n}$ be the associated canonical dot product.

Definition 23.7 The Hodge operator $* : \begin{Bmatrix} \Omega^{n-k} & \to \Omega^k \\ \ell & \to *\ell \end{Bmatrix}$ is defined by, for all orthonormal direct basis $(\vec{e}_1, ..., \vec{e}_n)$,

$$(*\ell)(\vec{e}_1, ..., \vec{e}_{n-k}) = \ell(\vec{e}_{n-k+1}, ..., \vec{e}_n). \tag{23.16}$$

E.g., $* : \Omega^1 \to \Omega^{n-1}$ is defined by, for all linear form ℓ by, for all orthonormal direct basis $(\vec{e}_1, ..., \vec{e}_n)$,

$$(*\ell)(\vec{e}_1, ..., \vec{e}_{n-1}) = \ell(\vec{e}_n). \tag{23.17}$$

23.4.2 Example: Faraday electromagnetic field

Consider a particle with mass m, electrical charge e, velocity $\vec{v}$, and momentum $\vec{p} = m\vec{v}$. Let $\vec{E}$ and $\vec{B}$ be the ambient electric and magnetic fields. The energy $\mathcal{E} = mc^2 + \frac{1}{2}mv^2$ and the electromagnetic force $e(\vec{E} + \vec{v} \wedge \vec{B})$ satisfy

$$\frac{d\mathcal{E}}{dt} = e\,\vec{E}.\vec{v} \quad\text{and}\quad \frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v} \wedge \vec{B}). \tag{23.18}$$

Choose a basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3) \in \vec{\mathbb{R}^3}$. With $\vec{B} = B_x\vec{e}_1 + B_y\vec{e}_t + B_z\vec{e}_3$ and $\vec{E} = E_x\vec{e}_1 + E_y\vec{e}_t + E_z\vec{e}_3$, write $\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$ and $\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$, and $R_{\vec{B}} = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}$ and $R_{\vec{E}} = \begin{pmatrix} 0 & E_z & -E_y \\ -E_z & 0 & E_x \\ E_y & -E_x & 0 \end{pmatrix}$

Thus:

$$\vec{v} \wedge \vec{B} = R_{\vec{B}}.\vec{v}, \quad\text{and}\quad \frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v} \wedge \vec{B}) = e\vec{E} + eR_{\vec{B}}.\vec{v}. \tag{23.19}$$

(R_B is the antisymmetric endomorphism characterizing the magnetic field, and $\vec{B}$ is the associated vector field, relative to the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, characterized by $R_B.\vec{w} = -\vec{B}\wedge\vec{w}$ for all $\vec{w}$.) Then consider the position, velocity and momentum quadri-vectors

$$\bar{\vec{x}} := \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \overset{\text{written}}{=} \begin{pmatrix} t \\ \vec{x} \end{pmatrix}, \quad \bar{\vec{v}} := \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix} \overset{\text{written}}{=} \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}, \quad \bar{\vec{p}} := \begin{pmatrix} \mathcal{E} \\ p_x \\ p_y \\ p_z \end{pmatrix} \overset{\text{written}}{=} \begin{pmatrix} \mathcal{E} \\ \vec{p} \end{pmatrix}, \tag{23.20}$$

Then the Faraday tensor F is defined by its matrix relative to the extended basis $(\bar{\vec{e}}_0, \bar{\vec{e}}_1, \bar{\vec{e}}_2, \bar{\vec{e}}_3) \in \mathbb{R}^4$ (with $\bar{\vec{e}}_0 = (1,0,0,0)$, $\bar{\vec{e}}_1 = (0, \vec{e}_1)$, ...)

$$[F] = [F^i_j] = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & [\vec{E}]^T \\ [\vec{E}] & R_{\vec{B}} \end{pmatrix}. \tag{23.21}$$

So (23.18) reads

$$\frac{d\bar{\vec{p}}}{dt} = e\,[F].\bar{\vec{v}}. \tag{23.22}$$

Then consider the Minkowski pseudo-metric given by $\eta = \sum_{i,j=0}^4 \eta_{ij} e^i \otimes e^j = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3$, that is,

$$[\eta] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{23.23}$$

and the associated Faraday tensor $F^\flat \in \mathcal{A}_2(\mathbb{R}^4)$

$$F^\flat = \eta.F, \qquad [F^\flat] = [F_{ij}] = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -[\vec{E}]^T \\ [\vec{E}] & R_{\vec{B}} \end{pmatrix}. \tag{23.24}$$

(Given by the contraction $\sum_{i,j=1}^4 F_{ij} e^i \otimes e^j = (\sum_{i,k=1}^4 \eta_{ik} e^i \otimes e^k)(\sum_{k,j=1}^4 F^k_j \vec{e}_k \otimes e^j)$.) That is,

$$F^\flat = -E_x\,dt \wedge dx - E_y\,dt \wedge dy - E_z\,dt \wedge dz + B_z\,dx \wedge dy + B_y\,dz \wedge dx + B_x\,dy \wedge dz. \tag{23.25}$$

Thus the Hodge operator gives

$$(*F)(\vec{e}_0, \vec{e}_1) = F^\flat(\vec{e}_2, \vec{e}_3), \quad (*F)(\vec{e}_0, \vec{e}_2) = -F^\flat(\vec{e}_1, \vec{e}_3),$$

the second equality because of the opposite orientation of $(\vec{e}_0, \vec{e}_2, \vec{e}_1, \vec{e}_3)$ relative to $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$, and

$$(*F)(\vec{e}_1, \vec{e}_2) = F^\flat(\vec{e}_0, \vec{e}_3), \quad (*F)(\vec{e}_1, \vec{e}_3) = -F^\flat(\vec{e}_0, \vec{e}_2), \quad (*F)(\vec{e}_2, \vec{e}_3) = F^\flat(\vec{e}_0, \vec{e}_1).$$

Thus,

$$[*F] = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & [\vec{B}]^T \\ -[\vec{B}] & -R_{\vec{E}} \end{pmatrix}. \tag{23.26}$$

The Hodge operator $*$ has switched the roles of $\vec{E}$ and $\vec{B}$ (and the sign), cf. (23.24).

We will also need the action of the exterior differential: With (23.25) we get

$$\begin{aligned} d_{\text{ext}}F^\flat = &(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}) dx\wedge dy\wedge dz \\ &+ (\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) dt\wedge dy\wedge dz \\ &+ (\frac{\partial B_y}{\partial t} + \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) dt\wedge dz\wedge dx \\ &+ (\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) dt\wedge dx\wedge dy, \end{aligned} \tag{23.27}$$

and

$$\begin{aligned} d_{\text{ext}}(*F) = &(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}) dx\wedge dy\wedge dz \\ &+ (\frac{\partial E_x}{\partial t} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z}) dt\wedge dy\wedge dz \\ &+ (\frac{\partial E_y}{\partial t} - \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x}) dt\wedge dz\wedge dx \\ &+ (\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y}) dt\wedge dx\wedge dy. \end{aligned} \tag{23.28}$$

With $F^\sharp = \eta^{-1}.F$ and $[F^\sharp] = [F^{ij}]$, since $[\eta]^{-1} = [\eta]$, we get :

$$F^\sharp = -E_x\, d\vec{e}_0\wedge d\vec{e}_1 - E_y\, d\vec{e}_0\wedge d\vec{e}_2 - E_z\, d\vec{e}_0\wedge d\vec{e}_3 + B_z\, d\vec{e}_1\wedge d\vec{e}_2 + B_y\, d\vec{e}_3\wedge d\vec{e}_1 + B_x\, d\vec{e}_2\wedge d\vec{e}_3. \tag{23.29}$$

And the divergence $\nabla\cdot F^\sharp$ is (derivation line by line)

$$\begin{aligned} \nabla\cdot F^\sharp &= \sum_{ij} F^{ij}{}_{,j}\vec{e}_i \\ &= -(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z})\vec{e}_0 \\ &\quad + (\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z})\vec{e}_1 + (\frac{\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z})\vec{e}_2 + (\frac{\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y})\vec{e}_3 \end{aligned} \tag{23.30}$$

23.4.3 Maxwell equations

Maxwell equations:

$$\begin{cases} \text{div}\vec{B} = 0, \\ \dfrac{\partial\vec{B}}{\partial t} + \nabla\wedge\vec{E} = \vec{0} \end{cases} \quad\text{and}\quad \begin{cases} \text{div}\vec{E} = 4\pi\rho, \\ \dfrac{\partial\vec{E}}{\partial t} - \nabla\wedge\vec{B} = -4\pi\vec{J}. \end{cases} \tag{23.31}$$

that is,

$$d_{\text{ext}}(*F) = 0, \quad \text{i.e.} \quad \nabla\cdot F^\sharp = 4\pi\bar{\vec{J}}, \tag{23.32}$$

where $\bar{\vec{J}} = (\rho, \vec{J}) \in \mathbb{R}^4$. And

$$d_{\text{ext}}(*F) = 4\pi(*\bar{\vec{J}}), \tag{23.33}$$

where $*\bar{\vec{J}} \in \mathcal{A}_3(\mathbb{R}^4)$ is given by,

$$*\bar{\vec{J}} = J^0\, dx\wedge dy\wedge dz - J^1\, dy\wedge dz\wedge dt + J^2\, dz\wedge dt\wedge dx - J^3\, dt\wedge dx\wedge dy \tag{23.34}$$

24 Second fundamental form: The curvature tensor $k \in T^0_2(S)$

24.1 Curvature

24.1.1 Positive curvature of a curve in $S \subset \mathbb{R}^n$

Let $c : s \in [0, L] \to p = c(s) \in \mathbb{R}^n$ be a regular curve in $\mathbb{R}^n$ parametrized with an intrinsic parameter s, that is, such that $||\vec{c}\,'(s)|| = 1$ for all s. With $p = c(s)$, let $\vec{v}(p) = \vec{c}\,'(s)$. The non negative curvature of c at $p = c(s)$ is

$$k(p) := ||\vec{c}\,''(s)|| = ||d\vec{v}(p).\vec{v}(p)||, \tag{24.1}$$

the last equation since $\vec{v}(\vec{c}(s)) = \vec{c}\,'(s)$.

24.1.2 Algebraic curvature of a curve in $S \subset \mathbb{R}^n$

Let S be a regular hyper-surface in $\mathbb{R}^n$. If $p \in S$, let $\vec{n}(p)$ be a (one of the two) normal unit vector to S at p (it defines the orientation of S).

Let $c : s \in [0, L] \to p = c(s) \in S$ be a geodesic in S parametrized with an intrinsic parameter s, that is, such that $||\vec{c}\,'(s)|| = 1$ for all s, and let $\vec{v}(p) = \vec{c}\,'(s) = \vec{v}_p$ at $p = c(s)$. Since $(\vec{c}\,'(s), \vec{c}\,'(s))_{\mathbb{R}^n} = 1$ we have $d\vec{v}(p).\vec{v}(p) \perp T_pS$, cf. (20.1), thus

$$\exists \kappa_p \in \mathbb{R}, \quad d\vec{v}(p).\vec{v}(p) = -\kappa_p\, \vec{n}(p), \quad \text{and} \quad \kappa_p = -(d\vec{v}(p).\vec{v}(p), \vec{n}(p))_{\mathbb{R}^n}. \tag{24.2}$$

Definition 24.1 The real κ_p is the algebraic curvature of the geodesic c at $p \in \text{Im}c$. (And $|\kappa_p| = ||\vec{c}\,''(s)||$ is the non negative curvature along Imc at $p = c(s)$, cf. (24.1).)

