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Virtual Power Principle: A Lie Covariant Approach. Applications to Non-Linear Elasticity, Turbulence, Visco-elasticity

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Abstract A covariant formulation of the virtual power principle based on Lie derivatives is proposed. The Lie covariant approach does not require an inner product and the Cauchy deformation tensor to start, but, at first order in a Galilean Euclidean setting, gives the usual linear results classically obtained with the Cauchy deformation tensor. The Lie approach may also enable to differentiate a fluid from a solid from an analytical point of view, and leads to propose a model for hysteresis. In the non-linear first order case we get covariant models for visco-elasticity, non linear fluids and non linear elasticity which differ from usual models. In the second order case, enriched modelizations are obtained.

Keywords Virtual power principle · Lie derivative · Visco-elasticity · Non Newtonian fluids · Non linear elasticity.

Contents

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1 Introduction

The starting point is the classical virtual power principle as stated by Germain [4]. This principle is based on the "frame invariance principle", cf. Truesdell and Noll [9] p. 43, which is given under the "isometric objectivity" hypothesis: "Change of frame are related by rigid transformations combined with a time-shift" (p. 41), and "the units of length and time are kept fixed" (p. 42).

However Marsden and Hughes [6] p. 22 indicate that the considered objectivity should be covariant: "The use of geometry in attempting to isolate the basic principles that are covariant—that is, that make intrinsic tensorial sense independent of a preferred coordinate system—automatically clears up several basic issues. For example, balance of linear momentum does note make tensorial sense as it stands. However, one can make covariant sense out of balance energy principles with no reference to rigid body motions. That is, the Noll and Green–Rivlin–Naghdi program can be done covariantly, although this is not obvious given the existing literature."

But unlike Marsden and Hughes who use a metric, and the Cauchy deformation tensor C seen as the pull-back of the metric, in this manuscript there is no use of any metric a priori: It is the objective covariant derivative of Lie that is used to start. (For the Lie derivative and its interpretation which motivates this manuscript, see appendix § C; And for a comparison with the Cauchy deformation tensor, see appendix § D).

The presentation is here limited to the affine space $\mathbb{R}^n$ to simplify the writings. After recalling the notations and the classical setting in § 2 and 3, the Lie derivative approach is introduced in § 4. The first order, linear, is given in § 5, and with a Galilean Euclidean setting the classical formulation is recovered. Applications to Newtonian fluids and elastic solids is proposed in § 6 and § 7. And for solids the Lie approach immediately gives a linear model for the elastic stress tensor: To be compared with the classical approach where the linear model is obtained by linearization of the (quadratic type) Cauchy deformation tensor $C = F^T.F$, see appendix § I. Moreover the Lie approach may give a way to differentiate fluids from solids, and thus, in the first order case give a hysteresis model, see § 8. And the Lie approach may also provide a simple characterization of hyper-elasticity, see § 9. Then the non-linear first order Lie virtual power principle is introduced in § 10. It produces models for non linear first order fluids (possibly turbulence), see § 11, non-linear first order elasticity, see § 12, and Maxwell type visco-elastic model, see § 13. Then the second order Lie virtual power principle is introduced in § 14, to get enriched models for non linear fluids, non linear elasticity, as well as second order models with some elasticity and some viscosity. And a quite long appendix is provided since, to the knowledge of the author, there is no obvious (short) reference in the existing literature concerning covariant objectivity and Lie derivative for classical mechanics.

2 Notation

We use $:=$ to mean "defined by". Time and space are decoupled (classical mechanic). The geometric affine space is $\mathbb{R}^n$, $n = 1$, 2 or 3. The associated vector space is also written $\mathbb{R}^n$ (context removes ambiguities), and is equipped with its usual topology. And $\mathbb{R}^{n*} = \mathcal{L}(\mathbb{R}^n;\mathbb{R})$ is its dual space (the space of linear forms on $\mathbb{R}^n$). A basis $(\mathbf{e}_i)_{i=1,...,n}$ is simply denoted $(\mathbf{e}_i)$. An observer defines a referential $\mathcal{R} = (O, (\mathbf{e}_i))$ (an origin and a basis, an origin of time and a timescale being implicit). Let $t_1 < t_2$. The observer locates the particles of an object *Obj* with the mapping, called the motion of *Obj* in $\mathcal{R}$,

$$\widetilde{\Phi} : \begin{cases} [t_1, t_2] \times Obj & \to \mathbb{R}^n \\ (t, P_{Obj}) & \to p_t = \widetilde{\Phi}(t, P_{Obj}) = O + \displaystyle\sum_{i=1}^n x^i \mathbf{e}_i, \end{cases} \tag{1}$$

where p_t is to position of P_{Obj} at t in $\mathcal{R}$. For t fixed, let $\widetilde{\Phi}_t(P_{Obj}) := \widetilde{\Phi}(t, P_{Obj})$. The configuration of *Obj* at t is $\Omega_t := \widetilde{\Phi}_t(Obj)$. And let $\mathcal{C} := \bigcup_{t\in[t_1,t_2]}(\{t\} \times \Omega_t)$, a subset in the standard Newtonian spacetime.

At t, the geometric space $\mathbb{R}^n$ is written $\mathbb{R}^n_t$ (on a differential manifold the tangent bundle $T\Omega_t$ is considered). Assuming $\widetilde{\Phi}$ is C^1 in time, the Eulerian velocity field $\widetilde{\mathbf{v}} : (t, p_t) \in \mathcal{C} \to ((t, p_t), \mathbf{v}(t, p_t)) \in \mathcal{C} \times \mathbb{R}^n$ is defined by

$$\mathbf{v}(t, p_t) := \lim_{h\to 0} \frac{\widetilde{\Phi}(t+h, P_{Obj}) - \widetilde{\Phi}(t, P_{Obj})}{h} = \frac{\partial \widetilde{\Phi}}{\partial t}(t, P_{Obj}) \in \mathbb{R}^n_t. \tag{2}$$

If no confusion arises then $\widetilde{\mathbf{v}}(t, p_t)$ is abbreviated as $\mathbf{v}(t, p_t)$ (and $\mathbf{v}(t, p_t)$ is drawn at (t, p_t)). Assuming $\widetilde{\Phi}$ is C^2 in time, the Eulerian acceleration field is defined with $p_t = \widetilde{\Phi}(t, P_{Obj})$ by $\boldsymbol{\gamma}(t, p_t) = \frac{\partial^2 \widetilde{\Phi}}{\partial t^2}(t, P_{Obj})$.

For an C^1 Eulerian function $\mathcal{E}$ (defined on $\mathcal{C}$), the space differential $d\mathcal{E}(t, p_t)$ of $\mathcal{E}$ at $(t, p_t) \in \{t\} \times \Omega_t$ is the differential $d\mathcal{E}_t(p_t)$ of $\mathcal{E}_t$ at p_t. (In differential geometry the tangent map $T\mathcal{E}_t$ of $\mathcal{E}_t$ is used, where $T\mathcal{E}_t(p_t) := (p_t, d\mathcal{E}_t(p_t))$, that is $T\mathcal{E}_t$ is the "full notation" of $d\mathcal{E}_t$.) And its material derivative $\frac{D\mathcal{E}}{Dt}$ is its derivative along a trajectory, that is, is the time derivative of the function $t \to \mathcal{E}(t, \widetilde{\Phi}(t, P_{Obj}))$ at $p_t = \widetilde{\Phi}(t, P_{Obj})$; So, $\frac{D\mathcal{E}}{Dt}(t, p_t) = \frac{\partial \mathcal{E}}{\partial t}(t, p_t) + d\mathcal{E}(t, p_t).\mathbf{v}(t, p_t)$. E.g., $\boldsymbol{\gamma} = \frac{D\mathbf{v}}{Dt}$.

Let $t_0 \in]t_1, t_2[$ be an initial time of computation as set by an observer. The notations of Marsden and Hughes [6] are used: Capital letters in Ω_{t_0} and tiny letters in Ω_t. And with $t \in [t_1, t_2]$, let $\Phi_t^{t_0} : \Omega_{t_0} \to \Omega_t$ be defined by $\Phi_t^{t_0} := \widetilde{\Phi}_t \circ (\widetilde{\Phi}_{t_0})^{-1}$, that is

$$\text{if} \quad p_{t_0} = P = \widetilde{\Phi}(t_0, P_{Obj}) \quad \text{then} \quad p_t = \Phi_t^{t_0}(P) := \widetilde{\Phi}(t, P_{Obj}). \tag{3}$$

And the motion relative to the configuration Ω_{t_0} is $\Phi^{t_0} : [t_1, t_2] \times \Omega_{t_0} \to \mathbb{R}^n$ defined by

$$(p_t =) \quad \Phi^{t_0}(t, P) := \Phi_t^{t_0}(P). \tag{4}$$

So, $\Phi^{t_0}(t, P) = \widetilde{\Phi}(t, P_{Obj}) = p_t$ when $P = \widetilde{\Phi}(t_0, P_{Obj})$.

The Lagrangian velocity $\mathbf{V}^{t_0} : [t_1, t_2] \times \Omega_{t_0} \to \mathbb{R}_t^n$ relative to t_0 is defined by $\mathbf{V}^{t_0}(t, P) := \frac{\partial \Phi^{t_0}}{\partial t}(t, P)$. Let $\mathbf{V}_t^{t_0}(P) := \mathbf{V}^{t_0}(t, P)$. (The mapping $\mathbf{V}_t^{t_0} : \Omega_{t_0} \to \mathbb{R}_t^n$ does not define a "vector field" but a two points (P and p_t) mapping, see Marsden and Hughes [6] (the vector $\mathbf{V}^{t_0}(t, P)$ is drawn at (t, p_t) since $\mathbf{V}^{t_0}(t, P) = \mathbf{v}(t, p_t)$ is the tangent vector at p_t along the trajectory $t \to \widetilde{\Phi}(t, P_{Obj}) = \Phi^{t_0}(t, P)$).

$\Phi_t^{t_0}$ will be assumed to be a C^2 diffeomorphism, and its differential $F_t^{t_0} := d\Phi_t^{t_0} : \Omega_{t_0} \to \mathcal{L}(\mathbb{R}_{t_0}^n; \mathbb{R}_t^n)$ is named the (covariant) deformation gradient between t_0 and t. ($F_t^{t_0}$ does not define a tensor but a two points (P and p_t) mapping, see [6].) Let F^{t_0} be defined by $F^{t_0}(t, P) := F_t^{t_0}(P)$.

Let $L(E; F)$ be the set of continuous linear mappings between two vector spaces E and F, and let $(\cdot, \cdot)_G$ (resp. $(\cdot, \cdot)_g$) be an inner product in $\mathbb{R}_{t_0}^n$ (resp. in $\mathbb{R}_t^n$). The transpose $(F_t^{t_0}(P))^T \in \mathcal{L}(\mathbb{R}_t^n; \mathbb{R}_{t_0}^n)$, relative to $(\cdot, \cdot)_G$ and $(\cdot, \cdot)_g$ is characterized by $((F_t^{t_0}(P))^T.\mathbf{w}_{p_t}, \mathbf{W}_P)_G = (\mathbf{w}_{p_t}, F_t^{t_0}(P).\mathbf{W}_P)_g$ for all $\mathbf{W}_P \in \mathbb{R}_{t_0}^n$ and $\mathbf{w}_{p_t} \in \mathbb{R}_t^n$, see [6]. And the deformation tensor relative to t_0, t, $(\cdot, \cdot)_G$ and $(\cdot, \cdot)_g$ is $C_t^{t_0} := (F_t^{t_0})^T \circ F_t^{t_0} : \Omega_{t_0} \to \mathcal{L}(\mathbb{R}_{t_0}^n; \mathbb{R}_{t_0}^n)$, written $C = F^T.F$.

Remark 2.1 To be able to impose "the same Euclidean basis at two distinct times t_0 and t" (and on two different tangent spaces) Marsden and Hughes [6] p. 57 use the shifter: In $\mathbb{R}^n$ it is the mapping $S_t^{t_0} : (P, \mathbf{W}_P) \in \Omega_{t_0} \times \mathbb{R}_{t_0}^n \to S_t^{t_0}(P, \mathbf{W}_P) = (p_t, \mathbf{w}_{p_t}) \in \Omega_t \times \mathbb{R}_t^n$ where $p_t = \Phi_t^{t_0}(P)$ and $\mathbf{w}_{p_t} := \mathbf{W}_P$ (parallel displacement in $\mathbb{R}^n$). In particular, using "the same Euclidean basis at t_0 and t" means using a Euclidean basis $(\mathbf{E}_i)$ in $\mathbb{R}_{t_0}^n$ and the basis $(\mathbf{e}_i)$ in $\mathbb{R}_t^n$ given by $(p_t, \mathbf{e}_i) = S_t^{t_0}(P, \mathbf{E}_i)$, so $\mathbf{e}_i = \mathbf{E}_i$.

