Objectivity in continuum mechanics, an introduction

Motions, Eulerian and Lagrangian variables and functions, deformation gradient, Lie derivatives, velocity-addition formula, Coriolis.

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In classical mechanics, there are two objectivities: 1- The covariant objectivity concerns the universal laws of physics required to be observer independent (true in any reference frame); This is a main topic in this manuscript. 2- The isometric objectivity concerns the constitutive laws of materials once expressed in a reference frame.

Covariant objectivity in continuum mechanics follows Maxwell's requirements, cf. [15] page 1: "2. (...) The formula at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result. (...) 10. (...) The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in mathematical progress, for it reduced the methods of geometry to calculations performed on numerical quantities. The position of a point is made to depend on the length of three lines which are always drawn in determinate directions (...) But for many purposes in physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other,..."

And see the (short) historical note given in the introduction of Abraham and Marsden book "Foundations of Mechanics" [1], about qualitative versus quantitative theory: "Mechanics begins with a long tradition of qualitative investigation culminating with KEPLER and GALILEO. Following this is the period of quantitative theory (1687-1889) characterized by concomitant developments in mechanics, mathematics, and the philosophy of science that are epitomized by the works of NEWTON, EULER, LAGRANGE, LAPLACE, HAMILTON, and JACOBI. (...) For celestial mechanics (...) resolution we owe to the genius of POINCARÉ, who resurrected the qualitative point of view (...) One advantage (...) is that by suppressing unnecessary coordinates the full generality of the theory becomes evident."

After having defined motions, Eulerian and Lagrangian variables and functions, we give the definition of the deformation gradient as a function. We then obtain a simple understanding of the Lie derivatives of vector fields which meet the needs of engineers. Then we get the velocity addition formula and verify that the Lie derivatives are (covariant) objective. Note that Cauchy would certainly have used the Lie derivatives if they had existed during his lifetime: To get a stress, Cauchy had to compare two vectors, whereas one vector is enough when using the derivatives of Lie.

We systematically start with qualitive definitions (observer independent), before quantifying with bases and/or Euclidean dot products (observer dependent). A fairly long appendix tries to give in one manuscript the definitions, properties and interpretations, usually scattered across several books (and not always that easy to find).

Contents

A quantity f being given then: g defined by « g equals f » is noted $g := f$.

Part I
Motions, Eulerian and Lagrangian descriptions, flows

1 Motions

The framework is classical mechanics, time being decoupled from space. $\mathbb{R}^3$ is the classical geometric affine space (the space we live in), and $(\vec{\mathbb{R}^3}, +, .) = \{\vec{pq} : p, q \in \mathbb{R}^3\} =^{\text{noted}} \vec{\mathbb{R}^3}$ is the associated vector space of bipoint vectors equipped with its usual rules. We also consider $\mathbb{R}$ and $\mathbb{R}^2$ as subspaces of $\mathbb{R}^3$, i.e. we consider $\mathbb{R}^n$ and $\vec{\mathbb{R}^n}$, $n = 1, 2, 3$.

1.1 Referential

Origin: An observer chooses an origin $\mathcal{O} \in \mathbb{R}^n$; Thus a point $p \in \mathbb{R}^n$ can be located by the observer thanks to the bipoint vector $\overrightarrow{\mathcal{O}p} = \vec{x} \in \vec{\mathbb{R}^n}$; Hence $p = \mathcal{O} + \vec{x}$, and $\vec{x} = \overrightarrow{\mathcal{O}p} =^{\text{noted}} p - \mathcal{O}$.

Another observer chooses an origin $\widetilde{\mathcal{O}} \in \mathbb{R}^n$; Thus the point p can also be located by this observer with the bipoint vector $\overrightarrow{\widetilde{\mathcal{O}}p} = \widetilde{\vec{x}} \in \vec{\mathbb{R}^n}$; So $p = \mathcal{O} + \vec{x} = \widetilde{\mathcal{O}} + \widetilde{\vec{x}}$, and $\widetilde{\vec{x}} = \overrightarrow{\mathcal{O}\widetilde{\mathcal{O}}} + \vec{x}$.

Cartesian coordinate system: A Cartesian coordinate system in the affine space $\mathbb{R}^n$ is a set $\mathcal{R}_{\text{Cart}} = (\mathcal{O}, (\vec{e}_i)_{i=1,...,n})$, where $\mathcal{O}$ is an origin and $(\vec{e}_i) := (\vec{e}_i)_{i=1,...,n}$ is a basis in $\vec{\mathbb{R}^n}$ chosen by the observer. Thus the location of a point $p \in \mathbb{R}^n$ can quantified by the observer $\exists \vec{x} \in \vec{\mathbb{R}^n}$ s.t.

$$p = \mathcal{O} + \vec{x} \quad \text{with} \quad \vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i, \quad \text{i.e.} \quad [\overrightarrow{\mathcal{O}p}]_{|\vec{e}} = [\vec{x}]_{|\vec{e}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \tag{1.1}$$

$[\vec{x}]_{|\vec{e}} = [\overrightarrow{\mathcal{O}p}]_{|\vec{e}}$ being the column matrix containing the components $x_i \in \mathbb{R}$ of $\overrightarrow{\mathcal{O}p} = \vec{x}$ in the basis $(\vec{e}_i)$. Another observer with his origin O_b and his Cartesian basis $(\vec{b}_i)_{i=1,...,n}$ make the Cartesian coordinate system $\mathcal{R}_{\text{Cart,b}} = (\mathcal{O}_b, (\vec{b}_i)_{i=1,...,n})$, and gets for the same position p in $\mathbb{R}^n$,

$$p = \mathcal{O}_b + \vec{y} \quad \text{with} \quad \vec{y} = \sum_{i=1}^{n} \widetilde{y}_i \widetilde{\vec{b}}_i, \quad \text{i.e.} \quad [\overrightarrow{\mathcal{O}_b p}]_{|\vec{b}} = [\vec{y}]_{|\vec{b}} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \tag{1.2}$$

$[\vec{y}]_{|\vec{b}} = [\overrightarrow{\mathcal{O}_b p}]_{|\vec{b}}$ being the column matrix containing the components $y_i \in \mathbb{R}$ of $\overrightarrow{\mathcal{O}_b p} = \vec{y}$ in the basis $(\vec{b}_i)$. And $\overrightarrow{\mathcal{O}_b p} = \overrightarrow{\mathcal{O}_b \mathcal{O}} + \overrightarrow{\mathcal{O}p}$, i.e. $\vec{y} = \overrightarrow{\widetilde{\mathcal{O}}\mathcal{O}} + \vec{x}$, gives the relation between $\vec{x}$ and $\vec{y}$ (drawing).

Chronology: A chronology (or temporal coordinate system) is a set $\mathcal{R}_{\text{time}} = (t_0, (\Delta t))$ chosen by an observer, where $t_0 \in \mathbb{R}$ is the time origin, and (Δt) is the time unit (a basis in $\vec{\mathbb{R}}$).

Referentiel: A referential $\mathcal{R}$ is the set

$$\mathcal{R} = (\mathcal{R}_{\text{time}}, \mathcal{R}_{\text{Cart}}) = (t_0, (\Delta t), \mathcal{O}, (\vec{e}_i)_{i=1,...,n}) = (\text{"chronologie"}, \text{"Cartesian coordinate system"}), \tag{1.3}$$

made of a chronology and a Cartesian coordinate system, chosen by an observer.

In the following, to simplify the writings, the same implicit chronology is used by all observers, and a referential $\mathcal{R} = (\mathcal{R}_{\text{time}}, \mathcal{R}_{\text{Cart}})$ will simply be noted as the reference frame $\mathcal{R} = (\mathcal{O}, (\vec{e}_i))$ (so $:= \mathcal{R}_{\text{Cart}}$).

1.2 Einstein's convention (duality notation)

Starting point: The classical notation x_i for the components of a vector $\vec{x}$ relative to a basis, cf. (1.1). Then the duality notion is introduced: $x_i =^{\text{noted}} x^i$ (enables to see the difference between a vector and a function when using components). So

$$\vec{x} = \underbrace{\sum_{i=1}^{n} x_i \vec{e}_i}_{\text{classic not.}} = \underbrace{\sum_{i=1}^{n} x^i \vec{e}_i}_{\text{duality not.}} , \quad \text{and} \quad [\vec{x}]_{|\vec{e}} \overset{\text{clas.}}{=} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{dual}}{=} \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} . \tag{1.4}$$

The duality notation is part of the Einstein's convention; Moreover Einstein's convention uses the notation $\sum_{i=1}^{n} x^i \vec{e}_i =^{\text{noted}} x^i \vec{e}_i$, i.e. the sum sign $\sum_{i=1}^{n}$ can be omitted when an index (i here) is used twice, once up and once down, details at § A.4. However this omission of the sum sign $\sum$ will not be made in this manuscript (to avoid ambiguities): The TEX-LATEX program makes it easy to print $\sum_{i=1}^{n}$.

Example 1.1 The height of a child is represented on a wall by a vertical bipoint vector $\vec{x}$ starting from the ground up to a pencil line. Question: What is the size of the child ?

Answer: It depends... on the observer (quantitative value = subjective result). E.g., an English observer chooses a vertical basis vector $\vec{a}_1$ which length is one English foot (ft). So he writes $\vec{x} = x_1 \vec{a}_1$, and for him the size of the child (size of $\vec{x}$) is x_1 in foot. E.g. $x_1 = 4$ means the child is 4 ft tall. A French observer chooses a vertical basis vector $\vec{b}_1$ which length is one metre (m). So he writes $\vec{x} = y_1 \vec{b}_1$, and for him the size of the child (size of $\vec{x}$) is y^1 metre. E.g., if $x_1 = 4$ then $y_1 \simeq 1.22$, since 1 ft := 0.3048 m: The child is both 4 and 1.22 tall... in foot or metre. This quantification is written $\vec{x} = 4$ ft = 1.22 m, where ft means $\vec{a}_1$ and m means $\vec{b}_1$ here. NB: The qualitative vector $\vec{x}$ is the same vector for all observers, not the quantitative values 4 or 1.22 (depends on a choice of a unit of measurement).

With duality notation: $\vec{x} = x^1 \vec{a}_1 = y^1 \vec{b}_1$, so if $x^1 = 4$ then $y^1 \simeq 1.22$. ∴

This manuscript insists on covariant objectivity; Thus an English engineer (and his foot) and a French engineer (and his metre) will be able to work together ... and be able to avoid crashes like that of the Mars Climate Orbiter probe, see remark A.17. And they will be able to use the results of Galileo, Descartes, Newton, Euler... who used their own unit of length, and knew nothing about the metre defined in 1793 and adopted in 1799 in France (after 6 years of measurements), and considered by the scientific community at the end of the ninetieth century... and couldn't explicitly use the "Euclidean dot products" either (which seems to have been defined mathematically by Grassmann around 1844).

1.3 Motion of an object

Let *Obj* be a "real object", or "material object", made of particles (e.g., the Moon: Exists independently of an observer). Let $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$.

Definition 1.2 The motion of *Obj* in $\mathbb{R}^n$ is the map

$$\widetilde{\Phi} : \begin{cases} [t_1, t_2] \times Obj & \to \mathbb{R}^n \\ (t, \underbrace{P_{Obj}}_{\text{particle}}) & \to \underbrace{p = \widetilde{\Phi}(t, P_{Obj})}_{\text{its position at } t \text{ in the Universe}} \end{cases} . \tag{1.5}$$

And t is the time variable, p is the space variable, and $(t, p) \in \mathbb{R} \times \mathbb{R}^n$ is the time-space variable. And $\widetilde{\Phi}$ is supposed to be C^2 in time.

With an origin $\mathcal{O}$ (observer dependent), the motion can be described with the bi-point vector

$$\vec{x} = \overrightarrow{\mathcal{O}\widetilde{\Phi}(t, P_{Obj})} = \overrightarrow{\mathcal{O}p} \overset{\text{noted}}{=} \widetilde{\vec{\varphi}}(t, P_{Obj}). \tag{1.6}$$

But then, two observers with different origins $\mathcal{O}$ and $\mathcal{O}_b$ have different description of the motion. Therefore, in the following we won't use $\widetilde{\vec{\varphi}}$. Then (quantification) with a Cartesian basis $(\vec{e}_i)$ to make a referential $\mathcal{R}$, we get (1.1).

1.4 Virtual and real motion

Definition 1.3 A virtual (or possible) motion of *Obj* is a function $\widetilde{\Phi}$ "regular enough for the calculations to be meaningful". Among all the virtual motions, the observed motion is called the real motion.

1.5 Hypotheses (Newton and Einstein)

Hypotheses of Newtonian mechanics (Galileo relativity) and general relativity (Einstein):

1- You can describe a phenomenon only at the actual time t and from the location p you are at (you have **no** gift of ubiquity in time or space);

2- You don't know the future;

3- You can use your memory, so use some past time t_0 and some past position p_{t_0};

4- You can use someone else memory (results of measurements) if you can communicate objectively.

1.6 Configurations

Fix $t \in [t_1, t_2]$, and define $\widetilde{\Phi}_t : \begin{cases} Obj & \to \mathbb{R}^n \\ P_{Obj} & \mapsto p = \widetilde{\Phi}_t(P_{Obj}) := \widetilde{\Phi}(t, P_{Obj}). \end{cases}$

Definition 1.4 The "configuration at t" of Obj is the range (or image) of $\widetilde{\Phi}_t$, i.e. is the subset of $\mathbb{R}^n$ (affine space) defined by

$$\Omega_t := \{p \in \mathbb{R}^n : \exists P_{Obj} \in Obj \text{ s.t. } p = \widetilde{\Phi}_t(P_{Obj})\} \overset{\text{noted}}{=} \widetilde{\Phi}_t(Obj) \overset{\text{noted}}{=} \text{Im}(\widetilde{\Phi}_t). \tag{1.7}$$

If t is the actual time then Ω_t is the actual (or current or Eulerian) configuration.

If t_0 is a time in the past then Ω_{t_0} is the past (or initial or Lagrangian) configuration.

Hypothesis: At any time t, Ω_t is supposed to be a "smooth domain" in $\mathbb{R}^n$, and the map $\widetilde{\Phi}_t$ is assumed to be one-to-one (= injective): Obj does not crash onto itself.

1.7 Definition of the Eulerian and Lagrangian variables

• If t is the actual time, then $p_t = \widetilde{\Phi}_t(P_{Obj}) \in \Omega_t$ is called the Eulerian variable relative to P_{Obj} and t.

• If t_0 is a time in the past, then $p_{t_0} = \widetilde{\Phi}_{t_0}(P_{Obj}) \in \Omega_{t_0}$ is called the Lagrangian variable relative to P_{Obj} and t_0. (A Lagrangian variable is a "past Eulerian variable"). (Two observers with two different origin of time t_0 and $t_0{}'$ get two different Lagrangian variable while they have the same Eulerian variable.)

1.8 Trajectories

Let $\widetilde{\Phi}$ be a motion of Obj, cf. (1.5), and $P_{Obj} \in Obj$ (a particle in Obj = e.g. the Moon).

Definition 1.5 The (parametric) trajectory of P_{Obj} is the function

$$\widetilde{\Phi}_{P_{Obj}} : \begin{cases} [t_1, t_2] & \to \mathbb{R}^n, \\ \quad t & \mapsto p(t) = \widetilde{\Phi}_{P_{Obj}}(t) := \widetilde{\Phi}(t, P_{Obj}) \quad \text{(position of } P_{Obj} \text{ at } t \text{ in the Universe)}. \end{cases} \tag{1.8}$$

Its geometric trajectory is the range (image) of $\widetilde{\Phi}_{P_{Obj}}$, i.e.

$$\text{geometric trajectory of } P_{Obj} := \{q \in \mathbb{R}^n : \exists t \in [t_1, t_2] \text{ s.t. } q = \widetilde{\Phi}_{P_{Obj}}(t)\} = \text{Im}(\widetilde{\Phi}_{P_{Obj}}) = \widetilde{\Phi}_{P_{Obj}}([t_1, t_2]). \tag{1.9}$$

1.9 Pointed vector, tangent space, fiber, vector field, bundle

(See e.g. Abraham–Marsden [1].) To deal with surfaces S in $\mathbb{R}^3$, e.g. with S = a sphere (and more generally with manifolds in $\mathbb{R}^n$), a vector cannot simply be a "bi-point vector connecting two points of S" (would get "through the surface"). A vector is defined to be tangent to S: Consider a "regular" curve $c : s \in]-\varepsilon, \varepsilon[\to c(s) \in S$ where S is a surface in an affine space, and the vector tangent to S at $c(0)$ is $\vec{w}(c(0)) = \lim_{h\to 0} \frac{c(h)-c(0)}{h}$ (it is defined with a parametrization of c in a general manifold); Considering all the possible curves, we get "all possible vectors on S".

Notation:

$$T_pS := \{\text{tangent vectors } \vec{w}_p \text{ at } S \text{ at } p\} = \text{The tangent space at } p \in S. \tag{1.10}$$

E.g., if S is a sphere in $\mathbb{R}^3$ and $p \in S$, then T_pS is its usual tangent plane at p at S.

E.g., particular case: If $S = \Omega$ is an open set in $\mathbb{R}^n$, then $T_pS = T_p\Omega = \vec{\mathbb{R}^n}$ is independent of p.

Definition 1.6

$$\text{The fiber at } p := \{p\} \times T_pS = \{ \underbrace{(p, \vec{w}_p)}_{\text{pointed vector}} \in \{p\} \times T_pS\}, \tag{1.11}$$

i.e., the fiber at p is the set of "pointed vectors at p", a pointed vector being the couple $(p, \vec{w}_p)$ made of the "base point" p and the vector $\vec{w}_p$ defined at p.

Drawing: A vector in $\vec{\mathbb{R}^n}$ can be drawn anywhere in $\mathbb{R}^n$; While a "pointed vectors at p" has to be drawn at the point p in $\mathbb{R}^n$.

If the context is clear, a pointed vector is simply noted $\widetilde{\vec{w}}(p) =^{\text{noted}} \vec{w}(p)$ (lighten the writing).

Particular case: If $S = \Omega$ is an open set in $\mathbb{R}^n$, then the fiber at p is $T_p\Omega = \{p\} \times \vec{\mathbb{R}^n}$.

Definition 1.7

$$\text{The tangent bundle } TS := \bigcup_{p \in S} (\{p\} \times T_pS), \tag{1.12}$$

that is, is the union of the fibers.

Definition 1.8 A vector field $\widetilde{\vec{w}}$ in S is a C^∞ function (or at least C^2 in the following)

$$\widetilde{\vec{w}} : \begin{cases} S & \to TS \\ p & \to \widetilde{\vec{w}}(p) = (p, \vec{w}(p)). \end{cases} \tag{1.13}$$

If the context is clear, a vector field is simply noted $\widetilde{\vec{w}} =^{\text{noted}} \vec{w}$ (lighten the writing).

2 Eulerian description (spatial description at actual time t)

2.1 The set of configurations

Let $\widetilde{\Phi}$ be a motion of *Obj*, cf. (1.5), and $\Omega_t = \widetilde{\Phi}_t(\mathit{Obj}) \subset \mathbb{R}^n$ be the configuration at t, cf. (1.7). The set of configurations is the subset $\mathcal{C} \subset \mathbb{R} \times \mathbb{R}^n$ (the "time-space") defined by

$$\begin{aligned} \mathcal{C} &:= \bigcup_{t \in [t_1, t_2]} (\{t\} \times \Omega_t) \quad (= \text{set in which you find particles in "time-space"}) \\ &= \{(t, p) \in \mathbb{R} \times \mathbb{R}^n : \exists (t, P_{\mathit{Obj}}) \in [t_1, t_2] \times \mathit{Obj},\ p = \widetilde{\Phi}(t, P_{\mathit{Obj}})\}, \end{aligned} \tag{2.1}$$

Question: Why don't we simply use $\bigcup_{t \in [t_1, t_2]} \Omega_t$ instead of $\mathcal{C} = \bigcup_{t \in [t_1, t_2]} (\{t\} \times \Omega_t)$?

Answer: $\mathcal{C}$ gives the film of the life of *Obj* = the succession of the photos Ω_t taken at each t; And Ω_t is obtained from $\mathcal{C}$ thanks to the pause feature at t. Whereas $\bigcup_{t \in [t_1, t_2]} \Omega_t \subset \mathbb{R}^n$ is the superposition of all the photos on the image $\bigcup_{t \in [t_1, t_2]} \Omega_t$... and we don't distinguish the past from the present.

2.2 Eulerian variables and functions

Definition 2.1 In short: A Eulerian function relative to *Obj* is a function, with $m \in \mathbb{N}^*$,

$$\mathcal{E}ul : \begin{cases} \mathcal{C} & \to \vec{\mathbb{R}^m} \text{ (or more generally a suitable set of tensors)} \\ (t, p) & \to \mathcal{E}ul(t, p), \end{cases} \tag{2.2}$$

the spatial variable p being the Eulerian variable.

In details: A function $\mathcal{E}ul$ being given as in (2.2), the associated Eulerian function $\widehat{\mathcal{E}ul}$ is the function

$$\widehat{\mathcal{E}ul} : \begin{cases} \mathcal{C} & \to \mathcal{C} \times \vec{\mathbb{R}^m} \text{ (or } \mathcal{C} \times \text{ some suitable set of tensors)} \\ (t, p) & \to \widehat{\mathcal{E}ul}(t, p) = ((t, p), \mathcal{E}ul(t, p)) = (\text{time-space position , value}), \end{cases} \tag{2.3}$$

and is called "a field of functions". So $\widehat{\mathcal{E}ul}(t, p)$ is the "pointed $\mathcal{E}ul(t, p)$" at (t, p) (in time-space).

So, the range $\mathrm{Im}(\widehat{\mathcal{E}ul}) = \widehat{\mathcal{E}ul}(\mathcal{C})$ of an Eulerian function $\widehat{\mathcal{E}ul}$ is the graph of $\mathcal{E}ul$. (Recall: The graph of a function $f : x \in A \to f(x) \in B$ is the subset $\{(x, f(x)) \in A \times B\} \subset A \times B$: gives the "drawing of f").

If there is no ambiguity, $\widehat{\mathcal{E}ul} =^{\text{noted}} \mathcal{E}ul$ for short.

At t, the Eulerian vector field at t is $\widehat{\mathcal{E}ul}_t : \begin{cases} \Omega_t & \to \Omega_t \times \vec{\mathbb{R}^n} \\ p & \to \widehat{\mathcal{E}ul}_t(p) := (p, \mathcal{E}ul_t(p)) = (\text{position} , \text{value}). \end{cases}$

Example 2.2 $\mathcal{E}ul(t,p) = \theta(t,p) \in \mathbb{R}$ = temperature of the particle P_{Obj} which is at t at $p = \widetilde{\Phi}(t, P_{Obj})$; ▟

Example 2.3 $\mathcal{E}ul(t,p) = \vec{u}(t,p) \in \vec{\mathbb{R}^n}$ = force applied on the particle P_{Obj} which is at t at p. ▟

Example 2.4 $\mathcal{E}ul(t,p) = d\vec{u}(t,p) \in \mathcal{L}(\vec{\mathbb{R}^n} : \vec{\mathbb{R}^n})$ = the differential at t at p of a Eulerian function $\vec{u}$. ▟

Question: Why introduce $\widehat{\mathcal{E}ul}$? Isn't $\mathcal{E}ul$ sufficient?
Answer: The "pointed value" $\widehat{\mathcal{E}ul}(t,p) = ((t,p), \mathcal{E}ul((t,p)))$ is drawn on the graph of $\mathcal{E}ul$.
E.g., at t at p the velocity vector $\vec{v}(t,p) \in \vec{\mathbb{R}^3}$ can be drawn anywhere, while the "pointed vector" $\widehat{\vec{v}}(t,p) = ((t,p); \vec{v}(t,p))$ is $\vec{v}(t,p)$ drawn at t at p (and $\widehat{\vec{v}}$ is called the velocity field).
Moreover (2.3) emphasizes the difference between a Eulerian vector field and a Lagrangian vector function, see (3.14).

Remark 2.5 E.g., the initial framework of Cauchy for his description of forces is Eulerian: The Cauchy stress vector $\vec{t} = \underline{\underline{\sigma}}.\vec{n}$ is considered at the actual time t at a point $p \in \Omega_t$. (It is not Lagrangian.) ▟

2.3 Eulerian velocity (spatial velocity) and speed

Definition 2.6 In short: Consider a particle P_{Obj} and its (regular) trajectory $\widetilde{\Phi}_{P_{Obj}} : t \to p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$, cf. (1.8). Its Eulerian velocity at t at $p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$ is

$$\vec{v}(t, p(t)) := \widetilde{\Phi}_{P_{Obj}}{}'(t) \overset{\text{noted}}{=} \frac{\partial \widetilde{\Phi}}{\partial t}(t, P_{Obj}), \quad \text{when} \quad p(t) = \widetilde{\Phi}_{P_{Obj}}(t), \tag{2.4}$$

i.e., $\vec{v}(t,p(t))$ is the tangent vector at t at $p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$ to the trajectory $\widetilde{\Phi}_{P_{Obj}}$. This defines the vector field (in short) $\vec{v} : \left\{ \begin{array}{ll} \mathcal{C} & \to \vec{\mathbb{R}^n} \\ (t, p_t) & \to \vec{v}(t, p_t) \end{array} \right\}$.

In details: cf. (2.3), the Eulerian velocity is the function $\widehat{\vec{v}} : \left\{ \begin{array}{ll} \mathcal{C} & \to \mathcal{C} \times \vec{\mathbb{R}^m} \\ (t,p) & \to \widehat{\vec{v}}(t,p) = ((t,p), \vec{v}(t,p)) \end{array} \right\}$ (pointed vector) where $\vec{v}(t,p)$ is given by (2.4).

Remark 2.7 $\frac{d\widetilde{\Phi}_{P_{Obj}}}{dt}(t) = \vec{v}(t, \widetilde{\Phi}_{P_{Obj}}(t))$, with $p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$, is often written

$$\frac{dp}{dt}(t) = \vec{v}(t, p(t)), \quad \text{or} \quad \frac{d\vec{x}}{dt}(t) = \vec{v}(t, \vec{x}(t)), \quad \text{or} \quad \frac{d\vec{x}}{dt} = \vec{v}(t, \vec{x}), \tag{2.5}$$

the two last notations when an origin O is chosen and $\vec{x}(t) = \overrightarrow{Op(t)}$. Such an equation is the prototype of an ODE (ordinary differential equation) solved with the Cauchy–Lipschitz theorem, see § 5. (A Lagrangian velocity does not produce an ODE, see (3.21).) ▟

Definition 2.8 If an observer chooses a Euclidean dot product $(\cdot,\cdot)_g$ (e.g. foot or metre built), the associated norm being $||.||_g$, then the length $||\vec{v}(t,p)||_g$ is the speed (or scalar velocity) of P_{Obj} (e.g. in ft/s or in m/s). And the context must remove the ambiguities: the "velocity" is either the vector velocity $\vec{v}(t,p) = \widetilde{\Phi}_{P_{Obj}}{}'(t)$ or the speed (the scalar velocity) $||\vec{v}(t,p)||_g$.

Exercice 2.9 Euclidean dot product $(\cdot,\cdot)_g$, $\vec{x}(t) = \overrightarrow{Op(t)}$, $\vec{T}(t) = \frac{\vec{x}\,'(t)}{||\vec{x}\,'(t)||_g}$, and $f(t) = ||\vec{x}\,'(t)||_g$ (speed). Prove : $\frac{df}{dt}(t) = (\vec{x}\,''(t), \vec{T}(t))_g =^{\text{noted}} \vec{x}\,''(t) \bullet \vec{T}(t)$ (= tangential acceleration).

Answer. 2-D and Euclidean basis: $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ gives $f(t) = (x'(t)^2 + y'(t)^2)^{\frac{1}{2}}$, thus $f'(t) = \frac{x'(t)x''(t)+y'(t)y''(t)}{f(t)} = \frac{\vec{r}\,'(t) \bullet \vec{r}\,''(t)}{||\vec{r}\,'(t)||}$. Idem in n-D. ▟

2.4 Spatial derivative of the Eulerian velocity

$t \in [t_1, t_2]$ is fixed, $\mathcal{E}ul$ is a given Eulerian function, and $\mathcal{E}ul_t : \left\{ \begin{array}{ll} \Omega_t & \to \vec{\mathbb{R}^m} \\ p & \to \mathcal{E}ul_t(p) := \mathcal{E}ul(t,p) \end{array} \right\}$ is C^1.

2.4.1 Definition

Recall: If Ω is an open set in $\mathbb{R}^n$ and if $f : \Omega \to \mathbb{R}$ is differentiable at p, then its differential at p is the linear form $df(p) \in \mathcal{L}(\vec{\mathbb{R}^n};\mathbb{R})$ (linear map with real values) defined by, for all $\vec{u} \in \vec{\mathbb{R}^n}$ (vector at p),

$$df(p).\vec{u} = \lim_{h\to 0} \frac{f(p+h\vec{u}) - f(p)}{h}. \tag{2.6}$$

This expression is the same for all observers (English, French...: There is no inner dot product here).

Definition 2.10 The space derivative of $\mathcal{E}ul$ at (t,p) is the differential $d\mathcal{E}ul_t$ at p, i.e., for all $t \in [t_1,t_2]$, all $p \in \Omega_t$ and all $\vec{w}_p \in \vec{\mathbb{R}}^n_t$ (vector at p),

$$(d\mathcal{E}ul_t(p).\vec{w}_p =) \quad \boxed{d\mathcal{E}ul(t,p).\vec{w}_p = \lim_{h\to 0} \frac{\mathcal{E}ul(t,p+h\vec{w}_p) - \mathcal{E}ul(t,p)}{h}} \overset{\text{noted}}{=} \frac{\partial \mathcal{E}ul}{\partial p}(t,p).\vec{w}_p. \tag{2.7}$$

In Ω_t (the photo at t), $d\mathcal{E}ul(t,p).\vec{w}_p$ gives the rate of variations of $\mathcal{E}ul_t$ at p in the direction $\vec{w}_p$.

E.g., at t, the space derivative $d\vec{v}$ of the Eulerian velocity field is defined by

$$d\vec{v}(t,p).\vec{w}_p = \lim_{h\to 0} \frac{\vec{v}(t,p+h\vec{w}_p) - \vec{v}(t,p)}{h} \quad (= d\vec{v}_t(p).\vec{w}_p). \tag{2.8}$$

Remark 2.11 In differential geometry, (2.6) is also written $\vec{u}(f)(p) = \frac{d}{dh} f(p+h\vec{u})_{|h=0}$; Don't use this notation if you are not at ease with differential geometry (where a vector is defined to be a derivation, so $\vec{u}[f]$ is the derivation of f by $\vec{u}$). ▪

2.4.2 The convective derivative $d\mathcal{E}ul.\vec{v}$

Definition 2.12 If $\vec{v}$ is the Eulerian velocity field, then $d\mathcal{E}ul.\vec{v}$ is called the convective derivative of $\mathcal{E}ul$.

2.4.3 Quantification in a basis: $df.\vec{u}$ is written $(\vec{u}.\vec{\text{grad}})f$

Quantification: Let $f : p \in \mathbb{R}^n \to f(p) \in \mathbb{R}$ be C^1. Let $(\vec{e}_i)$ be a basis in $\vec{\mathbb{R}^n}$. Let (usual definition)

$$\frac{\partial f}{\partial x_i}(p) := df(p).\vec{e}_i \quad \text{and} \quad [df(p)]_{|\vec{e}} = \left(\tfrac{\partial f}{\partial x_1}(p) \quad \dots \quad \tfrac{\partial f}{\partial x_n}(p) \right) \ \text{(line matrix)}. \tag{2.9}$$

(Recall: The matrix which represents a linear form is a line matrix.) And $[df(p)]_{|\vec{e}}$ is the Jacobian matrix of f at p relative to $(\vec{e}_i)$. So, with $\vec{u} = \sum_{i=1}^n u_i\vec{e}_i$ a vector at p, and with the usual matrix multiplication rule, we have

$$df(p).\vec{u} = [df(p)]_{|\vec{e}}.[\vec{u}]_{|\vec{e}} = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)u_i = \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}(p) \overset{\text{noted}}{=} (\vec{u}.\vec{\text{grad}})_{|e} f(p), \tag{2.10}$$

where $(\vec{u}.\vec{\text{grad}})_{|e} : C^1(\Omega;\mathbb{R}) \to C^0(\Omega;\mathbb{R})$ is the differential operator defined relative to a basis $(\vec{e}_i)$ by

$$(\vec{u}.\vec{\text{grad}})_{|e}(f) = \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}. \tag{2.11}$$

If the basis $(\vec{e}_i)$ is unambiguously imposed, then $(\vec{u}.\vec{\text{grad}})_{|e} =^{\text{noted}} \vec{u}.\vec{\text{grad}}$

For vector valued functions $\vec{f} : \Omega \to \vec{\mathbb{R}^m}$, the above steps apply to the components of $\vec{f}$ in a basis $(\vec{b}_i)$ in $\vec{\mathbb{R}^m}$: If $\vec{f} = \sum_{i=1}^m f_i\vec{b}_i$, i.e. $\vec{f}(p) = \sum_{i=1}^m f_i(p)\vec{b}_i$, then

$$(\vec{u}.\vec{\text{grad}})_{|e}(\vec{f}) = \sum_{i=1}^m (df_i.\vec{u})\vec{b}_i = \sum_{i=1}^m ((\vec{u}.\vec{\text{grad}})_{|e} f_i)\vec{b}_i = \sum_{i=1}^m \sum_{j=1}^n (u_j.\frac{\partial f_i}{\partial x_j})\vec{b}_i. \tag{2.12}$$

2.4.4 Representation relative to a Euclidean dot product: $\vec{\text{grad}} f$

An observer chooses a distance unit (foot, metre...) and uses the associated Euclidean dot product $(\cdot,\cdot)_g$.

Let Ω be an open set in $\mathbb{R}^n$, $f \in C^1(\Omega;\mathbb{R})$ (scalar valued function), and $p \in \Omega$. Then the $(\cdot,\cdot)_g$-Riesz representation vector of the differential form $df(p)$ is called the gradient of f at p relative to $(\cdot,\cdot)_g$, and named $\vec{\text{grad}}_g f(p) \in \vec{\mathbb{R}^n}$: It is defined by

$$\forall \vec u \in \vec{\mathbb{R}^n}, \quad (\vec{\text{grad}}_g f(p), \vec u)_g = df(p).\vec u, \quad \text{written} \quad \vec{\text{grad}} f \bullet \vec u = df.\vec u, \tag{2.13}$$

the last notation iff a Euclidean dot product $(\cdot,\cdot)_g$ is imposed to all observer (quite subjective: foot, metre ?).

(The first order Taylor expansion $f(p{+}h\vec u) = f(p) + h\,df(p).\vec u + o(h)$ can therefore, after a choice of an Euclidean dot product, be written $f(p{+}h\vec u) = f(p) + h\,\vec{\text{grad}}_g f(p) \bullet_g \vec u + o(h)$.)

Quantification: Let $(\vec e_i)$ be a Cartesian basis in $\mathbb{R}^n$. Then (2.13) gives $[df].[\vec u] = [\vec{\text{grad}} f]^T.[g].[\vec u]$, for all $\vec u \in \vec{\mathbb{R}}^n_t$ (more precisely $[df]_{|\vec e}.[\vec u]_{|\vec e} = [\vec{\text{grad}}_g f]^T_{|\vec e}.[g]_{|\vec e}.[\vec u]_{|\vec e}$), thus (since $[g]_{|\vec e}$ is symmetric)

$$[\vec{\text{grad}} f] = [g].[df]^T \quad \text{(column matrix)}. \tag{2.14}$$

I.e., if $\vec{\text{grad}} f = \sum_{i=1}^n a_i \vec e_i$ then $a_i = \sum_{j=1}^n g_{ij} \frac{\partial f}{\partial x_j}$ for all i. In particular, if $(\vec e_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis then $[\vec{\text{grad}} f] = [df]^T$.

With duality notations, $\vec{\text{grad}} f = \sum_{i=1}^n a^i \vec e_i$ and (2.14) gives $a^i = \sum_{j=1}^n g_{ij} \frac{\partial f}{\partial x^j}$: The Einstein convention is ***not*** satisfied (the index j is twice bottom), which is expected since the definition of $\vec{\text{grad}}_g f$ depends on a subjective choice (unit of length). In comparison, $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$ satisfies the Einstein convention (a differential is objective).

Mind the notations: The gradient $\vec{\text{grad}}_g f \overset{\text{noted}}{=} \vec{\text{grad}} f$ depends on $(\cdot,\cdot)_g$, cf. (2.13)-(2.14), while $(\vec u.\vec{\text{grad}})f$ does not (only depends on a basis), cf. (2.11) (historical notations...).

2.4.5 Vector valued functions

For vector valued functions $\vec f : \Omega \to \vec{\mathbb{R}^m}$, the above steps apply to the components f_i of $\vec f$ relative to a basis $(\vec b_i)$ in $\vec{\mathbb{R}^m}$... But, depending on the book you read:

1- Ambiguous: $d\vec f$, the differential of $\vec f$, is unfortunately also sometimes called the "gradient matrix" (although no Euclidean dot product is required).

2- Ambiguous: It could mean the differential... or the Jacobian matrix... or its transposed... because an orthonormal basis relative to an imposed Euclidean dot product is chosen (which one?) and then $[\vec{\text{grad}} f_i] = [df_i]^T$... And calculations confuses [.] and $[.]^T$...

3- Non ambiguous: In the objective framework of this manuscript, we will use the differential $d\vec f$ (objective) to begin with; And only after an explicit choice of bases $(\vec e_i)$ for quantitative purposes, the Jacobian matrix, which is $[d\vec f]_{|\vec e}$, will be used.

Exercice 2.13 A Euclidean framework being chosen, prove: $(\vec v.\vec{\text{grad}})\vec v = \frac12 \vec{\text{grad}}(||\vec v||^2) + \vec{\text{curl}}\vec v \wedge \vec v$.

Answer. Euclidean basis $(\vec E_i)$, Euclidean dot product $(\cdot,\cdot)_g \overset{\text{noted}}{=} (\cdot,\cdot)$, associated norm $||.||_g \overset{\text{noted}}{=} ||.||$. Thus $\vec v = \sum_{i=1}^n v_i \vec E_i$ gives $||\vec v||^2 = \sum_i v_i^2$, thus $\dfrac{\partial ||\vec v||^2}{\partial x_k} = \sum_i 2 v_i \dfrac{\partial v_i}{\partial x_k}$, for any $k = 1,2,3$. And, the first component of $\vec{\text{curl}}\vec v$ is $(\vec{\text{curl}}\vec v)_1 = \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3}$, idem for $(\vec{\text{curl}}\vec v)_2$ and $(\vec{\text{curl}}\vec v)_3$ (circular permutation). Thus (first component) $(\vec{\text{curl}}\vec v \wedge \vec v)_1 = (\dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1})v_3 - (\dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2})v_2$, idem for $(\vec{\text{curl}}\vec v \wedge \vec v)_2$ and $(\vec{\text{curl}}\vec v \wedge \vec v)_2$. Thus $(\frac12 \vec{\text{grad}}(||\vec v||^2) + \vec{\text{curl}}\vec v \wedge \vec v)_1 = v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_1} + v_3 \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} v_3 - \frac{\partial v_3}{\partial x_1} v_3 - \frac{\partial v_2}{\partial x_1} v_2 + \frac{\partial v_1}{\partial x_2} v_2 = v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} = (\vec v.\vec{\text{grad}})v_1$. Idem for the other components. ▪

2.5 Streamline (current line)

Fix $t \in \mathbb{R}$, and consider the photo $\Omega_t = \widetilde\Phi_t(Obj)$. Let $p_t \in \Omega_t$, $\varepsilon > 0$, and consider the spatial curve in Ω_t at p_t defined by:

$$c_{p_t} : \left\{ \begin{array}{rl}]-\varepsilon,\varepsilon[& \to \Omega_t \\ s & \to q = c_{p_t}(s) \end{array} \right\} \quad \text{s.t.} \quad c_{p_t}(0) = p_t. \tag{2.15}$$

So s is a curvilinear spatial coordinate (dimension of a length), and the graph of c_{p_t} is drawn in the photo Ω_t at t.

Definition 2.14 $\vec{v} : (t,p) \to \vec{v}(t,p)$ being the Eulerian velocity field of Obj, a streamline through a point $p_t \in \Omega_t$ is a (parametric) spatial curve c_{p_t} solution of the differential equation

$$\frac{dc_{p_t}}{ds}(s) = \vec{v}_t(c_{p_t}(s)) \quad \text{with} \quad c_{p_t}(0) = p_t. \tag{2.16}$$

And $\mathrm{Im}(c_{p_t})$ is the geometric associated streamline ($\subset \Omega_t$).

NB: (2.16) cannot be confused with (2.5): In (2.5) the variable is the time variable t, while in (2.16) the variable is the space variable s.

Usual notation: If an origin $\mathcal{O}$ is chosen at t by an observer and $\vec{x}(s) := \overrightarrow{\mathcal{O}c_{p_t}(s)}$, then (2.16) is written

$$\frac{d\vec{x}}{ds}(s) = \vec{v}_t(\vec{x}(s)) \quad \text{with} \quad \vec{x}(0) = \overrightarrow{\mathcal{O}p_t}. \tag{2.17}$$

Moreover, with a Cartesian basis $(\vec{e}_i))$ chosen at t by the observer, with $\vec{x}(s) = \sum_{i=1}^n x_i(s)\vec{e}_i$ we get $\frac{d\vec{x}}{ds}(s) = \sum_{i=1}^n \frac{dx_i}{ds}(s)\vec{e}_i$, and (2.17) reads as the differential system of n equations in $\mathbb{R}^n$

$$\forall i = 1, ..., n, \quad \frac{dx_i}{ds}(s) = v_i(t, x_1(s), ..., x_n(s)) \quad \text{with} \quad x_i(0) = (\overrightarrow{\mathcal{O}p_t})_i \tag{2.18}$$

(the n functions $x_i : s \to x_i(s)$ are the unknown). Also written

$$\frac{dx_1}{v_1} = ... = \frac{dx_n}{v_n} = ds, \tag{2.19}$$

which means: It is the differential system (2.18) of n equations and n unknowns which must be solved.

(With duality notations, $\frac{dx^i}{ds}(s) = v^i(t, x^1(s), ..., x^n(s))$ and $x^i(0) = (\overrightarrow{\mathcal{O}p_t})^i$ for all i.)

2.6 Material time derivative (dérivées particulaires)

2.6.1 Usual definition

Goal: To compute the variations of a Eulerian function $\mathcal{E}ul$ along the trajectory $\widetilde{\Phi}_{P_{Obj}}$ of a particle P_{Obj} (e.g. the temperature of a particle along its trajectory). So consider the function $g_{P_{Obj}}$ giving the values of $\mathcal{E}ul$ relative to a P_{Obj} along its trajectory:

$$g_{P_{Obj}}(t) := \mathcal{E}ul(t, p(t)) \quad \text{when} \quad p(t) := \widetilde{\Phi}_{P_{Obj}}(t). \tag{2.20}$$

Definition 2.15 The Material time derivative of $\mathcal{E}ul$ at $(t, p(t))$ is $g_{P_{Obj}}{}'(t) =^{\text{noted}} \frac{D\mathcal{E}ul}{Dt}(t, p(t))$.

So:

$$\frac{D\mathcal{E}ul}{Dt}(t, p(t)) := g_{P_{Obj}}{}'(t) \quad (= \lim_{h \to 0} \frac{\mathcal{E}ul(t{+}h, p(t{+}h)) - \mathcal{E}ul(t, p(t))}{h}). \tag{2.21}$$

Since $g_{P_{Obj}}(t) := \mathcal{E}ul(t, \widetilde{\Phi}_{P_{Obj}}(t))$ we get $g_{P_{Obj}}{}'(t) = \frac{\partial \mathcal{E}ul}{\partial t}(t, \widetilde{\Phi}_{P_{Obj}}(t)) + d\mathcal{E}ul(t, \widetilde{\Phi}_{P_{Obj}}(t)).\widetilde{\Phi}'_{P_{Obj}}(t)$, thus, having $\widetilde{\Phi}'_{P_{Obj}}(t) = \vec{v}(t, p(t))$ (Eulerian velocity), $\frac{D\mathcal{E}ul}{Dt}(t, p(t)) = \frac{\partial \mathcal{E}ul}{\partial t}(t, p(t)) + d\mathcal{E}ul(t, p(t)).\vec{v}(t, p(t))$:

$$\boxed{\frac{D\mathcal{E}ul}{Dt} := \frac{\partial \mathcal{E}ul}{\partial t} + d\mathcal{E}ul.\vec{v}}. \tag{2.22}$$

Proposition 2.16 *$\frac{D}{Dt}$ is a derivation: All the functions being Eulerian and C^1,*

- *Linearity:*

$$\frac{D(\mathcal{E}ul_1 + \lambda \mathcal{E}ul_2)}{Dt} = \frac{D\mathcal{E}ul_1}{Dt} + \lambda \frac{D\mathcal{E}ul_2}{Dt}. \tag{2.23}$$

- *Product rules: If $\mathcal{E}ul_1, \mathcal{E}ul_2$ are scalar valued functions then*

$$\frac{D(\mathcal{E}ul_1 \mathcal{E}ul_2)}{Dt} = \frac{D\mathcal{E}ul_1}{Dt} \mathcal{E}ul_2 + \mathcal{E}ul_1 \frac{D\mathcal{E}ul_2}{Dt}. \tag{2.24}$$

In paticular $\vec{w}$ is a vector field and T a compatible tensor (so $T.\vec{w}$ is meaningful) then

$$\frac{D(T.\vec{w})}{Dt} = \frac{DT}{Dt}.\vec{w} + T.\frac{D\vec{w}}{Dt}. \tag{2.25}$$

Proof. Let $i=1,2$, and g_i defined by $g_i(t):=\mathcal{E}ul_i(t,p(t))$ where $p(t)=\widetilde{\Phi}_{P_{Obj}}(t)$.

• $(g_1+\lambda g_2)'=g_1'+\lambda g_2'$ gives (2.23).

• On the one hand $\frac{D(T.\vec w)}{Dt}=\frac{\partial(T.\vec w)}{\partial t}+d(T.\vec w).\vec v=\frac{\partial T}{\partial t}.\vec w+T.\frac{\partial\vec w}{\partial t}+(dT.\vec v).\vec w+T.(d\vec w.\vec v)$, and on the other hand $\frac{DT}{Dt}.\vec w+T.\frac{D\vec w}{Dt}=(\frac{\partial T}{\partial t}+dT.\vec v).\vec w+T.(\frac{\partial\vec w}{\partial t}+d\vec w.\vec v)$. Thus (2.24)-(2.25). ▪

2.6.2 Commutativity issue

The Schwarz theorem tells: If $\mathcal{E}ul$ is C^2, the derivatives $d(\frac{\partial\mathcal{E}ul}{\partial t})$ and $\frac{\partial(d\mathcal{E}ul)}{\partial t}$ commute. But

Proposition 2.17 *The material time derivative $\frac{D}{Dt}$ does not commute with the partial derivation $\frac{\partial}{\partial t}$ or with the spatial derivative d, i.e. $\frac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt}\neq\frac{\partial(\frac{D\mathcal{E}ul}{Dt})}{\partial t}$ and $\frac{D(d\mathcal{E}ul)}{Dt}\neq d(\frac{D\mathcal{E}ul}{Dt})$ in general (because the variables t and p are not independent along a trajectory). We have, if $\mathcal{E}ul$ is C^2,*

$$\left.\begin{aligned}\frac{\partial(\frac{D\mathcal{E}ul}{Dt})}{\partial t}&=\frac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt}+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}\\&=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+d\frac{\partial\mathcal{E}ul}{\partial t}.\vec v+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t},\end{aligned}\right\}\text{ and }\left\{\begin{aligned}d(\frac{D\mathcal{E}ul}{Dt})&=\frac{D(d\mathcal{E}ul)}{Dt}+d\mathcal{E}ul.d\vec v\\&=\frac{\partial(d\mathcal{E}ul)}{\partial t}+d^2\mathcal{E}ul.\vec v+d\mathcal{E}ul.d\vec v.\end{aligned}\right.\tag{2.26}$$

Proof. $\frac{\partial\frac{D\mathcal{E}ul}{Dt}}{\partial t}=\frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec v)}{\partial t}=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec v+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}$. And $d\frac{D\mathcal{E}ul}{Dt}=d(\frac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec v)=\frac{\partial(d\mathcal{E}ul)}{\partial t}+d(d\mathcal{E}ul).\vec v+d\mathcal{E}ul.d\vec v=\frac{D(d\mathcal{E}ul)}{Dt}+d\mathcal{E}ul.d\vec v$, thus (2.26). ▪

Exercice 2.18 If $\mathcal{E}ul$ is C^2 and $\vec w$ is C^1, check $\frac{D(d\mathcal{E}ul.\vec w)}{Dt}=\frac{D(d\mathcal{E}ul)}{Dt}.\vec w+d\mathcal{E}ul.\frac{D\vec w}{Dt}$ (i.e. $\frac{D}{Dt}$ is a derivation), and

$$\begin{aligned}\frac{D(d\mathcal{E}ul.\vec w)}{Dt}&=d\frac{\partial\mathcal{E}ul}{\partial t}.\vec w+d\mathcal{E}ul.\frac{\partial\vec w}{\partial t}+(d(d\mathcal{E}ul).\vec v).\vec w+d\mathcal{E}ul.d\vec w.\vec v\\&=d\mathcal{E}ul.\frac{D\vec w}{Dt}+\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec w+d^2\mathcal{E}ul(\vec v,\vec w),\end{aligned}\tag{2.27}$$

and

$$\begin{aligned}\frac{D^2\mathcal{E}ul}{Dt^2}&=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+2d\frac{\partial\mathcal{E}ul}{\partial t}.\vec v+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}+(d(d\mathcal{E}ul).\vec v).\vec v+d\mathcal{E}ul.d\vec v.\vec v\\&=d\mathcal{E}ul.\frac{D\vec v}{Dt}+\frac{\partial^2\mathcal{E}ul}{\partial t^2}+d\frac{\partial\mathcal{E}ul}{\partial t}.\vec v+\frac{D(d\mathcal{E}ul)}{Dt}.\vec v.\end{aligned}\tag{2.28}$$

Answer. $\frac{D(d\mathcal{E}ul.\vec w)}{Dt}=\frac{\partial(d\mathcal{E}ul.\vec w)}{\partial t}+d(d\mathcal{E}ul.\vec w).\vec v=\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec w+d\mathcal{E}ul.\frac{\partial\vec w}{\partial t}+(d(d\mathcal{E}ul).\vec v).\vec w+d\mathcal{E}ul.d\vec w.\vec v=\frac{D(d\mathcal{E}ul)}{Dt}.\vec w+d\mathcal{E}ul.\frac{D\vec w}{Dt}$. And $\mathcal{E}ul\in C^2$ and Schwarz give $\frac{\partial(d\mathcal{E}ul)}{\partial t}=d(\frac{\partial\mathcal{E}ul}{\partial t})$ and $(d^2\mathcal{E}ul.\vec v).\vec w=d^2\mathcal{E}ul(\vec v,\vec w)$, hence (2.27). And

$$\begin{aligned}\frac{D^2\mathcal{E}ul}{Dt^2}=g''_{P_{Obj}}(t)=\frac{D\frac{D\mathcal{E}ul}{Dt}}{Dt}&=\frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec v)}{\partial t}+d(\frac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec v).\vec v\\&=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec v+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}+d\frac{\partial\mathcal{E}ul}{\partial t}.\vec v+(d^2\mathcal{E}ul.\vec v).\vec v+d\mathcal{E}ul.d\vec v.\vec v,\end{aligned}$$

with $\frac{\partial}{\partial t}\circ d=d\circ\frac{\partial}{\partial t}$ (Schwarz), $\frac{D(d\mathcal{E}ul)}{Dt}=\frac{\partial(d\mathcal{E}ul)}{\partial t}+d^2\mathcal{E}ul.\vec v$ and $d\mathcal{E}ul.\frac{D\vec v}{Dt}=d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}+d\mathcal{E}ul.d\vec v.\vec v$, hence (2.28). ▪

Exercice 2.19 Prove (2.26) with components.

Answer. $(\vec e_i)$ is a Cartesian basis. $\frac{\partial\frac{D\mathcal{E}ul}{Dt}}{\partial t}=\frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+\sum_i\frac{\partial\mathcal{E}ul}{\partial x^i}.v^i)}{\partial t}=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\sum_i\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^i}.v^i+\sum_i\frac{\partial\mathcal{E}ul}{\partial x^i}.\frac{\partial v^i}{\partial t}=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\sum_i\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^i}.v^i+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}$. And $\frac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt}=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\sum_i\frac{\partial\frac{\partial\mathcal{E}ul}{\partial t}}{\partial x^i}.v^i=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\sum_i\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^i}.v^i$.

And $d(\frac{D\mathcal{E}ul}{Dt}).\vec w=\sum_j\frac{\partial\frac{D\mathcal{E}ul}{Dt}}{\partial x^j}w^j=\sum_j\frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+\sum_i\frac{\partial\mathcal{E}ul}{\partial x^i}v^i)}{\partial x^j}w^j=\sum_j\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^j}w^j+\sum_{ij}\frac{\partial^2\mathcal{E}ul}{\partial x^i\partial x^j}v^iw^j+\sum_{ij}\frac{\partial\mathcal{E}ul}{\partial x^i}\frac{\partial v^i}{\partial x^j}w^j=\sum_j\frac{\partial^2\mathcal{E}ul}{\partial t\partial x^j}w^j+d^2\mathcal{E}ul(\vec v,\vec w)+d\mathcal{E}ul.d\vec v.\vec w$. And $\frac{D(d\mathcal{E}ul)}{Dt}.\vec w=(\frac{\partial(d\mathcal{E}ul)}{\partial t}+d(d\mathcal{E}ul).\vec v).\vec w=\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec w+d^2\mathcal{E}ul(\vec v,\vec w)=\sum_i\frac{\partial^2\mathcal{E}ul}{\partial x^i\partial t}w^i+d^2\mathcal{E}ul(\vec v,\vec w)$. Thus $d(\frac{D\mathcal{E}ul}{Dt}).\vec w=\frac{D(d\mathcal{E}ul)}{Dt}.\vec w+d\mathcal{E}ul.d\vec v.\vec w$ for all $\vec w$. ▪

2.6.3 Remark: About notations

• The notation $\frac{d}{dt}$ (lowercase letters) concerns a function of one variable, e.g. $\frac{dg}{dt}(t) := g'(t) := \lim_{h\to 0}\frac{g(t+h))-g(t)}{h}$;

• The notation $\frac{\partial}{\partial t}$ concerns a function with more than one variable, e.g. $\frac{\partial \mathcal{E}ul}{\partial t}(t,p) = \lim_{h\to 0}\frac{\mathcal{E}ul(t+h,p)-\mathcal{E}ul(t,p)}{h}$;

• The notation $\frac{D}{Dt}$ (capital letters) concerns a Eulerian function differentiated along a motion, cf. (2.21).

• Other notations, often practical but might be ambiguous if composed functions are considered:

$$\frac{d\mathcal{E}ul(t,p(t))}{dt} := g_{P_{Obj}}{}'(t) = \frac{D\mathcal{E}ul}{Dt}(t,p(t)), \quad \text{and} \quad \frac{d\mathcal{E}ul(t,p(t))}{dt}_{|t=t_0} := g_{P_{Obj}}{}'(t_0) = \frac{D\mathcal{E}ul}{Dt}(t_0,p(t_0)). \tag{2.29}$$

2.6.4 Definition bis: Time-space definition

Consider the affine time-space $\mathbb{R}\times\mathbb{R}^n$ and a C^1 function $f : (t,p) \in \mathbb{R}\times\mathbb{R}^n \to f(t,p)$.

Definition 2.20 The differential of f is called the "total differential", or "total derivative", and noted Df.

So, with $\vec{\mathbb{R}}\times\vec{\mathbb{R}^n}$ the associated time-space vector space, if $p_+ = (t,p) \in \mathbb{R}\times\mathbb{R}^n$ and $\vec{w}_+ = (w_0,\vec{w}) \in \vec{\mathbb{R}}\times\vec{\mathbb{R}^n}$ then, by definition of a differential, $Df(p_+).\vec{w}_+ := \lim_{h\to 0}\frac{f(p_++h\vec{w}_+)-f(p_+)}{h}$, i.e.

$$Df(t,p).(w_0,\vec{w}) := \lim_{h\to 0}\frac{f(t+hw_0, p+h\vec{w}) - f(t,p)}{h}. \tag{2.30}$$

Thus

$$Df(t,p) = \frac{\partial f}{\partial t}(t,p)\,dt + df(t,p). \tag{2.31}$$

(Recall: df is the space differentiation, so if $(\vec{e}_i)$ is a Cartesian basis then $df(t,p) = \frac{\partial f}{\partial x_1}(t,p)dx_1 + ... + \frac{\partial f}{\partial x_n}(t,p)dx_n$ and $\vec{w} = \sum_i w_i\vec{e}_i$ gives $Df(t,p).(w_0,w_1,...,w_n) = \frac{\partial f}{\partial t}(t,p)w_0 + \frac{\partial f}{\partial x_1}(t,p)w_1 + ... + \frac{\partial f}{\partial x_n}(t,p)w_n$).

Then consider the time-space trajectory

$$\widetilde{\Psi}_{P_{Obj}} : \begin{cases} [t_1,t_2] & \to \mathbb{R}\times\mathbb{R}^n \\ \quad t & \to \widetilde{\Psi}_{P_{Obj}}(t) := (t, \widetilde{\Phi}_{P_{Obj}}(t)) \quad (= (t,p(t))). \end{cases} \tag{2.32}$$

(So $\mathrm{Im}(\widetilde{\Psi}_{P_{Obj}}) = \mathrm{graph}(\widetilde{\Phi}_{P_{Obj}})$.) The tangent vector to this curve at t is

$$\widetilde{\Psi}_{P_{Obj}}{}'(t) = (1, \widetilde{\Phi}_{P_{Obj}}{}'(t)) = (1, \vec{v}(t,p(t)) \in \vec{\mathbb{R}}\times\vec{\mathbb{R}^n} \tag{2.33}$$

where $\vec{v}(t,p(t)) = \frac{d\widetilde{\Phi}_{P_{Obj}}}{dt}(t)$ is the Eulerian velocity at $(t,p(t))$. And (2.20) reads

$$g_{P_{Obj}}(t) = (\mathcal{E}ul \circ \widetilde{\Psi}_{P_{Obj}})(t) = \mathcal{E}ul(\widetilde{\Psi}_{P_{Obj}}(t)), \tag{2.34}$$

thus

$$g'_{P_{Obj}}(t) = D\mathcal{E}ul(\widetilde{\Psi}(t)).\widetilde{\Psi}_{P_{Obj}}{}'(t) = \frac{\partial \mathcal{E}ul}{\partial t}(t,p(t)).1 + d\mathcal{E}ul(t,p(t)).\vec{v}(t,p(t)) \overset{\text{noted}}{=} \frac{D\mathcal{E}ul}{Dt}(t,p(t)) : \tag{2.35}$$

We have (2.22): The material time derivative is the "total derivative" $D\mathcal{E}ul$ along the time-space trajectory $\widetilde{\Psi}_{P_{Obj}}$.

2.7 Eulerian acceleration

Definition 2.21 In short: If $\widetilde{\Phi}_{P_{Obj}}$ is C^2, then the Eulerian acceleration of the particle P_{Obj} which is at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$ is

$$\vec{\gamma}(t,p_t) := \widetilde{\Phi}_{P_{Obj}}{}''(t) \overset{\text{noted}}{=} \frac{\partial^2\widetilde{\Phi}}{\partial t^2}(t, P_{Obj}). \tag{2.36}$$

In details: as in (2.3), the Eulerian acceleration (vector) field $\widehat{\vec{\gamma}}$ is defined with (2.36) by

$$\widehat{\vec{\gamma}}(t,p_t) = ((t,p_t), \vec{\gamma}(t,p_t)) \in \mathcal{C}\times\vec{\mathbb{R}}^n_t \quad \text{(pointed vector)}. \tag{2.37}$$

Proposition 2.22

$$\boxed{\vec\gamma = \frac{D\vec v}{Dt} = \frac{\partial\vec v}{\partial t} + d\vec v.\vec v}. \tag{2.38}$$

And if $\vec v$ is C^2 then

$$d\vec\gamma = \frac{\partial(d\vec v)}{\partial t} + d^2\vec v.\vec v + d\vec v.d\vec v = \frac{D(d\vec v)}{Dt} + d\vec v.d\vec v. \tag{2.39}$$

Proof. With $g(t) = \vec v(t,p(t)) = \widetilde{\Phi}_{P_{Obj}}{}'(t)$ and (2.22) we get $\vec\gamma(t,p(t)) = g'(t) = \frac{D\vec v}{Dt}(t,p(t))$. And $\vec v$ being C^2, the Schwarz theorem gives $d\frac{\partial\vec v}{\partial t} = \frac{\partial(d\vec v)}{\partial t}$. ∎

Definition 2.23 If an observer chooses a Euclidean dot product $(\cdot,\cdot)_g$ (based on a foot, a metre...), the associated norm being $||.||_g$, then the length $||\vec\gamma(t,p_t)||_g$ is the (scalar) acceleration of P_{Obj}.

2.8 Time Taylor expansion of $\widetilde{\Phi}$

Let $P_{Obj} \in Obj$ and $t \in]t_1,t_2[$. Suppose $\widetilde{\Phi}_{P_{Obj}} \in C^2(]t_1,t_2[;\mathbb{R}^n)$. Its second-order (time) Taylor expansion of $\widetilde{\Phi}_{P_{Obj}}$ is, in the vicinity of a $t \in]t_1,t_2[$,

$$\widetilde{\Phi}_{P_{Obj}}(\tau) = \widetilde{\Phi}_{P_{Obj}}(t) + (\tau-t)\widetilde{\Phi}'_{P_{Obj}}(t) + \frac{(\tau-t)^2}{2}\widetilde{\Phi}''_{P_{Obj}}(t) + o((\tau-t)^2), \tag{2.40}$$

i.e.

$$p(\tau) = p(t) + (\tau-t)\vec v(t,p(t)) + \frac{(\tau-t)^2}{2}\vec\gamma(t,p(t)) + o((\tau-t)^2). \tag{2.41}$$

3 Lagrangian description = Motion from an initial configuration

Instead of working on Obj, an observer may prefer to work with an initial configuration $\Omega_{t_0} = \widetilde{\Phi}(t_0, Obj)$ of Obj (essential for elasticity): This is the “Lagrangian approach”. This Lagrangian approach is not objective: Two observers may choose two different initial (times and) configurations.

3.1 Initial configuration and Lagrangian “motion”

3.1.1 Definition

Obj is a material object, $\widetilde{\Phi} : [t_1,t_2[\times Obj \to \mathbb{R}^n$ is its motion, $\Omega_\tau = \widetilde{\Phi}_\tau(Obj)$ is its configuration at τ, $t_0 \in]t_1,t_2[$ is an “initial time”, and Ω_{t_0} is the initial configuration for the observer who chose t_0.

Definition 3.1 The motion of Obj relative to the initial configuration $\Omega_{t_0} = \widetilde{\Phi}(t_0, Obj)$ is the function

$$\Phi^{t_0} : \begin{cases} [t_1,t_2]\times\Omega_{t_0} \to \mathbb{R}^n \\ \quad (t,p_{t_0}) \mapsto p_t = \Phi^{t_0}(t,p_{t_0}) := \widetilde{\Phi}(t,P_{Obj}) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}(t_0,P_{Obj}). \end{cases} \tag{3.1}$$

So, $p_t = \Phi^{t_0}(t,p_{t_0}) := \widetilde{\Phi}(t,P_{Obj})$ is the position at t of the particle P_{Obj} which was at p_{t_0} at t_0. In particular $p_{t_0} = \Phi^{t_0}(t_0,p_{t_0}) := \widetilde{\Phi}(t_0,P_{Obj})$.