Definition 24.2 Considering all the geodesics in S, we have thus defined κ_p on unit vector $\vec{v}_p$ at p by

$$\kappa_p(\vec{v}_p) = -(d\vec{v}_p.\vec{v}_p, \vec{n}_p)_g \overset{named}{=} k(p)(\vec{v}(p), \vec{v}(p)). \tag{24.3}$$

And thus we have defined κ on vector fields $\vec{v} \in \Gamma(S)$ by

$$\kappa(\vec{v}) = -(d\vec{v}.\vec{v}, \vec{n})_g \overset{named}{=} k(\vec{v}, \vec{v}), \tag{24.4}$$

and the curvature of S along $\vec{v}$ is defined at p by $\kappa(\vec{v})(p) = k(p)(\vec{v}(p), \vec{v}(p))$ when $||\vec{v}|| = 1$. So,

$$\kappa(\vec{v}) = k(\vec{v}, \vec{v}) := -n^\flat.(d\vec{v}.\vec{v}) \quad (= -(d\vec{v}.\vec{v}, \vec{n})_g). \tag{24.5}$$

Interpretation: We will see that $k = dn^\flat$, cf. (24.9): so, if $n^\flat$ varies slowly along c, then the curvature is small (and the radius of curvature is large).

Corollary 24.3 *The geodesic equation (20.2) then reads in $\mathbb{R}^n$ (with intrinsic coordinates), cf. (24.2),*

$$\frac{d\vec{v}}{ds} = -k(\vec{v}, \vec{v})\vec{n}, \tag{24.6}$$

which means $(d\vec{v}(p).\vec{v}(p) =)$ $\frac{d(\vec{v}\circ c)}{ds}(s) = k(p)(\vec{v}(p), \vec{v}(p))\vec{n}(p)$ when $p = c(s)$.

Proof. In $\mathbb{R}^n$, with $p = \vec{c}(s)$ we have $\frac{d\vec{v}}{ds}(p) := \frac{d(\vec{v}\circ\vec{c})}{ds}(s) = d\vec{v}(p).\vec{v}(p)$, and (24.4) gives (24.6). ∎

24.1.3 Curvature tensor $k(\cdot, \cdot)$ for $S \subset \mathbb{R}^n$

Definition 24.4 The curvature tensor $k \in T^0_2(S)$ is defined (in S) by, for all $\vec{v}, \vec{w} \in \Gamma(S)$:

$$k(\vec{v}, \vec{w}) = -n^\flat.(d\vec{w}.\vec{v}) \quad (= -(\vec{n}, d\vec{w}.\vec{v})_g). \tag{24.7}$$

That is, $-k(\vec{w}, \vec{v})(p)$ gives the normal component of $d\vec{w}(p).\vec{v}(p)$ at S at p. (And $k(\vec{v}, \vec{v})$ defined in (24.4) is a particular case).

Proposition 24.5 *k is a symmetric tensor: For all $\vec{v}, \vec{w} \in \Gamma(S)$,*

$$k(\vec{v}, \vec{w}) = k(\vec{w}, \vec{v}), \quad \text{i.e.} \quad n^\flat.(d\vec{w}.\vec{v}) = n^\flat.(d\vec{v}.\vec{w}). \tag{24.8}$$

Proof. $[\vec{v}, \vec{w}] = d\vec{w}.\vec{v} - d\vec{v}.\vec{w} \in \Gamma(S)$, cf. (10.6), thus $n^\flat.[\vec{v}, \vec{w}] = 0 = n^\flat.(d\vec{w}.\vec{v}) - n^\flat.(d\vec{v}.\vec{w})$. ∎

24.2 $k = dn^\flat \in T^0_2(S)$

Proposition 24.6

$$k = dn^\flat, \quad \text{i.e.} \quad k(\vec{v}, \vec{w}) = dn^\flat(\vec{v}, \vec{w}),\ \forall \vec{v}, \vec{w} \in \Gamma(S). \tag{24.9}$$

and k is a $\binom{0}{2}$ tensor in S (measures the variations of $n^\flat$ on S). In particular, $dn^\flat$ is symmetric.

Proof. For all $\vec{w} \in \Gamma(S)$, $n^\flat.\vec{w} = 0$, thus $d(n^\flat.\vec{w}) = 0$, thus, for all $\vec{v} \in \Gamma(S)$,

$$(dn^\flat.\vec{v}).\vec{w} + n^\flat.(d\vec{w}.\vec{v}) = 0 = (dn^\flat.\vec{v}).\vec{w} - k(\vec{v}, \vec{w}),$$

thus $k(\vec{v}, \vec{w}) = (dn^\flat.\vec{v}).\vec{w}$. And (24.8) gives $(dn^\flat.\vec{v}).\vec{w} = (dn^\flat.\vec{w}).\vec{v}$, thus $(dn^\flat.\vec{v}).\vec{w} =^{\text{written}} dn^\flat(\vec{v}, \vec{w})$ with $dn^\flat$ symmetric, thus (24.9). Hence $k = dn^\flat$ is $\mathcal{F}(S)$-multilinear: it is a tensor. ∎

24.3 Components $k_{ij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^n}$

Let Φ be a coordinate system in S, $(\vec{e}_i)_{i=1,...,n-1}$ the coordinate basis, and

$$k = \sum_{i,j=1}^{n-1} k_{ij}\, e^i \otimes e^j. \tag{24.10}$$

So $k_{ij} = k(\vec{e}_i, \vec{e}_j)$ and $[k]_{|\vec{e}} = [k_{ij}]$ the $(n-1)*(n-1)$ matrix of k relative to $(\vec{e}_i)$.

Consider $\vec{n}(p) =^{\text{written}} \vec{e}_n(p)$ and $(\vec{e}_i(p))_{i=1,...,n}$. The Euclidean metric reads

$$g = \sum_{i,j=1}^{n} g_{ij}\, e^i \otimes e^j, \quad g_{ij}(p) = (\vec{e}_i(p), \vec{e}_i(p))_g, \quad \forall p \in S. \tag{24.11}$$

(In particular $g_{in} = 0 = g_{ni}$ for all $i = 1, ..., n-1$. And the Christoffel symbols are given by $d\vec{e}_j.\vec{e}_i = \sum_{k=1}^n \gamma_{ij}^k \vec{e}_k$.

Corollary 24.7 *For all $i, j = 1, ..., n-1$,*

$$k_{ij} = -\gamma_{ij}^n, \quad \text{and} \quad k_{ij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial q^n} \qquad (= \gamma_{ij}^n), \tag{24.12}$$

so, the curvature tensor is given thanks to the Euclidean metric.

Proof. (24.9) gives $dn^\flat.(\vec{e}_j, \vec{e}_i) = -n^\flat.(d\vec{e}_i.\vec{e}_j) = -\gamma_{ij}^n$. And (17.15) gives $\gamma_{jk}^n = \frac{1}{2}\sum_\ell g^{n\ell}(\frac{\partial g_{\ell j}}{\partial q^k} + \frac{\partial g_{\ell k}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell}) = \frac{1}{2}(\frac{\partial g_{nj}}{\partial q^k} + \frac{\partial g_{nk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^n}) = \frac{1}{2}(0 + 0 - \frac{\partial g_{jk}}{\partial q^n})$, thus (24.12). ∎

24.4 Example: Sphere

Example 24.8 Let $S = S(\vec{0}, R)$ in $\mathbb{R}^3$. GPS coordinates $p = \Phi(r, \theta, \varphi) = \begin{pmatrix} x = r\cos\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\sin\varphi \end{pmatrix}$, cf. (3.27). Basis $\vec{e}_j(p) = \frac{\partial\Phi}{\partial q^j}(r, \theta, \varphi)$ at $p = \Phi(r, \theta, \varphi)$ in $\mathbb{R}^3$. Basis $(\vec{f}_1(p), \vec{f}_2(p), \vec{f}_3(p)) = (\vec{e}_2(p), \vec{e}_3(p), \vec{e}_1(p))$ at $p = \Phi(r, \theta, \varphi)$. Thus

$$[g(p)]_{|\vec{f}} = [g_{ij}(p)] = \begin{pmatrix} r^2\cos^2\varphi & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{24.13}$$

Choose $\vec{n}(p) = +\vec{e}_1(p) = \vec{f}_3(p)$ at $p \in S$, thus $n^\flat(p) = dr(p)$. Thus $k_{ij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial q^n} = \frac{1}{2}\frac{\partial g_{ij}}{\partial r} = -\gamma_{ij}^n$ for $j = 1, 2$ and with the basis $(\vec{f}_1(p), \vec{f}_2(p))$ in T_pS:

$$[k(p)]_{\vec{f}} = \begin{pmatrix} R\cos^2\varphi & 0 \\ 0 & R \end{pmatrix} = [dn^\flat(p)]_{\vec{f}}, \tag{24.14}$$

i.e., $k(p) = R\cos^2\varphi\, d\theta(p) \otimes d\theta(p) + R\, d\varphi(p) \otimes d\varphi(p) = dn^\flat(p)$ (or see (3.42)-(3.43)).

E.g., the curvature of the sphere along a meridian (a geodesic in S) is

$$k(\frac{\vec{f}_2}{||\vec{f}_2||}, \frac{\vec{f}_2}{||\vec{f}_2||}) = k(\frac{\vec{e}_3}{||\vec{e}_3||}, \frac{\vec{e}_3}{||\vec{e}_3||}) = R\,\frac{1}{R^2} = \frac{1}{R},$$

thus the radius of curvature is $-R$, and the center of the sphere is $p - R.\vec{n}(p) = \vec{0}$. E.g., the curvature of the sphere along a parallel is

$$k(\frac{\vec{f}_1}{||\vec{f}_1||}, \frac{\vec{f}_1}{||\vec{f}_1||}) = k(\frac{\vec{e}_2}{||\vec{e}_2||}, \frac{\vec{e}_2}{||\vec{e}_2||}) = R\cos^2\varphi\,\frac{1}{R^2\cos^2\varphi} = \frac{1}{R},$$

which is the curvature of the sphere.

NB: This is not the curvature of a parallel: This is the curvature of the sphere at a point p which belong to many curves, in particular belongs to a geodesic tangent to $\vec{f}_1(p) = \vec{e}_2(p)$ at p (the only information used is the vector $\vec{f}_1(p)$ which is tangent to many curves at p).