Representation. Let $\mathbb{R}_t^{n*} = \mathcal{L}(\mathbb{R}_t^n; \mathbb{R})$ be the space of linear functions on $\mathbb{R}_t^n$. If $(\mathbf{e}_i)$ is a basis in $\mathbb{R}_t^n$, its dual basis (e^i) is the basis of $\mathbb{R}_t^{n*}$ made of the linear functions e^i characterized by $e^i(\mathbf{e}_j) = \delta_j^i$ (Kronecker symbol). The e^i being linear, $e^i(\mathbf{v})$ is written $e^i.\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}_t^n$. A basis in $\mathbb{R}_t^n$ will be supposed Cartesian to simplify. At t_0 a basis in $\mathbb{R}_{t_0}^n$ will be denoted $(\mathbf{E}_I)$ and its dual basis (E^I). At t in a referential $\mathcal{R} = (o_t, (\mathbf{e}_i))$, a point $p_t = \Phi_t^{t_0}(P)$ is located as the bipoint vector $\overrightarrow{o_t p_t} = \overrightarrow{o_t \Phi_t^{t_0}(P)} = \sum_{i=1}^n \Phi_t^i(P)\mathbf{e}_i$ in $\mathbb{R}_t^n$, and the deformation gradient is denoted

$$d\Phi_t^{t_0}(P) = F_t^{t_0}(P) = \sum_{i,J=1}^n F_J^i(t, P)\, \mathbf{e}_i \otimes E^J, \quad [F]_{|\mathbf{E},\mathbf{e}} = [F_J^i]. \tag{5}$$

So, $F_J^i(t, P) = d\Phi^i(t, P).\mathbf{E}_J$, and with Cartesian bases $F_J^i(t, P) = \frac{\partial \Phi^i}{\partial X^J}(t, P)$. And $[F_J^i(t, P)] = [F_t^{t_0}(P)]_{|\mathbf{E},\mathbf{e}} = [d\Phi_t^{t_0}(P)]_{|\mathbf{E},\mathbf{e}}$ is the Jacobian matrix of $\Phi_t^{t_0}$ at P relatively to the bases $(\mathbf{E}_I)$ and $(\mathbf{e}_i)$. Since $\Phi_t^{t_0}$ is supposed to be a diffeomorphism we have $d\Phi_t^{t_0}(P)^{-1} = (d\Phi_t^{t_0})^{-1}(p_t)$ when $p_t = \Phi_t^{t_0}(P)$, and we will denote $H_t^{t_0}(p_t) := (d\Phi_t^{t_0})^{-1}(p_t)$. And with the above bases:

$$H_t^{t_0}(p_t) = H^{t_0}(t, p_t) = \sum_{I,j=1}^n H_j^I(t, p_t)\mathbf{E}_I \otimes e^j, \quad [H]_{|\mathbf{e},\mathbf{E}} = [H_j^I]. \tag{6}$$

3 Classical virtual power principle

We use Germain's setting, see [4], with no volumic double forces to simplify. Let $(\cdot, \cdot)_g$ be a Euclidean inner product in $\mathbb{R}_t^n$. Let $\mathcal{V}$ be a vector space of "admissible velocity fields" (sufficiently regular vector fields for the mathematical expressions to be meaningful). The virtual power principle connects, at all times t, three linear functionals $\mathcal{V} \to \mathbb{R}$: The virtual power of external forces

$$\mathcal{P}_e(\mathbf{v}) = \int_{\Omega_t} (\mathbf{f}, \mathbf{v})_g \, d\Omega + \int_{\partial\Omega_t} (\mathbf{g}, \mathbf{v})_g \, d\sigma \tag{7}$$

where $\mathbf{f}$ and $\mathbf{g}$ are given vector fields, the virtual power of mass-acceleration

$$\mathcal{P}_a(\mathbf{v}) := \int_{\Omega_t} \rho(\boldsymbol{\gamma}, \mathbf{v})_g \, d\Omega \tag{8}$$

where ρ is the mass density and $\boldsymbol{\gamma} = \frac{D\mathbf{w}}{Dt}$ is the eulerian acceleration undergone by a material with velocity $\mathbf{w}$, and the virtual power of internal forces

$$\mathcal{P}_i(\mathbf{v}) = \int_{\Omega_t} p_{\text{int}}(\mathbf{v})\, d\Omega \tag{9}$$

where p_{int} is a linear form of $\mathbf{v}$ and its differentials. And the principle is: For any $\mathbf{v} \in \mathcal{V}$,

$$\mathcal{P}_i(\mathbf{v}) + \mathcal{P}_e(\mathbf{v}) = \mathcal{P}_a(\mathbf{v}). \tag{10}$$

4 Virtual power principle with Lie derivatives

The virtual power of external forces (7) and of mass-acceleration (8) are used; And the "virtual power of pressure"

$$\mathcal{P}_{pres}(\mathbf{v}) = \int_{\Omega_t} p \operatorname{div}\mathbf{v}\, d\Omega_t \tag{11}$$

is introduced, where p is a differentiable function, and $\operatorname{div}\mathbf{v}$ is the divergence of the velocity field $\mathbf{v}$. (The virtual power of pressure will enable a simple formulation for the virtual power of internal forces as is classically done for Maxwell visco-elastic material, see § 13. Also see § I.)

And the virtual power of internal forces (9) will be used with p_{int} (non linear in general) given with Lie derivatives as described in § 5 and following: This is the main purpose of this manuscript.

Then the principle of virtual power reads: for any admissible vector field $\mathbf{v}$,

$$\mathcal{P}_i(\mathbf{v}) + \mathcal{P}_{pres}(\mathbf{v}) + \mathcal{P}_e(\mathbf{v}) = \mathcal{P}_a(\mathbf{v}). \tag{12}$$

5 Lie virtual power of internal forces: Linear first order

(The Lie derivatives in classical mechanics are described in § C.)

5.1 First Order Formulation

First-order conjecture: 1- A material, occupying at t a domain Ω_t in $\mathbb{R}^n$, can be characterized by $3n$ vector fields $\mathbf{a}_{00j}, \mathbf{a}_{01j}, \mathbf{a}_{10j}$ and/or $3n$ one-forms $\alpha_{00}^j, \alpha_{01}^j, \alpha_{10}^j$. 2- At t, in a referential $\mathcal{R} = (O, (\mathbf{e}_i))$ and with (e^i) the dual basis of $(\mathbf{e}_i)$, the measured density of power of internal forces depends on an admissible virtual velocity field $\mathbf{v}$ acting on the material, and reads

$$p_{\text{int}\,1}(\mathbf{v}) = \sum_{j=1}^{n} e^j.\mathbf{a}_{00j} + \mathcal{L}_{\mathbf{v}} e^j.\mathbf{a}_{01j} + e^j.\mathcal{L}_{\mathbf{v}}\mathbf{a}_{10j} + \alpha_{00}^j.\mathbf{e}_j + \alpha_{01}^j.\mathcal{L}_{\mathbf{v}}\mathbf{e}_j + \mathcal{L}_{\mathbf{v}}\alpha_{10}^j.\mathbf{e}_j, \tag{13}$$

where $\mathcal{L}_{\mathbf{v}}\mathbf{a} = \frac{\partial \mathbf{a}}{\partial t} + d\mathbf{a}.\mathbf{v} - d\mathbf{v}.\mathbf{a}$ is the Lie derivative of a vector field $\mathbf{a}$, see (63), and $\mathcal{L}_{\mathbf{v}}\alpha = \frac{\partial \alpha}{\partial t} + d\alpha.\mathbf{v} + \alpha.d\mathbf{v}$ is the Lie derivative of a one-form α, see (64). (For special materials the number of vector fields and/or one-forms can be chosen to be greater than n the dimension of $\mathbb{R}^n$.)

5.2 Galilean setting

In a Galilean referential and with a Cartesian basis $(\mathbf{e}_i)$, we have $\mathcal{L}_{\mathbf{v}}\mathbf{e}_j = -d\mathbf{v}.\mathbf{e}_j$ and $\mathcal{L}_{\mathbf{v}} e^j = e^j.d\mathbf{v}$, and (13) reduces to $p_{\text{int}\,1}(\mathbf{v}) = \sum_{j=1}^{n} e^j.\mathbf{a}_{00j} + e^j.d\mathbf{v}.\mathbf{a}_{01j} + e^j.(\frac{\partial \mathbf{a}_{10j}}{\partial t} + d\mathbf{a}_{10j}.\mathbf{v} - d\mathbf{v}.\mathbf{a}_{10j}) + \alpha_{00}^j.\mathbf{e}_j - \alpha_{01}^j.d\mathbf{v}.\mathbf{e}_j + (\frac{\partial \alpha_{01}^j}{\partial t} + d\alpha_{01}^j.\mathbf{v} + \alpha_{01}^j.d\mathbf{v}).\mathbf{e}_j$. And the virtual power of the internal forces is assumed to vanish whenever $d\mathbf{v} = 0$ (Galilean referential). Thus we are left with $p_{\text{int}\,1}(\mathbf{v}) = \sum_{j=1}^{n} e^j.d\mathbf{v}.\mathbf{a}_{01j} - e^j.d\mathbf{v}.\mathbf{a}_{10j} - \alpha_{01}^j.d\mathbf{v}.\mathbf{e}_j + \alpha_{01}^j.d\mathbf{v}.\mathbf{e}_j$, that is,

$$p_{\text{int}\,1}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot d\mathbf{v}, \quad -\underline{\underline{\tau}}_1 = \sum_{j=1}^{n} \mathbf{a}_{1j} \otimes e^j + \mathbf{e}_j \otimes \alpha_1^j, \quad \begin{cases} \mathbf{a}_{1j} = \mathbf{a}_{01j} - \mathbf{a}_{10j}, \\ \alpha_1^j = \alpha_{01}^j - \alpha_{01}^j, \end{cases} \tag{14}$$

where $\odot$ is the double objective contraction between the $\binom{1}{1}$ tensors $\underline{\underline{\tau}}_1$ and $d\mathbf{v}$, see (46).

5.3 Galilean Euclidean setting

In the previous Galilean setting, the basis $(\mathbf{e}_i)$ is chosen to be Euclidean. Let $(\cdot,\cdot)_g$ be the associated inner product. Then the transposed $d\mathbf{v}^T$ of $d\mathbf{v}$ is defined (by $(d\mathbf{v}^T.\mathbf{u},\mathbf{w})_g = (d\mathbf{v}.\mathbf{w},\mathbf{u})_g$ for any $\mathbf{u},\mathbf{w}\in\mathbb{R}^n_t$). The internal power being independent of a rigid motion, i.e independent of $\frac{d\mathbf{v}-d\mathbf{v}^T}{2}$, (14) gives

$$p_{\mathrm{int}\,1}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot \frac{d\mathbf{v}+d\mathbf{v}^T}{2} = -\underline{\underline{\sigma}} \odot \frac{d\mathbf{v}+d\mathbf{v}^T}{2} = -\underline{\underline{\sigma}} \odot d\mathbf{v}, \quad \underline{\underline{\sigma}} = \frac{\underline{\underline{\tau}}_1 + \underline{\underline{\tau}}_1^T}{2}, \tag{15}$$

which is a classical expression.

6 Stokes Fluid

The "pressure forces" being taken into account in the virtual power of pressure, see (11), we look at the virtual internal power of "viscous forces". Consider a fluid animated with a movement $\widetilde{\Phi}$, see (1), let $p(\tau)=\widetilde{\Phi}(\tau,P_{Obj})$, and let $\mathbf{w}(\tau,p(\tau)) = \frac{\partial\widetilde{\Phi}}{\partial\tau}(\tau,P_{Obj})$ be its Eulerian velocity, see (2). Then, with $t\in]t_1,t_2[$ and τ in the vicinity of t, consider the associated movement Φ^t_τ, see (3), and its deformation gradient $F^t_\tau = d\Phi^t_\tau$ between t and τ. Since $\frac{\partial\Phi^t}{\partial\tau}(\tau,p_t) = \mathbf{w}(t,\Phi^t(\tau,p_t))$ we have $\frac{\partial F^t}{\partial\tau}(\tau,p_t) = d\mathbf{w}(t,p(\tau)).F^t(\tau,p_t)$.

Eventuality 1. Vector fields $\mathbf{a}_{1j}$ characterize the fluid, and the virtual power of internal forces results from their transport by the flow. Then (14) gives (Galilean setting)

$$p_{\mathrm{int}\,1}(\mathbf{v}) = \sum_{j=1}^n e^j.\mathcal{L}_{\mathbf{v}}\mathbf{a}_{1j} = -\underline{\underline{\tau}}_1 \odot d\mathbf{v}, \quad \underline{\underline{\tau}}_1 = -\sum_{j=1}^n \mathbf{a}_{1j}\otimes e^j. \tag{16}$$

Then consider that a Newtonian fluid (which is isotropic) is characterized by the n vector fields given by $\mathbf{a}_{1j}(\tau,p_\tau) := -2\mu\,\frac{\partial F^t}{\partial\tau}(\tau,p_t).\mathbf{e}_j$ when $\tau=t$ (no memory), that is,

$$\mathbf{a}_{1j}(t,p_t) = -2\mu\, d\mathbf{w}(t,p_t).\mathbf{e}_j. \tag{17}$$

(The matrix $[\underline{\underline{\tau}}_1]_{|\mathbf{e}} = 2\mu\,[d\mathbf{w}]_{|\mathbf{e}}$ is made of the columns $-[\mathbf{a}_{1j}]_{|\mathbf{e}} = 2\mu\,[\frac{\partial\mathbf{w}}{\partial x^j}]_{|\mathbf{e}}$.) Thus, in a Galilean Euclidean setting, we get $[\underline{\underline{\sigma}}]_{|\mathbf{e}} = 2\mu\frac{[d\mathbf{w}]_{|\mathbf{e}}+[d\mathbf{w}]^T_{|\mathbf{e}}}{2}$, see (15), which is the classical model.

E.g. if $\mathbf{a}_{1j}=\mathbf{a}_{10j}$, that is, if $p_{\mathrm{int}\,1}(\mathbf{v}) = \sum_{j=1}^n e^j.\mathcal{L}_{\mathbf{v}}\mathbf{a}_{10j}$, see (13) and (14), then the power is measured by the observer with the projections e^j. (And if $\mathbf{a}_{1j}=\mathbf{a}_{01j}$, that is if $p_{\mathrm{int}\,1}(\mathbf{v}) = \sum_{j=1}^n \mathbf{a}_{01j}.\mathcal{L}_{\mathbf{v}}e^j$, then the power is measured with the e^j immersed in the flow: will be used with the non linear approach.)

Eventuality 2. One-forms characterize the fluid, and their transport by a flow $\mathbf{v}$ give the internal power. In particular in a Galilean Euclidean setting with a Euclidean basis $(\mathbf{e}_i)$ in $\mathbb{R}^n_t$, (14) gives $p_{\mathrm{int}\,1}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot d\mathbf{v}$ where $\underline{\underline{\tau}}_1 = -\sum_{i=1}^n \mathbf{e}_i\otimes\alpha^i_1$. Here $\alpha^i_1 = -2\mu\, e^j.d\mathbf{w} = -2\mu\, dw^i$, and $[\underline{\underline{\tau}}_1]_{|\mathbf{e}}$ is the matrix made of the lines $-[\alpha^i_{10}]_{|\mathbf{e}} = 2\mu[dw^i]_{|\mathbf{e}}$.

7 Linear elasticity

Consider an elastic solid animated with a movement $\widetilde{\Phi}$, see (1), let $t_0\in]t_1,t_2[$, let Φ^{t_0} be the associated movement, see (4) and (3), and let $F^{t_0}_t = d\Phi^{t_0}_t$ be the covariant deformation gradient between t_0 and t.