Marsden and Hughes notations: Once an initial time t_0 has been chosen by an observer, then $\Phi^{t_0} =^{\text{noted}} \Phi$, then $p_{t_0} =^{\text{noted}} P$ (capital letter for positions at t_0) and $p_t =^{\text{noted}} p$ (lowercase letter for positions at t), so

$$p = \Phi(t,P) \ \in \Omega_t. \tag{3.2}$$

(When objectivity is under concern, we need to switch back to the notations Φ^{t_0}, p_{t_0} and p_t.)

NB: • Talking about the motion of a position p_{t_0} is absurd: A position in $\mathbb{R}^n$ does not move. Thus Φ^{t_0} has no existence without the definition, at first, of the motion $\widetilde{\Phi}$ of particles.

• The domain of definition of Φ^{t_0} depends on t_0 through Ω_{t_0}: The superscript ^{t_0} recalls it. And a late observer with initial time $t_0{}' > t_0$ defines $\Phi^{t_0{}'}$ which domain of definition is $[t_1,t_2]\times\Omega_{t_0{}'}$; And $\Phi^{t_0{}'} \neq \Phi^{t_0}$ in general because $\Omega_{t_0{}'} \neq \Omega_{t_0}$ in general.

• The following notation is also used:

$$\Phi^{t_0}(t,p_{t_0}) = \Phi(t;t_0,p_{t_0}). \tag{3.3}$$

(The couple (t_0,p_{t_0}) is "the initial condition", or t_0 and p_{t_0} are the initial conditions, see the § on flows).

• If a origin $\mathcal{O}\in\mathbb{R}^n$ is chosen by the observer, we may also use, with (1.6),

$$\vec{x}_{t_0} = \overrightarrow{\mathcal{O}p_{t_0}} = \vec{\varphi}^{t_0}(t_0,\vec{x}_{t_0}) = \vec{X} = \overrightarrow{\mathcal{O}P} \quad\text{and}\quad \vec{x}_t = \overrightarrow{\mathcal{O}p_t} = \vec{\varphi}^{t_0}(t,\vec{x}_{t_0}) = \vec{x} = \overrightarrow{\mathcal{O}p}. \tag{3.4}$$

3.1.2 Diffeomorphism between configurations

With (3.1), define

$$\Phi_t^{t_0} : \begin{cases} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} & \to p_t = \Phi_t^{t_0}(p_{t_0}) := \Phi^{t_0}(t,p_{t_0}). \end{cases} \tag{3.5}$$

Hypothesis: For all $t_0,t\in]t_1,t_2[$, the map $\Phi_t^{t_0}:\Omega_{t_0}\to\Omega_t$ is a C^k diffeomorphism (a C^k invertible function whose inverse is C^k), where $k\in\mathbb{N}^*$ depends on the required regularity.

Thus (3.5) gives $\widetilde{\Phi}_t(P_{Obj}) = \Phi_t^{t_0}(\widetilde{\Phi}_{t_0}(P_{Obj}))$, true for all $P_{Obj}\in Obj$, thus $\Phi_t^{t_0}\circ\widetilde{\Phi}_{t_0} = \widetilde{\Phi}_t$, i.e.

$$\boxed{\Phi_t^{t_0} := \widetilde{\Phi}_t\circ(\widetilde{\Phi}_{t_0})^{-1}}. \tag{3.6}$$

Thus, $\Phi_{t_0}^{t_0} = I$ and $\Phi_{t_0}^{t}\circ\Phi_t^{t_0} = (\widetilde{\Phi}_t\circ(\widetilde{\Phi}_{t_0})^{-1})\circ(\widetilde{\Phi}_{t_0}\circ(\widetilde{\Phi}_t)^{-1}) = I$ give

$$\Phi_{t_0}^{t} = (\Phi_t^{t_0})^{-1}. \tag{3.7}$$

3.1.3 Trajectories

Let $(t_0,p_{t_0})\in[t_1,t_2]\times\Omega_{t_0}$ (initial conditions) and with (3.1) define

$$\Phi_{p_{t_0}}^{t_0} : \begin{cases} [t_1,t_2] & \to \mathbb{R}^n \\ \qquad t & \mapsto p(t) = \Phi_{p_{t_0}}^{t_0}(t) := \widetilde{\Phi}_{P_{Obj}}(t) = \Phi^{t_0}(t,p_{t_0}) \quad\text{when}\quad p_{t_0} = \widetilde{\Phi}_{P_{Obj}}(t_0). \end{cases} \tag{3.8}$$

Definition 3.2 $\Phi_{p_{t_0}}^{t_0}$ is called the (parametric) "trajectory of p_{t_0}", which means: $\Phi_{p_{t_0}}^{t_0}$ is the trajectory of the particle P_{Obj} that is located at $p_{t_0} = \widetilde{\Phi}(t,P_{Obj})$ at t_0. And the geometric "trajectory of p_{t_0}" is

$$\mathrm{Im}(\Phi_{p_{t_0}}^{t_0}) = \Phi_{p_{t_0}}^{t_0}([t_1,t_2]) = \bigcup_{t\in[t_1,t_2]}\{\Phi_{p_{t_0}}^{t_0}(t)\} \quad (= \mathrm{Im}(\widetilde{\Phi}_{P_{Obj}})). \tag{3.9}$$

NB: The terminology "trajectory of p_{t_0}" is awkward, since a position p_{t_0} does not move: It is indeed the trajectory $\widetilde{\Phi}_{P_{Obj}}$ of a particle P_{Obj} which is at p_{t_0} at t_0 that must be understood.

3.1.4 Streaklines (lignes d'émission)

Take a film between t_0 and T (start and end).

Definition 3.3 Let Q be a fixed point in $\mathbb{R}^n$ (you see the point Q on each photo that make up the film). The streakline through Q is the set

$$\begin{aligned} E_{t_0,T}(Q) &= \{p\in\Omega : \exists\tau\in[t_0,T] : p = \Phi_T^{\tau}(Q) = (\Phi_\tau^T)^{-1}(Q)\} \\ &= \{p\in\Omega : \exists u\in[0,T{-}t_0] : p = \Phi_T^{T-u}(Q) = (\Phi_{T-u}^T)^{-1}(Q)\} \end{aligned} \tag{3.10}$$

= the set at T of the positions (a line in $\mathbb{R}^n$) of all the particles which were at Q at a $\tau\in[t_0,T]$.

Example 3.4 Smoke comes out of a chimney. Fix a camera nearby, choose a point Q at the top of the chimney where the particles are colored, and make a film. At T stop filming. Then (at time T) superimpose the photos in the film: The colored curve we see is the streakline. ♣

In other words $= \bigcup_{\tau\in[t_0,T]}\{\Phi_Q^\tau(T)\} = \bigcup_{u\in[0,T-t_0]}\{\Phi_Q^{T-u}(T)\}$.

3.2 Lagrangian variables and functions

3.2.1 Definition

Consider a motion $\widetilde{\Phi}$, cf. (1.5). An observer chose (subjective) a $t_0 \in [t_1, t_2]$ ("in the past"); So $\Omega_{t_0} = \widetilde{\Phi}(t_0, \mathit{Obj})$ is his initial configuration. Let $m \in \mathbb{N}^*$.

Definition 3.5 In short: A Lagrangian function, relative to Obj, $\widetilde{\Phi}$ and t_0, is a function

$$\mathcal{L}ag^{t_0} : \begin{cases} [t_1, t_2] \times \Omega_{t_0} & \to \mathbb{R}^{\vec{m}} \quad \text{(or, more generally, some adequat set)} \\ (t, p_{t_0}) & \to \mathcal{L}ag^{t_0}(t, p_{t_0}), \end{cases} \tag{3.11}$$

and p_{t_0} is called the Lagrangian variable relative to the (subjective) choice t_0.
(To compare with (2.2): A Eulerian function does not depend on any t_0.)

Example 3.6 Scalar values: $\mathcal{L}ag^{t_0}(t, p_{t_0}) = \Theta^{t_0}(t, p_{t_0}) =$ temperature at t at $p_t = \Phi_t^{t_0}(p_{t_0}) = \widetilde{\Phi}(t, P_{\mathit{Obj}})$ of the particle P_{Obj} that was at p_{t_0} at t_0. (So, continuing example 2.2, $\Theta^{t_0}(t, p_{t_0}) = \theta(t, p_t)$.) ▪

Example 3.7 Vectorial values: $\mathcal{L}ag^{t_0}(t, p_{t_0}) = \vec{U}^{t_0}(t, p_{t_0}) =$ force at t at $p_t = \Phi_t^{t_0}(p_{t_0}) = \widetilde{\Phi}(t, P_{\mathit{Obj}})$ acting on the particle P_{Obj} that was at p_{t_0} at t_0. (So, continuing example 2.3, $\vec{U}^{t_0}(t, p_{t_0}) = \vec{u}(t, p_t)$.) ▪

If t is fixed or if $p_{t_0} \in \Omega_{t_0}$ is fixed, then we define

$$\mathcal{L}ag_t^{t_0} : \begin{cases} \Omega_{t_0} & \to \mathbb{R}^{\vec{m}} \text{ (or, more generally, some adequat set)} \\ p_{t_0} & \to \mathcal{L}ag_t^{t_0}(p_{t_0}) := \mathcal{L}ag^{t_0}(t, p_{t_0}), \end{cases} \tag{3.12}$$

$$\mathcal{L}ag_{p_{t_0}}^{t_0} : \begin{cases} [t_1, t_2] & \to \mathbb{R}^{\vec{m}} \text{ (or, more generally, some adequat set)} \\ t & \to \mathcal{L}ag_{p_{t_0}}^{t_0}(t) := \mathcal{L}ag^{t_0}(t, p_{t_0}). \end{cases} \tag{3.13}$$

Remark 3.8 The position p_{t_0} is also sometimes called a "material point", which is counter intuitive: P_{Obj} (objective) is the material point, and p_{t_0} is just its spatial position at t_0 (subjective); And a Eulerian variable p_t is not called a "material point" at t...
By the way, the variable p_t is also called the "updated Lagrangian variable"... ▪

3.2.2 A Lagrangian function is a two point tensor

Definition 3.9 In details: $\mathcal{L}ag^{t_0}$ being defined in (3.11), a Lagrangian function is a function

$$\widetilde{\mathcal{L}ag}^{t_0} : \begin{cases} [t_1, t_2] \times \Omega_{t_0} & \to \mathcal{C} \times \mathbb{R}^{\vec{m}} \\ (t, p_{t_0}) & \to \widetilde{\mathcal{L}ag}^{t_0}(t, p_{t_0}) = ((t, p_t), \mathcal{L}ag^{t_0}(t, p_{t_0})) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \end{cases} \tag{3.14}$$

I.e. $\widetilde{\mathcal{L}ag}^{t_0}(t, p_{t_0}) = ((t, \Phi_t^{t_0}(p_{t_0})), \mathcal{L}ag^{t_0}(t, p_{t_0}))$. (And $\mathbb{R}^{\vec{m}}$ can be replaced by some set.)

Definition 3.10 (Marsden and Hughes [14].) A Lagrangian function is a "two point vector field" (or more generally a "two point tensor") in reference to the points $p_{t_0} \in \Omega_{t_0}$ (departure set) and $p_t \in \Omega_t$ (arrival set) where the value $\mathcal{L}ag^{t_0}(t, p_{t_0})$ is considered.

Interpretation: (3.14) tells that $\mathcal{L}ag^{t_0}(t, p_{t_0})$ is ***not*** represented at (t, p_{t_0}), but at (t, p_t): That is, having

$$\text{graph}(\mathcal{L}ag^{t_0}) = \{((t, p_{t_0}), \mathcal{L}ag^{t_0}(t, p_{t_0})) \quad \text{and} \quad \text{Im}(\widetilde{\mathcal{L}ag}^{t_0}) = \{((t, p_t), \mathcal{L}ag^{t_0}(t, p_{t_0}))\}, \tag{3.15}$$

we have

$$\text{Im}(\widetilde{\mathcal{L}ag}^{t_0}) \neq \text{graph}(\mathcal{L}ag^{t_0}) : \tag{3.16}$$

So a Lagrangian function does ***not*** define a tensor in the usual sense. To compare with the Eulerian function $\mathcal{E}ul$ which defines a tensor (in particular $\text{Im}(\widehat{\mathcal{E}ul}) = \text{graph}(\mathcal{E}ul)$), cf. (2.3).

3.3 Lagrangian function associated with a Eulerian function

3.3.1 Definition

Let $\widetilde{\Phi}$ be a motion, cf. (1.5). Let $\mathcal{E}ul$ be a Eulerian function, cf. (2.3). Let $t_0 \in [t_1, t_2]$.

Definition 3.11 The Lagrangian function $\mathcal{L}ag^{t_0}$ associated with the Eulerian function $\mathcal{E}ul$ is defined by, for all $(t, P_{Obj}) \in [t_1, t_2] \times Obj$,

$$\mathcal{L}ag^{t_0}(t, \widetilde{\Phi}(t_0, P_{Obj})) := \mathcal{E}ul(t, \widetilde{\Phi}(t, P_{Obj})), \tag{3.17}$$

i.e., for all $(t, p_{t_0}) \in [t_1, t_2] \times \Omega_{t_0}$,

$$\mathcal{L}ag^{t_0}(t, p_{t_0}) := \mathcal{E}ul(t, p_t), \quad \text{when} \quad p_t = \widetilde{\Phi}(t, P_{Obj}) = \Phi_t^{t_0}(p_t) \tag{3.18}$$

i.e., $\mathcal{L}ag^{t_0}(t, p_{t_0}) := \mathcal{E}ul(t, p_t)$ when $p_{t_0} = (\Phi_t^{t_0})^{-1}(p_t)$ for all $(t, p_t) \in \mathcal{C}$. In other words:

$$\boxed{\mathcal{L}ag_t^{t_0} := \mathcal{E}ul_t \circ \Phi_t^{t_0}}. \tag{3.19}$$

3.3.2 Remarks

• If you have a Lagrangian function, then you can associate the function

$$\mathcal{E}ul_t^{t_0} := \mathcal{L}ag_t^{t_0} \circ (\Phi_t^{t_0})^{-1} \tag{3.20}$$

which thus a priori depends on t_0. But, a Eulerian function is independent of any initial time t_0.

• For one measurement, there is only one Eulerian function $\mathcal{E}ul$, while there are as many associated Lagrangian function $\mathcal{L}ag^{t_0}$ as they are t_0 (as many as observers): The Lagrangian function $\mathcal{L}ag^{{t_0}'}$ of a late observer who chooses ${t_0}' > t_0$ is different from $\mathcal{L}ag^{t_0}$ since the domains of definition Ω_{t_0} and $\Omega_{{t_0}'}$ are different (in general).

3.4 Lagrangian velocity

3.4.1 Definition

Definition 3.12 In short: The Lagrangian velocity at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$ of the particle P_{Obj} is the function

$$\vec{V}^{t_0} : \begin{cases} \mathbb{R} \times \Omega_{t_0} & \to \vec{\mathbb{R}^n} \\ (t, p_{t_0}) & \to \vec{V}^{t_0}(t, p_{t_0}) := \widetilde{\Phi}_{P_{Obj}}{}'(t) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj}). \end{cases} \tag{3.21}$$

In details: With (3.21), the Lagrangian velocity is the two point vector field given by

$$\widehat{\vec{V}^{t_0}}(t, p_{t_0}) : \begin{cases} \mathbb{R} \times \Omega_{t_0} & \to \mathcal{C} \times \vec{\mathbb{R}^n} \\ (t, p_{t_0}) & \to \widehat{\vec{V}^{t_0}}(t, p_{t_0}) := ((t, p_t), \vec{V}^{t_0}(t, p_{t_0})), \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}). \end{cases} \tag{3.22}$$

Thus $\vec{V}^{t_0}(t, p_{t_0}) = \widetilde{\Phi}_{P_{Obj}}{}'(t) = \vec{v}(t, p_t)$ is the velocity at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$ of the particle P_{Obj} which was at $p_{t_0} = \widetilde{\Phi}(p_{t_0}, P_{Obj})$ at t_0; And $\vec{V}^{t_0}(t, p_{t_0})$ is ***not*** tangent to graph$(\vec{V}^{t_0})$, cf. (3.16): It is tangent to graph$(\vec{v})$ at (t, p_t).

If t is fixed, or if $p_{t_0} \in \Omega_{t_0}$ is fixed, then we define

$$\vec{V}_t^{t_0}(p_{t_0}) := \vec{V}^{t_0}(t, p_{t_0}), \quad \text{or} \quad \vec{V}_{p_{t_0}}^{t_0}(t) := \vec{V}^{t_0}(t, p_{t_0}). \tag{3.23}$$

Remark: A usual definition is given without explicit reference to a particle; It is, instead of (3.21),

$$\vec{V}^{t_0}(t, p_{t_0}) := \frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0}), \quad \forall (t, p_{t_0}) \in \mathbb{R} \times \Omega_{t_0}. \tag{3.24}$$

3.4.2 Lagrangian velocity versus Eulerian velocity

(3.21) and (2.4) give (alternative definition), with $p_\tau = \widetilde{\Phi}(\tau, P_{Obj})$,

$$\vec{V}^{t_0}(t, p_{t_0}) = \vec{v}(t, p_t) \quad \left(= \frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0}) = \widetilde{\Phi}_{P_{Obj}}{}'(t) = \text{velocity at } t \text{ at } p_t \text{ of } P_{Obj}\right). \tag{3.25}$$

In other words,

$$\boxed{\vec{V}_t^{t_0} = \vec{v}_t \circ \Phi_t^{t_0}}. \tag{3.26}$$

3.4.3 Relation between differentials

For C^2 motions (3.26) gives

$$d\vec{V}_t^{t_0}(p_{t_0}) = d\vec{v}_t(p_t).d\Phi_t^{t_0}(p_{t_0}) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \tag{3.27}$$

I.e., with

$$F_t^{t_0} = d\Phi_t^{t_0} \overset{\text{noted}}{=} \text{the deformation gradient relative to } t_0 \text{ and } t, \tag{3.28}$$

$$\boxed{d\vec{V}_t^{t_0}(p_{t_0}) = d\vec{v}_t(p_t).F_t^{t_0}(p_{t_0})} \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \tag{3.29}$$

Abusively written (dangerous notation: At what points, relative to what times?)

$$d\vec{V} = d\vec{v}.F. \tag{3.30}$$

3.4.4 Computation of $d\vec{v}$ called $L = \dot{F}.F^{-1}$ wih Lagrangian variables

Start with a Lagrangian velocity $\vec{V}^{t_0}$, then define the Eulerian velocity by, with $p_t = \Phi_t^{t_0}(p_{t_0})$,

$$\vec{v}^{t_0}(t, p_t) := \vec{V}^{t_0}(t, p_{t_0}), \tag{3.31}$$

(the Eulerian velocity thus depends on t_0 a priori), i.e. $\vec{v}^{t_0}(t, \Phi_t^{t_0}(p_{t_0})) = \frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0})$. Thus

$$d\vec{v}^{t_0}(t, p_t).d\Phi^{t_0}(t, p_{t_0}) = d(\frac{\partial \Phi^{t_0}}{\partial t})(t, p_{t_0}) = \frac{\partial (d\Phi^{t_0})}{\partial t}(t, p_{t_0}) = \frac{\partial F^{t_0}}{\partial t}(t, p_{t_0}), \tag{3.32}$$

with Φ^{t_0} C^2 for the second equality. Thus

$$d\vec{v}^{t_0}(t, p_t) = \frac{\partial F^{t_0}}{\partial t}(t, p_{t_0}).F^{t_0}(t, p_{t_0})^{-1}, \quad \text{written in short} \quad L := d\vec{v} = \dot{F}.F^{-1}, \tag{3.33}$$

but L thus "defined" is defined at what points? What times?

In books, it seems that L is Eulerian ($L(t, p_t) = d\vec{v}(t, p_t)$), not Lagrangian (not $L^{t_0}(t, p_{t_0}) = d\vec{v}(t, p_t)$).

Reminder: Start with Eulerian quantities and use Eulerian quantities as long as possible[1], which in particular say that $d\vec{v}$ doesn't depend on t_0.

3.5 Lagrangian acceleration

Let $P_{Obj} \in Obj$, $t_0, t \in \mathbb{R}$, $p_{t_0} = \widetilde{\Phi}_{P_{Obj}}(t_0)$ and $p_t = \widetilde{\Phi}_{P_{Obj}}(t)$ (positions of P_{Obj} at t_0 and t).

Definition 3.13 In short, the Lagrangian acceleration at t at p_t of the particle P_{Obj} is

$$\vec{\Gamma}^{t_0}(t, p_{t_0}) := \widetilde{\Phi}_{P_{Obj}}{}''(t) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}_{P_{Obj}}(t_0). \tag{3.34}$$

In other words

$$\vec{\Gamma}^{t_0}(t, p_{t_0}) := \vec{\gamma}(t, p_t) \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}), \tag{3.35}$$

where $\vec{\gamma}(t, p_t) = \widetilde{\Phi}_{P_{Obj}}{}''(t)$ is the Eulerian acceleration at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$, cf. (2.36).

In details, the Lagrangian acceleration is the "two point vector field" defined on $\mathbb{R} \times \Omega_{t_0}$ by

$$\widetilde{\vec{\Gamma}}^{t_0}(t, p_{t_0}) = ((t, p_t), \widetilde{\Phi}_{P_{Obj}}{}''(t)), \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}). \tag{3.36}$$

(To compare with (2.37).) In particular $\vec{\Gamma}^{t_0}(t, p_{t_0})$ is not drawn on the graph of $\vec{\Gamma}^{t_0}$ at (t, p_{t_0}), but on the graph of $\vec{\gamma}$ at (t, p_t).

[1] To get Eulerian results from Lagrangian computations can make the understanding of a Lie derivative quite difficult: To introduce the "so-called" Lie derivatives in classical mechanics you can find the following steps: 1- At t consider the Cauchy stress vector $\vec{t}$ (Eulerian), 2- then with a unit normal vector $\vec{n}$, define the associated Cauchy stress tensor $\underline{\underline{\sigma}}$ (satisfying $\vec{t} = \underline{\underline{\sigma}}.\vec{n}$), 3- then use the virtual power and the change of variables in integrals to be back into Ω_{t_0} to be able to work with Lagrangian variables, 4- then introduce the first Piola–Kirchhoff (two point) tensor $\mathbb{K}$, 5- then introduce the second Piola–Kirchhoff tensor $\mathbb{S}$ (endomorphism in Ω_{t_0}), 6- then differentiate $\mathbb{S}$ in Ω_{t_0} (in the Lagrangian variables although the initials variables are the Eulerian variables in Ω_t), 7- then back in Ω_t to get back to Eulerian functions (change of variables in integrals), 8- then you get some Jaumann or Truesdell or other so called Lie derivatives type terms, the appropriate choice among all these derivatives being quite obscure because the covariant objectivity has been forgotten en route... While, with simple Eulerian considerations, it requires a few lines to understand the (real) Lie derivative (Eulerian concept) and its simplicity, see § 9, and deduce second order covariant objective results.

If t is fixed, or if $p_{t_0} \in \Omega_{t_0}$ is fixed, then define

$$\vec\Gamma_t^{t_0}(p_{t_0}) := \vec\Gamma^{t_0}(t,p_{t_0}), \quad \text{and} \quad \vec\Gamma_{p_{t_0}}^{t_0}(t) := \vec\Gamma^{t_0}(t,p_{t_0}). \tag{3.37}$$

Thus

$$\vec\Gamma_t^{t_0} = \vec\gamma_t \circ \Phi_t^{t_0}, \quad \text{and} \quad d\vec\Gamma_t^{t_0}(p_{t_0}) = d\vec\gamma_t(p_t).F_t^{t_0}(p_{t_0}), \tag{3.38}$$

when $p_t = \Phi_t^{t_0}(p_{t_0})$ and $F_t^{t_0} := d\Phi_t^{t_0}$ (the deformation gradient).

Risky notation: $d\vec\Gamma = d\vec\gamma.F$ (points? times?).

3.6 Time Taylor expansion of Φ^{t_0}

Let $p_{t_0} \in \Omega_{t_0}$. Then, at second order,

$$\Phi_{p_{t_0}}^{t_0}(\tau) = \Phi_{p_{t_0}}^{t_0}(t) + (\tau{-}t)\Phi_{p_{t_0}}^{t_0}{}'(t) + \frac{(\tau{-}t)^2}{2}\Phi_{p_{t_0}}^{t_0}{}''(t) + o((\tau{-}t)^2), \tag{3.39}$$

that is, with $p(\tau) = \widetilde\Phi_{P_{Obj}}(\tau) = \Phi_\tau^{t_0}(p_{t_0})$,

$$p(\tau) = p(t) + (\tau{-}t)\vec V^{t_0}(t,p_{t_0}) + \frac{(\tau{-}t)^2}{2}\vec\Gamma^{t_0}(t,p_{t_0}) + o((\tau{-}t)^2). \tag{3.40}$$

NB: There are ***three*** times involved: t_0 (observer dependent), t and τ (for the Taylor expansion). To compare with (2.40)-(2.41): $p(\tau) = p(t) + (\tau{-}t)\vec v(t,p(t)) + \frac{(\tau-t)^2}{2}\vec\gamma(t,p(t)) + o((\tau{-}t)^2)$, independent of t_0.

3.7 A vector field that let itself be deformed by a motion

Consider a C^0 Eulerian vector field $\vec w : \begin{cases} \mathcal{C} & \to \vec{\mathbb{R}^n} \\ (t,p_t) & \to \vec w(t,p_t) \end{cases}$. Let $t_0 \in [t_1,t_2[$ and let $\vec w_{t_0} : \begin{cases} \Omega_{t_0} & \to \vec{\mathbb{R}^n} \\ p_{t_0} & \to \vec w_{t_0}(p_{t_0}) := \vec w(t_0,p_{t_0}) \end{cases}$ (vector field in Ω_{t_0}). Then define the (virtual) vector field

$$\vec w_{t_0*} : \begin{cases} \mathcal{C} & \to \vec{\mathbb{R}^n} \\ (t,p_t) & \to \vec w_{t_0*}(t,p_t) := d\Phi^{t_0}(t,p_{t_0}).\vec w_{t_0}(p_{t_0}), \quad \text{when} \quad p(t) = \Phi^{t_0}(t,p_{t_0}). \end{cases} \tag{3.41}$$

(The push-forward = result of the deformation of $\vec w_{t_0}$ by the motion, see figure 4.1.)

Proposition 3.14 *For C^2 motions, we have (time variation rate along a virtual trajectory)*

$$\frac{D\vec w_{t_0*}}{Dt} = d\vec v.\vec w_{t_0*}, \tag{3.42}$$

i.e. $\mathcal{L}_{\vec v}\vec w_{t_0*} = \vec 0$, where $\mathcal{L}_{\vec v}\vec u := \frac{D\vec u}{Dt} - d\vec v.\vec u$ ($= \frac{\partial \vec u}{\partial t} + d\vec u.\vec v - d\vec v.\vec u$) is the Lie derivative of a (unsteady) vector field $\vec u : \mathcal{C} \to \vec{\mathbb{R}^n}$ along $\vec v$.

Interpretation: *We will see that $\mathcal{L}_{\vec v}\vec w(t_0,p_{t_0}) = \lim_{t\to t_0} \frac{\vec w(t,p(t)) - \vec w_{t_0*}(t,p(t))}{h}$ measures the "resistance of $\vec w$ to a motion", see § 9.3.2; Thus the result $\mathcal{L}_{\vec v}\vec w_{t_0*}(t_0,p_{t_0}) = \vec 0$ is "obvious" ($= \lim_{t\to t_0} \frac{\vec w_{t_0*}(t,p(t)) - \vec w_{t_0*}(t,p(t))}{h}$): If $\vec w = \vec w_{t_0*}$ then the vector ("force") field $\vec w$ does not oppose any resistance to the flow.*

Proof. p_{t_0} being fixed, with $d\Phi^{t_0}(t,p_{t_0}) =^{\text{noted}} F(t)$ we have $\vec w_{t_0*}(t,p(t)) =^{(3.41)} F(t).\vec w_{t_0}(p_{t_0})$, thus $\frac{D\vec w_{t_0*}}{Dt}(t,p(t)) = F'(t).\vec w_{t_0}(p_{t_0}) = F'(t).F(t)^{-1}.\vec w_{t_0*}(t,p(t)) =^{(3.33)} d\vec v(t,p(t)).\vec w_{t_0*}(t,p(t))$, i.e. (3.42). ∎

4 Deformation gradient $F := d\Phi$

Consider a motion $\widetilde\Phi : \begin{cases} \mathbb{R}\times Obj & \to \mathbb{R}^n \\ (t,P_{Obj}) & \to p_t = \widetilde\Phi(t,P_{Obj}) \end{cases}$, $\Omega_t := \widetilde\Phi(t,Obj)$ the configuration of Obj at any t, fix t_0,t in $\mathbb{R}$, and let $\Phi_t^{t_0} : \begin{cases} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} = \widetilde\Phi(t_0,P_{Obj}) & \to p_t = \Phi_t^{t_0}(p_{t_0}) := \widetilde\Phi(t,P_{Obj}) \end{cases}$, supposed to be a C^1 diffeomorphism. Notations for calculations (quantification), to comply with practices:

1- Classical (unambiguous) notations as in Arnold, Germain: E.g., $(\vec{a}_i)$ and $(\vec{b}_i)$ are bases resp. in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$, $\vec{w}_{t_0}(p_{t_0}) = \sum_i w_{t_0,i}(p_{t_0})\vec{a}_i \in \vec{\mathbb{R}}^n_{t_0}$, $\vec{w}_{t,i}(p_t) = \sum_i w_{t,i}(p_t)\vec{b}_i \in \vec{\mathbb{R}}^n_t$; And

2- Marsden–Hughes duality notations: Capital letters at t_0, lower case letters at t, duality notation, e.g. $(\vec{E}_I)$ and $(\vec{e}_i)$ are bases resp. in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$, $\vec{W}(P) = \sum_I W^I(P)\vec{E}_I \in \vec{\mathbb{R}}^n_{t_0}$, $\vec{w}(p) = \sum_i w^i(p)\vec{e}_i \in \vec{\mathbb{R}}^n_t$.

4.1 Definitions

4.1.1 Definition of the deformation gradient F

Definition 4.1 The differential $d\Phi^{t_0}_t =^{\text{noted}} F^{t_0}_t : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t) \\ p_{t_0} & \to F^{t_0}_t(p_{t_0}) := d\Phi^{t_0}_t(p_{t_0}) \end{array} \right\}$ is called "the covariant deformation gradient between t_0 and t", or simply "the deformation gradient". And "the covariant deformation gradient at p_{t_0} between t_0 and t", or in short "the deformation gradient at p_{t_0}" is the linear map $F^{t_0}_t(p_{t_0}) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$, so defined by, for all $\vec{w}_{t_0}(p_{t_0}) \in \vec{\mathbb{R}}^n_{t_0}$ (vector at p_{t_0}),

$$\boxed{F^{t_0}_t(p_{t_0}).\vec{w}_{t_0}(p_{t_0}) := \lim_{h\to 0} \frac{\Phi^{t_0}_t(p_{t_0}+h\vec{w}_{t_0}(p_{t_0})) - \Phi^{t_0}_t(p_{t_0})}{h}} \overset{\text{noted}}{=} (\Phi^{t_0}_t)_*(\vec{w}_{t_0})(p_t) \overset{\text{noted}}{=} \vec{w}_{t_0*}(t,p_t), \tag{4.1}$$

with $p_t = \Phi^{t_0}_t(p_{t_0})$. See figure 4.1.

Marsden–Hughes notations: $\Phi := \Phi^{t_0}_t$, $F := d\Phi$, $P := p_{t_0}$, $\vec{W}(P) := \vec{w}_{t_0}(p_{t_0})$, $p = \Phi(P)$, thus

$$\boxed{F(P).\vec{W}(P) := \lim_{h\to 0} \frac{\Phi(P+h\vec{W}(P)) - \Phi(P)}{h}} \overset{\text{noted}}{=} \Phi_*\vec{W}(p) \overset{\text{noted}}{=} \vec{w}_*(p). \tag{4.2}$$

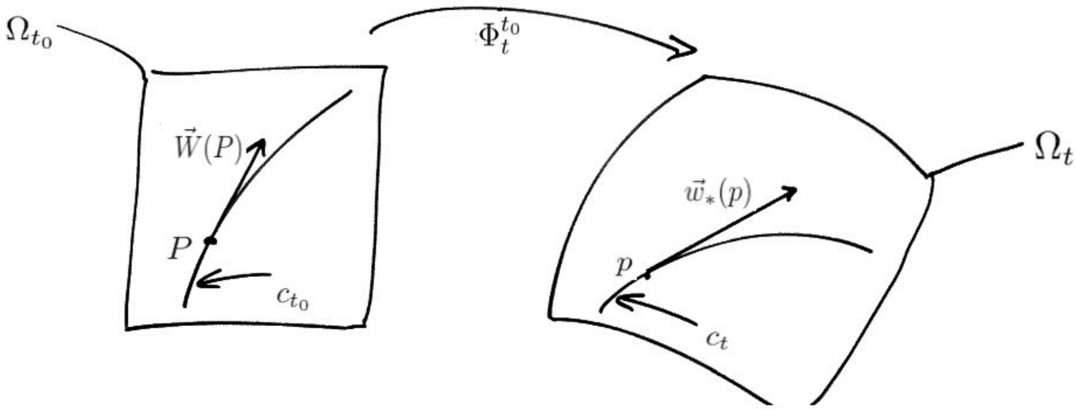


Figure 4.1: $\vec{w}$ is a Eulerian vector field. At t_0 define vector field $\vec{w}_{t_0}$ in Ω_{t_0} by $\vec{w}_{t_0}(p_{t_0}) := \vec{w}(t_0,p_{t_0})$. The (spatial) curve $c_{t_0} : s \to p_{t_0} = c_{t_0}(s)$ in Ω_{t_0} is an integral curve of $\vec{w}_{t_0}$, i.e. satisfies ${c_{t_0}}'(s) = \vec{w}_{t_0}(c_{t_0}(s))$. It is transformed by $\Phi^{t_0}_t$ into the (spatial) curve $c_t = \Phi^{t_0}_t \circ c_{t_0} : s \to p_t = c_t(s) {=} \Phi^{t_0}_t(c_{t_0}(s))$ in Ω_t; Hence $c_t'(s) = d\Phi^{t_0}_t(p_{t_0}).{c_{t_0}}'(s) = d\Phi^{t_0}_t(p_{t_0}).\vec{w}_{t_0}(p_{t_0}) =^{\text{noted}} \vec{w}_{t_0*}(t,p_t)$ is the tangent vector at c_t at p_t (the push-forward of $\vec{w}_{t_0}$ by $\Phi^{t_0}_t$). And $\vec{w}(t,p(t))$ (actual value) is also drawn.

NB: The "deformation gradient" $F^{t_0}_t = d\Phi^{t_0}_t$ is ***not*** a "gradient" (its definition does ***not*** need a Euclidean dot product); This lead to confusions when covariance-contravariance and objectivity are at stake. It would be simpler to stick to the name "$F^{t_0}_t$ = the differential of $\Phi^{t_0}_t$", but it is not the standard usage, except in thermodynamics: E.g., the differential dU of the internal energy U is ***not*** called "the gradient of U" (there is no meaningful inner dot product): It is just called "the differential of U"...

4.1.2 Push-forward (values of F)

Definition 4.2 Let $\vec{w}_{t_0} : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \vec{\mathbb{R}}^n_{t_0} \\ p_{t_0} & \to \vec{w}_{t_0}(p_{t_0}) \end{array} \right\}$ be a vector field in Ω_{t_0}. Its push-forward by $\Phi^{t_0}_t$ is the vector field $(\Phi^{t_0}_t)_*(\vec{w}_{t_0})$ in Ω_t defined by

$$(\Phi^{t_0}_t)_*\vec{w}_{t_0}(p_t) = F^{t_0}_t(p_{t_0}).\vec{w}_{t_0}(p_{t_0}) \overset{\text{noted}}{=} \vec{w}_{t_0*}(t,p_t) \quad \text{when} \quad p_t = \Phi^{t_0}_t(p_{t_0}). \tag{4.3}$$

See figure 4.1. Marsden notation: $\Phi_*\vec{W}(p) = F(P).\vec{W}(P) =^{\text{noted}} \vec{w}_*(p)$ when $p = \Phi^{t_0}_t(P)$.

In other words

$$(\Phi_t^{t_0})_*\vec{w}_{t_0} := (F_t^{t_0}.\vec{w}_{t_0}) \circ (\Phi_t^{t_0})^{-1}. \tag{4.4}$$

Marsden notation: $\Phi_*\vec{W} = (F.\vec{W}) \circ \Phi^{-1} = \vec{w}_*$.

4.1.3 F is a two point tensors

With (4.1), "the tangent map" is

$$\widehat{F_t^{t_0}} : \begin{cases} \Omega_{t_0} & \to \Omega_t \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t) \\ p_{t_0} & \to \widehat{F_t^{t_0}}(p_{t_0}) = (p_t, F_t^{t_0}(p_{t_0})) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \end{cases} \tag{4.5}$$

Definition 4.3 (Marsden–Hughes [14].) The function $\widehat{F_t^{t_0}}$ is called a two point tensor, referring to the points $p_{t_0} \in \Omega_{t_0}$ (departure set) and $p_t = \Phi_t^{t_0}(p_{t_0}) \in \Omega_t$ (arrival set where $\vec{w}_{t_0*}(t,p_t) = F_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0})$ is drawn). And in short $\widehat{F_t^{t_0}} =^{\text{noted}} F_t^{t_0}$ is said to be a two point tensor.

Remark 4.4 The name "two point tensor" is a shortcut than can create confusions and errors when dealing with the transposed: $F_t^{t_0}$ is not immediately a "tensor": A tensor is a multilinear form, so gives ***scalar*** results ($\in \mathbb{R}$), while $F(P) := F_t^{t_0}(P) =^{\text{noted}} F_P \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ gives ***vector*** results (in $\vec{\mathbb{R}}^n_t$). However F_P can be naturally and canonically associated with the bilinear form $\widetilde{F}_P \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t, \vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ defined by, for all $\vec{u}_P \in \vec{\mathbb{R}}^n_{t_0}$ and $\ell_p \in \vec{\mathbb{R}}^{n*}_t$, with $p = \Phi_t^{t_0}(P)$,

$$\widetilde{F}_P(\ell_p, \vec{u}_P) := \ell_p.F_P.\vec{u}_P \ (\in \mathbb{R}), \tag{4.6}$$

see § A.14, and it is $\widetilde{F}_P$ which defines the so-called "two point tensor".

But don't forget that the transposed of a linear form (F_P here) is ***not*** deduced from the transposed of the associated bilinear form ($\widetilde{F}_P$ here). So be careful with the word "transposed" and its two distinct definitions: The transposed of a bilinear form $b(\cdot,\cdot)$ is intrinsic to $b(\cdot,\cdot)$ (is objective), given by $b^T(\vec{u},\vec{w}) = b(\vec{w},\vec{u})$, while the transposed of a linear function L is not intrinsic to L (is subjective), given by $(L^T.\vec{u},\vec{w})_g = (L.\vec{w},\vec{u})_h$ where $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are inner dot products (additional tools) chosen by Human beings (L^T should be written L^T_{gh}). (Details in § A.8.2 and § A.12.1). ■

Remark 4.5 More generally for manifolds, the differential of $\Phi := \Phi_t^{t_0}$ at $P \in \Omega_{t_0}$ is $F(P) := d\Phi(P)$: $\begin{Bmatrix} T_P\Omega_{t_0} & \to T_p\Omega_t \\ \vec{W}(P) & \to \vec{w}_*(p) := d\Phi(P).\vec{W}(P) \end{Bmatrix}$ with $p = \Phi_t^{t_0}(P)$. And the tangent map is

$$T\Phi : \begin{cases} T\Omega_{t_0} & \to T\Omega_t \\ (P,\vec{W}(P)) & \to T\Phi(P,\vec{W}(P)) := (p, d\Phi(P).\vec{W}(P)) = (p,\vec{w}_*(p)), \quad \text{where} \quad p = \Phi_t^{t_0}(P), \end{cases} \tag{4.7}$$

called the associated two point tensor. ■

4.1.4 Evolution: Toward the Lie derivative (in continuum mechanics)

Consider a Eulerian vector field $\vec{w} : \begin{Bmatrix} \mathcal{C} = \bigcup_t(\{t\} \times \Omega_t) & \to \vec{\mathbb{R}}^n \\ (t,p) & \to \vec{w}(t,p) \end{Bmatrix}$, e.g. a "force field". Then, at t_0 consider $\vec{w}_{t_0} : \begin{Bmatrix} \Omega_{t_0} & \to \vec{\mathbb{R}}^n_{t_0} \\ p_{t_0} & \to \vec{w}_{t_0}(p_{t_0}) := \vec{w}(t_0,p_{t_0}) \end{Bmatrix}$. The push-forward of $\vec{w}_{t_0}$ by $\Phi_t^{t_0}$ is, cf. (4.2),

$$\vec{w}_{t_0*}(t,p(t)) = F_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0}), \quad \text{where} \quad p(t) = \Phi^{t_0}(t,p_{t_0}). \tag{4.8}$$

See figure 4.1. Then, without any ubiquity gift, at t at $p(t)$ we can compare $\vec{w}(t,p(t))$ (real value of $\vec{w}$ at t at $p(t)$) with $\vec{w}_{t_0*}(t,p(t))$ (transported memory along the trajectory). Thus the rate

$$\frac{\vec{w}(t,p(t)) - \vec{w}_{t_0*}(t,p(t))}{t-t_0} = \frac{\text{actual}(t,p(t)) - \text{memory}(t,p(t))}{t-t_0} \quad \text{is meaningful at } (t,p(t)) \tag{4.9}$$

(no ubiquity gift required). This rate gives, as $h \to 0$, the Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ (the rate of stress), and we will see at § 9.3 that $\mathcal{L}_{\vec{v}}\vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}$ (the $d\vec{v}$ term tells that a "non-uniform flow" acts on the stress).

4.1.5 Pull-back

Formally the pull-back is the push-forward with $(\Phi_t^{t_0})^{-1}$:

Definition 4.6 The pull-back $(\Phi_t^{t_0})^*\vec{w}_t$ of a vector field $\vec{w}_t$ defined on Ω_t is the vector field defined on Ω_{t_0} by, with $p_{t_0} = (\Phi_t^{t_0})^{-1}(p_t)$,

$$\vec{w}_t^*(t_0, p_{t_0}) = (\Phi_t^{t_0})^*\vec{w}_t(p_{t_0}) := (F_t^{t_0})^{-1}(p_t).\vec{w}_t(p_t), \quad \text{written} \quad \vec{W}^*(P) = F^{-1}(p).\vec{w}(p). \tag{4.10}$$

4.2 Quantification with bases

(Simple Cartesian framework.) $(\vec{a}_i)$ is a Cartesian basis in $\vec{\mathbb{R}}^n_{t_0}$, $(\vec{b}_i)$ is a Cartesian basis in $\vec{\mathbb{R}}^n_t$, o_t is an origin in $\mathbb{R}^n$ at t, $\Phi_t^{t_0} =^{\text{noted}} \Phi$ supposed C^1, $\varphi_i : \Omega_{t_0} \to \mathbb{R}$ is its components in the referential $(o_t, (\vec{b}_i))$:

$$\Phi(p_{t_0}) = o_t + \sum_{i=1}^n \varphi_i(p_{t_0})\vec{b}_i, \quad \text{i.e.} \quad \overrightarrow{o_t\Phi(p_{t_0})} = \sum_{i=1}^n \varphi_i(p_{t_0})\vec{b}_i. \tag{4.11}$$

Thus, with the classic notation $d\varphi_i(p_{t_0}).\vec{a}_j =^{\text{noted}} \frac{\partial\varphi_i}{\partial X_j}(p_{t_0})$ since $(\vec{a}_i)$ is a Cartesian basis, and $(\vec{b}_i)$ being a Cartesian basis,

$$d\Phi(p_{t_0}).\vec{a}_j = \sum_{i=1}^n (d\varphi_i(p_{t_0}).\vec{a}_j)\vec{b}_i = \sum_{i=1}^n \frac{\partial\varphi_i}{\partial X_j}(p_{t_0})\vec{b}_i, \quad \text{thus} \quad [d\Phi(p_{t_0})]_{[\vec{a},\vec{b}]} = [\frac{\partial\varphi_i}{\partial X_j}(p_{t_0})] = [F(p_{t_0})]_{[\vec{a},\vec{b}]},$$

$[d\Phi(p_{t_0})]_{[\vec{a},\vec{b}]} = [F(p_{t_0})]_{[\vec{a},\vec{b}]}$ being the Jacobian matrix of Φ at p_{t_0} relative to the chosen bases. In short:

$$d\Phi.\vec{a}_j = \sum_{i=1}^n \frac{\partial\varphi_i}{\partial X_j}\vec{b}_i, \quad \text{thus} \quad [d\Phi]_{[\vec{a},\vec{b}]} = [\frac{\partial\varphi_i}{\partial X_j}] = [F]_{[\vec{a},\vec{b}]} = [F_{ij}], \tag{4.12}$$

Thus, if $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ is a vector at p_{t_0} and $\vec{W} = \sum_{j=1}^n W_j\vec{a}_j$ then, by linearity of differentials,

$$d\Phi.\vec{W} = F.\vec{W} = \sum_{i=1}^n F_{ij}W_j\vec{b}_i, \quad \text{i.e.} \quad [F.\vec{W}]_{|\vec{b}} = [F]_{|\vec{a},\vec{b}}.[\vec{W}]_{|\vec{a}} \tag{4.13}$$

(more precisely: $F_t^{t_0}(p_{t_0}).\vec{W}(p_{t_0}) = \sum_{i=1}^n F_{ij}(p_{t_0})W_j(p_{t_0})\vec{b}_i$).

Similarly, for the second order derivative $d^2\Phi = dF$ (when Φ is C^2): With $\vec{U} = \sum_{j=1}^n U_j\vec{a}_j$ and $\vec{W} = \sum_{k=1}^n W_k\vec{a}_k$, and with $(\vec{a}_i)$ and $(\vec{b}_i)$ Cartesian bases, we get

$$dF(\vec{U},\vec{W}) = d^2\Phi(\vec{U},\vec{W}) = \sum_{i=1}^n d^2\varphi_i(\vec{U},\vec{W})\vec{b}_i = \sum_{i,j,k=1}^n \frac{\partial^2\varphi_i}{\partial X_j\partial X_k}U_jW_k\vec{b}_i = \sum_{i=1}^n \Big([\vec{U}]^T_{|\vec{a}}.[d^2\varphi_i]_{|\vec{a}}.[\vec{W}]_{|\vec{a}}\Big)\,\vec{b}_i, \tag{4.14}$$

$[d^2\varphi_i(p_{t_0})]_{|\vec{a}} = [\frac{\partial^2\varphi_i}{\partial X_j\partial X_k}(p_{t_0})]_{\substack{j=1,...,n\\k=1,...,n}}$ being the Hessian matrix of φ_i at p_{t_0} relative to the basis $(\vec{a}_i)$.

With Marsden duality notations:

$$\begin{aligned}
&\bullet\ p = \Phi(P) = o_t + \sum_{i=1}^n \varphi^i(P)\vec{e}_i, \quad F^i{}_J(P) = \frac{\partial\varphi^i}{\partial X^J}(P) \quad (= d\varphi^i(P).\vec{E}_J),\\
&\bullet\ F(P).\vec{W} = \sum_{i,J=1}^n F^i{}_J(P)\,W^J\vec{e}_i, \quad [F] = [F^i{}_J] = [d\Phi],\\
&\bullet\ dF(\vec{U},\vec{W}) = d^2\Phi(\vec{U},\vec{W}) = \sum_{i,J,K=1}^n \frac{\partial^2\varphi^i}{\partial X^J\partial X^K}U^JW^K\vec{e}_i = \sum_{i=1}^n \Big([\vec{U}]^T.[d^2\varphi^i].[\vec{W}]\Big)\,\vec{e}_i.
\end{aligned} \tag{4.15}$$

Remark 4.7 J, j are dummy variables when used in a summation: E.g., $df.\vec{W} = \sum_{j=1}^n \frac{\partial f}{\partial X^j}W^j = \sum_{J=1}^n \frac{\partial f}{\partial X^J}W^J = \sum_{\alpha=1}^n \frac{\partial f}{\partial X^\alpha}W^\alpha = \frac{\partial f}{\partial X^1}W^1 + \frac{\partial f}{\partial X^2}W^2 + ...$ (there is no uppercase for 1, 2...). And Marsden–Hughes notations (capital letters for the past) are not at all compulsory, classical notations being just as good and even preferable if you hesitate (because they are not misleading). See § A. ▪

4.3 The unfortunate notation $d\vec{x} = F.d\vec{X}$

4.3.1 Issue

(4.3), i.e. $\vec{w}_*(p) := F(P).\vec{W}(P)$, is sometimes written

$$d\vec{x} = F.d\vec{X} \ : \text{“a very unfortunate and misleading notation”} \tag{4.16}$$

which amounts to “confuse a length and a speed”... And you also the phrase “(4.16) is still true if $||d\vec{X}|| = 1$”... while $d\vec{X}$ is supposed to be small...

4.3.2 Where does this unfortunate notation come from?

The notation (4.16) comes from the first order Taylor expansion $\Phi(Q) = \Phi(P) + d\Phi_t^{t_0}(P).(Q{-}P) + o(||Q{-}P||)$, where $P, Q \in \Omega_{t_0}$, i.e., with $p = \Phi_t^{t_0}(P)$ and $q = \Phi_t^{t_0}(Q)$ and $h = ||Q{-}P||$,

$$q - p = F(P).(Q{-}P) + o(h), \quad \text{written} \quad \delta\vec{x} = F.\delta\vec{X} + o(\delta\vec{X}), \tag{4.17}$$

or $\vec{pq} = F(P).\vec{PQ} + o(h)$. So as $Q \to P$ we get $0 = 0$... Quite useless, isn't it?

While

$$\frac{q-p}{h} = F(P).\frac{Q-P}{h} + o(1) \quad \text{is useful:} \tag{4.18}$$

As $Q \to P$ we get $\vec{w}_* = F(P).\vec{W}$ which relates tangent vectors, see figure 4.1 Details:

4.3.3 Interpretation: Vector approach

Consider a spatial curve $c_{t_0} : \left\{ \begin{array}{rl} [s_1, s_2] & \to \Omega_{t_0} \\ s & \to P := c_{t_0}(s) \end{array} \right\}$ in Ω_{t_0}, cf. figure 4.1. It is deformed by $\Phi_t^{t_0}$ to become the spatial curve defined by $c_t := \Phi_t^{t_0} \circ c_{t_0} : \left\{ \begin{array}{rl} [s_1, s_2] & \to \Omega_t \\ s & \to p := c_t(s) = \Phi_t^{t_0}(c_{t_0}(s)). \end{array} \right.$ in Ω_t. Hence, relation between tangent vectors:

$$\frac{dc_t}{ds}(s) = d\Phi_t^{t_0}(c_{t_0}(s)).\frac{dc_{t_0}}{ds}(s), \quad \text{written} \quad \frac{d\vec{x}}{ds}(s) = F(X(s)).\frac{d\vec{X}}{ds}(s), \quad \text{written} \quad \frac{d\vec{x}}{ds} = F.\frac{d\vec{X}}{ds}, \tag{4.19}$$

But you ***can't*** simplify by ds to get $d\vec{x} = F.d\vec{X}$: It is absurd to confuse “a slope $\frac{d\vec{X}}{ds}(s)$” and “a length $\delta\vec{X} = p - q$”. Recall: With $P = c_{t_0}(s)$ and $p = c_t(s)$, (4.19) reads $\vec{w}_*(p) = F(P).\vec{W}(P)$, cf. (4.2).

NB: $||\frac{dc_{t_0}}{ds}(s)|| = ||\frac{d\vec{X}}{ds}(s)|| = 1$ is meaningful in (4.19): It means that the parametrization of the spatial curve c_{t_0} in Ω_{t_0} uses a curvilinear parameter s such that $||c_{t_0}{}'(s)|| = 1$ for all s, i.e. s.t. $||\vec{W}_P|| = 1$ in figure 4.1. You ***cannot*** simplify by ds: $||d\vec{X}|| = 1$ is absurd together with $d\vec{X}$ “small”.

4.3.4 Interpretation: Differential approach

(4.16) is a relation between differentials... if you adopt the correct notations; Let us do it: With (4.11),

$$\vec{x} = \vec{o_t p} = \overrightarrow{o_t\Phi_t^{t_0}(P)} = \sum_{i=1}^n \varphi_i(P)\vec{b}_i \overset{\text{noted}}{=} \sum_{i=1}^n x_i(P)\vec{b}_i, \quad \text{where} \quad \varphi_i \overset{\text{noted}}{=} x_i \quad \text{(function of } P). \tag{4.20}$$

Thus, with $(\pi_{ai}) = (dX_i)$ the (covariant) dual basis of $(\vec{a}_i)$ we get the system of n equations (functions):

$$d\Phi = F, \quad \text{i.e.} \quad \left\{ \begin{array}{l} d\varphi_1(P) = \sum_{j=1}^n \frac{\partial\varphi_1}{\partial X_j}(P)\, dX_j \\ \vdots \\ d\varphi_n(P) = \sum_{j=1}^n \frac{\partial\varphi_n}{\partial X_j}(P)\, dX_j \end{array} \right\}, \quad \text{which is noted} \quad d\vec{x} = F.d\vec{X}, \tag{4.21}$$

this last notation being often misunderstood[2]: It is nothing more than $d\Phi = F$ (coordinate free notation).

[2] Spivak [19] chapter 4: Classical differential geometers (and classical analysts) did not hesitate to talk about “infinitely small” changes dx^i of the coordinates x^i, just as Leibnitz had. No one wanted to admit that this was nonsense, because true results were obtained when these infinitely small quantities were divided into each other (provided one did it in the right way). Eventually it was realized that the closest one can come to describing an infinitely small change is to describe a direction in which this change is supposed to occur, i.e., a tangent vector. Since *df* is supposed to be the infinitesimal change of f under an infinitesimal change of the point, *df* must be a function of this change, which means that *df* should be a function on tangent vectors. The dX_i themselves then metamorphosed into functions, and it became clear that they must be distinguished from the tangent vectors $\partial/\partial X_i$. Once this realization came, it was only a matter of making new definitions, which preserved the old notation, and waiting for everybody to catch up.

4.3.5 The ambiguous notation $\overset{\bullet}{d\vec{x}} = \overset{\bullet}{F}.d\vec{X}$

The bad notation $d\vec{x} = F.d\vec{X}$ gives the unfortunate and misunderstood notations $\overset{\bullet}{d\vec{x}} = \overset{\bullet}{F}.d\vec{X}$, and then

$$\overset{\bullet}{d\vec{x}} = L.d\vec{x} \quad \text{where} \quad L = \overset{\bullet}{F}.F^{-1}. \tag{4.22}$$

Question: What is the meaning (and legitimate notation) of (4.22)?

Answer: $\overset{\bullet}{d\vec{x}} = L.d\vec{x}$ means

$$\boxed{\frac{D\vec{w}_{t_0*}}{Dt} = d\vec{v}.\vec{w}_{t_0*}} \quad = \text{evolution rate of tangent vectors along a trajectory} \tag{4.23}$$

see figure 4.1. Indeed, $\vec{w}_{t_0*}(t,p(t)) =^{(4.8)} F^{t_0}(t,p_{t_0}).\vec{w}_{t_0}(p_{t_0})$ gives

$$\frac{D\vec{w}_{t_0*}}{Dt}(t,p(t)) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).\vec{w}_{t_0}(p_{t_0}) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).(F_t^{t_0}(p_{t_0})^{-1}.\vec{w}_{t_0*}(t,p(t))), \tag{4.24}$$

i.e. $\frac{D\vec{w}_{t_0*}}{Dt}(t,p(t)) = d\vec{v}(t,p(t)).\vec{w}_{t_0*}(t,p(t))$, i.e. (4.23). In particular $\frac{D\vec{w}_{t_0*}}{Dt}(t_0,p_{t_0}) = d\vec{v}(t_0,p_{t_0}).\vec{w}_{t_0}(p_{t_0})$ is the evolution rate of tangent vectors at t_0 at p_{t_0}.

4.4 Change of coordinate system at t for F

$p_{t_0} \in \Omega_{t_0}$, $p_t = \Phi_t^{t_0}(p_{t_0}) \in \Omega_t$, $\vec{W}(p_{t_0}) \in \vec{\mathbb{R}}^n_{t_0}$, $\vec{w}(p_t) = F_t^{t_0}(p_{t_0}).\vec{W}(p_{t_0}) \in \vec{\mathbb{R}}^n_t$, written $\vec{w} = F.\vec{W}$.

4.4.1 Change of basis system at t for F

The observer at t_0 used a basis $(\vec{a}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$. At t, in $\vec{\mathbb{R}}^n_t$, a first observer chooses a Cartesian basis $(\vec{b}_{\text{old},i})$, and a second observer chooses a Cartesian basis $(\vec{b}_{\text{new},i})$. And $P = [P_{ij}]$ is the transition matrix from $(\vec{b}_{\text{old},i})$ to $(\vec{b}_{\text{new},i})$, i.e. $\vec{b}_{\text{new},j} = \sum_{i=1}^n P_{ij}\vec{b}_{\text{old},i}$ for all j. The change of basis formula in $\vec{\mathbb{R}}^n_t$ gives

$$[\vec{w}]_{|\vec{b}_{new}} = P^{-1}.[\vec{w}]_{|\vec{b}_{old}}, \quad \text{thus} \quad [F.\vec{W}]_{|\vec{b}_{new}} = P^{-1}.[F.\vec{W}]_{|\vec{b}_{old}}. \tag{4.25}$$

Thus $[F]_{|\vec{a},\vec{b}_{new}}.[\vec{W}]_{|\vec{a}} = P^{-1}.[F]_{|\vec{a},\vec{b}_{old}}.[\vec{W}]_{|\vec{a}}$, true for all $\vec{W}$, thus

$$\boxed{[F]_{|\vec{a},\vec{b}_{new}} = P^{-1}.[F]_{|\vec{a},\vec{b}_{old}}}. \tag{4.26}$$

Remark 4.8 (4.26) is ***not*** $[L]_{|new} = P^{-1}.[L]_{|old}.P$, the change of basis formula for endomorphisms, which would be nonsense since $F := F_t^{t_0}(p_{t_0}) : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ is not an endomorphism; (4.26) is just the usual change of basis formula $[\vec{w}]_{|\vec{b}_{new}} = P^{-1}.[\vec{w}]_{|\vec{b}_{old}}$ for vectors $\vec{w}$ in $\vec{\mathbb{R}}^n_t$ (contravariant vectors). ∎

4.4.2 Change of basis system at t_0 for F

The observer at t with his basis $(\vec{b}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$ wants to compare results of two observers à t_0: The first used a Cartesian basis $(\vec{a}_{\text{old},i})$ e.g. Bernoulli with one Switzerland foot, the second used a Cartesian basis $(\vec{a}_{\text{new},i})$ e.g. Timoshenko with the English foot. $P = [P_{ij}]$ being the transition matrix from $(\vec{a}_{\text{old},i})$ to $(\vec{a}_{\text{new},i})$, for any $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$,

$$[\vec{W}]_{|\vec{a}_{new}} = P^{-1}.[\vec{W}]_{|\vec{a}_{old}}. \tag{4.27}$$

And $F.\vec{W} = F.\vec{W}$ gives $[F.\vec{W}]_{|\vec{b}} = [F.\vec{W}]_{|\vec{b}}$, thus $[F]_{|\vec{a}_{new},\vec{b}}.[\vec{W}]_{|\vec{a}_{new}} = [F]_{|\vec{a}_{old},\vec{b}}.[\vec{W}]_{|\vec{a}_{old}}$, hence $[F]_{|\vec{a}_{new},\vec{b}}.P^{-1}.[\vec{W}]_{|\vec{a}_{old}} = [F]_{|\vec{a}_{old},\vec{b}}.[\vec{W}]_{|\vec{a}_{old}}$, for all $\vec{W}$. Thus $[F]_{|\vec{a}_{new},\vec{b}}.P^{-1} = [F]_{|\vec{a}_{old},\vec{b}}$, thus

$$\boxed{[F]_{|\vec{a}_{new},\vec{b}} = [F]_{|\vec{a}_{old},\vec{b}}.P}. \tag{4.28}$$

This is the change of basis formula for linear forms (covariant vectors), which is expected since here F is considered to be a linear function that acts on vectors in $\vec{\mathbb{R}}^n_{t_0}$.

Exercice 4.9 Detail the matrix calculation which gave (4.28) with Marsden's notations.

Answer. Let $F.\vec{E}_{old,J} = \sum_i F^i_{o,J}\vec{e}_i$ and $F.\vec{E}_{new,J} = \sum_i F^i_{n,J}\vec{e}_i$, and $\vec{W} = \sum_J W^J_o\vec{E}_{old,J} = \sum_J W^J_n\vec{E}_{new,J}$, and $Q = [Q^I_J] := P^{-1}$, so $[\vec{W}]_{\vec{E}_{new}} = Q.[\vec{W}]_{\vec{E}_{old}}$, i.e. $W^J_n = \sum_K Q^J_K W^K_o$ for all J. Thus $F.\vec{W} = \sum_{iJ} F^i_{n,J}W^J_n\vec{e}_i = \sum_{iJK} F^i_{n,J}Q^J_K W^K_o\vec{e}_i$ together with $F.\vec{W} = \sum_{iK} F^i_{o,K}W^K_o\vec{e}_i$, for all $\vec{W}$, thus $\sum_J F^i_{n,J}Q^J_K = F^i_{o,K}$ for all i,K, thus $[F]_{|\vec{E}_{new},\vec{e}}.Q = [F]_{|\vec{E}_{old},\vec{e}}$. ∎

4.5 Tensor notations: Warnings

As already noted, cf. (4.6), the linear map $F = d\Phi := d\Phi_t^{t_0}(p_{t_0}) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ is naturally canonically associated with the bipoint tensor $\widetilde{F} \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t, \vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ defined by, for all $(\ell, \vec{W}) \in \vec{\mathbb{R}}^{n*}_t \times \vec{\mathbb{R}}^n_{t_0}$,

$$\widetilde{F}(\ell, \vec{W}) := \ell.(F.\vec{W}). \tag{4.29}$$

Quantification: basis $(\vec{a}_i) \in \vec{\mathbb{R}}^n_{t_0}$ with its (covariant) dual basis (π_{ai}) in $\vec{\mathbb{R}}^{n*}_{t_0}$, origin $o_t \in \mathbb{R}^n$ and basis $(\vec{b}_i)$ in $\vec{\mathbb{R}}^n_t$ with its (covariant) dual basis (π_{bi}) in $\vec{\mathbb{R}}^{n*}_t$, $\overrightarrow{o_t\Phi} = \sum_{i=1}^n \varphi_i \vec{b}_i$, and $d\varphi_i.\vec{a}_j = \frac{\partial\varphi_i}{\partial X_j}$, i.e. $d\varphi_i = \sum_{j=1}^n \frac{\partial\varphi_i}{\partial X_j}\,\pi_{aj}$. Hence $d\Phi.\vec{a}_j = \sum_{i=1}^n (d\varphi_i.\vec{a}_j)\vec{b}_i$ and

$$F.\vec{a}_j = d\Phi.\vec{a}_j = \sum_{i=1}^n \frac{\partial\varphi_i}{\partial X_j}\vec{b}_i, \quad \text{thus} \quad \widetilde{F} = \sum_{i=1}^n \vec{b}_i \otimes d\varphi_i = \sum_{i,j=1}^n \frac{\partial\varphi_i}{\partial X_j}\,\vec{b}_i \otimes \pi_{aj}. \tag{4.30}$$

Indeed, $(\sum_{i,j=1}^n \frac{\partial\varphi_i}{\partial X_j}\,\vec{b}_i \otimes \pi_{aj})(\pi_{bk}, \vec{a}_\ell) = \sum_{i,j=1}^n \frac{\partial\varphi_i}{\partial X_j}\,(\vec{b}_i.\pi_{bk})(\pi_{aj}\vec{a}_\ell) = \sum_{i,j=1}^n \frac{\partial\varphi_i}{\partial X_j}\,\delta_{ik}\delta_{j\ell} = \frac{\partial\varphi_k}{\partial X_\ell}$, and $\pi_{bk}.(F.\vec{a}_\ell) = \pi_{bk}.(\sum_{i=1}^n \frac{\partial\varphi_i}{\partial X_\ell}\vec{b}_i) = \sum_{i=1}^n \frac{\partial\varphi_i}{\partial X_\ell}\pi_{bk}.\vec{b}_i = \sum_{i=1}^n \frac{\partial\varphi_i}{\partial X_\ell}\delta_{kj} = \frac{\partial\varphi_k}{\partial X_\ell}$: Equality for all k, ℓ.