Reminder: At a point p of the parallele = the curve $c(\theta) = \Phi(R,\theta,\varphi_0)$ we have $\vec{c}'(\theta) = R\cos\varphi \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$ and $\vec{c}''(\theta) = -R\cos\varphi \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$. And the osculating plane is the "horizontal affine plane" $(p, \mathrm{Vect}\{\vec{c}'(\theta), \vec{c}''(\theta)\})$ (contains the parallel, not $\vec{n}(p)$ the normal vector to the sphere); And the curvature of the parallel is $||\tilde{c}''(s)||$ where $\tilde{c}(s)$ is the parallel with an intrinsic parameter, that is $\tilde{c}(s) = \begin{pmatrix} x = R\cos\frac{s}{R\cos\varphi_0}\cos\varphi_0 \\ y = R\sin\frac{s}{R\cos\varphi_0}\cos\varphi_0 \\ z = R\sin\varphi_0 \end{pmatrix}$; So $\tilde{c}'(s) = \begin{pmatrix} -\sin\frac{s}{R\cos\varphi_0} \\ \cos\frac{s}{R\cos\varphi_0} \\ 0 \end{pmatrix}$, and $\tilde{c}''(s) = \begin{pmatrix} x = -\frac{1}{R\cos\varphi_0}\cos\frac{s}{R\cos\varphi_0} \\ y = -\frac{1}{R\cos\varphi_0}\sin\frac{s}{R\cos\varphi_0} \\ 0 \end{pmatrix}$, and $||\tilde{c}''(s)|| = \frac{1}{R\cos\varphi_0}$, so the radius of curvature of the parallel is $R\cos\varphi_0$. ▪

24.5 The associated curvature tensor $K \in T^1_1(S)$

Consider the curvature tensor $k \in T^0_2(S)$, cf. (24.7). With a Euclidean dot product $(\cdot,\cdot)_g$ and the Riesz representation theorem, let $K \in T^1_1(S)$ be defined by, for all $\vec{v},\vec{w} \in TS$,

$$(K.\vec{v},\vec{w})_g = k(\vec{v},\vec{w}). \tag{24.15}$$

that is,

$$g.K = k, \quad \text{i.e.} \quad K^\flat := k. \tag{24.16}$$

Indeed, with a basis $(\vec{e}_i)$, (24.15) gives $[\vec{w}]^T.[g].[K].[\vec{v}] = [\vec{v}]^T.[k].[\vec{w}]$, and the symmetry of k, cf. (24.8), gives (24.16). And $[K] = [g]^{-1}.[k]$.

(Component expressions: if $g = \sum_{i,j=1}^n g_{ij}e^i\otimes e^j$ and $K = \sum_{i,j=1}^n K^i_j\vec{e}_i\otimes e^j$ and $k = \sum_{i,j=1}^n k_{ij}e^i\otimes e^j$ then (24.15) and the symmetry of k give $(K.\vec{e}_j,\vec{e}_i)_g = k(\vec{e}_i,\vec{e}_j)$, that is, $\sum_\ell K^\ell_j(\vec{e}_\ell,\vec{e}_i)_g = k_{ij}$, that is, $\sum_\ell g_{i\ell}K^\ell_j = k_{ij}$ since g is symmetric, thus (24.16).)

Corollary 24.9 *k being symmetric, K is symmetric (relative to $(\cdot,\cdot)_g$), that is $K^T_g = K$: For all $\vec{v},\vec{w}$,*

$$(K.\vec{v},\vec{w})_g = (\vec{v},K.\vec{w})_g \quad (= k(\vec{v},\vec{w}) = k(\vec{w},\vec{v})). \tag{24.17}$$

And K gives the variations of $\vec{n}$ in S:

$$K = d\vec{n}, \tag{24.18}$$

i.e., for all $\vec{v},\vec{w} \in \Gamma(S)$,

$$(d\vec{n}.\vec{v},\vec{w})_g = k(\vec{v},\vec{w}) \quad (= (d\vec{n}^\flat.\vec{v}).\vec{w}). \tag{24.19}$$

Proof. (24.17) is given by (24.8).

Then $n^\flat.\vec{w} = (\vec{n},\vec{w})_g = g(\vec{n},\vec{w})$ gives

$$(dn^\flat.\vec{v}).\vec{w} + n^\flat.(d\vec{w}.\vec{v}) = (dg.\vec{v})(\vec{n},\vec{w}) + g(d\vec{n}.\vec{v},\vec{w}) + g(\vec{n},d\vec{w}.\vec{v}).$$

And $dg = 0$ (Euclidean metric), and $n^\flat.(d\vec{w}.\vec{v}) = (\vec{n},d\vec{w}.\vec{v})_g$, cf. (23.13), give (24.19). So, with (24.9),

$$(K.\vec{v},\vec{w})_{\mathbb{R}^n} = k(\vec{w},\vec{v}) = (dn^\flat.\vec{v}).\vec{w} = (d\vec{n}.\vec{v},\vec{w})_g, \tag{24.20}$$

i.e. (24.18) and (24.19). ▪

Example 24.10 Continuing example 24.8. $[k(p)]_{\vec{f}} = \begin{pmatrix} -R\cos^2\varphi & 0 \\ 0 & -R \end{pmatrix}$, cf. (24.14), $[g(p)]_{|\vec{f}} = [g_{ij}(p)] = \begin{pmatrix} R^2\cos^2\varphi & 0 \\ 0 & R^2 \end{pmatrix}$, thus $[g]^{-1} = \begin{pmatrix} \frac{1}{R^2\cos^2\varphi} & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix} = [g^{ij}(p)]$, thus

$$[K_p]_{|\vec{f}} = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix} = \frac{1}{R}I. \tag{24.21}$$

And the curvature of the sphere is $\frac{1}{R}$ (the radius of curvature is R). ▪

24.5.1 Principals, mean and Gaussian curvatures

Definition 24.11 Let $p \in S \subset \mathbb{R}^3$. The eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$ of the endomorphism $K(p)$ are the principal curvatures at $p \in S$, and their inverse $\frac{1}{\kappa_1}$ and $\frac{1}{\kappa_2}$ are the principal radius of curvatures.

The trace (first invariant)

$$\mathrm{Tr}(K_p) = \kappa_1 + \kappa_2 \tag{24.22}$$

is the mean curvature. The determinant (second invariant)

$$\det(K_p) = \kappa_1\kappa_2 \tag{24.23}$$

is the Gaussian curvature.

Remark 24.12 An alternative definition or the mean curvature is $\frac{1}{n-1}|\mathrm{Tr}(K)|$, which for the sphere gives $\frac{1}{R}$ (expected), cf. (24.21).

The non negative Gaussian curvature can be defined as $(|\det K|)^{\frac{1}{n-1}}$, which for the sphere gives $\frac{1}{R}$ (expected), cf. (24.21). ▪

Part VIII
Riemann curvature tensor

25 Geodesic deviation

Geodesics which are "somewhere parallel" do not remain "parallel" in a non-planar surface: E.g., two nearby meridians are parallel on the equator, but meet at the poles, see figure 21.1. This loss of the parallelism gives a way to measure the curvature of a surface "from inside the surface" (Gauss).

25.1 Family of geodesics and separation vector

Let S be a surface in $\mathbb{R}^n$, and ∇ be a usual Riemannian connection in $\mathbb{R}^n$ (the usual differential operator $\nabla = d$ in $\mathbb{R}^n$). Let $\varepsilon > 0$, $a < b$, and consider a regular geodesic family $(c_u)_{u\in[-\varepsilon,\varepsilon]}$ in S,

$$u \in [-\varepsilon, \varepsilon], \quad c_u : \left\{ \begin{array}{ll} [a,b] & \to S \\ s & \to p = c_u(s), \end{array} \right\}, \quad ||\vec{c}_u{}'(s)|| = 1, \forall s \in [a,b]. \tag{25.1}$$

The velocity at $p = c_u(s)$ along c_u is $\vec{v}_u(p) = c_u{}'(s)$, and $||\vec{v}_u(p)|| = 1$. And

$$c : \left\{ \begin{array}{ll} [a,b] \times [-\varepsilon, \varepsilon] & \to S \\ (s,u) & \to p = c(s,u) := c_u(s), \end{array} \right. \tag{25.2}$$

is a 2-D parametrized (sub-)surface in S, supposed regular. E.g., the family of meridians give the sphere. The velocity at $p = c_u(s)$ along c_u is then

$$\vec{v}_u(p) = c_u{}'(s) = \frac{\partial c}{\partial s}(s,u) \quad (\text{and } ||\vec{v}_u(p)|| = 1). \tag{25.3}$$

Definition 25.1 The transverse displacement at $s \in [a,b]$ is the curve

$$c_s : \left\{ \begin{array}{ll} [a,b] & \to S \\ u & \to c_s(u) := c(s,u) \quad (= c_u(s)). \end{array} \right. \tag{25.4}$$

And the transverse velocity at $p = c(s,u)$ is the velocity $\vec{w}_s(p)$ along c_s:

$$\vec{w}_s(p) = c_s{}'(u) = \frac{\partial c}{\partial u}(s,u). \tag{25.5}$$

And the transverse velocity at p is also called the separation vector at p.

Thus, the surface c being supposed regular, at $p = c(s,u)$,

$$\vec{f}_1(p) = \vec{v}_u(p) = \frac{\partial c}{\partial s}(s,u) \quad \text{and} \quad \vec{f}_2(p) = \vec{w}_s(p) = \frac{\partial c}{\partial u}(s,u) \tag{25.6}$$

are the basis tangent vectors to the 2-D surface c at p.

Example 25.2 $\mathbb{R}^2$, polar curves $\vec{c}_\theta(r) = \overrightarrow{Oc_\theta(r)} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$ (straight lines); Here $\vec{c}_\theta\,'(r) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ is unitary, so r is an intrinsic curvilinear coordinate. So $\vec{v}_\theta(p) = \vec{c}_\theta\,'(r) = \vec{e}_1(p)$, and $\vec{w}_r(p) = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} = \vec{e}_2(p)$ (the geodesics separate at 90°: $\vec{e}_1(p) \perp \vec{e}_2(p)$). ▪

Example 25.3 Family of meridians on the 2-D sphere in $\mathbb{R}^3$: Intrinsic parametrization $p = \vec{c}_\theta(s) = \vec{c}(s,\theta) = R\begin{pmatrix} \cos\theta\cos(\frac{s}{R}) \\ \sin\theta\cos(\frac{s}{R}) \\ \sin(\frac{s}{R}) \end{pmatrix}$ (indeed, $||\vec{c}_\theta\,'(s)|| = 1$ for all s), with $s \in]-R\frac{\pi}{2}, R\frac{\pi}{2}[$. So $\vec{v}_\theta(p) = \vec{c}_\theta\,'(s) = \begin{pmatrix} -\cos\theta\sin(\frac{s}{R}) \\ -\sin\theta\sin(\frac{s}{R}) \\ \cos(\frac{s}{R}) \end{pmatrix} = \frac{\vec{e}_3(p)}{R}$ (unit velocity along the meridian). In particular for $s = 0$ (on the equator), $\vec{v}(p) = \vec{c}_\theta\,'(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for all θ. And $\vec{w}_s(p) = \frac{\partial \vec{c}}{\partial \theta}(s,\theta) = R\cos(\frac{s}{R})\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = \vec{e}_2(p)$ (the geodesics separate at 90°). ▪

25.2 Separation velocity

Let $\vec{v}_u(p) =^{named} \vec{v}(p)$ and $\vec{w}_s(p) =^{named} \vec{w}(p)$.