Eventuality 1. One-forms characterize the elastic material (Germain's point of view see [3]), and their transport by the flow gives the internal power. Then (14) gives (Galilean setting)

$$p_{\mathrm{int}\,1}(\mathbf{v}) = \sum_{i=1}^n \mathcal{L}_{\mathbf{v}}\alpha^i_1.\mathbf{e}_i = -\underline{\underline{\tau}}_1 \odot d\mathbf{v}, \quad \underline{\underline{\tau}}_1 = -\sum_{i=1}^n \mathbf{e}_i\otimes\alpha^i_1. \tag{18}$$

Then consider that an isotropic elastic solid is characterized by n one-forms α^i_1 on Ω_t that are push-forwards of one-forms $\alpha^i_{1t_0}$ on Ω_{t_0}, see (56), that is, with $p_t = \Phi^{t_0}_t(P)$ and $H^{t_0}_t(p_t) = (F^{t_0}_t(P))^{-1}$,

$$\alpha^i_1(t,p_t) = \alpha^i_{1t_0}(P).H^{t_0}_t(p_t). \tag{19}$$

Then $\underline{\underline{\tau}}_1(t,p_t) = -\sum_{i=1}^n(\mathbf{e}_i\otimes\alpha^i_{1t_0}(P)).H^{t_0}(t,p_t)$; And with $(\mathbf{E}_i)$ a Cartesian basis in $\mathbb{R}^n_{t_0}$, choose $\alpha^i_{1t_0} = E^i$ (isotropic material), so that, with (6), $\underline{\underline{\tau}}_1(t,p_t) = -\sum_{i=1}^n(\mathbf{e}_i\otimes E^i).H^{t_0}(t,p_t) = -\sum_{i,k,j=1}^n H^k_j(t,p_t)(\mathbf{e}_i\otimes E^i).(\mathbf{E}_k\otimes e^j) = -\sum_{i,j=1}^n H^i_j(t,p_t)\mathbf{e}_i\otimes e^j$, that is,

$$\underline{\underline{\tau}}_1(t,p_t) = -\sum_{i,j=1}^n H^i_j(t,p_t)\mathbf{e}_i\otimes e^j, \quad\text{and}\quad [\underline{\underline{\tau}}_1]_{|\mathbf{e}} = -2\tilde{\mu}[H^{t_0}]_{|\mathbf{e},\mathbf{E}}. \tag{20}$$

So, in a Galilean Euclidean setting, with $(\mathbf{E}_i)$ a Euclidean basis, we get $[\underline{\underline{\sigma}}]_{|\mathbf{e}} = -2\tilde{\mu}\frac{[H^{t_0}]_{|\mathbf{e},\mathbf{E}}+[H^{t_0}]^T_{|\mathbf{e},\mathbf{E}}}{2}$, see (15), classical expression for small displacements, except for the trace part, see § I.

E.g. if $\alpha_1^j = \alpha_{10}^j$, that is if $p_{\text{int}\,1}(\mathbf{v}) = \sum_{j=1}^n \mathcal{L}_\mathbf{v}\alpha_{10}^j.\mathbf{e}_j$, then the power is measured by the observer with the Euclidean projections along the $\mathbf{e}_j$. (And if $\alpha_1^j = \alpha_{01}^j$, that is if $p_{\text{int}\,1}(\mathbf{v}) = \sum_{j=1}^n \alpha_{01}^j.\mathcal{L}_\mathbf{v}\mathbf{e}_j$, then the power is measured with the $\mathbf{e}_j$ immersed in the flow; It will be used for visco-elasticity, see (31).)

Eventuality 2. Vector fields characterize the solid, and their transports give the internal power. In particular in a Galilean Euclidean setting with a Euclidean basis $(\mathbf{e}_i)$ in $\mathbb{R}^n_t$, (14) gives

$$p_{\text{int}\,1}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot d\mathbf{v}, \quad \underline{\underline{\tau}}_1 = -\sum_{j=1}^n \mathbf{a}_{1j} \otimes e^j. \tag{21}$$

Then suppose that the $\mathbf{a}_{1j}$ are the push-forwards of vector fields $\mathbf{a}_{jt_0}(P)$ see (54), that is, with $p_t = \Phi_t^{t_0}(P)$, $\mathbf{a}_{1j}(t,p_t) = F_t^{t_0}(P).\mathbf{a}_{jt_0}(P)$. Then $\underline{\underline{\tau}}_1(t,p_t) = -F^{t_0}(t,P).\sum_{j=1}^n \mathbf{a}_{jt_0}(P)\otimes e^j$. With a Euclidean base $(\mathbf{E}_i)$ in $\mathbb{R}^n_{t_0}$, then for isotropic homogeneous elasticity we may choose $\mathbf{a}_{jt_0} = -2\tilde{\mu}\mathbf{E}_j$, then, see (5):

$$\underline{\underline{\tau}}_1(t,p_t) = F^{t_0}(t,P).\sum_{j=1}^n \mathbf{E}_j \otimes e^j, \quad \text{and} \quad [\underline{\underline{\tau}}_1]_{|\mathbf{e}} = 2\tilde{\mu}[F^{t_0}]_{|\mathbf{E},\mathbf{e}}. \tag{22}$$

Thus, in a Galilean Euclidean setting, with $(\mathbf{E}_i)$ a Euclidean basis, we get $[\underline{\underline{\sigma}}] = 2\tilde{\mu}\frac{[F^{t_0}]+[F^{t_0}]^T}{2}$, classical expression for small displacements, except for the trace part, see § I.

8 Fluids vs solids, and hysteresis

For a Stokes fluid, a modelization with vector fields can be considered, see (16). And for an elastic solid, a modelization with one-forms can be considered, see (18). Thus fluids would be differentiated from solids in a non-algebraic way. And "mixed linear materials" could be considered with

$$p_{\text{int}\,1}(\mathbf{v}) = c_1\sum_{j=1}^n e^j.\mathcal{L}_\mathbf{v}\mathbf{a}_{10j} + c_2\sum_{i=1}^n \mathcal{L}_\mathbf{v}\alpha_{10}^i.\mathbf{e}_i = -c_1\,\underline{\underline{\tau}}_1 \odot d\mathbf{v} - c_2\,\underline{\underline{\kappa}}_1 \odot d\mathbf{v}, \tag{23}$$

the last equality in a Cartesian setting where $\underline{\underline{\tau}}_1 = -\sum_{j=1}^n \mathbf{a}_{10j}\otimes e^j$ characterizes the fluid part, $\underline{\underline{\kappa}}_1 = -\sum_{i=1}^n \mathbf{e}_i\otimes\alpha_{10}^i$ characterizes the solid part, and $c_1, c_2 \in \mathbb{R}$. This could model simple visco-elasticity or hysteresis.

9 About hyper-elasticity

Hyper-elasticity aims to find a "stored energy function" see Marsden and Hughes [6] ("énergie volumique des déformations élastiques" see Germain [5]). The classical starting point is a Euclidean setting and the Piola–Kirchhoff tensor $PK = J\underline{\underline{\sigma}}.F^{-T}$, simplified notation of $PK_t^{t_0}(P) = J_t^{t_0}(P)\underline{\underline{\sigma}}(t,p_t).(F_t^{t_0}(P))^{-T}$, where $p_t = \Phi_t^{t_0}(P)$, $J = \det(F)$, F^{-T} is the inverse of the transpose, $\underline{\underline{\sigma}}$ is the Cauchy stress tensor at (t,p_t) built from the Cauchy stress vector at (t,p_t) and the unit orthonormal vector $\mathbf{n}(t,p_t)$ to $\partial\Omega_t$ (depends on the metric). Thus PK measures the force "per unit of undistorted area" (by change of variables in the integrals). And the material is said to be hyper-elastic if $PK^{t_0}(t,P) = \widehat{PK}^{t_0}(t,F^{t_0}(t,P))$ and if there exists a function $\mathcal{W}^{t_0}$ s.t., with (5), $[\widehat{PK}^{t_0}] = [\frac{\partial\mathcal{W}^{t_0}}{\partial F^i_j}]$. This derivation in terms of the components of the two-point tensor F is quite intriguing (see [6]).

With the Lie derivative approach, and the introduction of the virtual power of pressure, see (11), the "hyper-elastic potential" for isotropic homogeneous elasticity can be considered to be simply $\mathcal{W}_t^{t_0} := 2\tilde{\mu}\Phi_t^{t_0}$: Then $d\mathcal{W}_t^{t_0} = 2\tilde{\mu}F_t^{t_0}$, and then $[\underline{\underline{\tau}}_1(t,p_t)]_{|\mathbf{e}} = [d\mathcal{W}^{t_0}(t,P)]_{|\mathbf{E},\mathbf{e}} = 2\tilde{\mu}[F^{t_0}(t,P)]_{|\mathbf{E},\mathbf{e}}$, see (22). But the preference may go to

$$\mathcal{W}_t^{t_0} := -2\tilde{\mu}(\Phi_t^{t_0})^{-1} : \Omega_t \to \Omega_{t_0} \tag{24}$$

so that $d\mathcal{W}_t^{t_0}(p_t) = -2\tilde{\mu}H_t^{t_0}(p_t)$, and $[\underline{\underline{\tau}}_1(t,.)]_{|\mathbf{e}} = [d\mathcal{W}_t^{t_0}]_{|\mathbf{e},\mathbf{E}} = -2\tilde{\mu}[H_t^{t_0}]_{|\mathbf{e},\mathbf{E}}$, see (20). Here $\underline{\underline{\tau}}_1(t,.)$ and $d\mathcal{W}_t^{t_0}$, in (24), are defined at p_t, and $\mathcal{W}_t^{t_0}$ refers to the past (values at t_0 from Ω_t) which is the usual approach of Galilean or general relativity.

10 Lie virtual power of internal forces: Non linear first order

(13) is enriched: 1'- The material can be characterized by the preceding fields and by n vector fields $\mathbf{a}_{11j}$ on Ω_t and/or n one-forms α^j_{11} on Ω_t. 2- For a velocity field $\mathbf{v}$, at t, we get the density of the power of internal forces (first order nonlinear)

$$p_{\text{int}\,1n}(\mathbf{v}) = p_{\text{int}\,1}(\mathbf{v}) + \sum_{j=1}^{n} \mathcal{L}_{\mathbf{v}}\alpha^j_{11}.\mathcal{L}_{\mathbf{v}}\mathbf{e}_j + \mathcal{L}_{\mathbf{v}}e^j.\mathcal{L}_{\mathbf{v}}\mathbf{a}_{11j}. \tag{25}$$

In a Galilean setting, the assumption of zero internal power if $d\mathbf{v} = 0$ gives with (14):

$$p_{\text{int}\,1n}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot d\mathbf{v} + (\frac{\partial \alpha^j_{11}}{\partial t} + d\alpha^j_{11}.\mathbf{v} + \alpha^j_{11}.d\mathbf{v}).(-d\mathbf{v}.\mathbf{e}_j) + e^j.d\mathbf{v}.(\frac{\partial \mathbf{a}_{11j}}{\partial t} + d\mathbf{a}_{11j}.\mathbf{v} - d\mathbf{v}.\mathbf{a}_{11j}).$$

In (25), in addition to $p_{\text{int}\,1}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot d\mathbf{v}$, it appears:

the unsteady linear term $-\frac{\partial \alpha^j_{11}}{\partial t}.d\mathbf{v}.\mathbf{e}_j + e^j.d\mathbf{v}.\frac{\partial \mathbf{a}_{11j}}{\partial t}$,
the non-linear term $-(d\alpha^j_{11}.\mathbf{v}).d\mathbf{v}.\mathbf{e}_j + e^j.d\mathbf{v}.(d\mathbf{a}_{11j}.\mathbf{v})$, non-linear in $\mathbf{v}$ and $d\mathbf{v}$,
and the non-linear term $-\alpha^j_{11}.d\mathbf{v}.d\mathbf{v}.\mathbf{e}_j - e^j.d\mathbf{v}.d\mathbf{v}.\mathbf{a}_{11j}$, non-linear in $d\mathbf{v}.d\mathbf{v}$.

11 Non Newtonian fluids 1

$2n$ vector fields $\mathbf{a}_{1j}$ and $\mathbf{a}_{11j}$ characterize a non linear first order fluid (non Newtonian, turbulence type model?). Then, with (16) and $\underline{\underline{\tau}}_1 = -\sum_{j=1}^n \mathbf{a}_{1j} \otimes e^j$,

$$p_{\text{int}\,1n}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot d\mathbf{v} + \sum_{j=1}^{n}(\frac{\partial \mathbf{a}_{11j}}{\partial t} \otimes e^j + (d\mathbf{a}_{11j}.\mathbf{v}) \otimes e^j) \odot d\mathbf{v} - (\mathbf{a}_{11j} \otimes e^j) \odot (d\mathbf{v}.d\mathbf{v}). \tag{26}$$

And (for matrix computation) with $\underline{\underline{\tau}}_{11} = -\sum_{j=1}^n \mathbf{a}_{11j} \otimes e^j$ we get

$$p_{\text{int}\,1n}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot d\mathbf{v} - (\frac{\partial \underline{\underline{\tau}}_{11}}{\partial t} + d\underline{\underline{\tau}}_{11}.\mathbf{v} - d\mathbf{v}.\underline{\underline{\tau}}_{11}) \odot d\mathbf{v}. \tag{27}$$

(The expression in parentheses, that is $\frac{\partial \underline{\underline{\tau}}_{11}}{\partial t} + d\underline{\underline{\tau}}_{11}.\mathbf{v} - d\mathbf{v}.\underline{\underline{\tau}}_{11}$, is not a Lie derivative of $\underline{\underline{\tau}}_{11}$ see (65): Here it is the Lie derivative of the vector fields $\mathbf{a}_{11j}$ that are considered.)

12 Non linear elasticity 1

$2n$ one-forms α^j_1 and α^j_{11} characterize non linear first order elasticity. Then, with (18) and $\underline{\underline{\tau}}_1 = -\sum_{j=1}^n \mathbf{e}_j \otimes \alpha^j_1$,

$$p_{\text{int}\,1n}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot d\mathbf{v} - \sum_{j=1}^{n}(\mathbf{e}_j \otimes (\frac{\partial \alpha^j_{11}}{\partial t} + d\alpha^j_{11}.\mathbf{v} + \alpha^j_{11}.d\mathbf{v})) \odot d\mathbf{v}. \tag{28}$$

And (for matrix computation), with $\underline{\underline{\tau}}_{11} = \sum_{j=1}^n \mathbf{e}_j \otimes \alpha^j_{11}$ we get

$$p_{\text{int}\,1n}(\mathbf{v}) = -\underline{\underline{\tau}}_1 \odot d\mathbf{v} - (\frac{\partial \underline{\underline{\tau}}_{11}}{\partial t} + d\underline{\underline{\tau}}_{11}.\mathbf{v} + \underline{\underline{\tau}}_{11}.d\mathbf{v}) \odot d\mathbf{v}. \tag{29}$$

(The expression in parentheses, that is $\frac{\partial \underline{\underline{\tau}}_{11}}{\partial t} + d\underline{\underline{\tau}}_{11}.\mathbf{v} + \underline{\underline{\tau}}_{11}.d\mathbf{v}$, is not a Lie derivative of $\underline{\underline{\tau}}_{11}$ see (65): Here it is the Lie derivative of the one-forms α^i_{11} that are considered.)

13 Visco-elastic fluids 1

Galilean Euclidean setting. The classical model for a visco-elastic fluids of Maxwell reads

$$\underline{\underline{\tau}} + \lambda \mathcal{L}_{\mathbf{v}}\underline{\underline{\tau}} = \mu \frac{d\mathbf{v}+d\mathbf{v}^T}{2}, \tag{30}$$

where $\underline{\underline{\tau}}$ is a stress tensor, $\mathbf{v}$ the velocity field of the material, λ an elasticity constant, μ a viscosity constant, and $\mathcal{L}_{\mathbf{v}}\underline{\underline{\tau}}$ a Lie derivative. And the pressure p is classically introduced with (11). In (30), the derivative of Lie $\mathcal{L}_{\mathbf{v}}\underline{\underline{\tau}}$ is considered with the velocity $\mathbf{v}$ of the material (not a virtual velocity), and can the Jaumann or the upper-convected or the lower-convected Lie derivative, see (65); Or can be a linear combination of such tensors to improve the numerical results (which otherwise are not convincing), even if such a linear combination is absurd as far as objectivity is concerned (addition of vector fields and one-forms).