So $\widetilde{F}(\ell, \vec{W}) = \sum_{ij} \frac{\partial\varphi_i}{\partial X_j}\ell_i W^j$ when $F.\vec{W} = \sum_i \frac{\partial\varphi_i}{\partial X_j} W_j \vec{b}_i$, for all $\ell = \sum_i \ell_i \pi_{bi}$ and $\vec{W} = \sum_j W_j \vec{a}_j$.

And similarly, $d^2\varphi_i(\vec{a}_j, \vec{a}_k) = \frac{\partial^2\varphi_i}{\partial X_j \partial X_k}$ for all j,k, i.e. $d^2\varphi_i = \sum_{j,k=1}^n \frac{\partial^2\varphi_i}{\partial X_j\partial X_k}\,\pi_{aj}\otimes\pi_{ak}$, and

$$\begin{aligned} &dF(.,.) = \sum_{i=1}^n d^2\varphi_i(.,.)\vec{b}_i \overset{\text{noted}}{=} d\widetilde{F} = \sum_{i=1}^n \vec{b}_i \otimes d^2\varphi_i = \sum_{i,j,k=1}^n \frac{\partial^2\varphi_i}{\partial X_j \partial X_k}\,\vec{b}_i \otimes (\pi_{aj}\otimes\pi_{ak}), \quad \text{so} \\ &dF(\vec{U}, \vec{W}) = d\widetilde{F}(\vec{U}, \vec{W}) = \sum_{i=1}^n d^2\varphi_i(\vec{U}, \vec{W})\vec{b}_i = \sum_{i,j,k=1}^n \frac{\partial^2\varphi_i}{\partial X_j \partial X_k} U_j W_k \vec{b}_i. \end{aligned} \tag{4.31}$$

Marsden duality notations: $d\varphi^i = \sum_J \frac{\partial\varphi^i}{\partial X^J}\,dX^J$, $F^i{}_J = \frac{\partial\varphi^i}{\partial X^J}$, $F.\vec{E}_J = \sum_i F^i{}_J\vec{e}_i$, $\widetilde{F} = \sum_i \vec{e}_i \otimes d\varphi^i = \sum_{iJ} F^i{}_J\,\vec{e}_i \otimes dX^J$, $d^2\varphi^i = \sum_{JK} \frac{\partial^2\varphi^i}{\partial X^J\partial X^K}\,dX^J \otimes dX^K$, $dF = \sum_i \vec{e}_i \otimes d^2\varphi^i = \sum_{iJK} \frac{\partial^2\varphi^i}{\partial X^J\partial X^K}\,\vec{e}_i \otimes dX^J \otimes dX^K$.

Warning 1: The tensor notation can be misleading, e.g. if you use the transposed, see remark 4.4. So, you should always use the standard $F.\vec{a}_j = \sum_{j=1}^n F_{ij}\vec{b}_i$ notation (vector value), i.e. $F.\vec{E}_J = \sum_{i,j=1}^n F^i{}_J \vec{e}_i$ with Marsden notations. And avoid the use of $\widetilde{F}$, i.e. of $\widetilde{F}(\ell, \vec{W})$ (scalar value).

Warning 2: You can't use $\vec{a}_j$ instead of π_{aj} in (4.30), i.e. you can't use $\widehat{F} = \sum_{i,j=1}^n \frac{\partial\varphi_i}{\partial X_j}\,\vec{b}_i \otimes \vec{a}_j$ instead of $\widetilde{F}$ in (4.30), because there is no canonical natural isomorphism between $\vec{\mathbb{R}}^n$ and $\mathbb{R}^{n*}$: E.g. $\vec{a}_{new,j} = \sum_i P_{ij}\vec{a}_{old,i}$ while $\pi_{anew,i} = \sum_j Q_{ij}\pi_{aold,j}$ where $Q = P^{-1}$, see (A.27), and you get $\sum_{ik} \widehat{F}_{n,ij}\,\vec{b}_i \otimes \vec{a}_{new,j} = \widehat{F} = \sum_{ik} \widehat{F}_{o,ik}\,\vec{b}_i \otimes \vec{a}_{old,k} = \sum_{ijk} \widehat{F}_{o,ik}Q_{jk}\,\vec{b}_i \otimes \vec{a}_{new,j} = \sum_{ij}([\widehat{F}_o].Q^T)_{ij}\,\vec{b}_i \otimes \vec{a}_{new,j}$, thus $[\widehat{F}]_{|\vec{b},\vec{a}_{new}} = [\widehat{F}]_{|\vec{b},\vec{a}_{old}}.P^{-T}$, which is ***not*** (4.28). So it is wrong if you want to compare Euler's results with those of Newton, Lagrange, Cauchy... because they didn't use the same unit of measurement.

In other words, an inner dot product can't be confused with a matrix product, e.g. you never talk about the "trace" $\sum_i g_{ii}$ of an inner dot product $g(\cdot,\cdot) : E \times E \to \mathbb{R}$ (not invariant), while the trace of an endomorphism (linear $E \to E$) is useful and invariant, see § A.10.

Warning 3: In some manuscripts you find the notation $F = d\Phi =^{\text{noted}} \Phi \otimes \nabla_X$. It does not help to understand what F is (it is the differential $d\Phi$), and must be avoided as far as objectivity is concerned:

• A differentiation is ***not*** a tensor operation, see the fundamental example S.1, so why use the tensor product notation $\Phi \otimes \nabla_X$, when the standard notation $d\Phi$ (or if you use a basis $(\vec{b}_i)$ in $\vec{\mathbb{R}}^n_t$ the notation $d\Phi(.) = \sum_{i=1}^n d\varphi_i(.)\vec{b}_i$) is legitimate, explicit and easy to use?

• It could be misinterpreted because in mechanics ∇f is often understood to be a vector (contravariant) while the differential df is covariant (unmissable in thermodynamics because you can't use gradients).

• It gives the confusing notation $\Phi \otimes \nabla_X \otimes \nabla_X$, instead of the legitimate, explicit and easy to use notation $d^2\Phi$ (or if you use a basis $(\vec{b}_i)$ in $\vec{\mathbb{R}}^n_t$ the notation $d^2\Phi(.,.) = \sum_{i=1}^n d^2\varphi_i(.,.)\vec{b}_i$).

4.6 Spatial Taylor expansion of F

$\Phi_t^{t_0} =^{\text{noted}} \Phi$ is supposed to be C^3 for all t_0, t, and $F = d\Phi$. Then, in Ω_t, with $P \in \Omega_{t_0}$ and $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ vector at P, $\Phi(P{+}h\vec{W}) = \Phi(P) + h\,F(P).\vec{W} + \frac{h^2}{2}\,dF(P)(\vec{W}, \vec{W}) + o(h)$, and

$$F(P{+}h\vec{W}) = F(P) + h\,dF(P).\vec{W} + \frac{h^2}{2}\,d^2F(P)(\vec{W}, \vec{W}) + o(h^2). \tag{4.32}$$

4.7 Time Taylor expansion of F

t_0 is fixed, Φ^{t_0} is supposed to be C^3, $p_{t_0} = \widetilde\Phi(t_0, P_{Obj})$, $p_t = \widetilde\Phi(t_0, P_{Obj}) = \Phi^{t_0}(t, p_{t_0})$, and $\vec V^{t_0}(t, p_{t_0}) = \frac{\partial\Phi^{t_0}}{\partial t}(t, p_{t_0}) = \vec v(t, p_t) = \vec v(t, \Phi^{t_0}(t, p_{t_0}))$ (Lagrangian and Eulerian velocities), p_{t_0} is fixed, $\Phi^{t_0}_{p_{t_0}}(t) := \Phi^{t_0}(t, p_{t_0})$, and $F^{t_0}_{p_{t_0}}(t) := F^{t_0}(t, p_{t_0}) = d\Phi^{t_0}(t, p_{t_0})$. Hence

$$F^{t_0}_{p_{t_0}}{}'(t) = \frac{\partial F^{t_0}}{\partial t}(t, p_{t_0}) = \frac{\partial(d\Phi^{t_0})}{\partial t}(t, p_{t_0}) = d(\frac{\partial\Phi^{t_0}}{\partial t})(t, p_{t_0}) = d\vec V^{t_0}(t, p_{t_0}) = d\vec v(t, p(t)).F^{t_0}_{p_{t_0}}(t) \qquad (4.33)$$

(in short $\overset{\bullet}{F} = d\vec V = d\vec v.F$). Thus the first order time Taylor expansion $F^{t_0}_{p_{t_0}}(t{+}h) = F^{t_0}_{p_{t_0}}(t) + h\,F^{t_0}_{p_{t_0}}{}'(t) + o(h)$ near t gives

$$\begin{aligned} F^{t_0}_{p_{t_0}}(t{+}h) &= F^{t_0}_{p_{t_0}}(t) + h\, d\vec V^{t_0}_{p_{t_0}}(t) + o(h) \\ &= \Big(I + h\, d\vec v(t, p(t))\Big).F^{t_0}_{p_{t_0}}(t) + o(h). \end{aligned} \qquad (4.34)$$

NB: They are ***three*** times are involved: t and $t{+}h$ as usual, and t_0 (observer dependent) through $F := F^{t_0}_{p_{t_0}}$ and $\vec V := \vec V^{t_0}_{p_{t_0}}$, as in (3.39). This Taylor expansion requires Lagrangian variables (requires Φ^{t_0}).

And, with $\vec A^{t_0}(t, p_{t_0}) = \frac{\partial^2\Phi^{t_0}}{\partial t^2}(t, p_{t_0}) = \vec\gamma(t, p(t)) == \vec\gamma(t, \Phi^{t_0}(t, p_{t_0}))$ (Lagrangian and Eulerian accelerations),

$$F^{t_0}_{p_{t_0}}{}''(t) = \frac{\partial^2 F^{t_0}}{\partial t^2}(t, p_{t_0}) = \frac{\partial^2(d\Phi^{t_0})}{\partial t^2}(t, p_{t_0}) = d(\frac{\partial^2\Phi^{t_0}}{\partial t^2})(t, p_{t_0}) = d\vec A^{t_0}(t, p_{t_0}) = d\vec\gamma(t, p_t).F(t) \qquad (4.35)$$

(in short $\overset{\bullet\bullet}{F} = d\vec A = d\vec\gamma.F$). Thus (second order time Taylor expansion of $F^{t_0}_{p_{t_0}}$ near t):

$$\begin{aligned} F^{t_0}_{p_{t_0}}(t{+}h) &= F^{t_0}_{p_{t_0}}(t) + h\, d\vec V^{t_0}_{p_{t_0}}(t) + \frac{h^2}{2}\, d\vec A^{t_0}_{p_{t_0}}(t) + o(h^2) \\ &= \Big(I + h\, d\vec v(t, p(t)) + \frac{h^2}{2}\, d\vec\gamma(t, p(t))\Big).F^{t_0}_{p_{t_0}}(t) + o(h^2). \end{aligned} \qquad (4.36)$$

In particular with $t = t_0$: Then $F^{t_0}_{p_{t_0}}(t_0) = I$, thus

$$\begin{aligned} F^{t_0}_{p_{t_0}}(t_0{+}h) &= I + h\, d\vec V^{t_0}_{p_{t_0}}(t_0) + \frac{h^2}{2}\, d\vec A^{t_0}_{p_{t_0}}(t_0) + o(h^2) \\ &= \Big(I + h\, d\vec v(t_0, p_{t_0}) + \frac{h^2}{2}\, d\vec\gamma(t_0, p_{t_0})\Big) + o(h^2). \end{aligned} \qquad (4.37)$$

Remark 4.10 $\gamma = \frac{\partial\vec v}{\partial t} + d\vec v.\vec v$ is not linear in $\vec v$. Idem,

$$d\vec\gamma = d(\frac{D\vec v}{Dt}) = d(\frac{\partial\vec v}{\partial t} + d\vec v.\vec v) = d\frac{\partial\vec v}{\partial t} + d^2\vec v.\vec v + d\vec v.d\vec v \quad (= \frac{D(d\vec v)}{Dt} + d\vec v.d\vec v) \qquad (4.38)$$

is non linear in $\vec v$, and gives $F^{t_0}_{p_{t_0}}{}''(t) = (d\frac{\partial\vec v}{\partial t} + d^2\vec v.\vec v + d\vec v.d\vec v)(t, p_t).F^{t_0}_{p_{t_0}}(t)$, non linear in $\vec v$. ▪

Exercice 4.11 Directly check that (short notation) $F' = d\vec v.F$ gives $F'' = d\vec\gamma.F$.

Answer. $F'(t) = d\vec v(t, p(t)).F(t)$ gives $F''(t) = \frac{D(d\vec v)}{Dt}(t, p(t)).F(t) + d\vec v(t, p(t)).F'(t)$ with $\frac{D(d\vec v)}{Dt} = d\vec\gamma - d\vec v.\vec v$, cf. (4.38), thus $F''(t) = (d\vec\gamma - d\vec v.d\vec v)(t, p(t)).F(t) + d\vec v(t, p(t)).d\vec v(t, p(t)).F(t) = d\vec\gamma(t, p(t)).F(t)$. ▪

4.8 Homogeneous and isotropic material

Let $P \in \Omega_{t_0}$, let $F^{t_0}_t(P) := d\Phi^{t_0}_t(P)$; Suppose that the "Cauchy stress vector" $\vec f_t(p_t)$ à t at $p_t = \Phi^{t_0}_t(P)$ only depends on P and on $F^{t_0}_t(P)$ the first gradient at P, i.e. there exists a function $\vec{\text{fun}}$ such that

$$\vec f_t(p_t) = \vec{\text{fun}}(P, F^{t_0}_t(P)). \qquad (4.39)$$

Definition 4.12 A material is homogeneous iff $\vec{\text{fun}}$ doesn't depend on the first variable P of $\vec{\text{fun}}$, i.e., iff, for all $P \in \Omega_{t_0}$,

$$\vec{\text{fun}}(P, F^{t_0}_t(P)) = \vec{\text{fun}}(F^{t_0}_t(P)) \quad (= \vec f_t(p_t)). \qquad (4.40)$$

(Same mechanical property at any point.)

Definition 4.13 (Isotropy.) Consider a Euclidean dot product, the same at all time. A material is isotropic at $P \in \Omega_{t_0}$ iff $\vec{\text{fun}}$ is independent of the direction you consider, i.e., iff, for any rotation $R_{t_0}(P)$ in $\vec{\mathbb{R}}^n_{t_0}$,

$$\vec{\text{fun}}(P, F^{t_0}_t(P) = \vec{\text{fun}}(P, F^{t_0}_t(P).R_{t_0}(P)) \quad (= \vec{f}_t(p_t)). \tag{4.41}$$

(Mechanical property unchanged when rotating the material first.)

Definition 4.14 A material is isotropic homogeneous iff it is isotropic and homogeneous.

4.9 The inverse of the deformation gradient

$((\Phi^{t_0}_t)^{-1} \circ \Phi^{t_0}_t)(P) = P$ gives, with $p = \Phi^{t_0}_t(P)$,

$$d(\Phi^{t_0}_t)^{-1}(p).d\Phi^{t_0}_t(P) = I_{t_0}, \quad \text{thus} \quad d(\Phi^{t_0}_t)^{-1}(p) = d\Phi^{t_0}_t(P)^{-1} = F^{t_0}_t(P)^{-1}, \tag{4.42}$$

where $F^{t_0}_t = d\Phi^{t_0}_t$ is the deformation gradient. We have thus define the two point tensor

$$H^{t_0}_t := (F^{t_0}_t)^{-1} : \begin{cases} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0}) \\ p & \to \boxed{H^{t_0}_t(p) = (F^{t_0}_t)^{-1}(p) := (F^{t_0}_t(P))^{-1}} \quad \text{when} \quad p = \Phi^{t_0}_t(P). \end{cases} \tag{4.43}$$

So

$$H^{t_0}_t(p).\vec{w}(p) = (F^{t_0}_t)^{-1}(p).\vec{w}(p) := F^{t_0}_t(P)^{-1}.\vec{w}(p) \in \vec{\mathbb{R}}^n_{t_0}, \quad \text{in short} \quad H.\vec{w} = F^{-1}.\vec{w}, \tag{4.44}$$

for all $\vec{w}(p) \in \vec{\mathbb{R}}^n_t$ vector at p. This defines, with $p_t = \Phi^{t_0}(t, P)$,

$$H^{t_0} : \begin{cases} \mathcal{C} = \bigcup_t (\{t\} \times \Omega_t) & \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0}) \\ (t, p_t) & \to H^{t_0}(t, p_t) := H^{t_0}_t(p_t) = (F^{t_0}(t, P))^{-1}. \end{cases} \tag{4.45}$$

NB: H^{t_0} looks like a Eulerian map, but isn't: H^{t_0} depends on a initial time t_0 and is a two point tensor (starts in $\vec{\mathbb{R}}^n_t$, arrives in $\vec{\mathbb{R}}^n_{t_0}$). We will however use the material time derivative $\frac{D}{Dt}$ notation in this case, that is, we define, along a trajectory $t \to p(t) = \Phi^{t_0}(t, P)$,

$$\frac{DH^{t_0}}{Dt}(t, p(t)) := \frac{\partial H^{t_0}}{\partial t}(t, p(t)) + dH^{t_0}(t, p(t)).\vec{v}(t, p(t)), \quad \text{i.e.} \quad \frac{DH^{t_0}}{Dt} = \frac{\partial H^{t_0}}{\partial t} + dH^{t_0}.\vec{v}, \tag{4.46}$$

which is the time derivative $g'(t)$ of the function $g : t \to g(t) = H^{t_0}(t, \Phi^{t_0}(t, P))$ (i.e. $g(t) = H^{t_0}(t, p(t))$).

Hence, with $p(t) = \Phi^{t_0}(t, P)$ and $H^{t_0}(t, p(t)).F^{t_0}(t, P) = I_{t_0}$, written $H.F = I$, we get

$$\frac{DH}{Dt}.F + H.\frac{\partial F}{\partial t} = 0, \quad \text{thus} \quad \boxed{\frac{DH}{Dt} = -H.d\vec{v}}, \tag{4.47}$$

since $\frac{\partial F}{\partial t}(t, P).F^{-1}(t, p(t)) = d\vec{v}(t, p(t))$ cf. (4.33).

Exercice 4.15 With $\vec{w}_{t_0*}(t, p(t)) = F^{t_0}(t, P).\vec{W}(P)$, i.e. $H^{t_0}(t, p(t)).\vec{w}_{t_0*}(t, p(t)) = \vec{W}(P)$, when $p(t) = \Phi^{t_0}(t, P)$, prove (4.47).

Answer. $\frac{D\vec{w}_{t_0*}}{Dt}(t, p(t)) = d\vec{v}(t, p(t)).\vec{w}_{t_0*}(t, p(t))$, cf. (4.23); And $(H^{t_0}.\vec{w}_{t_0*})(t, p(t)) = \vec{W}(P)$ gives $\frac{DH^{t_0}}{Dt}.\vec{w}_{t_0*} + H^{t_0}.\frac{D\vec{w}_{t_0*}}{Dt} = 0$; Thus $\frac{DH^{t_0}}{Dt}.\vec{w}_{t_0*} + H^{t_0}.d\vec{v}.\vec{w}_{t_0*} = 0$, thus $\frac{DH}{Dt} = -H.d\vec{v}$. ▪

Exercice 4.16 Prove: $H^{t_0}_t = H^{t_0}_{t_1} \circ H^{t_1}_t$ and $\frac{DH^{t_0}}{Dt}(t, p(t)) = H^{t_0}_{t_1}(p_{t_1}).\frac{DH^{t_1}}{Dt}(t, p(t))$ for all t_0, t_1 with $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0})$.

Answer. We have $\Phi^{t_0}_t(p_{t_0}) = \Phi^{t_1}_t(\Phi^{t_0}_{t_1}(p_{t_0}))$, cf. (5.18), hence $F^{t_0}_t(p_{t_0}) = F^{t_1}_t(p_{t_1}).F^{t_0}_{t_1}(p_{t_0})$, thus $F^{t_0}_t(p_{t_0})^{-1} = F^{t_0}_{t_1}(p_{t_0})^{-1}.F^{t_1}_t(p_{t_1})^{-1}$, i.e. $H^{t_0}_t(p_t) = H^{t_0}_{t_1}(p_{t_1}).H^{t_1}_t(p(t))$, thus, $H^{t_0}(t, p(t)) = H^{t_0}_{t_1}(p_{t_1}).H^{t_1}(t, p(t))$, thus $\frac{DH^{t_0}}{Dt}(t, p(t)) = H^{t_0}_{t_1}(p_{t_1}).\frac{DH^{t_1}}{Dt}(t, p(t))$. ▪

5 Flow

5.1 Introduction: Motion versus flow

- Motion: A motion $\widetilde{\Phi} : (t, P_{Obj}) \to p_t = \widetilde{\Phi}(t, P_{Obj})$ locates at t a particle P_{Obj} in the affine space $\mathbb{R}^n$, cf. (1.5); From which the Eulerian velocity field $\vec{v}$ is deduced: $\vec{v}(t, p_t) := \frac{d\widetilde{\Phi}_{P_{Obj}}}{dt}(t, P_{Obj})$, cf. (2.4).
- Flow: A flow starts with a Eulerian velocity field $\vec{v}$, from which we deduce a motion by solving the ODE (ordinary differential equation) $\frac{d\Phi}{dt}(t) = \vec{v}(t, \Phi(t))$.

5.2 Definition

Let $\vec{v}:\left\{\begin{array}{ll}\mathbb{R}\times\mathbb{R}^n & \to \vec{\mathbb{R}^n}\\ (t,p) & \to \vec{v}(t,p)\end{array}\right\}$ be a unstationary vector field (e.g., a Eulerian velocity field which definition domain is $\mathcal{C}=\bigcup_{t\in[t_1,t_2]}(\{t\}\times\Omega_t))$. We look for maps $\Phi:\left\{\begin{array}{ll}\mathbb{R} & \to \mathbb{R}^n\\ t & \to p=\Phi(t)\end{array}\right\}$ which are locally (i.e. in the vicinity of some t_0) solutions of the ODE (ordinary differential equation)

$$\frac{d\Phi}{dt}(t)=\vec{v}(t,\Phi(t)),\quad \text{also written}\quad \frac{dp}{dt}(t)=\vec{v}(t,p(t)),\quad \text{or}\quad \frac{d\vec{x}}{dt}(t)=\vec{v}(t,\vec{x}(t)) \tag{5.1}$$

where $\vec{x}(t)=\overrightarrow{\mathcal{O}p(t)}$ after a choice of an origin. Also written $\frac{dp}{dt}=\vec{v}(t,p)$ or $\frac{d\vec{x}}{dt}=\vec{v}(t,\vec{x})$.

Definition 5.1 A solution Φ of (5.1) is a flow of $\vec{v}$; Also called an integral curve of $\vec{v}$ since (5.1) also reads $\Phi(t)=\int_{\tau=t_1}^{t}\vec{v}(\tau,\Phi(\tau))\,d\tau+\Phi(t_1)$.

Remark 5.2 Improper notation for (5.1):

$$\frac{dp}{dt}(t)\overset{\text{noted}}{=}\frac{dp(t)}{dt}\qquad (=\vec{v}(t,p(t))). \tag{5.2}$$

Question: If the notation $\frac{dp(t)}{dt}$ is used, then what is the meaning of $\frac{dp(f(t))}{dt}$?

Answer: It means, either $\frac{dp}{dt}(f(t))$, or $\frac{d(p\circ f)}{dt}(t)=\frac{dp}{dt}(f(t))\frac{df}{dt}(t)$: Ambiguous. So it is better to use $\frac{dp}{dt}(t)$, and to avoid $\frac{dp(t)}{dt}$, unless the context is clear (no composite functions). ▪

5.3 Cauchy–Lipschitz theorem

Let (t_0,p_{t_0}) be in the definition domain of $\vec{v}$. We look for Φ solution of "the ODE with initial condition (t_0,p_{t_0})", in some vicinity of t_0, i.e. such that

$$\frac{d\Phi}{dt}(t)=\vec{v}(t,\Phi(t))\quad\text{and}\quad \Phi(t_0)=p_{t_0}. \tag{5.3}$$

(The couple (t_0,p_{t_0}) is the initial condition, and the values t_0 and p_{t_0} are the initial conditions.)

Definition 5.3 Let $t_1,t_2\in\mathbb{R}$, $t_1<t_2$. Let Ω be an open set in $\mathbb{R}^n$ and $\overline{\Omega}$ its closure supposed to be a regular domain. Let $||.||$ be a norm in $\vec{\mathbb{R}^n}$. A continuous map $\vec{v}:[t_1,t_2]\times\overline{\Omega}\to\vec{\mathbb{R}^n}$ is Lipschitzian iff it is "space Lipschitzian, uniformly in time", that is, iff

$$\exists k>0,\ \forall t\in[t_1,t_2],\ \forall p,q\in\overline{\Omega},\ ||\vec{v}(t,q)-\vec{v}(t,p)||\le k||q-p||. \tag{5.4}$$

So, $\frac{||\vec{v}_t(q)-\vec{v}_t(p)||}{||q-p||}\le k$, for all t and all $p\ne q$ (the variations of $\vec{v}$ are bounded in space, uniformly in time).

Theorem 5.4 (and definifion) *(Cauchy-Lipschitz). If $\vec{v}:[t_1,t_2]\times\overline{\Omega}\to\vec{\mathbb{R}^n}$ is Lipschitzian and $(t_0,p_{t_0})\in]t_1,t_2[\times\Omega$, then there exists $\varepsilon=\varepsilon_{t_0,p_{t_0}}>0$ s.t. (5.3) has a unique solution $\Phi:]t_0-\varepsilon,t_0+\varepsilon[\to\mathbb{R}^n$, noted $\Phi_{p_{t_0}}^{t_0}$:*

$$\frac{d\Phi_{p_{t_0}}^{t_0}}{dt}(t)=\vec{v}(t,\Phi_{p_{t_0}}^{t_0}(t))\quad\textit{and}\quad \Phi_{p_{t_0}}^{t_0}(t_0)=p_{t_0}. \tag{5.5}$$

Moreover, if $\vec{v}$ is C^k then Φ is C^{k+1}.

Proof. See e.g. Arnold [2], or any ODE course. In particular $||\vec{v}||_\infty:=\sup_{t\in]t_0-\varepsilon,t_0+\varepsilon[,\ p\in\Omega}||\vec{v}(t,p)||_{\mathbb{R}^n}$ (maximum speed) exists since $\vec{v}\in C^0$ on the compact $[t_1,t_2]\times\overline{\Omega})$, see definition 5.3, hence we can choose $\varepsilon=\min(t_0-t_1,t_2-t_0,\frac{d(p_{t_0},\partial\Omega)}{||\vec{v}||_\infty})$ (the time needed to reach the border $\partial\Omega$ from p_{t_0}). ▪

We have thus defined the function, also called "a flow",

$$\Phi : \begin{cases}]t_1,t_2[\times]t_1,t_2[\times\Omega_{t_0} & \to \Omega \\ \qquad (t,t_0,p_{t_0}) & \to p = \Phi(t,t_0,p_{t_0}) := \Phi^{t_0}_{p_{t_0}}(t) \overset{\text{noted}}{=} \Phi(t;t_0,p_{t_0}). \end{cases} \tag{5.6}$$

And (5.5) reads

$$\frac{\partial\Phi}{\partial t}(t;t_0,p_{t_0}) = \vec{v}(t,\Phi(t;t_0,p_{t_0})), \quad \text{with} \quad \Phi(t_0;t_0,p_{t_0}) = p_{t_0}. \tag{5.7}$$

We have thus defined the function, also called "a flow",

$$\Phi^{t_0} : \begin{cases} [t_0{-}\varepsilon, t_0{+}\varepsilon] \times \Omega_{t_0} & \to \mathbb{R}^n \\ \qquad (t,p_{t_0}) & \to p = \Phi^{t_0}(t,p_{t_0}) := \Phi^{t_0}_{p_{t_0}}(t) : \end{cases} \tag{5.8}$$

And (5.5) reads

$$\frac{\partial\Phi^{t_0}}{\partial t}(t,p_{t_0}) = \vec{v}(t,\Phi^{t_0}(t,p_{t_0})), \quad \text{and} \quad \Phi^{t_0}(t_0,p_{t_0}) = p_{t_0}. \tag{5.9}$$

Other definition and notation (can be ambiguous): $\Phi_{t;t_0} = \Phi^{t_0}_t : \Omega_{t_0} \to \mathbb{R}^n$, and (5.7) is written

$$\frac{d\Phi_{t;t_0}(p_{t_0})}{dt} = \vec{v}(t,\Phi_{t;t_0}(p_{t_0})), \quad \text{and} \quad \Phi_{t_0;t_0}(p_{t_0}) = p_{t_0}. \tag{5.10}$$

Theorem 5.5 *Let $\vec{v}$ be Lipschitzian, let $t_0 \in]t_1,t_2[$, and let Ω_{t_0} be an open set s.t. $\Omega_{t_0} \subset\subset \Omega$ (i.e. there exists a compact set $K \in \mathbb{R}^n$ s.t. $\Omega_{t_0} \subset K \subset \Omega$). Then there exists $\varepsilon > 0$ s.t. a flow Φ^{t_0} exists on $]t_0{-}\varepsilon, t_0{+}\varepsilon[\times\Omega_{t_0}$.*

Proof. Let $d = d(K, \mathbb{R}^n{-}\Omega)$ (la distance of K to the border of Ω.

Let $||\vec{v}||_\infty := \sup_{t\in[t_1,t_2],p\in\overline{\Omega}} ||\vec{v}(t,p)||_{\mathbb{R}^n}$ (exists since $\vec{v} \in C^0$ on the compact $[t_1,t_2]\times\overline{\Omega}$).

Let $\varepsilon = \min(t_0{-}t_1, t_2{-}t_0, \frac{d}{||\vec{v}||_\infty})$ (less that the minimum time to reach the border from K at maximum speed $||v||_\infty$).

Let $p_{t_0} \in K$ and $t \in]t_0{-}\varepsilon, t_0{+}\varepsilon[$. Then $\Phi^{t_0}_{p_{t_0}}$ exists, cf.theorem 5.4, and $||\Phi^{t_0}_{p_{t_0}}(t) - \Phi^{t_0}_{p_{t_0}}(t_0)||_{\mathbb{R}^n} \le [t-t_0| \sup_{\tau\in]t_0-\varepsilon,t_0+\varepsilon[}(||(\Phi^{t_0}_{p_{t_0}})'(\tau)||_{\mathbb{R}^n})$ (mean value theorem since, $\vec{v}$ being C^0, Φ is C^1). Thus $||\Phi^{t_0}_{p_{t_0}}(t) - \Phi^{t_0}_{p_{t_0}}(t_0)||_{\mathbb{R}^n} \le [t-t_0|\,||v||_\infty$, thus $\Phi^{t_0}_{p_{t_0}}(t) \in \Omega$. Thus $\Phi^{t_0}_{p_{t_0}}$ exists on $]t_0{-}\varepsilon, t_0{+}\varepsilon[$, for all $p_{t_0} \in K$. ∎

Remark 5.6 The definition of a flow starts with a Eulerian velocity (independent of any initial time), and then, due to the introduction of initial conditions, leads to the Lagrangian functions Φ^{t_0}, cf. (5.8). Once again, Lagrangian functions are the result of Eulerian functions. ∎

5.4 Examples

Example 1 $\mathbb{R}^2$ with an origin $\mathcal{O}$, a Euclidean basis $(\vec{e}_1,\vec{e}_2)$ and $\Omega = [0,2]\times[0,1]$ (observation window). Let $p \in \mathbb{R}^2$, $\overrightarrow{\mathcal{O}p} =^{\text{noted}} \vec{x} = x\vec{e}_1 + y\vec{e}_2 =^{\text{noted}} (x,y)$. Let $t_1 = -1$, $t_2 = 1$, $t_0 \in]t_1,t_2[$, $a,b \in \mathbb{R}$, $a \neq 0$, and

$$\vec{v}(t,p) = \begin{cases} v^1(t,x,y) = ay, \\ v^2(t,x,y) = b\sin(t{-}t_0). \end{cases} \tag{5.11}$$

($b = 0$ corresponds to the stationary case = shear flow.) $\vec{x}(t_0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \overrightarrow{\mathcal{O}\Phi^{t_0}_{p_{t_0}}(t)}$ and (5.9) give

$$\begin{cases} \dfrac{dx}{dt}(t) = v^1(t,x(t),y(t)) = ay(t), \\ \dfrac{dy}{dt}(t) = v^2(t,x(t),y(t)) = b\sin(t{-}t_0), \end{cases} \quad \text{with} \quad \begin{cases} x(t_0) = x_0, \\ y(t_0) = y_0. \end{cases} \tag{5.12}$$

Thus

$$\vec{x}(t) = \overrightarrow{\mathcal{O}p(t)} = \overrightarrow{\mathcal{O}\Phi^{t_0}_{p_{t_0}}(t)} = \begin{pmatrix} x(t) = x_0 + a(y_0+b)(t{-}t_0) - ab\sin(t{-}t_0) \\ y(t) = y_0 + b - b\cos(t{-}t_0) \end{pmatrix}. \tag{5.13}$$

Example 2 Similar framework. Let $\omega > 0$ and consider (spin vector field)

$$\vec{v}(t,x,y) = \begin{pmatrix} -\omega y \\ \omega x \end{pmatrix} = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \overset{\text{noted}}{=} \vec{v}(x,y). \tag{5.14}$$

With $\overrightarrow{Op_{t_0}} = \vec{x}_{t_0} = \begin{pmatrix} x_{t_0} \\ y_{t_0} \end{pmatrix}$, $r_{t_0} = \sqrt{x_{t_0}^2 + y_{t_0}^2}$, and θ_0 s.t. $\vec{x}_{t_0} = \begin{pmatrix} x_{t_0} = r_{t_0}\cos(\omega t_0) \\ y_{t_0} = r_{t_0}\sin(\omega t_0) \end{pmatrix}$, the solution $\Phi^{t_0}_{p_{t_0}}$ of (5.9) is

$$\vec{x}(t) = \overrightarrow{Op(t)} = \overrightarrow{O\Phi^{t_0}_{p_{t_0}}(t)} = \begin{pmatrix} x(t) = r_{t_0}\cos(\omega t) \\ y(t) = r_{t_0}\sin(\omega t) \end{pmatrix}. \tag{5.15}$$

Indeed, $\begin{pmatrix} \frac{\partial x}{\partial t}(t,\vec{x}_0) \\ \frac{\partial y}{\partial t}(t,\vec{x}_0) \end{pmatrix} = \begin{pmatrix} v^1(t,x(t,\vec{x}_0),y(t,\vec{x}_0)) \\ v^2(t,x(t,\vec{x}_0),y(t,\vec{x}_0)) \end{pmatrix} = \begin{pmatrix} -\omega y(t,\vec{x}_0) \\ \omega x(t,\vec{x}_0) \end{pmatrix}$, thus $\frac{\partial x}{\partial t}(t,\vec{x}_0) = -\omega y(t,\vec{x}_0)$ and $\frac{\partial y}{\partial t}(t,\vec{x}_0) = \omega x(t,\vec{x}_0)$, thus $\frac{\partial^2 y}{\partial t^2}(t,\vec{x}_0) = -\omega^2 y(t,\vec{x}_0)$, hence y; Idem for x. Here $d\vec{v}(t,x,y) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \omega \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$ is the $\pi/2$-rotation composed with the homothety with ratio ω.

5.5 Composition of flows

Let $\vec{v}$ be a vector field on $\mathbb{R} \times \Omega$ and $\Phi^{t_0}_{p_{t_0}}$ solution of (5.5). We use the notations

$$p_t = \Phi^{t_0}_t(p_{t_0}) = \Phi_{t;t_0}(p_{t_0}) := \Phi^{t_0}_{p_{t_0}}(t) = \Phi^{t_0}(t,p_{t_0}) = \Phi(t;t_0,p_{t_0}) = \Phi_{t_0,p_{t_0}}(t). \tag{5.16}$$

5.5.1 Law of composition of flows (determinism)

Proposition 5.7 *For all $t_0, t_1, t_2 \in \mathbb{R}$, we have (determinism)*

$$\Phi^{t_1}_{t_2} \circ \Phi^{t_0}_{t_1} = \Phi^{t_0}_{t_2}, \quad \text{i.e.} \quad \Phi_{t_2;t_1} \circ \Phi_{t_1;t_0} = \Phi_{t_2;t_0}. \tag{5.17}$$

("The composition of the photos gives the film"). So,

$$p_{t_2} = \Phi^{t_1}_{t_2}(p_{t_1}) = \Phi^{t_0}_{t_2}(p_{t_0}) \quad \textit{when} \quad p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0}), \tag{5.18}$$

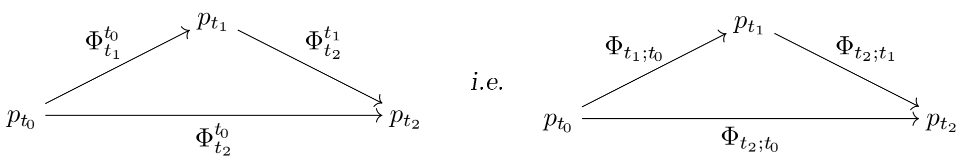
i.e.,

$$p_{t_2} = \Phi_{t_2;t_1}(p_{t_1}) = \Phi_{t_2;t_0}(p_{t_0}) \quad \textit{when} \quad p_{t_1} = \Phi_{t_1;t_0}(p_{t_0}). \tag{5.19}$$

Thus

$$d\Phi^{t_1}_{t_2}(p_{t_1}).d\Phi^{t_0}_{t_1}(p_{t_0}) = d\Phi^{t_0}_{t_2}(p_{t_0}), \quad \text{i.e.} \quad d\Phi_{t_2;t_1}(p_{t_1}).d\Phi_{t_1;t_0}(p_{t_0}) = d\Phi_{t_2;t_0}(p_{t_0}). \tag{5.20}$$

Summary with commutative diagrams:



Proof. Let $p_{t_1} = \Phi^{t_0}_{p_{t_0}}(t_1)$. (5.9) gives

$$\left\{ \begin{aligned} \frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t) = \vec{v}(t, \Phi^{t_0}_{p_{t_0}}(t)), \\ \frac{d\Phi^{t_1}_{p_{t_1}}}{dt}(t) = \vec{v}(t, \Phi^{t_1}_{p_{t_1}}(t)), \end{aligned} \right\} \quad \text{with} \quad p_{t_1} = \Phi^{t_0}_{p_{t_0}}(t_1) = \Phi^{t_1}_{p_{t_1}}(t_1).$$

Thus $\Phi^{t_0}_{p_{t_0}}$ and $\Phi^{t_1}_{p_{t_1}}$ satisfy the same ODE with the same value at t_1; Thus they are equal (uniqueness thanks to Cauchy–Lipschitz theorem), thus $\Phi^{t_1}_{p_{t_1}}(t) = \Phi^{t_0}_{p_{t_0}}(t)$ when $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0})$, that is, $\Phi^{t_1}_t(p_{t_1}) = \Phi^{t_0}_t(p_{t_0})$ when $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0})$, which is (5.17) for any $t = t_2$. Thus (5.20). ∎

Corollary 5.8 *A flow is compatible with the motion $\widetilde{\Phi}$ of an object Obj: (3.6) gives $\Phi^{t_1}_{t_2} \circ \Phi^{t_0}_{t_1} = (\widetilde{\Phi}_{t_2} \circ (\widetilde{\Phi}_{t_1})^{-1}) \circ (\widetilde{\Phi}_{t_1} \circ (\widetilde{\Phi}_{t_0})^{-1}) = \widetilde{\Phi}_{t_2} \circ (\widetilde{\Phi}_{t_0})^{-1} = \Phi^{t_0}_{t_2}$, that is (5.17).*

5.5.2 Stationnary case

Definition 5.9 $\vec{v}$ is a stationary vector field iff $\frac{\partial \vec{v}}{\partial t}=0$; And then $\vec{v}(t,p) =^{\text{noted}} \vec{v}(p)$. And the associated flow Φ^{t_0} which satisfies

$$\frac{\partial \Phi^{t_0}}{\partial t}(t,p_{t_0}) = \vec{v}(p_t) \quad \text{when} \quad p_t = \Phi^{t_0}(t,p_{t_0}), \tag{5.21}$$

is said to be stationary.

Proposition 5.10 *If $\vec{v}$ is a stationary vector field then, for all t_0,t_1,h, when meaningful (h small enough and t_1 close enough to t_0),*

$$\Phi^{t_1}_{t_1+h} = \Phi^{t_0}_{t_0+h}, \quad \text{i.e.} \quad \Phi_{t_1+h;t_1} = \Phi_{t_0+h;t_0}, \tag{5.22}$$

i.e. $\Phi^{t_1}_{t_1+h}(q) = \Phi^{t_0}_{t_0+h}(q)$, i.e. $\Phi(t_1{+}h;t_1,q) = \Phi(t_0{+}h;t_0,q)$ for all $q \in \Omega_{t_0}$ (see theorem 5.5). In other words,

$$\Phi^{t_0+h}_{t_1+h} = \Phi^{t_0}_{t_1}, \quad \text{i.e.} \quad \Phi_{t_1+h;t_0+h} = \Phi_{t_1;t_0}, \tag{5.23}$$

i.e. $\Phi^{t_0+h}_{t_1+h}(q) = \Phi^{t_0}_{t_1}(q)$, i.e. $\Phi(t_1{+}h;t_0{+}h,q) = \Phi(t_1;t_0,q)$ for all $q \in \Omega_{t_0}$.

Proof. Let $q \in \Omega_{t_0}$, $\alpha(h) = \Phi^{t_0}_{t_0+h}(q) = \Phi^{t_0}_q(t_0{+}h)$ and $\beta(h) = \Phi^{t_1}_{t_1+h}(q) = \Phi^{t_1}_q(t_1{+}h)$.

Thus $\alpha'(h) = \frac{d\Phi^{t_0}_q}{dt}(t_0{+}h) = \vec{v}(t_0{+}h, \Phi^{t_0}_q(t_0{+}h)) = \vec{v}(\Phi^{t_0}_q(t_0{+}h)) = \vec{v}(\alpha(h))$ (stationary flow), and $\beta'(h) = \frac{d\Phi_{t_1 q}}{dt}(t_1{+}h) = \vec{v}(t_1{+}h, \Phi^{t_1}_q(t_1{+}h)) = \vec{v}(\Phi^{t_1}_q(t_1{+}h)) = \vec{v}(\beta(h))$ (stationary flow).

Thus α and β satisfy the same ODE with the same initial condition $\alpha(0) = \beta(0) = q$. Thus $\alpha = \beta$. Hence (5.22). Thus, with $h = t_1{-}t_0$, i.e. with $t_1 = t_0{+}h$ and $t_0{+}h = t_1$, we get (5.23). ∎

Corollary 5.11 *If $\vec{v}$ is a stationary vector field, cf. (5.21), then*

$$d\Phi^{t_0}_t(p_{t_0}).\vec{v}(p_{t_0}) = \vec{v}(p_t) \quad \text{when} \quad p_t = \Phi^{t_0}_t(p_{t_0}), \tag{5.24}$$

that is, if $\vec{v}$ is stationary, then $\vec{v}$ is transported (push-forwarded by $\Phi^{t_0}_t$) along itself.

Proof. (5.18), $t_2 = t_1{+}s$ and $t_1 = t_0{+}s$ give $\Phi^{t_0+s}_{t_1+s}(\Phi^{t_0}_{t_0+s}(p_{t_0})) = \Phi^{t_0}_{t_1+s}(p_{t_0})$, and $\vec{v}$ is stationary, thus $\Phi^{t_0}_{t_1}(\Phi^{t_0}_{t_0+s}(p_{t_0})) = \Phi^{t_0}_{t_1+s}(p_{t_0})$, i.e. $\Phi(t_1;t_0,\Phi_{t_0,p_{t_0}}(t_0{+}s)) = \Phi_{t_0,p_{t_0}}(t_1{+}s)$, thus ($s$ derivative)

$$d\Phi(t_1;t_0,\Phi(t_0{+}s;t_0,p_{t_0})).{\Phi_{t_0,p_{t_0}}}'(t_0{+}s) = {\Phi_{t_0,p_{t_0}}}'(t_1{+}s),$$

thus $d\Phi^{t_0}_{t_1}(\Phi(t_0{+}s;t_0,p_{t_0})).\vec{v}(t_0{+}s,\Phi_{t_0,p_{t_0}}(t_0{+}s)) = \vec{v}(t_1{+}s,\Phi_{t_0,p_{t_0}}(t_1{+}s))$. Thus with $s=0$, and $\vec{v}$ being stationary, $d\Phi^{t_0}_{t_1}(\Phi(t_0;t_0,p_{t_0})).\vec{v}(\Phi_{t_0,p_{t_0}}(t_0)) = \vec{v}(\Phi_{t_0,p_{t_0}}(t_1))$, thus (5.24). ∎

5.6 Velocity on the trajectory traveled in the opposite direction

Let $t_0,t_1 \in \mathbb{R}$, $t_1 > t_0$, and $p_{t_0} \in \mathbb{R}^n$. Consider the trajectory $\Phi^{t_0}_{p_{t_0}} : \left\{ \begin{array}{ll} [t_0,t_1] & \to \mathbb{R}^n \\ t & \to p(t) = \Phi^{t_0}_{p_{t_0}}(t) \end{array} \right\}$. So p_{t_0} is the beginning of the trajectory, $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0})$ at the end, $\vec{v}(t,p(t)) = \frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t)$ being the velocity.

Define the trajectory traveled in the opposite direction, i.e. define

$$\Psi^{t_1}_{p_{t_1}} : \left\{ \begin{array}{ll} [t_0,t_1] & \to \mathbb{R}^n \\ u & \to q(u) = \Psi^{t_1}_{p_{t_1}}(u) := \Phi^{t_0}_{p_{t_0}}(t_0{+}t_1{-}u) = \Phi^{t_0}_{p_{t_0}}(t) = p(t) \quad \text{when} \quad t = t_0{+}t_1{-}u. \end{array} \right. \tag{5.25}$$

In particular $q(t_0) = \Psi^{t_1}_{p_{t_1}}(t_0) = \Phi^{t_0}_{p_{t_0}}(t_1) = p(t_1)$ and $q(t_1) = \Psi^{t_1}_{p_{t_1}}(t_1) = \Phi^{t_0}_{p_{t_0}}(t_0) = p(t_0)$.

Proposition 5.12 *The velocity on the trajectory traveled in the opposite direction is the opposite of the velocity on the initial trajectory:*

$$\frac{d\Psi^{t_1}_{p_{t_1}}}{du}(u) = q'(u) = -p'(t) = -\vec{v}(t,p(t)) \quad \text{when} \quad t = t_0{+}t_1{-}u, \tag{5.26}$$

Proof. $\Psi^{t_1}_{p_{t_1}}(u) = \Phi^{t_0}_{p_{t_0}}(t_0{+}t_1{-}u)$ gives $\frac{d\Psi^{t_1}_{p_{t_1}}}{du}(u) = -\frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t_0{+}t_1{-}u) = -\vec{v}(t_0{+}t_1{-}u, \Phi^{t_0}_{p_{t_0}}(t_0{+}t_1{-}u)) = -\vec{v}(t,\Phi^{t_0}_{p_{t_0}}(t))$ when $t = t_0{+}t_1{-}u$. ∎

5.7 Variation of the flow as a function of the initial time

5.7.1 Ambiguous and non ambiguous notations

Let $\Phi : (t,u,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \Phi(t,u,p) \in \mathbb{R}^n$ be a C^1 function. The partial derivatives are

$$\partial_1\Phi(t,u,p) := \lim_{h\to 0} \frac{\Phi(t{+}h,u,p)-\Phi(t,u,p)}{h}, \tag{5.27}$$

$$\partial_2\Phi(t,u,p) := \lim_{h\to 0} \frac{\Phi(t,u{+}h,p)-\Phi(t,u,p)}{h}, \tag{5.28}$$

and $\partial_3\Phi(t,u,p)$, defined for all $\vec{w}\in\vec{\mathbb{R}^n}$ (a vector at p) by,

$$\partial_3\Phi(t,u,p).\vec{w} := \lim_{h\to 0} \frac{\Phi(t,u,p{+}h\vec{w})-\Phi(t,u,p)}{h} \overset{\text{noted}}{=} d\Phi(t,u,p).\vec{w}, \tag{5.29}$$

When the name of the first variable is systematically noted t, then

$$\partial_1\Phi(t,u,p) \overset{\text{noted}}{=} \frac{\partial\Phi}{\partial t}(t,u,p) \overset{\text{noted}}{=} \frac{\partial\Phi(t,u,p)}{\partial t}. \tag{5.30}$$

NB: This notation can be ambiguous: What is the meaning of $\frac{\partial\Phi}{\partial t}(t;t,p)$? In ambiguous situations, use the notation $\partial_1\Phi$, or (if no composed functions inside) use $\frac{\partial\Phi(t,u,p)}{\partial t}_{|u=t}$ (so t is the derivation variable, and after the calculation you take $u=t$).

When the name of the second variable is systematically noted u, then

$$\partial_2\Phi(t,u,p) \overset{\text{noted}}{=} \frac{\partial\Phi}{\partial u}(t,u,p) \overset{\text{noted}}{=} \frac{\partial\Phi(t,u,p)}{\partial u}. \tag{5.31}$$

NB: Idem this notation can be ambiguous: What is the meaning of $\frac{\partial\Phi}{\partial u}(u;u,p)$? In ambiguous situations, use the notation $\partial_2\Phi$, or use $\frac{\partial\Phi(t,u,p)}{\partial u}_{|t=u}$.

When the name of the third variable is systematically a space variable noted p, then

$$\partial_3\Phi(t,u,p) \overset{\text{noted}}{=} d\Phi(t,u,p) \overset{\text{noted}}{=} \frac{\partial\Phi}{\partial p}(t,u,p) \overset{\text{noted}}{=} \frac{\partial\Phi(t,u,p)}{\partial p}. \tag{5.32}$$

5.7.2 Variation of the flow as a function of the initial time

The law of composition of the flows gives (5.19) gives $\Phi(t;u,\Phi(u;t_0,p_0)) = \Phi(t;t_0,p_0)$. Thus the derivative in u gives

$$\begin{aligned} &\partial_2\Phi(t;u,\Phi(u;t_0,p_0)) + d\Phi(t;u,\Phi(u;t_0,p_0)).\partial_1\Phi(u;t_0,p_0) = 0,\\ \text{i.e.}\quad &\partial_2\Phi(t;u,p(u)) = -d\Phi(t;u,p(u)).\vec{v}(u,p(u)) \quad\text{when}\quad p(u)=\Phi(u;t_0,p_0). \end{aligned} \tag{5.33}$$

In particular $u=t_0$ gives, for all $(t,t_0,p_0)\in\mathbb{R}^2\times\Omega_{t_0}$,

$$(\frac{\partial\Phi(t;t_0,p_0)}{\partial t_0} =) \quad \partial_2\Phi(t;t_0,p_0) = -d\Phi(t;t_0,p_0).\vec{v}(t_0,p_0). \tag{5.34}$$

In particular

$$(\frac{d\Phi(t;t_0,p_0)}{dt_0}_{|t=t_0} =) \quad \partial_2\Phi(t_0;t_0,p_0) = -\vec{v}(t_0,p_0). \tag{5.35}$$

Part II
Push-forward

6 Push-forward

The general tool to describe "transport" is "push-forward by a motion" (the "take with you" operator), cf. § 4.1 and figure 4.1. The push-forward also gives the tool needed to understand the velocity addition formula: In that case, the push-forward is the translator between observers. The push-forward can also be used to write coordinate systems. As usual, we start with qualitative results (observer independent results); Then, quantitative results are deduced.

6.1 Definition

$\mathcal{E}$ and $\mathcal{F}$ are affine spaces, E and F are the associated vector spaces equipped with norms $||.||_E$ and $||.||_F$ with $\dim E = \dim F = n$, $\mathcal{U}_\mathcal{E}$ and $\mathcal{U}_\mathcal{F}$ are open sets in the affine space $\mathcal{E}$ and $\mathcal{F}$, or possibly the vector spaces E and F, and

$$\Psi : \begin{cases} \mathcal{U}_\mathcal{E} & \to \mathcal{U}_\mathcal{F} \\ p_\mathcal{E} & \to p_\mathcal{F} = \Psi(p_\mathcal{E}) \quad \text{is a diffeomorphism} \end{cases} \tag{6.1}$$

(a C^1 invertible map which inverse is C^1), called the push-forward, and Ψ^{-1} is the pull-back (push-forward with Ψ^{-1}).

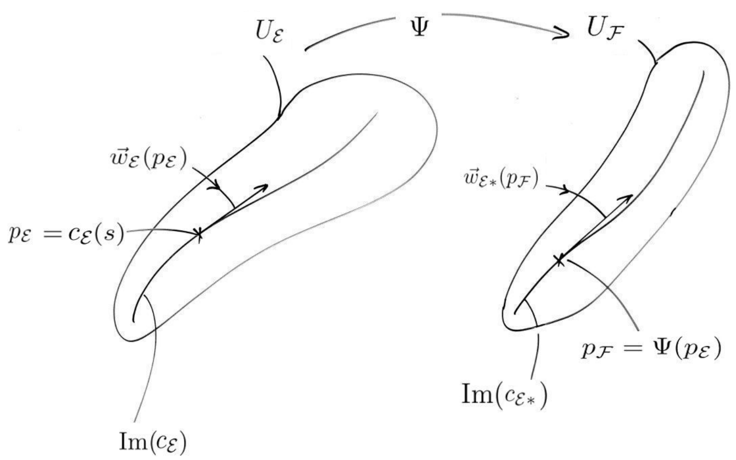


Figure 6.1: $c_\mathcal{E} : s \to p_\mathcal{E} = c_\mathcal{E}(s)$ is a curve in $\mathcal{U}_\mathcal{E}$. Push-forwarded by Ψ it becomes the curve $c_{\mathcal{E}*} := \Psi \circ c_\mathcal{E}$ in $\mathcal{U}_\mathcal{F}$. The tangent vector at $p_\mathcal{E} = c_\mathcal{E}(s)$ is $\vec{w}_\mathcal{E}(p_\mathcal{E}) = c_\mathcal{E}{}'(s)$, and the tangent vector at $p_\mathcal{F} = c_\mathcal{F}(s) = \Psi(c_\mathcal{E}(s))$ is $\vec{w}_{\mathcal{E}*}(p_\mathcal{F}) = c_\mathcal{F}{}'(s) = d\Psi(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E})$. Other illustation: See figure 4.1.

Example: $\Psi = \Phi_t^{t_0} : \Omega_{t_0} \to \Omega_t$, the motion that transforms Ω_{t_0} into Ω_t, cf. (3.5).
Example: $\Psi : U_E \to U_F$ a coordinate system, see example 6.11.
Example: $\Psi = \Theta_t : \mathcal{R}_B \to \mathcal{R}_A$, a change of referential at t (change of observer), see § 10.

6.2 Push-forward and pull-back of points

Definition 6.1 If $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$ (a point in $\mathcal{U}_\mathcal{E}$) then its push-forward by Ψ is the point

$$p_\mathcal{F} = \boxed{\Psi_* p_\mathcal{E} := \Psi(p_\mathcal{E})} = p_{\mathcal{E}*} \ \in \mathcal{U}_\mathcal{F}, \tag{6.2}$$

see figure 6.1, the last notation if Ψ is implicit. And if $p_\mathcal{F} \in \mathcal{U}_\mathcal{F}$ then its pull-back by Ψ is the point

$$p_\mathcal{E} = \boxed{\Psi^* p_\mathcal{F} := \Psi^{-1}(p_\mathcal{F})} = p_\mathcal{F}{}^* \ \in \mathcal{U}_\mathcal{E}. \tag{6.3}$$

We immediately have $\Psi^* \circ \Psi_* = I$.

The notations $_*$ for push-forward and * for pull-back have been proposed by Spivak; Also see Abraham and Marsden [1] (second edition) who adopt this notation.

6.3 Push-forward and pull-back of curves

We push-forward (and pull-back) the points on a curve:

Definition 6.2 Let $c_{\mathcal{E}} : \left\{ \begin{array}{ll}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal{E}} \\ s & \to p_{\mathcal{E}} = c_{\mathcal{E}}(s) \end{array} \right\}$ be a curve in $\mathcal{U}_{\mathcal{E}}$. Its push-forward by Ψ is the curve

$$\Psi_* c_{\mathcal{E}} := \Psi \circ c_{\mathcal{E}} : \begin{cases}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal{F}} \\ s & \to p_{\mathcal{F}} = \Psi_* c_{\mathcal{E}}(s) := \Psi(c_{\mathcal{E}}(s)) \overset{\text{noted}}{=} c_{\mathcal{E}*}(s) \quad (= \Psi(p_{\mathcal{E}})), \end{cases} \tag{6.4}$$

see figure 6.1. ($\Psi_* c_{\mathcal{E}} =^{\text{noted}} c_{\mathcal{E}*}$ when Ψ is implicit.) This defines

$$\Psi_* : \begin{cases} \mathcal{F}(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{E}}) & \to \mathcal{F}(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{F}}) \\ c_{\mathcal{E}} & \to \Psi_*(c_{\mathcal{E}}) := \Psi \circ c_{\mathcal{E}} \overset{\text{noted}}{=} \Psi_* c_{\mathcal{E}} = c_{\mathcal{E}*}. \end{cases} \tag{6.5}$$

Definition 6.3 Let $c_{\mathcal{F}} : \left\{ \begin{array}{ll}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal{F}} \\ s & \to p_{\mathcal{F}} = c_{\mathcal{F}}(s) \end{array} \right\}$ is a curve in $\mathcal{U}_{\mathcal{F}}$. Its pull-back by Ψ is

$$\Psi^* c_{\mathcal{F}} := \Psi^{-1} \circ c_{\mathcal{E}} \begin{cases}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal{E}} \\ s & \to p_{\mathcal{E}} = \Psi^* c_{\mathcal{F}}(s) := \Psi^{-1}(c_{\mathcal{F}}(s)) \overset{\text{noted}}{=} c_{\mathcal{F}}{}^*(s) \quad (= \Psi^{-1}(p_{\mathcal{F}})). \end{cases} \tag{6.6}$$

We have thus defined

$$\Psi^* : \begin{cases} \mathcal{F}(C^1(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{F}}) & \to \mathcal{F}(C^1(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{E}}) \\ c_{\mathcal{F}} & \to \Psi^*(c_{\mathcal{F}}) := \Psi^{-1} \circ c_{\mathcal{F}} \overset{\text{noted}}{=} \Psi^* c_{\mathcal{F}} = c_{\mathcal{F}}{}^*. \end{cases} \tag{6.7}$$

6.4 Push-forward and pull-back of scalar functions

6.4.1 Definitions

Definition 6.4 Let $f_{\mathcal{E}} : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{E}} & \to \mathbb{R} \\ p_{\mathcal{E}} & \to f_{\mathcal{E}}(p_{\mathcal{E}}) \end{array} \right\}$ (scalar valued function). Its push-forward by Ψ is the (scalar valued) function

$$\Psi_* f_{\mathcal{E}} := f_{\mathcal{E}} \circ \Psi^{-1} : \begin{cases} \mathcal{U}_{\mathcal{F}} & \to \mathbb{R} \\ p_{\mathcal{F}} & \to \Psi_* f_{\mathcal{E}}(p_{\mathcal{F}}) := f_{\mathcal{E}}(p_{\mathcal{E}}) \overset{\text{noted}}{=} f_{\mathcal{E}*}(p_{\mathcal{F}}) \quad \text{when} \quad p_{\mathcal{E}} = \Psi^{-1}(p_{\mathcal{F}}), \end{cases} \tag{6.8}$$

(noted $f_{\mathcal{E}*}$ when Ψ is implicit), i.e. $\Psi_* f_{\mathcal{E}}(\Psi_* p_{\mathcal{E}}) := f_{\mathcal{E}}(p_{\mathcal{E}})$, or $f_{\mathcal{E}*}(p_{\mathcal{E}*}) := f_{\mathcal{E}}(p_{\mathcal{E}})$ when $p_{\mathcal{E}*} = \Psi(p_{\mathcal{E}})$. We have thus defined

$$\Psi_* : \begin{cases} \mathcal{F}(\mathcal{U}_{\mathcal{E}};\mathbb{R}) & \to \mathcal{F}(\mathcal{U}_{\mathcal{F}};\mathbb{R}) \\ f_{\mathcal{E}} & \to f_{\mathcal{F}} := \Psi_*(f_{\mathcal{E}}) = f_{\mathcal{E}} \circ \Psi^{-1} \overset{\text{noted}}{=} \Psi_* f_{\mathcal{E}}, \end{cases} \tag{6.9}$$

the notation $\Psi_*(f_{\mathcal{E}}) = \Psi_* f_{\mathcal{E}}$ since Ψ_* is linear: $((f_{\mathcal{E}} + \lambda g_{\mathcal{E}}) \circ \Psi^{-1})(p_{\mathcal{F}}) = (f_{\mathcal{E}} + \lambda g_{\mathcal{E}})(p_{\mathcal{E}}) = f_{\mathcal{E}}(p_{\mathcal{E}}) + \lambda g_{\mathcal{E}}(p_{\mathcal{E}}) = (f_{\mathcal{E}} \circ \Psi^{-1})(p_{\mathcal{F}}) + \lambda (g_{\mathcal{E}} \circ \Psi^{-1})(p_{\mathcal{F}})$ gives $\Psi_*(f_{\mathcal{E}} + \lambda g_{\mathcal{E}}) = \Psi_*(f_{\mathcal{E}}) + \lambda \Psi_*(g_{\mathcal{E}})$.

Definition 6.5 Let $f_{\mathcal{F}} : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{F}} & \to \mathbb{R} \\ p_{\mathcal{F}} & \to f_{\mathcal{F}}(p_{\mathcal{F}}) \end{array} \right\}$. Its pull-back by Ψ is the push-forward by Ψ^{-1}, i.e. is

$$\Psi^* f_{\mathcal{F}} := f_{\mathcal{F}} \circ \Psi : \begin{cases} \mathcal{U}_{\mathcal{E}} & \to \mathbb{R} \\ p_{\mathcal{E}} & \to \Psi^* f_{\mathcal{F}}(p_{\mathcal{E}}) := f_{\mathcal{F}}(p_{\mathcal{F}}) \overset{\text{noted}}{=} f_{\mathcal{F}}{}^*(p_{\mathcal{E}}) \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}), \end{cases} \tag{6.10}$$

i.e. $\Psi^* f_{\mathcal{F}}(\Psi^* p_{\mathcal{F}}) := f_{\mathcal{F}}(p_{\mathcal{F}})$, i.e. $f_{\mathcal{F}}{}^*(p_{\mathcal{F}}{}^*) := f_{\mathcal{F}}(p_{\mathcal{F}})$ when $p_{\mathcal{F}} = \Psi^*(p_{\mathcal{F}})$. We have thus defined

$$\Psi^* : \begin{cases} \mathcal{F}(\mathcal{U}_{\mathcal{F}};\mathbb{R}) & \to \mathcal{F}(\mathcal{U}_{\mathcal{E}};\mathbb{R}) \\ f_{\mathcal{F}} & \to \Psi^*(f_{\mathcal{F}}) = f_{\mathcal{F}}{}^* := f_{\mathcal{F}} \circ \Psi \overset{\text{noted}}{=} \Psi^* f_{\mathcal{F}}. \end{cases} \tag{6.11}$$

We immediately have $\Psi^* \circ \Psi_* = I$ and $\Psi_* \circ \Psi^* = I$ (the first I is the identity in $\mathcal{F}(\mathcal{U}_{\mathcal{E}};\mathbb{R})$, the second I is the identity in $\mathcal{F}(\mathcal{U}_{\mathcal{F}};\mathbb{R})$).