Definition 25.4 The separation velocity at $p = c(s,u)$ in S is the vector

$$\text{Proj}_{T_pS}(d\vec{w}(p).\vec{v}(p) = \nabla_{\vec{v}}\vec{w}(p) = \frac{D\vec{w}}{ds}(p) \quad (= \text{Proj}_{T_pS}(\frac{\partial \frac{\partial c}{\partial u}}{\partial s}(s,u)) = \text{Proj}_{T_pS}(\frac{\partial^2 c}{\partial s \partial u}(s,u))). \tag{25.7}$$

(Measures the variations in S of $\vec{w}$ along a geodesic $c_u : s \to c_u(s)$.)

c being regular, we also have

$$d\vec{w}.\vec{v} = \frac{\partial \frac{\partial c}{\partial u}}{\partial s}(s,u) = \frac{\partial \frac{\partial c}{\partial s}}{\partial u}(s,u) = d\vec{v}.\vec{w}. \tag{25.8}$$

(Also obtained with (25.6): basis of a coordinate system.)

Example 25.5 Continuing example 25.2: $d\vec{e}_2.\vec{e}_1(p) = (\nabla_{\vec{v}}\vec{w})(p) = \frac{\partial \frac{\partial \vec{c}}{\partial \theta}}{\partial r}(s,\theta) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \frac{\vec{e}_2}{r}$, and the separation velocity is orthogonal to the radial geodesic c_θ. ▪

Example 25.6 Continuing example 25.3: $d\vec{e}_3(p).\frac{\vec{e}_2(p)}{R} = \frac{\partial \frac{\partial \vec{c}}{\partial \theta}}{\partial s}(s,\theta) = -\sin(\frac{s}{R})\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = -\frac{1}{R}\tan\varphi\vec{e}_2(p) = (\nabla_{\vec{v}}\vec{w})(p)$, which varies along the meridian θ (and vanishes at the equator). ▪

Exercise 25.7 Consider a family $(\vec{c}_u)_{u\in[-\varepsilon,\varepsilon]}$ of parallel straight lines (intrinsic coordinate) in a plane surface. Prove

$$\frac{D\vec{w}}{ds}(p_f) = 0 \tag{25.9}$$

i.e., the separation velocity vanishes (parallel straight lines remain parallel).

Answer. Let $\vec{z}_0 = \vec{c}_u'(s) = \frac{\partial \vec{c}}{\partial s}(u,s)$ the (unitary) common tangent vector to all curves, so, for all $s \in [a,b]$,

$$(\vec{c}_u(s) =) \quad \vec{c}(u,s) = s\vec{z}_0 + \vec{c}(u,0). \tag{25.10}$$

Thus $\frac{\partial \vec{c}}{\partial u}(u,s) = \frac{\partial \vec{c}}{\partial u}(u,0)$, and $\frac{\partial \frac{\partial \vec{c}}{\partial u}}{\partial s}(u,s) = \vec{0} = \frac{\partial \frac{\partial \vec{c}}{\partial s}}{\partial u}(u,s)$, thus $\frac{D\vec{w}}{ds}(p_f) = \frac{\partial \frac{\partial \vec{c}}{\partial s}}{\partial u}(0,s) = \vec{0}$. ▪

25.3 Separation acceleration (relative-acceleration)

Definition 25.8 The separation acceleration field along the fiducial geodesic $c_0 : s \to c_0(s)$ is, at $p = \vec{c}(0,s)$,

$$\nabla_{\vec{v}}(\nabla_{\vec{v}}\vec{w})(p) = \frac{D^2\vec{w}}{ds^2}(p) = \text{called the relative-acceleration} \tag{25.11}$$

thus $= \mathrm{Proj}_{T_pS}\frac{\partial \mathrm{Proj}_{T_pS}(\frac{\partial^2 c}{\partial s \partial u}(s,u))}{\partial s}(s,u)$.

Example 25.9 Continuing example 25.5: $\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{w}(p) = \vec{0}$ (a vector space is flat). ▪

Example 25.10 Continuing example 25.6: $(\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{w})(p) = \frac{\partial^2 \frac{\partial \vec{c}}{\partial \theta}}{\partial s^2}(s,\theta) = -\frac{1}{R}\cos(\frac{s}{R})\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$, which varies along the meridian θ (and its norm is $\frac{1}{R}$ at the equator where the meridians are parallel). ▪

26 Riemann curvature tensor

26.1 Lie bracket $[\nabla_{\vec{v}}, \nabla_{\vec{w}}]$ on $\Gamma(S)$

Let S be a manifold, let ∇ be connection, cf. definition 11.1, and let $\tilde{\nabla}$ be the differential operator on scalar functions $f \in \mathcal{F}(S)$, that is, $\tilde{\nabla}_{\vec{v}} f = df.\vec{v} = \mathcal{L}^0_{\vec{v}}(f)$, cf. (5.4).

Definition 26.1 Let ∇ be a connection in S. The Lie bracket $[\nabla_{\vec{v}}, \nabla_{\vec{w}}] : \Gamma(S) \to \Gamma(S)$ is defined by

$$\begin{aligned} [\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) &:= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) \\ &= (\nabla_{\vec{v}} \circ \nabla_{\vec{w}} - \nabla_{\vec{w}} \circ \nabla_{\vec{v}})(\vec{u}) \overset{\text{written}}{=} (\nabla_{\vec{v}}\nabla_{\vec{w}} - \nabla_{\vec{w}}\nabla_{\vec{v}})(\vec{u}). \end{aligned} \tag{26.1}$$

In the following will only consider connections $\left\{ \begin{array}{ll} \nabla : \Gamma(S) \times \Gamma(S) & \to \Gamma(S) \\ \qquad (\vec{v}, \vec{w}) & \to \nabla(\vec{v}, \vec{w}) = \nabla_{\vec{v}}\vec{w} \end{array} \right\}$, cf. (11.1), which are torsion-free (like Riemannian connections), i.e. connections ∇ related to $\left\{ \begin{array}{ll} \tilde{\nabla} : \Gamma(S) \times \mathcal{F}(S) & \to \mathcal{F}(S) \\ \qquad (\vec{v}, f) & \to \tilde{\nabla}_{\vec{v}}(f) := df.\vec{v} \end{array} \right\}$, cf. (5.5), through the relation

$$\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v} = \tilde{\nabla}_{\vec{v}} \circ \tilde{\nabla}_{\vec{w}} - \tilde{\nabla}_{\vec{w}} \circ \tilde{\nabla}_{\vec{v}} \quad (= [\vec{v}, \vec{w}]), \tag{26.2}$$

cf. (11.11) (the vector field $\vec{u} = \nabla_{\vec{v}}\vec{w}$ is identified with the derivation operator $\tilde{\nabla}_{\vec{u}}$ on $\mathcal{F}(S)$, cf. (4.7)). That is, we will only consider connections ∇ such that, for all $f \in \mathcal{F}(S)$,

$$(\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v})(f) = d(df.\vec{w}).\vec{v}) - d(df.\vec{v}).\vec{w}) \tag{26.3}$$

(Expression in a holonomic basis: The Christoffel symbols disappear in $[\vec{v}, \vec{w}]$, cf. (11.20).)

Example 26.2 And in S, with the Riemannian connection $\nabla_{\vec{v}}\vec{w} = \mathrm{Proj}_{TS}(d\vec{w}.\vec{v})$, we get, for all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$,

$$\begin{aligned} [\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) \\ &= \mathrm{Proj}_{TS}\Big(d(\mathrm{Proj}_{TS}(d\vec{u}.\vec{w}).\vec{v}) - d(\mathrm{Proj}_{TS}(d\vec{u}.\vec{v}).\vec{w}\Big). \end{aligned} \tag{26.4}$$

(So the derivations of the projection $\mathrm{Proj}_{TS}(d\vec{u}.\vec{w})$ and $\mathrm{Proj}_{TS}(d\vec{u}.\vec{v})$ are considered before being differentiated.) E.g., on the 2-D sphere in $\mathbb{R}^3$, with $\vec{u} = \vec{w} = \vec{e}_3$ and $\vec{v} = \vec{e}_2$, and with $d\vec{e}_3(p).\vec{e}_3(p) = -r\,\vec{e}_1(p)$ and $d\vec{e}_2(p).\vec{e}_3(p) = -\tan\varphi\,\vec{e}_2(p)$, cf. (3.34)-(3.35), we get $\nabla_{\vec{e}_3}\vec{e}_3 = \mathrm{Proj}_{TS}(d\vec{e}_3.\vec{e}_3) = \vec{0}$, and $\nabla_{\vec{e}_2}\vec{e}_3 = \mathrm{Proj}_{TS}(d\vec{e}_3.\vec{e}_2) = -\tan\varphi\,\vec{e}_2 = \nabla_{\vec{e}_3}\vec{e}_2$, thus, $\nabla_{\vec{e}_2}(\nabla_{\vec{e}_3}\vec{e}_3) = \nabla_{\vec{e}_2}(\vec{0}) = \vec{0}$, and $\nabla_{\vec{e}_3}(\nabla_{\vec{e}_2}\vec{e}_3) = \mathrm{Proj}_{TS}(d(-\tan\varphi\vec{e}_2).\vec{e}_3) = -\mathrm{Proj}_{TS}(\frac{\partial(\tan\varphi\,\vec{e}_2)}{\partial\varphi}) = -\mathrm{Proj}_{TS}((1+\tan^2\varphi)\vec{e}_2 - \tan\varphi\,\vec{e}_2) = -\vec{e}_2$. Hence,

$$[\nabla_{\vec{e}_2}, \nabla_{\vec{e}_3}](\vec{e}_3) = \vec{e}_2. \tag{26.5}$$

NB: $[\nabla_{\vec{e}_2}, \nabla_{\vec{e}_3}] : \Gamma(S) \to \Gamma(S)$ is obviously different from $[\vec{e}_2, \vec{e}_3] : \mathcal{F}(S) \to \mathcal{F}(S)$. And moreover we have $[\vec{e}_2, \vec{e}_3] = 0$, cf. (5.12). ▪

Proposition 26.3 *The Lie bracket operator* $B : (\vec{v},\vec{w}) \in \Gamma(S)^2 \to B(\vec{v},\vec{w}) = [\nabla_{\vec{v}},\nabla_{\vec{w}}] \in \mathcal{F}(\Gamma(S);\Gamma(S))$ *is* $\mathbb{R}$*-bilinear and antisymmetric. But is not* $\mathcal{F}(S)$*-bilinear (not tensorial because of a* df *term): For all* $f \in \mathcal{F}(S)$ *and all* $\vec{u},\vec{v},\vec{w} \in \Gamma(S)$,

$$[\nabla_{f\vec{v}},\nabla_{\vec{w}}](\vec{u}) = f\,[\nabla_{\vec{v}},\nabla_{\vec{w}}](\vec{u}) - (df.\vec{w})\nabla_{\vec{v}}\vec{u}, \tag{26.6}$$

and,

$$[\nabla_{\vec{v}},\nabla_{\vec{w}}](f\vec{u}) = f[\nabla_{\vec{v}},\nabla_{\vec{w}}](\vec{u}) + ([\vec{v},\vec{w}](f))\vec{u}. \tag{26.7}$$

Proof. For all $\vec{u},\vec{v},\vec{w},\vec{z} \in \Gamma(S)$, $B(\vec{v},\vec{w})(\vec{u}) = -B(\vec{w},\vec{v})(\vec{u})$ (trivial). For all $\vec{u},\vec{v},\vec{w},\vec{z} \in \Gamma(S)$:

$$\begin{aligned} B(\vec{v}+\vec{z},\vec{w})(\vec{u}) &= \nabla_{\vec{v}+\vec{z}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}+\vec{z}}\vec{u}), \quad \text{cf. (26.1)},\\ &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) + \nabla_{\vec{z}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u} + \nabla_{\vec{z}}\vec{u}),\\ &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) + \nabla_{\vec{z}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{z}}\vec{u}),\\ &= B(\vec{v},\vec{w})(\vec{u}) + B(\vec{z},\vec{w})(\vec{u}). \end{aligned}$$

For all $f \in \mathcal{F}(S)$, $\vec{u},\vec{v},\vec{w} \in \Gamma(S)$:

$$\begin{aligned} B(f\vec{v},\vec{w})(\vec{u}) &= \nabla_{f\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{f\vec{v}}\vec{u}), \quad \text{cf. (26.1)},\\ &= f\nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(f\nabla_{\vec{v}}\vec{u}), \quad \text{cf. (11.2)},\\ &= f\nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - (df.\vec{v})\nabla_{\vec{v}}\vec{u} - f\nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}),\\ &= fB(\vec{v},\vec{w})(\vec{u}) - (df.\vec{v})\nabla_{\vec{v}}\vec{u}. \end{aligned} \tag{26.8}$$

Thus B is $\mathbb{R}$-bilinear (take f = constant), but not $\mathcal{F}(S)$-bilinear since $(df.\vec{v}) \neq 0$ in general.