A possible covariant model could be, with $\mathbf{v}$ a virtual velocity,

$$p_{\text{int}\,1n}(\mathbf{v}) = \sum_{j=1}^{n} e^j.\mathcal{L}_{\mathbf{v}}\mathbf{a}_{1j} + \sum_{j=1}^{n} \mathcal{L}_{\mathbf{v}}\alpha_{11}^j.\mathcal{L}_{\mathbf{v}}\mathbf{e}_j = -\underline{\underline{\tau}}_1 \odot d\mathbf{v} - (\frac{\partial \underline{\underline{\tau}}_{11}}{\partial t} + d\underline{\underline{\tau}}.\mathbf{v} + \underline{\underline{\tau}}_{11}.d\mathbf{v}) \odot d\mathbf{v}, \tag{31}$$

obtained by using the Lie derivatives of vector fields for the fluid part and the Lie derivative of one forms for the elastic part, see (16) and (18), the last equality being given in a Galilean setting with $\underline{\underline{\tau}}_1 = -\sum_{j=1}^n \mathbf{a}_{1j} \otimes e^j$, see (16), and $\underline{\underline{\tau}}_{11} = \sum_{j=1}^n \mathbf{e}_j \otimes \alpha_{11}^i$, see the last term in (29).

14 Lie virtual power of internal forces: Second order

(25) is enriched with vector fields $\mathbf{a}_{xyj}$ and one-forms α_{xy}^j and the second ordre Lie derivatives
$\alpha_{02}^j.\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\mathbf{e}_j)$, $\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\alpha_{20}^j).\mathbf{e}_j$, $\mathcal{L}_{\mathbf{v}}\alpha_{12}^j.\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\mathbf{e}_j)$, $\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\alpha_{21}^j).\mathcal{L}_{\mathbf{v}}\mathbf{e}_j$, $\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\alpha_{22}^j).\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\mathbf{e}_j)$, and
$e^j.\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\mathbf{a}_{02j})$, $\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}e^j).\mathbf{a}_{20j}$, $\mathcal{L}_{\mathbf{v}}e^j.\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\mathbf{a}_{12j})$, $\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}e^j).\mathcal{L}_{\mathbf{v}}\mathbf{a}_{21j}$, $\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}e^j).\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\mathbf{a}_{22j})$.
And, with (25), the measured power density of the internal forces reads:

$$p_{\text{int}\,2}(\mathbf{v}) = p_{\text{int}\,1n}(\mathbf{v}) + \sum \text{the above second order terms.} \tag{32}$$

The second order Lie derivative of a vector field $\mathbf{w}$ is $\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\mathbf{w}) = \frac{\partial^2\mathbf{w}}{\partial t^2} + 2d\frac{\partial \mathbf{w}}{\partial t}.\mathbf{v} - 2d\mathbf{v}.\frac{\partial \mathbf{w}}{\partial t} + d\mathbf{w}.\frac{\partial \mathbf{v}}{\partial t} - \frac{\partial d\mathbf{v}}{\partial t}.\mathbf{w} + (d^2\mathbf{w}.\mathbf{v}).\mathbf{v} + d\mathbf{w}.d\mathbf{v}.\mathbf{v} - 2d\mathbf{v}.d\mathbf{w}.\mathbf{v} - (d^2\mathbf{v}.\mathbf{v}).\mathbf{w} + d\mathbf{v}.d\mathbf{v}.\mathbf{w}$, computed with (63). And the second order Lie derivative of a one-form α is $\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\alpha) = \frac{\partial^2\alpha}{\partial t^2} + 2d\frac{\partial\alpha}{\partial t}.\mathbf{v} + 2\frac{\partial\alpha}{\partial t}.d\mathbf{v} + d\alpha.\frac{\partial \mathbf{v}}{\partial t} + \alpha.\frac{\partial d\mathbf{v}}{\partial t} + d^2\alpha(\mathbf{v},\mathbf{v}) + d\alpha.(d\mathbf{v}.\mathbf{v}) + 2(d\alpha.\mathbf{v}).d\mathbf{v} + \alpha.(d^2\mathbf{v}.\mathbf{v}) + (\alpha.d\mathbf{v}).d\mathbf{v}$, computed with (56).

(Generalization: Higher order Lie derivative can also be considered.)

15 Non Newtonian fluids 2

E.g. (26) is enriched to get e.g. ("one pure second order fluids")

$$p_{\text{int}\,2}(\mathbf{v}) = \sum_{j=1}^{n} e^j.\mathcal{L}_{\mathbf{v}}\mathbf{a}_{1j} + \mathcal{L}_{\mathbf{v}}e^j.\mathcal{L}_{\mathbf{v}}\mathbf{a}_{11j} + e^j.\mathcal{L}_{\mathbf{v}}(\mathcal{L}_{\mathbf{v}}\mathbf{a}_{20j}). \tag{33}$$

(And for a "pure fluid" any other Lie derivative of vector fields can be added.)

Galilean setting: The internal power vanishing if $d\mathbf{v}=0$ and $\frac{\partial \mathbf{v}}{\partial t}=0$, we get

$$\begin{aligned} p_{\text{int}\,2}(\mathbf{v}) = p_{\text{int}\,1n}(\mathbf{v}) - \sum_{j=1}^{n} & 2e^j.d\mathbf{v}.\frac{\partial \mathbf{a}_{20j}}{\partial t} + e^j.d\mathbf{a}_{20j}.\frac{\partial \mathbf{v}}{\partial t} - e^j.\frac{\partial d\mathbf{v}}{\partial t}.\mathbf{a}_{20j} \\ & + e^j.d\mathbf{a}_{20j}.d\mathbf{v}.\mathbf{v} - 2e^j.d\mathbf{v}.d\mathbf{a}_{20j}.\mathbf{v} - e^j.(d^2\mathbf{v}.\mathbf{v}).\mathbf{a}_{20j} + e^j.d\mathbf{v}.d\mathbf{v}.\mathbf{a}_{20j}, \end{aligned} \tag{34}$$

which is a "non linear second gradient" (non Newtonian) expression. With $\underline{\underline{\tau}} = \sum_{j=1}^n \mathbf{a}_{1j} \otimes e^j$ and $\underline{\underline{\kappa}}_2 = \sum_{j=1}^n \mathbf{a}_{20j} \otimes e^j$ we obtain (for matrix computation with the generic notation $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + dT.\mathbf{v}$):

$$p_{\text{int}\,2}(\mathbf{v}) = p_{\text{int}\,1n}(\mathbf{v}) - \Big(\underline{\underline{\kappa}}_2 \odot (\frac{\partial d\mathbf{v}}{\partial t} + d^2\mathbf{v}.\mathbf{v} - d\mathbf{v}.d\mathbf{v}) + d\underline{\underline{\kappa}}_2.\frac{D\mathbf{v}}{Dt} + 2\frac{D\underline{\underline{\kappa}}_2}{Dt} \odot d\mathbf{v}\Big). \tag{35}$$

16 Elasticity 2

E.g. (28) is enriched to get e.g. ("a pure second order elastic material")

$$p_{\text{int}\,2}(\mathbf{v}) = \sum_{i=1}^{n} \mathcal{L}_\mathbf{v}\alpha_1^i.\mathbf{e}_i + \mathcal{L}_\mathbf{v}\alpha_{11}^j.\mathcal{L}_\mathbf{v}\mathbf{e}_j + \mathcal{L}_\mathbf{v}(\mathcal{L}_\mathbf{v}\alpha_{20}^i).\mathbf{e}_i. \tag{36}$$

(And for a "pure elastic material" any other Lie derivative of one-forms can be added.)

Galilean setting: The internal power vanishing if $d\mathbf{v} = 0$ and $\frac{\partial \mathbf{v}}{\partial t} = 0$, we get

$$\begin{aligned} p_{\text{int}\,2}(\mathbf{v}) = p_{\text{int}\,1n}(\mathbf{v}) + \sum_{j=1}^{n} 2\frac{\partial \alpha_{20}^i}{\partial t}.d\mathbf{v}.\mathbf{e}_i + d\alpha_{20}^i.\frac{\partial \mathbf{v}}{\partial t}.\mathbf{e}_i + \alpha_{20}^i.\frac{\partial d\mathbf{v}}{\partial t}.\mathbf{e}_i \\ + d\alpha_{20}^i.(d\mathbf{v}.\mathbf{v}).\mathbf{e}_i + 2(d\alpha_{20}^i.\mathbf{v}).d\mathbf{v}.\mathbf{e}_i + \alpha_{20}^i.(d^2\mathbf{v}.\mathbf{v}).\mathbf{e}_i + (\alpha_{20}^i.d\mathbf{v}).d\mathbf{v}.\mathbf{e}_i, \end{aligned} \tag{37}$$

which is a "non linear second gradient" expression. With $\underline{\underline{\kappa}}_2 = -\sum_{i=1}^n \mathbf{e}_i \otimes \alpha_{20}^i$, we get (for matrix computation):

$$p_{\text{int}\,2}(\mathbf{v}) = p_{\text{int}\,1n}(\mathbf{v}) - \Big(\underline{\underline{\kappa}}_2 \,\,⦂\, (\frac{\partial d\mathbf{v}}{\partial t} + d^2\mathbf{v}.\mathbf{v} + d\mathbf{v}.d\mathbf{v}) + d\underline{\underline{\kappa}}_2.\frac{D\mathbf{v}}{Dt} + 2\frac{D\underline{\underline{\kappa}}_2}{Dt} \,⦂\, d\mathbf{v}\Big). \tag{38}$$

With the Eulerian acceleration $\boldsymbol{\gamma} = \frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + d\mathbf{v}.\mathbf{v}$, that gives $d\boldsymbol{\gamma} = \frac{\partial (d\mathbf{v})}{\partial t} + d^2\mathbf{v}.\mathbf{v} + d\mathbf{v}.d\mathbf{v}$, we have

$$p_{\text{int}\,2}(\mathbf{v}) = p_{\text{int}\,1n}(\mathbf{v}) - \Big(\underline{\underline{\kappa}}_2 \,⦂\, d\boldsymbol{\gamma} + d\underline{\underline{\kappa}}_2.\boldsymbol{\gamma} + 2\frac{D\underline{\underline{\kappa}}_2}{Dt} \,⦂\, d\mathbf{v}\Big). \tag{39}$$

(For comparison purposes, the second-order Taylor development of the Cauchy tensor $C = F^T.F$ in the neighborhood of t_0 reads $C^{t_0}(t_0{+}h, P) = (I + h\,(d\mathbf{v} + d\mathbf{v}^T) + \frac{h^2}{2}(2d\mathbf{v}^T.d\mathbf{v} + d\boldsymbol{\gamma} + d\boldsymbol{\gamma}^T))(t_0, P) + o(h^2)$.)

17 Visco-elasticity 2

Second-order visco-elastic materials (not purely fluid or elastic) can be obtained with other Lie combinations.

A Tensors and objective contractions

Let E be a finite dimensional (to simplify) real vector space, and let $E^* := \mathcal{L}(E;\mathbb{R})$ be the dual space of E (the space of linear functions). An element $\ell \in E^*$ being linear, $\ell(\mathbf{v})$ is written $\ell.\mathbf{v}$, for all $\mathbf{v} \in E$. Let $E^{**} := \mathcal{L}(E^*;\mathbb{R})$ be the bidual of E, and let $\mathcal{J}_1 : \mathbf{v} \in E \to v = \mathcal{J}_1(\mathbf{v}) \in E^{**}$ be defined by $v(\ell) := \ell.\mathbf{v}$ for all $\ell \in E^*$. Then $\mathcal{J}_1$ is a natural canonical isomorphism, see e.g. Spivak [8], where canonical means that the definition only uses the constant 1 which is the identity element in multiplication and is the "simplest possible" (quite blurry), and natural means that $\mathcal{J}_1$ is independent of the observer. Thus E and E^{**} are identified, and $v = \mathcal{J}_1(\mathbf{v})$ is written $v = \mathbf{v}$. And, by linearity of v, $v(\ell)$ is written $v.\ell$ or $\mathbf{v}.\ell$. (There is no natural isomorphism between E and E^*, see e.g. [8]: An isomorphism exists but depends on an observer, e.g. depends on a basis, or e.g. depends on an inner product see (67) and (69)).

Let E and F be two finite dimensional real spaces, and $\dim E = n$ and $\dim F = m$. Let $\mathcal{L}(E;F)$ be the set of linear mappings from E to F. Let $(\mathbf{a}_i)$ be a basis in E. Then a linear map $L \in \mathcal{L}(E;F)$ is characterized by the vectors $L.\mathbf{a}_j$. Let $(\mathbf{b}_i)$ be a basis in F, and let (b^i) be its dual basis. And let $L^i{}_j$ be the components of $L.\mathbf{a}_j$ in the basis $(\mathbf{b}_i)$, that is,

$$L.\mathbf{a}_j = \sum_{i=1}^{m}\sum_{j=1}^{n} L^i{}_j \mathbf{b}_i, \quad \text{so} \quad L^i{}_j = b^i.(L.\mathbf{a}_j). \tag{40}$$

And we write

$$L = \sum_{i=1}^{m}\sum_{j=1}^{n} L^i{}_j \mathbf{b}_i \otimes a^j. \tag{41}$$

This is coherent with the usual tensorial product: Let $\mathcal{L}(F^*, E;\mathbb{R})$ the space of bilinear forms on $F^* \times E$. If $(\mathbf{v}_F, \ell_E) \in F \times E^*$, then their tensorial product $\mathbf{v}_F \otimes \ell_E$ is the bilinear form in $\mathcal{L}(F^*, E;\mathbb{R})$ defined by $(\mathbf{v}_F \otimes \ell_E)(\ell_F, \mathbf{v}_E) := \mathbf{v}_F(\ell_F)\,\ell_E(\mathbf{v}_E) = \ell_F(\mathbf{v}_F)\,\ell_E(\mathbf{v}_E)$, thanks to $\mathcal{J}_1$, that is $(\mathbf{v}_F \otimes \ell_E)(\ell_F, \mathbf{v}_E) := (\ell_F.\mathbf{v}_F)\,(\ell_E.\mathbf{v}_E)$. And we have the natural canonical isomorphism $\mathcal{J}_2 : \widetilde{L} \in \mathcal{L}(F^*, E;\mathbb{R}) \to L = \mathcal{J}_2(\widetilde{L}) \in \mathcal{L}(E;F)$ defined by, see [8],

$$\forall(\ell_F, \mathbf{u}) \in F^* \times E, \quad \ell_F.(L.\mathbf{u}) := \widetilde{L}(\ell_F, \mathbf{u}). \tag{42}$$

Thus if $\widetilde{L} = \sum_{i=1}^m\sum_{j=1}^n L^i{}_j \mathbf{b}_i \otimes a^j$, then $L^i{}_j = \widetilde{L}(b^i, \mathbf{a}_j)$, then $L^i{}_j = b^i.L.\mathbf{a}_j$ where $L = \mathcal{J}_2^{-1}(\widetilde{L})$, which gives (40). And $[L^i{}_j]$ is the matrix $[L]_{|\mathbf{a},\mathbf{b}}$ of $L \in \mathcal{L}(E;F)$ as well as the matrix $[\widetilde{L}]_{|\mathbf{a},\mathbf{b}}$ of $\widetilde{L} \in \mathcal{L}(F^*, E;\mathbb{R})$ when $L = \mathcal{J}_2^{-1}(\widetilde{L})$, relatively to the bases $(\mathbf{a}_i)$ and $(\mathbf{b}_i)$. And in any case, the Einstein convention is satisfied. In this manuscript this is applied e.g. with the deformation gradient $L = F_t^{t_0}(P) = d\Phi_t^{t_0}(P) : \mathbb{R}^n_{t_0} \to \mathbb{R}^n_t$ represented with bases in (5), thanks to $\mathcal{J}_2$.