NB: We used the same notations Ψ_* and Ψ^* than for the push-forward and pull-backs of points: The context removes ambiguities.

6.4.2 Interpretation: Why is it useful?

E.g.: Let $\widetilde{\Phi} : \mathbb{R} \times Obj \to \mathbb{R}^n$ be a motion of an object Obj. An observer records the temperature θ at all $t \in [t_0, T]$ and all $p = \widetilde{\Phi}(t, Obj)$: He gets $\theta : \left\{ \begin{array}{ll} \mathcal{C} = \bigcup_t (\{t\} \times \Omega_t) & \to \mathbb{R} \\ (t,p) & \to \theta(t,p) \end{array} \right\}$ a Eulerian scalar valued function, cf. (2.2). Then he chooses an initial time t_0 and considers the associated motion Φ^{t_0}, cf. (3.1), and considers $\theta_{t_0} : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \mathbb{R} \\ p_{t_0} & \to \theta_{t_0}(p_{t_0}) := \theta(t_0, p_{t_0}) \end{array} \right\}$ (snapshot of the temperatures at t_0 in Ω_{t_0}). The push-forward of θ_{t_0} by $\Phi_t^{t_0}$ is $(\Phi_t^{t_0})_* \theta_{t_0} := \theta_{t_0} \circ (\Phi_t^{t_0})^{-1}$ defines the "memory function"

$$(\Phi_t^{t_0})_* \theta_{t_0} : \left\{ \begin{array}{ll} \Omega_t & \to \mathbb{R} \\ p_t & \to (\Phi_t^{t_0})_* \theta_{t_0}(p_t) := \theta_{t_0}(p_{t_0}) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}), \end{array} \right. \tag{6.12}$$

And he writes $(\Phi_t^{t_0})_* \theta_{t_0}(p_t) =^{\text{noted}} \theta_{t_0 *}(t, p_t)$, so the memory transported is at t at p_t (along a trajectory) by

$$\theta_{t_0 *}(t, p(t)) = \theta_{t_0}(p_{t_0}). \tag{6.13}$$

Question: Why do we introduce $\theta_{t_0 *}$ since we have θ_{t_0}?

Answer: An observer does not have the gift of temporal and/or spatial ubiquity; He has to do with values at the actual time t and position p_t where he is (Newton and Einstein's point of view). So, when he was at t_0 at p_{t_0} the observer wrote the value $\theta_{t_0}(p_{t_0})$ on a piece of paper (for memory), puts the piece of paper is his pocket, then once at t at $p(t) = \Phi^{t_0}(t, p_{t_0})$, he takes the paper out of his pocket, and renames the value he reads as $\theta_{t_0 *}(t, p_t)$ because he is now at t at p_t. And, now at t at p_t, he can compare the past and present value. In particular the rate

$$\frac{\theta(t, p(t)) - \theta_{t_0 *}(t, p(t))}{t - t_0} = \frac{\text{actual}(t, p(t)) - \text{memory}_*(t, p(t))}{t - t_0} \tag{6.14}$$

is physically meaningful for one observer at t at p_t (no ubiquity gift required). For scalar value functions, we get the usual rate $\frac{\theta(t,p(t)) - \theta(t_0, p(t_0))}{t - t_0}$, but it isn't that simple for vector valued functions.

And the limit $t \to t_0$ in (6.14) defines the Lie derivative for scalar valued functions.

6.5 Push-forward and pull-back of vector fields

This is one of the most important concept for mechanical engineers.

6.5.1 A definition by approximation

Elementary introduction. Let $p_{\mathcal{E}}$ and $q_{\mathcal{E}}$ be points in $\mathcal{U}_{\mathcal{E}}$, and let $p_{\mathcal{F}} = p_{\mathcal{E}*} = \Psi(p_{\mathcal{E}})$ and $q_{\mathcal{F}} = q_{\mathcal{E}*} = \Psi(q_{\mathcal{E}})$ in $\mathcal{U}_{\mathcal{F}}$ be the push-forwards by Ψ cf. (6.1). The first order Taylor expansion gives

$$(\Psi(q_{\mathcal{E}}) - \Psi(p_{\mathcal{E}}) =) \quad q_{\mathcal{F}} - p_{\mathcal{F}} = d\Psi(p_{\mathcal{E}}).(q_{\mathcal{E}} - p_{\mathcal{E}}) + o(||q_{\mathcal{E}} - p_{\mathcal{E}}||_E), \tag{6.15}$$

thus,

$$\frac{\overrightarrow{p_{\mathcal{F}} q_{\mathcal{F}}}}{||\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}||_E} = d\Psi(p_{\mathcal{E}}).\frac{\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}}{||\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}||_E} + o(1). \tag{6.16}$$

And the definition of the push-forward is obtained by "neglecting" the $o(1)$ (limit as $q_{\mathcal{E}} \to p_{\mathcal{E}}$):

Definition 6.6 If $\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \in E$ is a vector at $p_{\mathcal{E}} \in U$ then its push-forward by Ψ is the vector $\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) =^{\text{noted}} \vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) =^{\text{noted}} \Psi_* \vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) \in F$ defined at $p_{\mathcal{F}} = p_{\mathcal{E}*} = \Psi(p_{\mathcal{E}}) \in \mathcal{U}_{\mathcal{F}}$ by

$$\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) = \boxed{\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) := d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})} \overset{\text{noted}}{=} \Psi_* \vec{w}_{\mathcal{E}}(p_{\mathcal{F}}). \tag{6.17}$$

6.5.2 The definition of the push-forward of a vector field

To fully grasp the definition, and to avoid making interpretation errors as in § 4.3 (the unfortunate notation $d\vec{x} = F.d\vec{X}$), we use the following definition of "a vector": It is a "tangent vector to a curve" (needed for surfaces and manifolds). Details:

• Let $c_{\mathcal{E}} : \left\{ \begin{array}{rl}]-\varepsilon,\varepsilon[& \to \mathcal{U}_{\mathcal{E}} \\ s & \to p_{\mathcal{E}} = c_{\mathcal{E}}(s) \end{array} \right\}$ be a C^1 curve in $\mathcal{U}_{\mathcal{E}}$. Its tangent vector at $p_{\mathcal{E}} = c_{\mathcal{E}}(s)$ is

$$\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) := c_{\mathcal{E}}'(s) \quad (= \lim_{h\to 0} \frac{c_{\mathcal{E}}(s+h) - c_{\mathcal{E}}(s)}{h}), \tag{6.18}$$

see figure 6.1. This defines the function $\vec{w}_{\mathcal{E}} : \left\{ \begin{array}{rl} \text{Im}(c_{\mathcal{E}}) & \to E \\ p_{\mathcal{E}} & \to \vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \end{array} \right\}$ called a vector field along $\text{Im}(c_{\mathcal{E}}) \subset \mathcal{U}_{\mathcal{E}}$.

• The push-forward of $c_{\mathcal{E}}$ by Ψ being the image curve $c_{\mathcal{E}*} = \Psi \circ c_{\mathcal{E}}$ (the curve transformed by Ψ) cf. (6.4), its tangent vector at $p_{\mathcal{F}} = c_{\mathcal{E}*}(s)$ is

$$\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) := c_{\mathcal{E}*}'(s) \quad \text{thus} \quad = d\Psi(p_{\mathcal{E}}).c_{\mathcal{E}}'(s) = d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}). \tag{6.19}$$

Thus we have defined the vector field $\vec{w}_{\mathcal{E}*}$ along $\text{Im}(c_{\mathcal{E}*})$ called the push-forward of $\vec{w}_{\mathcal{E}}$ by Ψ.

With all the integral curves of a vector field defined in $\mathcal{U}_{\mathcal{E}}$, we get:

Definition 6.7 The push-forward by Ψ of a C^0 vector field $\vec{w}_{\mathcal{E}} : \left\{ \begin{array}{rl} \mathcal{U}_{\mathcal{E}} & \to E \\ p_{\mathcal{E}} & \to \vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \end{array} \right\}$ is the vector field

$$\Psi_*\vec{w}_{\mathcal{E}} = \vec{w}_{\mathcal{E}*} : \left\{ \begin{array}{rl} \mathcal{U}_{\mathcal{F}} & \to F \\ p_{\mathcal{F}} & \to \boxed{\Psi_*\vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) := d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})} \overset{\text{noted}}{=} \vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}), \end{array} \right. \tag{6.20}$$

see figure 6.1. ($\Psi_*\vec{w}_{\mathcal{E}} =^{\text{noted}} \vec{w}_{\mathcal{E}*}$ if Ψ is implicit). In other words,

$$\Psi_*\vec{w}_{\mathcal{E}} := (d\Psi.\vec{w}_{\mathcal{E}}) \circ \Psi^{-1}. \tag{6.21}$$

This defines the map $\Psi_* : \left\{ \begin{array}{rl} C^\infty(\mathcal{U}_{\mathcal{E}}; E) & \to C^\infty(\mathcal{U}_{\mathcal{F}}; F) \\ \vec{w}_{\mathcal{E}} & \to \Psi_*(\vec{w}_{\mathcal{E}}) := \Psi_*\vec{w}_{\mathcal{E}} = \vec{w}_{\mathcal{E}*} \end{array} \right\}$. (We use the same notation Ψ_* as in definition 6.4 for scalar valued functions: The context removes ambiguity.)

Remark 6.8 Unlike scalar functions, cf. § 6.4.2: At t_0 at p_{t_0} you cannot just draw a vector $\vec{w}_{t_0}(p_{t_0})$ on a piece of paper, put the paper in your pocket, then let yourself be carried by the flow $\Psi = \Phi_t^{t_0}$ (push-forward), then, once arrived at t at p_t, take the paper out of your pocket and read it to get the push-forward: The direction and length of the vector $\vec{w}_{t_0*}(t, p_t)$ are modified by the flow (a vector is not just a collection of scalar components). ▪

Exercice 6.9 Prove:

$$\vec{c_{\mathcal{E}}}''(s) = d\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}), \tag{6.22}$$

and

$$d\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}) = d\Psi(p_{\mathcal{E}}).d\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) + d^2\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}), \tag{6.23}$$

and

$$c_{\mathcal{E}*}''(s) = d\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}).\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) \quad (= d\Psi(p_{\mathcal{E}}).\vec{c_{\mathcal{E}}}''(s) + d^2\Psi(p_{\mathcal{E}}).\vec{c_{\mathcal{E}}}'(s).\vec{c_{\mathcal{E}}}'(s)). \tag{6.24}$$

Answer. $\vec{c_{\mathcal{E}}}'(s) = \vec{w}_{\mathcal{E}}(c_{\mathcal{E}}(s))$ gives $\vec{c_{\mathcal{E}}}''(s) = d\vec{w}_{\mathcal{E}}(c_{\mathcal{E}}(s)).\vec{c_{\mathcal{E}}}'(s)$, hence (6.22).

$\vec{w}_{\mathcal{E}*}(\Psi(p_{\mathcal{E}})) = d\Phi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})$ by definition of $\vec{w}_{\mathcal{E}*}$, hence (6.23).

$c_{\mathcal{F}}(s) = \Psi(c_{\mathcal{E}}(s))$ gives $\vec{c_{\mathcal{F}}}'(s) = d\Psi(c_{\mathcal{E}}(s)).\vec{c_{\mathcal{E}}}'(s) = d\Psi(c_{\mathcal{E}}(s)).\vec{w}_{\mathcal{E}}(c_{\mathcal{E}}(s)) = \vec{w}_{\mathcal{E}*}(c_{\mathcal{F}}(s))$. Thus $\vec{c_{\mathcal{F}}}''(s) = (d^2\Psi(c_{\mathcal{E}}(s)).\vec{c_{\mathcal{E}}}'(s)).\vec{c_{\mathcal{E}}}'(s) + d\Psi(c_{\mathcal{E}}(s)).\vec{c_{\mathcal{E}}}''(s) = d\vec{w}_{\mathcal{E}*}(c_{\mathcal{F}}(s)).\vec{c_{\mathcal{F}}}'(s)$, hence (6.24). ▪

6.5.3 Pull-back of a vector field

Definition 6.10 If $\vec{w}_{\mathcal{F}} : \left\{ \begin{array}{rl} \mathcal{U}_{\mathcal{F}} & \to F \\ p_{\mathcal{F}} & \to \vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) \end{array} \right\}$ is a vector field on $\mathcal{U}_{\mathcal{F}}$, then its pull-back by Ψ is the push-forward by Ψ^{-1}, i.e. is the vector field on $\mathcal{U}_{\mathcal{E}}$ defined by

$$\Psi^*\vec{w}_{\mathcal{F}} : \left\{ \begin{array}{rl} \mathcal{U}_{\mathcal{E}} & \to E \\ p_{\mathcal{E}} & \to \boxed{\Psi^*\vec{w}_{\mathcal{F}}(p_{\mathcal{E}}) := d\Psi^{-1}(p_{\mathcal{F}}).\vec{w}_{\mathcal{F}}(p_{\mathcal{F}})} \overset{\text{noted}}{=} \vec{w}_{\mathcal{F}}{}^*(p_{\mathcal{E}}), \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}). \end{array} \right. \tag{6.25}$$

In other words,

$$\Psi^*\vec{w}_{\mathcal{F}} := (d\Psi^{-1}.\vec{w}_{\mathcal{F}}) \circ \Psi \overset{\text{noted}}{=} \vec{w}_{\mathcal{F}}{}^*. \tag{6.26}$$

And we get

$$\Psi^* \circ \Psi_* = I \quad \text{and} \quad \Psi_* \circ \Psi^* = I. \tag{6.27}$$

Indeed, $\Psi^*(\Psi_*\vec{w}_{\mathcal{E}})(p_{\mathcal{E}}) = d\Psi^{-1}(p_{\mathcal{F}}).\Psi_*\vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) = d\Psi^{-1}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) = \vec{w}_{\mathcal{E}}(p_{\mathcal{E}})$, for all $p_{\mathcal{E}}$. Idem for the second equality. ∎

6.6 Quantification with bases

6.6.1 Usual result

$(\vec{a}_i)$ is a Cartesian basis in E, $O_{\mathcal{F}}$ and $(\vec{b}_i)$ are an origin in $\mathcal{F}$ and a Cartesian basis in F, $p_{\mathcal{E}} \in \mathcal{U}_{\mathcal{E}}$,

$$p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}) = O_{\mathcal{F}} + \sum_{i=1}^n \psi_i(p_{\mathcal{E}})\,\vec{b}_i, \quad \text{i.e.} \quad [\overrightarrow{O_{\mathcal{F}}p_{\mathcal{F}}}]_{|\vec{b}} = \begin{pmatrix} \psi_1(p_{\mathcal{E}}) \\ \vdots \\ \psi_n(p_{\mathcal{E}}) \end{pmatrix}. \tag{6.28}$$

Then, if $\vec{w}_{\mathcal{E}}$ is a vector field in $\mathcal{U}_{\mathcal{E}}$ and $\vec{w}_{\mathcal{E}} = \sum_i w_j\vec{a}_i$, we get $\Psi_*\vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) = d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) = \sum_{i=1}^n (d\psi_i(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}))\,\vec{b}_i = \sum_{i,j=1}^n w_j(p_{\mathcal{E}})(d\psi_i(p_{\mathcal{E}}).\vec{a}_j)\,\vec{b}_i = \sum_{i,j=1}^n \frac{\partial\psi_i}{\partial x_j}(p_{\mathcal{E}})w_j(p_{\mathcal{E}})\,\vec{b}_i$, so

$$[\Psi_*\vec{w}_{\mathcal{E}}(p_{\mathcal{F}})]_{|\vec{b}} = [d\Psi(p_{\mathcal{E}})]_{|\vec{a},\vec{b}}.[\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})]_{|\vec{a}}, \tag{6.29}$$

where $[d\Psi(p_{\mathcal{E}})]_{|\vec{a},\vec{b}} = [d\psi_i(p_{\mathcal{E}}).\vec{a}_j] =^{\text{noted}} [\frac{\partial\psi_i}{\partial x_j}(p_{\mathcal{E}})]$ is the Jacobian matrix.

6.6.2 Example: Polar coordinate system

Example 6.11 Change of coordinate system interpreted as a push-forward: Paradigmatic example of the polar coordinate system (model generalized for the parametrization of any manifold).

Parametric Cartesian vector space $\mathbb{R} \times \mathbb{R} =^{\text{noted}} \vec{\mathbb{R}}^2_p = \{\vec{q} = (r,\theta)\}$, with its canonical basis $(\vec{a}_1, \vec{a}_2)$, and $\vec{q} = r\vec{a}_1 + \theta\vec{a}_2 =^{\text{noted}} (r,\theta)$, so $[\vec{q}]_{|\vec{a}} = \begin{pmatrix} r \\ \theta \end{pmatrix}$. Geometric affine space $\mathbb{R}^2$ (of positions), $p \in \mathbb{R}^2$, associated vector space $\vec{\mathbb{R}^2}$, $O \in \mathbb{R}^2$ (origin), $\vec{x} = \overrightarrow{Op}$, and a Euclidean basis $(\vec{b}_1, \vec{b}_2)$ in $\vec{\mathbb{R}^2}$. The "polar coordinate system" is the associated map $\Psi : \left\{ \begin{array}{ll} \vec{\mathbb{R}}^*_+ \times \mathbb{R} \subset \vec{\mathbb{R}}^2_p & \to \mathbb{R}^2 \\ \vec{q} = (r,\theta) & \to \vec{x} = \Psi(\vec{q}) = \Psi(r,\theta), \end{array} \right\}$ defined by

$$\vec{x} = \Psi(\vec{q}) := r\cos\theta\,\vec{b}_1 + r\sin\theta\,\vec{b}_2, \quad \text{i.e.} \quad [\vec{x}]_{|\vec{b}} = \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}. \tag{6.30}$$

The i-th coordinate line at $\vec{q}$ in $\vec{\mathbb{R}}^2_p$ (parametric space) is the straight line $\vec{c}_{\vec{q},i} : \left\{ \begin{array}{ll} \mathbb{R} & \to \vec{\mathbb{R}}^2_p \\ s & \to \vec{c}_{\vec{q},i}(s) = \vec{q} + s\vec{a}_i \end{array} \right\}$, and its tangent vector at $\vec{c}_{\vec{q},i}(s)$ is $\vec{c}_{\vec{q},i}{}'(s) = \vec{a}_i$ for all s. This line is transformed by Ψ into the curve $\Psi_*(c_{q,i}) = \Psi \circ \vec{c}_{\vec{q},i} =^{\text{noted}} c_{\vec{x},i} : \left\{ \begin{array}{ll} \mathbb{R} & \to \mathbb{R}^2 \\ s & \to c_{\vec{x},i}(s) = \Psi(\vec{q} + s\vec{a}_i) \end{array} \right\}$ (in particular $c_{\vec{x},i}(0) = \vec{x}$). So

$$[\overrightarrow{Oc_{\vec{x},1}(s)}]_{|\vec{b}} = \begin{pmatrix} (r+s)\cos\theta \\ (r+s)\sin\theta \end{pmatrix} \quad \text{(straight line)}, \quad \text{and} \quad [\overrightarrow{Oc_{\vec{x},2}(s)}]_{|\vec{b}} = \begin{pmatrix} r\cos(\theta+s) \\ r\sin(\theta+s) \end{pmatrix} \quad \text{(circle)}. \tag{6.31}$$

And the tangent vector at $c_{\vec{x},i}(s)$ is $c_{\vec{x},i}{}'(s) =^{\text{noted}} \vec{a}_{i*}(\vec{x})$ (push-forward by Ψ), so

$$\begin{aligned} \vec{a}_{1*}(\vec{x}) &:= \Psi_*\vec{a}_1(\vec{x}) = d\Psi(\vec{q}).\vec{a}_1 = \lim_{h\to 0} \frac{\Psi(\vec{q}+h\vec{a}_1) - \Psi(\vec{q})}{h} = \lim_{h\to 0} \frac{\Psi(r+h,\theta) - \Psi(r,\theta)}{h} = \frac{\partial\Psi}{\partial r}(\vec{q}), \\ \vec{a}_{2*}(\vec{x}) &:= \Psi_*\vec{a}_2(\vec{x}) = d\Psi(\vec{q}).\vec{a}_2 = \lim_{h\to 0} \frac{\Psi(\vec{q}+h\vec{a}_2) - \Psi(\vec{q})}{h} = \lim_{h\to 0} \frac{\Psi(r,\theta+h) - \Psi(r,\theta)}{h} = \frac{\partial\Psi}{\partial\theta}(\vec{q}), \end{aligned} \tag{6.32}$$

Thus

$$\vec{a}_{1*}(\vec{x}) = \cos\theta\vec{b}_1 + \sin\theta\vec{b}_2 \quad \text{and} \quad \vec{a}_{2*}(\vec{x}) = -r\sin\theta\vec{b}_1 + r\cos\theta\vec{b}_2, \tag{6.33}$$

i.e.

$$[\vec{a}_{1*}(\vec{x})]_{|\vec{b}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad [\vec{a}_{2*}(\vec{x})]_{|\vec{b}} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}. \tag{6.34}$$

The basis $(\vec{a}_{1*}(\vec{x}), \vec{a}_{2*}(\vec{x}))$ is called the basis of the polar coordinate system at $\vec{x}$ (it is orthogonal but not orthonormal since $||\vec{a}_{2*}(\vec{x})|| = r \neq 1$ in general); And $[d\Psi(\vec{q})]_{|\vec{a},\vec{b}} = \big([\frac{\partial\Psi}{\partial r}(\vec{q})]_{|\vec{b}} \;\; [\frac{\partial\Psi}{\partial\theta}(\vec{q})]_{|\vec{b}} \big) = \big([\vec{a}_{1*}(\vec{x})]_{|\vec{b}} \;\; [\vec{a}_{2*}(\vec{x})]_{|\vec{b}} \big) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = [\frac{\partial\Psi^i}{\partial q^j}(\vec{q})]$ is the Jacobian matrix of Ψ at $\vec{q}$.

And the dual basis of the polar system basis $(\vec{a}_{1*}(\vec{x}),\vec{a}_{2*}(\vec{x}))$ is called $(dq_1(\vec{x}),dq_2(\vec{x}))$ (defined by $dq_i(\vec{x}).\vec{a}_{j*}(\vec{x})=\delta_{ij}$), so

$$dq_1(\vec{x})=\cos\theta\,dx_1+\sin\theta\,dx_2 \quad\text{and}\quad dq_2(\vec{x})=-\frac{1}{r}\sin\theta\,dx_1+\frac{1}{r}\cos\theta\,dx_2, \tag{6.35}$$

i.e. $[dq_1(\vec{x})]_{|\vec{b}}=(\cos\theta \quad \sin\theta)$ and $[dq_2(\vec{x})]_{|\vec{b}}=-\frac{1}{r}(\sin\theta \quad \cos\theta)$ (row matrices) when $\vec{x}=\Psi(\vec{q})$. ▪

Remark 6.12 The components $\gamma_{ij}^k(\vec{x})$ of the vector $d\vec{a}_{j*}(\vec{x}).\vec{a}_{i*}(\vec{x})\in\vec{\mathbb{R}}^2$ in the basis $(\vec{a}_{i*}(\vec{x}))$ are the Christoffel symbols of the polar coordinate system (with duality notations as it is usually presented):

$$d\vec{a}_{j*}(\vec{x}).\vec{a}_{i*}(\vec{x})=\sum_{k=1}^{n}\gamma_{ij}^k(\vec{x})\vec{a}_{k*}(\vec{x}). \tag{6.36}$$

At $\vec{x}=\Psi(\vec{q})$, with $\vec{a}_{j*}(\vec{x})=d\Psi(\vec{q}).\vec{a}_j$, i.e. $(\vec{a}_{j*}\circ\Psi)(\vec{q})=\frac{\partial\Psi}{\partial q^j}$, we get

$$d\vec{a}_{j*}(\vec{x}).\vec{a}_{i*}(\vec{x})=\frac{\partial^2\Psi}{\partial q^i\partial q^j}(\vec{q})=d\vec{a}_{i*}(\vec{x}).\vec{a}_{j*}(\vec{x}),\quad\text{so}\quad \gamma_{ij}^k=\gamma_{ji}^k \tag{6.37}$$

for all i,j (symmetry of the bottom indices as soon as Ψ is C^2).

Here for the polar coordinates, $\frac{\partial\Psi}{\partial r}(\vec{q})=\cos\theta\vec{b}_1+\sin\theta\vec{b}_2$ gives $\frac{\partial^2\Psi}{\partial r^2}(\vec{q})=\vec{0}$, thus $\gamma_{11}^1=\gamma_{11}^2=0$, and $\frac{\partial^2\Psi}{\partial\theta\partial r}(\vec{q})=-\sin\theta\vec{b}_1+\cos\theta\vec{b}_2=\frac{1}{r}\vec{a}_{2*}(\vec{x})$, thus $\gamma_{12}^1=0=\gamma_{21}^1$ and $\gamma_{12}^2=\frac{1}{r}=\gamma_{21}^2$. And $\frac{\partial\Psi}{\partial\theta}(\vec{q})=-r\sin\theta\vec{b}_1+r\cos\theta\vec{b}_2$ gives $\frac{\partial^2\Psi}{\partial\theta^2}(\vec{q})=-r\cos\theta\vec{b}_1-r\sin\theta\vec{b}_2=-r\vec{a}_{1*}(\vec{x})$, thus $\gamma_{22}^1=-r$ and $\gamma_{22}^2=0$. ▪

Remark 6.13 The (widely used) normalized polar coordinate basis $(\vec{n}_1(\vec{x}),\vec{n}_2(\vec{x}))=(\vec{a}_{1*}(\vec{x}),\frac{1}{r}\vec{a}_{2*}(\vec{x}))$ is not holonomic, i.e. is not the basis of a coordinate system (and its use makes higher derivation formulas complicated). Indeed $\vec{n}_2(\vec{x})=\frac{1}{r}\vec{a}_{2*}(\vec{x})$ gives $d\vec{n}_2(\vec{x}).\vec{n}_1(\vec{x})=(d(\frac{1}{r})(\vec{x}).\vec{n}_1(\vec{x}))\vec{a}_{2*}(\vec{x})+\frac{1}{r}d\vec{a}_{2*}(\vec{x}).\vec{n}_1(\vec{x})$, and $\vec{n}_1(\vec{x})=\vec{a}_{1*}(\vec{x})$ gives $d\vec{n}_1(\vec{x}).\vec{n}_2(\vec{x})=d\vec{a}_{1*}(\vec{x}).(\frac{1}{r}\vec{a}_{2*})$, thus $d\vec{n}_2(\vec{x}).\vec{n}_1(\vec{x})-d\vec{n}_1(\vec{x}).\vec{n}_2(\vec{x})=(d(\frac{1}{r})(\vec{x}).\vec{n}_1(\vec{x}))\vec{a}_{2*}(\vec{x})\neq\vec{0}$, since $\frac{1}{r}=(x^2+y^2)^{-\frac{1}{2}}$ gives $d(\frac{1}{r})(\vec{x}).\vec{n}_1(\vec{x})=(-x(x^2+y^2)^{-\frac{3}{2}}\quad -y(x^2+y^2)^{-\frac{3}{2}}).\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}=\frac{1}{r^3}(-r\cos^2\theta-r\sin^2\theta)=\frac{-1}{r^2}\neq 0$. ▪

Remark 6.14 (Pay attention to the notations.) Let $f:\vec{q}\in\vec{\mathbb{R}}_p^2\to f(\vec{q})\in\mathbb{R}$ be C^2. Call g its push-forward by Ψ, i.e. $g:\vec{x}\in\mathbb{R}^2\to g(\vec{x})=f(\vec{q})\in\mathbb{R}$ when $\vec{x}=\Psi(\vec{q})$. So $f(\vec{q})=(g\circ\Psi)(\vec{q})$and

$$df(\vec{q}).\vec{a}_j=dg(\Psi(\vec{q})).d\Psi(\vec{q}).\vec{a}_j=dg(\vec{x}).\vec{a}_{j*}(\vec{x}). \tag{6.38}$$

With $df(\vec{q}).\vec{a}_j=^{\text{noted}}\frac{\partial f}{\partial q^j}(\vec{q})$ and $dg(\vec{x}).\vec{b}_j=^{\text{noted}}\frac{\partial g}{\partial x^j}(\vec{x})$ and $\vec{a}_{j*}(\vec{x})=d\Psi(\vec{q}).\vec{a}_j=\sum_i\frac{\partial\Psi^i}{\partial q^j}(\vec{q})\vec{a}_j$, we get

$$\frac{\partial f}{\partial q^j}(\vec{q})=\sum_i\frac{\partial g}{\partial x^i}(\vec{x})\frac{\partial\Psi^i}{\partial q^j}(\vec{q})\overset{\text{noted}}{=}\frac{\partial g}{\partial q^j}(\vec{x})\ \ldots\ (!!) \tag{6.39}$$

Mind this notation!! g is a function of $\vec{x}$, not of $\vec{q}$, so $\frac{\partial g}{\partial q^i}(\vec{x})\overset{\text{means}}{=}\frac{\partial f}{\partial q^i}(\vec{q})$, i.e. $\frac{\partial g}{\partial q^i}(\vec{x})\overset{\text{means}}{=}\frac{\partial(g\circ\Psi)}{\partial q^i}(\vec{q})$... which is $[df(\vec{q})]=[dg(\vec{x})].[d\Psi(\vec{q})$...

Then (with f and Ψ C^2)

$$\begin{aligned}\frac{\partial\frac{\partial g}{\partial q^i}}{\partial q^j}(\vec{x})\overset{\text{means}}{=}\frac{\partial\frac{\partial(g\circ\Psi)}{\partial q^i}}{\partial q^j}(\vec{q})&=d(dg.\vec{a}_{i*})(\vec{x}).d\Psi(\vec{q}).\vec{a}_j=d(dg.\vec{a}_{i*})(\vec{x}).\vec{a}_{j^*}(\vec{x})\\ &=d((dg(\vec{x}).\vec{a}_{j^*}(\vec{x})).\vec{a}_{i*}(\vec{x})+dg(\vec{x}).(d\vec{a}_{i*}(\vec{x}).\vec{a}_j(\vec{x}))\overset{\text{noted}}{=}\frac{\partial^2 g}{\partial q^i\partial q^j}(\vec{x}).\end{aligned} \tag{6.40}$$

So

$$\frac{\partial^2 g}{\partial q^i\partial q^j}(\vec{x})\overset{\text{means}}{=}d^2g(\vec{x})(\vec{a}_{i*}(\vec{x}),\vec{a}_{j*}(\vec{x}))+\sum_{k=1}^{n}\frac{\partial g}{\partial x^k}(\vec{x})\gamma_{ij}^k(\vec{x})\vec{a}_k(\vec{x}), \tag{6.41}$$

and $\frac{\partial^2 g}{\partial q^i\partial q^j}(\vec{x})$ is ***not*** reduced to $d^2g(\vec{x})(\vec{a}_{i*}(\vec{x}),\vec{a}_{j*}(\vec{x}))$ (the Christoffel symbols have appeared): First order derivatives $\frac{\partial g}{\partial x^k}$ are still alive. (Contrary to $\frac{\partial^2 g}{\partial x^i\partial x^j}(\vec{x})=d^2g(\vec{x})(\vec{b}_i,\vec{b}_j)$ with a Cartesian basis $(\vec{b}_i)$.)

NB: The independent variables r and θ don't have the same dimension (a length and an angle): There is no physical meaningful inner dot product in the parameter space $\vec{\mathbb{R}}_p^2=\mathbb{R}\times\mathbb{R}=\{(r,\theta)\}$, but this space is very useful... (As in thermodynamics: No meaningful inner dot product in the (T,P) space.) ▪

7 Push-forward and pull-back of differential forms

7.1 Definition

Setting of § 6.1. Consider a differential form $\alpha_{\mathcal{E}} : \left\{\begin{matrix} \mathcal{U}_{\mathcal{E}} & \to E^* = \mathcal{L}(E;\mathbb{R}) \\ p_{\mathcal{E}} & \to \alpha_{\mathcal{E}}(p_{\mathcal{E}}) \end{matrix}\right\}$ on $\mathcal{U}_{\mathcal{E}}$ (a field of linear forms), and a vector field $\vec{w}_{\mathcal{E}} : \left\{\begin{matrix} \mathcal{U}_{\mathcal{E}} & \to E \\ p_{\mathcal{E}} & \to \vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \end{matrix}\right\}$. Hence

$$f_{\mathcal{E}} = \alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}} : \begin{cases} \mathcal{U}_{\mathcal{E}} & \to \mathbb{R} \\ p_{\mathcal{E}} & \to f_{\mathcal{E}}(p_{\mathcal{E}}) = (\alpha.\vec{w}_{\mathcal{E}})(p_{\mathcal{E}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \end{cases}$$

is a scalar valued function (value of $\vec{w}_{\mathcal{E}}$ given by $\alpha_{\mathcal{E}}$). And (6.8) gives (push-forward $f_{\mathcal{E}} = \alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}}$ by Ψ)

$$\Psi_*(\alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}})(p_{\mathcal{F}}) = (\alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}})(p_{\mathcal{E}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}). \tag{7.1}$$

With $\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) = d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})$ cf. (6.20) (push-forward of $\vec{w}_{\mathcal{E}}$), we get

$$\Psi_*(\alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}})(p_{\mathcal{F}}) = \underbrace{\alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi(p_{\mathcal{E}})^{-1}}_{=\text{noted } \alpha_{\mathcal{E}*}(p_{\mathcal{F}})}.\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}) \ : \tag{7.2}$$

Definition 7.1 The push-forward of a differential form $\alpha_{\mathcal{E}} \in \Omega^1(\mathcal{U}_{\mathcal{E}})$ is the differential form $\in \Omega^1(\mathcal{U}_{\mathcal{F}})$ given by

$$\Psi_*\alpha_{\mathcal{E}} : \begin{cases} \mathcal{U}_{\mathcal{F}} & \to F^* = \mathcal{L}(F;\mathbb{R}) \\ p_{\mathcal{F}} & \to \boxed{\Psi_*\alpha_{\mathcal{E}}(p_{\mathcal{F}}) := \alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi(p_{\mathcal{E}})^{-1}} \overset{\text{noted}}{=} \alpha_{\mathcal{E}*}(p_{\mathcal{F}}) \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}), \end{cases} \tag{7.3}$$

the last notation when Ψ is implicit. In other words, $\Psi_*\alpha_{\mathcal{E}}(p_{\mathcal{F}}) = \alpha_{\mathcal{E}}(\Psi^{-1}(p_{\mathcal{F}})).d\Psi^{-1}(p_{\mathcal{F}})$, i.e.

$$\Psi_*\alpha_{\mathcal{E}} := (\alpha_{\mathcal{E}} \circ \Psi^{-1}).d\Psi^{-1}. \tag{7.4}$$

(Once again, we used the same notation Ψ_* than for the push-forward of vector fields and functions: The context removes any ambiguities.)

Remark 7.2 We cannot always see a vector field (e.g. we can't see an internal force field): To know it we need to measure it with a well defined tool, the tool being here a differential form; And the definition 7.1 is a compatbility definition so that we can recover the push-forward of the vector field. ■

Definition 7.3 The pull-forward of a a differential form $\alpha_{\mathcal{F}} \in \Omega^1(\mathcal{U}_{\mathcal{F}})$ is the differential form

$$\Psi^*\alpha_{\mathcal{F}} : \begin{cases} \mathcal{U}_{\mathcal{E}} & \to \mathcal{L}(E;\mathbb{R}) \\ p_{\mathcal{E}} & \to \Psi^*\alpha_{\mathcal{F}}(p_{\mathcal{E}}) := \alpha_{\mathcal{F}}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}) \overset{\text{noted}}{=} {\alpha_{\mathcal{F}}}^*(p_{\mathcal{E}}) \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}), \end{cases} \tag{7.5}$$

In other words,

$$\Psi^*\alpha_{\mathcal{F}} := (\alpha_{\mathcal{F}} \circ \Psi).d\Psi. \tag{7.6}$$

(For an alternative definition, see remark 7.5.)

Proposition 7.4 *For all $\alpha_{\mathcal{E}} \in \Omega^1(\mathcal{U}_{\mathcal{E}})$ and $\alpha_{\mathcal{F}} \in \Omega^1(\mathcal{U}_{\mathcal{F}})$ (differential forms), and $\vec{w}_{\mathcal{E}} \in \Gamma(\mathcal{U}_{\mathcal{E}})$ and $\vec{w}_{\mathcal{F}} \in \Gamma(\mathcal{U}_{\mathcal{F}})$ (vector fields), we have (objectivity result)*

$$(\Psi_*\alpha_{\mathcal{E}})(p_{\mathcal{F}}).\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).(\Psi_*\vec{w}_{\mathcal{F}})(p_{\mathcal{E}}) \quad \textit{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}), \tag{7.7}$$

i.e. $\alpha_{\mathcal{E}*}(p_{\mathcal{F}}).\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).{\vec{w}_{\mathcal{F}}}^*(p_{\mathcal{E}})$. In particular with $\alpha_{\mathcal{E}} = df$ (exact differential form) where $f \in C^1(\mathcal{U}_{\mathcal{E}};\mathbb{R})$,

$$d(\Psi_*f) = \Psi_*(df). \tag{7.8}$$

(This commutativity result is very particular to the case $\alpha = df$: In general $d(\Psi_*T) \neq \Psi_*(dT)$ for a tensor of order ≥ 2, see e.g. (8.19)).

Proof. $\alpha_{\mathcal{E}*}(p_\mathcal{F}).\vec{w}_\mathcal{F}(p_\mathcal{F}) = (\alpha_\mathcal{E}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F})).\vec{w}_\mathcal{F}(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).(d\Psi^{-1}(p_\mathcal{F}).\vec{w}_\mathcal{F}(p_\mathcal{F})) = \alpha_\mathcal{E}(p_\mathcal{E}).\vec{w}^*_\mathcal{F}(p_\mathcal{E})$, for all $p_\mathcal{F} = \Psi(p_\mathcal{E}) \in \mathcal{U}_\mathcal{F}$.

And $\Psi_* f(p_\mathcal{F}) := f(p_\mathcal{E}) = f(\Psi^{-1}(p_\mathcal{F}))$, thus $d(\Psi_* f)(p_\mathcal{F}) = df(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}) = \Psi_*(df)(p_\mathcal{F})$. ∎

And we have

$$\Psi^* \circ \Psi_* = I \quad \text{and} \quad \Psi_* \circ \Psi^* = I. \tag{7.9}$$

Indeed $\Psi^*(\Psi_*\alpha_\mathcal{E})(p_\mathcal{E}) = \Psi_*\alpha_\mathcal{E}(p_\mathcal{F}).d\Psi(p_\mathcal{E}) = \alpha_\mathcal{E}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}).d\Psi(p_\mathcal{E}) = \alpha_\mathcal{E}(p_\mathcal{E})$. Idem for $\Psi_* \circ \Psi^* = I$.

Remark 7.5 The pull-back $\alpha_\mathcal{F}{}^*$ can also be defined thanks to the natural canonical isomorphism $\left\{\begin{array}{ll}\mathcal{L}(E;F) & \to \mathcal{L}(F^*;E^*)\\ L & \to L^*\end{array}\right\}$ given by $L^*(\ell_F).\vec{u}_E = \ell_F.(L.\vec{u}_E)$ for all $(\vec{u}_E, \ell_F) \in E \times F^*$, and $L^*(\ell_F) = \ell_F.L$ is called the pull-back of ℓ_F by L. In particular with $\ell_F = \alpha_\mathcal{F}(p_\mathcal{F})$ and $L = d\Psi(p_\mathcal{E})$ we get $d\Psi(p_\mathcal{E})^*(\alpha_\mathcal{F}(p_\mathcal{F})) = \alpha_\mathcal{F}(p_\mathcal{F}).d\Psi(p_\mathcal{E})$, i.e. (7.5). ∎

7.2 Incompatibility: Riesz representation and push-forward

A push-forward is independent of any inner dot product: It is objective.

But here we introduce inner dot products $(\cdot,\cdot)_g$ in E and $(\cdot,\cdot)_h$ in F, e.g. Euclidean dot products in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$ (observer dependent therefore subjective), because some mechanical engineers can't begin with their beloved Euclidean dot products.

Let $\alpha_\mathcal{E} \in \Omega^1(\mathcal{U}_\mathcal{E})$ and call $\beta_\mathcal{F} := \Psi_*\alpha_\mathcal{E}$ its push-forward by Ψ, i.e.

$$\beta_\mathcal{F}(p_\mathcal{F}) := \alpha_\mathcal{E}(p_\mathcal{E}).d\Psi(p_\mathcal{E})^{-1} \quad \text{when} \quad p_\mathcal{F} = \Psi(p_\mathcal{E}). \tag{7.10}$$

Then call $\vec{a}_g(p_\mathcal{E}) \in E$ and $\vec{b}_h(p_\mathcal{F}) \in F$ the $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$-Riesz representation vectors of $\alpha_\mathcal{E}$ and $\beta_\mathcal{F}$, so, for all $\vec{u}_\mathcal{E} \in \Gamma(\mathcal{U}_\mathcal{E})$ and all $\vec{w}_\mathcal{F} \in \Gamma(\mathcal{U}_\mathcal{F})$, in short,

$$\alpha_\mathcal{E}.\vec{u}_\mathcal{E} = (\vec{a}_g, \vec{u}_\mathcal{E})_g, \quad \text{and} \quad \beta_\mathcal{F}.\vec{w}_\mathcal{F} = (\vec{b}_h, \vec{w}_\mathcal{F})_h, \tag{7.11}$$

which means $\alpha_\mathcal{E}(p_\mathcal{E}).\vec{u}_\mathcal{E}(p_\mathcal{E}) = (\vec{a}_g(p_\mathcal{E}), \vec{u}_\mathcal{E}(p_\mathcal{E}))_g$ and $\beta_\mathcal{F}(p_\mathcal{F}).\vec{w}_\mathcal{F}(p_\mathcal{F}) = (\vec{b}_h(p_\mathcal{F}), \vec{w}_\mathcal{F}(p_\mathcal{F}))_h$, for all $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$ and $p_\mathcal{F} \in \mathcal{U}_\mathcal{F}$. This defines the vector fields $\vec{a}_g \in \Gamma(\mathcal{U}_\mathcal{E})$ and $\vec{b}_h \in \Gamma(\mathcal{U}_\mathcal{F})$.

Proposition 7.6 *$\vec{b}_h \neq \Psi_*\vec{a}_g$ in general (although $\beta_\mathcal{F} = \Psi_*\alpha_\mathcal{E}$), because*

$$\begin{aligned}\vec{b}_h(p_\mathcal{F}) &= d\Psi(p_\mathcal{E})^{-T}.\vec{a}_g(p_\mathcal{E})\\ &\neq d\Psi(p_\mathcal{E}).\vec{a}_g(p_\mathcal{E}) \ \ \textit{in general}\end{aligned} \tag{7.12}$$

(unless $d\Psi(p_\mathcal{E})^{-T} = d\Psi(p_\mathcal{E})$, i.e. $d\Psi(p_\mathcal{E})^T.d\Psi(p_\mathcal{E})^{-1} = I$, as a rigid body motion).

So the Riesz representation vector of the push-forwarded linear form is not the push-forwarded representation vector of the linear form push-forwarded.

This is not a surprise: A push-forward is independent of any inner dot product, while a Riesz representation vector depends on a chosen inner dot product (E.g. Euclidean foot? metre?).

So, as long as possible (not before you need to quantify), you should avoid using a Riesz representation vector, i.e. you should use the original (the qualitative differential form) as long as possible, and delay the use of a representative (quantification with which dot product?) as late as possible.

Proof. Recall: The transposed relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ of the linear map $d\Psi(p_\mathcal{E}) \in \mathcal{L}(E;F)$ is the linear map $d\Psi(p_\mathcal{E})^T_{gh} =^{\text{noted}} d\Psi(p_\mathcal{E})^T \in \mathcal{L}(F;E)$ defined by, for all $\vec{u}_\mathcal{E} \in E$ and $\vec{w}_\mathcal{F} \in F$ vectors at $p_\mathcal{E}$ and $p_\mathcal{F}$, cf. (A.66),

$$(d\Psi(p_\mathcal{E})^T.\vec{w}_\mathcal{F}, \vec{u}_\mathcal{E})_g = (\vec{w}_\mathcal{F}, d\Psi(p_\mathcal{E}).\vec{u}_\mathcal{E})_h. \tag{7.13}$$

(7.11) gives, with $p_\mathcal{F} = \Psi(p_\mathcal{E})$,

$$\begin{aligned}(\vec{a}_g(p_\mathcal{E}), \vec{u}_\mathcal{E})_g &= \alpha_\mathcal{E}(p_\mathcal{E}).\vec{u}_\mathcal{E} = \big(\beta_\mathcal{F}(p_\mathcal{F}).d\Psi(p_\mathcal{E})\big).\vec{u}_\mathcal{E} = \beta_\mathcal{F}(p_\mathcal{F}).\big(d\Psi(p_\mathcal{E}).\vec{u}_\mathcal{E}\big)\\ &= (\vec{b}_h(p_\mathcal{F}), d\Psi(p_\mathcal{E}).\vec{u}_\mathcal{E})_h = (d\Psi(p_\mathcal{E})^T.\vec{b}_h(p_\mathcal{F}), \vec{u}_\mathcal{E})_g,\end{aligned} \tag{7.14}$$

true for all $\vec{u}_\mathcal{E}$, thus $\vec{a}_g(p_\mathcal{E}) = d\Psi(p_\mathcal{E})^T.\vec{b}_h(p_\mathcal{F})$, thus (7.12). ∎

8 Push-forward and pull-back of tensors

To lighten the presentation, we only deal with order 1 and 2 tensors. Similar approach for any tensor.

8.1 Push-forward and pull-back of order 1 tensors

Proposition 8.1 *If T is either a vector field or a differential form, then its push-forward satisfies, for all ξ vector field or differential form (when required) in $\mathcal{U}_\mathcal{F}$,*

$$\textit{in short:}\quad (\Psi_*T)(\xi) = T(\Psi^*\xi), \quad \textit{written} \quad \Psi_*T(.) = T(\Psi^*.), \tag{8.1}$$

i.e. $(\Psi_*T)(p_\mathcal{F}).\xi(p_\mathcal{F}) = T(p_\mathcal{E}).\Psi^*\xi(p_\mathcal{E})$ *when* $p_\mathcal{F} = \Psi(p_\mathcal{E})$. *Similarly:*

$$\textit{in short:}\quad (\Psi^*T)(\xi) = T(\Psi_*\xi), \quad \textit{written} \quad \Psi^*T(.) = T(\Psi_*.), \tag{8.2}$$

i.e. $(\Psi^*T)(p_\mathcal{E}).\xi(p_\mathcal{E}) = T(p_\mathcal{F}).\Psi_*\xi(p_\mathcal{F})$ *when* $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Proof. • Case $T = \alpha_\mathcal{E} \in \Omega^1(\mathcal{U}_\mathcal{E})$ (differential form = a $\binom{0}{1}$ tensor), then here $\xi = \vec{w}_\mathcal{F} \in \Gamma(\mathcal{U}_\mathcal{F})$ and we have to check: $(\Psi_*\alpha_\mathcal{E})(p_\mathcal{F}).\vec{w}_\mathcal{F}(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).\Psi^*\vec{w}_\mathcal{F}(p_\mathcal{E})$, i.e. $(\alpha_\mathcal{E}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{E})).\vec{w}_\mathcal{F}(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).(d\Psi^{-1}(p_\mathcal{E}).\vec{w}_\mathcal{F}(p_\mathcal{F}))$: True.

• Case $T = \vec{w}_\mathcal{E} \in \Gamma(\mathcal{U}_\mathcal{E})$ (vector field $\simeq$ a $\binom{1}{0}$ tensor), then here $\xi = \alpha_\mathcal{F} \in \Omega^1(\mathcal{U}_\mathcal{F})$ we have to check: $(\Psi_*\vec{w}_\mathcal{E})(p_\mathcal{F}).\alpha_\mathcal{F}(p_\mathcal{F}) = \vec{w}_\mathcal{E}(p_\mathcal{E}).\Psi^*(\alpha_\mathcal{F})(p_\mathcal{E})$, where we implicitly use to the natural canonical isomorphism $\mathcal{J} : \left\{ \begin{array}{ll} E & \to E^{**} \\ \vec{w} & \to w \overset{\text{noted}}{=} \vec{w} \end{array} \right\}$ defined by $w(\ell) = \ell.\vec{w}$ for all $\ell \in E^*$. So we have to check: $\alpha_\mathcal{F}(p_\mathcal{F}).(\Psi_*\vec{w}_\mathcal{E})(p_\mathcal{F}) = \Psi^*(\alpha_\mathcal{F})(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E})$, i.e. $\alpha_\mathcal{F}(p_\mathcal{F}).(d\Psi(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E})) = (\alpha_\mathcal{F}(p_\mathcal{F}).d\Psi(p_\mathcal{E})^{-1}).\vec{w}_\mathcal{E})(p_\mathcal{E})$: True.

For (8.2), use Ψ^{-1} instead of Ψ. ∎

8.2 Push-forward and pull-back of order 2 tensors

Definition 8.2 Let T be an order 2 tensor in $\mathcal{U}_\mathcal{E}$. Its push-forward by Ψ is the order 2 tensor Ψ_*T in $\mathcal{U}_\mathcal{F}$ defined by, for all ξ_1, ξ_2 vector field or differential form (when required) in $\mathcal{U}_\mathcal{F}$,

$$\text{in short:}\quad \Psi_*T(\xi_1,\xi_2) := T(\Psi^*\xi_1, \Psi^*\xi_2) \quad \text{written} \quad \Psi_*T(\cdot,\cdot) := T(\Psi^*\cdot, \Psi^*\cdot), \tag{8.3}$$

i.e. $\Psi_*T(p_\mathcal{F})(\xi_1(p_\mathcal{F}), \xi_2(p_\mathcal{F})) := T(p_\mathcal{E})(\Psi^*\xi_1(p_\mathcal{E}), \Psi^*\xi_2(p_\mathcal{E}))$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Let T be an order 2 tensor in $\mathcal{U}_\mathcal{F}$. Its pull-back by Ψ is the order 2 tensor Ψ^*T in $\mathcal{U}_\mathcal{E}$ defined by, for all ξ_1, ξ_2 vector field or differential form (when required) in $\mathcal{U}_\mathcal{E}$,

$$\text{in short:}\quad \Psi^*T(\xi_1,\xi_2) := T(\Psi_*\xi_1, \Psi_*\xi_2) \quad \text{written} \quad \Psi^*T(\cdot,\cdot) := T(\Psi_*\cdot, \Psi_*\cdot), \tag{8.4}$$

i.e., $\Psi^*T(p_\mathcal{E})(\xi_1(p_\mathcal{E}), \xi_2(p_\mathcal{E})) := T(p_\mathcal{F})(\Psi_*\xi_1(p_\mathcal{F}), \Psi_*\xi_2(p_\mathcal{F}))$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Example 8.3 If $T \in T_2^0(\mathcal{U}_\mathcal{E})$ (e.g., a metric) then, for all vector fields $\vec{w}_1, \vec{w}_2$ in $\mathcal{U}_\mathcal{F}$,

$$T_*(\vec{w}_1, \vec{w}_2) \overset{(8.3)}{=} T(\vec{w}_1{}^*, \vec{w}_2{}^*) = T(d\Psi^{-1}.\vec{w}_1, d\Psi^{-1}.\vec{w}_2), \tag{8.5}$$

i.e., $T_*(p_\mathcal{F})(\vec{w}_1(p_\mathcal{F}), \vec{w}_2(p_\mathcal{F})) = T(p_\mathcal{E})(d\Psi^{-1}(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F}), d\Psi^{-1}(p_\mathcal{F}).\vec{w}_2(p_\mathcal{F}))$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Expression with bases $(\vec{a}_i)$ in E and $(\vec{b}_i)$ in F: In short we have $(T_*)_{ij} = T_*(\vec{b}_i, \vec{b}_j) = T(\vec{b}_i{}^*, \vec{b}_j{}^*) = [\vec{b}_i^*]^T_{|\vec{a}}.[T]_{|\vec{a}}.[\vec{b}_j^*]_{|\vec{a}} = ([\vec{b}_i]^T_{|\vec{b}}.[d\Psi]^{-T}_{|\vec{a},\vec{b}}).[T]_{|\vec{a}}.([d\Psi]^{-1}_{|\vec{a},\vec{b}}.[\vec{b}_j]_{|\vec{b}}) = ([d\Psi]^{-T}_{|\vec{a},\vec{b}}.[T]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}})_{ij}$, thus

$$[T_*]_{|\vec{b}} = [d\Psi]^{-T}_{|\vec{a},\vec{b}}.[T]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}}, \tag{8.6}$$

which means $[(\Psi_*T)(p_\mathcal{F})]_{|\vec{b}} = ([d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}})^{-T}.[T(p_\mathcal{E})]_{|\vec{a}}.([d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}})^{-1}$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Particular case of an elementary tensor $T = \alpha_1 \otimes \alpha_2 \in T_2^0(\mathcal{U}_\mathcal{E})$, where $\alpha_1, \alpha_2 \in \Omega^1(\mathcal{U}_\mathcal{E})$, so $T(\vec{u}_1, \vec{u}_2) = (\alpha_1 \otimes \alpha_2)(\vec{u}_1, \vec{u}_2) = (\alpha_1.\vec{u}_1)(\alpha_2.\vec{u}_2)$: For all $\vec{w}_1, \vec{w}_2 \in \Gamma(\mathcal{U}_\mathcal{F})$,

$$(\alpha_1 \otimes \alpha_2)_*(\vec{w}_1, \vec{w}_2) \overset{(8.3)}{=} (\alpha_1 \otimes \alpha_2)(\vec{w}_1^*, \vec{w}_2^*) = (\alpha_1.\vec{w}_1^*)(\alpha_2.\vec{w}_2^*) \overset{(7.7)}{=} (\alpha_{1*}.\vec{w}_1)(\alpha_{2*}.\vec{w}_2), \tag{8.7}$$

thus

$$(\alpha_1 \otimes \alpha_2)_* = \alpha_{1*} \otimes \alpha_{2*}. \tag{8.8}$$

(And any tensor is a finite sum of elementary tensors.)

And for the pull-back: For all vector fields $\vec{u}_1, \vec{u}_2$ in $\mathcal{U}_\mathcal{E}$,

$$T^*(\vec{u}_1, \vec{u}_2) \overset{(8.3)}{=} T(\vec{u}_{1*}, \vec{u}_{2*}) = T(d\Psi.\vec{u}_1, d\Psi.\vec{u}_2). \tag{8.9}$$

▪

Example 8.4 If $T \in T_1^1(\mathcal{U}_\mathcal{E})$ then for all vector fields $\vec{w} \in \Gamma(\mathcal{U}_\mathcal{F})$ and differential forms $\beta \in \Omega^1(\mathcal{U}_\mathcal{F})$,

$$T_*(\beta, \vec{w}) = T(\beta^*, \vec{w}^*) = T(\beta.d\Psi, d\Psi^{-1}.\vec{w}), \tag{8.10}$$

i.e., $T_*(p_\mathcal{F})(\beta(p_\mathcal{F}), \vec{w}(p_\mathcal{F})) = T(p_\mathcal{E})(\beta(p_\mathcal{F}).d\Psi(p_\mathcal{E}), d\Psi^{-1}(p_\mathcal{F}).\vec{w}(p_\mathcal{F}))$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

For the elementary tensor $T = \vec{u} \otimes \alpha \in T_1^1(\mathcal{U}_\mathcal{E})$, made of the vector field $\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$ and of the differential form $\alpha \in \Omega^1(\mathcal{U}_\mathcal{E})$: For all $\beta, \vec{w} \in \Omega^1(\mathcal{U}_\mathcal{F}) \times \Gamma(\mathcal{U}_\mathcal{F})$, in short,

$$(\vec{u} \otimes \alpha)_*(\beta, \vec{w}) \overset{(8.3)}{=} (\vec{u} \otimes \alpha)(\beta^*, \vec{w}^*) = (\vec{u}.\beta^*)(\alpha.\vec{w}^*) \overset{(7.7)}{=} (\vec{u}_*.\beta)(\alpha_*.\vec{w}) = (\vec{u}_* \otimes \alpha_*)(\beta, \vec{w}), \tag{8.11}$$

thus

$$(\vec{u} \otimes \alpha)_* = \vec{u}_* \otimes \alpha_*. \tag{8.12}$$

Expression with bases $(\vec{a}_i)$ in E and $(\vec{b}_i)$ in F: In short we have $(T_*)_{ij} = T_*(b^i, \vec{b}_j) = T(\Psi^*(b^i), \Psi^*(\vec{b}_j)) = [\Psi^*(b^i)].[T].[\Psi^*(\vec{b}_j)] = [b^i].[d\Psi].[T].[d\Psi^{-1}].[\vec{b}_j] = ([d\Psi].[T].[d\Psi^{-1}])_{ij}$, thus

$$[T_*]_{|\vec{b}} = [d\Psi]_{|\vec{a},\vec{b}}.[T]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}}, \tag{8.13}$$

which means $[(\Psi_*T)(p_\mathcal{F})]_{|\vec{b}} = [d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}}.[T(p_\mathcal{E})]_{|\vec{a}}.[d\Psi(p_\mathcal{E})]^{-1}_{|\vec{a},\vec{b}}$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$. ▪

8.3 Push-forward and pull-back of endomorphisms

We have the natural canonical isomorphism

$$\mathcal{J}_2 : \begin{cases} \mathcal{L}(E;E) & \to \mathcal{L}(E^*, E; \mathbb{R}) \\ \qquad L & \to T_L = \mathcal{J}_2(L) \quad \text{where} \quad T_L(\alpha, \vec{u}) := \alpha.L.\vec{u}, \quad \forall (\alpha, \vec{u}) \in E^* \times E. \end{cases} \tag{8.14}$$

Thus $\Psi_*T_L(m, \vec{w}) = T_L(\Psi^*m, \Psi^*\vec{w}) = (\Psi^*m).L.(\Psi^*\vec{w}) = m.d\Psi.L.d\Psi^{-1}.\vec{w}$, thus:

Definition 8.5 The push-forward by Ψ of a field of endomorphisms L on $\mathcal{U}_\mathcal{E}$ is the field of endomorphisms $\Psi_*L = L_*$ on $\mathcal{U}_\mathcal{F}$ defined by

$$\text{in short:} \quad \Psi_*L = \boxed{L_* = d\Psi.L.d\Psi^{-1}}, \tag{8.15}$$

i.e., $L_*(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).L(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F})$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Thus with bases we get $[L_*]_{|\vec{b}} = [d\Psi]_{|\vec{a},\vec{b}}.[L]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}}$, "as in (8.13)".

Example 8.6 Elementary field of endomorphisms $L = (\mathcal{J}_2)^{-1}(\vec{u} \otimes \alpha)$, where $\vec{u} \in \Gamma(E)$ and $\alpha \in \Omega^1(E)$: So $T_L = \vec{u} \otimes \alpha$ and $L.\vec{u}_2 = (\alpha.\vec{u}_2)\vec{u}$ for all $\vec{u}_2 \in \Gamma(\mathcal{U}_\mathcal{E})$). Thus $L_*.\vec{w}_2 = d\Psi.L.d\Psi^{-1}.\vec{w}_2 = d\Psi.L.\vec{w}_2{}^* = (\alpha.\vec{w}_2{}^*)d\Psi.\vec{u} = (\alpha_*.\vec{w}_2)\vec{u}_*$ for all $\vec{w}_2 \in \Gamma(E)$, thus $(T_L)_* = \vec{u}_* \otimes \alpha_*$. ▪

Definition 8.7 Let L be a field of endomorphisms on $\mathcal{U}_\mathcal{F}$. Its pull-back by Ψ is the field of endomorphisms $\Psi^*L = L^*$ on $\mathcal{U}_\mathcal{E}$ defined by

$$\text{in short:} \quad \Psi^*L = \boxed{L^* = d\Psi^{-1}.L.d\Psi}, \tag{8.16}$$

i.e., $L^*(p_\mathcal{E}) = d\Psi^{-1}(p_\mathcal{F}).L(p_\mathcal{F}).d\Psi(p_\mathcal{E})$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

8.4 Application to derivatives of vector fields

$\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$ is a C^1 vector field in $\mathcal{U}_\mathcal{E}$), $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$, so $d\vec{u} : \mathcal{U}_\mathcal{E} \to \mathcal{L}(E;E)$ (given by $d\vec{u}(p_\mathcal{E}).\vec{w}(p_\mathcal{E}) = \lim_{h\to 0} \frac{\vec{u}(p_\mathcal{E}+h\vec{w}(p_\mathcal{E}))-\vec{u}(p_\mathcal{E})}{h}$ for all $\vec{w} \in \Gamma(\mathcal{U}_\mathcal{E})$). Thus its push-forward:

$$((d\vec{u})_* =) \quad \Psi_*(d\vec{u}) = d\Psi.d\vec{u}.d\Psi^{-1} \tag{8.17}$$

i.e. $(d\vec{u})_*(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).d\vec{u}(p_\mathcal{E}).d\Psi(p_\mathcal{E})^{-1}$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

8.5 $\Psi_*(d\vec{u})$ versus $d(\Psi_*\vec{u})$: No commutativity

Here Ψ is C^2, $\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$, $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$, $p_\mathcal{F} = \Psi(p_\mathcal{E})$, so $\Psi_*\vec{u}(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).\vec{u}(p_\mathcal{E}) = (d\Psi(\Psi^{-1}(p_\mathcal{F})).(\vec{u}(\Psi^{-1}(p_\mathcal{F}))$, and, for all $\vec{w} \in \Gamma(\mathcal{U}_\mathcal{F})$,

$$d(\Psi_*\vec{u})(p_\mathcal{F}).\vec{w}(p_\mathcal{F}) = (d^2\Psi(p_\mathcal{E}).(d\Psi^{-1}(p_\mathcal{F}).\vec{w}(p_\mathcal{F}))).\vec{u}(p_\mathcal{E}) + d\Psi(p_\mathcal{E}).d\vec{u}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}).\vec{w}(p_\mathcal{F}), \qquad (8.18)$$

with $\Psi_*(d\vec{u})(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).d\vec{u}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F})$, thus, in short,

$$d(\Psi_*\vec{u}).\vec{w} = \Psi_*(d\vec{u}).\vec{w} + d^2\Psi(\Psi^*\vec{w},\vec{u}) \neq \Psi_*(d\vec{u}) \quad \text{in general.} \qquad (8.19)$$

So the differentiation d and the push-forward $_*$ do not commute ($d(\Psi_*\vec{u}) = \Psi_*(d\vec{u})$ iff Ψ is affine).