And for all $f \in \mathcal{F}(S)$, $\vec{u},\vec{v},\vec{w} \in \Gamma(S)$:

$$\begin{aligned} [\nabla_{\vec{v}},\nabla_{\vec{w}}](f\vec{u}) &= \nabla_{\vec{v}}(\nabla_{\vec{w}}(f\vec{u})) - \nabla_{\vec{w}}(\nabla_{\vec{v}}(f\vec{u}))\\ &= \nabla_{\vec{v}}((df.\vec{w})\vec{u} + f\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}((df.\vec{v})\vec{u} + f\nabla_{\vec{v}}\vec{u})\\ &= (d(df.\vec{w}).\vec{v})\vec{u} + (df.\vec{w})\nabla_{\vec{v}}\vec{u} + (df.\vec{v})\nabla_{\vec{w}}\vec{u} + f\nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u})\\ &\quad - (d(df.\vec{v}).\vec{w})\vec{u} - (df.\vec{v})\nabla_{\vec{w}}\vec{u} - (df.\vec{w})\nabla_{\vec{v}}\vec{u} - f\nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u})\\ &= ([\vec{v},\vec{w}](f))\vec{u} + f[\nabla_{\vec{v}},\nabla_{\vec{w}}](\vec{u}) \end{aligned} \tag{26.9}$$

∴

Example 26.4 Usual Riemannian connection ∇ on a surface S in $\mathbb{R}^n$, that is, $\nabla_{\vec{w}}\vec{u} = \text{Proj}_{TS}(d\vec{u}.\vec{w})$: We get

$$\begin{aligned} [\nabla_{\vec{v}},\nabla_{\vec{w}}](\vec{u}) &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u})\\ &= \text{Proj}_{TS}\Big(d(\text{Proj}_{TS}(d\vec{u}.\vec{w})).\vec{v}\Big) - \text{Proj}_{TS}\Big(d(\text{Proj}_{TS}(d\vec{u}.\vec{v})).\vec{w}\Big). \end{aligned} \tag{26.10}$$

To compare with

$$\nabla_{[\vec{v},\vec{w}]}\vec{u} = \text{Proj}_{TS}(d\vec{u}.[\vec{v},\vec{w}]) = \text{Proj}_{TS}(d\vec{u}.(d\vec{w}.\vec{v})) - \text{Proj}_{TS}(d\vec{u}.(d\vec{v}.\vec{w})). \tag{26.11}$$

∴

Proposition 26.5

$$[\nabla_{\vec{v}},\nabla_{\vec{w}}](\vec{u}) \neq \nabla_{[\vec{v},\vec{w}]}\vec{u} \quad \textit{in general.} \tag{26.12}$$

In particular for the usual Riemann connection in S *a 2-D surface in* $\mathbb{R}^3$*, if* $(\vec{e}_i)$ *is a coordinate basis, then* $[\vec{e}_i,\vec{e}_j] = 0$ *but*

$$[\nabla_{\vec{e}_i},\nabla_{\vec{e}_j}] \neq \vec{0} \quad \textit{in general.} \tag{26.13}$$

Proof. On the 2-D sphere in $\mathbb{R}^3$, and $\vec{e}_2,\vec{e}_3$ the coordinate basis, (26.5) gives $[\nabla_{\vec{e}_2},\nabla_{\vec{e}_3}](\vec{e}_3) = \vec{e}_2$, whereas $[\vec{e}_2,\vec{e}_3] = \vec{0}$ (basis of a coordinate system) and then $\nabla_{[\vec{e}_2,\vec{e}_3]}(\vec{e}_3) = 0$. ∴

26.2 Gravitation forces (tidal forces)

Proposition 26.6 *If ∇ is a torsion-free connection, then the separation vector $\vec{w}$, cf. (25.5), satisfies*

$$(\frac{D^2\vec{w}}{ds^2}=)\quad \underbrace{\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{w}}_{\text{relative-acceleration}} \quad = - \quad \underbrace{[\nabla_{\vec{w}},\nabla_{\vec{v}}]\vec{v}}_{\text{gravitation force}} \quad (= -\nabla_{\vec{w}}\nabla_{\vec{v}}\vec{v} - \nabla_{\vec{v}}\nabla_{\vec{w}}\vec{v}). \tag{26.14}$$

(The "separation acceleration" = "relative-acceleration": gives a measure of gravity.)

Proof. $\nabla_{\vec{v}}\vec{w} = \nabla_{\vec{w}}\vec{v} = \frac{\partial^2 c}{\partial s \partial u} = \frac{\partial^2 c}{\partial u \partial s}$, cf. (25.5), thus $\nabla_{\vec{v}}(\nabla_{\vec{v}}\vec{w}) = \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{v})$.

And $s \to c_u(s)$ being a geodesic, $\nabla_{\vec{v}}\vec{v} = \vec{0}$. Thus $\nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{v}) = 0$.

Thus

$$\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{w} + [\nabla_{\vec{w}},\nabla_{\vec{v}}]\vec{v} = \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{v}) + \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{v}) - \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{v}) = 0.$$

▪

Remark 26.7 The Riemann curvature tensor is defined by $R(\vec{a},\vec{b},\vec{c}) := [\nabla_{\vec{a}},\nabla_{\vec{b}}]\vec{c} - \nabla_{[\vec{a},\vec{b}]}\vec{c}$, see (26.19). In particular, here $[\vec{w},\vec{v}]=0$, cf. (25.6), and $R(\vec{w},\vec{v},\vec{v}) = [\nabla_{\vec{w}},\nabla_{\vec{v}}]\vec{v}$ describe the gravitation field. ▪

26.3 Riemann curvature on $\Gamma(S)$

If $\vec{v},\vec{w} \in \Gamma(S)$ then the Riemann curvature operation $\rho(\vec{v},\vec{w})$ relative to $\vec{v}$ and $\vec{w}$ is the map

$$\rho(\vec{v},\vec{w}) : \begin{cases} \Gamma(S) & \to \Gamma(S) \\ \quad \vec{u} & \to \rho(\vec{v},\vec{w})(\vec{u}) := [\nabla_{\vec{v}},\nabla_{\vec{w}}](\vec{u}) - \nabla_{[\vec{v},\vec{w}]}\vec{u}, \end{cases} \tag{26.15}$$

that is,

$$\begin{aligned} \rho(\vec{v},\vec{w})(\vec{u}) &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) - \nabla_{[\vec{v},\vec{w}]}\vec{u} \\ &\overset{\text{written}}{=} \nabla_{\vec{v}}\nabla_{\vec{w}}\vec{u} - \nabla_{\vec{w}}\nabla_{\vec{v}}\vec{u} - \nabla_{[\vec{v},\vec{w}]}\vec{u}. \end{aligned} \tag{26.16}$$

The Riemann curvature operator is the map

$$\rho : \begin{cases} \Gamma(S)\times\Gamma(S) & \to \mathcal{F}(\Gamma(S);\Gamma(S)) \\ \quad (\vec{v},\vec{w}) & \to \rho(\vec{v},\vec{w}) \end{cases} \tag{26.17}$$

The Riemann curvature is the map

$$R : \begin{cases} (\Gamma(S))^3 & \to \Gamma(S) \\ (\vec{u},\vec{v},\vec{w}) & \to R(\vec{u},\vec{v},\vec{w}) := \rho(\vec{v},\vec{w})(\vec{u}) \quad (= (\nabla_{\vec{v}}\nabla_{\vec{w}} - \nabla_{\vec{w}}\nabla_{\vec{v}} - \nabla_{[\vec{v},\vec{w}]})(\vec{u})) \end{cases} \tag{26.18}$$

(NB: (26.18) is the definition found in Misner–Thorne–Wheeler [15] (to get "usual" symmetries).) It is also the name of the associated tensor $\mathcal{R} \in T^1_3(S)$ defined by

$$\mathcal{R} : \begin{cases} \Omega^1(S)\times(\Gamma(S))^3 & \to \mathbb{R} \\ \quad (\alpha,\vec{u},\vec{v},\vec{w}) & \to \mathcal{R}(\alpha,\vec{u},\vec{v},\vec{w}) := \alpha.R(\vec{u},\vec{v},\vec{w}) \quad (= \alpha.[\nabla_{\vec{v}},\nabla_{\vec{w}}](\vec{u}) - \alpha.\nabla_{[\vec{v},\vec{w}]}(\vec{u}). \end{cases} \tag{26.19}$$

Proposition 26.8 *$\mathcal{R}$ defined in (26.19) is a $\binom{1}{3}$ tensor: $\mathcal{R} \in T^1_3(S)$. And R defined in (26.18) is also said to be a tensor, as well as ρ defined in (26.17). And ρ is antisymmetric, that is, for all $\vec{v},\vec{w} \in \Gamma(S)$,*

$$\rho(\vec{v},\vec{w}) = -\rho(\vec{w},\vec{v}), \tag{26.20}$$

that is, for all $\vec{u},\vec{v},\vec{w} \in \Gamma(S)$,

$$R(\vec{u},\vec{v},\vec{w}) = -R(\vec{u},\vec{w},\vec{v}) \quad (\textit{antisymmetry for the last to slots}), \tag{26.21}$$

that is, for all $\alpha \in \Omega^1(S)$ and $\vec{u},\vec{v},\vec{w} \in \Gamma(S)$,

$$\mathcal{R}(\alpha,\vec{u},\vec{v},\vec{w}) = -\mathcal{R}(\alpha,\vec{u},\vec{w},\vec{v}) \quad (\textit{antisymmetry for the last two slots}). \tag{26.22}$$

Proof. We have to prove that $\mathcal{R}$ is $\mathcal{F}(S)$-multilinear , that is, $\mathcal{R}(...,z_1+z_2,...) = \mathcal{R}(...,z_1,...) + \mathcal{R}(...,z_2,...)$ and $\mathcal{R}(...,f,...) = f\mathcal{R}(...,z,...)$ for all $f \in \mathcal{F}(S)$ and z, z_1, z_2 in $\Omega^1(S)$ or $\Gamma(S)$ where appropriate.