And, with $\mathcal{J}_2$, we have then defined the contraction of a bilinear mapping $\widetilde{L} \in \mathcal{L}(F^*, E; \mathbb{R})$ with a vector $\mathbf{u} \in E$: The definition being $\widetilde{L}.\mathbf{u} := L.\mathbf{u}$ when $L = \mathcal{J}_2^{-1}(\widetilde{L})$.

Let $r, s \in \mathbb{N}$ s.t. $r+s \geq 1$. The set $\mathcal{L}^r_s(E) := \mathcal{L}(\underbrace{E^*, ..., E^*}_{r \text{ times}}, \underbrace{E, ..., E}_{s \text{ times}}; \mathbb{R})$ of $\mathbb{R}$-multilinear forms is called the set of uniform tensors of type $\binom{r}{s}$ on E. And $\mathcal{L}^0_0(E) := \mathbb{R}$. An elementary uniform tensor in $\mathcal{L}^r_s(E)$ is a tensor $u_1 \otimes ... \otimes u_r \otimes \ell_1 \otimes ... \otimes \ell_s$, where $u_i \in E^{**}$ and $\ell_i \in E^*$, which value on $(m_1, ..., m_r, \mathbf{v}_1, ..., \mathbf{v}_s) \in (E^*)^r \times E^s$ is $(u_1.m_1)...(u_r.m_r)(\ell_1.\mathbf{v}_1)...(\ell_s.\mathbf{v}_s)$. And with $u_i = \mathcal{J}_1(\mathbf{u}_i)$ this tensor is written $\mathbf{u}_1 \otimes ... \otimes \mathbf{u}_r \otimes \ell_1 \otimes ... \otimes \ell_s$.

If $\widetilde{L}_1 \in \mathcal{L}^{r_1}_{s_1}(E)$ and $\widetilde{L}_2 \in \mathcal{L}^{r_2}_{s_2}(E)$ then their tensor product is the tensor $\widetilde{L}_1 \otimes \widetilde{L}_2 \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}(E)$ defined by:

$$(\widetilde{L}_1 \otimes \widetilde{L}_2)(\ell_{1,1}, ..., \ell_{2,1}, ..., \mathbf{u}_{1,1}, ..., \mathbf{u}_{2,1}, ...) := \widetilde{L}_1(\ell_{1,1}, ..., \mathbf{u}_{1,1}, ...)\widetilde{L}_2(\ell_{2,1}, ..., \mathbf{u}_{2,1}, ...). \tag{43}$$

If $\widetilde{L}_1 \in \mathcal{L}^{r_1}_{s_1}(E)$, $\widetilde{L}_2 \in \mathcal{L}^{r_2}_{s_2}(E)$, $\ell \in \mathcal{L}^0_1(E) = E^*$, $u \in \mathcal{L}^1_0(E) = E^{**}$, $\mathbf{u} = \mathcal{J}^{-1}(u) \in E$, and u written $\mathbf{u}$, the objective tensor contraction of $\widetilde{L}_1 \otimes \ell \in \mathcal{L}^{r_1}_{s_1+1}(E)$ with $\mathbf{u} \otimes \widetilde{L}_2 \in \mathcal{L}^{r_2+1}_{s_2}(E)$ is defined by

$$(\widetilde{L}_1 \otimes \ell).(\mathbf{u} \otimes \widetilde{L}_2) := (\ell.\mathbf{u})\, \widetilde{L}_1 \otimes \widetilde{L}_2 \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}(E). \tag{44}$$

And the objective tensor contraction of $\widetilde{L}_1 \otimes \mathbf{u}$ with $\ell \otimes \widetilde{L}_2$ is defined by $(\widetilde{L}_1 \otimes \mathbf{u}).(\ell \otimes \widetilde{L}_2) = (\mathbf{u}.\ell)\, \widetilde{L}_1 \otimes \widetilde{L}_2 = (\ell.\mathbf{u})\, \widetilde{L}_1 \otimes \widetilde{L}_2$.

The objective double tensor contraction $\odot$, for compatible tensors, results from the simple contraction applied twice. E.g. the double objective tensor contraction of $\widetilde{L}_1 \otimes \ell_{1,1} \otimes \mathbf{u}_{1,2}$ and $\ell_{2,1} \otimes \mathbf{u}_{2,2} \otimes \widetilde{L}_2$ is defined by

$$(\widetilde{L}_1 \otimes \ell_{1,1} \otimes \mathbf{u}_{1,2}) \odot (\ell_{2,1} \otimes \mathbf{u}_{2,2} \otimes \widetilde{L}_2) = (\mathbf{u}_{1,2}.\ell_{2,1})(\ell_{1,1}.\mathbf{u}_{2,2})\, \widetilde{L}_1 \otimes \widetilde{L}_2. \tag{45}$$

Representation with a basis $(\mathbf{e}_i)$ of E. E.g. with $S, T \in \mathcal{L}^1_1(E)$, $S = \sum_{j=1}^n S^i_j \mathbf{e}_i \otimes e^j$ and $T = \sum_{j=1}^n T^i_j \mathbf{e}_i \otimes e^j$, we get

$$S.T = \sum_{i,j,k=1}^n S^i_k T^k_j \mathbf{e}_i \otimes e^j, \quad \text{and} \quad S \odot T = \sum_{i,j=1}^n S^i_j T^j_i, \tag{46}$$

that is $S \odot T = \mathrm{Tr}(S.T)$ (the trace of $S.T$ considered to be an endomorphism thanks to $\mathcal{J}_2$), which value is independent of the chosen basis (Einstein convention is satisfied). Thus, if $\mathbf{u} \in E$, $\ell \in E^*$, $T \in \mathcal{L}^1_1(E)$, then $(\mathbf{u} \otimes \ell) \odot T = \ell.T.\mathbf{u}$; With a basis $(\mathbf{e}_i)$, and $\mathbf{u} = \sum_{i=1}^n u^i \mathbf{e}_i$, $\ell = \sum_{i=1}^n \ell_i e^i$, $T = \sum_{j=1}^n T^i_j \mathbf{e}_i \otimes e^j$, we have $(\mathbf{u} \otimes \ell) \odot T = \ell.T.\mathbf{u} = \sum_{i,j,k=1}^n \ell_i T^i_j u^j$.

Let $A = [A^i_j]$ and $B = [B^i_j]$ be two square $n * n$ matrices. The double matrix contraction of A and B is defined by:

$$A : B := \sum_{i,j=1}^n A^i_j B^i_j. \tag{47}$$

This is not an objective contraction (the Einstein convention is not satisfied). E.g. $A = [S]_{|\mathbf{e}}$ and $B = [T]_{|\mathbf{e}}$ give $[S]_{|\mathbf{e}} : [T]_{|\mathbf{e}} = A : B$, value that depends on the choice of the basis. To be compared with $S \odot T = A^T : B$, see (46).

Let $\mathcal{E}$ be an affine space and E be the associated vector space. Let $\Omega_\mathcal{E}$ be an open set in $\mathcal{E}$. Let $\mathcal{F}(\Omega_\mathcal{E}; \mathbb{R})$ be the set of real valued functions on $\Omega_\mathcal{E}$. A tensor of type $\binom{r}{s}$ on $\Omega_\mathcal{E}$, see Abraham and Marsden [1], is a mapping $\widetilde{T} : p \in \Omega_\mathcal{E} \to \widetilde{T}(p) = (p, T(p)) \in \Omega_\mathcal{E} \times \mathcal{L}^r_s(\Omega_\mathcal{E})$ that is $\mathcal{F}(\Omega_\mathcal{E}; \mathbb{R})$-multilinear; That is, for all $f \in \mathcal{F}(\Omega_\mathcal{E}; \mathbb{R})$ and all z_1, z_2 vector fields or one forms on $\Omega_\mathcal{E}$ where appropriate, we have

$$T(..., f z_1 + z_2, ...) = f\, T(..., z_1, ...) + T(..., z_2, ...). \tag{48}$$

(That is, for all $p \in \Omega_\mathcal{E}$, $T(p)(..., f(p) z_1(p) + z_2(p), ...) = f(p)\, T(p)(..., z_1(p), ...) + T(p)(..., z_2(p), ...)$.) If there is no ambiguity then $\widetilde{T}$ is simply written T. The set of tensors of type $\binom{r}{s}$ on $\Omega_\mathcal{E}$ is denoted $T^r_s(\Omega_\mathcal{E})$. And $T^0_0(\Omega_\mathcal{E}) := \mathcal{F}(\Omega_\mathcal{E}; \mathbb{R})$. Example: $T^0_1(\Omega_\mathcal{E})$ is identified with the set $\Omega^1(\Omega_\mathcal{E})$ of one-forms on $\Omega_\mathcal{E}$, and $T^1_0(\Omega_\mathcal{E})$ is identified with the set $\Gamma(\Omega_\mathcal{E})$ of vector fields on $\Omega_\mathcal{E}$ thanks to $\mathcal{J}_1$. Counter example: A derivation operator ∇ is $\mathbb{R}$-linear, but is not $\mathcal{F}(\Omega_\mathcal{E}; \mathbb{R})$-linear because (48) is not satisfied since $\nabla(f z_1) = (\nabla f)\, z_1 + f\, (\nabla z_1) \neq f\, \nabla z_1$ if f is not constant (∇ is not a tensor but is a spray, see [1]).

Note that the initial concept is "vector fields"; Then "one-forms" are introduced, which are functions of vector fields; And then "tensors" are introduced, which are functions of one forms and velocity fields.

For a complete mathematical introduction (with differential geometry) see e.g. Spivak [8], Abraham and Marsden [1], Arnold [2], Marsden and Hughes [6].

B Push-forward and pull-back

Push-forwards and pull-backs enable to define, among others, the velocity addition formula, objectivity, and Lie derivatives. We follow Abraham and Marsden [1], here in a simplified affine framework.

Let $\mathcal{E}$ and $\mathcal{F}$ be affine sets, and let E and F be the associated vector space supposed to be normed, let $\Omega_\mathcal{E}$ and $\Omega_\mathcal{F}$ be open sets in $\mathcal{E}$ and $\mathcal{F}$, and let $\Psi : \Omega_\mathcal{E} \to \Omega_\mathcal{F}$ be a diffeomorphism. In this manuscript E and F are finite dimensional, and Ψ will be either a motion $\Phi^{t_0}_t : \Omega_{t_0} \to \Omega_t$, see (3), or a translator $\Theta_t : \mathcal{R}_B \to \mathcal{R}_A$, see (74).

The push-forward of a function $f_\mathcal{E} : \Omega_\mathcal{E} \to \mathbb{R}$ by Ψ is the function $\Psi_* f_\mathcal{E} = f_{\mathcal{E}*} : \Omega_\mathcal{F} \to \mathbb{R}$ defined by $f_{\mathcal{E}*} := f_\mathcal{E} \circ (\Psi)^{-1}$, so, with $p_\mathcal{F} = \Psi(p_\mathcal{E})$,

$$f_{\mathcal{E}*}(p_\mathcal{F}) := f_\mathcal{E}(p_\mathcal{E}). \tag{49}$$

The pull-back of a function $f_\mathcal{F} : \Omega_\mathcal{F} \to \mathbb{R}$ by Ψ is the function $\Psi^* f_\mathcal{F} = (\Psi^{-1})_* f_\mathcal{F} = f^*_\mathcal{F}$, that is $f^*_\mathcal{F} = f_\mathcal{F} \circ \Psi$, so $f^*_\mathcal{F}(p_\mathcal{E}) = f_\mathcal{F}(p_\mathcal{F})$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Let

$$c_\mathcal{E} : \begin{cases}]s_1, s_2[\to \Omega_\mathcal{E} \\ \quad s \to p_\mathcal{E} = c_\mathcal{E}(s), \end{cases} \tag{50}$$

be a regular curve in $\Omega_{\mathcal{E}}$. The push-forward of $c_{\mathcal{E}}$ by Ψ is the curve $\Psi_* c_{\mathcal{E}} = c_{\mathcal{E}*}$ defined by $c_{\mathcal{E}*} := \Psi \circ c_{\mathcal{E}}$, that is,

$$c_{\mathcal{E}*} : \begin{cases}]s_1, s_2[& \to \Omega_{\mathcal{F}} \\ s & \to p_{\mathcal{F}} = c_{\mathcal{E}*}(s) := \Psi(c_{\mathcal{E}}(s)) \quad (= \Psi(p_{\mathcal{E}})). \end{cases} \tag{51}$$

Let $c_{\mathcal{F}} : s \in]s_1, s_2[\to c_{\mathcal{F}}(s) \in \Omega_{\mathcal{F}}$ be a regular curve. The pull-back of $c_{\mathcal{F}}$ by Ψ is the curve $\Psi^* c_{\mathcal{F}} = c_{\mathcal{F}}^* := (\Psi^{-1})_* c_{\mathcal{F}}$, that is $c_{\mathcal{F}}^* = \Psi^{-1} \circ c_{\mathcal{F}}$.