8.6 Application to derivative of differential forms

Let $\alpha \in \Omega^1(\mathcal{U}_\mathcal{E})$ (a differential form on $\mathcal{U}_\mathcal{E}$). Its derivative $d\alpha : \mathcal{U}_\mathcal{E} \to \mathcal{L}(E;E^*)$ is given by $d\alpha(p_\mathcal{E}).\vec{u}(p_\mathcal{E}) = \lim_{h\to 0} \frac{\alpha(p_\mathcal{E}+h\vec{u}(p_\mathcal{E}))-\alpha(p_\mathcal{E})}{h} \in E^*$, for all $\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$, i.e., for all $\vec{u}_1, \vec{u}_2 \in \Gamma(\mathcal{U}_\mathcal{E})$,

$$(d\alpha(p_\mathcal{E}).\vec{u}_1(p_\mathcal{E})).\vec{u}_2(p_\mathcal{E}) = \lim_{h\to 0} \frac{\alpha(p_\mathcal{E}+h\vec{u}_1(p_\mathcal{E})).\vec{u}_2(p_\mathcal{E}) - (\alpha(p_\mathcal{E}).\vec{u}_1(p_\mathcal{E})).\vec{u}_2(p_\mathcal{E})}{h} \in \mathbb{R}. \qquad (8.20)$$

With the natural canonical isomorphism $\mathcal{L}(E;E^*) \simeq \mathcal{L}(E,E;\mathbb{R})$, cf. (U.16) with $E^{**} \simeq E$, we can write $d\alpha(p_\mathcal{E})(\vec{u}_1(p_\mathcal{E})).\vec{u}_2(p_\mathcal{E}) = d\alpha(p_\mathcal{E})(\vec{u}_1(p_\mathcal{E}),\vec{u}_2(p_\mathcal{E}))$, i.e.

$$d\alpha(\vec{u}_1).\vec{u}_2 = d\alpha(\vec{u}_1,\vec{u}_2). \qquad (8.21)$$

Thus the push-forward $\Psi_*(d\alpha) =^{\text{noted}} (d\alpha)_*$ of $d\alpha$, is given by, for all $\vec{w}_1, \vec{w}_2 \in \Gamma(\mathcal{U}_\mathcal{F})$, in short,

$$(d\alpha)_*(\vec{w}_1,\vec{w}_2) = d\alpha(\vec{w}_1^*,\vec{w}_2^*), \qquad (8.22)$$

i.e., with $p_\mathcal{F} = \Psi(p_\mathcal{E})$, $(d\alpha)_*(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F})).\vec{w}_2(p_\mathcal{F}) = (d\alpha(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F})).d\Psi^{-1}(p_\mathcal{F}).\vec{w}_2(p_\mathcal{F})$.

In particular, $(d^2f)_*(\vec{w}_1,\vec{w}_2) = d^2f(d\Psi^{-1}.\vec{w}_1, d\Psi^{-1}.\vec{w}_2)$ $(= d^2f(\vec{w}_1^*,\vec{w}_2^*))$.

8.7 $\Psi_*(d\alpha)$ versus $d(\Psi_*\alpha)$: No commutativity

Here Ψ is C^2, $\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$, $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$ and $p_\mathcal{F} = \Psi(p_\mathcal{E})$. We have $\Psi_*\alpha(p_\mathcal{F}) = \alpha(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}) = \alpha(\Psi^{-1}(p_\mathcal{F})).d\Psi^{-1}(p_\mathcal{F})$, thus, for all $\vec{w}_1 \in \Gamma(\mathcal{U}_\mathcal{F})$,

$$d(\psi_*\alpha)(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F}) = (d\alpha(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F})).d\Psi^{-1}(p_\mathcal{F}) + \alpha(p_\mathcal{E}).d^2\Psi^{-1}(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F}) \in F^*, \qquad (8.23)$$

thus, for all $\vec{w}_1, \vec{w}_2 \in \Gamma(\mathcal{U}_\mathcal{F})$, in short

$$d(\psi_*\alpha)(\vec{w}_1,\vec{w}_2) = d\alpha(d\Psi^{-1}.\vec{w}_1, d\Psi^{-1}.\vec{w}_2) + \alpha.d^2\Psi^{-1}(\vec{w}_1,\vec{w}_2) \neq d\alpha(\vec{w}_1^*,\vec{w}_2^*) \quad \text{in general.} \qquad (8.24)$$

So the differentiation d and the push-forward $_*$ do not commute ($d(\Psi_*\alpha) = \Psi_*(d\alpha)$ iff Ψ is affine).

Part III
Lie derivative

9 Lie derivative

9.0 Purpose and first results

9.0.1 Purpose?

Cauchy's approach may be insufficient, e.g.:

1. - Cauchy's approach needs to compare **two** vectors deformed by a motion, thanks to a Euclidean dot product $(\cdot,\cdot)_g$ and the deformation gradient F; Recall, the Cauchy deformation tensor C is defined by comparing $(\vec{u},\vec{w})_g$ and $(\vec{u}_*,\vec{w}_*)_g$ where $\vec{u}_* = F.\vec{u}$ and $\vec{w}_* = F.\vec{w}$ are the deformed vectors by the motion (the push-forwards independent of a stress): We have $(\vec{u}_*,\vec{w}_*)_g - (\vec{u},\vec{w})_g = ((C-I).\vec{u},\vec{w})_g$. It is a quantitative approach (needs a chosen Euclidean dot product: foot? metre?).

   - Cauchy's approach is a first order method (dedicated to linear material): Only the first order Taylor expansion of the motion is used: Only $d\Phi = F$ is used (the "slope"), not $d^2\Phi = dF$ (the "curvature") or higher derivatives.
2. - The Lie derivative $\mathcal{L}_{\vec{v}}\vec{u}$ of a vector field $\vec{u}$ measures the resistance of **one** vector field $\vec{u}$ submitted to a motion.

   - Lie's approach "naturally" applies to non-linear materials thanks to second order Lie derivatives which uses the second order Taylor expansion of the motion.

   - Lie's approach is qualitative. So no Euclidean dot product are required to begin with. (The quantification in a Galilean Euclidean framework for the first order approximation will give the usual results of Cauchy's approach.)

   - In a non planar surface S, you need the Lie derivative if you want to derive along a trajectory.

   (Cauchy died in 1857, and Lie was born in 1842.)

9.0.2 Basic results

With $\vec{v}$ the Eulerian velocity of the motion:

The Lie derivative $\mathcal{L}_{\vec{v}}f$ of a Eulerian scalar valued function f is the material derivative

$$\mathcal{L}_{\vec{v}}f = \frac{Df}{Dt}. \tag{9.1}$$

The Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ of a Eulerian vector field $\vec{w}$ is more than just the material derivative $\frac{D\vec{w}}{Dt}$:

$$\mathcal{L}_{\vec{v}}\vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}. \tag{9.2}$$

In particular the $-d\vec{v}.\vec{w}$ term in $\mathcal{L}_{\vec{v}}\vec{w}$ tells: The spatial variations $d\vec{v}$ of $\vec{v}$ act on the evolution of the stress (anticipated, $d\vec{v} = \vec{0}$ meaning $\vec{v} = \vec{cst}$).

(9.1)-(9.2) enable to define the Lie derivatives of tensors of any type and order.

9.1 Definition

9.1.1 Issue (ubiquity gift)...

$\widetilde{\Phi}$ is supposed to be regular. $\vec{v}(t,p(t)) = \frac{\partial\widetilde{\Phi}}{\partial t}(t,P_{Obj})$ is the Eulerian velocity at t at $p(t) = \widetilde{\Phi}(t,P_{Obj})$. Recall: If $\mathcal{E}ul$ is a Eulerian function then its material time derivative is

$$\frac{D\mathcal{E}ul}{Dt}(t,p(t)) = \lim_{h\to 0}\frac{\mathcal{E}ul(t+h,p(t+h)) - \mathcal{E}ul(t,p(t))}{h}. \tag{9.3}$$

Issue: The rate $\frac{\mathcal{E}ul(t+h,p(t+h))-\mathcal{E}ul(t,p(t))}{h}$ raises questions:

1- The difference $\mathcal{E}ul(t+h,p(t+h)) - \mathcal{E}ul(t,p(t))$ requires the time and space ubiquity gift to be calculated by an observer, since it mixes two distinct times, t and $t+h$, and two distinct locations, $p(t)$ and $p(t+h)$.

2- The difference $\mathcal{E}ul(t+h,p(t+h)) - \mathcal{E}ul(t,p(t))$ can be impossible: E.g. if $\mathcal{E}ul = \vec{w}$ is a vector field in a "non planar surface considered on its own" (manifold) then $\mathcal{E}ul(t+h,p(t+h))$ and $\mathcal{E}ul(t,p(t))$ don't belong to the same (tangent) vector space, so the difference $\vec{w}(t+h,p(t+h)) - \vec{w}(t,p(t))$ is meaningless.

9.1.2 ...Toward a solution (without ubiquity gift)...

To compare $\mathcal{E}ul(t{+}h, p(t{+}h))$ and $\mathcal{E}ul(t, p(t))$ (to get the evolution of $\mathcal{E}ul$ along a trajectory), you need the duration h to get from t to $t{+}h$ and to move from $p(t)$ to $p(t{+}h)$. So, you must:

- take the value $\mathcal{E}ul(t, p_t))$ with you (for memory),
- move along the considered trajectory, and doing so, the value $\mathcal{E}ul(t, p_t)$ has possibly changed to, with $\tau = t{+}h$,

$$((\Phi^t_\tau)_*\mathcal{E}ul_t)(p_\tau) \overset{\text{noted}}{=} \mathcal{E}ul_{t*}(\tau, p_\tau) \quad \text{(push-forward);} \tag{9.4}$$

- And now, at (τ, p_τ) where you are, you can compare the actual value $\mathcal{E}ul(\tau, p_\tau)$ with the value $\mathcal{E}ul_{t*}(\tau, p_\tau)$ you arrived with (the transported memory), thus the difference

$$\mathcal{E}ul(\tau, p_\tau) - \mathcal{E}ul_{t*}(\tau, p_\tau) \tag{9.5}$$

is meaningful for a human being since it is computed at a unique time τ and at a unique point p_τ (no gift of ubiquity required).

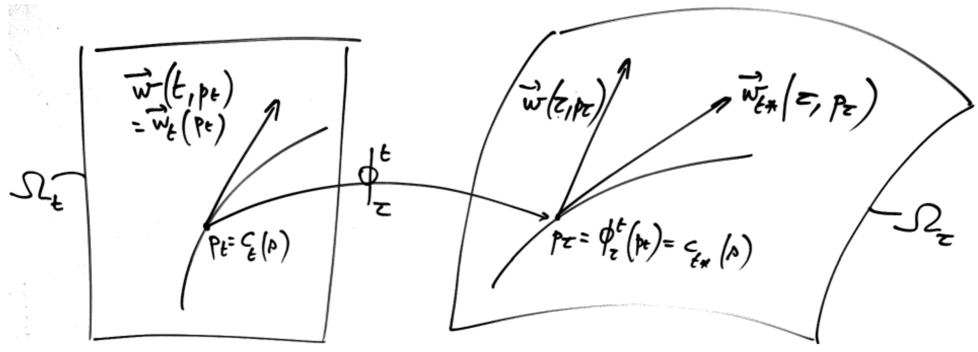


Figure 9.1: To compute (9.5) with $\mathcal{E}ul = \vec{w}$ a (Eulerian) vector field: At t define the vector field $\vec{w}_t$ in Ω_t by $\vec{w}_t(p_t) := \vec{w}(t, p_t)$. The (spatial) curve $c_t : s \to p_t = c_t(s)$ in Ω_t is an integral curve of $\vec{w}_t$, i.e. satisfies $c_t{}'(s) = \vec{w}_t(c_t(s))$. c_t is transformed by Φ^t_τ into the (spatial) curve $c_\tau = \Phi^t_\tau \circ c_t : s \to p_\tau = c_\tau(s) = \Phi^t_\tau(c_t(s))$ in Ω_τ; Hence $c_\tau{}'(s) = d\Phi^t_\tau(p_t).c'(s) = d\Phi^t_\tau(p_t).\vec{w}_t(p_t) =^{\text{noted}} \vec{w}_{t*}(\tau, p_\tau)$ is the tangent vector at c_τ at p_τ (push-forward). Thus the difference $\vec{w}(\tau, p_\tau) - \vec{w}_{t*}(\tau, p_\tau)$ can be computed by a human being, i.e. without ubiquity gift.

9.1.3 ... The Lie derivative, first definition

Motion $\widetilde{\Phi} : (t, P_{Obj}) \to p(t) = \widetilde{\Phi}(t, P_{Obj})$, Eulerian velocity given by $\vec{v}(t, p(t)) = \frac{\partial \widetilde{\Phi}}{\partial t}(t, P_{Obj})$ (velocity of P_{Obj} at t). Eulerian function $\mathcal{E}ul$, and $\mathcal{E}ul_t(p_t) := \mathcal{E}ul(t, p_t)$, and $\mathcal{E}ul_{t*}(\tau, p_\tau) := ((\Phi^t_\tau)_*\mathcal{E}ul_t)(p_\tau)$, cf. (9.4).

Definition 9.1 The Lie derivative $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ along $\vec{v}$ of an Eulerian function $\mathcal{E}ul$ is the Eulerian function $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ defined by, at t at $p_t := p(t) = \widetilde{\Phi}(t, P_{Obj})$,

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t, p_t) := \lim_{\tau \to t} \frac{\mathcal{E}ul_\tau(p_\tau) - ((\Phi^t_\tau)_*\mathcal{E}ul_t)(p_\tau)}{\tau - t} = \lim_{\tau \to t} \frac{\mathcal{E}ul(\tau, p_\tau) - \mathcal{E}ul_{t*}(\tau, p_\tau)}{\tau - t}. \tag{9.6}$$

Interpretation: $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ measures the rate of change of $\mathcal{E}ul$ along a trajectory:

- $\mathcal{E}ul(\tau, p_\tau)$ is the value of $\mathcal{E}ul$ at τ at p_τ, see figure 9.1 with $\mathcal{E}ul = \vec{w}$.
- $\mathcal{E}ul_{t*}(\tau, p_\tau) = ((\Phi^t_\tau)_*\mathcal{E}ul_t)(\tau, p_\tau)$ is exclusively strain related (kinematic): It is the memory transported by the flow.

In other words, with g defined by

$$g(\tau) = ((\Phi^t_\tau)_*\mathcal{E}ul_t)(p(\tau)) \tag{9.7}$$

(in particular $g(t) = \mathcal{E}ul_t(p_t)$):

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t, p_t) := g'(t) = \lim_{\tau \to t} \frac{g(\tau) - g(t)}{\tau - t} \overset{\text{also written}}{=} \frac{d((\Phi^t_\tau)_*\mathcal{E}ul_t)(p(\tau))}{d\tau}_{|\tau = t}. \tag{9.8}$$

Remark 9.2 More precise definition, as in (2.3):

$$\widetilde{\mathcal{L}}_{\vec{v}}\mathcal{E}ul(t, p_t) := ((t, p_t), \mathcal{L}_{\vec{v}}\mathcal{E}ul(t, p_t)) \quad \text{(pointed function at } (t, p_t)), \tag{9.9}$$

And, to lighten the notation, $\widetilde{\mathcal{L}}_{\vec{v}}\mathcal{E}ul(t, p_t) =^{\text{noted}} \mathcal{L}_{\vec{v}}\mathcal{E}ul(t, p_t)$ (second component of $\widetilde{\mathcal{L}}_{\vec{v}}\mathcal{E}ul(t, p_t)$). ▪

9.1.4 A more general definition

The rate in (9.6) has to be slightly modified to be adequate in all situations: $\mathcal{E}ul(\tau,p_\tau)-\mathcal{E}ul_{t*}(\tau,p_\tau)$ is computed at (τ,p_τ) which moves as $\tau\to t$, and on a "non-planar manifold" this is problematic (the tangent plane changes with τ). The "natural" definition is to arrive with the memory (you can't rejuvenate):

Definition 9.3 The Lie derivative $\mathcal{L}_{\vec v}\mathcal{E}ul$ of an Eulerian function $\mathcal{E}ul$ along $\vec v$ is the Eulerian function $\mathcal{L}_{\vec v}\mathcal{E}ul$ defined by, at t at $p_t=\widetilde\Phi_{P_{Obj}}(t)$,

$$\mathcal{L}_{\vec v}\mathcal{E}ul(t,p_t):=\lim_{h\to0}\frac{\mathcal{E}ul_t(p_t)-(\Phi_t^{t-h})_*\mathcal{E}ul_{t-h}(p_t)}{h}=\lim_{\tau\to t}\frac{\mathcal{E}ul_t(p_t)-(\Phi_t^{\tau})_*\mathcal{E}ul_{\tau}(p_t)}{t-\tau}. \tag{9.10}$$

(The rate is calculated at (t,p_t), and a human being can't rejuvenate so he takes $h>0$, i.e. $\tau<t$.)

In other words, with $\bar g$ defined by

$$\bar g(\tau)=((\Phi_t^\tau)_*\mathcal{E}ul_\tau)(p_t) \tag{9.11}$$

(in particular $\bar g(t)=\mathcal{E}ul(t,p_t)$):

$$\mathcal{L}_{\vec v}\mathcal{E}ul(t,p_t):=\bar g'(t)=\lim_{\tau\to t}\frac{\bar g(t)-\bar g(\tau)}{t-\tau}=\lim_{\tau\to t}\frac{\bar g(\tau)-\bar g(t)}{\tau-t}\overset{\text{also written}}{=}\frac{d((\Phi_t^\tau)_*\mathcal{E}ul_\tau)(p_t)}{d\tau}_{|\tau=t}. \tag{9.12}$$

Here the observer must:

- At $\tau=t-h$ at $p(\tau)=p(t-h)=\widetilde\Phi_{P_{Obj}}(t-h)$, take the value $\mathcal{E}ul(\tau,p(\tau))$ (memory),
- move along the trajectory $\widetilde\Phi_{P_{Obj}}$,
- once at t at $p_t=\widetilde\Phi_{P_{Obj}}(t)$, the memory turned into $((\Phi_t^\tau)_*\mathcal{E}ul_\tau)(p_t)$,
- which can be compared with $\mathcal{E}ul(t,p_t)$ without any ubiquity gift.

Exercice 9.4 Prove: (9.6) and (9.10) are equivalent.

Answer. With $(\Phi^t_{t+h})^*.(\Phi^t_{t+h})_*=I$, (9.6) gives $\mathcal{L}_{\vec v}\mathcal{E}ul(t,p_t)=\lim_{h\to0}\frac{(\Phi^t_{t+h})^*\mathcal{E}ul(t,p_t)-\mathcal{E}ul_t(t,p_t)}{h}=\lim_{h\to0}\frac{(\Phi^t_{t-h})^*\mathcal{E}ul_{t-h})(p_t)-\mathcal{E}ul_t(p_t)}{-h}=\lim_{h\to0}\frac{\mathcal{E}ul_t(p_t)-((\Phi^t_{t-h})^*\mathcal{E}ul_{t-h})(p_t)}{h}$, and use $(\Phi^t_{t-h})^*=(\Phi_t^{t-h})_*$. ∎

9.1.5 Equivalent definition (differential geometry)

Definition 9.5 The Lie derivative of a Eulerian function $\mathcal{E}ul$ along a flow of Eulerian velocity $\vec v$ is the Eulerian function $\mathcal{L}_{\vec v}\mathcal{E}ul$ defined at (t,p_t) by

$$\mathcal{L}_{\vec v}\mathcal{E}ul(t,p_t):=\lim_{\tau\to t}\frac{((\Phi^t_\tau)^*\mathcal{E}ul_\tau)(p_t)-\mathcal{E}ul(t,p_t)}{\tau-t}=\lim_{h\to0}\frac{((\Phi^t_{t+h})^*\mathcal{E}ul_{t+h})(p_t)-\mathcal{E}ul(t,p_t)}{h}. \tag{9.13}$$

In other words, with $\hat g$ defined by

$$\hat g(\tau)=((\Phi^t_\tau)^*\mathcal{E}ul_\tau)(p_t) \tag{9.14}$$

(in particular $\hat g(t)=\mathcal{E}ul(t,p_t)$):

$$\mathcal{L}_{\vec v}\mathcal{E}ul(t,p_t):=\hat g'(t)=\lim_{\tau\to t}\frac{\hat g(\tau)-\hat g(t)}{\tau-t}\overset{\text{also written}}{=}\frac{d((\Phi^t_\tau)^*\mathcal{E}ul_\tau)(p_t)}{d\tau}_{|\tau=t}. \tag{9.15}$$

Exercice 9.6 Prove: (9.10) and (9.13) are equivalent.

Answer. (9.13) also reads $\mathcal{L}_{\vec v}\mathcal{E}ul(t,p_t)=\lim_{h\to0}\frac{((\Phi^t_{t-h})^*\mathcal{E}ul_{t-h})(p_t)-\mathcal{E}ul_t(p_t)}{-h}$, and $(\Phi^t_{t-h})^*.(\Phi_t^{t-h})_*=I$. ∎

9.2 Lie derivative of a scalar function

Let f be a C^1 Eulerian scalar valued function. With $(\Phi_t^{t-h})_*f_{t-h}(p_t)=f_{t-h}(p(t-h))$, cf. (6.10), we get

$$\mathcal{L}_{\vec v}f(t,p_t)\overset{(9.10)}{=}\lim_{h\to0}\frac{f(t,p_t)-f(t-h,p(t-h))}{h},\quad\text{i.e.}\quad \boxed{\mathcal{L}_{\vec v}f=\frac{Df}{Dt}}=\frac{\partial f}{\partial t}+df.\vec v. \tag{9.16}$$

So, for scalar functions, the Lie derivative is the material derivative.

Interpretation: $\mathcal{L}_{\vec v}f$ measures the rate of change of f along a trajectory.

Proposition 9.7 *$\mathcal{L}_{\vec v} f = 0$ iff f is constant along any trajectory (the real value is the memory value):*

$$\mathcal{L}_{\vec v} f = 0 \quad\Longleftrightarrow\quad \forall t,\tau \in [t_0,T],\ (\Phi^t_{\tau*}) f_t(p_\tau) = f(t,p(t)) \ \text{ when } \ p_\tau = \Phi^t_\tau(p_t), \tag{9.17}$$

i.e. iff $f(t,p(t)) = f(t_0,p_{t_0})$ when $p(t) = \Phi^{t_0}(t,p_{t_0})$, i.e. iff f let itself be carried by the flow (unchanged).

Proof. Let $p(t) = \widetilde\Phi(t,P_{Obj}) = p_t$ for all t, so $p(\tau) = \widetilde\Phi(\tau,P_{Obj}) = p_\tau = \Phi^t_{t+h}(p_t) = \Phi^t(\tau,p_t)$.

$\Leftarrow$: If $f_\tau = (\Phi^t_{t+h})_* f_t$, then $f_\tau(p_\tau) = f_t(p_t)$, thus $\lim_{\tau\to t} \frac{f(\tau,p(\tau)) - f(t,p(t))}{\tau - t} = 0$, that is, $\frac{Df}{Dt} = 0$.

$\Rightarrow$: If $\frac{Df}{Dt} = 0$ then $f(t,p(t))$ is a constant function on the trajectory $t \to \widetilde\Phi(t,P_{Obj})$, for any particle P_{Obj}, so $f(\tau,p(\tau)) = f(t,p_t)$ when $p(\tau) = \Phi^t_{t+h}(p_t)$, that is, $f(\tau,p_\tau) = (\Phi^t_{t+h})_* f_t(p_\tau)$. ∎

Exercice 9.8 Prove: $\mathcal{L}_{\vec v}(\mathcal{L}_{\vec v} f) = \frac{D^2 f}{Dt^2} = \frac{\partial^2 f}{\partial t^2} + 2d(\frac{\partial f}{\partial t}).\vec v + d^2 f(\vec v,\vec v) + df.(\frac{\partial \vec v}{\partial t} + d\vec v)$.

Answer. See (2.28). ∎

9.3 Lie derivative of a vector field

9.3.1 Formula

Proposition 9.9 *Let $\vec w$ be a C^1 (Eulerian) vector field. We have*

$$\boxed{\mathcal{L}_{\vec v}\vec w = \frac{D\vec w}{Dt} - d\vec v.\vec w} = \frac{\partial \vec w}{\partial t} + d\vec w.\vec v - d\vec v.\vec w. \tag{9.18}$$

So the Lie derivative is not reduced to the material derivative $\frac{D\vec w}{Dt}$ (unless $d\vec v = 0$, i.e. unless $\vec v$ is uniform): The spatial variations $d\vec v$ of $\vec v$ influences the rate of stress: $\vec v$ tries to bend $\vec w$ (which is expected).

Proof. Here (9.14) reads $\vec g(\tau) = d\Phi^t_\tau(p_t)^{-1}.\vec w(\tau,p(\tau))$, and (9.15) reads $\vec g\,'(t) = \mathcal{L}_{\vec v}\vec w(t,p(t))$. Since $\vec w(\tau,p(\tau)) = d\Phi^t_\tau(p_t).\vec g(\tau) = d\Phi^t(\tau,p_t).\vec g(\tau)$ we get

$$\frac{D\vec w}{D\tau}(\tau,p(\tau)) = \underbrace{\frac{\partial(d\Phi^t)}{\partial\tau}(\tau,p_t)}_{d\vec v(\tau,p(\tau)).d\Phi^t(\tau,p_t)} . \underbrace{\vec g(\tau)}_{d\Phi^t(\tau,p_t)^{-1}.\vec w(\tau,p(\tau))} + \underbrace{d\Phi^t(\tau,p_t)}_{F^t_\tau(p_t)} . \underbrace{\vec g\,'(\tau)}_{\mathcal{L}_{\vec v}\vec w(\tau,p(\tau))} \tag{9.19}$$

Thus $\frac{D\vec w}{Dt}(t,p_t) = d\vec v(t,p_t).\vec w(t,p_t) + I.\mathcal{L}_{\vec v}\vec w(t,p_t)$, thus (9.18). ∎

Quantification: Basis $(\vec e_i)$, $\vec v = \sum_i v_i \vec e_i$, $\vec w = \sum_i w_i \vec e_i$, $d\vec v.\vec e_j = \sum_{ij} v_{i|j}\vec e_i$, $d\vec w.\vec e_j = \sum_{ij} w_{i|j}\vec e_i$; Then

$$\mathcal{L}_{\vec v}\vec w = \sum_{i=1}^n \frac{\partial w_i}{\partial t}\vec e_i + \sum_{i,j=1}^n w_{i|j} v_j \vec e_i - \sum_{i,j=1}^n v_{i|j} w_j \vec e_i. \tag{9.20}$$

So, with $[\cdot] := [\cdot]_{|\vec e}$,

$$\boxed{[\mathcal{L}_{\vec v}\vec w] = [\frac{D\vec w}{Dt}] - [d\vec v].[\vec w]} \quad (= [\frac{\partial \vec w}{\partial t}] + [d\vec w.\vec v] - [d\vec v].[\vec w]). \tag{9.21}$$

(And $[d\vec w.\vec v] = [d\vec w].[\vec v]$.) Duality notations: $\mathcal{L}_{\vec v}\vec w = \sum_i \frac{\partial w^i}{\partial t}\vec e_i + \sum_{ij} w^i_{|j} v^j \vec e_i - \sum_{ij} v^i_{|j} w^j \vec e_i$.

9.3.2 Interpretation: Flow resistance measurement

Proposition 9.10 *Φ^{t_0} is supposed to be a C^2 motion and a C^1 diffeomorphism in space, and $\vec w$ is a vector field.*

$$\mathcal{L}_{\vec v}\vec w = 0 \quad\Longleftrightarrow\quad \forall t \in [t_0,T],\ \ \vec w_t = (\Phi^{t_0}_t)_* \vec w_{t_0}. \tag{9.22}$$

i.e., $\frac{D\vec w}{Dt} = d\vec v.\vec w \Leftrightarrow$ the actual vector $\vec w(t,p(t))$ is equal to $F^{t_0}_t(p_{t_0}).\vec w_{t_0}(p_{t_0}) = \vec w_{t_0*}(t,p(t))$ the deformed vector by the flow, see figure 9.1. So: The Lie derivative $\mathcal{L}_{\vec v}\vec w$ vanishes iff $\vec w$ does not resist the flow (let itself be deformed by the flow), i.e. iff $\vec w(t,p_t) = \vec w_{t_0*}(t,p_t)$.

Proof. We have $\mathcal{L}_{\vec v}\vec w = \frac{D\vec w}{Dt} - d\vec v.\vec w$ and $\frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}) = d\vec v(t,p(t)).F_t^{t_0}(p_{t_0})$, cf. (3.33).

$\Leftarrow$ (derivation): Suppose $\vec w(t,p(t)) = F^{t_0}(t,p_{t_0}).\vec w(t_0,p_{t_0})$ when $p(t) = \Phi_t^{t_0}(p_{t_0})$. Then $\frac{D\vec w}{Dt}(t,p(t)) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).\vec w(t_0,p_{t_0}) = (d\vec v(t,p(t)).F_t^{t_0}(p_{t_0})).(F_t^{t_0}(p_{t_0})^{-1}.\vec w(t,p(t))) = d\vec v(t,p(t)).\vec w(t,p(t))$, thus $\frac{D\vec w}{Dt} - d\vec v.\vec w = 0$. (See proposition 3.14.)

$\Rightarrow$ (integration): Suppose $\frac{D\vec w}{Dt} = d\vec v.\vec w$. Let $\vec f(t) = (F_t^{t_0}(p_{t_0}))^{-1}.\vec w(t,p(t))$ (= pull-back $(\Phi_t^{t_0})^*\vec w(t_0,p_{t_0})$) when $p(t) = \Phi^{t_0}(t,p_{t_0})$; So $\vec w(t,p(t)) = F^{t_0}(t,p_{t_0}).\vec f(t)$ and $\frac{D\vec w}{Dt}(t,p(t)) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).\vec f(t) + F_t^{t_0}(p_{t_0}).\vec f\,'(t) = d\vec v(t,p(t)).F_t^{t_0}(p_{t_0}).\vec f(t) + F_t^{t_0}(p_{t_0}).\vec f\,'(t) = d\vec v(t,p(t)).\vec w(t,p(t)) + F_t^{t_0}(p_{t_0}).\vec f\,'(t) =^{hyp.} \frac{D\vec w}{Dt}(t,p(t)) + F_t^{t_0}(p_{t_0}).\vec f\,'(t)$ for all t; Thus $F_t^{t_0}(p_{t_0}).\vec f\,'(t) = \vec 0$, thus $\vec f\,'(t) = \vec 0$ (because $\Phi_t^{t_0}$ is a diffeomorphism), thus $\vec f(t) = \vec f(t_0)$, i.e. $\vec w_t = (\Phi_t^{t_0})_*\vec w_{t_0}$, for all t. ∎

9.3.3 Autonomous Lie derivative and Lie bracket

The Lie bracket of two vector fields $\vec v$ and $\vec w$ is

$$[\vec v,\vec w] := d\vec w.\vec v - d\vec v.\vec w \overset{\text{noted}}{=} \mathcal{L}^0_{\vec v}\vec w. \tag{9.23}$$

And $\mathcal{L}^0_{\vec v}\vec w = [\vec v,\vec w]$ is called the autonomous Lie derivative of $\vec w$ along $\vec v$. Thus

$$\mathcal{L}_{\vec v}\vec w = \frac{\partial \vec w}{\partial t} + [\vec v,\vec w] = \frac{\partial \vec w}{\partial t} + \mathcal{L}^0_{\vec v}\vec w. \tag{9.24}$$

NB: $\mathcal{L}^0_{\vec v}\vec w$ is used when $\vec v$ et $\vec w$ are stationary vector fields, thus does not concern objectivity: A stationary vector field in a referential is not necessary stationary in another (moving) referential.

9.4 Examples

9.4.1 Lie Derivative of a vector field along itself

(9.18) with $\vec w = \vec v$ gives $\mathcal{L}_{\vec v}\vec v = \frac{\partial \vec v}{\partial t}$. In particular, if $\vec v$ is a stationary vector field then $\mathcal{L}_{\vec v}\vec v = \vec 0$ $(= [\vec v,\vec v])$.

9.4.2 Lie derivative along a uniform flow

Here $d\vec v = 0$, thus

$$\mathcal{L}_{\vec v}\vec w = \frac{D\vec w}{Dt} = \frac{\partial \vec w}{\partial t} + d\vec w.\vec v \quad (\text{when } d\vec v = 0). \tag{9.25}$$

Here the flow is rectilinear ($d\vec v = 0$): there is no curvature (of the flow) to influence the stress on $\vec w$.

Moreover, if $\vec w$ is stationary, that is $\frac{\partial \vec w}{\partial t} = 0$, then $\mathcal{L}_{\vec v}\vec w = d\vec w.\vec v =$ the directional derivative $\frac{\partial \vec w}{\partial \vec v}$ of the vector field $\vec w$ in the direction $\vec v$.

9.4.3 Lie derivative of a uniform vector field

Here $d\vec w(t,p) = 0$, thus

$$\mathcal{L}_{\vec v}\vec w = \frac{\partial \vec w}{\partial t} - d\vec v.\vec w \quad (\text{when } d\vec w = 0), \tag{9.26}$$

thus the stress on $\vec w$ is due to the space variations of $\vec v$. Moreover, is $\vec w$ is stationary then $\mathcal{L}_{\vec v}\vec w = -d\vec v.\vec w$.

9.4.4 Uniaxial stretch of an elastic material

- Strain. With $[\overrightarrow{OP}]_{|\vec e} = [\vec X]_{|\vec e} = \begin{pmatrix} X \\ Y \end{pmatrix}$, with $\xi > 0$, $t \geq t_0$, $p(t) = \Phi^{t_0}(t,P)$ and $[\vec x]_{|\vec e} = [\overrightarrow{Op(t)}]_{|\vec e}$:

$$[\vec x]_{|\vec e} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} + \xi(t-t_0)\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X(1+\xi(t-t_0)) \\ Y \end{pmatrix}. \tag{9.27}$$

- Eulerian velocity $\vec v(t,p) = \begin{pmatrix} \xi X \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\xi}{1+\xi(t-t_0)}x \\ 0 \end{pmatrix}$, $d\vec v(t,p) = \begin{pmatrix} \frac{\xi}{1+\xi(t-t_0)} & 0 \\ 0 & 0 \end{pmatrix}$ (independent of p).

• Deformation gradient (independent of P), with $\kappa_t = \xi(t-t_0)$:

$$F_t = d\Phi_t^{t_0}(P) = \begin{pmatrix} 1+\kappa_t & 0 \\ 0 & 1 \end{pmatrix} = I + \kappa_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{9.28}$$

Infinitesimal strain tensor, with $F_t^T = F_t$ here:

$$\underline{\underline{\varepsilon}}_t^{t_0}(P) = F_t - I = \kappa_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \underline{\underline{\varepsilon}}_t. \tag{9.29}$$

• Stress. Constitutive law = Linear isotropic elasticity:

$$\underline{\underline{\sigma}}_t(p_t) = \lambda \mathrm{Tr}(\underline{\underline{\varepsilon}}_t) I + 2\mu \underline{\underline{\varepsilon}}_t = \kappa_t \begin{pmatrix} \lambda+2\mu & 0 \\ 0 & \lambda \end{pmatrix} = \underline{\underline{\sigma}}_t. \tag{9.30}$$

Cauchy stress vector $\vec{T}$ on a surface at p with normal $\vec{n}_t(p) = \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} = \vec{n}$:

$$\vec{T}_t(p_t) = \underline{\underline{\sigma}}_t.\vec{n} = \kappa_t \begin{pmatrix} (\lambda+2\mu)n_1 \\ \lambda n_2 \end{pmatrix} = \xi(t-t_0) \begin{pmatrix} (\lambda+2\mu)n_1 \\ \lambda n_2 \end{pmatrix} = \vec{T}_t. \tag{9.31}$$

• Push-forwards: $\vec{T}_{t_0}(p_{t_0}) = 0$, thus $F_{t_0+h}^{t_0}(p_{t_0}).\vec{T}_{t_0}(p_{t_0}) = \vec{0}$.
• Lie derivative:

$$\mathcal{L}_{\vec{v}}\vec{T}(t_0, p_{t_0}) = \lim_{t\to t_0} \frac{\vec{T}_t(p_t) - F_t^{t_0}(p_{t_0}).\vec{T}_{t_0}(p_{t_0})}{t-t_0} = \xi \begin{pmatrix} (\lambda+2\mu)n_1 \\ \lambda n_2 \end{pmatrix} \quad \text{(rate of stress at } (t_0, p_{t_0})). \tag{9.32}$$

• Generic computation with $\mathcal{L}_{\vec{v}}\vec{T} = \frac{\partial \vec{T}}{\partial t} + d\vec{T}.\vec{v} - d\vec{v}.\vec{T}$: (9.31) gives $\frac{\partial \vec{T}}{\partial t} = \xi \begin{pmatrix} (\lambda+2\mu)\, n^1 \\ \lambda\, n^2 \end{pmatrix}$ and $d\vec{T} = 0$ and $d\vec{v}_t.\vec{T}_t = \begin{pmatrix} \frac{\xi}{1+\xi(t-t_0)} & 0 \\ 0 & 0 \end{pmatrix}.\xi(t-t_0) \begin{pmatrix} (\lambda+2\mu)\, n^1 \\ \lambda\, n^2 \end{pmatrix} = \frac{\xi^2(t-t_0)}{1+\xi(t-t_0)} \begin{pmatrix} (\lambda+2\mu)\, n^1 \\ 0 \end{pmatrix}$. In particular, $d\vec{v}(t_0, p_{t_0}).\vec{T}(t_0, p_{t_0}) = \vec{0}$. Thus $\mathcal{L}_{\vec{v}}\vec{T}(t_0, p_{t_0}) = \xi \begin{pmatrix} (\lambda+2\mu)\, n^1 \\ \lambda\, n^2 \end{pmatrix}$ = rate of stress at the initial (t_0, p_{t_0}).

9.4.5 Simple shear of an elastic material

Fixed Euclidean basis $(\vec{e}_1, \vec{e}_2)$ in $\mathbb{R}^2$ at all time. Initial configuration $\Omega_{t_0} = [0, L_1]\otimes[0, L_2]$. Initial position: $[\overrightarrow{OP}]_{\vec{e}} = [\overrightarrow{Op_{t_0}}]_{\vec{e}} = [\vec{X}]_{\vec{e}} = \begin{pmatrix} X \\ Y \end{pmatrix} =^{\text{noted}} \vec{X}$. Position at t: $p_t = \Phi_t^{t_0}(p_{t_0})$, $[\vec{x}]_{|\vec{e}} = [\overrightarrow{Op(t)}]_{|\vec{e}} =^{\text{noted}} \vec{x}$. Let $\xi \in \mathbb{R}^*$, and

$$\vec{x} = \begin{pmatrix} x = \varphi^1(t, X, Y) \\ y = \varphi^2(t, X, Y) \end{pmatrix} = \begin{pmatrix} X + \xi(t-t_0)Y \\ Y \end{pmatrix} = \begin{pmatrix} X + \kappa(t)Y \\ Y \end{pmatrix} \quad \text{where} \quad \kappa(t) = \xi(t-t_0) = \kappa_t. \tag{9.33}$$

• Deformation gradient:

$$d\Phi_t^{t_0}(P) = \begin{pmatrix} 1 & \kappa_t \\ 0 & 1 \end{pmatrix} = F_t^{t_0}, \quad \text{thus} \quad F_t^{t_0} - I = \kappa_t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9.34}$$

• Lagrangian velocity $\vec{V}_t(p_{t_0}) = \begin{pmatrix} \xi Y \\ 0 \end{pmatrix}$, thus $d\vec{V}_t(p_{t_0}) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}$.
• Eulerian velocity $\vec{v}_t(p_t) = \vec{V}_t(p_{t_0}) = \begin{pmatrix} \xi y \\ 0 \end{pmatrix}$, thus $d\vec{v}_t(p_t) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}$. (We check that $d\vec{v}.F = d\vec{V}$.)
• Infinitesimal strain tensor:

$$\underline{\underline{\varepsilon}}_t^{t_0}(P) = \frac{F_t^{t_0}(P) - I + (F_t^{t_0}(P) - I)^T}{2} = \frac{\kappa_t}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{\varepsilon}}_t^{t_0} \overset{\text{noted}}{=} \underline{\underline{\varepsilon}}_t. \tag{9.35}$$

• Stress. Constitutive law, usual linear isotropic elasticity (requires a Euclidean dot product):

$$\underline{\underline{\sigma}}(t, p_t) = \lambda \mathrm{Tr}(\underline{\underline{\varepsilon}}_t) I + 2\mu \underline{\underline{\varepsilon}}_t = \mu\kappa_t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{\sigma}}_t. \tag{9.36}$$

Cauchy stress vector $\vec{T}(t,p_t)$ (at t at p_t) on a surface at p with normal $\vec{n}_t(p) = \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} = \vec{n}$:

$$\vec{T}_t = \underline{\underline{\sigma}}_t.\vec{n} = \mu\kappa_t \begin{pmatrix} n^2 \\ n^1 \end{pmatrix} = \mu\xi(t{-}t_0) \begin{pmatrix} n^2 \\ n^1 \end{pmatrix} = \vec{T}(t) \quad \text{(stress independent of } p_t). \tag{9.37}$$

• Lie derivative, with $\vec{T}_{t_0} = \vec{0}$:

$$\mathcal{L}_{\vec{v}}\vec{T}(t_0,p_{t_0}) = \lim_{t\to t_0} \frac{\vec{T}_t(p_t) - F_t^{t_0}(p_{t_0}).\vec{T}_{t_0}(p_{t_0})}{t-t_0} = \mu\xi \begin{pmatrix} n^2 \\ n^1 \end{pmatrix} \quad \text{(rate of stress at } (t_0,p_{t_0})). \tag{9.38}$$

• Generic computation: $\mathcal{L}_{\vec{v}}\vec{T} = \frac{\partial \vec{T}}{\partial t} + d\vec{T}.\vec{v} - d\vec{v}.\vec{T}$. (9.37) gives $\frac{\partial \vec{T}}{\partial t}(t,p) = \mu\xi \begin{pmatrix} n^2 \\ n^1 \end{pmatrix}$ and $d\vec{T} = 0$. With $d\vec{v}_{t_0}.\vec{T}_{t_0} = \vec{0}$. Thus $\mathcal{L}_{\vec{v}}\vec{T}(t_0,p_{t_0}) = \mu\xi \begin{pmatrix} n^2 \\ n^1 \end{pmatrix}$.

9.4.6 Shear flow

Stationary shear field, see (5.11) with $\alpha = 0$ and $t_0 = 0$ (or see (9.33) with $\xi = \lambda$):

$$\vec{v}(x,y) = \begin{cases} v^1(x,y) = \lambda y, \\ v^2(x,y) = 0, \end{cases} \qquad d\vec{v}(x,y) = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}. \tag{9.39}$$

Let $\vec{w}(t,p) = \begin{pmatrix} 0 \\ b \end{pmatrix} = \vec{w}(t_0,p_{t_0})$ (constant in time and uniform in space). Then $\mathcal{L}_{\vec{v}}\vec{w} = -d\vec{v}.\vec{w} = \begin{pmatrix} -\lambda b \\ 0 \end{pmatrix}$ measures "the resistance to deformation due to the flow". See figure 9.2, the virtual vector $\vec{w}_*(t,p) = d\Phi(t_0,p_{t_0}).\vec{w}(t_0,p_{t_0})$ being the vector that would have let itself be carried by the flow (the push-forward).

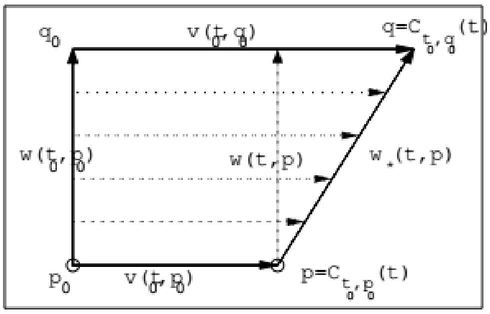


Figure 9.2: Shear flow, cf. (9.39), with $\vec{w}$ constant and uniform. $\mathcal{L}_{\vec{v}}\vec{w}$ measures the resistance to the deformation.

9.4.7 Spin

Rotating flow: Continuing (5.14):

$$\vec{v}(x,y) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad d\vec{v}(x,y) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \omega\, Rot(\pi/2). \tag{9.40}$$

In particular $d^2\vec{v} = 0$. With $\vec{w} = \vec{w}_0$ constant and uniform we get

$$\mathcal{L}_{\vec{v}}\vec{w}_0 = -d\vec{v}(p).\vec{w}_0 = -\omega\, Rot(\pi/2).\vec{w}_0 \qquad (\perp \begin{pmatrix} a \\ b \end{pmatrix} = \vec{w}_0). \tag{9.41}$$

gives "the force at which $\vec{w}$ refuses to turn with the flow".

9.4.8 Second order Lie derivative

Exercice 9.11 Let $\vec{v},\vec{w}$ be C^2 and $\vec{g}(t) = (\Phi_\tau^{t*}\vec{w})(t,p_t) = d\Phi_\tau^t(p_t)^{-1}.\vec{w}(\tau,p(\tau))$ when $p(\tau) = \Phi^t(\tau,p_t)$. We have $\mathcal{L}_{\vec{v}}\vec{w}(t,p(t)) =^{(9.13)} \vec{g}\,'(t)$. Prove $\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w})(t,p(t)) = \vec{g}\,''(t)$, i.e.:

$$\begin{aligned}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w}) &= \frac{D^2\vec{w}}{Dt^2} - 2d\vec{v}.\frac{D\vec{w}}{Dt} - \frac{D(d\vec{v})}{Dt}.\vec{w} + d\vec{v}.d\vec{v}.\vec{w}\\ &= \frac{\partial^2\vec{w}}{\partial t^2} + 2d\frac{\partial\vec{w}}{\partial t}.\vec{v} - 2d\vec{v}.\frac{\partial\vec{w}}{\partial t} + d\vec{w}.\frac{\partial\vec{v}}{\partial t} - d\frac{\partial\vec{v}}{\partial t}.\vec{w}\\ &\quad + (d^2\vec{w}.\vec{v}).\vec{v} + d\vec{w}.d\vec{v}.\vec{v} - 2d\vec{v}.d\vec{w}.\vec{v} - (d^2\vec{v}.\vec{v}).\vec{w} + d\vec{v}.d\vec{v}.\vec{w}\end{aligned} \tag{9.42}$$

Answer.

$$\begin{aligned}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w}) &= \frac{D(\mathcal{L}_{\vec{v}}\vec{w})}{Dt} - d\vec{v}.(\mathcal{L}_{\vec{v}}\vec{w}) = \frac{D(\frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w})}{Dt} - d\vec{v}.(\frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w})\\ &= \frac{D^2\vec{w}}{Dt^2} - \frac{D(d\vec{v})}{Dt}.\vec{w} - d\vec{v}.\frac{D\vec{w}}{Dt} - d\vec{v}.\frac{D\vec{w}}{Dt} + d\vec{v}.d\vec{v}.\vec{w},\end{aligned}$$

with (2.26)-(2.27)-(2.28). ▪

9.5 Lie derivative of a differential form

When the Lie derivative of a vector field $\vec{w}$ cannot be obtained by direct measurements, you need to use a "measuring device" (Germain: To know the weight of a suitcase you have to lift it: You use work).

Here we consider a measuring device which is a differential form α. So, if $\vec{w}$ is a vector field then $f = \alpha.\vec{v}$ is a scalar function, and (9.16) gives $\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) = \frac{D(\alpha.\vec{w})}{Dt} = \frac{D\alpha}{Dt}.\vec{w} + \alpha.\frac{D\vec{w}}{Dt}$, thus

$$\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) = \underbrace{\frac{D\alpha}{Dt}.\vec{w} + \alpha.d\vec{v}.\vec{w}}_{\to(\mathcal{L}_{\vec{v}}\alpha).\vec{w}} + \underbrace{\alpha.\frac{D\vec{w}}{Dt} - \alpha.d\vec{v}.\vec{w}}_{=\alpha.\mathcal{L}_{\vec{v}}\vec{w}} : \tag{9.43}$$

Definition 9.12 Let α be a differential form. The Lie derivative of α along $\vec{v}$ is the differential form

$$\boxed{\mathcal{L}_{\vec{v}}\alpha := \frac{D\alpha}{Dt} + \alpha.d\vec{v}} = \frac{\partial\alpha}{\partial t} + d\alpha.\vec{v} + \alpha.d\vec{v}. \tag{9.44}$$

(An equivalent definition is given at (9.50).) I.e., for all vector field $\vec{w}$,

$$\mathcal{L}_{\vec{v}}\alpha.\vec{w} := \frac{D\alpha}{Dt}.\vec{w} + \alpha.d\vec{v}.\vec{w} \quad (= \frac{\partial\alpha}{\partial t}.\vec{w} + (d\alpha.\vec{v}).\vec{w} + \alpha.d\vec{v}.\vec{w}). \tag{9.45}$$

The definition of $\mathcal{L}_{\vec{v}}\alpha$, cf. (9.44), immediately gives the "derivation property"

$$\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) = (\mathcal{L}_{\vec{v}}\alpha).\vec{w} + \alpha.(\mathcal{L}_{\vec{v}}\vec{w}) \quad \text{(i.e. } \mathcal{L}_{\vec{v}} \text{ is a derivation)}. \tag{9.46}$$

Quantification: Relative to a basis $(\vec{e}_i)$ and with $[\cdot] := [\cdot]_{|\vec{e}}$,

$$\boxed{[\mathcal{L}_{\vec{v}}\alpha] = [\frac{D\alpha}{Dt}] + [\alpha].[d\vec{v}]} \quad \text{(row matrix)} \quad = [\frac{\partial\alpha}{\partial t}] + [d\alpha.\vec{v}] + [\alpha].[d\vec{v}]. \tag{9.47}$$

Thus

$$[\mathcal{L}_{\vec{v}}\alpha.\vec{w}] = [\mathcal{L}_{\vec{v}}\alpha].[\vec{w}] = [\frac{\partial\alpha}{\partial t}].[\vec{w}] + [d\alpha.\vec{v}].[\vec{w}] + [\alpha].[d\vec{v}].[\vec{w}]. \tag{9.48}$$

Exercice 9.13 Prove (9.47) with components. And prove $[d\alpha.\vec{v}] = [\vec{v}]^T.[d\alpha]^T$ (row matrix), thus $[d\alpha.\vec{v}].[\vec{w}] = [\vec{v}]^T.[d\alpha]^T.[\vec{w}] = [\vec{w}]^T.[d\alpha].[\vec{v}]$.

Answer. Basis $(\vec{e}_i)$, dual basis (π_{ei}), thus (9.44) gives $[\mathcal{L}_{\vec{v}}\alpha] = [\frac{D\alpha}{Dt}] + [\alpha.d\vec{v}]$. Let $\alpha = \sum_i \alpha_i\pi_{ei}$, $\vec{v} = \sum_i v_i\vec{e}_i$, $d\vec{v} = \sum_{ij} v_{i|j}\vec{e}_i\otimes\pi_{ej}$ (tensorial writing convenient for calculations), i.e. $[d\vec{v}]_{|\vec{e}} = [v_{i|j}]$, thus $\alpha.d\vec{v} = \sum_{ij}\alpha_i v_{i|j}\pi_{ej}$, thus $[\alpha.d\vec{v}]_{|\pi_e} = [\alpha]_{|\pi_e}.[d\vec{v}]_{|\vec{e}}$ (row matrix). And $d\alpha = \sum_{ij}\alpha_{i|j}\pi_{ei}\otimes\pi_{ej}$, i.e. $[d\alpha]_{|\pi_e} = [\alpha_{i|j}]$, gives $d\alpha.\vec{v} = \sum_{ij}\alpha_{i|j}v_j\pi_{ei} = \sum_{ij} v_i\alpha_{j|i}\pi_{ej}$, and $[d\alpha.\vec{v}]_{|\pi_e}$ is a row matrix ($d\alpha.\vec{v}$ is a differential form), thus $[d\alpha.\vec{v}]_{|\pi_e} = [\vec{v}]_{|\vec{e}}^T.[d\alpha]_{|\pi_e}^T$. (Or compute $(d\alpha.\vec{v}).\vec{w} = \sum_{ij}\alpha_{i|j}v_jw_i = [\vec{w}]_{|\vec{e}}^T.[d\alpha]_{|\vec{e}}.[\vec{v}]_{|\vec{e}} = [\vec{v}]_{|\vec{e}}^T.[d\alpha]_{|\pi_e}^T.[\vec{w}]_{|\vec{e}}$.) ▪

Exercice 9.14 Let α be a differential form, and let $\alpha_t(p) := \alpha(t,p)$. Prove, when $\Phi_t^{t_0}$ is a diffeomorphism,

$$\mathcal{L}_{\vec v}\alpha = 0 \quad\Longleftrightarrow\quad \forall t\in[t_0,T],\ \alpha_t = (\Phi_t^{t_0})_*\alpha_{t_0}. \tag{9.49}$$

I.e.: $\frac{D\alpha}{Dt} = -\alpha.d\vec v \iff \alpha_t(p_t) = \alpha_{t_0}(p_{t_0}).F_t^{t_0}(p_{t_0})^{-1}$ for all t, when $p_t = \Phi_t^{t_0}(p_{t_0})$.

Answer. $\Leftarrow$: If $\alpha_t(p(t)) = \alpha_{t_0}(p_{t_0}).F_t^{t_0}(p_{t_0})^{-1}$, then $\alpha(t,p(t)).F^{t_0}(t,p_{t_0}) = \alpha_{t_0}(p_{t_0})$, thus $\frac{D\alpha}{Dt}(t,p_t).F_t^{t_0}(p_{t_0}) + \alpha_t(p_t).\frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}) = 0$, thus $\frac{D\alpha}{Dt}(t,p(t)).F_t^{t_0}(p_{t_0}) + \alpha_t(p_t).d\vec v(t,p_t).F_t^{t_0}(p_{t_0}) = 0$, thus $\mathcal{L}_{\vec v}\alpha = 0$, since $\Phi_t^{t_0}$ is a diffeomorphism.

$\Rightarrow$: If $\beta(t) := (\Phi_t^{t_0})_*\alpha_{t_0}(p_{t_0}) = \alpha_t(p(t)).F_t^{t_0}(p_{t_0})$ (pull-back at (t_0,p_{t_0})), then $\beta(t) = \alpha(t,p(t)).F^{t_0}(t,p_{t_0})$, thus $\beta'(t) = \frac{D\alpha}{Dt}(t,p_t).F_t^{t_0}(p_{t_0}) + \alpha(t,p_t).d\vec v(t,p_t).F_t^{t_0}(p_{t_0}) = 0$ (hypothesis $\mathcal{L}_{\vec v}\alpha = 0$), thus $\beta(t) = \beta(t_0) = \alpha_{t_0}(p_{t_0})$. ∴

Remark 9.15 A definition equivalent to (9.44) is, cf. (9.13),

$$\begin{aligned}\mathcal{L}_{\vec v}\alpha(t,p_t) &:= \lim_{\tau\to t}\frac{(\Phi_\tau^t)^*\alpha_\tau(p_t)-\alpha_t(p_t)}{\tau-t} \quad \Big(=\lim_{\tau\to t}\frac{\alpha_\tau(p_\tau).d\Phi_\tau^t(p_t)-\alpha_t(p_t)}{\tau-t}\Big)\\ &\overset{\text{noted}}{=} \frac{D(\Phi_\tau^{t*}\alpha_\tau(p_t))}{D\tau}_{|\tau=t} \overset{\text{noted}}{=} \frac{D(\alpha_\tau^*(p_t))}{D\tau}_{|\tau=t} \quad \Big(=\frac{D(\alpha_\tau(p_\tau).d\Phi_\tau^t(p_t))}{D\tau}_{|\tau=t}\Big).\end{aligned} \tag{9.50}$$

Indeed, if $\beta(\tau) = (\Phi_\tau^t)^*\alpha_\tau(p_t) = \alpha_\tau(p_\tau).d\Phi_\tau^t(p_t)$, then $\beta'(\tau)$ and then $\tau = t$ give (9.44). ∴

Exercice 9.16 $\vec v$ and α being C^2, prove:

$$\begin{aligned}\mathcal{L}_{\vec v}(\mathcal{L}_{\vec v}\alpha) = {}& \frac{\partial^2\alpha}{\partial t^2} + 2d\frac{\partial\alpha}{\partial t}.\vec v + 2\frac{\partial\alpha}{\partial t}.d\vec v + d\alpha.\frac{\partial\vec v}{\partial t} + \alpha.\frac{\partial d\vec v}{\partial t}\\ &+ (d^2\alpha.\vec v).\vec v + d\alpha.(d\vec v.\vec v) + 2(d\alpha.\vec v).d\vec v + \alpha.(d^2\vec v.\vec v) + (\alpha.d\vec v).d\vec v.\end{aligned} \tag{9.51}$$

Answer. (9.44) gives

$$\begin{aligned}\mathcal{L}_{\vec v}(\mathcal{L}_{\vec v}\alpha) &= \mathcal{L}_{\vec v}(\frac{\partial\alpha}{\partial t}) + \mathcal{L}_{\vec v}(d\alpha.\vec v) + \mathcal{L}_{\vec v}(\alpha.d\vec v)\\ &= \frac{\partial^2\alpha}{\partial t^2} + d\frac{\partial\alpha}{\partial t}.\vec v + \frac{\partial\alpha}{\partial t}.d\vec v + \frac{\partial(d\alpha.\vec v)}{\partial t} + d(d\alpha.\vec v).\vec v + (d\alpha.\vec v).d\vec v + \frac{\partial(\alpha.d\vec v)}{\partial t} + d(\alpha.d\vec v).\vec v + (\alpha.d\vec v).d\vec v\\ &= \frac{\partial^2\alpha}{\partial t^2} + d\frac{\partial\alpha}{\partial t}.\vec v + \frac{\partial\alpha}{\partial t}.d\vec v + \frac{\partial d\alpha}{\partial t}.\vec v + d\alpha.\frac{\partial\vec v}{\partial t} + (d^2\alpha.\vec v).\vec v + d\alpha.(d\vec v.\vec v) + (d\alpha.\vec v).d\vec v\\ &\quad + \frac{\partial\alpha}{\partial t}.d\vec v + \alpha.\frac{\partial d\vec v}{\partial t} + (d\alpha.\vec v).d\vec v + \alpha.d^2\vec v.\vec v + (\alpha.d\vec v).d\vec v\\ &= \frac{\partial^2\alpha}{\partial t^2} + 2d\frac{\partial\alpha}{\partial t}.\vec v + 2\frac{\partial\alpha}{\partial t}.d\vec v + d\alpha.\frac{\partial\vec v}{\partial t} + (d^2\alpha.\vec v).\vec v + d\alpha.(d\vec v.\vec v) + 2(d\alpha.\vec v).d\vec v + \alpha.\frac{\partial d\vec v}{\partial t}\\ &\quad + \alpha.(d^2\vec v.\vec v) + (\alpha.d\vec v).d\vec v.\end{aligned}$$

∴

9.6 Incompatibility with Riesz representation vectors

The Lie derivative has nothing to do with any inner dot product (the Lie derivative does not compare two vectors, contrary to a Cauchy type approach).

Here we introduce a Euclidean dot product $(\cdot,\cdot)_g$ and show that the Lie derivative of a linear form α is not trivially deduced from the Lie derivative of a Riesz representation vector of α (which one?). (Same issue as at § 7.2.)

Let α be a Eulerian differential form; Then let $\vec a_g(t,p)\in\vec{\mathbb{R}^n}$ be the $(\cdot,\cdot)_g$-Riesz representation vector of the linear form $\alpha(t,p)\in\mathbb{R}^{n*}$: So, for all Eulerian vector field $\vec w$,

$$\alpha.\vec w = (\vec a_g,\vec w)_g \qquad (= \vec a_g \bullet_g \vec w), \tag{9.52}$$

which means $\alpha(t,p).\vec w(t,p) = (\vec a_g(t,p),\vec w(t,p))_g$ at all admissible (t,p). This defines the Eulerian vector field $\vec a_g$ (not intrinsic to α: $\vec a_g$ depends on the choice of $(\cdot,\cdot)_g$, cf. (F.13)).

Proposition 9.17 *For all* $\vec v,\vec w\in\vec{\mathbb{R}^n}$,

$$\frac{\partial\alpha}{\partial t}.\vec w = (\frac{\partial\vec a_g}{\partial t},\vec w)_g,\quad (d\alpha.\vec v).\vec w = (d\vec a_g.\vec v,\vec w)_g,\quad \frac{D\alpha}{Dt}.\vec w = (\frac{D\vec a_g}{Dt},\vec w)_g. \tag{9.53}$$

Thus

$$\mathcal{L}_{\vec v}\alpha.\vec w = (\mathcal{L}_{\vec v}\vec a_g,\vec w)_g + (\vec a_g,(d\vec v+d\vec v^T).\vec w)_g, \quad \text{and} \quad \boxed{\mathcal{L}_{\vec v}\alpha.\vec w \neq (\mathcal{L}_{\vec v}\vec a_g,\vec w)_g} \quad \text{in general.} \tag{9.54}$$

So $\mathcal{L}_{\vec v}\vec a_g$ is ***not*** *the Riesz representation vector of $\mathcal{L}_{\vec v}\alpha$ (but for solid body motions). (Expected: A Lie derivative is covariant objective, see § 11.4, and the use of an inner dot product ruins this objectivity.)*

Proof. A Euclidean dot product $g(\cdot,\cdot)$ is bilinear constant and uniform, thus:

$\alpha.\vec{w} = (\vec{a}_g, \vec{w})_g$ gives $\frac{\partial\alpha}{\partial t}.\vec{w} + \alpha.\frac{\partial\vec{w}}{\partial t} = (\frac{\partial\vec{a}_g}{\partial t}, \vec{w})_g + (\vec{a}_g, \frac{\partial\vec{w}}{\partial t})_g$, with $\alpha.\frac{\partial\vec{w}}{\partial t} = (\vec{a}_g, \frac{\partial\vec{w}}{\partial t})_g$, thus we are left with $\frac{\partial\alpha}{\partial t}.\vec{w} = (\frac{\partial\vec{a}_g}{\partial t}, \vec{w})_g$, for all $\vec{w}$.

$\alpha.\vec{w} = (\vec{a}_g, \vec{w})_g$ gives $d(\alpha.\vec{w}).\vec{v} = d(\vec{a}_g, \vec{w})_g.\vec{v}$ for all $\vec{v}, \vec{w}$, thus $(d\alpha.\vec{v}).\vec{w} + \alpha.(d\vec{w}.\vec{v}) = (d\vec{a}_g.\vec{v}, \vec{w})_g + (\vec{a}_g, d\vec{w}.\vec{v})_g$, with $\alpha.(d\vec{w}.\vec{v}) = (\vec{a}_g, d\vec{w}.\vec{v})_g$, thus we are left with $(d\alpha.\vec{v}).\vec{w} = (d\vec{a}_g.\vec{v}, \vec{w})_g$.

Thus $\frac{D\alpha}{Dt}.\vec{w} = (\frac{D\vec{a}_g}{Dt}, \vec{w})_g$.

Thus $(\mathcal{L}_{\vec{v}}\alpha).\vec{w} = \frac{D\alpha}{Dt}.\vec{w} + \alpha.d\vec{v}.\vec{w} = (\frac{D\vec{a}_g}{Dt}, \vec{w})_g + (\vec{a}_g, d\vec{v}.\vec{w})_g = (\mathcal{L}_{\vec{v}}\vec{a}_g + d\vec{v}.\vec{a}_g, \vec{w})_g + (d\vec{v}_g^T.\vec{a}_g, \vec{w})_g$. ▪

Remark 9.18 Chorus: a “differential form” (measuring instrument, covariant) should not be confused with a “vector field” (object to be measured, contravariant); Thus, the use of a dot product (which one?) and the Riesz representation theorem should be restricted for computational purposes, after an objective equation has been established. See also remark F.12. ▪

9.7 Lie derivative of a tensor

The Lie derivative of any tensor of order ≥ 2 is defined thanks to

$$\mathcal{L}_{\vec{v}}(T \otimes S) = (\mathcal{L}_{\vec{v}}T) \otimes S + T \otimes (\mathcal{L}_{\vec{v}}S) \quad \text{(derivation formula).} \tag{9.55}$$

(Or direct definition: $\mathcal{L}_{\vec{v}}T(t_0, p_{t_0}) = \frac{D((\Phi_t^{t_0})^* T_t)(p_{t_0})}{Dt}_{|t=t_0}$).