0- $\mathcal{R}(...,z_1+z_2,...) = \mathcal{R}(...,z_1,...) + \mathcal{R}(...,z_2,...)$ is trivial, for all z_1, z_2 in $\Omega^1(S)$ or $\Gamma(S)$ where appropriate. Then for $f \in \mathcal{F}(S)$,

1- $\mathcal{R}(f\alpha,\vec{u},\vec{v},\vec{w}) = f\mathcal{R}(\alpha,\vec{u},\vec{v},\vec{w})$ is trivial ($= f\,\alpha.R(\vec{u},\vec{v},\vec{w})$).

2- Then (10.5) gives $[f\vec{v},\vec{w}] = f([\vec{v},\vec{w}] - (df.\vec{w})\vec{v}$, and (26.8) gives $\rho(f\vec{v},\vec{w})(\vec{u}) = B(f\vec{v},\vec{w})\vec{u} - [f\vec{v},\vec{w}](\vec{u}) = f\,B(\vec{v},\vec{w})\vec{u} - (df.\vec{w})\nabla_{\vec{v}}\vec{u} - (f([\vec{v},\vec{w}] - (df.\vec{w})\vec{v}) = f(B(\vec{v},\vec{w})\vec{u} - [\vec{v},\vec{w}](\vec{u})) = f\rho(\vec{v},\vec{w})(\vec{u})$. Then $R(\vec{u},f\vec{v},\vec{w}) = f\,R(\vec{u},\vec{v},\vec{w})$.

3- Then ρ being antisymmetric, $\rho(\vec{v},f\vec{w})(\vec{u}) = -\rho(f\vec{w},\vec{v})(\vec{u}) = -f\rho(\vec{w},\vec{v})(\vec{u}) = f\rho(\vec{v},\vec{w})(\vec{u})$. Then $R(\vec{u},\vec{v},f\vec{w}) = f\,R(\vec{u},\vec{v},\vec{w})$.

4- And, (26.7) and (11.3) give $\rho(\vec{v},\vec{w})(f\vec{u}) = [\nabla_{\vec{v}},\nabla_{\vec{w}}](f\vec{u}) - \nabla_{[\vec{v},\vec{w}]}(f\vec{u}) = ([\vec{v},\vec{w}](f))\vec{u} + f[\nabla_{\vec{v}},\nabla_{\vec{w}}](\vec{u}) - ([\vec{v},\vec{w}](f))\vec{u} - f\nabla_{[\vec{v},\vec{w}]}(\vec{u}) = f[\nabla_{\vec{v}},\nabla_{\vec{w}}](\vec{u}) - f\nabla_{[\vec{v},\vec{w}]}(\vec{u}) = f\,\rho(\vec{v},\vec{w})(\vec{u})$. Then $R(f\vec{u},\vec{v},\vec{w}) = f\,R(\vec{u},\vec{v},\vec{w})$. ∎

26.4 Expression in a coordinate system

With the coordinate basis $(\vec{e}_i)$, let $R^i_{jk\ell}$ be the components of the tensor $\mathcal{R} \in T^1_3$:

$$\mathcal{R} = \sum_{i,j,k,\ell=1}^{m} R^i_{jk\ell}\vec{e}_i \otimes e^j \otimes e^k \otimes e^\ell, \quad \text{i.e.} \quad R^i_{jk\ell} = \mathcal{R}(e^i,\vec{e}_j,\vec{e}_k,\vec{e}_\ell). \tag{26.23}$$

Thus $R^i_{jk\ell} = e^i.R(\vec{e}_j,\vec{e}_k,\vec{e}_\ell) = e^i.\rho(\vec{e}_k,\vec{e}_\ell).\vec{e}_j$ and

$$R(\vec{e}_j,\vec{e}_k,\vec{e}_\ell) = \sum_{i=1}^{m} R^i_{jk\ell}\vec{e}_i, \quad \text{and} \quad \rho(\vec{e}_k,\vec{e}_\ell) = \sum_{i,j=1}^{m} R^i_{jk\ell}\vec{e}_i \otimes e^j. \tag{26.24}$$

In particular, (26.22) gives $\mathcal{R}(e^i,\vec{e}_j,\vec{e}_k,\vec{e}_\ell) = -\mathcal{R}(e^i,\vec{e}_j,\vec{e}_\ell,\vec{e}_k)$, that is

$$R^i_{jk\ell} = -R^i_{j\ell k} \quad \text{(last two indices antisymmetry)}. \tag{26.25}$$

Proposition 26.9 *With the Chistoffel symbols $\gamma^i_{jk} = e^i.\nabla_{\vec{e}_j}\vec{e}_k$, we get*

$$\nabla_{\vec{e}_k}\nabla_{\vec{e}_\ell}\vec{e}_j = \sum_i (\frac{\partial \gamma^i_{\ell j}}{\partial q^k} + \sum_\nu \gamma^\nu_{\ell j}\gamma^i_{k\nu})\vec{e}_i. \tag{26.26}$$

Then,

$$R^i_{jk\ell} = \frac{\partial \gamma^i_{j\ell}}{\partial q^k} - \frac{\partial \gamma^i_{jk}}{\partial q^\ell} + \sum_\nu \gamma^i_{k\nu}\gamma^\nu_{j\ell} - \sum_\nu \gamma^i_{\ell\nu}\gamma^\nu_{jk}. \tag{26.27}$$

Proof. $\nabla_{\vec{e}_\ell}\vec{e}_j = \sum_i \gamma^i_{\ell j}\vec{e}_i$ gives

$$\nabla_{\vec{e}_k}(\nabla_{\vec{e}_\ell}\vec{e}_j) = \sum_i (d\gamma^i_{\ell j}.\vec{e}_k)\vec{e}_i + \sum_i \gamma^i_{\ell j}\nabla_{\vec{e}_k}\vec{e}_i = \sum_i \frac{\partial \gamma^i_{\ell j}}{\partial q^k}\vec{e}_i + \sum_{i\nu} \gamma^i_{\ell j}\gamma^\nu_{ki}\vec{e}_\nu$$

Thus $\nabla_{\vec{e}_k}\nabla_{\vec{e}_\ell}\vec{e}_j - \nabla_{\vec{e}_\ell}\nabla_{\vec{e}_k}\vec{e}_j$ and $[\vec{e}_k,\vec{e}_\ell] = 0$ and $\gamma^a_{bc} = \gamma^a_{cb}$ give (26.27). ∎

Remark 26.10 $R(\vec{e}_j,\vec{e}_k,\vec{e}_\ell) = \sum_{i=1}^m R^i_{jk\ell}\vec{e}_i$ is also written $R(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}) = \sum_{i=1}^m R^i_{jk\ell}\frac{\partial}{\partial x^i}$. ∎

26.5 Flat space: $R = 0$

Corollary 26.11 *Let ∇ be a usual Riemannian metric. If S is a flat surface in $\mathbb{R}^n$, then $R = 0$.*

Proof. Choose a Cartesian basis: Thus $\gamma^i_{jk} = 0$, and (26.27) gives the result. ∎

Definition 26.12 A flat manifold is a manifold such that $R = 0$.

26.6 Bianchi identities

Proposition 26.13 *For all $\vec{u},\vec{v},\vec{w}\in\Gamma(S)$:*
First Bianchi identity (cyclic permutation):

$$R(\vec{u},\vec{v},\vec{w})+R(\vec{w},\vec{u},\vec{v})+R(\vec{v},\vec{w},\vec{u})=0,\quad i.e.\quad R^i_{jk\ell}+R^i_{k\ell j}+R^i_{\ell jk}=0\overset{\text{written}}{=}R^i_{[jk\ell]}. \tag{26.28}$$

Second Bianchi identity: If ∇ is torsion free then, for all $\vec{z}\in\Gamma(S)$,

$$(\nabla_{\vec{z}}\rho)(\vec{v},\vec{w})+(\nabla_{\vec{w}}\rho)(\vec{z},\vec{v})+(\nabla_{\vec{v}}\rho)(\vec{w},\vec{z})=0, \tag{26.29}$$

that is, $(\nabla_{\vec{z}}\rho)(\vec{v},\vec{w}).\vec{u}+(\nabla_{\vec{w}}\rho)(\vec{z},\vec{v}).\vec{u}+(\nabla_{\vec{v}}\rho)(\vec{w},\vec{z}).\vec{u}=0$. And

$$R^i_{jk\ell|\nu}+R^i_{j\ell\nu|k}+R^i_{j\nu k|\ell}=0\overset{\text{written}}{=}R^i_{j[k\ell|\nu]}. \tag{26.30}$$

Proof. The Jacobi identity (10.10) reads $\nabla_{\vec{v}}([\vec{w},\vec{u}])-\nabla_{[\vec{v},\vec{w}]}\vec{u}+\nabla_{\vec{w}}([\vec{u},\vec{v}])-\nabla_{[\vec{w},\vec{u}]}\vec{v}+\nabla_{\vec{u}}([\vec{v},\vec{w}])-\nabla_{[\vec{u},\vec{v}]}\vec{w}=0$. Thus, with (26.18) we have $R(\vec{u},\vec{v},\vec{w})+R(\vec{v},\vec{w},\vec{u})+R(\vec{w},\vec{u},\vec{v})$
$=\nabla_{\vec{v}}\nabla_{\vec{w}}\vec{u}-\nabla_{\vec{w}}\nabla_{\vec{v}}\vec{u}-\nabla_{[\vec{v},\vec{w}]}\vec{u}+\nabla_{\vec{w}}\nabla_{\vec{u}}\vec{v}-\nabla_{\vec{u}}\nabla_{\vec{w}}\vec{v}-\nabla_{[\vec{w},\vec{u}]}\vec{v}+\nabla_{\vec{u}}\nabla_{\vec{v}}\vec{w}-\nabla_{\vec{v}}\nabla_{\vec{u}}\vec{w}-\nabla_{[\vec{u},\vec{v}]}\vec{w}$
$=\nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}-\nabla_{\vec{u}}\vec{w})+\nabla_{\vec{w}}(\nabla_{\vec{u}}\vec{v}-\nabla_{\vec{v}}\vec{u})+\nabla_{\vec{u}}(\nabla_{\vec{v}}\vec{w}-\nabla_{\vec{w}}\vec{v})-\nabla_{[\vec{v},\vec{w}]}\vec{u}-\nabla_{[\vec{w},\vec{u}]}\vec{v}-\nabla_{[\vec{u},\vec{v}]}\vec{w}$
$=\nabla_{\vec{v}}([\vec{w},\vec{u}])+\nabla_{\vec{w}}([\vec{u},\vec{v}])+\nabla_{\vec{u}}([\vec{v},\vec{w}])-\nabla_{[\vec{v},\vec{w}]}\vec{u}-\nabla_{[\vec{w},\vec{u}]}\vec{v}-\nabla_{[\vec{u},\vec{v}]}\vec{w}=0$ (Jacobi identity), i.e. (26.28).

