Lie derivative

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Motion

Let *Obj* be a "real" object made of particles P_{Obj}. Let $\mathcal{R}$ be a referential.

Definition: A motion is a map $\widetilde{\Phi} : \left\{ \begin{array}{ll} \mathbb{R} \times Obj & \to \mathcal{R} \\ (t, P_{Obj}) & \to p = \widetilde{\Phi}(t, P_{Obj}) \end{array} \right\}$ that locates a particle P_{Obj} in $\mathcal{R}$.

Notation: The Eulerian velocity field $\vec{v}$ is defined by:

$$\text{if} \quad \underbrace{p = \widetilde{\Phi}(t, P_{Obj})}_{\text{position of } P_{Obj} \text{ at } t} \quad \text{then} \quad \underbrace{\vec{v}(t,p) := \frac{\partial \widetilde{\Phi}}{\partial t}(t, P_{Obj})}_{\text{velocity of } P_{Obj} \text{ at } t}. \tag{1}$$

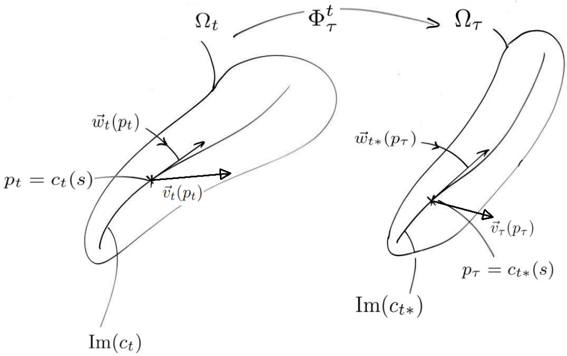
Notation: $\Omega_t = \widetilde{\Phi}(t, P_{Obj})$ = configuration at t (= the photo of *Obj* at t).

Notation: τ being close to t, let

$$\Phi^t_\tau : \left\{ \begin{array}{rl} \Omega_t & \to \Omega_\tau \\ p_t = \widetilde{\Phi}(t, P_{Obj}) & \to p_\tau = \Phi^t_\tau(p_t) := \widetilde{\Phi}(\tau, P_{Obj}). \end{array} \right.$$

(See figure page 4.)

figure



- $p_t = \widetilde{\Phi}(t, P_{Obj})$ = position of a particle P_{Obj} at t, and $\vec{v}_t(p_t)$ = velocity of P_{Obj} at t.
- c_t = a line in Ω_t passing through p_t, and $\vec{w}_t(p_t)$ = tangent vector at p_t.
- $c_{t*} := \Phi^t_\tau(c_t)$ = the transported line by the motion, $p_\tau = \Phi^t_\tau(p_t)$.
- $\vec{w}_{t*}(p_\tau) = d\Phi^t_\tau(p_t).\vec{w}(p_t)$ = tangent vector at c_{t*} at p_τ = the push-forward by Φ^t_τ. In short:

$$\vec{w}_* = F.\vec{w}.$$

- So: $\vec{w}_{t*}(p_\tau)$ results from a "strain" (a motion): Not linked to a constitutive law (not linked to a "stress").

Push-forward

- Let $c_t : \begin{Bmatrix} \mathbb{R} & \to \Omega_t \\ s & \to p_t = c_t(s) \end{Bmatrix}$ be a curve in Ω_t (so s is a space curvilinear coordinate, not a time coordinate), and let (see figure page 4)

$$\vec{w}_t(p_t) := (c_t)'(s) = \text{tangent vector to } \mathrm{Im}(c_t) \text{ at } p_t. \tag{2}$$

- Let $c_{t*} := \Phi^t_\tau \circ c_t : \begin{Bmatrix} \mathbb{R} & \to \Omega_\tau \\ s & \to p_\tau = c_{t*}(s) := \Phi^t_\tau(p_t) \end{Bmatrix}$ be the transported curve by the motion Φ^t_τ, and let (see figure page 4)

$$\vec{w}_{t*}(p_\tau) := (c_{t*})'(s) = \text{tangent vector to } \mathrm{Im}(c_{t*}) \text{ at } p_\tau. \tag{3}$$

- **Definition:** The vector field $\vec{w}_{t*}$ (defined in Ω_τ), which is the result of the deformation of $\vec{w}_t$ by the motion, is called the push-forward of the vector field $\vec{w}_t$ by the motion Φ^t_τ.

- Let $F^t_\tau = d\Phi^t_\tau$ (deformation tensor); Then $c_{t*}(s) := (\Phi^t_\tau \circ c_t)(s)$ gives

$$\vec{w}_{t*}(p_\tau) := F^t_\tau(p_t).\vec{w}_t(p_t) \quad \text{at} \quad p_\tau = \Phi^t_\tau(p_t). \tag{4}$$

This is the “push-forward by a motion formula” for vector fields.