9.7.1 Lie derivative of a mixed tensor

Let $T_m \in T_1^1(\Omega)$, and T_m is called a mixed tensor; Its Lie derivative, called the Jaumann derivative, is given by

$$\boxed{\mathcal{L}_{\vec{v}}T_m = \frac{DT_m}{Dt} - d\vec{v}.T_m + T_m.d\vec{v}} = \frac{\partial T_m}{\partial t} + dT_m.\vec{v} - d\vec{v}.T_m + T_m.d\vec{v}. \tag{9.56}$$

Can be checked with an elementary tensor $T = \vec{w} \otimes \alpha$: we have $d(\vec{w} \otimes \alpha).\vec{v} = (d\vec{w}.\vec{v}) \otimes \alpha + \vec{w} \otimes (d\alpha.\vec{v})$ and $(d\vec{v}.\vec{w}) \otimes \alpha = d\vec{v}.(\vec{w} \otimes \alpha)$, and $\vec{w} \otimes (\alpha.d\vec{v}) = (\vec{w} \otimes \alpha).d\vec{v}$, thus (9.55) gives $\mathcal{L}_{\vec{v}}(\vec{w} \otimes \alpha) = (\mathcal{L}_{\vec{v}}\vec{w}) \otimes \alpha + \vec{w} \otimes (\mathcal{L}_{\vec{v}}\alpha)$
$= \frac{\partial\vec{w}}{\partial t} \otimes \alpha + (d\vec{w}.\vec{v}) \otimes \alpha - (d\vec{v}.\vec{w}) \otimes \alpha + \vec{w} \otimes \frac{\partial\alpha}{\partial t} + \vec{w} \otimes (d\alpha.\vec{v}) + \vec{w} \otimes (\alpha.d\vec{v})$
$= \frac{\partial\vec{w}\otimes\alpha}{\partial t} + d(\vec{w} \otimes \alpha).\vec{v} - d\vec{v}.(\vec{w} \otimes \alpha) + (\vec{w} \otimes \alpha).d\vec{v}$.

Quantification. Relative to a basis $(\vec{e}_i)$:

$$[\mathcal{L}_{\vec{v}}T_m] = [\frac{DT_m}{Dt}] - [d\vec{v}].[T_m] + [T_m].[d\vec{v}] \tag{9.57}$$

(the signs $\mp$ are mixed). “Mixed” also refers to positions of indices (up and down with duality notations): $T_m = \sum_{i,j=1}^n T^i{}_j \vec{e}_i \otimes e^j$ with the dual basis (e^i), i.e. $[T_m]_{|\vec{e}} = [T^i{}_j]$.

Exercice 9.19 With components, prove (9.57).

Answer. $\frac{\partial T_m}{\partial t} = \sum_{ij} \frac{\partial T^i{}_j}{\partial t} \vec{e}_i \otimes e^j$, $dT_m = \sum_{ijk} T^i{}_{j|k} \vec{e}_i \otimes e^j \otimes e^k$, $\vec{v} = \sum_i v^i \vec{e}_i$, $d\vec{v} = \sum_{ij} v^i_{|j} \vec{e}_i \otimes e^j$, thus $dT_m.\vec{v} = \sum_{ijk} T^i{}_{j|k} v^k \vec{e}_i \otimes e^j$, $d\vec{v}.T_m = \sum_{ijk} v^i_{|k} T^k{}_j \vec{e}_i \otimes e^j$, $T_m.d\vec{v} = \sum_{ijk} T^i{}_k v^k_{|j} \vec{e}_i \otimes e^j$. ▪

9.7.2 Lie derivative of a up-tensor

Recall: If $L \in \mathcal{L}(E; F)$ (a linear map) then its adjoint $L^* \in \mathcal{L}(F^*; E^*)$ is defined by, cf. § A.13,

$$\forall m \in F^*, \quad \boxed{L^*.m := m.L}, \quad \text{i.e.,} \quad \forall m, \vec{u} \in (F^* \times E), \quad (L^*.m).\vec{u} = m.L.\vec{u}. \tag{9.58}$$

(There is no inner dot product involved here.) In particular, $d\vec{v}^*.m := m.d\vec{v}$ for all $m \in \vec{\mathbb{R}}_t^{n*}$, i.e. $(d\vec{v}^*.m).\vec{u} = (m.d\vec{v}).\vec{u} = m.(d\vec{v}.\vec{u})$ for all $m \in \vec{\mathbb{R}}_t^{n*}$ and all $\vec{u} \in \vec{\mathbb{R}}_t^n$.

Let $T_u \in T_0^2(\Omega)$, and T_u is called a up tensor; Its Lie derivative is called the upper-convected (Maxwell) derivative or the Oldroyd derivative and is given by

$$\boxed{\mathcal{L}_{\vec{v}}T_u = \frac{DT_u}{Dt} - d\vec{v}.T_u - T_u.d\vec{v}^*} = \frac{\partial T_u}{\partial t} + dT_u.\vec{v} - d\vec{v}.T_u - T_u.d\vec{v}^*. \tag{9.59}$$

Can be checked with an elementary tensor $T = \vec{u} \otimes \vec{w}$ and $\mathcal{L}_{\vec{v}}(\vec{u} \otimes \vec{w}) = (\mathcal{L}_{\vec{v}}\vec{u}) \otimes \vec{w} + \vec{u} \otimes (\mathcal{L}_{\vec{v}}\vec{w})$.

Quantification. Relative to a basis $(\vec{e}_i)$:

$$[\mathcal{L}_{\vec{v}} T_u] = [\frac{DT_u}{Dt}] - [d\vec{v}].[T_u] - [T_u].[d\vec{v}]^T. \tag{9.60}$$

"up" also refers to positions of indices (with duality notations): $T_u = \sum_{i,j=1}^n T^{ij} \vec{e}_i \otimes \vec{e}_j$ with the dual basis (e^i), i.e. $[T_u]_{|\vec{e}} = [T^{ij}]$.

Exercice 9.20 With components, prove (9.59).

Answer. $\frac{\partial T_u}{\partial t} = \sum_{ij} \frac{\partial T^{ij}}{\partial t} \vec{e}_i \otimes \vec{e}_j$, $dT_u = \sum_{ijk} T^{ij}_{|k} \vec{e}_i \otimes \vec{e}_j \otimes e^k$, $\vec{v} = \sum_i v^i \vec{e}_i$, $d\vec{v} = \sum_{ij} v^i_{|j} \vec{e}_i \otimes e^j$, $d\vec{v}^* = \sum_{ij} v^j_{|i} e^i \otimes \vec{e}_j$, thus $dT_u.\vec{v} = \sum_{ijk} T^{ij}_{|k} v^k \vec{e}_i \otimes e^j$, $d\vec{v}.T_u = \sum_{ijk} v^i_{|k} T^{kj} \vec{e}_i \otimes \vec{e}_j$, $T_u.d\vec{v}^* = \sum_{ijk} T^{ik} v^j_{|k} e^i \otimes \vec{e}_j$. ▪

9.7.3 Lie derivative of a down-tensor

Let $T_d \in T^0_2(\Omega)$, and T_d is called a down tensor; The Lie derivative is called the lower-convected Maxwell derivative and is given by

$$\boxed{\mathcal{L}_{\vec{v}} T_d = \frac{DT_d}{Dt} + T_d.d\vec{v} + d\vec{v}^*.T_d} = \frac{\partial T_d}{\partial t} + dT_d.\vec{v} + T_d.d\vec{v} + d\vec{v}^*.T_d. \tag{9.61}$$

Can be checked with an elementary tensor $T = \ell \otimes m$ and $\mathcal{L}_{\vec{v}}(\ell \otimes m) = (\mathcal{L}_{\vec{v}}\ell) \otimes m + \ell \otimes (\mathcal{L}_{\vec{v}} m)$.

Quantification. Relative to a basis $(\vec{e}_i)$:

$$[\mathcal{L}_{\vec{v}} T_d] = [\frac{DT_d}{Dt}] + [T_d].[d\vec{v}] + [d\vec{v}]^T.[T_d]. \tag{9.62}$$

"down" also refers to positions of indices (with duality notations): $T_d = \sum_{i,j=1}^n T_{ij} e^i \otimes e^j$ with the dual basis (e^i), i.e. $[T_d]_{|\vec{e}} = [T_{ij}]$.

Exercice 9.21 With components, prove (9.62).

Answer. $\frac{\partial T_d}{\partial t} = \sum_{ij} \frac{\partial T_{ij}}{\partial t} e^i \otimes e^j$, $dT_d = \sum_{ijk} T_{ij|k} e^i \otimes e^j \otimes e^k$, $\vec{v} = \sum_i v^i \vec{e}_i$, $d\vec{v} = \sum_{ij} v^i_{|j} \vec{e}_i \otimes e^j$, $d\vec{v}^* = \sum_{ij} v^j_{|i} e^i \otimes \vec{e}_j$, thus $dT_d.\vec{v} = \sum_{ijk} T_{ij|k} v^k e^i \otimes e^j$, $T_d.d\vec{v} = \sum_{ijk} T_{ik} v^k_{|j} e^i \otimes \vec{e}_j$, $d\vec{v}^*.T_d = \sum_{ijk} v^k_{|i} T_{kj} e^i \otimes \vec{e}_j$. ▪

Example 9.22 Let $g = (\cdot,\cdot)_g \in T^0_2(\Omega)$ be a constant and uniform metric (a unique inner dot product for all t, p, e.g., a Euclidean dot product at all t). Then $\frac{Dg}{Dt} = 0$, thus $\mathcal{L}_{\vec{v}} g = 0 + g.d\vec{v} + d\vec{v}^*.g$, thus $[\mathcal{L}_{\vec{v}} g] = [g].[d\vec{v}] + [d\vec{v}]^T.[g]$. ▪

Part IV
Velocity-addition formula

10 Change of referential and velocity-addition formula

10.0 Issue and result (summary)

Issue: The velocity-addition formula is usually written (classical mechanics)

$$\vec{v}_A = \vec{v}_D + \vec{v}_B, \quad \text{i.e.} \quad \text{absolute velocity} = \text{(drive+relative) velocities}, \tag{10.1}$$

where $\vec{v}_A$ and $\vec{v}_D$ being described by an observer A in his referential $\mathcal{R}_A = (O_A, (\vec{A}_i))$ and $\vec{v}_B$ being described by an observer B in his referential $\mathcal{R}_B = (O_B, (\vec{B}_i))$. Hence (10.1) is problematic (inconsistent):

- $\vec{v}_A$ and $\vec{v}_D$ are quantified in the basis $(\vec{A}_i)$, e.g. in foot/s, chosen by the absolute observer,
- $\vec{v}_B$ is quantified in the another basis $(\vec{B}_i)$, e.g. in metre/s, chosen by the relative observer;

Thus, in (10.1), $\vec{v}_B + \vec{v}_D$ adds metre/s and foot/s... relative to different bases... Absurd. So:

Question: What are we missing (and how should (10.1) be written, or what does it really mean)?
Answer: We miss a link = the translator between A and B:

Summary (full details in the following paragraphs): *Obj* is an object and $\widetilde{\Phi} : (t, P_{Obj}) \in [t_1, t_2] \times Obj \to p(t) = \widetilde{\Phi}(t, P_{Obj}) \in \mathbb{R}^n$ is its motion. It is *quantified* by A in his referential $\mathcal{R}_A$ thanks to $\vec{\varphi}_A : (t, P_{Obj}) \to \vec{x}_A(t) = \vec{\varphi}_A(t, P_{Obj}) := [\overrightarrow{O_A\widetilde{\Phi}(t, P_{Obj})}]_{|\vec{A}}$ (stored components in $(\vec{A}_i)$), and by B in his referential $\mathcal{R}_B$ thanks to $\vec{\varphi}_B : (t, P_{Obj}) \to \vec{x}_B(t) = \vec{\varphi}_B(t, P_{Obj}) := [\overrightarrow{O_B\widetilde{\Phi}(t, P_{Obj})}]_{|\vec{B}}$ (stored components in $(\vec{B}_i)$). At any t, the translator Θ_t connects $\vec{x}_{At} := \vec{x}_A(t) = \vec{\varphi}_A(t, P_{Obj})$ and $\vec{x}_{Bt} := \vec{x}_B(t) = \vec{\varphi}_B(t, P_{Obj})$:

$$\vec{x}_{At} = \Theta_t(\vec{x}_{Bt}), \quad \text{so} \quad \vec{\varphi}_A(t, P_{Obj}) = \Theta(t, \vec{\varphi}_B(t, P_{Obj})). \tag{10.2}$$

Thus (time differentiation)

$$\underbrace{\frac{\partial \vec{\varphi}_A}{\partial t}(t, P_{Obj})}_{\text{absolute velocity } \vec{v}_A(t, \vec{x}_{At})} = \underbrace{\frac{\partial \Theta}{\partial t}(t, \vec{\varphi}_B(t, P_{Obj}))}_{\text{drive velocity } \vec{v}_D(t, \vec{x}_{At})} + \underbrace{d\Theta(t, \vec{\varphi}_B(t, P_{Obj})).\frac{\partial \vec{\varphi}_B}{\partial t}(t, P_{Obj})}_{\text{translated velocity } \vec{v}_{B*}(t, \vec{x}_{At})}. \tag{10.3}$$

Which gives "the velocity-addition formula": For observer A,

$$\boxed{\vec{v}_A = \vec{v}_D + \vec{v}_{B*}}, \quad \text{where} \quad \vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) \ \text{ at } \ \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}), \tag{10.4}$$

i.e.: Absolute velocity = Drive velocity + Translated relative velocity.

Example 10.1 • Translation motion of $\mathcal{R}_B$ in $\mathcal{R}_A$ with $\vec{B}_i = \lambda \vec{A}_i$ (e.g. $\lambda \simeq 3.28$ when $\vec{A}_i$ in foot and $\vec{B}_i$ in meter). Here $d\Theta_t = \lambda I$, hence $\vec{v}_{At}(\vec{x}_{At}) = \vec{v}_{Dt}(\vec{x}_{At}) + \lambda\, \vec{v}_{Bt}(\vec{x}_{Bt})$, which is the expected relation ("sum of the velocities with the good units", e.g. foot/s).
• "Rotation" of $\mathcal{R}_B$ in $\mathcal{R}_A$: See § 10.12 (motion of the Earth around the Sun). ▪

Then (10.3) gives (time differentiation), with $\frac{D(\frac{\partial \Theta}{\partial t})}{Dt}(t, \vec{x}_B(t)) = \frac{\partial(\frac{\partial \Theta}{\partial t})}{\partial t}(t, \vec{x}_{Bt}) + d(\frac{\partial \Theta}{\partial t})(t, \vec{x}_{Bt}).\vec{v}_B(t, \vec{x}_{Bt})$, and in the classical case $d^2\Theta_t = 0$,

$$\underbrace{\vec{\gamma}_{At}(\vec{x}_{At})}_{\text{Absolute acc.}} = \underbrace{\frac{\partial^2 \Theta}{\partial t^2}(t, \vec{x}_{Bt})}_{\text{Drive acc. } \vec{\gamma}_D(t, \vec{x}_{At})} + \underbrace{2\, d\frac{\partial \Theta}{\partial t}(t, \vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})}_{\text{Coriolis acc. } \vec{\gamma}_C(t, \vec{x}_{At})} + \underbrace{d\Theta_t(\vec{x}_{Bt}).\frac{\partial^2 \vec{\varphi}_B}{\partial t^2}(t, P_{Obj})}_{\text{Translated acc. } \vec{\gamma}_{B*}(t, \vec{x}_{At})}, \tag{10.5}$$

Which gives "the acceleration-addition formula": For observer A,

$$\boxed{\vec{\gamma}_A = \vec{\gamma}_D + \vec{\gamma}_C + \vec{\gamma}_{B*}}, \quad \text{where} \quad \vec{\gamma}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{\gamma}_{Bt}(\vec{x}_{Bt}) \ \text{ at } \ \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}),, \tag{10.6}$$

i.e.: Absolute acceleration = (Drive + Coriolis + Translated relative) accelerations.

And $\frac{\partial \Theta}{\partial t}(t, \vec{x}_{Bt}) = \vec{v}_{Dt}(\Theta_t(\vec{x}_{Bt}))$ gives $d(\frac{\partial \Theta}{\partial t})(t, \vec{x}_{Bt}))) = d\vec{v}_{Dt}(\Theta_t(\vec{x}_{Bt})).d\Theta_t(\vec{x}_{Bt})$, thus $\vec{\gamma}_{Ct}(\vec{x}_{At}) = 2\, d\vec{v}_{Dt}(\vec{x}_{At}).\vec{v}_{Bt*}(\vec{x}_{At})$, thus, e.g. for the motion of the Earth around the Sun,

$$\vec{\gamma}_{Ct} = 2\, \vec{\omega}_{Dt} \times \vec{v}_{Bt*}. \tag{10.7}$$

10.1 Referentials and "matrix motions"

10.1.1 Motion of *Obj* in our classical Universe

Classical mechanics framework: Time and space are decoupled, all the observers share the same time origin and unit (e.g. the second) and live in "our Universe" modeled as the affine space $\mathbb{R}^3$ with its usual associated vector space $\vec{\mathbb{R}^3}$ (bi-point vectors). More generally, the affine space is $\mathbb{R}^n$ associated to the vector space $\vec{\mathbb{R}^n}$, $n \in \{1,2,3\}$.

Obj is an object. Its (regular) motion (in our Universe) during a time interval $[t_1,t_2]$ is the function

$$\widetilde{\Phi} : \begin{cases} [t_1,t_2] \times Obj & \to \mathbb{R}^n \\ \quad (t, P_{Obj}) & \to p_t = \widetilde{\Phi}(t, P_{Obj}) = \text{position at } t \text{ of the particle } P_{Obj}. \end{cases} \tag{10.8}$$

With $\Omega_t = \widetilde{\Phi}(t, Obj) \subset \mathbb{R}^n$, its Eulerian velocities and accelerations vector fields are the functions $\vec{v}$ and $\vec{\gamma} : \bigcup_{t\in[t_1,t_2]}(\{t\} \times \Omega_t) \to \vec{\mathbb{R}^n}$ defined by, at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$,

$$\vec{v}(t,p_t) = \frac{\partial\widetilde{\Phi}}{\partial t}(t,P_{Obj}) \quad\text{and}\quad \vec{\gamma}(t,p_t) = \frac{\partial^2\widetilde{\Phi}}{\partial^2 t}(t,P_{Obj}). \tag{10.9}$$

10.1.2 Absolute and relative referentials ...

• An observer A, called the absolute observer, chooses a rigid object $Obj\mathcal{R}_A$ in the Universe, chooses four particles which are at t at $O_{At}, P_{A1t}, P_{A2t}, P_{A3t} \in Obj\mathcal{R}_A$ s.t. the bi-point vectors $\vec{A}_{it} := \overrightarrow{O_{At}P_{Ait}}$ make a basis in $\vec{\mathbb{R}^n}$ at t. He has thus built his (Cartesian) referential $\mathcal{R}_{At} = (O_{At}, (\vec{A}_{it}))$ at t, called the absolute referential at t. And $\mathcal{R}_{At}$ is supposed fixed relative to A, so is written $\mathcal{R}_A = (O_A, (\vec{A}_i))$ when used by A.

E.g. $Obj\mathcal{R}_A$ is the "Sun extended to infinity", and at t, O_{At} is the position of the center of the Sun in the Universe and $(\vec{A}_{it}) = (\overrightarrow{O_{At}P_{Ait}})$ is a Euclidean basis in foot fixed relative to stars.

• An observer B, called the relative observer, proceeds similarly: He builds his Cartesian referential $\mathcal{R}_{Bt} = (O_{Bt}, (\vec{B}_{it}))$, called the relative referential, and written $\mathcal{R}_B = (O_B, (\vec{B}_i))$ when used by B.

E.g. $Obj\mathcal{R}_B$ is the "Earth extended to infinity", and at t, O_{Bt} is the position of the center of the Earth and $(\vec{B}_{it}) = (\overrightarrow{O_{Bt}P_{Bit}})$ is a Euclidean basis in metre fixed relative to the Earth.

10.1.3 ... Matrix representations of a vector ...

$\mathcal{M}_{n1}$ is the abstract vector space of $n*1$ real column matrices and $\vec{E}_1 = \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$, ..., $\vec{E}_n = \begin{pmatrix}0\\ \vdots\\0\\1\end{pmatrix}$ makes its canonical basis $(\vec{E}_i)$ (with 0 and 1 the identity addition and multiplication elements in the field $\mathbb{R}$). In particular $[\vec{A}_{it}]_{|\vec{A}} = \vec{E}_i = [\vec{B}_{it}]_{|\vec{B}}$.

With the only purpose to know which observer builds column matrices, $\mathcal{M}_{n1}$ is called $\mathcal{M}_{n1}$(A) when used by A and $\mathcal{M}_{n1}$(B) when used by B. So, at t, a position $p_t \in \mathbb{R}^n$ is stored by A as the column matrix in $\mathcal{M}_{n1}$ given by the components x_{Ati} of the vector $\overrightarrow{O_{At}p_t} = \sum_{i=1}^n x_{Ati}\vec{A}_{it}$, idem for B:

$$\vec{x}_{At} := [\overrightarrow{O_{At}p_t}]_{|\vec{A}} = \begin{pmatrix}x_{At1}\\ \vdots\\ x_{Atn}\end{pmatrix} \in \mathcal{M}_{n1}(\text{A}), \quad\text{and}\quad \vec{x}_{Bt} := [\overrightarrow{O_{Bt}p_t}]_{|\vec{B}} = \begin{pmatrix}x_{Bt1}\\ \vdots\\ x_{Btn}\end{pmatrix} \in \mathcal{M}_{n1}(\text{B}). \tag{10.10}$$

10.1.4 ... Absolute and relative "motions" of *Obj* (quantification)

(10.10) defines the "absolute motion" $\vec{\varphi}_A$ and the "relative motion" $\vec{\varphi}_B$ of *Obj* (matrix valued):

$$\vec{\varphi}_A : \begin{cases} [t_1,t_2]\times Obj & \to \mathcal{M}_{n1}(\text{A}) \\ \quad (t,P_{Obj}) & \to \vec{x}_{At} = \boxed{\vec{\varphi}_A(t,P_{Obj}) := [\overrightarrow{O_A\widetilde{\Phi}(t,P_{Obj})}]_{|\vec{A}}} \overset{\text{noted}}{=} \vec{x}_A(t) = [\overrightarrow{O_Ap(t)}]_{|\vec{A}}, \end{cases} \tag{10.11}$$

$$\vec{\varphi}_B : \begin{cases} [t_1,t_2]\times Obj & \to \mathcal{M}_{n1}(\text{B}) \\ \quad (t,P_{Obj}) & \to \vec{x}_{Bt} = \boxed{\vec{\varphi}_B(t,P_{Obj}) := [\overrightarrow{O_B\widetilde{\Phi}(t,P_{Obj})}]_{|\vec{B}}} \overset{\text{noted}}{=} \vec{x}_B(t) = [\overrightarrow{O_Bp(t)}]_{|\vec{B}}. \end{cases} \tag{10.12}$$

And (10.9) gives the "absolute" and "relative" velocities and accelerations of P_{Obj} (column matrices):

$$\boxed{\vec{v}_A(t,\vec{x}_{At}) := [\vec{v}(t,p_t)]_{|\vec{A}}} \quad\text{and}\quad \vec{\gamma}_A(t,\vec{x}_{At}) := [\vec{\gamma}(t,p_t)]_{|\vec{A}}, \quad\text{when}\quad \vec{x}_{At} := [\overrightarrow{O_Ap_t}]_{|\vec{A}}, \tag{10.13}$$

$$\boxed{\vec{v}_B(t,\vec{x}_{Bt}) := [\vec{v}(t,p_t)]_{|\vec{B}}} \quad\text{and}\quad \vec{\gamma}_B(t,\vec{x}_{Bt}) := [\vec{\gamma}(t,p_t)]_{|\vec{B}}, \quad\text{when}\quad \vec{x}_{Bt} := [\overrightarrow{O_Bp_t}]_{|\vec{B}}. \tag{10.14}$$

These definitions are consistent:

Proposition 10.2 $\vec{v}_A(t,\vec{x}_{At}) = \frac{\partial\vec{\varphi}_A}{\partial t}(t,P_{Obj})$ $(= \vec{x}_A{}'(t))$, *and* $\vec{v}_B(t,\vec{x}_{Bt}) = \frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj})$ $(= \vec{x}_B{}'(t))$.

Proof. Let $\vec{A}_i(t) := \vec{A}_{it}$, $O_A(t) := O_{At}$. Then $[\vec{A}_i'(t)]_{|\vec{A}} = [\lim_{h\to0}\frac{\vec{A}_i(t+h)-\vec{A}_i(t)}{h}]_{|\vec{A}} = \lim_{h\to0}\frac{[\vec{A}_i(t+h)]_{|\vec{A}}-[\vec{A}_i(t)]_{|\vec{A}}}{h} = \lim_{h\to0}\frac{\vec{E}_i-\vec{E}_i}{h} = \vec{0} \in \mathcal{M}_{n1}$ since $\vec{A}_{it}$ is fixed for A. And $[O_A{}_i'(t)]_{|\vec{A}} = [\lim_{h\to0}\frac{O_{A_i}(t+h)-O_{A_i}(t)}{h}]_{|\vec{A}} = \lim_{h\to0}\frac{[\overrightarrow{O_{A_i}(t)O_{A_i}(t+h)}]_{|\vec{A}}}{h} = \lim_{h\to0}\frac{\vec{0}}{h} = \vec{0} \in \mathcal{M}_{n1}$ since O_{At} is fixed for A. And $\widetilde{\Phi}(t,P_{Obj}) = O_A(t) + \overrightarrow{O_A(t)\widetilde{\Phi}(t,P_{Obj})}$, thus $[\widetilde{\Phi}(t,P_{Obj})]_{|\vec{A}} = [O_{A_i}(t)]_{|\vec{A}} + \vec{\varphi}_A(t,P_{Obj})$, thus $\vec{v}_A(t,\vec{x}_{At}) =^{(10.13)} [\vec{v}(t,p_t)]_{|\vec{A}} =^{(10.9)} [\frac{\partial\widetilde{\Phi}}{\partial t}(t,P_{Obj})]_{|\vec{A}} = [O_A{}_i'(t)]_{|\vec{A}} + \frac{\partial\vec{\varphi}_A}{\partial t}(t,P_{Obj}) = \vec{0} + \frac{\partial\vec{\varphi}_A}{\partial t}(t,P_{Obj})$. Idem for B. ∎

Exercice 10.3 t is fixed. Let $p \in \mathbb{R}^n$ (point), $\vec{x}_A := [\overrightarrow{O_Ap}]_{|\vec{A}} \in \mathcal{M}_{n1}$, $\vec{u} : \mathbb{R}^n \to \vec{\mathbb{R}^n}$ be a C^1 vector field in $\mathbb{R}^n$, and define $\vec{u}_A(\vec{x}_A) := [\vec{u}(p)]_{|\vec{A}}$ (so $\vec{u}_A$ is a matrix field in $\mathcal{M}_{n1}$). Prove: $[d\vec{u}(p)]_{|\vec{A}} = d\vec{u}_A(\vec{x}_A)$ (endomorphism in $\mathcal{M}_{n1}$(A)), i.e. $d\vec{u}_A(\vec{x}_A).[\vec{w}]_{|\vec{A}} = [d\vec{u}(p)]_{|\vec{A}}.[\vec{w}]_{|\vec{A}}$ $(= [d\vec{u}(p).\vec{w}]_{|\vec{A}})$ for all $\vec{w} \in \vec{\mathbb{R}^n}$.

Answer. A point $p+h\vec{w} \in \mathbb{R}^n$ is stored by A as the matrix $[\overrightarrow{O_Ap}+h\vec{w}]_{|\vec{A}} = [\overrightarrow{O_Ap}]_{|\vec{A}} + h[\vec{w}]_{|\vec{A}} = \vec{x}_A + h[\vec{w}]_{|\vec{A}}$. Thus $d\vec{u}_A(\vec{x}_A).[\vec{w}]_{|\vec{A}} = \lim_{h\to0}\frac{\vec{u}_A(\vec{x}_A+h[\vec{w}]_{|\vec{A}})-\vec{u}_A(\vec{x}_A)}{h} = \lim_{h\to0}\frac{[\vec{u}(p+h\vec{w})]_{|\vec{A}}-[\vec{u}(p)]_{|\vec{A}}}{h} = \lim_{h\to0}\frac{[\vec{u}(p+h\vec{w})-\vec{w}(p)]_{|\vec{A}}}{h} = [\lim_{h\to0}\frac{\vec{u}(p+h\vec{w})-\vec{w}(p)}{h}]_{|\vec{A}} = [d\vec{u}(p).\vec{w}]_{|\vec{A}} = [d\vec{u}(p)]_{|\vec{A}}.[\vec{w}]_{|\vec{A}}$, true for all $\vec{w}$. ∎

Exercice 10.4 Call Q_t the transition matrix from $(\vec{A}_{it})$ to $(\vec{B}_{it})$ at t. Prove $\vec{x}_{At} = [\overrightarrow{O_AO_{Bt}}]_{|\vec{A}} + Q_t.\vec{x}_{Bt}$.

Answer. $[\vec{x}]_{|\vec{B}} = Q_t^{-1}.[\vec{x}]_{|\vec{A}}$ for all $\vec{x} \in \vec{\mathbb{R}^n}$ (change of basis formula) gives $\vec{x}_{At} = [\overrightarrow{O_Ap_t}]_{|\vec{A}} = [\overrightarrow{O_AO_{Bt}} + \overrightarrow{O_{Bt}p_t}]_{|\vec{A}} = [\overrightarrow{O_AO_{Bt}}]_{|\vec{A}} + [\overrightarrow{O_{Bt}p_t}]_{|\vec{A}} = [\overrightarrow{O_AO_{Bt}}]_{|\vec{A}} + Q_t.[\overrightarrow{O_{Bt}p_t}]_{|\vec{B}} = [\overrightarrow{O_AO_{Bt}}]_{|\vec{A}} + Q_t.\vec{x}_{Bt}$. ∎

10.1.5 Motion of $\mathcal{R}_B$ ...

Particular case $Obj = Obj\mathcal{R}_B$ in (10.8): The motion of $Obj\mathcal{R}_B$, also called the motion of $\mathcal{R}_B$, is

$$\widetilde{\Phi}_{\mathcal{R}_B} : \begin{cases} [t_1,t_2]\times Obj\mathcal{R}_B & \to \mathbb{R}^n \\ (t,Q_{\mathcal{R}_B}) & \to q_t = \widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B}), \end{cases} \tag{10.15}$$

So the Eulerian velocity and acceleration of a particle $Q_{\mathcal{R}_B} \in Obj\mathcal{R}_B$ are, at t at $q_t = \widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})$,

$$\vec{v}_{\mathcal{R}_B}(t,q_t) = \frac{\partial\widetilde{\Phi}_{\mathcal{R}_B}}{\partial t}(t,Q_{\mathcal{R}_B}) \quad\text{and}\quad \vec{\gamma}_{\mathcal{R}_B}(t,q_t) = \frac{\partial^2\widetilde{\Phi}_{\mathcal{R}_B}}{\partial^2 t}(t,Q_{\mathcal{R}_B}). \tag{10.16}$$

10.1.6 ... Drive and static "motions of $\mathcal{R}_B$"

The drive $\vec{\varphi}_D$ and static $\vec{\varphi}_S$ "motions" of $\mathcal{R}_B$ are the names given to $\vec{\varphi}_A$ and $\vec{\varphi}_B$ when $Obj = Obj\mathcal{R}_B$:

$$\vec{\varphi}_D : \begin{cases} [t_1,t_2]\times Obj\mathcal{R}_B & \to \mathcal{M}_{n1}(\text{A}) \\ (t,Q_{\mathcal{R}_B}) & \to \vec{\varphi}_D(t,Q_{\mathcal{R}_B}) := [\overrightarrow{O_Aq(t)}]_{|\vec{A}} = [\overrightarrow{O_A\widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec{A}} \overset{\text{noted}}{=} \vec{y}_D(t), \end{cases} \tag{10.17}$$

and

$$\vec{\varphi}_S : \begin{cases} Obj\mathcal{R}_B & \to \mathcal{M}_{n1}(\text{B}) \\ Q_{\mathcal{R}_B} & \to \vec{\varphi}_S(Q_{\mathcal{R}_B}) := [\overrightarrow{O_Bq(t)}]_{|\vec{B}} = [\overrightarrow{O_B\widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec{B}} \overset{\text{noted}}{=} \vec{y}_S. \end{cases} \tag{10.18}$$

($\vec{\varphi}_S$ is independent of t since $Q_{\mathcal{R}_B}$ is fixed in $\mathcal{R}_B$.) So the drive and static velocity of $Q_{\mathcal{R}_B} \in Obj\mathcal{R}_B$ at t are

$$\begin{cases} \vec{v}_D(t,\vec{y}_{Dt}) := [\vec{v}_{\mathcal{R}_B}(t,q_t)]_{|\vec{A}} = \dfrac{\partial\vec{\varphi}_D}{\partial t}(t,P_{Obj}) \quad\text{when}\quad \vec{y}_{Dt} := [\overrightarrow{O_Aq_t}]_{|\vec{A}}, \\ \vec{v}_S(t,\vec{y}_S) := [\vec{v}_{\mathcal{R}_B}(t,q_t)]_{|\vec{B}} = \vec{0} \quad \text{(null matrix since } Q_{\mathcal{R}_B} \text{ is fixed in } \mathcal{R}_B). \end{cases} \tag{10.19}$$

And the drive and static accelerations are $\vec{\gamma}_D(t,\vec{y}_{Dt}) = [\vec{\gamma}_{\mathcal{R}_B}(t,q_t)]_{|\vec{A}} = \frac{\partial^2\vec{\varphi}_D}{\partial t^2}(t,P_{Obj})$ and $\vec{\gamma}_S(t,\vec{y}_S) = \vec{0}$.

Exercice 10.5 Why introduce $\vec\varphi_S$ (static)?

Answer. You can't confuse a particle $Q_{\mathcal{R}_B}$ with stored values: matrix $\vec y_{Dt}$ by A and matrix $\vec y_S$ by B. ▪

10.2 The translator Θ_t

Definition 10.6 At t, the translator Θ_t connects the informations stored by A with that stored by B:

$$\Theta_t : \left\{\begin{array}{rl} \mathcal{M}_{n1}(\mathrm{B}) & \to \mathcal{M}_{n1}(\mathrm{A}) \\ \vec y_S & \to \vec y_{Dt} = \Theta_t(\vec y_S) \end{array}\right\} \quad \text{when} \quad \left\{\begin{array}{l} \vec y_S = \vec\varphi_S(Q_{\mathcal{R}_B}) \ \ (= [\overrightarrow{O_B\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec B}), \text{ and} \\ \vec y_{Dt} = \vec\varphi_D(t,Q_{\mathcal{R}_B}) \ \ (= [\overrightarrow{O_A\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec A}). \end{array}\right. \tag{10.20}$$

I.e. Θ_t is the "translator at t from B to A", or the "inter-referential function at t from $\mathcal{R}_B$ to $\mathcal{R}_A$", i.e. translates the "matrix position" stored by B to the corresponding "matrix position" stored by A. So, for all $Q_{\mathcal{R}_B} \in Obj_{\mathcal{R}_B}$,

$$\vec\varphi_{Dt}(Q_{\mathcal{R}_B}) = \Theta_t(\vec\varphi_S(Q_{\mathcal{R}_B})), \quad \text{i.e.} \quad \boxed{\vec\varphi_{Dt} = \Theta_t \circ \vec\varphi_S}, \tag{10.21}$$

i.e.

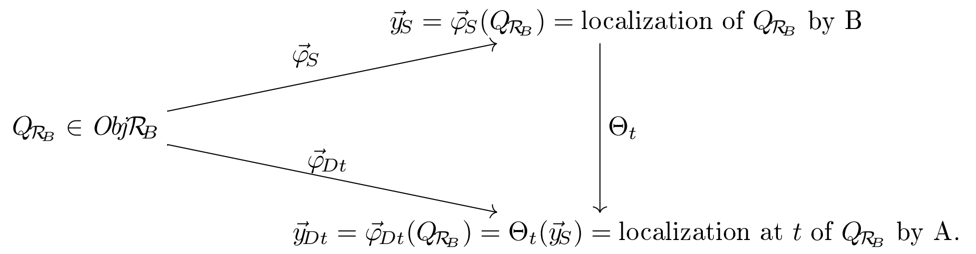
$$\Theta_t := \vec\varphi_{Dt} \circ \vec\varphi_S^{-1} : \left\{\begin{array}{rl} \mathcal{M}_{n1}(\mathrm{B}) & \to \mathcal{M}_{n1}(\mathrm{A}) \\ \vec y_S & \to \vec y_{Dt} = \Theta_t(\vec y_S) := \vec\varphi_{Dt}(\vec\varphi_S^{-1}(\vec y_S)). \end{array}\right. \tag{10.22}$$

E.g. $[\overrightarrow{O_AO_{Bt}}]_{|\vec A} = \Theta_t([\overrightarrow{O_BO_{Bt}}]_{|\vec B})$ (for the particle Q_{O_B} chosen by B to be define his origin O_{Bt} at t), so

$$[\overrightarrow{O_AO_{Bt}}]_{|\vec A} = \Theta_t(\vec 0) = \text{position of } O_{Bt} \text{ as stored by } A \text{ at } t. \tag{10.23}$$

In other words, Θ_t is defined such that the following diagram commutes:

(10.24)



Application to particles of Obj**:** Let $p_t := \widetilde\Phi(t,P_{Obj})$ = position at t in the Universe of a particle $P_{Obj} \in Obj$. Let $Q_{\mathcal{R}_B} \in Obj_{\mathcal{R}_B}$ be the particle which is at t at $q_t = p_t$, i.e. s.t. $\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B}) = \widetilde\Phi(t,P_{Obj})$, i.e. $Q_{\mathcal{R}_B} := (\widetilde\Phi_{\mathcal{R}_{Bt}})^{-1}(p_t)$. So $\vec x_{At} = [\overrightarrow{O_Ap_t}]_{\vec A} = [\overrightarrow{O_Aq_t}]_{\vec A} = \vec y_{Dt}$ and $\vec x_{Bt} = [\overrightarrow{O_Bp_t}]_{\vec B} = [\overrightarrow{O_Bq_t}]_{\vec B} = \vec y_S$, and (10.22) gives

$$\vec x_{At} = \Theta_t(\vec x_{Bt}), \quad \text{i.e.} \quad \vec\varphi_{At}(P_{Obj}) = \Theta_t(\vec\varphi_{Bt}(P_{Obj})), \quad \text{i.e.} \quad \boxed{\vec\varphi_{At} = \Theta_t \circ \vec\varphi_{Bt}}. \tag{10.25}$$

In other words, the diagram (10.23) commutes with Obj, $\vec\varphi_{Bt}$ and $\vec\varphi_{At}$ in place of $Obj_{\mathcal{R}_B}$, $\vec\varphi_S$ and $\vec\varphi_{Dt}$.

10.3 The differential $d\Theta_t$, and push-forward of vector fields

Fix t and let $\vec y_S \in \mathcal{M}_{n1}(\mathrm{B})$. Recall: Θ_t being supposed C^1, the differential $d\Theta_t(\vec y_S)$ is defined by

$$d\Theta_t(\vec y_S) : \left\{\begin{array}{rl} \mathcal{M}_{n1}(\mathrm{B}) & \to \mathcal{M}_{n1}(\mathrm{A}) \\ \vec w_S & \to d\Theta_t(\vec y_S).\vec w_S = \displaystyle\lim_{h\to 0} \frac{\Theta_t(\vec y_S + h\vec w_S) - \Theta_t(\vec y_S)}{h}. \end{array}\right. \tag{10.26}$$

And if $\vec w_S : \left\{\begin{array}{rl} \mathcal{M}_{n1}(\mathrm{B}) & \to \mathcal{M}_{n1}(\mathrm{B}) \\ \vec y_S & \to \vec w_S(\vec y_S) \end{array}\right\}$ is a vector field in $\mathcal{M}_{n1}(\mathrm{B})$ then its push-forward by Θ_t is the vector field $\Theta_{t*}\vec w_S = \vec w_{St*} : \left\{\begin{array}{rl} \mathcal{M}_{n1}(\mathrm{A}) & \to \mathcal{M}_{n1}(\mathrm{A}) \\ \vec y_{Dt} & \to \vec w_{St*}(\vec y_{Dt}) \end{array}\right\}$ in $\mathcal{M}_{n1}(\mathrm{A})$ defined by

$$\vec w_{St*}(\vec y_{Dt}) := d\Theta_t(\vec y_S).\vec w_S(\vec y_S) \quad \text{when} \quad \vec y_{Dt} = \Theta_t(\vec y_S). \tag{10.27}$$

I.e., $\vec w_{St*}([\overrightarrow{O_Aq_t}]_{|\vec A}) = d\Theta_t([\overrightarrow{O_Bq_t}]_{|\vec B}).\vec w_S([\overrightarrow{O_Bq_t}]_{|\vec B})$ for all $Q_{\mathcal{R}_B} \in Obj_{\mathcal{R}_B}$ with $q_t = \widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B}) \in \mathbb{R}^n$.

10.4 Θ_t is affine in classical mechanics

Proposition 10.7 *t being fixed, $\Theta_t : \mathcal{M}_{nl}(B) \to \mathcal{M}_{nl}(A)$ is affine: For all $\vec{y}_{S0}, \vec{y}_{S1} \in \mathcal{M}_{nl}(B)$ and all $u \in \mathbb{R}$,*

$$\Theta_t((1-u)\vec{y}_{S0} + u\,\vec{y}_{S1}) = (1-u)\Theta_t(\vec{y}_{S0}) + u\,\Theta_t(\vec{y}_{S1}). \tag{10.28}$$

Thus

$$d\Theta_t(\vec{y}_{S0}) \overset{noted}{=} d\Theta_t \text{ is independent of } \vec{y}_{S0}, \tag{10.29}$$

and

$$\Theta_t(\vec{y}_{S1}) = \Theta_t(\vec{y}_{S0}) + d\Theta_t.(\vec{y}_{S1}-\vec{y}_{S0}), \quad \text{i.e.} \quad \vec{y}_{Dt1} = \vec{y}_{Dt0} + d\Theta_t.(\vec{y}_{S1}-\vec{y}_{S0}) \tag{10.30}$$

where $\vec{y}_{Dt0} = \Theta_t(\vec{y}_{S0})$ and $\vec{y}_{Dt1} = \Theta_t(\vec{y}_{S1})$. In other words, for all $Q_{B0}, Q_{B1} \in Obj\mathcal{R}_B$ and all $u \in \mathbb{R}$, with $q_0 = \widetilde{\Phi}_{\mathcal{R}_{Bt}}(Q_{B0})$, $q_1 = \widetilde{\Phi}_{\mathcal{R}_{Bt}}(Q_{B1}) \in \mathbb{R}^n$, $\vec{y}_{S0} = [\overrightarrow{O_B q_0}]_{|\vec{B}}$, $\vec{y}_{S1} = [\overrightarrow{O_B q_1}]_{|\vec{B}}$, $\vec{y}_{Dt0} = [\overrightarrow{O_A q_0}]_{|\vec{A}}$, $\vec{y}_{Dt1} = [\overrightarrow{O_A q_1}]_{|\vec{A}}$,

$$\Theta_t([\overrightarrow{O_B q_0}]_{|\vec{B}} + u\,[\overrightarrow{q_0 q_1}]_{|\vec{B}}) = [\overrightarrow{O_A q_0}]_{|\vec{A}} + u\,[\overrightarrow{q_0 q_1}]_{|\vec{A}}, \quad \text{i.e.} \quad [\overrightarrow{q_0 q_1}]_{|\vec{A}} = d\Theta_t.[\overrightarrow{q_0 q_1}]_{|\vec{B}}. \tag{10.31}$$

In particular

$$[\vec{B}_{it}]_{\vec{A}} = d\Theta_t.[\vec{B}_i]_{|\vec{B}}. \tag{10.32}$$

Proof. At t, consider a straight line of particles (possible in classical mechanic) spotted along $q : u \to q(u) = q_0 + u\,\overrightarrow{q_0 q_1}$ in $\mathbb{R}^n$. In particular, $q(0) = q_0$ and $q(1) = q_1$. Let $\vec{y}_S(u) = [\overrightarrow{O_B q(u)}]_{|\vec{B}}$ and $\vec{y}_{Dt}(u) = [\overrightarrow{O_A q(u)}]_{|\vec{A}}$. With $[\overrightarrow{O_B q(u)}]_{|\vec{B}} = [\overrightarrow{O_B q_0}+u\,\overrightarrow{q_0 q_1}]_{|\vec{B}} = [(1-u)\overrightarrow{O_B q_0}+u\,\overrightarrow{O_B q_1}]_{|\vec{B}} = (1-u)[\overrightarrow{O_B q_0}]_{|\vec{B}}+u\,[\overrightarrow{O_B q_1}]_{|\vec{B}}$, idem with $\vec{y}_{Dt}$, we get

$$\vec{y}_S(u) = (1-u)\vec{y}_{S0} + u\,\vec{y}_{S1} \quad \text{and} \quad \vec{y}_{Dt}(u) = (1-u)\vec{y}_{Dt0} + u\vec{y}_{Dt1} \quad (\text{straight lines in } \mathcal{M}_{nl}) \tag{10.33}$$

where $\vec{y}_{S0} = \vec{y}_S(0) = [\overrightarrow{O_B q_0}]_{|\vec{B}}$, $\vec{y}_{S1} = \vec{y}_S(1) = [\overrightarrow{O_B q_1}]_{|\vec{B}}$, $\vec{y}_{Dt0} = \vec{y}_{Dt}(0) = [\overrightarrow{O_A q_0}]_{|\vec{A}}$, $\vec{y}_{Dt1} = \vec{y}_{Dt}(1) = [\overrightarrow{O_A q_1}]_{|\vec{A}}$. And (10.20) gives $\Theta_t(\vec{y}_S(u)) = \vec{y}_{Dt}(u)$ for all u, thus (10.33) gives $\Theta_t((1-u)\vec{y}_{S0} + u\vec{y}_{S1}) = (1-u)\vec{y}_{Dt0} + u\vec{y}_{Dt1}$, thus (10.28): Θ_t is affine. Thus (10.29) and (10.30).

And with $\vec{y}_{S0} = [O_B O_{Bt}]_{\vec{B}} = \vec{0}$ and $\vec{y}_{Si} = [O_B P_{Bit}]_{\vec{B}} = [\vec{B}_i]_{|\vec{B}}$, we have $\vec{y}_{Dt0} = [O_A O_{Bt}]_{|\vec{A}}$ and $\vec{y}_{Dti} = [O_A P_{Bit}]_{|\vec{A}}$, thus $[O_{Bt} P_{Bit}]_{|\vec{A}} =^{(10.30)} d\Theta_t.[O_B P_{Bit}]_{|\vec{B}}$, i.e. (10.32). ∎

Exercice 10.8 Call $Q_t = [Q_{t,ij}]$ the transition matrix from $(\vec{A}_{it})$ to $(\vec{B}_{it})$ in $\vec{\mathbb{R}^n}$. Prove

$$[d\Theta_t]_{|\vec{E}} = Q_t, \quad \text{i.e.} \quad \forall j,\ d\Theta_t.\vec{E}_j = \sum_{i=1}^n Q_{t,ij}\vec{E}_i \ \ (= \vec{E}_{jt*}). \tag{10.34}$$

Answer. The transition matrix Q_t is defined by $\vec{B}_{jt} = \sum_{i=1}^n Q_{t,ij}\vec{A}_{it}$. Thus $[\vec{B}_{jt}]_{|\vec{A}} = \sum_{i=1}^n Q_{t,ij}[\vec{A}_{it}]_{|\vec{A}} = \sum_{i=1}^n Q_{t,ij}\vec{E}_i$. With $[\vec{B}_{jt}]_{|\vec{A}} =^{(10.32)} d\Theta_t.[\vec{B}_{jt}]_{|\vec{B}} = d\Theta_t.\vec{E}_j$. Thus $d\Theta_t.\vec{E}_j = \sum_{i=1}^n Q_{t,ij}\vec{E}_i$, thus $[d\Theta_t]_{|\vec{E}} = Q_t$. ∎

10.5 Translated velocities

Definition 10.9 The translated velocities and translated accelerations from B to A, called the translated relative velocities and translated relative accelerations, are the push-forwards by Θ_t of the relative velocity and the relative accelerations: At t at p_t,

$$\boxed{\vec{v}_{Bt*}(\vec{x}_{At}) := d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})} \text{ and } \boxed{\vec{\gamma}_{Bt*}(\vec{x}_{At}) := d\Theta_t(\vec{x}_{Bt}).\vec{\gamma}_{Bt}(\vec{x}_{Bt})} \text{ when } \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}). \tag{10.35}$$

In other words: $\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).[\vec{v}_t(p_t)]_{\vec{B}}$ and $\vec{\gamma}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).[\vec{\gamma}_t(p_t)]_{\vec{B}}$ in $\mathcal{M}_{nl}(A)$.

In particular if Θ_t is affine, with $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$:

$$\vec{v}_{Bt*}(\vec{x}_{At}) := d\Theta_t.\vec{v}_{Bt}(\vec{x}_{Bt}) \quad \text{and} \quad \vec{\gamma}_{Bt*}(\vec{x}_{At}) = d\Theta_t.\vec{\gamma}_{Bt}(\vec{x}_{Bt}). \tag{10.36}$$

Exercice 10.10 $(\vec{A}_i)$ and $(\vec{B}_i)$ are Euclidean bases (e.g. in foot and metre), $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_B$ are the associated Euclidean dot products, $\lambda = ||\vec{B}_i||_A$ (e.g. $\simeq 3.28$), so $(\cdot,\cdot)_A = \lambda^2(\cdot,\cdot)_B$. And $(\cdot,\cdot)_{\rm can}$ is the canonical inner dot product in $\mathcal{M}_{nl}$ (defined by $(\vec{E}_i,\vec{E}_j)_{\rm can} = \delta_{ij}$ for all i,j). Suppose Θ_t is affine and let $\vec{E}_{it*} := d\Theta_t.\vec{E}_i$ (push-forward by Θ_t). Prove:

$$\forall i,j,\ \ (\vec{E}_{it*},\vec{E}_{jt*})_{\rm can} = \lambda^2\delta_{ij}, \quad \text{and} \quad d\Theta_t{}^T.d\Theta_t = \lambda^2 I. \tag{10.37}$$

Answer. $(\vec{B}_i)$ is a Euclidean basis for B, thus is a Euclidean orthogonal basis for all observers; It is seen at t as $(\vec{B}_{it})$ by A with $||\vec{B}_{it}||_A = \lambda$ for all i, so $(\cdot,\cdot)_A = \lambda^2(\cdot,\cdot)_B$. And $\vec{E}_{it*} = d\Theta_t.\vec{E}_i = d\Theta_t.[\vec{B}_{it}]_{|\vec{B}} =^{(10.32)} [\vec{B}_{it}]_{|\vec{A}}$. Thus $(\vec{E}_{it*},\vec{E}_{jt*})_{\rm can} = [\vec{E}_{it*}]^T.[\vec{E}_{jt*}] = [\vec{B}_{it}]_{|\vec{A}}^T.[\vec{B}_{jt}]_{|\vec{A}} = (\vec{B}_{it},\vec{B}_{jt})_A = \lambda^2(\vec{B}_{it},\vec{B}_{jt})_B = \lambda^2\delta_{ij}$, thus $(10.37)_1$; Hence $\lambda^2(\vec{E}_i,\vec{E}_j)_{\rm can} = \lambda^2\delta_{ij} = (\vec{E}_{it*},\vec{E}_{jt*})_{\rm can} = (d\Theta_t.\vec{E}_i, d\Theta_t.\vec{E}_j)_{\rm can} = (d\Theta_t{}^T.d\Theta_t.\vec{E}_i,\vec{E}_j)_{\rm can}$, true for all i,j, thus $d\Theta_t{}^T.d\Theta_t = \lambda^2 I$, thus $(10.37)_2$. ∎

10.6 Definition of Θ

Definition 10.11 The translator from B to A is the function Θ defined with (10.22) by

$$\Theta : \begin{cases} [t_1,t_2]\times\mathcal{M}_{nl}(\mathrm{B}) & \to \mathcal{M}_{nl}(\mathrm{A}) \\ (t,\vec{y}_S) & \to \vec{y}_D(t) = \boxed{\Theta(t,\vec{y}_S) := \Theta_t(\vec{y}_S)} = \vec{y}_{Dt}. \end{cases} \tag{10.38}$$

So, for all $Q_{\mathcal{R}_B} \in Obj\mathcal{R}_B$ and all t,

$$\Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B})) = \vec{\varphi}_D(t,Q_{\mathcal{R}_B}). \tag{10.39}$$

E.g., $\Theta(t,\vec{0}) = [\overrightarrow{O_AO_B(t)}]_{|\vec{A}}$, cf. (10.23).

Remark 10.12 The translator Θ looks like a motion, but is not: A motion is characterized in one referential and connects one particle to its positions; While Θ connects two referentials: It is an "inter-referential" function. ∎

10.7 The "Θ-velocity" is the drive velocity

Definition 10.13 The "Θ-velocity" and "Θ-acceleration" $\vec{v}_\Theta$, $\vec{\gamma}_\Theta : [t_1,t_2]\times\mathcal{M}_{nl}(\mathrm{A}) \to \vec{\mathbb{R}^n}$ are defined by (Eulerian type definition), at t at $\vec{y}_{Dt} = \Theta(t,\vec{y}_S)$,

$$\begin{cases} \vec{v}_\Theta(t,\vec{y}_{Dt}) := \dfrac{\partial\Theta}{\partial t}(t,\vec{y}_S), \\ \vec{\gamma}_\Theta(t,\vec{y}_{Dt}) = \dfrac{\partial^2\Theta}{\partial t^2}(t,\vec{y}_S). \end{cases} \tag{10.40}$$

(Recall: $\frac{\partial\Theta}{\partial t}(t,\vec{y}_S) = \lim_{h\to 0}\frac{\Theta(t+h,\vec{y}_S)-\Theta(t,\vec{y}_S)}{h} \in \mathcal{M}_{nl}(\mathrm{A})$.)

Proposition 10.14

$$\boxed{\vec{v}_\Theta = \vec{v}_D} \quad \textit{and} \quad \boxed{\vec{\gamma}_\Theta = \vec{\gamma}_D}, \tag{10.41}$$

i.e. $\vec{v}_\Theta(t,\vec{y}) = \vec{v}_D(t,\vec{y})$ *and* $\vec{\gamma}_\Theta(t,\vec{y}) = \vec{\gamma}_D(t,\vec{y})$ *in* $\mathcal{M}_{nl}(\mathrm{A})$*, for all* $t\in[t_1,t_2]$ *and all* $\vec{y}\in\mathcal{M}_{nl}(\mathrm{A})$.

Proof. $\vec{\varphi}_D(t,Q_{\mathcal{R}_B}) =^{(10.22)} \Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B}))$, for all t and $Q_{\mathcal{R}_B}$, gives

$$\frac{\partial\vec{\varphi}_D}{\partial t}(t,Q_{\mathcal{R}_B}) = \frac{\partial\Theta}{\partial t}(t,\vec{\varphi}_S(Q_{\mathcal{R}_B})), \quad \text{i.e.} \quad \vec{v}_D(t,\vec{\varphi}_D(t,Q_{\mathcal{R}_B})) = \vec{v}_\Theta(t,\Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B}))), \tag{10.42}$$

i.e. $\vec{v}_D(t,\vec{y}_{Dt}) = \vec{v}_\Theta(t,\Theta_t(\vec{y}_S)) = \vec{v}_\Theta(t,\vec{y}_{Dt})$. Idem with $\frac{\partial^2}{\partial t^2}$. ∎

10.8 The velocity-addition formula

(10.25) gives

$$\vec\varphi_A(t,P_{Obj}) = \Theta(t,\vec\varphi_B(t,P_{Obj})). \tag{10.43}$$

Thus

$$\underbrace{\frac{\partial\vec\varphi_A}{\partial t}(t,P_{Obj})}_{\vec v_{At}(\vec x_{At})} = \underbrace{\frac{\partial\Theta}{\partial t}(t,\vec\varphi_B(t,P_{Obj}))}_{\overset{(10.41)}{=}\vec v_{Dt}(\vec x_{At})} + \underbrace{d\Theta(t,\vec\varphi_B(t,P_{Obj})).\frac{\partial\vec\varphi_B}{\partial t}(t,P_{Obj})}_{=d\Theta_t(\vec x_{Bt}).\vec v_{Bt}(\vec x_{Bt})\overset{(10.35)}{=}\vec v_{Bt*}(\vec x_{At})}, \tag{10.44}$$

i.e. $\vec v_{At}(\vec x_{At}) = \vec v_{Dt}(\vec x_{At}) + \vec v_{Bt*}(\vec x_{At})$, where $\vec x_{Bt} = \vec\varphi_B(t,P_{Obj})$ and $\vec x_{At} = \vec\varphi_A(t,P_{Obj}) = \Theta_t(\vec x_{Bt})$, Hence:

$$\boxed{\vec v_{At} = \vec v_{Dt} + \vec v_{Bt*}} = \text{the velocity-addition formula in } \mathcal{R}_A, \tag{10.45}$$

which reads:

$$\begin{aligned}\text{absolute velocity} &= \text{drive velocity}\\ &\quad + \text{translated relative velocity from B to A.}\end{aligned} \tag{10.46}$$

In other words (relation between the numerical values of the velocities stored by A and B),

$$[\vec v_t(p_t)]_{|\vec A} = [\vec v_{\mathcal{R}_{Bt}}(p_t)]_{|\vec A} + d\Theta_t([\overrightarrow{O_{Bt}p_t}]_{|\vec B}).[\vec v_t(p_t)]_{\vec B}. \tag{10.47}$$

10.9 Coriolis acceleration, and the acceleration-addition formula

(10.44) gives

$$\begin{aligned}\underbrace{\frac{\partial^2\vec\varphi_A}{\partial t^2}(t,P_{Obj})}_{\vec\gamma_{At}(\vec x_{At})} = {} & \underbrace{\frac{\partial^2\Theta}{\partial t^2}(t,\vec x_{Bt})}_{\vec\gamma_{Dt}(\vec x_{At})} + d\frac{\partial\Theta}{\partial t}(t,\vec x_{Bt}).\frac{\partial\vec\varphi_B}{\partial t}(t,P_{Obj})\\ & + \Big(\frac{\partial(d\Theta)}{\partial t}(t,\vec x_{Bt}) + d^2\Theta_t(\vec x_{Bt}).\frac{\partial\vec\varphi_B}{\partial t}(t,P_{Obj})\Big).\frac{\partial\vec\varphi_B}{\partial t}(t,P_{Obj}) + \underbrace{d\Theta_t(\vec x_{Bt}).\frac{\partial^2\vec\varphi_B}{\partial t^2}(t,P_{Obj})}_{\vec\gamma_{Bt*}(\vec x_{At})},\end{aligned} \tag{10.48}$$

i.e.

$$\vec\gamma_{At}(\vec x_{At}) = \vec\gamma_{Dt}(\vec x_{At}) + \vec\gamma_{Ct}(\vec x_{At}) + \vec\gamma_{Bt*}(\vec x_{At}), \tag{10.49}$$

where (with $\Theta\in C^2$):

Definition 10.15 At t, the Coriolis acceleration $\vec\gamma_{Ct}$ is, at $\vec x_{At}$,

$$\vec\gamma_{Ct}(\vec x_{At}) = 2\,d\vec v_{Dt}(\vec x_{At}).\vec v_{Bt*}(\vec x_{At}) + d^2\Theta(t,\vec x_{Bt})(\vec v_{Bt}(\vec x_{Bt}),\vec v_{Bt}(\vec x_{Bt})). \tag{10.50}$$

And the Coriolis acceleration $\vec\gamma_C$ at t at $\vec x_{At}$ is $\vec\gamma_C(t,\vec x_{At}) := \vec\gamma_{Ct}(\vec x_{At})$.

Hence:

$$\boxed{\vec\gamma_{At} = \vec\gamma_{Dt} + \vec\gamma_{Ct} + \vec\gamma_{Bt*}} = \text{the acceleration-addition formula in } \mathcal{R}_A, \tag{10.51}$$

which reads:

$$\text{absolute acceleration} = (\text{drive} + \text{Coriolis} + \text{translated})\text{ accelerations.} \tag{10.52}$$

Particular case Θ_t affine ($d^2\Theta_t = 0$): At t, the Coriolis acceleration $\vec\gamma_{Ct}$ at $\vec x_{At}$ is

$$\vec\gamma_{Ct}(\vec x_{At}) = 2\,d\vec v_{Dt}(\vec x_{At}).\vec v_{Bt*}(\vec x_{At}), \quad \text{i.e.} \quad \boxed{\vec\gamma_{Ct} = 2d\vec v_{Dt}.\vec v_{Bt*}}. \tag{10.53}$$

10.10 With an initial time

Let $t_0, t \in \mathbb{R}$. Consider the Lagrangian associated function $\Phi_t^{t_0}$ with the motion $\widetilde{\Phi}$ of *Obj*:

$$\Phi_t^{t_0} : \begin{cases} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj}) & \to p_t = \Phi_t^{t_0}(p_{t_0}) := \widetilde{\Phi}(t, P_{Obj}). \end{cases} \tag{10.54}$$

And, with $\vec{x}_{At} = \vec{\varphi}_A(t, P_{Obj}) = [\overrightarrow{O_A p_t}]_{|\vec{A}}$ and $\vec{x}_{Bt} = \vec{\varphi}_B(t, P_{Obj}) = [\overrightarrow{O_B p_t}]_{|\vec{B}}$, define the "matrix motions" $\vec{\varphi}_{At}^{t_0} : \mathcal{M}_{n1}(A) \to \mathcal{M}_{n1}(A)$ and $\vec{\varphi}_{Bt}^{t_0} : \mathcal{M}_{n1}(B) \to \mathcal{M}_{n1}(B)$ by

$$\begin{cases} \vec{\varphi}_{At}^{t_0}(\vec{x}_{At_0}) := \vec{x}_{At} \quad (= [\overrightarrow{O_A \widetilde{\Phi}(t, P_{Obj})}]_{|\vec{A}} = [\overrightarrow{O_A \Phi_t^{t_0}(p_{t_0})}]_{|\vec{A}} = \vec{\varphi}_{At}(P_{Obj})), \\ \vec{\varphi}_{Bt}^{t_0}(\vec{x}_{Bt_0}) := \vec{x}_{Bt} \quad (= [\overrightarrow{O_B \widetilde{\Phi}(t, P_{Obj})}]_{|\vec{B}} = [\overrightarrow{O_B \Phi_t^{t_0}(p_{t_0})}]_{|\vec{B}} = \vec{\varphi}_{Bt}(P_{Obj})). \end{cases} \tag{10.55}$$

And $\Theta_t(\vec{x}_{Bt}) = \vec{x}_{At}$, i.e. $\Theta_t(\vec{\varphi}_{Bt}^{t_0}(\vec{x}_{Bt_0})) = \vec{\varphi}_{At}^{t_0}(\vec{x}_{At_0})$ with $\vec{x}_{At_0} = \Theta_{t_0}(\vec{x}_{Bt_0})$, thus

$$\boxed{\Theta_t \circ \vec{\varphi}_{Bt}^{t_0} = \vec{\varphi}_{At}^{t_0} \circ \Theta_{t_0}} : \mathcal{M}_{n1}(B) \to \mathcal{M}_{n1}(A). \tag{10.56}$$

In other words, the following diagram commutes:

$$\begin{array}{ccccc} & & \vec{x}_{Bt_0} = \vec{\varphi}_B(t_0, P_{Obj}) & \xrightarrow{\ \vec{\varphi}_{Bt}^{t_0}\ } & \vec{x}_{Bt} = \vec{\varphi}_{Bt}^{t_0}(\vec{x}_{Bt_0}) \\ & \nearrow^{\vec{\varphi}_{Bt_0}} & \downarrow \Theta_{t_0} & & \downarrow \Theta_t \\ P_{Obj} \in Obj & & & & \\ & \searrow_{\vec{\varphi}_{At_0}} & & & \\ & & \vec{x}_{At_0} = \vec{\varphi}_A(t_0, P_{Obj}) = \Theta_{t_0}(\vec{x}_{Bt_0}) & \xrightarrow{\ \vec{\varphi}_{At}^{t_0}\ } & \vec{x}_{At} = \vec{\varphi}_{At}^{t_0}(\vec{x}_{At_0}) = \Theta_t(\vec{x}_{Bt}). \end{array} \tag{10.57}$$

Thus, for any vector field $\vec{u}_{Bt_0}$ in $\mathcal{R}_B$,

$$\underbrace{d\Theta_t(\vec{x}_{Bt})}_{\text{(translation at t)}} . \underbrace{d\vec{\varphi}_{Bt}^{t_0}(\vec{x}_{Bt_0}).\vec{u}_{Bt_0}(\vec{x}_{Bt_0})}_{\text{(deformation from } t_0 \text{ to t)}} = \underbrace{d\vec{\varphi}_{At}^{t_0}(\vec{x}_{At_0})}_{\text{(deformation from } t_0 \text{ to t)}} . \underbrace{d\Theta_{t_0}(\vec{x}_{Bt_0}).\vec{u}_{Bt_0}(\vec{x}_{Bt_0})}_{\text{(translation at } t_0)} . \tag{10.58}$$

Exercice 10.16 Redo the above steps with $Obj\mathcal{R}_B$ instead of *Obj*.