Then (See https://math.stackexchange.com/questions/1494262/direct-proof-of-the-second-bianchi-identity) $\nabla_{\vec{z}}(\rho(\vec{v},\vec{w}).\vec{u})=(\nabla_{\vec{z}}\rho)(\vec{v},\vec{w}).\vec{u}+\rho(\nabla_{\vec{z}}\vec{v},\vec{w}).\vec{u}+\rho(\vec{v},\nabla_{\vec{z}}\vec{w}).\vec{u}+\rho(\vec{v},\vec{w}).\nabla_{\vec{z}}\vec{u}$, i.e.,

$$\begin{aligned}(\nabla_{\vec{z}}\rho)(\vec{v},\vec{w}).\vec{u}&=\nabla_{\vec{z}}(\rho(\vec{v},\vec{w}).\vec{u})-\rho(\nabla_{\vec{z}}\vec{v},\vec{w}).\vec{u}-\rho(\vec{v},\nabla_{\vec{z}}\vec{w}).\vec{u}-\rho(\vec{v},\vec{w}).\nabla_{\vec{z}}\vec{u},\\&=\nabla_{\vec{z}}(\rho(\vec{v},\vec{w}).\vec{u})+\rho(\nabla_{\vec{z}}\vec{w},\vec{v}).\vec{u}-\rho(\nabla_{\vec{z}}\vec{v},\vec{w}).\vec{u}-\rho(\vec{v},\vec{w}).\nabla_{\vec{z}}\vec{u},\end{aligned} \tag{26.31}$$

ρ being antisymmetric. And for the first term in (26.31),

- $\rho(\vec{v},\vec{w}).\vec{u}=\nabla_{\vec{v}}\nabla_{\vec{w}}\vec{u}-\nabla_{\vec{w}}\nabla_{\vec{v}}\vec{u}-\nabla_{[\vec{v},\vec{w}]}\vec{u}$ gives

$$\nabla_{\vec{z}}(\rho(\vec{v},\vec{w}).\vec{u})=\nabla_{\vec{z}}\nabla_{\vec{v}}\nabla_{\vec{w}}\vec{u}-\nabla_{\vec{z}}\nabla_{\vec{w}}\nabla_{\vec{v}}\vec{u}-\nabla_{\vec{z}}\nabla_{[\vec{v},\vec{w}]}\vec{u},$$

with $\nabla_{\vec{z}}\nabla_{[\vec{v},\vec{w}]}\vec{u}-\nabla_{[\vec{v},\vec{w}]}\nabla_{\vec{z}}\vec{u}-\nabla_{[\vec{z},[\vec{v},\vec{w}]]}\vec{u}=\rho(\vec{z},[\vec{v},\vec{w}]).\vec{u}$, so

$$\nabla_{\vec{z}}(\rho(\vec{v},\vec{w}).\vec{u})=\nabla_{\vec{z}}\nabla_{\vec{v}}\nabla_{\vec{w}}\vec{u}-\nabla_{\vec{z}}\nabla_{\vec{w}}\nabla_{\vec{v}}\vec{u}-\nabla_{[\vec{v},\vec{w}]}\nabla_{\vec{z}}\vec{u}-\nabla_{[\vec{z},[\vec{v},\vec{w}]]}\vec{u}-\rho(\vec{z},[\vec{v},\vec{w}]).\vec{u}.$$

Thus (circular permutation),

$$\begin{aligned}&\nabla_{\vec{z}}\,(\rho(\vec{v},\vec{w}).\vec{u})+\nabla_{\vec{w}}(\rho(\vec{z},\vec{v}).\vec{u})+\nabla_{\vec{v}}(\rho(\vec{w},\vec{z}).\vec{u})\\&=\nabla_{\vec{z}}\nabla_{\vec{v}}\nabla_{\vec{w}}\vec{u}-\nabla_{\vec{z}}\nabla_{\vec{w}}\nabla_{\vec{v}}\vec{u}+\nabla_{\vec{w}}\nabla_{\vec{z}}\nabla_{\vec{v}}\vec{u}-\nabla_{\vec{w}}\nabla_{\vec{v}}\nabla_{\vec{z}}\vec{u}+\nabla_{\vec{v}}\nabla_{\vec{w}}\nabla_{\vec{z}}\vec{u}-\nabla_{\vec{v}}\nabla_{\vec{z}}\nabla_{\vec{w}}\vec{u}\\&\quad-\nabla_{[\vec{v},\vec{w}]}\nabla_{\vec{z}}\vec{u}-\nabla_{[\vec{z},\vec{v}]}\nabla_{\vec{w}}\vec{u}-\nabla_{[\vec{w},\vec{z}]}\nabla_{\vec{v}}\vec{u}-\nabla_{[\vec{z},[\vec{v},\vec{w}]]}\vec{u}-\nabla_{[\vec{w},[\vec{z},\vec{v}]]}\vec{u}-\nabla_{[\vec{v},[\vec{w},\vec{z}]]}\vec{u}\\&\quad-\rho(\vec{z},[\vec{v},\vec{w}]).\vec{u}-\rho(\vec{w},[\vec{z},\vec{v}]).\vec{u}-\rho(\vec{v},[\vec{w},\vec{z}]).\vec{u}\\&=[\nabla_{\vec{z}},\nabla_{\vec{v}}]\nabla_{\vec{w}}\vec{u}+[\nabla_{\vec{w}},\nabla_{\vec{z}}]\nabla_{\vec{v}}\vec{u}+[\nabla_{\vec{v}},\nabla_{\vec{w}}]\nabla_{\vec{z}}\vec{u}\\&\quad-\nabla_{[\vec{v},\vec{w}]}\nabla_{\vec{z}}\vec{u}-\nabla_{[\vec{z},\vec{v}]}\nabla_{\vec{w}}\vec{u}-\nabla_{[\vec{w},\vec{z}]}\nabla_{\vec{v}}\vec{u}-0(\text{Jacobi identity})\\&\quad-\rho(\vec{z},[\vec{v},\vec{w}]).\vec{u}-\rho(\vec{w},[\vec{z},\vec{v}]).\vec{u}-\rho(\vec{v},[\vec{w},\vec{z}]).\vec{u}\\&=\rho(\vec{z},\vec{v})\nabla_{\vec{w}}\vec{u}+\rho(\vec{w},\vec{z})\nabla_{\vec{v}}\vec{u}+\rho(\vec{v},\vec{w})\nabla_{\vec{z}}\vec{u}\\&\quad-\rho(\vec{z},[\vec{v},\vec{w}]).\vec{u}-\rho(\vec{w},[\vec{z},\vec{v}]).\vec{u}-\rho(\vec{v},[\vec{w},\vec{z}]).\vec{u}.\end{aligned}$$

- And for the last three terms in (26.31), (circular permutation),

$$\begin{aligned}&\rho(\nabla_{\vec{z}}\vec{w},\vec{v}).\vec{u}-\rho(\nabla_{\vec{z}}\vec{v},\vec{w}).\vec{u}-\rho(\vec{v},\vec{w}).\nabla_{\vec{z}}\vec{u}+\rho(\nabla_{\vec{v}}\vec{z},\vec{w}).\vec{u}-\rho(\nabla_{\vec{v}}\vec{w},\vec{z}).\vec{u}-\rho(\vec{z},\vec{v}).\nabla_{\vec{w}}\vec{u}\\&\quad+\rho(\nabla_{\vec{w}}\vec{v},\vec{z}).\vec{u}-\rho(\nabla_{\vec{w}}\vec{z},\vec{v}).\vec{u}-\rho(\vec{w},\vec{z}).\nabla_{\vec{v}}\vec{u}\\&=\rho(\nabla_{\vec{z}}\vec{w}-\nabla_{\vec{w}}\vec{z},\vec{v}).\vec{u}+\rho(\nabla_{\vec{v}}\vec{z}-\nabla_{\vec{z}}\vec{v},\vec{w}).\vec{u}+\rho(\nabla_{\vec{w}}\vec{v}-\nabla_{\vec{v}}\vec{w},\vec{z}).\vec{u}\\&\quad-\rho(\vec{v},\vec{w}).\nabla_{\vec{z}}\vec{u}-\rho(\vec{z},\vec{v}).\nabla_{\vec{w}}\vec{u}-\rho(\vec{w},\vec{z}).\nabla_{\vec{v}}\vec{u}\\&=\rho([\vec{z},\vec{w}]).\vec{u}+\rho([\vec{v},\vec{z}],\vec{w}).\vec{u}+\rho([\vec{w},\vec{v}],\vec{z}).\vec{u}\ (\text{torsion-free})\\&\quad-\rho(\vec{v},\vec{w}).\nabla_{\vec{z}}\vec{u}-\rho(\vec{z},\vec{v}).\nabla_{\vec{w}}\vec{u}-\rho(\vec{w},\vec{z}).\nabla_{\vec{v}}\vec{u}.\end{aligned}$$

Hence (26.29) (sum of the two above equations with ρ antisymmetric). And (26.27) gives (26.30). ∎

26.6.1 Example with the sphere

2-D in $\mathbb{R}^3$, usual Riemannian connection, and $(\vec{e}_2,\vec{e}_3)$ renamed $(\vec{e}_1,\vec{e}_2)$. So, with (3.33)-(3.34)-(3.35) (and Proj_{TS}), $\nabla_{\vec{e}_1}\vec{e}_1 = \cos\varphi\sin\varphi\vec{e}_2$, $\nabla_{\vec{e}_1}\vec{e}_2 = -\tan\varphi\vec{e}_1 = \nabla_{\vec{e}_2}\vec{e}_1$, $\nabla_{\vec{e}_2}\vec{e}_2 = \vec{0}$. Thus:

$\nabla_{\vec{e}_1}(\nabla_{\vec{e}_1}\vec{e}_1) = \nabla_{\vec{e}_1}(\cos\varphi\sin\varphi\vec{e}_2) = \cos\varphi\sin\varphi\nabla_{\vec{e}_1}\vec{e}_2 = -\sin^2\varphi\vec{e}_1$,
$\nabla_{\vec{e}_1}(\nabla_{\vec{e}_1}\vec{e}_2) = \nabla_{\vec{e}_1}(\nabla_{\vec{e}_2}\vec{e}_1) = -\nabla_{\vec{e}_1}(\tan\varphi\vec{e}_1) = -\tan\varphi\nabla_{\vec{e}_1}(\vec{e}_1) = -\sin^2\varphi\vec{e}_2$,
$\nabla_{\vec{e}_1}(\nabla_{\vec{e}_2}\vec{e}_2) = \nabla_{\vec{e}_1}(\vec{0}) = \vec{0}$,
$\nabla_{\vec{e}_2}(\nabla_{\vec{e}_1}\vec{e}_1) = \nabla_{\vec{e}_2}(\cos\varphi\sin\varphi\vec{e}_2) = (-\sin^2\varphi+\cos^2\varphi)\vec{e}_2+\vec{0}$,
$\nabla_{\vec{e}_2}(\nabla_{\vec{e}_1}\vec{e}_2) = \nabla_{\vec{e}_2}(\nabla_{\vec{e}_2}\vec{e}_1) = -\nabla_{\vec{e}_2}(\tan\varphi\vec{e}_1) = -(1+\tan^2\varphi)\vec{e}_1+\tan^2\varphi\vec{e}_1 = -\vec{e}_1$,
$\nabla_{\vec{e}_2}(\nabla_{\vec{e}_2}\vec{e}_2) = \vec{0}$.