We use the definition of a vector field given by the tangent vectors to a curve (see e.g. [1] for equivalent definitions). For a regular curve $c_{\mathcal{E}}$ as in (50), its tangent vector at $p_{\mathcal{E}} = c_{\mathcal{E}}(s)$ is

$$\mathbf{w}_{\mathcal{E}}(p_{\mathcal{E}}) := c_{\mathcal{E}}{}'(s). \tag{52}$$

That defines the vector field $\mathbf{w}_{\mathcal{E}}$ on $\text{Im}(c_{\mathcal{E}})$. The push-forward of $\mathbf{w}_{\mathcal{E}}$ by Ψ is the vector field $\Psi_* \mathbf{w}_{\mathcal{E}} = \mathbf{w}_{\mathcal{E}*}$ on $\text{Im}(c_{\mathcal{E}*})$ made of the tangent vectors to the push-forward curve $c_{\mathcal{E}*}$, see (51), that is, $\mathbf{w}_{\mathcal{E}*}(p_{\mathcal{F}}) := c_{\mathcal{E}*}{}'(s)$ when $p_{\mathcal{F}} = c_{\mathcal{E}*}(s)$. So, since (51) gives $c_{\mathcal{E}*}{}'(s) = d\Psi(c_{\mathcal{E}}(s)).c_{\mathcal{E}}{}'(s)$, with $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$ we have

$$\mathbf{w}_{\mathcal{E}*}(p_{\mathcal{F}}) = d\Psi(p_{\mathcal{E}}).\mathbf{w}_{\mathcal{E}}(p_{\mathcal{E}}). \tag{53}$$

(That is, $\mathbf{w}_{\mathcal{E}*} = (d\Psi.\mathbf{w}_{\mathcal{E}}) \circ \Psi^{-1}$.) And a family of regular curves in $\Omega_{\mathcal{E}}$ gives a vector field $\mathbf{w}_{\mathcal{E}}$ on $\Omega_{\mathcal{E}}$ and its push-forward $\mathbf{w}_{\mathcal{E}*}$ on $\Omega_{\mathcal{F}}$. The pull-back of a vector field $\mathbf{w}_{\mathcal{F}} : \Omega_{\mathcal{F}} \to F$ by Ψ is the vector field $\Psi^* \mathbf{w}_{\mathcal{F}} = \mathbf{w}_{\mathcal{F}}^* := (\Psi^{-1})_* \mathbf{w}_{\mathcal{F}}$, that is, with $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$,

$$\mathbf{w}_{\mathcal{F}}^*(p_{\mathcal{E}}) := d\Psi(p_{\mathcal{E}})^{-1}.\mathbf{w}_{\mathcal{F}}(p_{\mathcal{F}}). \tag{54}$$

The push-forward of a one-form $\alpha_{\mathcal{E}} \in E^*$ by Ψ is the one-form $\Psi^* \alpha_{\mathcal{E}} = \alpha_{\mathcal{E}*}$ defined with $f_{\mathcal{E}} = \alpha_{\mathcal{E}}.\mathbf{w}_{\mathcal{E}}$, (49) and (53); So, with $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$,

$$\alpha_{\mathcal{E}*}(p_{\mathcal{F}}) := \alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi(p_{\mathcal{E}})^{-1}. \tag{55}$$

And the pull-back of a one-form $\alpha_{\mathcal{F}} \in F^*$ by Ψ is the one-form $\Psi^* \alpha_{\mathcal{F}} = (\Psi^{-1})_* \alpha_{\mathcal{F}} = \alpha_{\mathcal{F}}^*$, that is, with $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$,

$$\alpha_{\mathcal{F}}^*(p_{\mathcal{E}}) := \alpha_{\mathcal{F}}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}). \tag{56}$$

This pull-back can also be defined thanks to the natural canonical isomorphism $L \in \mathcal{L}(E;F) \to L^* \in \mathcal{L}(F^*;E^*)$ defined by $L^*(\ell_F).\mathbf{u}_{\mathcal{E}} = \ell_F.(L.\mathbf{u}_{\mathcal{E}})$; So $L^*(\ell_F) = \ell_F.L$; And $L^*(\ell_F) \in E^*$ is denoted ℓ_F^* and named the pull-back of ℓ_F by L. So that $\ell_F^* = \ell_F.L$. And (56) is obtained with $\ell_F = \alpha_{\mathcal{F}}(p_{\mathcal{F}})$ and $L = d\Psi(p_{\mathcal{E}})$, and $\ell_F^* = \alpha_{\mathcal{F}}(p_{\mathcal{F}})^*$ is denoted $\alpha_{\mathcal{F}}^*(p_{\mathcal{E}})$.

Let $T_{\mathcal{E}}$ be a $\binom{r}{s}$ tensor on $\Omega_{\mathcal{E}}$, with $r, s \geq 1$. Its push-forward by Ψ is the $\binom{r}{s}$ tensor $\Psi_* T_{\mathcal{E}} = T_{\mathcal{E}*}$ on $\Omega_{\mathcal{F}}$ defined by

$$T_{\mathcal{E}*}(\alpha_{1\mathcal{F}}, ..., \alpha_{r\mathcal{F}}, \mathbf{w}_{1\mathcal{F}}, ..., \mathbf{w}_{s\mathcal{F}}) = T_{\mathcal{E}}(\alpha_{1\mathcal{F}}^*, ..., \alpha_{r\mathcal{F}}^*, \mathbf{w}_{1\mathcal{F}}^*, ..., \mathbf{w}_{s\mathcal{F}}^*). \tag{57}$$

Let $T_{\mathcal{F}}$ be a $\binom{r}{s}$ tensor on $\Omega_{\mathcal{F}}$, with $r, s \geq 1$. Its pull-back by Ψ is the $\binom{r}{s}$ tensor $T_{\mathcal{F}}^*$ on $\Omega_{\mathcal{E}}$ defined by

$$T_{\mathcal{F}}^*(\alpha_{1\mathcal{E}}, ..., \alpha_{r\mathcal{E}}, \mathbf{w}_{1\mathcal{E}}, ..., \mathbf{w}_{s\mathcal{E}}) = T_{\mathcal{F}}(\alpha_{1\mathcal{E}*}, ..., \alpha_{r\mathcal{E}*}, \mathbf{w}_{1\mathcal{E}*}, ..., \mathbf{w}_{s\mathcal{E}*}). \tag{58}$$

C Lie derivatives: Introduction and interpretation

(Classical mechanics.) Let $\widetilde{\Phi}$ be a motion, see (1), $\mathbf{v}$ be the associated Eulerian velocity, see (2), $t, \tau \in \mathbb{R}$, Φ_τ^t the associated motion, see (4) and (3), and $F_\tau^t = d\Phi_\tau^t$. With reference to § B, let $\Psi = \Phi_\tau^t$, $\Omega_{\mathcal{E}} = \Omega_t$, $\Omega_{\mathcal{F}} = \Omega_\tau$, $p_{\mathcal{E}} = p_t$ and $p_{\mathcal{F}} = p_\tau = \Phi_\tau^t(p_t)$. If $f_\tau : \Omega_\tau \to \mathbb{R}$ is a function then its pull-back $(\Phi_\tau^t)^* f_\tau = f_\tau^{t*}$ is given by $f_\tau^{t*}(p_t) = f_\tau(p_\tau)$. If $\mathbf{w}_\tau$ is a vector field in Ω_τ then its pull-back $(\Phi_\tau^t)^* \mathbf{w}_\tau = \mathbf{w}_\tau^{t*}$ is given by $\mathbf{w}_\tau^{t*}(p_t) = F_\tau^t(p_t)^{-1}.\mathbf{w}_\tau(p_\tau)$. If α_τ is one-form in Ω_τ then its pull-back $(\Phi_\tau^t)^* \alpha_\tau = \alpha_\tau^{t*}$ is given by $\alpha_\tau^{t*}(p_t) = \alpha_\tau(p_\tau).F_\tau^t(p_t)$. And we define $f_\tau^*(t, p_t) := f_\tau^{t*}(p_t)$, $\mathbf{w}_\tau^*(t, p_t) := \mathbf{w}_\tau^{t*}(p_t)$, and $\alpha_\tau^*(t, p_t) := \alpha_\tau^{t*}(p_t)$.

The Lie derivative of a real-valued function along the flow of velocity $\mathbf{v}$ (or along the motion $\widetilde{\Phi}$) at (t, p_t) is defined by

$$\mathcal{L}_{\mathbf{v}} f(t, p_t) := \lim_{\tau \to t} \frac{f_\tau^*(t, p_t) - f(t, p_t)}{\tau - t}. \tag{59}$$

Thus, with $p_\tau = \Phi_\tau^t(p_t)$,

$$\mathcal{L}_{\mathbf{v}} f(t, p_t) = \lim_{\tau \to t} \frac{f(\tau, p_\tau) - f(t, p_t)}{\tau - t} = \frac{Df}{Dt}(t, p_t). \tag{60}$$

Interpretation: An observer does not have the gift of spatial and/or temporal ubiquity, thus cannot trivially compare values at a two distinct points or instants: He needs time (from t to τ) and a displacement (from p_t to p_τ) to compare $f(\tau, p_\tau)$ and $f(t, p_t)$ (to compute the difference $f(\tau, p_\tau) - f(t, p_t)$). So (60) is a computation consequence of (59).

The Lie derivative $\mathcal{L}_{\mathbf{v}} \mathbf{w}$ of a vector field $\mathbf{w}$ along the flow of velocity $\mathbf{v}$ is defined by

$$\mathcal{L}_{\mathbf{v}} \mathbf{w}(t, p_t) := \lim_{\tau \to t} \frac{\mathbf{w}_\tau^*(t, p_t) - \mathbf{w}(t, p_t)}{\tau - t}, \tag{61}$$

and the difference $\mathbf{w}^{t*}(t, p_t) - \mathbf{w}(t, p_t)$ is computed at a single time t and at a single point p_t (does not require ubiquity). In $\mathbb{R}^n$, $\mathcal{L}_{\mathbf{v}} \mathbf{w}$ is equivalently defined by

$$\mathcal{L}_{\mathbf{v}} \mathbf{w}(t, p_t) := \lim_{\tau \to t} \frac{\mathbf{w}(\tau, p_\tau) - \mathbf{w}_*^t(\tau, p_\tau)}{\tau - t}. \tag{62}$$

Interpretation: E.g. in $\mathbb{R}^n$ with (62): at τ and p_τ, the numerator $\mathbf{w}(\tau, p_\tau) - \mathbf{w}_*^t(\tau, p_\tau)$ compares the true value $\mathbf{w}(\tau, p_\tau)$ of $\mathbf{w}$ at (τ, p_τ) with the value $\mathbf{w}_*^t(\tau, p_\tau)$ which corresponds to "the vector that would have let itself transported by the motion" (the push-forward). So the Lie derivative $\mathcal{L}_{\mathbf{v}}\mathbf{w}(t,p_t)$, which is the limit of the rate $\frac{\mathbf{w}(\tau,p_\tau)-\mathbf{w}_*^t(\tau,p_\tau)}{\tau-t}$, measure the "resistance" of $\mathbf{w}$ "submitted to the motion $\widetilde{\Phi}$" ("submitted to the flow"). The Lie Virtual Power Principle proposed in this manuscript is based on this property. (For comparison with the classical approach see § D.)

And (61) gives:

$$\mathcal{L}_{\mathbf{v}}\mathbf{w} = \frac{D\mathbf{w}}{Dt} - d\mathbf{v}.\mathbf{w} \quad (= \frac{\partial \mathbf{w}}{\partial t} + d\mathbf{w}.\mathbf{v} - d\mathbf{v}.\mathbf{w}). \tag{63}$$

Indeed with $p_\tau = \Phi_\tau^t(p_t)$ and $\mathbf{g} : \tau \to \mathbf{g}(\tau) = d\Phi_\tau^t(p_t)^{-1}.\mathbf{w}(\tau,p_\tau)$ we have $\mathcal{L}_{\mathbf{v}}\mathbf{w}(t,p_t) = \mathbf{g}'(t)$, see (61); And $d\Phi^t(\tau,p_t).\mathbf{g}(\tau) = \mathbf{w}(\tau,p_\tau)$ leads to $d\frac{\partial\Phi^t}{\partial\tau}(\tau,p_t).\mathbf{g}(\tau) + d\Phi^t(\tau,p_t).\mathbf{g}'(\tau) = \frac{D\mathbf{w}}{D\tau}(\tau,p_\tau)$, therefore with $\tau = t$ we get $d\mathbf{v}(t,p_t).\mathbf{w}(t,p_t) + \mathbf{g}'(t) = \frac{D\mathbf{w}}{Dt}(t,p_t)$, thus (63). Note that, for a vector field $\mathbf{w}$ submitted to the flow, the term $-d\mathbf{v}.\mathbf{w}$ in $\mathcal{L}_{\mathbf{v}}\mathbf{w}$ takes into account the influence of non uniform flows $\mathbf{v}$ (flows s.t. $d\mathbf{v} \neq 0$) on $\mathbf{w}$.

The Lie derivative $\mathcal{L}_{\mathbf{v}}\alpha$ of a one-form α along the flow of velocity $\mathbf{v}$ is the one-form deduced for example from (63) and (60) with the derivation property $\mathcal{L}_{\mathbf{v}}(\alpha.\mathbf{w}) = \mathcal{L}_{\mathbf{v}}\alpha.\mathbf{w} + \alpha.\mathcal{L}_{\mathbf{v}}\mathbf{w}$. Thus

$$\mathcal{L}_{\mathbf{v}}\alpha = \frac{D\alpha}{Dt} + \alpha.d\mathbf{v} \quad (= \frac{\partial\alpha}{\partial t} + d\alpha.\mathbf{v} + \alpha.d\mathbf{v}). \tag{64}$$

The same result is obtained with the mathematical definition $\mathcal{L}_{\mathbf{v}}\alpha(t,p_t) := \lim_{\tau\to t} \frac{\alpha_\tau^*(t,p_t)-\alpha(t,p_t)}{\tau-t}$.

And for tensors of order 2, $\underline{\underline{\kappa_m}} \in T_1^1(\Omega_t)$ (mixed), $\underline{\underline{\kappa_u}} \in T_0^2(\Omega_t)$ (up) and $\underline{\underline{\kappa_d}} \in T_2^0(\Omega_t)$ (down), the generic derivation property $\mathcal{L}_{\mathbf{v}}(a\otimes b) = (\mathcal{L}_{\mathbf{v}}a)\otimes b + a\otimes\mathcal{L}_{\mathbf{v}}b$ (or definition with pull-backs) gives

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}}\underline{\underline{\kappa_m}} &= \frac{D\underline{\underline{\kappa_m}}}{Dt} + \underline{\underline{\kappa_m}}.d\mathbf{v} - d\mathbf{v}.\underline{\underline{\kappa_m}} \ \in T_1^1(\Omega_t) \quad \text{(Jaumann)},\\
\mathcal{L}_{\mathbf{v}}\underline{\underline{\kappa_u}} &= \frac{D\underline{\underline{\kappa_u}}}{Dt} - \underline{\underline{\kappa_u}}.(d\mathbf{v})^* - d\mathbf{v}.\underline{\underline{\kappa_u}} \ \in T_0^2(\Omega_t) \quad \text{(Maxwell upper-convected)},\\
\mathcal{L}_{\mathbf{v}}\underline{\underline{\kappa_d}} &= \frac{D\underline{\underline{\kappa_d}}}{Dt} + \underline{\underline{\kappa_d}}.d\mathbf{v} + (d\mathbf{v})^*.\underline{\underline{\kappa_d}} \ \in T_2^0(\Omega_t) \quad \text{(Maxwell lower-convected)},
\end{aligned} \tag{65}$$

where, for $\underline{\underline{\sigma_m}} \in T_1^1(\Omega_t)$, the adjoint tensor $\underline{\underline{\sigma_m}}^* \in T_1^1(\Omega_t)^*$ is defined by $\underline{\underline{\sigma_m}}^*(\mathbf{u},\ell) = \underline{\underline{\sigma_m}}(\ell,\mathbf{u})$, and $\frac{D\underline{\underline{\kappa}}}{Dt} = \frac{\partial\underline{\underline{\kappa_d}}}{\partial t} + d\underline{\underline{\kappa}}.\mathbf{v}$ (generic notation).