Answer. Consider the Lagrangian associated function $\Phi_{RBt}^{t_0}$ with the motion $\widetilde{\Phi}_{\mathcal{R}_B}$ of $Obj\mathcal{R}_B$:

$$\Phi_{RBt}^{t_0} : \left\{ \begin{array}{ll} \Omega_{RBt_0} = \mathbb{R}^n & \to \Omega_{RBt} = \mathbb{R}^n \\ q_{t_0} = \widetilde{\Phi}_{\mathcal{R}_B}(t_0, Q_{\mathcal{R}_B}) & \to q = \Phi_{RBt}^{t_0}(q_{t_0}) := \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B}), \end{array} \right\} \tag{10.59}$$

then define the "matrix motions" $\vec{\varphi}_{Dt}^{t_0} : \mathcal{M}_{n1}(A) \to \mathcal{M}_{n1}(A)$ and $\vec{\varphi}_{St}^{t_0} : \mathcal{M}_{n1}(B) \to \mathcal{M}_{n1}(B)$ by

$$\begin{cases} \vec{\varphi}_{Dt}^{t_0}(\vec{y}_{Dt0}) := \vec{y}_{Dt} \quad (= [\overrightarrow{O_A \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}]_{|\vec{A}} = [\overrightarrow{O_A \Phi_{RBt}^{t_0}(p_{t_0})}]_{|\vec{A}} = \vec{\varphi}_{Dt}(Q_{\mathcal{R}_B})), \\ \vec{\varphi}_{St}^{t_0}(\vec{y}_S) := \vec{y}_S \quad (= [\overrightarrow{O_B \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}]_{|\vec{B}} = [\overrightarrow{O_B \Phi_{RBt}^{t_0}(q_{t_0})}]_{|\vec{B}} = \vec{\varphi}_S(Q_{\mathcal{R}_B})), \end{cases} \tag{10.60}$$

Thus $\vec{\varphi}_S$ is a time-shift, which is also abusively noted $\vec{\varphi}_{St}^{t_0} = I$ (algebraic identity). So with $\Theta_t(\vec{y}_S) = \vec{y}_{Dt}$ we get $\Theta_t(\vec{\varphi}_{Dt}^{t_0}(\vec{y}_S)) = \vec{\varphi}_{Dt}^{t_0}(\vec{y}_{Dt0})$, with $\vec{y}_{Dt0} = \Theta_{t_0}(\vec{y}_S)$, thus

$$\boxed{\Theta_t \circ \vec{\varphi}_{St}^{t_0} = \vec{\varphi}_{Dt}^{t_0} \circ \Theta_{t_0}} : \mathcal{M}_{n1}(B) \to \mathcal{M}_{n1}(A) \tag{10.61}$$

(also abusively written $\Theta_t = \vec{\varphi}_{Dt}^{t_0} \circ \Theta_{t_0}$). In other words, the following diagram commutes:

$$\begin{array}{ccccc} & & \vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B}) & \xrightarrow{\ \vec{\varphi}_{St}^{t_0} = \text{time shift}\ } & \vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B}) \\ & \nearrow^{\vec{\varphi}_S} & \downarrow \Theta_{t_0} & & \downarrow \Theta_t \\ Q_{\mathcal{R}_B} \in Obj\mathcal{R}_B & & & & \\ & \searrow_{\vec{\varphi}_D^{t_0}} & & & \\ & & \vec{y}_{Dt0} = \vec{\varphi}_{Dt_0}(Q_{\mathcal{R}_B}) = \Theta_{t_0}(\vec{y}_S) & \xrightarrow{\ \vec{\varphi}_{Dt}^{t_0}\ } & \vec{y}_{Dt} = \vec{\varphi}_{Dt}(Q_{\mathcal{R}_B}) = \vec{\varphi}_{Dt}^{t_0}(\vec{y}_{Dt0}) = \Theta_t(\vec{y}_S). \end{array} \tag{10.62}$$

And (10.61) gives, for any $\vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B})$ and all vector field $\vec{u}_S$ (static in $\mathcal{R}_B$), with $\vec{y}_{Dt0} = \Theta_{t_0}(\vec{y}_S)$,

$$\underbrace{d\Theta_t(\vec{y}_S)}_{\text{(translation at t)}} . \underbrace{d\vec{\varphi}_{St}^{t_0}(\vec{y}_S).\vec{u}_S(\vec{y}_S)}_{\text{(time shift from } t_0 \text{ to t)}} = \underbrace{d\vec{\varphi}_{Dt}^{t_0}(\vec{y}_{Dt0})}_{\text{(Drive motion from } t_0 \text{ to t)}} . \underbrace{d\Theta_{t_0}(\vec{y}_S).\vec{u}_S(\vec{y}_S)}_{\text{(translation at } t_0)} . \tag{10.63}$$

▪

10.11 Drive and Coriolis forces

10.11.1 Fundamental principal: requires a Galilean referential

Second Newton's law of motion (fundamental principle of dynamics): In a Galilean referential, the sum of the external forces $\vec{f}$ on an object is equal to its mass multiplied by its acceleration:

$$\sum \text{external} \vec{f} = m\vec{\gamma} \quad \text{(in a Galilean referential)}. \tag{10.64}$$

Question: And in a Non Galilean referential?

Answer: Then you have to add "observer dependent forces", i.e. you have to add "apparent forces" due to the motion of the non Galilean observer. Indeed, the motion of an object in our Universe does not care about the observer motion (his accelerations and velocities).

See e.g. `https://www.youtube.com/watch?v=_36MiCUS1ro` for a carousel (a merry-go-round),

See e.g. `https://www.youtube.com/watch?v=aeY9tY9vKgs` for tornadoes.

10.11.2 Drive + Coriolis forces = the inertial force

Consider $\vec{f}(t,p_t)$ = the sum of the external forces acting on P_{Obj} at t at $p_t = \widetilde{\Phi}(t, P_{Obj})$.

In a Galilean referential $\mathcal{R}_A$, Newton laws (10.64) means

$$[\vec{f}_t(p_t)]_{|\vec{A}} = m\,[\vec{\gamma}_t(p_t)]_{|\vec{A}}, \quad \text{written} \quad \boxed{\vec{f}_{At}(\vec{x}_{At}) = m\,\vec{\gamma}_{At}(\vec{x}_{At})} \quad (\in \mathcal{M}_{n1}), \tag{10.65}$$

with $\vec{x}_{At} := [\overrightarrow{O_A p_t}]_{|\vec{A}}$, $\vec{f}_{At}(\vec{x}_{At}) := [\vec{f}_t(p_t)]_{|\vec{A}}$ and $\vec{\gamma}_{At}(\vec{x}_{At}) = [\vec{\gamma}_t(p_t)]_{|\vec{A}}$. With $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, the acceleration addition formula gives $\vec{f}_{At}(\vec{x}_{At}) = m(d\Theta_t.\vec{\gamma}_B(\vec{x}_{Bt}) + \vec{\gamma}_{Dt}(\vec{x}_{At}) + \vec{\gamma}_{Ct}(\vec{x}_{At})) \in \mathcal{R}_A$, thus, in $\mathcal{R}_B$,

$$\underbrace{d\Theta_t^{-1}.\vec{f}_{At}(\vec{x}_{At})}_{\vec{f}_{At^*}(\vec{x}_{Bt}) = \vec{f}_{Bt}(\vec{x}_{Bt})} = m\,\vec{\gamma}_B(\vec{x}_{Bt}) + \underbrace{m\,d\Theta_t^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At})}_{m\,\vec{\gamma}_{Dt}{}^*(\vec{x}_{Bt})} + \underbrace{m\,d\Theta_t^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At})}_{m\,\vec{\gamma}_{Ct}{}^*(\vec{x}_{Bt})}, \tag{10.66}$$

and $d\Theta_t^{-1}.[\vec{f}_t(p_t)]_{|\vec{A}} = d\Theta_t^{-1}.\vec{f}_{At}(\vec{x}_{At}) =^{(10.32)} [\vec{f}_t(p_t)]_{|\vec{B}} =^{\text{noted}} \vec{f}_{Bt}(\vec{x}_{Bt})$ is the external forces as quantified by B at t, cf. (10.32) (with Θ_t supposed to be affine). And with the pull-back notation, cf. (10.32):

Definition 10.17 For B at t at p_t, with $\vec{x}_{Bt} = [\overrightarrow{O_{Bt} p_t}]_{|\vec{B}}$ in $\mathcal{M}_{n1}$(B):

- The drive force $\vec{f}_{BDt}(\vec{x}_{Bt}) := -m\,\vec{\gamma}_{Dt}{}^*(\vec{x}_{Bt}) \quad (= -m\,d\Theta_t^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At}))$.
- The Coriolis force $\vec{f}_{BCt}(\vec{x}_{Bt}) := -m\,\vec{\gamma}_{Ct}{}^*(\vec{x}_{Bt}) \quad (= -m\,d\Theta_t^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At}))$. (10.67)
- The inertial (or fictitious) force $:= \vec{f}_{BDt}(\vec{x}_{Bt}) + \vec{f}_{BCt}(\vec{x}_{Bt}) = -m\,(\vec{\gamma}_{Dt}{}^* + \vec{\gamma}_{Ct}{}^*)(\vec{x}_{Bt})$.

Then (10.66) gives the fundamental principle quantified in $\mathcal{R}_B$ (non Galilean referential):

$$\boxed{\vec{f}_{Bt}(\vec{x}_{Bt}) + \vec{f}_{BDt}(\vec{x}_{Bt}) + \vec{f}_{BCt}(\vec{x}_{Bt}) = m\,\vec{\gamma}_B(\vec{x}_{Bt})}, \tag{10.68}$$

i.e., at t, in $\mathcal{R}_B$: The (external + Drive + Coriolis) forces = m times the acceleration.

10.12 Summary for "Sun and Earth" (and Coriolis forces on the Earth)

Illustation with a simplified (circular) motion of the (spherical) Earth around the Sun.

1. **Referentials.**

1.1. Relative referential $\mathcal{R}_B = (O_B, (\vec{B}_1, \vec{B}_2, \vec{B}_3))$ chosen by the observer B fixed on the Earth, where $O_{Bt} = \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{O_B})$ is the position of the particle Q_{O_B} at the center of the Earth, written O_B by B (fixed for B), and $(\vec{B}_{1t}, \vec{B}_{2t}, \vec{B}_{3t})$ is a Euclidean basis (e.g. built with the metre) fixed in the Earth, written $(\vec{B}_1, \vec{B}_2, \vec{B}_3)$ by B (fixed for B), with $\vec{B}_3$ chosen to be along the rotation axis of the Earth and oriented from the south pole to the north pole; And $(\cdot,\cdot)_B$ is the associated Euclidean dot product. So, a fixed particle $Q_{\mathcal{R}_B}$ in the Earth at longitude $\theta_{Q_{\mathcal{R}_B}} \in]-\pi,\pi]$ and latitude $\varphi_{Q_{\mathcal{R}_B}} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is referenced by observer B as the matrix $\vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B}) = [\overrightarrow{O_B \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}]_{|\vec{B}} = R_B \begin{pmatrix} \cos(\theta_{Q_{\mathcal{R}_B}})\cos(\varphi_{Q_{\mathcal{R}_B}}) \\ \sin(\theta_{Q_{\mathcal{R}_B}})\cos(\varphi_{Q_{\mathcal{R}_B}}) \\ \sin(\varphi_{Q_{\mathcal{R}_B}}) \end{pmatrix}$ where $R_B = ||\overrightarrow{O_B \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}||_B$ is the distance between Q_{O_B} and $Q_{\mathcal{R}_B}$ (e.g. if $Q_{\mathcal{R}_B}$ is on the surface of the Earth then $R_B \simeq 6371$ km).

1.2. Initial Galilean referential $\mathcal{R}_{A0} = (O_{A0}, (\vec{A}_1, \vec{A}_2, \vec{A}_3))$: O_{A0} is at the center of the Sun and $(\vec{A}_1, \vec{A}_2, \vec{A}_3)$ is a Euclidean basis (e.g. built with the foot) fixed relative to the stars, such that $\vec{A}_3 = \mu\vec{B}_3$ with $\mu > 0$ (e.g. $\mu = 0.3048$ and $\lambda = \frac{1}{\mu} \simeq 3.28$); And $(\cdot,\cdot)_A$ is the associated Euclidean dot product.

1.3. Deduced absolute Galilean referential $\mathcal{R}_A = (O_{At}, (\vec{A}_1, \vec{A}_2, \vec{A}_3))$ chosen by observer A, where $O_{At} = O_{Bt}$, written O_A by A (fixed for A). Since it takes more that 365 days for Q_{O_B} to complete a rotation around the Sun, the motion of Q_{O_B} will be considered to be rectilinear at constant velocity "in a short interval of time" sufficient for the computation of the Coriolis acceleration with "sufficient accuracy" (simplifies the calculations).

(If A prefers to work with the initial Galilean referential $\mathcal{R}_{A0}$, then the absolute matrix motion $\vec{\varphi}_A(t, P_{Obj}) = [\overrightarrow{O_A\widetilde{\Phi}(t, P_{Obj})}]_{|\vec{A}}$ has to be replaced by $\vec{\varphi}_A(t, P_{Obj}) = [\overrightarrow{O_{A0}O_B(t)}]_{|\vec{A}} + [\overrightarrow{O_B(t)\widetilde{\Phi}(t, P_{Obj})}]_{|\vec{A}}$, idem for the drive motion $\vec{\varphi}_D$.)

2. **Drive motion.**

2.1. The motion $t \to q = \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})$ of a particle $Q_{\mathcal{R}_B}$ fixed on Earth is stored by A as the drive motion $\vec{\varphi}_D$ given by (matrix valued), with ω the angular velocity of the Earth in $\mathcal{R}_A$,

$$\vec{y}_D(t) = \vec{\varphi}_D(t, Q_{\mathcal{R}_B}) = R_A(Q_{\mathcal{R}_B}) \begin{pmatrix} \cos(\omega t)\cos\varphi_{Q_{\mathcal{R}_B}} \\ \sin(\omega t)\cos\varphi_{Q_{\mathcal{R}_B}} \\ \sin\varphi_{Q_{\mathcal{R}_B}} \end{pmatrix} = [\overrightarrow{O_A q(t)}]_{|\vec{A}} = \begin{pmatrix} y_{D1}(t) \\ y_{D2}(t) \\ y_{D3} \end{pmatrix}, \tag{10.69}$$

where $R_A(Q_{\mathcal{R}_B}) = ||\overrightarrow{Q_{O_B}Q_{\mathcal{R}_B}}||_{|\vec{A}}$ is the distance between Q_{O_B} and $Q_{\mathcal{R}_B}$ for A (e.g. $R_A \simeq 20902231$ foot if $Q_{\mathcal{R}_B}$ is on the surface of the Earth). (And (ωt) by replaced by $(\alpha_0{+}\omega(t{-}t_0))$ to be more general.)

2.2. Drive velocity: With $\vec{\omega}_D := \omega\vec{A}_3$,

$$\vec{v}_D(t, \vec{y}_D(t)) = \vec{y}_D{}'(t) = \omega R_A \begin{pmatrix} -\sin(\omega t)\cos\varphi_{Q_{\mathcal{R}_B}} \\ \cos(\omega t)\cos\varphi_{Q_{\mathcal{R}_B}} \\ 0 \end{pmatrix} = \omega \begin{pmatrix} -y_2(t) \\ y_1(t) \\ 0 \end{pmatrix} = \omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .\vec{y}_D(t) = \vec{\omega}_D \wedge \vec{y}_D(t). \tag{10.70}$$

2.3. Drive acceleration:

$$\vec{\gamma}_D(t, \vec{y}_{Dt}) = \vec{y}_D{}''(t) = \vec{\omega}_D \wedge \vec{y}_D{}'(t) = \vec{\omega}_D \wedge \vec{v}_D(t, \vec{y}_{Dt}) = \vec{\omega}_D \wedge (\vec{\omega}_D \wedge \vec{y}_D(t)) = -\omega^2 \begin{pmatrix} y_{D1}(t) \\ y_{D2}(t) \\ 0 \end{pmatrix} \tag{10.71}$$

= the usual centrifugal acceleration (in a plane parallel to the equatorial plane, drawing).

2.4. Differential of the drive velocity (time and space independent here): (10.70) gives

$$d\vec{v}_D(t, \vec{y}_{Dt}) = d\vec{v}_D = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \vec{\omega}_D \wedge . \tag{10.72}$$

3. **Translator.**

3.1. Here $O_{At} = O_{Bt}$, thus $\Theta_t(\vec{0}) = \vec{0}$ (with $[\vec{0}] =^{\text{noted}} \vec{0} =$ the null matrix), cf. (10.23).

3.2. Calculation of $d\Theta_t$. With Θ_t affine, $d\Theta_t.[\vec{B}_{it}]_{|\vec{B}} = [\vec{B}_{it}]_{|\vec{A}}$. Thus $\vec{B}_3 = \lambda\vec{A}_3$ (hypothesis) and $d\Theta_t.[\vec{B}_3]_{|\vec{B}} = [\vec{B}_{3t}]_{|\vec{A}}$ give $d\Theta_t.\vec{E}_3 = \lambda\vec{E}_3$ where $(\vec{E}_i)$ is the canonical basis in $\mathcal{M}_{n1}$. Then let $Q_{Bi} \in Obj_{\mathcal{R}_B}$ be the Earth particle which position $q_{ti} = \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{Bi})$ makes $\vec{B}_{it} := \overrightarrow{O_{Bt}q_{ti}}$. So, $\vec{B}_1$ and $\vec{B}_2$ being in the equatorial plane, (10.69) gives $d\Theta_t.\vec{E}_1 = d\Theta_t.[\vec{B}_1]_{|\vec{B}} = [\vec{B}_1]_{|\vec{A}} = [\overrightarrow{O_A q_{t1}}]_{|\vec{A}} = \lambda \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix}$, and $d\Theta_t.\vec{E}_2 = d\Theta_t.[\vec{B}_2]_{|\vec{B}} = [\vec{B}_2]_{|\vec{A}} = [\overrightarrow{O_A q_{t2}}]_{|\vec{A}} = \lambda \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix}$. Thus $[d\Theta_t]_{|\vec{E}} = \lambda \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ = the expected rotation matrix expanded by λ (change of unit of measurement).

3.3. Calculation of Θ_t (affine): $\Theta_t(\vec{y}_S) = \Theta_t(\vec{0}) + d\Theta_t.\vec{y}_S$, so, with $O_{At} = O_{Bt}$ here,

$$\vec{y}_{Dt} := \Theta_t(\vec{y}_S) = d\Theta_t.\vec{y}_S \tag{10.73}$$

4. **Motions of *Obj*.**

4.1. B quantifies the motion $\widetilde{\Phi}$ of *Obj*, i.e. he stores the relative motion $\vec{\varphi}_B$ of *Obj*, and the relative velocities and accelerations $\vec{v}_{Bt}$ and $\vec{\gamma}_B$ (matrices), cf. (10.12)-(10.14).

4.2. Translations for A: With $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$,

$$\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) \quad \text{and} \quad \vec{\gamma}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{\gamma}_{Bt}(\vec{x}_{Bt}). \tag{10.74}$$

5. **Drive force** (apparent force in $\mathcal{R}_B$ due to the motion of B):

$$\vec{f}_{BDt}(\vec{x}_{Bt}) = -m\, d\Theta_t{}^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At}) \overset{(10.71)}{=} \lambda m\omega^2 d\Theta_t{}^{-1}.\begin{pmatrix} x_{A1}(t) \\ x_{A2}(t) \\ 0 \end{pmatrix} \overset{(10.73)}{=} \lambda m\omega^2 \begin{pmatrix} x_{B1}(t) \\ x_{B2}(t) \\ 0 \end{pmatrix}, \tag{10.75}$$

centrifugal force (in a "parallel plane" at latitude of P_{Obj}).

6. **Coriolis acceleration** (apparent acceleration due to the motion of B):

$$\vec{\gamma}_{Ct}(\vec{x}_{At}) = 2\, d\vec{v}_{Dt}.(d\Theta_t.\vec{v}_{Bt}(\vec{x}_{Bt})) = 2\, d\Theta_t.d\vec{v}_{Dt}.\vec{v}_{Bt}(\vec{x}_{Bt}) \tag{10.76}$$

because $d\Theta_t$ commutes with $d\vec{v}_{Dt}$ (composition of "rotations along the same south-north axis" which reads as $e^{i\omega t}.e^{i\frac{\pi}{2}} = e^{i\frac{\pi}{2}}e^{i\omega t} = e^{i(\frac{\pi}{2}+\omega t)}$ in the equatorial plane).

7. **Coriolis force** (apparent force due to the motion of B):

$$\vec{f}_{BCt}(\vec{x}_{Bt}) = -m\, d\Theta_t{}^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At}) = -2m\, d\vec{v}_{Dt}.\vec{v}_{Bt}(\vec{x}_{Bt}) = -2m\vec{\omega} \wedge \vec{v}_{Bt}(\vec{x}_{Bt}). \tag{10.77}$$

11 Objectivities

Goal: To give an objective expression of the laws of mechanics; As Maxwell [15] said: "The formula at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result".

Generic notation: if a function z is given as $z(t,x)$, then $z_t(x) := z(t,x)$, and conversely.

11.1 "Isometric objectivity" and "Frame Invariance Principle"

This manuscript is not intended to describe "isometric objectivity":

"Isometric objectivity" is the framework in which the "principle of material frame-indifference" ("frame invariance principle") is settled, principle which states that "Rigid body motions should not affect the stress constitutive law of a material". E.g., Truesdell–Noll [22] p. 41:

« Constitutive equations must be invariant under changes of frame of reference. »

Or Germain [11] :

« Axiom of power of internal forces. The virtual power of the "internal forces" acting on a system S for a given virtual motion is an objective quantity; i.e., it has the same value whatever be the frame in which the motion is observed. »

NB: Both of these affirmations are limited to "isometric changes of frame" (the same metric for all), as Truesdell–Noll [22] page 42-43 explain: The "isometric objectivity" concern one observer who defines his Euclidean dot product and consider only orthonormal change of bases to validate a constitutive law.

If you want to interpret "isometric objectivity" in the "covariant objectivity" framework, then "isometric objectivity" corresponds to a dictatorial management: One observer with his Euclidean referential (e.g. based on the English foot), imposes his unit of length to all other users (isometry hypothesis). (Note: The metre was not adopted by the scientific community until after 1875.)

Moreover, isometric objectivity leads to despise the difference between covariance and contravariance, due to the uncontrolled use of the Riesz representation theorem.

Remark 11.1 Marsden and Hughes [14] p. 8 use this isometric framework to begin with. But, pages 22 and 163, they write that a "good modelization" has to be "covariant objective" (observer independent) to begin with; And they propose a covariant modelization for elasticity at § 3.3. ▪

11.2 Definition and characterization of the covariant objectivity

11.2.1 Framework of classical mechanics

Framework of classical mechanics to simplify. Consider two observers A and B and their referentials $\mathcal{R}_A = (O_A, (\vec{A}_i))$ and $\mathcal{R}_B = (O_B, (\vec{B}_i))$. E.g., $(\vec{A}_i)$ and $(\vec{B}_i)$ are Euclidean bases in foot and metre, $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_B$ is their associated Euclidean dot products. And Θ is the translator, cf. (10.20).

Consider a regular motion $\widetilde{\Phi}$ of an object Obj, $p_t = \widetilde{\Phi}(t, P_{Obj}) \in \mathbb{R}^n$ the position at t of a particle in our Universe, $\Omega_t = \widetilde{\Phi}(t, Obj)$ the configuration at t, and $\mathcal{C} = \bigcup_{t\in[a,b]}(\{t\}\times\Omega_t)$ the set of configurations. And $\vec{x}_{At} := [\overrightarrow{O_Ap_t}]_{|\vec{A}} \in \mathcal{M}_{n1}(\text{A})$ and $\vec{x}_{Bt} := [\overrightarrow{O_Bp_t}]_{|\vec{B}} \in \mathcal{M}_{n1}(\text{B})$ are the stored components of p_t relative to the chosen referentials, $\mathcal{M}_{n1}(\text{A})$ and $\mathcal{M}_{n1}(\text{B})$ being the spaces of $n*1$ matrices as referred to by A and B.

11.2.2 Covariant objectivity of a scalar function

Let $f : \left\{\begin{array}{ll} \mathcal{C} & \to \mathbb{R} \\ (t,p_t) & \to f(t,p_t) \end{array}\right\}$ be a Eulerian scalar function (e.g., a temperature field). f is quantified by A and B as the functions $f_A : \left\{\begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(\text{A}) & \to \mathbb{R} \\ (t,\vec{x}_{At}) & \to f_A(t,\vec{x}_{At}) := f(t,p_t) \end{array}\right\}$ and $f_B : \left\{\begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(\text{B}) & \to \mathbb{R} \\ (t,\vec{x}_{Bt}) & \to f_B(t,\vec{x}_{Bt}) := f(t,p_t) \end{array}\right\}$.

Definition 11.2 f is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all t,

$$f_{At}(\vec{x}_{At}) = f_{Bt}(\vec{x}_{Bt}) \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}), \tag{11.1}$$

i.e. $f_{At} = f_{Bt*}$ is the push-forward of f_{Bt} by Θ_t cf. (6.8).

11.2.3 Covariant objectivity of a vector field

Let $\vec{w}$: $\left\{\begin{array}{ll} \mathcal{C} & \to \vec{\mathbb{R}^n} \\ (t,p_t) & \to \vec{w}(t,p_t) \end{array}\right\}$ be a Eulerian vector field (e.g., a force field). $\vec{w}$ is quantified by A and B as the functions $\vec{w}_A$: $\left\{\begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(A) & \to \mathcal{M}_{n1}(A) \\ (t,\vec{x}_{At}) & \to \vec{w}_A(t,\vec{x}_{At}) := [\vec{w}(t,p_t)]_{\vec{A}} \end{array}\right\}$ and $\vec{w}_B$: $\left\{\begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(B) & \to \mathcal{M}_{n1}(B) \\ (t,\vec{x}_{Bt}) & \to \vec{w}_B(t,\vec{x}_{Bt}) := [\vec{w}(t,p_t)]_{\vec{B}} \end{array}\right\}$. So $\vec{w}_A(t,\vec{x}_{At})$ and $\vec{w}_B(t,\vec{x}_{Bt})$ are the column matrices of the components of $\vec{w}(t,p_t)$ in $\mathcal{R}_A$ and $\mathcal{R}_B$.

Definition 11.3 $\vec{w}$ is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all t,

$$\vec{w}_{At}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}) \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}), \tag{11.2}$$

i.e. $\vec{w}_{At} = \vec{w}_{Bt*}$ is the push-forward of $\vec{w}_{Bt}$ by Θ_t cf. (6.20).

Example 11.4 Fundamental counter-example: A Eulerian velocity field is not objective, cf. (10.45), because of the drive velocity $\vec{v}_D \neq \vec{0}$ in general. Neither is a Eulerian acceleration field, cf. (10.51). ▟

Example 11.5 The field of gravitational forces (external forces) is objective covariant. ▟

11.2.4 Covariant objectivity of a differential form

Let α : $\left\{\begin{array}{ll} \mathcal{C} & \to \mathbb{R}^{n*} \\ (t,p_t) & \to \alpha(t,p_t) \end{array}\right\}$ be a Eulerian differential form (e.g. a measuring device used to get the internal power). α is quantified by A and B as the functions α_A : $\left\{\begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(A) & \to \mathcal{M}_{n1}(A) \\ (t,\vec{x}_{At}) & \to \alpha_A(t,\vec{x}_{At}) := [\alpha(t,p_t)]_{\vec{A}} \end{array}\right\}$ and α_B : $\left\{\begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1}(B) & \to \mathcal{M}_{n1}(B) \\ (t,\vec{x}_{Bt}) & \to \alpha_B(t,\vec{x}_{Bt}) := [\alpha(t,p_t)]_{\vec{B}} \end{array}\right\}$. So $\alpha_A(t,\vec{x}_{At})$ and $\alpha_B(t,\vec{x}_{Bt})$ are the row matrices of the components of $\alpha(t,p_t)$ in $\mathcal{R}_A$ and $\mathcal{R}_B$.

Definition 11.6 α is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all t,

$$\alpha_{At}(\vec{x}_{At}) = \alpha_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}). \tag{11.3}$$

i.e. $\alpha_{At} = \alpha_{Bt*}$ is the push-forward of α_{Bt} by Θ_t cf. (7.3).

NB: (11.3) and (11.2) are compatible: If $\vec{w}$ is an objective vector field and if α is an objective differential form, then the scalar function $\alpha.\vec{w}$ is objective:

$$\alpha_{At}(\vec{x}_{At}).\vec{w}_{At}(\vec{x}_{At}) = \alpha_{Bt}(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}) \quad (= (\alpha(t,p_t).\vec{w}(t,p_t)), \tag{11.4}$$

since $\alpha_{At}(\vec{x}_{At}).\vec{w}_{At}(\vec{x}_{At}) = (\alpha_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1}).(d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})) = \alpha_{Bt}(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})$.

11.2.5 Covariant objectivity of tensors

A tensor acts on both vector fields and differential forms, and its objectivity is deduced from the previous §.

So, let T be a (Eulerian) tensor corresponding to a "physical quantity". The observers A and B describe T as being the functions T_A and T_B.

Definition 11.7 T is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all t,

$$T_{At}(\vec{x}_{At}) = T_{Bt*}(\vec{x}_{At}) \tag{11.5}$$

i.e. T_{At} is the push-forward of T_{Bt} by Θ_t.

(Recall: $T_{Bt*}(\vec{x}_{At})(\alpha_1(\vec{x}_{At}),...,\vec{w}_1(\vec{x}_{At})) := T_{Bt}(\vec{x}_{Bt})(\alpha_1{}^*(\vec{x}_{Bt}),...,\vec{w}_1{}^*(\vec{x}_{Bt})).$)

Example 11.8 (Non covariant objectivity of a differential $d\vec{w}$) Let $\vec{w}$ be an objective vector field, seen as $\vec{w}_A$ by A and $\vec{w}_B$ by B; So $\vec{w}_{At}(\vec{x}_{At}) =^{(11.2)} d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})$ when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, thus

$$d\vec{w}_{At}(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}) + (d^2\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})), \tag{11.6}$$

hence

$$\begin{aligned} d\vec{w}_{At}(\vec{x}_{At}) &= d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} + (d^2\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})).d\Theta_t(\vec{x}_{Bt})^{-1} \\ &\neq d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} \quad \text{when} \quad d^2\Theta_t \neq 0. \end{aligned} \tag{11.7}$$

Thus $d\vec{w}$ is not covariant objective in general. However in classical mechanics for "change of Cartesian referentials" Θ_t is affine, so $d^2\Theta_t = 0$, and in particular $d\vec{w}$ is objective when $\vec{w}$ is. And

$$\begin{aligned} &(d^2\vec{w}_{At}(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt})).d\Theta_t(\vec{x}_{Bt}) + d\vec{w}_{At}(\vec{x}_{At}).d^2\Theta_t(\vec{x}_{Bt}) \\ &= d\Theta_t(\vec{x}_{Bt}).d^2\vec{w}_{Bt}(\vec{x}_{Bt}) + 2\,d^2\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}) + d^3\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}). \end{aligned} \tag{11.8}$$

Thus $d^2\vec{w}$ is not covariant objective in general (but if Θ_t is affine then $d^2\vec{w}$ is objective if $\vec{w}$ is). ▪

11.3 Non objectivity of the velocities

11.3.1 Eulerian velocity $\vec{v}$: not covariant (and not isometric) objective

Velocity addition formala: With $\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{w}(\vec{x}_{Bt})$ when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, cf. (10.45),

$$\begin{aligned} \vec{v}_{At}(\vec{x}_{At}) &= \vec{v}_{Bt*}(\vec{x}_{At}) + \vec{v}_{Dt}(\vec{x}_{At}) \\ &\neq \vec{v}_{Bt*}(\vec{x}_{At}) \quad \text{when} \quad \vec{v}_{Dt}(\vec{x}_{At}) \neq \vec{0}, \end{aligned} \tag{11.9}$$

thus a Eulerian velocity field is not covariant objective (and not isometric objective).

11.3.2 $d\vec{v}$ is not objective

The velocity addition formula, $(\vec{v}_{At} - \vec{v}_{Dt})(\vec{x}_{At}) = \vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})$ when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, gives

$$d(\vec{v}_{At} - \vec{v}_{Dt})(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d\Theta_t(\vec{x}_{Bt}).d\vec{v}_{Bt}(\vec{x}_{Bt}) + d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}), \tag{11.10}$$

thus $d\vec{v}$ is neither covariant objective nor isometric objective because of $d\vec{v}_D$:

$$d\vec{v}_{At}(\vec{x}_{At}) = d\vec{v}_{Bt*}(\vec{x}_{At}) + d\vec{v}_{Dt}(\vec{x}_{At}) + d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} \neq d\vec{v}_{Bt*}(\vec{x}_{At}) \quad \text{in general.} \tag{11.11}$$

Remark 11.9 Recall: "Isometric objective" implies
- The use of the same Euclidean metric in $\mathcal{R}_B$ and $\mathcal{R}_A$, i.e. $(\cdot,\cdot)_A = (\cdot,\cdot)_B$,
- $\widetilde{\Phi}_{\mathcal{R}_B}$ (motion of $\mathcal{R}_B$) is a solid body motion, and
- Θ_t is affine (so $d^2\Theta_t = 0$ for all t). ▪

Exercice 11.10 Prove, with Q_t the (orthonormal) transition matrix from $(\vec{A}_i)$ to $(\vec{B}_i)$:

$$[d\vec{v}_t]_{|\vec{B}} = Q_t.[d\vec{v}_t]_{|\vec{A}}.Q_t^{-1} + Q'(t).Q_t^{-1}, \quad \text{written} \quad [L]_{|\vec{B}} = Q.[L]_{|\vec{A}}.Q^T + \dot{Q}.Q^T. \tag{11.12}$$

(Used in classical mechanics courses, to prove that $d\vec{v}$ isn't "isometric objective" because of $\dot{Q}.Q^T$.)

Answer. $t_0, t \in \mathbb{R}$, $p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj})$, $p_t = \widetilde{\Phi}(t, P_{Obj}) = \Phi_t^{t_0}(p_{t_0})$, $\vec{v}(t, p_t) = \frac{\partial\widetilde{\Phi}}{\partial t}(t, P_{Obj})$, and $F_t^{t_0}(p_{t_0}) = d\Phi_t^{t_0}(p_{t_0})$. So $\vec{v}(t, \Phi_t^{t_0}(p_{t_0})) = \frac{\partial\Phi_{p_{t_0}}^{t_0}}{\partial t}(t, p_{t_0})$, thus $d\vec{v}(t,p_t).F_{p_{t_0}}^{t_0}(t) = \frac{\partial F_{p_{t_0}}^{t_0}}{\partial t}(t)$. And (4.26), with $F_{p_{t_0}}^{t_0} =^{\text{noted}} F$, gives $[F(t)]_{|\vec{a}_{t_0},\vec{B}} = Q(t).[F(t)]_{|\vec{a}_{t_0},\vec{A}}$, thus $[F'(t)]_{|\vec{a}_{t_0},\vec{B}} = Q'(t).[F(t)]_{|\vec{a}_{t_0},\vec{A}} + Q(t).[F'(t)]_{|\vec{a}_{t_0},\vec{A}}$. Thus $[d\vec{v}(t,p_t)]_{|\vec{B}} = [F_{p_{t_0}}^{t_0}{}'(t).F_{p_{t_0}}^{t_0}(t)]_{|\vec{B}} = [F_{p_{t_0}}^{t_0}{}'(t)]_{|\vec{B}}.[F_{p_{t_0}}^{t_0}(t)]_{|\vec{B}} = (Q'(t).[F(t)]_{|\vec{a}_{t_0},\vec{A}} + Q(t).[F'(t)]_{|\vec{a}_{t_0},\vec{A}}).[F(t)]^{-1}_{|\vec{a}_{t_0},\vec{A}}.Q(t)^{-1} = Q'(t).Q(t)^{-1} + Q(t).[F'(t)]_{|\vec{a}_{t_0},\vec{A}}.[F(t)]^{-1}_{|\vec{a}_{t_0},\vec{A}}.Q(t)^{-1} = Q'(t).Q(t)^{-1} + Q(t).[d\vec{v}(t,p_t)]_{|\vec{A}}.Q(t)^{-1}$. And cf. (3.33). ▪

Exercice 11.11 Prove that $d^2\vec{v}$ is "isometric objective" when $\widetilde{\Phi}_{\mathcal{R}_B}$ is a rigid body motion.

Answer. (11.8) with $\vec{v}_A - \vec{v}_D$ instead of $\vec{w}_A$, and $\vec{v}_B$ instead of $\vec{w}_B$ give, in an "isometric objective" framework,

$$d^2(\vec{v}_{At} - \vec{v}_{Dt})(\vec{x}_{At}).(\vec{u}_{Bt*}, \vec{w}_{Bt*}) = d\Theta_t(\vec{x}_{Bt}).d^2\vec{v}_{Bt}(\vec{x}_{Bt})(\vec{u}_B, \vec{w}_B). \tag{11.13}$$

Here $d^2\vec{v}_{Dt} = 0$ (rigid body motion), thus $d^2\vec{v}$ is "isometric objective". ▪

11.3.3 $d\vec{v}+d\vec{v}^T$ is "isometric objective"

Proposition 11.12 *If $\widetilde{\Phi}_{\mathcal{R}_B}$ is a rigid body motion then $d\vec{v}_t+d\vec{v}_t^T$ is "isometric objective"*

$$d\vec{v}_{At}+d\vec{v}_{At}^T=(d\vec{v}_{Bt}+d\vec{v}_{Bt}^T)_*. \tag{11.14}$$

(Isometric framework: The rate of deformation tensor is independent of an added added rigid motion.)

Proof. $Q.Q^T=I$ gives $\dot{Q}.Q^T+(\dot{Q}.Q^T)^T=0$, then apply (11.12). ▪

Exercice 11.13 Prove that $\Omega=\frac{d\vec{v}-d\vec{v}^T}{2}$ is not isometric objective.

Answer. (11.11) gives $d\vec{v}_{At}^T=d\vec{v}_{Bt*}^T+d\vec{v}_{Dt}^T$, thus $\frac{d\vec{v}_{At}-d\vec{v}_{At}^T}{2}=\frac{d\vec{v}_{Bt*}-d\vec{v}_{Bt*}^T}{2}+\frac{d\vec{v}_{Dt}-d\vec{v}_{Dt}^T}{2}\neq\frac{d\vec{v}_{Bt*}-d\vec{v}_{Bt*}^T}{2}$, even if $\widetilde{\Phi}_{\mathcal{R}_B}$ is a solid body motion (then $\frac{d\vec{v}_{Dt}-d\vec{v}_{Dt}^T}{2}=\vec{\omega}\wedge$ is a rotation time a dilation). ▪

11.3.4 Lagrangian velocities

The Lagrangian velocities do not define a vector field, cf. § 3.2.2. Thus asking about the objectivity of Lagrangian velocities is meaningless.

11.4 The Lie derivatives are covariant objective

Framework of § 10. In particular we have the velocity-addition formula $\vec{v}_{At}=\vec{v}_{Bt*}+\vec{v}_{Dt}$ in $\mathcal{R}_A$ where $\vec{v}_{Bt*}(\vec{x}_{At})=d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})$ and $\vec{x}_{Bt}=\Theta_t(\vec{x}_{At})$, cf. (10.45).

The objectivity under concern is the covariant objectivity (no inner dot product or basis required). The Lie derivatives are also called "objective rates" because they are covariant objectives. Easy proofs.

11.4.1 Scalar functions

Proposition 11.14 *If f be a covariant objective function, cf. (11.1), then its Lie derivative $\mathcal{L}_{\vec{v}}f$ is covariant objective:*

$$\mathcal{L}_{\vec{v}_A}f_A=\Theta_*(\mathcal{L}_{\vec{v}_B}f_B),\quad \text{i.e.}\quad \mathcal{L}_{\vec{v}_A}f_A(t,\vec{x}_{At})=\mathcal{L}_{\vec{v}_B}f_B(t,\vec{x}_{Bt})\quad \text{when}\quad \vec{x}_{At}=\Theta_t(\vec{x}_{Bt}), \tag{11.15}$$

i.e., $\frac{Df_A}{Dt}(t,\vec{x}_{At})=\frac{Df_B}{Dt}(t,\vec{x}_{Bt})$, i.e. $(\frac{\partial f_A}{\partial t}+df_A.\vec{v}_A)(t,\vec{x}_{At})=(\frac{\partial f_B}{\partial t}+df_B.\vec{v}_B)(t,\vec{x}_{Bt})$.

Proof. Consider the motion $t\to p(t)=\widetilde{\Phi}(tP_{Obj})$ of a particle P_{Obj}, and $\vec{x}_A(t)=[\overrightarrow{O_Ap(t)}]_{|\vec{A}}$ and $\vec{x}_B(t)=[\overrightarrow{O_Bp(t)}]_{|\vec{B}}$. With f objective, (11.1) gives $f_B(t,\vec{x}_B(t))=f_A(t,\Theta(t,\vec{x}_B(t)))$ $(=f_A(t,\vec{x}_A(t)))$, thus

$$\begin{aligned}\frac{Df_B}{Dt}(t,\vec{x}_B(t))&=\frac{\partial f_A}{\partial t}(t,\vec{x}_A(t))+df_{At}(\vec{x}_A(t)).(\underbrace{\frac{\partial\Theta}{\partial t}(t,\vec{x}_B(t))}_{\vec{v}_{Dt}(\vec{x}_{At})}+\underbrace{d\Theta_t(\vec{x}_B(t)).\vec{v}_{Bt}(\vec{x}_B(t)))}_{\vec{v}_{Bt*}(\vec{x}_{At})}\\&=\frac{\partial f_A}{\partial t}(t,\vec{x}_{At})+df_{At}(\vec{x}_{At}).\vec{v}_{At}(\vec{x}_{At})=\frac{Df_A}{Dt}(t,\vec{x}_{At}),\end{aligned} \tag{11.16}$$

thanks to velocity addiction formula $\vec{v}_{At}=\vec{v}_{Bt*}+\vec{v}_{Dt}$. ▪

11.4.2 Vector fields

Proposition 11.15 *Let $\vec{w}$ be a covariant objective vector field, cf. (11.2). Then its Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ is covariant objective:*

$$\mathcal{L}_{\vec{v}_A}\vec{w}_A=\Theta_*(\mathcal{L}_{\vec{v}_B}\vec{w}_B), \tag{11.17}$$

i.e., when $\vec{x}_{At}=\Theta_t(\vec{x}_{Bt})$,

$$\mathcal{L}_{\vec{v}_A}\vec{w}_A(t,\vec{x}_{At})=d\Theta_t(\vec{x}_{Bt}).\mathcal{L}_{\vec{v}_B}\vec{w}_B(t,\vec{x}_{Bt}), \tag{11.18}$$

i.e.,

$$(\frac{D\vec{w}_A}{Dt}-d\vec{v}_A.\vec{w}_A)(t,\vec{x}_{At})=d\Theta(t,\vec{x}_{Bt}).(\frac{D\vec{w}_B}{Dt}-d\vec{v}_B.\vec{w}_B)(t,\vec{x}_{Bt}), \tag{11.19}$$

i.e.,

$$(\frac{\partial\vec{w}_A}{\partial t}+d\vec{w}_A.\vec{v}_A-d\vec{v}_A.\vec{w}_A)(t,\vec{x}_{At})=d\Theta(t,\vec{x}_{Bt}).(\frac{\partial\vec{w}_B}{\partial t}+d\vec{w}_B.\vec{v}_B-d\vec{v}_B.\vec{w}_B)(t,\vec{x}_{Bt}). \tag{11.20}$$

But the partial, convected, material, and Lie autonomous derivatives are not covariant objective (not *even isometric objective because of the drive velocity $\vec{v}_D$): We have*

$$(d\vec{w}_{At}.(\vec{v}_{At}-\vec{v}_{Dt}))(\vec{x}_{At}) = (d\Theta_t.(d\vec{w}_{Bt}.\vec{v}_{Bt}) + (d^2\Theta_t.\vec{w}_{Bt}).\vec{v}_{Bt})(\vec{x}_{Bt}), \tag{11.21}$$

$$(d(\vec{v}_{At}-\vec{v}_{Dt}).\vec{w}_{At})(\vec{x}_{At}) = (d\Theta_t.(d\vec{v}_{Bt}.\vec{w}_{Bt}) + (d^2\Theta_t.\vec{v}_{Bt}).\vec{w}_{Bt})(\vec{x}_{Bt}), \tag{11.22}$$

$$(d(\vec{v}_{At}-\vec{v}_{Dt}).(\vec{v}_{At}-\vec{v}_{Dt}))(\vec{x}_{At}) = (d\Theta_t.(d\vec{v}_{Bt}.\vec{v}_{Bt}) + d^2\Theta_t(\vec{v}_{Bt},\vec{v}_{Bt}))(\vec{x}_{Bt}), \tag{11.23}$$

$$\mathcal{L}^0_{(\vec{v}_{At}-\vec{v}_{Dt})}\vec{w}_{At}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\mathcal{L}^0_{\vec{v}_{Bt}}\vec{w}_{Bt}(\vec{x}_{Bt}), \tag{11.24}$$

$$\frac{\partial \vec{w}_A}{\partial t}(t,\vec{x}_{At}) + \mathcal{L}^0_{\vec{v}_D}\vec{w}_{At}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\frac{\partial \vec{w}_B}{\partial t}(t,\vec{x}_{Bt}), \tag{11.25}$$

$$\frac{D\vec{w}_A}{Dt}(t,\vec{x}_{At}) - d\vec{v}_{Dt}.\vec{w}_{At}(\vec{x}_{At}) = d\Theta_t.(\vec{x}_{Bt}).\frac{D\vec{w}_B}{Dt}(t,\vec{x}_{Bt}) + d^2\Theta_t(\vec{v}_{Bt},\vec{w}_{Bt})(\vec{x}_{Bt}), \tag{11.26}$$

$$\frac{\partial(\vec{v}_A-\vec{v}_D)}{\partial t}(t,\vec{x}_{At}) + \mathcal{L}^0_{\vec{v}_D}(\vec{v}_A-\vec{v}_D)(t,\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\frac{\partial \vec{v}_B}{\partial t}(t,\vec{x}_{Bt}). \tag{11.27}$$

Proof. • $\vec{w}_{At}(\Theta_t(\vec{x}_{Bt})) = d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})$ gives

$$d\vec{w}_{At}(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d^2\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}) + d\Theta_t(\vec{x}_{Bt}).d\vec{w}_B(\vec{x}_{Bt}), \tag{11.28}$$

thus, with $d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) = (\vec{v}_{At}-\vec{v}_{Dt})(\vec{x}_{At}) = \vec{v}_{Bt*}(\vec{x}_{At})$ (velocity-addition formula),

$$d\vec{w}_{At}(\vec{x}_{At}).(\vec{v}_{At}-\vec{v}_{Dt})(\vec{x}_{At}) = (d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})).\vec{w}_{Bt}(\vec{x}_{Bt}) + d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}),$$

hence (11.21). In particular $d\vec{w}_{At}(\vec{x}_{At}).\vec{v}_{At}(\vec{x}_{At}) \neq d\Theta_t(\vec{x}_{Bt}).(d\vec{w}_{Bt}(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}))$ (the vector field $d\vec{w}.\vec{v}$ is not objective).

• $(\vec{v}_{At}-\vec{v}_{Dt})(\Theta_t(\vec{x}_{Bt})) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})$ gives

$$d(\vec{v}_{At}-\vec{v}_{Dt})(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) + d\Theta_t(\vec{x}_{Bt}).d\vec{v}_{Bt}(\vec{x}_{Bt}),$$

so, applied to $\vec{w}_{Bt}$ (resp. $\vec{v}_{Bt}$), we get (11.22) (resp. (11.23)). Hence (11.24).

• If $\vec{x}_{At} = \Theta_t(\vec{x}_B)$, then $\vec{w}_A(t,\Theta(t,\vec{x}_B)) = d\Theta(t,\vec{x}_B).\vec{w}_B(t,\vec{x}_B)$, so, with $\frac{\partial\Theta}{\partial t}(t,\vec{x}_B) = \vec{v}_{\Theta t}(\vec{x}_{At})$, we get

$$\begin{aligned}\frac{\partial \vec{w}_A}{\partial t}(t,\vec{x}_{At}) + d\vec{w}_{At}(\vec{x}_{At}).\vec{v}_{\Theta t}(\vec{x}_{At}) &= d\frac{\partial\Theta}{\partial t}(t,\vec{x}_B).\vec{w}_{Bt}(\vec{x}_B) + d\Theta_t(\vec{x}_B).\frac{\partial \vec{w}_B}{\partial t}(t,\vec{x}_B)\\ &= (d\vec{v}_{\Theta t}(\vec{x}_{At}).d\Theta_t(\vec{x}_B)).\vec{w}_{Bt}(\vec{x}_B) + d\Theta_t(\vec{x}_B).\frac{\partial \vec{w}_B}{\partial t}(t,\vec{x}_B),\end{aligned}$$

Thus (11.25) since $\vec{v}_\Theta = \vec{v}_D$; Then (11.21) gives (11.26).

• $\vec{v}_{B*}(t,\Theta(t,\vec{x}_B)) = d\Theta(t,\vec{x}_B).\vec{v}_B(t,\vec{x}_B)$ gives

$$\frac{\partial \vec{v}_{B*}}{\partial t}(t,\vec{x}_{At}) + d\vec{v}_{B*}(\vec{x}_{At}).\vec{v}_\Theta(t,\vec{x}_{At}) = \underbrace{\frac{\partial d\Theta}{\partial t}(t,\vec{x}_B)}_{d\vec{v}_{\Theta t}(\vec{x}_{At}).d\Theta_t(\vec{x}_B)}.\vec{v}_{Bt}(\vec{x}_B) + d\Theta(t,\vec{x}_B).\frac{\partial \vec{v}_B}{\partial t}(t,\vec{x}_B,)$$

since $\frac{\partial d\Theta}{\partial t}(t,\vec{x}_B) = d(\frac{\partial\Theta}{\partial t})(t,\vec{x}_B)$ and $\frac{\partial\Theta}{\partial t}(t,\vec{x}_B) = \vec{v}_\Theta(t,\vec{x}_{At}) = \vec{v}_{\Theta_t}(\Theta_t(\vec{x}_B))$; hence (11.27). ∎

11.4.3 Tensors

Proposition 11.16 *It T is a covariant objective tensor, then its Lie derivatives are covariant objectives:*

$$\mathcal{L}_{\vec{v}_A}T_A = \Theta_*(\mathcal{L}_{\vec{v}_B}T_B). \tag{11.29}$$

Proof. Corollary of (11.15) and (11.18) to get $\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) = (\mathcal{L}_{\vec{v}}\alpha).\vec{w} + \alpha.(\mathcal{L}_{\vec{v}}\vec{w})$; Then use $\mathcal{L}_{\vec{v}}(t_1\otimes t_2) = (\mathcal{L}_{\vec{v}}t_1)\otimes t_2 + t_1\otimes(\mathcal{L}_{\vec{v}}t_2)$. ∎

11.5 Taylor expansions and ubiquity gift

11.5.1 First order Taylor expansion and ubiquity issue

Let $\vec{w} : \mathbb{R}\times\mathbb{R}^n \to \vec{\mathbb{R}^n}$ be regular and $p(t) = \Phi^{t_0}(t, p_{t_0})$. With $\vec{f}(t) = \vec{w}(t, p(t))$, $\vec{f}(t) = \vec{f}(t_0)+(t-t_0)\,\vec{f}'(t_0)+o(t-t_0)$ (first order Taylor expansion), we get

$$\vec{w}(t, p(t)) = \vec{w}(t_0, p_{t_0}) + h\,\frac{D\vec{w}}{Dt}(t_0, p_{t_0}) + o(t-t_0). \tag{11.30}$$

Issue: The left hand side $\vec{w}(t, p(t))$ lives in $T_{p_t}(\Omega_t)$ while the right hand side (calculation) $\vec{w}(t_0, p_{t_0}) + h\,\frac{D\vec{w}}{Dt}(t_0, p_{t_0})$ lives in $T_{p_{t_0}}(\Omega_{t_0})$. Thus (11.30) is meaningless: To be meaningful, the $\vec{w}(t, p(t))$ term should first be pull-backed by $\Phi_t^{t_0}(p_{t_0})$ to be compared with $\vec{w}(t_0, p_{t_0})$ (or the $\vec{w}(t_0, p_{t_0})$ term should first be push-forwarded by $\Phi_t^{t_0}(p_{t_0})$ to be compared with $\vec{w}(t, p_t)$). E.g., in a non-planar manifold (e.g. in a surface in $\mathbb{R}^3$), $\vec{w}(t, p_t)$ and $\vec{w}(t_0, p_{t_0})$ don't belong to the same vector space (the "tangent spaces" $T_{p_t}(\Omega_t)$ and $T_{p_{t_0}}(\Omega_{t_0})$ are different in general).

Ok with Lie: The Lie derivative uses the pull-back:

$$\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) \overset{(9.13)}{=} \frac{d\Phi_t^{t_0}(p_{t_0})^{-1}.\vec{w}(t, p(t)) - \vec{w}(t_0, p_{t_0})}{t - t_0} + o(1) \tag{11.31}$$

is an equation in $T_{p_{t_0}}(\Omega_{t_0})$. We have obtained the first order Taylor expansion in $T_{p_{t_0}}(\Omega_{t_0})$: With $h = t-t_0$:

$$(\Phi_t^{t_0 *}\vec{w}(t_0, p_{t_0}) =) \quad d\Phi_t^{t_0}(p_{t_0})^{-1}.\vec{w}(t, p(t)) = \vec{w}(t_0, p_{t_0}) + h\,\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) + o(h). \tag{11.32}$$

Or with push-forwards, we have obtained the first order Taylor expansion in $T_{p_t}(\Omega_t)$:

$$\begin{aligned}\vec{w}(t, p(t)) &= d\Phi_t^{t_0}(p_{t_0}).(\vec{w}(t_0, p_{t_0}) + h\,\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) + o(h))\\ &= d\Phi_t^{t_0}(p_{t_0}).\vec{w}(t_0, p_{t_0}) + h\,d\Phi_t^{t_0}(p_{t_0}).\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) + o(h)\\ &= (\Phi_{t*}^{t_0}\vec{w})(t, p(t)) + h\,\Phi_{t*}^{t_0}(\mathcal{L}_{\vec{v}}\vec{w})(t, p(t)) + o(h).\end{aligned} \tag{11.33}$$

Proposition 11.17 *In $\mathbb{R}^n$, with the gift of ubiquity, (11.33) gives (11.30) (of course).*

Interpretation: Because ubiquity gifts don't exist, (11.30) is meaningless while (11.33) is meaningful; Which tells that "The Lie derivative is the meaningful derivative in physical sciences".

Proof. With $d\Phi^{t_0}(t_0{+}h, p_{t_0}) \overset{(4.37)}{=} I + h\,d\vec{v}(t_0, p_{t_0}) + o(h)$ and $\mathcal{L}_{\vec{v}}\vec{w} \overset{(9.18)}{=} \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}$, (11.33) gives

$$\begin{aligned}\vec{w}(t, p(t)) &= \underbrace{d\Phi_t^{t_0}(p_{t_0})}_{(I + h\,d\vec{v}(t_0, p_{t_0}) + o(h))} \,.\, \underbrace{(\vec{w}(t_0, p_{t_0}) + h\,\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}))}_{(\vec{w} + h\,(\frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}))(t_0, p_{t_0}) + o(h)} + o(h)\\ &= (\vec{w} + h\,(\frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}) + h\,d\vec{v}.\vec{w})(t, p(t)) + o(h),\end{aligned}$$

which is (11.30). ∎

11.5.2 Second order Taylor expansion

In $\mathbb{R}^n$, with $\vec{w} \in C^2$ let $\vec{f} : \left\{\begin{array}{rl}]t_0-\varepsilon, t_0+\varepsilon[& \to \vec{\mathbb{R}^n} \\ t & \to \vec{f}(t) := \vec{w}(t, p(t)) \end{array}\right\}$. Thus $\vec{f}$ is C^2, and $\vec{f}(t) = \vec{f}(t_0) + h\,\vec{f}'(t_0) + \frac{h^2}{2}\vec{f}''(t_0) + o(h^2)$ where $h = t-t_0$ (second order Taylor expansion). Thus, near (t_0, p_{t_0}),

$$\vec{w}(t, p(t)) = (\vec{w} + h\,\frac{D\vec{w}}{Dt} + \frac{h^2}{2}\,\frac{D^2\vec{w}}{Dt^2})(t_0, p(t_0)) + o(h^2). \tag{11.34}$$

Once again there is a ubiquity issue. Without ubiquity gifts, we have "the second order Taylor expansion:

$$\Phi_t^{t_0 *}\vec{w}(t, p(t)) = \big(\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w})\big)(t_0, p_{t_0}) + o(h^2), \tag{11.35}$$

i.e. $d\Phi_t^{t_0}(p_{t_0})^{-1}.\vec{w}(t, p(t)) = \big(\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w})\big)(t_0, p_{t_0}) + o(h^2)$ (pull-back),
i.e. $\vec{w}(t, p(t)) = \Phi_{t*}^{t_0}(\vec{w} + h\,\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w}))(t, p(t)) + o(h^2)$ (push-forward). Indeed:

Proposition 11.18 *In $\mathbb{R}^n$, with the gift of ubiquity, (11.35) gives (11.34).*

Proof. (4.36) gives $F^{t_0}_{p_{t_0}}(t) = I_{t_0} + h\,d\vec{v}(t_0,p_{t_0}) + \frac{h^2}{2}\,d\vec{\gamma}(t_0,p_{t_0}) + o(h^2)$. Thus, omitting the reference to (t_0,p_{t_0}) to lighten the writing, (11.35) gives

$$\begin{aligned} &d\Phi^{t_0}_t(p_{t_0}).(\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{v}}\vec{w} + o(h^2)) \\ &= \Big(I + h\,d\vec{v} + \frac{h^2}{2}\,d(\frac{D\vec{v}}{Dt}) + o(h^2)\Big).\Big(\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{v}}\vec{w} + o(h^2)\Big) \end{aligned} \tag{11.36}$$

The h^0 term is $I.\vec{w} = \vec{w}$. The h term is $\mathcal{L}_{\vec{v}}\vec{w} + d\vec{v}.\vec{w} = \frac{D\vec{w}}{Dt}$. The h^2 term is the sum of

- $\frac{1}{2}\mathcal{L}_{\vec{v}}\mathcal{L}_{\vec{v}}\vec{w} = \frac{1}{2}(\frac{D^2\vec{w}}{Dt^2} - 2\,d\vec{v}.\frac{D\vec{w}}{Dt} - \frac{D(d\vec{v})}{Dt}.\vec{w} + d\vec{v}.d\vec{v}.\vec{w})$, cf.(9.42),
- $d\vec{v}.\mathcal{L}_{\vec{v}}\vec{w} = d\vec{v}.\frac{D\vec{w}}{Dt} - d\vec{v}.d\vec{v}.\vec{w} = \frac{1}{2}(2d\vec{v}.\frac{D\vec{w}}{Dt} - 2d\vec{v}.d\vec{v}.\vec{w})$,
- $\frac{1}{2}d(\frac{D\vec{v}}{Dt}).\vec{w} = \frac{1}{2}(\frac{D(d\vec{v})}{Dt}.\vec{w} + d\vec{v}.d\vec{v}.\vec{w})$, cf.(2.26),

which indeed gives $\frac{D^2\vec{w}}{Dt^2}$. ∎

11.5.3 Higher order Taylor expansion

Exercice 11.19 Let $\vec{w} \in C^n$ and $\mathcal{L}^{(n)}_{\vec{v}} = \mathcal{L}_{\vec{v}} \circ ... \circ \mathcal{L}_{\vec{v}}$ (n-times). For all $n \in \mathbb{N}^*$, prove (Taylor expansion)

$$\vec{w}(t,p(t)) = d\Phi^{t_0}_t(p_{t_0}).(\vec{w} + (t{-}t_0)\mathcal{L}_{\vec{v}}\vec{w} + ... + \frac{(t{-}t_0)^n}{n!}\mathcal{L}^{(n)}_{\vec{v}}\vec{w})(t_0,p_{t_0}) + o((t{-}t_0)^n), \tag{11.37}$$

i.e. $F^{t_0}_t(p_{t_0})^{-1}.\vec{w}(t,p(t)) = \big(\sum_{k=0}^n \frac{(t-t_0)^k}{k!}(\mathcal{L}_{\vec{v}})^{(k)}\vec{w}\big)(t_0,p_{t_0}) + o((t{-}t_0)^n)$ in $T_{p_{t_0}}(\Omega_{t_0})$.

Answer. (Proof similar to one of the classical proof of Taylor's theorem.) t_0 and p_{t_0} are fixed, $p(t) = \Phi^{t_0}(t,p_{t_0})$, and $H^{t_0}(t,p(t)) := H^{t_0}_t(p(t)) := F^{t_0}_t(p_{t_0})^{-1}$. With

$$\vec{f}_{\vec{w},n}(t) = (H^{t_0}.\vec{w})(t,p(t)) - (\vec{w} + (t{-}t_0)\mathcal{L}_{\vec{v}}\vec{w} + ... + \frac{(t{-}t_0)^n}{n!}\mathcal{L}^{(n)}_{\vec{v}}\vec{w})(t_0,p_{t_0}), \tag{11.38}$$

we have to prove: $f_{\vec{w},n}(t) = o((t{-}t_0)^n)$ (which means $\forall\varepsilon>0$, $\exists h>0$, $\forall t \in [t_0{-}h, t_0{+}h]$, $||\vec{f}_{\vec{w},n}(t)||_g \le \varepsilon$).

Recurrence hypothesis: With $n \in \mathbb{N}^*$, for all $\vec{w} \in C^n$, $||\vec{f}_{\vec{w},n}(t)||_g = o((t{-}t_0)^n)$.

This is true for n=1, cf. (11.32). Suppose it is true for n.

Let $\vec{w} \in C^{n+1}$. With $\frac{DH^{t_0}}{Dt} = -H^{t_0}.d\vec{v}$, cf. (4.47), we get

$$\begin{aligned} \vec{f}_{\vec{w},n+1}{}'(t) &= (-H^{t_0}.d\vec{v}.\vec{w} + H^{t_0}.\frac{D\vec{w}}{Dt})(t,p(t)) - (0 + \mathcal{L}_{\vec{v}}\vec{w} + ... + \frac{(t{-}t_0)^n}{n!}\mathcal{L}^{(n+1)}_{\vec{v}}\vec{w})\Big)(t_0,p_{t_0}) \\ &= (H^{t_0}.\mathcal{L}_{\vec{v}}\vec{w})(t,p(t)) - (\mathcal{L}_{\vec{v}}\vec{w} + ... + \frac{(t{-}t_0)^n}{n!}\mathcal{L}^n_{\vec{v}}.\mathcal{L}_{\vec{v}}\vec{w})\Big)(t_0,p_{t_0}) = \vec{f}_{\mathcal{L}_{\vec{v}}\vec{w},n}(t). \end{aligned} \tag{11.39}$$

And the mean value theorem tells $\frac{||\vec{f}_{\vec{w},n+1}(t)-\vec{f}_{\vec{w},n+1}(t_0)||_g}{|t-t_0|} \le \sup_{\tau\in[t_0-h,t_0+h]} ||\vec{f}_{\vec{w},n+1}{}'(\tau)||_g$; And $\vec{f}_{\vec{w},n+1}(t_0) =^{(11.35)} \vec{0}$, thus $\frac{||\vec{f}_{\vec{w},n+1}(t)||_g}{|t-t_0|} \le \sup_{\tau\in[t_0-h,t_0+h]} ||\vec{f}_{\mathcal{L}_{\vec{v}}\vec{w},n}(\tau)||_g$. And, $\mathcal{L}_{\vec{v}}\vec{w} \in C^n$, hence the recurrence hypothesis tells: $||\vec{f}_{\mathcal{L}_{\vec{v}}\vec{w},n}(t)||_g = o((t{-}t_0)^n)$. Thus $\frac{||\vec{f}_{\vec{w},n+1}(t)||_g}{|t-t_0|} = o((t{-}t_0)^n$, thus $||\vec{f}_{\vec{w},n+1}(t)||_g = o((t{-}t_0)^{n+1}$. ∎

12 The virtual power principle

(See e.g. Germain [10]).