Thus $\rho(\vec{e}_1,\vec{e}_1) = \rho(\vec{e}_2,\vec{e}_2) = 0$, and with $[\vec{e}_i,\vec{e}_j]=0$ for all i,j (basis of a coordinate system) we get

$$\begin{cases} R(\vec{e}_1,\vec{e}_2,\vec{e}_2) = \rho(\vec{e}_1,\vec{e}_2)\vec{e}_1 = \nabla_{\vec{e}_1}(\nabla_{\vec{e}_2}\vec{e}_1) - \nabla_{\vec{e}_2}(\nabla_{\vec{e}_1}\vec{e}_1) = -\cos^2\varphi\vec{e}_2 = R^2_{122}\vec{e}_2 \\ R(\vec{e}_2,\vec{e}_2,\vec{e}_2) = \rho(\vec{e}_1,\vec{e}_2)\vec{e}_2 = \nabla_{\vec{e}_1}(\nabla_{\vec{e}_2}\vec{e}_2) - \nabla_{\vec{e}_2}(\nabla_{\vec{e}_1}\vec{e}_2) = \vec{e}_1 = R^1_{222}\vec{e}_1. \end{cases} \tag{26.32}$$

(The Riemann tensor does not vanish: The sphere is not flat!).

26.7 Remarkable Theorem of Gauss (theorema egregium)

With a Riemannian metric and the associated connection, cf. Lévi-Civita theorem 22.23:

Theorem 26.14 (theorema egregium) *The Riemann tensor only depends on the metric in the manifold (does not depend on the metric in a supposed affine space containing the manifold).*

Proof. (17.14) tells that the γ^i_{jk} are function of the metric, thus, so are the $R^i_{jk\ell}$, cf. (26.27). ∎

26.8 Metric and $R^\flat$ associated tensor

With a Riemannian metric $(\cdot,\cdot)_g$ and the associated connection ∇, cf. Lévi-Civita theorem 22.23:

26.8.1 Associated covariant Riemann tensor

Definition 26.15 $\mathcal{R} \in T^1_3(S)$ being the Riemann tensor in S, the associated covariant Riemann tensor $\mathcal{R}^\flat \in T^0_4(S)$ is defined by

$$\mathcal{R}^\flat(\vec{a},\vec{u},\vec{v},\vec{w}) = (\vec{a},R(\vec{u},\vec{v},\vec{w}))_g \quad (= (\vec{a},\rho(\vec{v},\vec{w}).\vec{u})_g). \tag{26.33}$$

In other words, if $a^\flat \in \Omega^1(S)$ (differential form) and if $\vec{a} \in \Gamma(S)$ (vector field) is the $(\cdot,\cdot)_g$-Riesz representation of α, that is,

$$a^\flat.\vec{b} = (\vec{a},\vec{b})_g, \tag{26.34}$$

for all $\vec{b} \in \Gamma(S)$, then

$$\mathcal{R}^\flat(\vec{a},\vec{u},\vec{v},\vec{w}) := \mathcal{R}(a^\flat,\vec{u},\vec{v},\vec{w}) \quad (= a^\flat.R(\vec{u},\vec{v},\vec{w})), \tag{26.35}$$

Quantification: With a basis $(\vec{e}_i)$ of a coordinate system, let $R_{ijk\ell} = \mathcal{R}^\flat(\vec{e}_i,\vec{e}_j,\vec{e}_k,\vec{e}_\ell)$ be the components of $\mathcal{R}^\flat$:

$$\mathcal{R}^\flat = \sum_{ijk\ell} R_{ijk\ell}e^i \otimes e^j \otimes e^k \otimes e^\ell. \tag{26.36}$$

Then, with $R(\vec{e}_j,\vec{e}_k,\vec{e}_\ell) = \sum_i R^i_{jk\ell}\vec{e}_i$ and $g = \sum_{\alpha,\beta} g_{\alpha\beta}e^\alpha \otimes e^\beta$, we get

$$R_{ijk\ell} = \sum_\alpha g_{i\alpha}R^\alpha_{jk\ell} \quad (= g(\vec{e}_i,R(\vec{e}_j,\vec{e}_k,\vec{e}_\ell)), \tag{26.37}$$

since $R_{ijk\ell} = \mathcal{R}^\flat(\vec{e}_i,\vec{e}_j,\vec{e}_k,\vec{e}_\ell) = g(\vec{e}_i,R(\vec{e}_j,\vec{e}_k,\vec{e}_\ell)) = g(\vec{e}_i,\sum_\alpha R^\alpha_{jk\ell}\vec{e}_\alpha) = \sum_\alpha R^\alpha_{jk\ell}g_{i\alpha}$.

Thus, (26.27) gives

$$R_{ijk\ell} = \sum_\alpha g_{i\alpha}(\frac{\partial\gamma^\alpha_{j\ell}}{\partial q^k} - \frac{\partial\gamma^\alpha_{jk}}{\partial q^\ell} + \sum_\nu \gamma^\alpha_{k\nu}\gamma^\nu_{j\ell} - \sum_\nu \gamma^\alpha_{\ell\nu}\gamma^\nu_{jk}). \tag{26.38}$$

26.8.2 Anti-symmetries of $\mathcal{R}^\flat$

(26.21) gives $a^\flat.R(\vec u,\vec v,\vec w) = -a^\flat.R(\vec u,\vec w,\vec v)$, thus $\mathcal{R}^\flat(\vec a,\vec u,\vec v,\vec w) = -\mathcal{R}^\flat(\vec a,\vec u,\vec w,\vec v)$, and

$$R_{ijk\ell} = -R_{ij\ell k} \quad \text{(antisymmetry of the last 2 indices).} \tag{26.39}$$

Proposition 26.16 *$a^\flat.(\rho(\vec v,\vec w).\vec u) = -u^\flat.(\rho(\vec v,\vec w).\vec a)$, i.e. $(\vec a, \rho(\vec v,\vec w).\vec u)_g = -(\vec u, \rho(\vec v,\vec w).\vec a)_g$, that is*

$$\mathcal{R}(a^\flat,\vec u,\vec v,\vec w) = -\mathcal{R}(u^\flat,\vec a,\vec v,\vec w), \quad \textit{and} \quad R_{ijk\ell} = -R_{jik\ell} \quad \textit{(antisymmetry of the first 2 indices).} \tag{26.40}$$

Proof. At a point $p\in S$, choose a orthonormal coordinate system Φ, that is, such that the $\vec e_i(p) = \frac{\partial\Phi}{\partial q^i}(\vec q)$ satisfy $g(p)(\vec e_i(p),\vec e_j(p)) = \delta^i_j$ (the coordinate lines are orthonormal geodesics at p). Then, at p, the Christoffel symbols vanish (not their derivatives), and (26.38) gives

$$R_{ijk\ell} = \sum_\alpha g_{i\alpha}(\frac{\partial\gamma^\alpha_{j\ell}}{\partial q^k} - \frac{\partial\gamma^\alpha_{jk}}{\partial q^\ell})$$

And (17.15), that is $\gamma^\alpha_{jk} = \frac12\sum_\beta g^{\alpha\beta}(\frac{\partial g_{\beta j}}{\partial q^k} + \frac{\partial g_{\beta k}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\beta})$, give, at p,

$$d\gamma^\alpha_{jk}.\vec e_\ell = \frac{\partial\gamma^\alpha_{jk}}{\partial q^\ell} = \frac12\sum_\beta g^{\alpha\beta}(\frac{\partial^2 g_{\beta j}}{\partial q^k\partial q^\ell} + \frac{\partial^2 g_{\beta k}}{\partial q^j\partial q^\ell} - \frac{\partial^2 g_{jk}}{\partial q^\beta\partial q^\ell})$$

since $dg^{\alpha\beta}.\vec e_\ell = 0$ (thanks to $\nabla g = 0$ at p). Thus

$$\frac{\partial\gamma^\alpha_{j\ell}}{\partial q^k} - \frac{\partial\gamma^\alpha_{jk}}{\partial q^\ell} = \frac12\sum_\beta g^{\alpha\beta}(\frac{\partial^2 g_{\beta j}}{\partial q^\ell\partial q^k} + \frac{\partial^2 g_{\beta\ell}}{\partial q^j\partial q^k} - \frac{\partial^2 g_{j\ell}}{\partial q^\beta\partial q^k} - \frac{\partial^2 g_{\beta j}}{\partial q^k\partial q^\ell} - \frac{\partial^2 g_{\beta k}}{\partial q^j\partial q^\ell} + \frac{\partial^2 g_{jk}}{\partial q^\beta\partial q^\ell}).$$

And $\frac{\partial^2 g_{\beta j}}{\partial q^\ell\partial q^k} = \frac{\partial^2 g_{\beta j}}{\partial q^k\partial q^\ell}$ (Schwarz equality) and at p, $g_{ij} = \delta_{ij}$ (orthonormal system), thus thus

$$R_{ijk\ell} = \frac{\partial^2 g_{i\ell}}{\partial q^j\partial q^k} - \frac{\partial^2 g_{j\ell}}{\partial q^i\partial q^k} - \frac{\partial^2 g_{ik}}{\partial q^j\partial q^\ell} + \frac{\partial^2 g_{jk}}{\partial q^i\partial q^\ell}.$$

Thus, at p, $R_{ijk\ell} = -R_{jik\ell}$, thus $\mathcal{R}^\flat(\vec a,\vec u,\vec v,\vec w) = -\mathcal{R}^\flat(\vec u,\vec a,\vec w,\vec v)$ for all $\vec a,\vec u,\vec v,\vec w$. ∎

26.9 Example in $\mathbb{R}^3$

Let S be a 2-D surface in $\mathbb{R}^3$. Then $[R_{ijk\ell}]_{i,j,k,\ell=1,2}$ is a set of $2^4 = 16$ scalars. The k,ℓ antisymmetry and the i,j antisymmetry gives $2*2 = 4$ independent scalars.

Proposition 26.17 *Let K be the mean curvature, cf. (24.22). Then, for all $i,j,k,\ell = 1,2$,*

$$R_{ijk\ell} = K(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}). \tag{26.41}$$

26.10 Ricci tensor and scalar curvature

Let $(\vec e_i)$ be a coordinate basis.

Definition 26.18 Ricci tensor $(Ric)^\flat \in T^0_2(S)$ is obtained by contracting the first and third index of $R^i_{jk\ell}$:

$$(Ric)^\flat = \sum_{j,\ell=1}^m (Ric)^\flat_{j\ell} e^j\otimes e^\ell \quad \text{where} \quad (Ric)^\flat_{j\ell} = \sum_{i=1}^m R^i_{ji\ell} \overset{\text{written}}{=} R_{j\ell}. \tag{26.42}$$

And the associated tensor $(Ric)\in T^1_1(S)$ is defined by

$$((Ric)\vec a,\vec u)_g = (Ric)^\flat(\vec a,\vec u). \tag{26.43}$$

Thus

$$(Ric) = \sum_{j,\ell=1}^m (Ric)^j_\ell \vec e_j\otimes e^\ell \quad \text{where} \quad (Ric)^j_\ell = \sum_\alpha g^{j\alpha}R_{\alpha\ell}. \tag{26.44}$$

Definition 26.19 The scalar Riemannian curvature is $R = \mathrm{Tr}(Ric) = \sum_{i=1}^m (Ric)^i_i$.

Thus,

$$R = \sum_j (Ric)^j_j = \sum_{j\alpha} g^{j\alpha}R_{\alpha j} = \sum_{ij\alpha} g^{j\alpha}R^i_{\alpha ij}. \tag{26.45}$$

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