Although an Eulerian velocity field $\mathbf{v}$ is not objective, see § G and H, we have: If T is an objective tensor, then its Lie derivative $\mathcal{L}_{\mathbf{v}}T$ is an objective tensor; see e.g. Marsden and Hughes [6] p.101. (But partial derivatives and material derivative are not objective in general.) Here objectivity refers to "covariant objectivity", see (78).

D Lie derivative versus deformation tensor

The deformation tensor $C = F^T.F$ between t_0 and t is used to measure the relative deformation between two vectors thanks to the use of two Euclidean inner products, $(\cdot,\cdot)_G$ at t_0 and $(\cdot,\cdot)_g$ at t (see e.g. Marsden and Hughes [6]): With $P \in \Omega_{t_0}$, $p_t = \Phi_t^{t_0}(P)$, $F_t^{t_0}(P) := d\Phi_t^{t_0}(P)$, $C_t^{t_0}(P) = (F_t^{t_0}(P))^T.F_t^{t_0}(P)$, $\mathbf{W}_i(P) \in \mathbb{R}^n_{t_0}$ and $\mathbf{w}_{i*}(p_t) = F_t^{t_0}(P).\mathbf{W}_i(P)$ the push-forward of $\mathbf{W}_i$ by $\Phi_t^{t_0}$ see (53), we have :

$$(\mathbf{w}_{1*}(p_t),\mathbf{w}_{2*}(p_t))_g = (C_t^{t_0}(P).\mathbf{W}_1(P),\mathbf{W}_2(P))_G. \tag{66}$$

This value is compared with $(\mathbf{W}_1(P),\mathbf{W}_2(P))_G$ in classical mechanics. N.B.: The deformation tensor C compares two vectors that have let themselves deformed by the flow, see (66), since $\mathbf{w}_{1*}$ and $\mathbf{w}_{2*}$ are the push-forwards by the flow.

While the Lie derivative of a vector field $\mathbf{w}$ measures the resistance of a single vector field $\mathbf{w}$ submitted to a flow, see interpretation of (62), and does not require a priori the use of inner products (Euclidean or not) since there is no comparison between two vectors.

E Change of Riesz representation vector

The Riesz representation theorem is often implicitly used in classical mechanics under isometric objectivity hypotheses (see § H). Unfortunately this causes problems since covariance cannot be confused with contravariance. E.g., Misner, Thorne, Wheeler [7] box 2.1: "Without it [the distinction between covariance and contravariance], one cannot know whether a vector is meant or the very different geometric object that is a 1-form."

Let E be a normed vector space and E^* the set of linear continuous functions $E \to \mathbb{R}$. There is no natural canonical isomorphism between E and its dual E^*, see e.g. Spivak [8], but if an observer introduces an inner product then an isomorphism (depending on the observer) is obtained:

Theorem E.1 (Riesz representation theorem) *If $(\cdot,\cdot)_g$ is an inner product in E so that E is a Hilbert space, then*

$$\forall \ell \in E^*, \quad \exists! \boldsymbol{\ell}_g \in E, \quad \forall \mathbf{w} \in E, \quad \ell.\mathbf{w} = (\boldsymbol{\ell}_g,\mathbf{w})_g. \tag{67}$$

Moreover $||\boldsymbol{\ell}_g||_g = ||\ell||_{E^*}$. And the vector $\boldsymbol{\ell}_g$ is called the Riesz representative vector of the linear form ℓ relatively to the inner product $(\cdot,\cdot)_g$, or the $(\cdot,\cdot)_g$-Riesz representative vector of ℓ.

Proof: If $\ell = 0$ then $\boldsymbol{\ell}_g = 0$. If $\ell \neq 0$, choose a $\mathbf{v} \notin \mathrm{Ker}\ell = \{\mathbf{v} \in E : \ell(\mathbf{v}) = 0\} = \ell^{-1}(\{0\})$ (closed sub-space since ℓ is continuous). Let $\mathbf{v}_0$ be the $(\cdot,\cdot)_g$-orthogonal projection of $\mathbf{v}$ on $\mathrm{Ker}\ell$, and let $\mathbf{n} = \frac{\mathbf{v}-\mathbf{v}_0}{||\mathbf{v}-\mathbf{v}_0||_g}$ (a $||.||_g$-unitary vector normal to $\mathrm{Ker}\ell$). Then take $\boldsymbol{\ell}_g = \ell(\mathbf{n})\mathbf{n}$ to get (67) and $||\boldsymbol{\ell}_g||_g = \sup_{||\mathbf{w}||_E=1} |\ell.\mathbf{w}|$ $(||\ell||_{E^*})$ thanks to the Cauchy–Schwarz inequality. And uniqueness is trivial.

Corollary E.2 (Change of Riesz representation vector) *Let $\ell \in E^*$. Let $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ be two inner products, and let $\boldsymbol{\ell}_a$ and $\boldsymbol{\ell}_b$ be the $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$-Riesz representative of ℓ. Then*

$$\forall \mathbf{w} \in E, \quad (\boldsymbol{\ell}_a, \mathbf{w})_a = (\boldsymbol{\ell}_b, \mathbf{w})_b. \tag{68}$$

Proof: For all $\mathbf{w}$, (67) gives $\ell.\mathbf{w} = (\boldsymbol{\ell}_a, \mathbf{w})_a$ and $\ell.\mathbf{w} = (\boldsymbol{\ell}_b, \mathbf{w})_b$, thus (68).

Example E.3 Let $(\cdot,\cdot)_a$ be the Euclidean inner product built with the English foot (ft) and $(\cdot,\cdot)_b$ be the Euclidean inner product built with the meter (m). We have 1 ft $= \mu$ m with $\mu = 0.3048$. And $(\cdot,\cdot)_b = \mu^2\,(\cdot,\cdot)_a$ (and $||.||_b = \mu||.||_a$), thus (68) gives $(\boldsymbol{\ell}_a, \mathbf{w})_a = \mu^2(\boldsymbol{\ell}_b, \mathbf{w})_a$ for all $\mathbf{w}$, therefore:

$$\boldsymbol{\ell}_a = \mu^2 \boldsymbol{\ell}_b \tag{69}$$

and the vector $\boldsymbol{\ell}_a$ is μ^2 times smaller (more than ten times smaller) than the vector $\boldsymbol{\ell}_b$. So the Riesz representation vector depends on the choice of the inner product (an inner product is a measuring tool, and a change of tool changes the result).

F Incompatibilities with Riesz representation vectors

We have just seen that the Riesz representation vector depends on the observer, see e.g. (69). But we also have e.g.:

1- Incompatibility with push-forwards. Let α be in $(\mathbb{R}^n_{t_0})^*$ and let $\alpha_* = \alpha.{F^{t_0}_t}^{-1} \in (\mathbb{R}^n_t)^*$ be its push-forward by $\Phi^{t_0}_t$, see (55). Let $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ be inner products in $\mathbb{R}^n_{t_0}$ and $\mathbb{R}^n_t$. Let $\boldsymbol{\alpha}_G \in \mathbb{R}^n_{t_0}$ and $\boldsymbol{\alpha}_{*g} \in \mathbb{R}^n_t$ be the $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$-Riesz representation vectors of α and α_*, see (67). And let $\boldsymbol{\alpha}_{G*} \in \mathbb{R}^n_t$ be the push-forward of $\boldsymbol{\alpha}_G$ by $\Phi^{t_0}_t$, see (53). So, with $p = \Phi^{t_0}_t(P)$ and $\mathbf{w}_p \in \mathbb{R}^n_t$, we have $(\boldsymbol{\alpha}_{*g}(p), \mathbf{w}_p)_g = \alpha_*(p).\mathbf{w}_p = (\alpha(P).F^{t_0}_t(P)^{-1}).\mathbf{w}_p = \alpha(P).(F^{t_0}_t(P)^{-1}.\mathbf{w}_p) = (\boldsymbol{\alpha}_G(P), F^{t_0}_t(P)^{-1}.\mathbf{w}_p)_G = (F^{t_0}_t(P)^{-T}.\boldsymbol{\alpha}_G(P), \mathbf{w}_p)_g$, true for all $\mathbf{w}_p$, thus

$$\boldsymbol{\alpha}_{*g}(p) = F^{t_0}_t(P)^{-T}.\boldsymbol{\alpha}_G(p). \tag{70}$$

So $\boldsymbol{\alpha}_{*g}$ is not the push-forward of $\boldsymbol{\alpha}_G$, that is $\boldsymbol{\alpha}_{*g}(p) \neq F^{t_0}_t(P).\boldsymbol{\alpha}_G(p)$, unless $\Phi^{t_0}_t$ is the motion of a solid. Thus the Riesz representation vectors should not be used if push-forwards of one-forms are needed.

2- Incompatibility with Lie derivative. Let β be in $(\mathbb{R}^n_t)^*$. Let $(\cdot,\cdot)_g$ be an inner product in $\mathbb{R}^n_t$, and let $\boldsymbol{\beta}_g$ be the $(\cdot,\cdot)_g$-Riesz representation vector of β, that is $\beta.\mathbf{w} = (\boldsymbol{\beta}_g, \mathbf{w})_g$ for any $\mathbf{w}$. Then we have $\mathcal{L}_\mathbf{v}\beta.\mathbf{w} \neq (\mathcal{L}_\mathbf{v}\boldsymbol{\beta}_g, \mathbf{w})_g$, unless Φ is the motion of a solid, that is we have $\frac{D\beta}{Dt}.\mathbf{w} + \beta.d\mathbf{v}.\mathbf{w} \neq (\frac{D\boldsymbol{\beta}_g}{Dt}, \mathbf{w})_g - (\boldsymbol{\beta}_g, d\mathbf{v}.\mathbf{w})_g$ in general. We have:

$$\mathcal{L}_\mathbf{v}\beta.\mathbf{w} = (\mathcal{L}_\mathbf{v}\boldsymbol{\beta}_g, \mathbf{w})_g + (\boldsymbol{\beta}_g, (d\mathbf{v} + d\mathbf{v}^T).\mathbf{w})_g \tag{71}$$

since $\frac{D\beta}{Dt}.\mathbf{w} + \beta.d\mathbf{v}.\mathbf{w} = (\frac{D\boldsymbol{\beta}_g}{Dt}, \mathbf{w})_g + (\boldsymbol{\beta}_g, d\mathbf{v}.\mathbf{w})_g$. Thus the Riesz representation vectors cannot be used if Lie derivative of one-forms are needed.

G Velocity addition formula

(This § is needed to define objectivity, see the next §.) Classical mechanics setting. The observers use the same time scale. An observer A defines a referential $\mathcal{R}_A = (O_A, (\mathbf{A}_i))$ and an observer B defines a referential $\mathcal{R}_B = (O_B, (\mathbf{B}_i))$, with $(\mathbf{A}_i)$ and $(\mathbf{B}_i)$ Cartesian bases. They observe an object Obj in a time interval $[t_1, t_2]$. Let $\widetilde{\Phi}_A : [t_1, t_2] \times \mathit{Obj} \to \mathcal{R}_A$ and $\widetilde{\Phi}_B : [t_1, t_2] \times \mathit{Obj} \to \mathcal{R}_B$ be the motions of Obj as described by A and B, and let $\mathbf{v}_A$ and $\mathbf{v}_B$ be the associated Eulerian velocity fields, see (1) and (2), that is,

$$\begin{cases} p_{At} = \widetilde{\Phi}_A(t, P_{\mathit{Obj}}) = O_A + \displaystyle\sum_{i=1}^n x^i_{At}\mathbf{A}_i, \\ \mathbf{v}_A(t, p_{At}) = \dfrac{\partial \widetilde{\Phi}_A}{\partial t}(t, P_{\mathit{Obj}}), \end{cases} \qquad \begin{cases} p_{Bt} = \widetilde{\Phi}_B(t, P_{\mathit{Obj}}) = O_B + \displaystyle\sum_{i=1}^n x^i_{Bt}\mathbf{B}_i, \\ \mathbf{v}_B(t, p_{Bt}) = \dfrac{\partial \widetilde{\Phi}_B}{\partial t}(t, P_{\mathit{Obj}}). \end{cases} \tag{72}$$

Let Obj_B be the object used by B to define his referential (e.g. Obj_B = Earth). Let $\widetilde{\Psi}_A : [t_1, t_2] \times \mathit{Obj}_B \to \mathcal{R}_A$ and $\widetilde{\Psi}_B : \mathit{Obj}_B \to \mathcal{R}_B$ (static) be the motions of Obj_B described by A and B, and let $\mathbf{w}_A$ and $\mathbf{w}_B$ be the associated Eulerian velocity fields, see (1) and (2), that is,

$$\begin{cases} q_{At} = \widetilde{\Psi}_A(t, Q_{ObjB}) = O_A + \displaystyle\sum_{i=1}^n y^i_{At}\mathbf{A}_i, \\ \mathbf{w}_A(t, q_{At}) = \dfrac{\partial \widetilde{\Psi}_A}{\partial t}(t, Q_{ObjB}), \end{cases} \qquad \begin{cases} q_B = \widetilde{\Psi}_B(Q_{ObjB}) = O_B + \displaystyle\sum_{i=1}^n y^i_B\mathbf{B}_i, \\ \mathbf{w}_B(t, q_B) = \mathbf{0}. \end{cases} \tag{73}$$

For t fixed, let $\widetilde{\Phi}_{At}(P_{\mathit{Obj}}) := \widetilde{\Phi}_A(t, P_{\mathit{Obj}})$ and $\widetilde{\Phi}_{Bt}(P_{\mathit{Obj}}) := \widetilde{\Phi}_B(t, P_{\mathit{Obj}})$, and $\widetilde{\Psi}_{At}(P_{\mathit{Obj}}) := \widetilde{\Psi}_A(t, P_{\mathit{Obj}})$.