Lie derivative: Definition, interpretation

- Let $\vec{w}(t,p)$ be a ("force") vector field defined at any time t and any point $p \in \Omega_t$ (so $\vec{w}$ is a Eulerian vector field).

- **Definition:** With $\vec{v}$ the velocity field of the motion, cf (1), the Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ of $\vec{w}$ along $\vec{v}$ at t at p_t is, with $p_\tau = \Phi^t_\tau(p_t)$,

$$\mathcal{L}_{\vec{v}}\vec{w}(t,p_t) = \lim_{\tau\to t} \frac{\vec{w}_\tau(p_\tau) - \vec{w}_{t*}(p_\tau)}{\tau - t}.$$

- Interpretation: At τ at p_τ the numerator gives the difference between
  - → the true value $\vec{w}_\tau(p_\tau)$ of $\vec{w}$ at τ at p_τ, and
  - → the virtual value $\vec{w}_{t*}(p_\tau) = F^t_\tau(p_t).\vec{w}_t(p_t)$ (the push-forward), cf (4): If $\vec{w}$ had allowed itself to be distorted by the flow (had not resisted the flow), then $\vec{w}_{t*}(p_\tau)$ would have been be the value of $\vec{w}$ at τ at p_τ.

Hence, $\mathcal{L}_{\vec{v}}\vec{w}$ gives "a rate of stress on $\vec{w}$" due to the flow.

- Computation:

$$\mathcal{L}_{\vec{v}}\vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w} \quad \left(= \frac{\partial \vec{w}}{\partial t} + d\vec{w}.\vec{v} - d\vec{v}.\vec{w}\right).$$

(The spatial variations $d\vec{v}$ of $\vec{v}$ influence "the stress on $\vec{w}$": Expected.)

Comparison with the Cauchy deformation tensor

• Let $F^t_\tau := d\Phi^t_\tau =^{\text{written}} F =$ the deformation gradient between t and τ.
Let $(\cdot,\cdot)_g$ be a Euclidean dot product in $\mathbb{R}^n$.
Let $C := F^T.F =$ the Cauchy–Green deformation tensor.
Let $\vec{u}_t, \vec{w}_t$ be vector fields at t. So:

$$\begin{aligned}(\vec{u}_{t*}(p_\tau), \vec{w}_{t*}(p_\tau))_g &= (F(p_t).\vec{u}_t(p_t), F(p_t).\vec{w}_t(p_t))_g \\ &= (C(p_t).\vec{u}_t(p_t), \vec{w}_t(p_t))_g,\end{aligned}$$

which is compared with $(\vec{u}_t(p_t), \vec{w}_t(p_t))_g$ (the value at t), that is, in short,

$$\vec{u}_* \bullet \vec{w}_* - \vec{u} \bullet \vec{w} = (C - I).\vec{u} \bullet \vec{w}.$$

Thus C enables to compare the relative deformation of **two** vectors which have let themselves be deformed by the motion (since we used the push-forwards $\vec{u}_* = F.\vec{u}$ and $\vec{w}_* = F.\vec{w}$).

• The Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ of a vector field $\vec{w}$ measures the resistance of **one** vector field $\vec{w}$ submitted to a motion: It seems suitable for the measurement of the stress due to a flow. Moreover $\mathcal{L}_{\vec{v}}\vec{w}$ is objective covariant.

And $\mathcal{L}_{\vec{v}}\vec{w}$ does not require the use a priori of some dot product (Euclidean or not), which is the Cauchy–Green tensor approach, since here there is no comparison between two vectors $\vec{u}$ and $\vec{w}$: Just one vector $\vec{w}$ (and a motion).

Dot product and absence of objectivity

• There is no natural canonical isomorphism between $\mathbb{R}^n$ and its dual $(\mathbb{R}^n)^* = \mathcal{L}(\mathbb{R}^n;\mathbb{R})$: A linear form (covariant) cannot be identified with a "Riesz representation vector" (contravariant).
NB: A Riesz representation vector is obtained thanks to the use of a dot product. Eg, with a Euclidean dot product, a Riesz representation vector depends on the chosen unit of measure (meter? foot?) with which the Euclidean dot product was made: So a Riesz representation vector is not (covariant) objective (it depends on a choice of an observer).

• A Riesz representation vector is not compatible with push-forwards: The push-forward of a linear form (covariant) is not represented by the push-forward of its Riesz representation vector (contravariant).

• A Riesz representation vector is not compatible with the use of the Lie derivative: The push-forward of the Lie derivative of a differential form (covariant) is not represented by the push-forward of its Riesz representation vector field (contravariant).