The mapping $\widetilde{\Phi}_A$, $\widetilde{\Psi}_A$ in $\mathcal{R}_A$ and $\widetilde{\Phi}_B$, $\widetilde{\Psi}_B$ in $\mathcal{R}_B$ are motions: They are defined by one observer in his referential. The translation mapping at t from B to A (the translator) is the "inter-referential" diffeomorphism $\Theta_t := \widetilde{\Psi}_{At} \circ \widetilde{\Psi}_B^{-1}$, and we denote

$$\Theta_t : \left\{ \begin{array}{ll} \mathcal{R}_B & \to \mathcal{R}_A \\ q_B = O_B + \displaystyle\sum_{i=1}^n y^i_B\mathbf{B}_i & \to q_{At} = \Theta_t(q_B) = O_A + \displaystyle\sum_{i=1}^n y^i_{At}\mathbf{A}_i \end{array} \right\}. \tag{74}$$

So, $\widetilde{\Psi}_{At}(Q_{ObjB}) = \Theta_t(\widetilde{\Psi}_B(Q_{ObjB}))$ for all $Q_{ObjB} \in Obj_B$: If a particle Q_{ObjB} of Obj_B is located at $q_B = \widetilde{\Psi}_B(Q_{ObjB}) \in \mathcal{R}_B$ by the observer B, then the observer A locates Q_{ObjB} at t at $\widetilde{\Psi}_{At}(Q_{ObjB}) = q_{At} = \Theta_t(q_B) \in \mathcal{R}_A$. E.g. $\Theta_t(O_B)$ is "the position of O_B in $\mathcal{R}_A$ at t"; And thus the push-forward $d\Theta_t(O_B).\mathbf{B}_i$ of $\mathbf{B}_i$ by Θ_t is "the basis vector $\mathbf{B}_i$ as seen by A at $\Theta_t(O_B)$" (see § B with the diffeomorphism $\Psi = \Theta_t$ and $c_\mathcal{E} =$ the i-th coordinate line in $\mathcal{R}_B$).

With (74) define $\Theta : [t_1, t_2] \times \mathcal{R}_B \to \mathcal{R}_A$ by $\Theta(t, q_B) := \Theta_t(q_B)$. And define the "$\Theta$-Eulerian velocity" at $q_{At} \in \mathcal{R}_A$ by:

$$\mathbf{w}_\Theta(t, q_{At}) = \frac{\partial \Theta}{\partial t}(t, q_B) \quad \text{if} \quad q_{At} = \Theta_t(q_B). \tag{75}$$

Note that $\mathbf{w}_\Theta$ looks like a Eulerian velocity but is not since Θ is not a motion (Θ_t is an inter-referential mapping).

Interpretation of $\mathbf{w}_\Theta$: With $\widetilde{\Psi}_A(t, Q_{ObjB}) = \Theta(t, \widetilde{\Psi}_B(Q_{ObjB}))$ and $q_B = \widetilde{\Psi}_B(Q_{ObjB})$ we get $\frac{\partial \widetilde{\Psi}_A}{\partial t}(t, Q_{ObjB}) = \frac{\partial \Theta}{\partial t}(t, q_B)$, thus, with (73) and (75) we have

$$\mathbf{w}_A(t, q_{At}) = \mathbf{w}_\Theta(t, q_{At}), \tag{76}$$

equality in $\mathcal{R}_A$; And the "Θ-Eulerian velocity" $\mathbf{w}_\Theta(t, q_{At})$ is the Eulerian velocity $\mathbf{w}_A(t, q_{At})$ of the particle $Q_{ObjB} \in Obj_B$ that is at t at $q_{At} = \widetilde{\Psi}_{At}(Q_{ObjB}) = \Theta_t(q_B) \in \mathcal{R}_A$.

Velocity addition formula. We also have $\widetilde{\Phi}_{At} = \Theta_t \circ \widetilde{\Phi}_{Bt}$, that is, with (72), $p_{At} = \Theta_t(p_{Bt})$ (inter-referential relations between positions at t). So $\widetilde{\Phi}_A(t, P_{Obj}) = \Theta(t, \widetilde{\Phi}_B(t, P_{Obj}))$, and with $p_{Bt} = \widetilde{\Phi}_B(t, P_{Obj})$ we get $\frac{\partial \widetilde{\Phi}_A}{\partial t}(t, P_{Obj}) = \frac{\partial \Theta}{\partial t}(t, p_{Bt}) + d\Theta(t, p_{Bt}).\frac{\partial \widetilde{\Phi}_B}{\partial t}(t, P_{Obj})$, that is, $\mathbf{v}_A(t, p_{At}) = \mathbf{w}_\Theta(t, p_{At}) + d\Theta(t, p_{Bt}).\mathbf{v}_B(t, p_{Bt})$, that is with (76),

$$\mathbf{v}_{At}(p_{At}) = \mathbf{v}_{Bt*}(p_{At}) + \mathbf{w}_{At}(p_{At}) \quad \text{where} \quad \mathbf{v}_{Bt*}(p_{At}) := d\Theta_t(p_{Bt}).\mathbf{v}_{Bt}(p_{Bt}). \tag{77}$$

This formula is the velocity addition formula. Interpretation: At t, if A is the "absolute observer" and B is the "relative observer" then the formula reads: "($\mathbf{v}_{At}$ the absolute velocity) = ($\mathbf{v}_{Bt*}$ the relative velocity translated for A) + ($\mathbf{w}_{At}$ the velocity of the coordinate system of B in $\mathcal{R}_A$)", equality in $\mathcal{R}_A$.

H Covariant and isometric objectivities

(Mainly from Marsden and Hughes [6].) Setting of the previous §.

Definition H.1 Let $\mathbf{u}_{Bt}$ be a vector field in $\mathcal{R}_B$ at t as described by the observer B. Let $\mathbf{u}_{Bt*}$ be its translation for A, that is, the push-forward of $\mathbf{u}_{Bt}$ by Θ_t (so $\mathbf{u}_{Bt*}(p_{At}) = d\Theta_t(p_{Bt}).\mathbf{u}_{Bt}(p_{Bt})$ when $p_{At} = \Theta_t(p_{Bt})$, see (53)). Then $\mathbf{u}_{Bt*}$ is called the objective transform of $\mathbf{u}_{Bt}$ by Θ_t. More generally, at t the objective transform of a tensor T_{Bt} in $\mathcal{R}_B$ is its push-forward T_{Bt*} by Θ_t.

Definition H.2 Consider "a quantity" that can be described as a vector field by any observer, quantity written $\mathbf{u}$. At t, A describes $\mathbf{u}$ as the vector field $\mathbf{u}_{At}$ in $\mathcal{R}_A$, and B describes $\mathbf{u}$ as the vector field $\mathbf{u}_{Bt}$ in $\mathcal{R}_B$. Then $\mathbf{u}$ is (covariant) objective if and only if, for all t and all observers A and B, $\mathbf{u}_{At}$ is the objective transform of $\mathbf{u}_{Bt}$ by Θ_t, that is,

$$\mathbf{u}_{At} = \mathbf{u}_{Bt*}. \tag{78}$$

In other words, $\mathbf{u}$ is objective iff, for all t and all observers A and B, whenever $q_{At} = \Theta_t(q_B)$ we have $\mathbf{u}_{At}(q_{At}) = d\Theta_t(q_B).\mathbf{u}_{Bt}(q_B)$. Similarly, a tensor T is objective iff $T_{At} = T_{Bt*}$ for all t and all observers.

Counter-example: a velocity field $\mathbf{v}$ is not objective, see (77): $\mathbf{w}_{At} \neq 0$ in general. So the objective vector fields in the definition H.2 refer, e.g., to the "force fields" $\mathbf{f}$ of Newton's fundamental law $\sum \mathbf{f} = m\boldsymbol{\gamma}$ where $\boldsymbol{\gamma} = \frac{D\mathbf{v}}{Dt}$ (acceleration).

Definition H.3 To work in an objective covariant framework means to consider any diffeomorphism Θ_t (any translator).

Definition H.4 To work in an "objective isometric" framework means to only consider observers A and B using a unique metric $(\cdot,\cdot)_g$ (isometric setting), that is, to only consider the Θ_t that are isometries relatively to one chosen inner product $(\cdot,\cdot)_g$ (and $(\Theta_t.\mathbf{u}_1, \Theta_t.\mathbf{u}_2)_g = (\mathbf{u}_1, \mathbf{u}_2)_g$ for any $\mathbf{u}_1, \mathbf{u}_2$). (And in classical mechanics the chosen metric $(\cdot,\cdot)_g$ is often assumed to a Euclidean metric.)

E.g. an English observer using feet and a french observer using meters have no choice but to use covariant objectivity if they want to communicate (they use different metrics). And in general relativity "objective isometry" is meaningless.

I About elasticity and classical formulations

The appendix follows Germain [5], apart from the remarks where $\mathrm{Tr}(F)$ is replaced by $\det(F)$.

• In $\mathbb{R}^n_{t_0}$. The abbreviated notations $C = F^T.F$ (deformation tensor) and $E = \frac{C-I}{2}$ (Green-Lagrange tensor) stand for the endomorphisms in $\mathbb{R}^n_{t_0}$ relative to t_0, t and $P \in \Omega_{t_0}$ defined by $C^{t_0}_t(P) = F^{t_0}_t(P)^T.F^{t_0}_t(P)$ and $E^{t_0}_t(P) = \frac{C^{t_0}_t(P) - I_{t_0}}{2}$ where I_{t_0} is the identity in $\mathbb{R}^n_{t_0}$. The classical isotropic homogeneous elasticity is e.g. stated as,

$$\underline{\underline{\sigma}} = \lambda \mathrm{Tr}(E)\, I + 2\mu E, \tag{79}$$

abbreviated notation for the endomorphism $\underline{\underline{\sigma}}^{t_0}_t(P) = \lambda \mathrm{Tr}(E^{t_0}_t(P))\, I + 2\mu E^{t_0}_t(P)$ in $\mathbb{R}^n_{t_0}$.

Small deformations: The linearization $E = \frac{C-I}{2} \simeq \frac{F+F^T}{2} - I$ gives (linear elasticity)

$$\underline{\underline{\sigma}} \simeq \lambda \mathrm{Tr}(\underline{\underline{\varepsilon}})\, I + 2\mu \underline{\underline{\varepsilon}}, \quad \underline{\underline{\varepsilon}} = \frac{F+F^T}{2} - I. \tag{80}$$

But $\underline{\underline{\varepsilon}}$ has no functional meaning because $F_t^{t_0}(P) : \mathbb{R}^n_{t_0} \to \mathbb{R}^n_t$, $F_t^{t_0}(P)^T : \mathbb{R}^n_t \to \mathbb{R}^n_{t_0}$, and $I = I_{t_0}$ stays in $\mathbb{R}^n_{t_0}$, so that the sum $\frac{F_t^{t_0} + F_t^{t_0\,T}}{2} - I_{t_0}$ is not a function (and $\mathrm{Tr}(\varepsilon)$ has no functional meaning either), unless the shifter is introduced, see remark 2.1, and/or (80) is considered in the matrix sense after having chosen a unique Euclidean basis at t_0 and t.

Remark relative to the introduction of the virtual power of pressure (11): Relatively to Euclidean bases, the volume change at $P \in \Omega_{t_0}$ is $J = \det(F) = (\det(C))^{\frac{1}{2}}$, where J, F and C stands for $J_t^{t_0}(P)$, $F_t^{t_0}(P)$ and $C_t^{t_0}(P)$. And for small displacements $\mathrm{Tr}(C - I) \simeq \det(C) - 1$: Indeed with an orthonormal basis of diagonalization of C we have $[C] = \mathrm{diag}(1+\varepsilon_1, ..., 1+\varepsilon_n)$, and then $\mathrm{Tr}(C) = n + \sum_i \varepsilon_i$ and $\det(C) = 1 + \sum_i \varepsilon_i + o(\varepsilon)$ in the neighborhood of $\varepsilon = 0$ where $\varepsilon = \max(|\varepsilon_i|)$. So $\mathrm{Tr}(E) = \frac{\mathrm{Tr}(C)-n}{2} \simeq \frac{\det(C)-1}{2}$. Then, for small displacements, (79) can be replaced by

$$\underline{\underline{\sigma}} = \frac{\lambda}{2}(\det(C) - 1)\, I + \mu(C - I). \tag{81}$$

And $\det(C) = \det(F^T.F) = \det(F)^2$, and $\det(F)^2 - 1 = (\det(F) - I)(\det(F) + I) \simeq 2(\det(F) - I)$ for small displacements. Then a pressure term can be considered with (11).

• In $\mathbb{R}^n_t$. Instead of the deformation tensor $C = F^T.F$ we may prefer to use the Finger tensor $\underline{\underline{b}} = F.F^T$, and more precisely its inverse $\underline{\underline{b}}^{-1} = H^T.H$ where $H = F^{-1}$. Unabbreviated notation: At t and $p_t = \Phi_t^{t_0}(P)$, $H_t^{t_0}(p_t) = F_t^{t_0}(P)^{-1}$ and $(\underline{\underline{b}}_t^{t_0})^{-1}(p_t) = H_t^{t_0}(p_t)^T.H_t^{t_0}(p_t)$. And instead of the Green–Lagrange tensor, we may use the Euler–Almansi tensor defined by

$$\underline{\underline{a}} = \frac{I_t - \underline{\underline{b}}^{-1}}{2} = \frac{I_t - H^T.H}{2}, \tag{82}$$

simplified notation of the endomorphism $\underline{\underline{a}}_t^{t_0}(p_t) = \frac{I_t - (\underline{\underline{b}}_t^{t_0})^{-1}(p_t)}{2}$ in $\mathbb{R}^n_t$. The classical elasticity is then e.g. stated as, with $\lambda, \mu \in \mathbb{R}$,

$$\underline{\underline{\sigma}} = \lambda \mathrm{Tr}(\underline{\underline{a}})\, I + 2\mu \underline{\underline{a}}, \tag{83}$$

to compare with (79). And here $\underline{\underline{\sigma}} \,\mathbin{0}\, d\mathbf{v}$ is meaningful (double objective contraction in $\mathbb{R}^n_t$). For small displacements, and for matrix computation, (82) yields $[\underline{\underline{a}}] \simeq [I] - \frac{[H]+[H]^T}{2}$ (linearization), and (83) reads (linear elasticity)

$$[\underline{\underline{\sigma}}] = \lambda \mathrm{Tr}([\underline{\underline{a}}])\, I + 2\mu[\underline{\underline{a}}] \quad \text{with} \quad [\underline{\underline{a}}] = [I] - \frac{[H] + [H]^T}{2}]. \tag{84}$$

Since $[F^{t_0}(t, P)] - [I] = [I] - [H^{t_0}(t, p(t))] + o(t{-}t_0)$ if $p(t) = \Phi^{t_0}(t, P)$, we get back to (80). And, as for (81), instead of (83) we can consider, for small displacements,

$$\underline{\underline{\sigma}} = \frac{\lambda}{2}(1 - \det(\underline{\underline{b}}^{-1}))\, I + \mu(I - \underline{\underline{b}}^{-1}). \tag{85}$$

(Or with $(1 - \det(H))$ instead of $\mathrm{Tr}(I - H)$.) And a pressure term can be considered with (11).

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