12.1 Newton fundamental laws

Consider $N \ge 1$ distinct particles P_{Obj_i}, $i = 1,...,N$. The set $\{P_{Obj_1},...,P_{Obj_N}\}$ is called a body. The particle P_{Obj_i} is at t at $p_i \in \mathbb{R}^n$ and its mass is m_i, and $p_j \ne p_j$ for all $i \ne j$. At t, each particle P_{Obj_i} is subject to an acceleration $\vec{\gamma}_t(p_i) =^{\text{noted}} \vec{\gamma}_i$, an internal forces $\vec{f}_{t,p_j}(p_i) =^{\text{noted}} \vec{f}_{ji}$ due to the other P_{Obj_j} and $\vec{f}_{ii} = \vec{0}$ for all i, and an external force $\vec{f}_t(p_i) =^{\text{noted}} \vec{f}_i$ (external to the body).

Newton postulates: There exists a Galilean referential $\mathcal{R}_a$ (called absolute) s.t. at any t:

- 1st law (Galileo law of inertia): "a body not acted upon remains at constant speed". (12.1)
- 2nd law: $\forall i = 1, ..., N: \ m_i \vec{\gamma}_i = \vec{f}_i + \sum_{j=1}^{N} \vec{f}_{ji} \ \ (= \vec{f}_i + \sum_{j \neq i} \vec{f}_{ji})$. (12.2)
- 3rd law (law of action and reaction): $\forall i, j = 1, ..., N: \ \vec{f}_{ji} = -\vec{f}_{ij}$ and $\overrightarrow{p_ip_j} \parallel \vec{f}_{ij}$. (12.3)

If $N = 1$ (one particle), then (12.2) reads $m\vec{\gamma} = \vec{f}$ and (12.3) is trivial.
And the 2nd and 3rd laws apply to any subset of $\{p_1, ..., p_N\}$ (on any sub-body).

12.2 D'Alembert formulation

12.2.1 The formulation, discrete framework

With the above discrete vectors fields $\vec{\gamma}_t, \vec{f}_t, \vec{f}_{t,p_j} : \{p_1, ..., p_N\} \to \vec{\mathbb{R}^3}$, consider a discrete vector field $\vec{u}_t : p \in \{p_1, ..., p_N\} \to \vec{u}_t(p) \in \vec{\mathbb{R}^3}$ (virtual velocity field at t), and let $\vec{u}_t(p_i) =^{\text{noted}} \vec{u}_i$. Choose a Euclidean dot product $(\cdot, \cdot)_g =^{\text{noted}} . \bullet .$ in $\vec{\mathbb{R}^3}$. The scalars

$$\mathcal{P}_a(\vec{u}) = \sum_{i=1}^{N} m_i \vec{\gamma}_i \bullet \vec{u}_i, \quad \mathcal{P}_e(\vec{u}) = \sum_{i=1}^{N} \vec{f}_i \bullet \vec{u}_i, \quad \mathcal{P}_{int}(\vec{u}) = \sum_{i=1}^{N} (\sum_{j=1}^{N} \vec{f}_{ji}) \bullet \vec{u}_i, \tag{12.4}$$

are the acceleration virtual power, the external virtual power, the internal virtual power. Since $\vec{f}_{ii} = \vec{0}$ for all i, we also have $\mathcal{P}_{int}(\vec{u}) = \sum_i (\sum_{j \neq i} \vec{f}_{ji}) \bullet \vec{u}_i$.

If there is just one particle then $\mathcal{P}_a(\vec{u}) = m\vec{\gamma} \bullet \vec{u}$, $\mathcal{P}_e(\vec{u}) = \vec{f} \bullet \vec{u}$, and $\mathcal{P}_{int}(\vec{u}) = 0$.

D'Alembert virtual power formulation[3] (variational formulation of 2nd and 3rd Newton's laws). There exists a Galilean referential $\mathcal{R}_a$ s.t., together with Galileo's law of inertia, at any t,

$$\forall \vec{u} \in \mathcal{F}(\{p_1, ..., p_N\}; \vec{\mathbb{R}^3}), \quad \mathcal{P}_a(\vec{u}) = \mathcal{P}_e(\vec{u}) + \mathcal{P}_{int}(\vec{u}). \tag{12.5}$$

Interpretation: To measure a force on a Obj you need to move it (Germain: "to know the weight of a suitcase you have to move it: Looking at it is not enough"), i.e. you need D'Alembert's formulation.

Proposition 12.1 *1- (12.2) is equivalent to (12.5).*
2- (12.3) is equivalent to: $\mathcal{P}_{int}(\vec{u}) = 0$ for all discrete rigid body velocity field $\vec{u} \in \mathcal{F}(\{p_1, ..., p_N\}; \vec{\mathbb{R}^3})$.

Proof. 1- (12.2) $\Leftrightarrow \Big(m_i\vec{\gamma}_i - \vec{f}_i - \sum_{j \neq i} \vec{f}_{ji} = \vec{0}$ for all $i \Big) \Leftrightarrow \Big((m_i\vec{\gamma}_i - \vec{f}_i - \sum_{j \neq i} \vec{f}_{ji}) \bullet \vec{u}_i = 0$ for all $\vec{u}_i \Big) \Leftrightarrow \Big(\sum_i (m_i\vec{\gamma}_i - \vec{f}_i - \sum_{j \neq i} \vec{f}_{ji}) \bullet \vec{u}_i = 0$ for all $(\vec{u}_i)_{i=1,...,N} \Big) \Leftrightarrow \Big(\mathcal{P}_a(\vec{u}) - \mathcal{P}_e(\vec{u}) - \mathcal{P}_{int}(\vec{u}) = 0$ for all $\vec{u} \in (\vec{\mathbb{R}^3})^N \Big)$.

2- Consider the two particles at p_1 and p_2 (others are outside the body $\{p_1, p_2\}$). A rigid body motion of $\{p_1, p_2\}$ is characterized by $\vec{u}_2 = \vec{u}_1 + \vec{\omega} \times \overrightarrow{p_1p_2}$ (after the choice of a Euclidean basis needed to define the vector product $\times$). With $\vec{f}_{ii} = \vec{0}$, the internal virtual power is $\mathcal{P}_{int}(\vec{u}) = \vec{f}_{21} \bullet \vec{u}_1 + \vec{f}_{12} \bullet \vec{u}_2 = (\vec{f}_{21} + \vec{f}_{12}) \bullet \vec{u}_1 + \vec{f}_{12} \bullet (\vec{\omega} \times \overrightarrow{p_1p_2}) = (\vec{f}_{21} + \vec{f}_{12}) \bullet \vec{u}_1 + \vec{\omega} \bullet (\overrightarrow{p_1p_2} \times \vec{f}_{12})$.

21- Suppose (12.3), i.e. $\vec{f}_{21} + \vec{f}_{12} = \vec{0}$ and $\overrightarrow{p_1p_2} \times \vec{f}_{12}$: A rigid body motion of $\{p_1, p_2\}$ gives $\mathcal{P}_{int}(\vec{u}) = 0+0$.

22- Suppose $\mathcal{P}_{int}(\vec{u}) = 0$ for all rigid body motion of $\{p_1, p_2\}$: So $(\vec{f}_{21} + \vec{f}_{12}) \bullet \vec{u}_1 + \vec{f}_{12} \bullet (\vec{\omega} \times \overrightarrow{p_1p_2}) = 0$ for all $\vec{u}_1, \vec{\omega}$. In particular $\vec{\omega} = \vec{0}$ (translation) gives $(\vec{f}_{21} + \vec{f}_{12}) \bullet \vec{u}_1 = 0$ for all $\vec{u}_1$, thus $\vec{f}_{21} + \vec{f}_{12} = \vec{0}$. We are left with $\vec{f}_{12} \bullet (\vec{\omega} \times \overrightarrow{p_1p_2}) = \vec{0} = \vec{\omega} \bullet (\overrightarrow{p_1p_2} \times \vec{f}_{12})$ for all $\vec{\omega}$, thus $\overrightarrow{p_1p_2} \times \vec{f}_{12} = \vec{0}$.

23- Idem for any two particles at p_i and p_j (instead of p_1 and p_2), for all i, j. And a rigid body motion of $\{p_1, ..., p_n\}$ implies a rigid body motion of any $\{p_i, p_j\}$. ∎

[3] Or D'Alembert, Lagrange, Euler, ... virtual power formulation

12.2.2 $L^2(\Omega)$ framework

At t, let Ω be a regular domain in $\mathbb{R}^3$ (a simply connected bounded open set in $\mathbb{R}^n$ with a C^∞ border). Consider the space of finite energy scalar valued functions with its usual inner dot product and norm:

$$\begin{aligned} &L^2(\Omega) = \{u : \Omega \to \mathbb{R} \text{ s.t. } \int_{p\in\Omega} u(p)^2\,d\Omega < \infty\},\\ &(u,w)_{L^2} := \int_{p\in\Omega} u(p)w(p)\,d\Omega \quad\text{and}\quad ||u||_{L^2}^2 = (u,u)_{L^2} = \int_{p\in\Omega} u(p)^2\,d\Omega. \end{aligned} \tag{12.6}$$

Let $T_0^1(\Omega)$ be the space of regular vector fields $\vec u : p \in \Omega \to \vec u(p) \in \vec{\mathbb{R}^3}$ (simplified notations). Choose a Euclidean dot product $.\bullet.$ in $\vec{\mathbb{R}^3}$ with its associated norm $||.||$, and consider the space of finite energy vector fields with its usual inner dot product and norm:

$$\begin{aligned} &L^2(\Omega)^3 = \{\vec u \in T_0^1(\Omega) \text{ s.t. } \int_{p\in\Omega} ||\vec u(p)||^2\,d\Omega < \infty\},\\ &(\vec u,\vec w)_{L^2} := \int_{p\in\Omega} \vec u(p)\bullet\vec w(p)\,d\Omega \overset{\text{noted}}{=} \int_\Omega \vec u\bullet\vec w\,d\Omega \quad\text{and}\quad ||\vec u||_{L^2}^2 = (\vec u,\vec u)_{L^2}. \end{aligned} \tag{12.7}$$

12.2.3 D'Alembert formulation, continuous framework

In (12.4), replace the sum sign $\sum$ by the sum sign $\int$, i.e., with ρ the mass density, define the acceleration, external and internal virtual powers by, for all $\vec u \in T_0^1(\Omega)$,

$$\mathcal{P}_a(\vec u) := \int_\Omega \rho\vec\gamma\bullet\vec u\,d\Omega,\quad \mathcal{P}_e(\vec u) := \int_\Omega \vec f\bullet\vec u\,d\Omega,\quad \mathcal{P}_{int}(\vec u) := \int_\Omega p_{int}(\vec u)\,d\Omega, \tag{12.8}$$

where $p_{int} : \vec u \in T_0^1(\Omega) \to p_{int}(\vec u) \in \mathcal{F}(\Omega;\mathbb{R})$ (so $p_{int}(\vec u) : p \in \Omega \to p_{int}(\vec u)(p) \in \mathbb{R}$).

D'Alembert virtual power formulation. There exists a Galilean referential $\mathcal{R}_a$ in which, at any t,

$$\forall \vec u \in T_0^1(\Omega),\quad \mathcal{P}_a(\vec u) = \mathcal{P}_e(\vec u) + \mathcal{P}_{int}(\vec u). \tag{12.9}$$

12.2.4 Remark: Rigid body motion and Germain's notations

With a Euclidean basis $(\vec e_i)$, the associated Euclidean dot product $.\bullet.$ and the associated vector product $\times$, let

$$\begin{aligned} \mathcal{SC} = \text{the screws} &:= \{\vec u \in T_0^1(\Omega) : \exists\vec\omega \in \vec{\mathbb{R}^3},\ \forall p,q \in \Omega,\ \vec u(q) = \vec u(p) + \vec\omega\times\vec{pq}\}\\ &= \{\vec u \in T_0^1(\Omega) : \exists\vec\omega \in \vec{\mathbb{R}^3},\ \forall q \in \Omega,\ \vec u(q) = \vec u(O) + \vec\omega\times\vec{Oq}\}, \end{aligned} \tag{12.10}$$

where $O \in \mathbb{R}^3$ (an origin): $\vec u \in \mathcal{SC}$ is affine and $\vec u \in T_0^1(\Omega)$ has implicitly been extended to $\vec u \in T_0^1(\mathbb{R}^3)$ (infinite rigid body) so that $\vec u(O)$ is meaningful. (The equality in (12.10) because $\vec u$ is affine.)

$\dim(\mathcal{SC}) = 6$ because $\vec u(O)$ and $\vec\omega$ characterize a screw $\vec u$.

Vocabulary: A screw which is the velocity field of a rigid body motion is called a twist or a kinematic screw or a distributor; And a screw which gives the moment of forces is called a wrench.

Germain's notations:

• A twist in noted $\widehat{\vec u}$ (the hat for virtual), and, with $\widehat{\vec u}(q) = \widehat{\vec u}(p) + \widehat{\vec\omega}\times\vec{pq}$, the twist $\widehat{\vec u}$ is represented by $\{\widehat C\} = \begin{pmatrix} [\widehat{\vec u}(p)]_{|\vec e} \\ [\widehat{\vec\omega}]_{|\vec e} \end{pmatrix} \overset{\text{noted}}{=} \begin{pmatrix} \widehat{\vec u}(p) \\ \widehat{\vec\omega} \end{pmatrix}$ (a $6*1$ matrix made of the reduction elements of $\widehat{\vec u}$ at p).

• A wrench is noted $\vec m$, and, with $\vec m(q) = \vec m(p) + \vec F\times\vec{pq}$, the wrench $\vec m$ is represented by the matrix $[\mathcal{F}] = \begin{pmatrix} [\vec F]_{|\vec e} \\ [\vec m(p)]_{|\vec e} \end{pmatrix} \overset{\text{noted}}{=} \begin{pmatrix} \vec F \\ \vec m(p) \end{pmatrix}$ in that order.

Let $\mathcal{SC}'$ be the dual of $\mathcal{SC}$, i.e. the set of linear forms $\ell : \mathcal{SC} \to \mathbb{R}$ (continuous since $\dim(\mathcal{SC}) < \infty$). If $\mathcal{SC}$ = the twists then $\mathcal{SC}'$ = the wrenches, and if $\mathcal{SC}$ = the wrenches then $\mathcal{SC}'$ = the twists (thanks to the natural canonical isomorphism $\mathcal{SC}'' \sim \mathcal{SC}$).

Proposition 12.2 *Suppose Ω bounded, let $p \in \Omega$. If $\ell \in \mathcal{SC}'$ then $\exists(\vec F, \vec m(p)) \in (\vec{\mathbb{R}^3})^2$ s.t. $\forall\widehat{\vec u} \in \mathcal{SC}$, with $\vec u(q) = \widehat{\vec u}(p) + \widehat{\vec\omega}\times\vec{pq}$ we have*

$$\ell.\vec u = \vec F\bullet\widehat{\vec u}(p) + \vec m(p)\bullet\widehat{\vec\omega} \overset{noted}{=} [\mathcal{F}]\bullet\{\widehat C\} \tag{12.11}$$

(Germain's notation where the $.\bullet.$ notation in $[\mathcal{F}]\bullet\{\widehat C\}$ is the canonical inner dot product in $\mathcal{M}_{61}$ the space or $6*1$ matrices). So if $\widehat{\vec u} \in \mathcal{SC}$ is a twist then ℓ can be represented by a wrench.

(In particular, in a Galilean Euclidean setting, $\mathcal{P}_{int}$ restricted to $\mathcal{SC}$ is the null function $\mathcal{P}_{int} = 0$.)

Proof. Ω bounded implies $\mathcal{SC} \subset L^2(\Omega)$: Indeed, if $\vec{u} \in \mathcal{SC}$ then $\int_\Omega ||\vec{u}(p)||^2\,d\Omega = \int_\Omega ||\vec{u}(0) + \vec{\omega} \times \overrightarrow{Op}||^2\,d\Omega \le 2\int_\Omega ||\vec{u}(0)||^2 + ||\vec{\omega}||\,||\overrightarrow{Op}||^2\,d\Omega \le 2(||\vec{u}(0)||^2 + C||\vec{\omega}||)|\Omega|$, with $|\Omega| = \int_\Omega d\Omega$ the volume of Ω and $C = \sup_{p\in\Omega} ||\overrightarrow{Op}||^2$ finite since Ω is bounded, thus $||\vec{u}||_{L^2} < \infty$, thus $\vec{u} \in L^2(\Omega)^3$.

Hence $\mathcal{SC}$ is a sub-vector space of $L^2(\Omega)$: Indeed, $\vec{u}, \vec{v} \in \mathcal{SC}$ with $\vec{u}(q) = \vec{u}(p) + \vec{\omega}_{\vec{u}} \times \overrightarrow{pq}$ and $\vec{v}(q) = \vec{v}(p) + \vec{\omega}_{\vec{v}} \times \overrightarrow{pq}$, give $(\vec{u}+\lambda\vec{v})(q) = \vec{u}(q) + \lambda\vec{v}(q) = \vec{u}(p) + \vec{\omega}_{\vec{u}} \times \overrightarrow{pq} + \lambda\vec{v}(p) + \vec{\omega}_{\vec{v}} \times \overrightarrow{pq} = (\vec{u}+\lambda\vec{v})(p) + (\vec{\omega}_{\vec{u}} + \lambda\vec{\omega}_{\vec{v}}) \times \overrightarrow{pq} = (\vec{u}+\lambda\vec{v})(p) + \vec{\omega}_{\vec{u}+\lambda\vec{v}} \times \overrightarrow{pq}$ where $\vec{\omega}_{\vec{u}+\lambda\vec{v}} := \vec{\omega}_{\vec{u}} + \lambda\vec{\omega}_{\vec{v}} \in \vec{\mathbb{R}}^3$.

And $\mathcal{SC}$ is finite dimensional ($\dim \mathcal{SC} = 6$), thus $\mathcal{SC}$ is a closed sub-vector space in $L^2(\Omega)$, thus $(\mathcal{SC}, (\cdot,\cdot)_{L^2})$ is a Hilbert space, and a linear $\ell : \mathcal{SC} \to \mathbb{R}$ is continuous. Hence we can apply the $(\cdot,\cdot)_{L^2}$-Riesz representation theorem: If $\ell \in \mathcal{SC}'$, then

$$\exists \vec{\ell} \in \mathcal{SC},\ \forall \vec{u} \in \mathcal{SC},\ \ell(\vec{u}) = (\vec{\ell}, \vec{u})_{L^2} = \int_{q\in\Omega} \vec{\ell}(q) \bullet \vec{u}(q)\,d\Omega \tag{12.12}$$

with the Euclidean dot product $.\bullet.$ relative to a chosen Euclidean basis $(\vec{e}_i)$ in $\vec{\mathbb{R}}^3$. And $\vec{u}(q) = \vec{u}(p) + \vec{\omega} \times \overrightarrow{pq}$ thus $\ell(\vec{u}) = \int_{q\in\Omega} \vec{\ell}(q) \bullet (\vec{u}(p) + \vec{\omega} \times \overrightarrow{pq})\,d\Omega = \int_{q\in\Omega} \vec{\ell}(q) \bullet \vec{u}(p)\,d\Omega + \int_{q\in\Omega} \vec{\ell}(q) \bullet (\vec{\omega} \times \overrightarrow{pq})\,d\Omega$ for all p, thus

$$\ell(\vec{u}) = \vec{F} \bullet \vec{u}(p) + \vec{\omega} \bullet \vec{m}_e(p) \quad \text{where} \quad \vec{F} = \int_{q\in\Omega} \vec{\ell}(q)\,d\Omega \quad \text{and} \quad \vec{m}(p) = \int_{q\in\Omega} \overrightarrow{pq} \times \vec{\ell}(q)\,d\Omega. \tag{12.13}$$

Thus (12.11). ∎

12.2.5 First order linear hypothesis

Let (for scalar valued functions)

$$H^1(\Omega) := \{u \in L^2(\Omega) : \forall j = 1,...,n,\ \frac{\partial u}{\partial x_j} \in L^2(\Omega)\} = \{u \in L^2(\Omega) : \vec{\text{grad}}u \in L^2(\Omega)^n\}. \tag{12.14}$$

$H^1(\Omega)$ is needed in continuum mechanics when "deformation gradients" are considered. Let $(\cdot,\cdot)_{H^1}$ and $||.||_{H^1}$ be the usual dot product and associated norm in $H^1(\Omega)$:

$$(u,v)_{H^1} := (u,v)_{L^2(\Omega)} + (\vec{\text{grad}}u, \vec{\text{grad}}v)_{L^2}, \quad ||u||_{H^1} := \sqrt{(u,u)_{H^1}}. \tag{12.15}$$

Thus $(H^1(\Omega), (\cdot,\cdot)_{H^1})$ is a Hilbert space (Riesz-Fisher theorem).

The dual space of $H^1(\Omega)$ is $H^1(\Omega)'$ = the space of continuous linear forms $\ell : H^1(\Omega) \to \mathbb{R}$. We have, see (V.16): $\ell \in H^1(\Omega)'$ iff $\exists (f,\vec{g}) \in L^2(\Omega) \times L^2(\Omega)^n$ s.t., $\forall \psi \in H^1(\Omega)$,

$$\ell(\psi) = (f,\psi)_{L^2} + (\vec{g}, \vec{\text{grad}}\psi)_{L^2} = \int_\Omega f\psi + \vec{g} \bullet \vec{\text{grad}}\psi\,d\Omega. \tag{12.16}$$

For vector valued functions,

$$H^1(\Omega)^n = \{\vec{u} \in L^2(\Omega)^n : \text{grad}\vec{u} \in L^2(\Omega)^{n^2}\} := \{\vec{u} = \sum_{i=1}^n u_i\vec{e}_i : \forall i = 1,...,n,\ u_i \in H^1(\Omega)\} \tag{12.17}$$

equipped with its usual inner dot product defined by, when $\vec{u} = \sum_i u_i\vec{e}_i$, $\vec{v} = \sum_i v_i\vec{e}_i$,

$$(\vec{u},\vec{v})_{H^1} := \sum_{i=1}^n (u_i,v_i)_{L^2} + \sum_{i,j=1}^n (\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j})_{L^2} \overset{\text{noted}}{=} (\vec{u},\vec{v})_{L^2} + (\text{grad}\vec{u}, \text{grad}\vec{v})_{L^2} \tag{12.18}$$

$= \int_\Omega \vec{u} \bullet \vec{v} + \vec{\text{grad}}\vec{u} : \vec{\text{grad}}\vec{v}\,d\Omega$ where $\vec{\text{grad}}\vec{u} : \vec{\text{grad}}\vec{v} := [d\vec{u}]_{|\vec{e}} : [d\vec{v}]_{|\vec{e}} = \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}$ (double matrix contraction).

Let $H^1(\Omega)^{n\prime}$ be the dual of $H^1(\Omega)^n$, i.e. the set of linear continuous forms $\mathcal{P} : H^1(\Omega)^n \to \mathbb{R}$. (12.16) leads to: $\mathcal{P} \in H^1(\Omega)^{n\prime}$ iff $\exists(\vec{f}, \underline{\underline{\sigma}}) \in L^2(\Omega)^n \times L^2(\Omega)^{n^2}$ s.t., $\forall \vec{v} \in H^1(\Omega)^n$,

$$\mathcal{P}(\vec{v}) = (\vec{f},\vec{v})_{L^2} + (\underline{\underline{\sigma}}, \nabla\vec{v})_{L^2} = \int_\Omega \vec{f} \bullet \vec{v} + \underline{\underline{\sigma}} : \text{grad}\vec{v}\,d\Omega. \tag{12.19}$$

I.e. $\mathcal{P}(\vec{v}) := \int_\Omega \sum_i f_i(p)v_i(p) + \sum_{ij} \sigma_{ij}(p)\frac{\partial v_i}{\partial x_j}(p)\,d\Omega$ when $\vec{v} = \sum_i v_i\vec{e}_i$, $\vec{f} = \sum_i f_i\vec{e}_i$, $[\underline{\underline{\sigma}}]_{|\vec{e}} = [\sigma_{ij}]$.

Galilean Euclidean referential: For $\mathcal{P} = \mathcal{P}_{int}$, $\mathcal{P}_{int}(\vec{v}) = 0$ for any $\vec{v}$ s.t. $d\vec{v} = 0$ (i.e. $\vec{v} = \vec{cste}$), true for all subset in Ω, thus $\vec{f}(p) \bullet \vec{v} = \vec{0}$ for all p and $\vec{v}$, thus $\vec{f}(p) = \vec{0}$ for all p, thus $\vec{f} = \vec{0}$, thus

$$\boxed{\mathcal{P}_{int}(\vec{v}) = \int_\Omega \underline{\underline{\sigma}} : \text{grad}\vec{v}\, d\Omega}. \tag{12.20}$$

And $\mathcal{P}_{int}(\vec{v}) = 0$ for any $\vec{v}$ s.t. $d\vec{v} + d\vec{v}^T = 0$ (rotation), true for all subset in Ω, thus $\underline{\underline{\sigma}} : \frac{d\vec{v} - d\vec{v}^T}{2} = 0$, and we are left with the usual

$$\mathcal{P}_{int}(\vec{v}) = \int_\Omega \underline{\underline{\sigma}} : \frac{d\vec{v} + d\vec{v}^T}{2}\, d\Omega. \tag{12.21}$$

Then, an integration by parts gives, with abusive notations (matrix calculations),

$$\mathcal{P}_{int}(\vec{u}) = -\int_\Omega \text{div}\underline{\underline{\sigma}} \bullet \vec{v}\, d\Omega + \int_\Gamma (\underline{\underline{\sigma}}.\vec{n}) \bullet \vec{v}\, d\Omega. \tag{12.22}$$

Example 12.3 Pressure in a perfect fluid: $\vec{f} = \vec{0}$ and $\underline{\underline{\sigma}} = prI$ where $pr \in L^2(\Omega)$ (pressure), thus

$$\mathcal{P}(\vec{v}) = \int_\Omega pr\ \text{div}\vec{v}\, d\Omega = -\int_\Omega \vec{\text{grad}}pr \bullet \vec{v}\, d\Omega + \int_\Gamma pr\, \vec{v} \bullet \vec{n}\, d\Gamma. \tag{12.23}$$

Germain's notations: $\mathcal{P}(\widehat{\vec{v}}) = \int_\Omega p\ \text{div}\widehat{\vec{v}}\, d\Omega$ with p the pressure and $\widehat{\vec{v}}$ a virtual velocity. ∴

12.2.6 Second order linear hypothesis

Generalization to

$$H^2(\Omega) := \{u \in L^2(\Omega) : \vec{\text{grad}}u \in L^2(\Omega)^n,\ d^2u \in L^2(\Omega)^{n^2}\}. \tag{12.24}$$

with its inner dot product $(u,v)_{H^2} = (u,v)_{L^2} + (\vec{\text{grad}}u, \vec{\text{grad}}v)_{L^2} + (d^2u, d^2v)_{L^2}$ and associated norm $||u||_{H^2} = \sqrt{(u,u)_{H^2}}$. And idem with $H^2(\Omega)^n$.

Second order linear formulation: $\mathcal{P} \in (H^2(\Omega)^n)'$ iff $\exists(\vec{f}, \underline{\underline{\sigma}}, \underline{\underline{\underline{\chi}}}) \in L^2(\Omega)^n \times L^2(\Omega)^{n^2} \times L^2(\Omega)^{n^3}$ s.t.

$$\mathcal{P}(\vec{u}) = (\vec{f}, \vec{u})_{L^2} + (\underline{\underline{\sigma}}, \nabla\vec{u})_{L^2} + (\underline{\underline{\underline{\chi}}}, d^2\vec{u})_{L^2} \tag{12.25}$$

for all $\vec{u} \in H^2(\Omega)^n$. Gives "micropolar materials". See Germain [11].

12.2.7 Issue: The linear hypothesis

The hypothesis (conjecture) "$\mathcal{P}_{int}$ is linear for a second order formulation" raises questions:

A linearity hypothesis enables to do nice simple mathematics thanks to duality; It is used by Germain[11] (who liked mathematics and duality) to define micromorphic materials, cf (12.25). And linearity yields simple computations.

But in "real life" are all "materials" linear?

In mechanics we learn that a constitutive law is useful if and only if the deduced calculations give good approximations of the results obtained by experiments.

So, question: Does the second order linear hypothesis give convincing results (apart from the theoretical micromorphic materials)?

If not, why not consider non linear mathematics, in particular, why not consider non linear internal virtual power? In fact non linearity is proposed in elementary mathematics, e.g. with the second (or higher) order Taylor expansion.

This is proposed in the next § (non linearity of $\mathcal{P}_{int}$).

12.3 Virtual power formulation with Lie derivatives

The Lie derivatives being the "natural derivatives" (and being covariant objective), it is tempting to use them to build the internal virtual power. Let us do it: The flow $\vec{v}$ will act on the Cauchy stress vector $\vec{T}$ to give $\mathcal{L}_{\vec{v}}\vec{T}$ (first order rate of stress), $\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{T})$ (second order), ...

12.3.1 First order approximation with Lie derivatives and classic result

$\vec{v}$ and $\vec{T}$ are the velocity and Cauchy stress vector fields. The rate of stress of $\vec{T}$ along $\vec{v}$ is the Lie derivative

$$\mathcal{L}_{\vec{v}}\vec{T} = \frac{D\vec{T}}{Dt} - d\vec{v}.\vec{T} = \frac{\partial\vec{T}}{\partial t} + d\vec{T}.\vec{v} - d\vec{v}.\vec{T}, \tag{12.26}$$

cf. § 9.3 and 9.5. To measure $\mathcal{L}_{\vec{v}}\vec{T}$, choose a differential form α, and define the internal virtual power

$$\mathcal{P}_{int}(\alpha, \vec{v}, \vec{T}) := \int_\Omega \alpha.\mathcal{L}_{\vec{v}}\vec{T}\, d\Omega, \tag{12.27}$$

with $\alpha.\mathcal{L}_{\vec{v}}\vec{T} = \alpha.\frac{\partial\vec{T}}{\partial t} + \alpha.d\vec{T}.\vec{v} - \alpha.d\vec{v}.\vec{T}$.

Galilean referential framework: The internal power vanishes when $d\vec{v} = 0$ (motion of translation), true for all subset of Ω, hence $\alpha.(\frac{\partial\vec{T}}{\partial t} + d\vec{T}.\vec{v})$ vanishes in (12.27), and we are left with

$$\mathcal{P}_{int}(\alpha, \vec{v}, \vec{T}) = -\int_\Omega \alpha.d\vec{v}.\vec{T}\, d\Omega = -\int_\Omega \underline{\underline{\tau}}_\alpha \mathbin{(\!:\!)} d\vec{v}\, d\Omega, \quad \text{where} \quad \underline{\underline{\tau}}_\alpha := \vec{T}\otimes\alpha. \tag{12.28}$$

Recall: $\underline{\underline{\tau}} \mathbin{(\!:\!)} d\vec{v} := \mathrm{Tr}(\underline{\underline{\tau}}.d\vec{v})$ is the double objective contraction between the $\binom{1}{1}$ tensors $\underline{\underline{\tau}}$ and $d\vec{v}$ (no basis and no inner dot product required a priori: covariant objective approach).

Then choose a Euclidean basis $(\vec{e}_i)$, with its covariant dual basis (e^i) and associated Euclidean dot product $.\bullet.$ (isometric framework); With $\vec{n}$ the exterior Euclidean normal unit vector field on Γ, we get

$$\mathcal{P}_{int}(\alpha, \vec{v}, \vec{T}) = \int_\Omega \widetilde{\mathrm{div}}\underline{\underline{\tau}}_\alpha.\vec{v} - \int_\Gamma (\underline{\underline{\tau}}_\alpha.\vec{v})\bullet\vec{n}\, d\Gamma, \quad \text{where} \quad \underline{\underline{\tau}}_\alpha := \vec{T}\otimes\alpha. \tag{12.29}$$

Recall: $\widetilde{\mathrm{div}}\underline{\underline{\tau}}_\alpha$ is the objective divergence of a $\binom{1}{1}$ tensor, cf. (T.59), and $\underline{\underline{\tau}}_\alpha.\vec{v} = (\alpha.\vec{v})\vec{T}$.

Then choose a uniform measuring tool α. Hence $\mathrm{div}\underline{\underline{\tau}}_\alpha = (\mathrm{div}\vec{T})\alpha$ and $\underline{\underline{\tau}}_\alpha.\vec{v} = (\alpha.\vec{v})\vec{T}$, hence

$$\mathcal{P}_{int}(\alpha, \vec{v}, \vec{T}) = \int_\Omega (\mathrm{div}\vec{T})(\alpha.\vec{v})\, d\Omega - \int_\Gamma (\alpha.\vec{v})\,\vec{T}\bullet\vec{n}\, d\Gamma, \tag{12.30}$$

the use of the Cauchy stress vector $\vec{T}$ being explicit. (Take $\alpha = e^i$ for a measurement along $\vec{e}_i$.)

Classic formulation recovered. $\vec{T} = \sum_i T^i\vec{e}_i$ and $\alpha = \sum_i \alpha_i e^i$ give $[\underline{\underline{\tau}}_\alpha]_{|\vec{e}} := [T^i\alpha_j]$, and $\vec{v} = \sum_i v^i\vec{e}_i$ gives $d\vec{v} = \sum_{ij}\frac{\partial v^i}{\partial x^j}\vec{e}_i\otimes e^j$ and $\alpha.d\vec{v}.\vec{T} = \sum_{ij}\alpha_i\frac{\partial v^i}{\partial x^j}T^j$, thus

$$\begin{aligned}\mathcal{P}_{int}(\alpha, \vec{v}, \vec{T}) &\overset{(12.28)}{=} -\int_\Omega \underline{\underline{\sigma}} : [d\vec{v}]_{|\vec{e}}\, d\Omega, \quad \text{where} \quad \underline{\underline{\sigma}}_\alpha := [\underline{\underline{\tau}}_\alpha]^T_{|\vec{e}} = [\alpha_i T^j] \ \text{(matrix)},\\ &\overset{(12.29)}{=} \int_\Omega \mathrm{div}\underline{\underline{\sigma}}\bullet[\vec{v}]_{|\vec{e}} - \int_\Gamma (\underline{\underline{\sigma}}.[\vec{n}]_{|\vec{e}})\bullet[\vec{v}]_{|\vec{e}}\, d\Gamma, \quad \text{where} \quad \mathrm{div}\underline{\underline{\sigma}} = \begin{pmatrix}\sum_{j=1}^n \frac{\partial\sigma_{1j}}{\partial x^j}\\ \vdots \\ \sum_{j=1}^n \frac{\partial\sigma_{nj}}{\partial x^j}\end{pmatrix}\end{aligned} \tag{12.31}$$

is the divergence of a "tensor" in mechanics (in fact divergence of a matrix), and where $.\bullet.$ is also the notation of the canonical dot product in the space $\mathcal{M}_{n1}$ of $n*1$ matrices. Here α is uniform thus $\sum_j\frac{\partial\sigma_{ij}}{\partial x^j} = \sum_j\alpha_i\frac{\partial T^j}{\partial x^j} = \alpha_i\mathrm{div}\vec{T}$ and $\mathrm{div}\underline{\underline{\sigma}} = (\mathrm{div}\vec{T})[\alpha]^T_{|\vec{e}}$. And (12.31) is abusively written (classic)

$$\mathcal{P}_{int}(\alpha, \vec{v}, \vec{T}) = -\int_\Omega \underline{\underline{\sigma}} : d\vec{v}\, d\Omega = \int_\Omega \mathrm{div}\underline{\underline{\sigma}}\bullet\vec{v} - \int_\Gamma (\underline{\underline{\sigma}}.\vec{n})\bullet\vec{v}\, d\Gamma. \tag{12.32}$$

Exercice 12.4 Write (12.28)-(12.29)-(12.30) with components in a Euclidean framework.

Answer. $\vec{T} = \sum_i T^i\vec{e}_i$, $\alpha = \sum_i\alpha_i e^i$ where (e^i) is the (covariant) dual basis of $(\vec{e}_i)$, $\underline{\underline{\tau}}_\alpha = \sum_{ij}\tau^i_j\vec{e}_i\otimes e^j = \sum_{ij}T^i\alpha_j\vec{e}_i\otimes e^j$, $\vec{v} = \sum_i v^i\vec{e}_i$, $d\vec{v} = \sum_{ij}\frac{\partial v^i}{\partial x^j}\vec{e}_i\otimes e^j$, $\alpha.d\vec{v}.\vec{T} = \sum_{ij}\alpha_i\frac{\partial v^i}{\partial x^j}T^j$, $\underline{\underline{\tau}}.d\vec{v} = \sum_{ijk}\tau^i_k\frac{\partial v^k}{\partial x^j}\vec{e}_i\otimes e^j$, $\underline{\underline{\tau}} \mathbin{(\!:\!)} d\vec{v} = \sum_{ij}\tau^i_j\frac{\partial v^j}{\partial x^i} = \sum_{ij}T^i\alpha_j\frac{\partial v^j}{\partial x^i}$, $d\underline{\underline{\tau}}_\alpha = \sum_{ijk}\frac{\partial\tau^i_j}{\partial x^k}\vec{e}_i\otimes e^j\otimes e^k$, $\widetilde{\mathrm{div}}(\underline{\underline{\tau}}_\alpha) = \sum_{ij}\frac{\partial\tau^i_j}{\partial x^i}e^j$, $\widetilde{\mathrm{div}}(\underline{\underline{\tau}}_\alpha).\vec{v} = \sum_{ij}\frac{\partial\tau^i_j}{\partial x^i}v^j$, $\vec{n} = \sum_i n^i\vec{e}_i$, $\vec{T}\bullet\vec{n} = \sum_i T^i n^i$, $\alpha.\vec{v} = \sum_i\alpha_i v^i$, $\mathcal{P}_{int} = -\sum_{i,j=1}^n\int_\Omega\tau^i_j\frac{\partial v^j}{\partial x^i}\, d\Omega = \sum_{i,j=1}^n\int_\Omega\frac{\partial\tau^i_j}{\partial x^i}v^j\, d\Omega - \sum_{i,j=1}^n\int_\Gamma\tau^i_j v^j n^i\, d\Gamma$. And (12.30) with α uniform, so $\frac{\partial\tau^i_j}{\partial x^i} = \frac{\partial T^i}{\partial x^i}\alpha_j$, $\mathrm{div}\vec{T} = \sum_i\frac{\partial T^i}{\partial x^i}$, $\mathrm{div}\vec{T}(\alpha.\vec{v}) = \sum_i\frac{\partial T^i}{\partial x^i}\sum_j\alpha_j iv^j$. ▪

12.3.2 Second order approximation with Lie derivatives

We add the second order Lie derivative $\mathcal{L}_{\vec v}(\mathcal{L}_{\vec v}\vec T_2)$ of a vector field $\vec T_2$ (instead of the first order Lie derivative $\mathcal{L}_{\vec v}\underline{\underline{\sigma}}$ of a tensor $\underline{\underline{\sigma}}$, cf. e.g. the Jaumann derivative) to get, for all $\vec v$,

$$\mathcal{P}_{int}(\alpha,\vec v,\vec T,\vec T_2)=\int_\Omega \alpha.(\mathcal{L}_{\vec v}\vec T+\mathcal{L}_{\vec v}^{(2)}\vec T_2)\,d\Omega, \tag{12.33}$$

where

$$\begin{aligned}\mathcal{L}_{\vec v}^{(2)}\vec T_2=\mathcal{L}_{\vec v}(\mathcal{L}_{\vec v}\vec T_2)\overset{(9.42)}{=}\ &\frac{\partial^2\vec T_2}{\partial t^2}+2d\frac{\partial \vec T_2}{\partial t}.\vec v-2d\vec v.\frac{\partial \vec T_2}{\partial t}+d\vec T_2.\frac{\partial\vec v}{\partial t}-d\frac{\partial \vec v}{\partial t}.\vec T_2\\ &+(d^2\vec T_2.\vec v).\vec v+d\vec T_2.d\vec v.\vec v-2d\vec v.d\vec T_2.\vec v-(d^2\vec v.\vec v).\vec T_2+d\vec v.d\vec v.\vec T_2.\end{aligned} \tag{12.34}$$

A simple choice is $\vec T_2=c\vec T$, $c\in\mathbb{R}$, to take into account, together with the first order variation $\mathcal{L}_{\vec v}\vec T$, the second order variation $c\mathcal{L}_{\vec v}^{(2)}\vec T$.

Galilean framework: $\mathcal{P}_{int}$ vanishes if $d\vec v=0$, thus moreover choosing a stationary $\vec v$ (so $\frac{\partial \vec v}{\partial t}=\vec 0$),

$$\begin{aligned}\mathcal{P}_{int}(...)&=\int_\Omega \alpha.(-d\vec v.\vec T-2d\vec v.\frac{\partial\vec T_2}{\partial t}+d\vec T_2.d\vec v.\vec v-2d\vec v.d\vec T_2.\vec v-(d^2\vec v.\vec v).\vec T_2+d\vec v.d\vec v.\vec T_2)\,d\Omega\\ &=\int_\Omega \alpha.(-d\vec v.\vec T-2d\vec v.\frac{D\vec T_2}{Dt}+d\vec T_2.d\vec v.\vec v-(d^2\vec v.\vec v).\vec T_2+d\vec v.d\vec v.\vec T_2)\,d\Omega.\end{aligned} \tag{12.35}$$

Then define $\underline{\underline{\tau}}:=\vec T\otimes\alpha$ and $\underline{\underline{\tau}}_2:=\vec T_2\otimes\alpha$ (for constitutive laws) and choose α uniform: We get

$$\mathcal{P}_{int}(...)=\int_\Omega -\underline{\underline{\tau}}:d\vec v-2\frac{D\underline{\underline{\tau}}_2}{Dt}:d\vec v+d\underline{\underline{\tau}}_2:(d\vec v.\vec v)\,d\Omega+\underline{\underline{\tau}}_2:(d\vec v.d\vec v-d^2\vec v.\vec v). \tag{12.36}$$

NB: The result (12.36) is given with tensors $\underline{\underline{\tau}}$ and $\underline{\underline{\tau}}_2$ to be able to compare classical results, e.g. with Jaumann derivatives (Lie derivative of $\binom{1}{1}$ tensor). But recall that here we only have Lie derivatives of the vector fields $\vec T$ and $\vec T_2$ (no Lie derivative of order 2 tensors).

(For an initial approach, see https://arxiv.org/abs/2301.01056 .)

Because $\mathcal{L}_{\vec v}^{(2)}\vec T_2$ is not linear in $\vec v$, this gives a non-linear virtual power in $\vec v$, which in fact could be expected: Not linear in $\vec v$, like any second order Taylor type approximation. It is linear in α.

12.3.3 Non linear first order approximation with Lie derivatives

Technically simpler than the second order approximation: Add to (12.27) a differential form α_1 (a measuring tool) imbedded in the flow to measure some internal force $\vec T_1$ subject to the flow:

$$\mathcal{P}_{int}(\alpha,\alpha_1,\vec v,\vec T,\vec T_1)=\int_\Omega \alpha.\mathcal{L}_{\vec v}\vec T+\mathcal{L}_{\vec v}\alpha_1.(\mathcal{L}_{\vec v}\vec T_1)\,d\Omega. \tag{12.37}$$

A first choice is $\alpha_1=\alpha$ and $\vec T_1=\vec T$. Recall: $\mathcal{L}_{\vec v}\alpha_1=\frac{\partial\alpha_1}{\partial t}+d\alpha_1.\vec v+\alpha.d\vec v$.

Then choose α_1 uniform and stationary, so $\mathcal{L}_{\vec v}\alpha_1=\alpha_1.d\vec v$, and

$$\mathcal{P}_{int}(...)=\int_\Omega \alpha.(\frac{\partial\vec T}{\partial t}+d\vec T.\vec v-d\vec v.\vec T)+\alpha_1.d\vec v.(\frac{\partial\vec T_1}{\partial t}+d\vec T_1.\vec v-d\vec v.\vec T_1)\,d\Omega. \tag{12.38}$$

It is linear in α and α_1, and we have a non linear $d\vec v.d\vec v$ term in $\vec v$.

The internal power has to vanish whenever $d\vec v=0$, true for all subset of Ω, hence the $\alpha.(\frac{\partial\vec T}{\partial t}+d\vec T.\vec v)$ term vanishes, and, with $\underline{\underline{\tau}}:=\vec T\otimes\alpha$ and $\underline{\underline{\tau}}_1:=\vec T_1\otimes\alpha_1$, we are left with

$$\begin{aligned}\mathcal{P}_{int}(...)&=\int_\Omega -\alpha.d\vec v.\vec T+\alpha_1.d\vec v.(\frac{D\vec T_1}{Dt}-d\vec v.\vec T_1)\,d\Omega\\ &=\int_\Omega -\underline{\underline{\tau}}:d\vec v+\frac{D\underline{\underline{\tau}}_1}{Dt}:d\vec v-\underline{\underline{\tau}}_1:(d\vec v.d\vec v)\,d\Omega.\end{aligned} \tag{12.39}$$

Recall: Only Lie derivatives of the vector fields $\vec T$ and $\vec T_1$ are used (no Lie derivative of order 2 tensors).

Part V

"Studying Mathematics I had hoped to penetrate the essence of truth...
... But all I was learning was cheap calculating tricks."
Bertrand Russell (beginning of the 20th century)

Isn't this still too often the case in continuum mechanics? (*"Studying Continuum Mechanics I had hoped to penetrate the essence of truth... But all I was learning was cheap calculating tricks."*)

It is mainly due to the lack of basic math definitions:

What is a motion? A Eulerian variable? A Lagrangian variable?
Why domain and codomain of a function are rarely mentioned (hence errors and misunderstandings)?
What is a "canonical", a "Cartesian", a "Euclidean" basis?
What is a transposed (of what)?
What is pseudo-vector versus a vector?
What are covariant and contravariant vectors?
Why a linear function can't be identified with a vector?
Why a endomorphism $E \to E$ can't be identified with a bilinear form $E \times E \to \mathbb{R}$?
What is the difference between a differential and a gradient?
What is the definition of Einstein's convention?
What is a tensor?
Why the infinitesimal tensor $\underline{\underline{\varepsilon}}$ is not a tensor?
What is the Lie derivative? And why is it "The natural derivative in continuum mechanics"?
What is a distribution?
What does $\frac{\partial W}{\partial F_{ij}}$ mean (derivation relative to components)?
⋮

(*"This is the big advantage of not giving definitions: It allows you to say anything... and be sure that you don't understand what you are talking about." Quote from one of my teachers.*)

Appendix

In this appendix, we give standard simple definitions and results, useful in mechanics, often scattered in the existing literature, and sometimes difficult to find. Hence no ambiguity will be possible; And we avoid notations which are of no use and add to confusion (some notations can be nightmarish when not understood, or misused, or made for calculus tricks, or come like a bull in a china-shop).

All the definitions apply to electromagnetism, chemistry, quantum mechanics, general relativity... and continuum mechanics (solids, fluids, thermodynamics...): The same math apply to everyone.

A Classical and duality notations

A.1 Contravariant vector and basis

A.1.1 Contravariant vectors, covariant vectors

Let $(E,+,.) =^{\text{noted}} E$ be a finite dimension real vector space (= a linear space on the field $\mathbb{R}$).

Definition A.1 An element $\vec{x} \in E$ is called a vector, or a "contravariant vector".

A vector is a vector... So why is it also called a "contravariant vector"?

Historical answer: Because of the change of basis formula $[\vec{x}]_{|new} = P^{-1}.[\vec{x}]_{|old}$, see (A.29), which uses P^{-1}, P being the transition matrix.

Definition A.2 Let $\mathcal{L}(E;\mathbb{R}) =^{\text{noted}} E^*$ be the space of linear scalar valued functions on E, called the space of linear forms on E. An element $\ell \in E^*$ (a linear form) is called a covariant vector.

So a covariant vector is the name given to a function $E \to \mathbb{R}$ which is linear.

Why a linear form is called a "covariant vector"?

Historical answer: Because of the change of basis formula $[\ell]_{|new} = [\ell]_{|old}.P$, which uses P, see (A.29).

To remember: A covariant vector is a linear form ℓ that gives values to vectors $\vec{v}$: value $\ell(\vec{v}) \in \mathbb{R}$. So a covariant vector (a linear form) is a measuring tool for vectors.

A.1.2 Basis

Recall (definitions):

• n vectors $\vec{e}_1, ..., \vec{e}_n \in E$ are linearly independent iff for all $\lambda, ..., \lambda_n \in \mathbb{R}$ the equality $\sum_{i=1}^n \lambda_i \vec{e}_i = \vec{0}$ implies $\lambda_i = 0$ for all $i = 1, ..., n$. (So n vectors $\vec{e}_1, ..., \vec{e}_n \in E$ are linearly dependent iff there exists $i \in [1, n]_{\mathbb{N}}$ and $\lambda_1, ..., \lambda_{i-1}, \lambda_{i+1}, ...\lambda_n \in \mathbb{R}$ s.t. $\vec{e}_i = \sum_{j\neq i} \lambda_j \vec{e}_j$.)

• n vectors $\vec{e}_1, ..., \vec{e}_n \in E$ span E iff : $\forall \vec{x} \in E$, $\exists \lambda_1, ..., \lambda_n \in \mathbb{R}$ s.t. $\vec{x} = \sum_{i=1}^n \lambda_i \vec{e}_i$.

• A basis in E is a set $\{\vec{e}_1, ..., \vec{e}_n\} \subset E$ made of n linearly independent vectors which span E: In which case the dimension of E is n.

A.1.3 Canonical basis

Consider the usual field $\mathbb{R}$ and the Cartesian product $\mathbb{R} \times ... \times \mathbb{R}$, n times. The canonical basis is

$$\vec{A}_1 = (1, 0, ..., 0), \; ..., \; \vec{A}_n = (0, ..., 0, 1), \tag{A.1}$$

with 0 the addition identity element used $n-1$ times, and 1 the multiplication identity element used once.

Remark A.3 Consider the 3-D geometric space "we live in", and the associated vector space $\vec{\mathbb{R}^3}$ of "bi-point vectors". There is no canonical basis in $\vec{\mathbb{R}^3}$: What would the identity element 1 mean? 1 metre? 1 foot? And there is no "intrinsic" preferred direction to define $\vec{e}_1$.

However $\vec{\mathbb{R}^3}$ is isomorphic to the mathematical Cartesian product $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. But such an isomorphism is not "canonical" (or "intrinsic" to $\vec{\mathbb{R}^3}$); For example an isomorphism $\mathcal{J} : \vec{\mathbb{R}^3} \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is defined after the choice of a basis $(\vec{e}_1, \vec{e}_1, \vec{e}_3)$ by some observer (English, French...) by $\mathcal{J}(\vec{e}_i) = \vec{A}_i$. ∎

A.1.4 Cartesian basis

(René Descartes 1596-1650.) Let $n = 1, 2, 3$, let $\mathbb{R}^n$ be the usual affine space (space of points), and let $\vec{\mathbb{R}^n} = (\vec{\mathbb{R}^n}, +, .)$ be the associated usual real vector space of bipoint vectors.

Let $p \in \mathbb{R}^n$, and let $(\vec{e}_i(p))$ be a basis at p (see e.g. the polar coordinate system, example 6.11).

A Cartesian basis in $\vec{\mathbb{R}^n}$ is a basis independent of p (the same at all p), and then $(\vec{e}_i(p)) =^{\text{noted}} (\vec{e}_i)$.

A Euclidean basis is a particular Cartesian basis described in § B.1.

A.2 Representation of a vector relative to a basis

There are to equivalent notation systems:

• the classical notation (non ambiguous), e.g. used by Arnold [3] and Germain [10], and

• the duality notation (can be ambiguous because of misuses), e.g. used by Marsden and Hughes [14].

Both classical and duality notation are equally good, but if you have any doubt, use the classical notations.

Definition A.4 Let $\vec{x} \in E$ and let $(\vec{e}_i)$ be a basis in E. The components of $\vec{x}$ relative to the basis $(\vec{e}_i)$ (in the basis $(\vec{e}_i)$) are the n real numbers $x_1, ..., x_n$ (classical notation) also named $x^1, ..., x^n$ (duality notation) such that

$$\vec{x} = \underbrace{x_1\vec{e}_1 + ... + x_n\vec{e}_n}_{\text{clas.}} = \underbrace{x^1\vec{e}_1 + ... + x^n\vec{e}_n}_{\text{dual}}, \quad \text{i.e.} \quad [\vec{x}]_{|\vec{e}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \tag{A.2}$$

$[\vec{x}]_{|\vec{e}}$ being the column matrix representing $\vec{x}$ relative to the basis $(\vec{e}_i)$. (Of course $x_i = x^i$ for all i.) And the column matrix $[\vec{x}]_{|\vec{e}}$ is simply named $[\vec{x}]$ if one chosen basis is imposed to all. With the sum sign:

$$\vec{x} = \underbrace{\sum_{i=1}^n x_i\vec{e}_i}_{\text{clas.}} = \underbrace{\sum_{i=1}^n x^i\vec{e}_i}_{\text{dual}} \quad (= \sum_{J=1}^n x_J\vec{e}_J = \sum_{\alpha=1}^n x^\alpha\vec{e}_\alpha). \tag{A.3}$$

The index in a summation is a dummy index; And with the Einstein's convention (which uses the duality notation) the sum sign $\sum$ can be omitted: $\vec{x} = \sum_{j=1}^n x^j\vec{e}_j =^{\text{noted}} x^j\vec{e}_j = x^i\vec{e}_i = x^J\vec{e}_J = x^\alpha\vec{e}_\alpha$. This omission was motivated by the difficulty of printing $\sum_{j=1}^n$ in the early 20th century. We won't omit the $\sum$ sign in the following, thanks to TEX-LATEX which makes writing it simple.

Example A.5 In $\vec{\mathbb{R}^2}$, let $\vec{x} = 3\vec{e}_1 + 4\vec{e}_2 = \sum_{i=1}^2 x_i\vec{e}_i = \sum_{i=1}^2 x^i\vec{e}_i$, so $x_1{=}x^1{=}3$ and $x_2{=}x^2{=}4$. And $[\vec{x}]_{|\vec{e}} = 3[\vec{e}_1]_{|\vec{e}} + 4[\vec{e}_2]_{|\vec{e}} = \sum_{i=1}^2 x_i[\vec{e}_i]_{|\vec{e}} = \sum_{i=1}^2 x^i[\vec{e}_i]_{|\vec{e}}$. In particular, with $\delta^i_j = \delta_{ij} := \begin{Bmatrix} = 1 \text{ if } i{=}j \\ = 0 \text{ if } i{\neq}j \end{Bmatrix}$ (Kronecker),

$$\vec{e}_j = \underbrace{\sum_{i=1}^n \delta_{ij}\vec{e}_i}_{\text{clas.}} = \underbrace{\sum_{i=1}^n \delta^i_j\vec{e}_i}_{\text{dual}}, \quad \text{i.e.} \quad [\vec{e}_1]_{|\vec{e}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \dots, \ [\vec{e}_n]_{|\vec{e}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{A.4}$$

that is, the components of $\vec{e}_j$ in $(\vec{e}_i)$ are δ_{ij} with classical notations, and δ^i_j with duality notations. ▪

Definition A.6 The basis $([\vec{e}_j]_{\vec{e}})$ is called the canonical basis of the vector space $\mathcal{M}_{n1}$ of $n * 1$ column matrices. A column matrix $[\vec{x}]_{|\vec{e}}$ is also called a "column vector", or a pseudo-vector.

Remark A.7 NB: A "column vector" is not a "bi-point vector of our geometric space", but just a matrix (a collection of real numbers) relative to the choice of a basis. See the change of basis formula (A.29) where the same vector is represented by two different "column vectors" (two column matrices). ▪

A.3 Dual basis

General usual notations: If E and F are vector spaces then $(\mathcal{F}(E;F), +, .) =^{\text{noted}} \mathcal{F}(E;F)$ is the usual real vector space of functions with the internal addition $(f,g) \to f{+}g$ defined by $(f{+}g)(x) := f(x){+}g(x)$ and the external multiplication $(\lambda, f) \to \lambda.f$ defined by $(\lambda.f)(x) := \lambda(f(x))$, for all $f, g \in \mathcal{F}(E;F)$, $x \in E$, $\lambda \in \mathbb{R}$. And $\lambda.f =^{\text{noted}} \lambda f$ for all $f \in \mathcal{F}(E;F)$ and $\lambda \in \mathbb{R}$.

A.3.1 Linear forms = "Covariant vectors"

Definition A.8 E being a real vector space, the set $E^* := \mathcal{L}(E;\mathbb{R})$ of linear real valued functions is called the dual of E:

$$E^* := \mathcal{L}(E;\mathbb{R}) = \text{the dual of } E. \tag{A.5}$$

An element $\ell \in E^*$ is called a linear form. A linear form ℓ in E^* is also called a "covariant vector".

NB: Co-variant refers to:

1- The action of a function ℓ on a vector $\vec{u}$ that gives the real $\ell(\vec{u})$, the calculation of $\ell(\vec{u})$ being called a co-variant calculation, and

2- The change of coordinate formula $[\ell]_{new} = [\ell]_{|old}.P$, see (A.29) (covariant formula).

Property: E^* is a vector space, sub-space of $\mathcal{F}(E;\mathbb{R})$ (trivial check).

Notation: If $\ell \in E^*$ then

$$\forall \vec{u} \in E, \quad \ell(\vec{u}) \overset{\text{noted}}{=} \ell.\vec{u}. \tag{A.6}$$

The dot in $\ell.\vec{u}$ in (A.6) is "the distributivity dot" since linearity $\ell(\vec{u} + \lambda\vec{v}) = \ell(\vec{u}) + \lambda\ell(\vec{v})$ follows the distributivity rule: $\ell.(\vec{u} + \lambda\vec{v}) = \ell.\vec{u} + \lambda\ell.\vec{v}$.

Also written $\ell(\vec{u}) =^{\text{noted}} \langle \ell, \vec{u} \rangle_{E^*,E}$ where $\langle .,. \rangle_{E^*,E}$ is the duality bracket.

NB: The dot in $\ell.\vec{u}$ is ***not*** an inner dot product (since $\ell \notin E$ while $\vec{u} \in E$).

Remark A.9 More precisely, E^* as defined in (A.5) is the algebraic dual of E. If E is infinite dimensional, then we may need to define a norm $||.||_E$ for which E is a Banach space. E.g. $E = L^2(\Omega)$ and $||f||^2_{L^2(\Omega)} := \int_\Omega f(\vec{x})^2\,d\Omega$. In that case E^* is the name given to the set of continuous linear forms on E, called the topological dual of E: It is essential in continuum mechanics.

(If E is finite dimensional then all norms are equivalent and a linear form is continuous.) ▪

Remark A.10 E^* being a vector space, an element $\ell \in E^*$ is indeed a vector. But E^* has no existence if E has not been specified first! And $\ell \in E^*$ can't be confused with a vector $\vec{u} \in E$ since there is no natural canonical isomorphism between E and E^* (no "intrinsic representation"), see § U.2. So if you want to represent a $\ell \in E^*$ by a vector then you need a tool which is observer dependent; E.g. you need some inner dot product (observer dependent) if you apply the Riesz-representation theorem, or you need to specify a basis (observer dependent) to represent ℓ with its matrix of components (in the dual basis). ▪

Remark A.11 (continuing.) Misner–Thorne–Wheeler [16], box 2.1, insist: "Without it [the distinction between covariance and contravariance, one cannot know whether a vector is meant or the very different object that is a linear form." ▪

A.3.2 Covariant dual basis (= the functions that give components of a vector)

Notation: If $\vec{u}_1, ..., \vec{u}_k$ are vectors in E, then let $\text{Vect}\{\vec{u}_1, ..., \vec{u}_k\}$ be the vector space spanned by $\vec{u}_1, ..., \vec{u}_k$.
Let E be a finite dimensional vector space, and let $(\vec{e}_i)_{i=1,...,n}$ be a basis in E

Definition A.12 Let $i \in [1,n]_{\mathbb{N}}$. The scalar projection on $\text{Vect}\{\vec{e}_i\}$ parallel to $\text{Vect}\{\vec{e}_1, ..., \vec{e}_{i-1}, \vec{e}_{i+1}, ..., \vec{e}_n\}$ is the linear form named $\pi_{ei} \in E^*$ with the classical notation, named $e^i \in E^*$ with the duality notation, defined by, for all i,j,

$$\begin{cases} \text{clas. not.}: & \pi_{ei}(\vec{e}_j) = \delta_{ij}, \quad \text{i.e.} \quad \pi_{ei}.\vec{e}_j = \delta_{ij}, \\ \text{dual not.}: & e^i(\vec{e}_j) = \delta^i_j, \quad \text{i.e.} \quad e^i.\vec{e}_j = \delta^i_j. \end{cases} \tag{A.7}$$

(The dual basis $(\pi_{ei}) = (e^i)$ is intrinsic to the $(\vec{e}_i)$: The same for an English and a French observer...)

Thus, if $\vec{x} = \sum_{j=1}^n x_j \vec{e}_j = \sum_{j=1}^n x^j \vec{e}_j$ (classical or duality notations), $\pi_{ei} = e^i$ being linear, we have $\pi_{ei}(\vec{x}) = \sum_{j=1}^n x_j\, \pi_{ei}(\vec{e}_j) = \sum_{i=1}^n x_i\, \delta_{ij} = x_j$, so

$$\pi_{ei}.\vec{x} \overset{\text{clas.}}{=} x_i \quad = \quad e^i.\vec{x} \overset{\text{dual}}{=} x^i \quad = \text{the } i\text{-th component of } \vec{x}, \tag{A.8}$$

i-th component relative to the basis $(\vec{e}_i)$, see figure A.1.

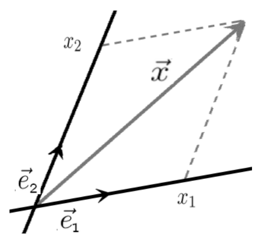


Figure A.1: Parallel projections: $\pi_{e1}(\vec{x}) = x_1$ and $\pi_{e2}(\vec{x}) = x_2$ (dual not.: $e^1(\vec{x}) = x^1$ and $e^2(\vec{x}) = x^2$).

NB: Fundamental: There can't be any intrinsic (objective) notion of orthogonality in E because orthogonality depends on the choice of an inner dot product (subjective). And $\pi_{e1}.\vec{x}$ is not an inner product because $\pi_{ei} = e^i \in E^*$ and $\vec{x} \in E$ do not belong to a same vector space.

Proposition A.13 and definition . $(\pi_{ei})_{i=1,...,n} = (e^i)_{i=1,...,n} =^{noted} (\pi_{ei}) = (e^i)$ *is a basis in* E^*, *called the (covariant) dual basis of the basis* $(\vec{e}_i)$. *Thus* $\dim E^* = n$. *And for all* $\ell \in E^*$ *the reals* $\ell_i := \ell.\vec{e}_i$ *are the components of* ℓ *in the basis dual basis:*

$$\ell \overset{\text{clas.}}{=} \sum_{i=1}^n \ell_i \pi_{ei} \overset{\text{dual}}{=} \sum_{i=1}^n \ell_i e^i \quad \text{where} \quad \ell_i = \ell.\vec{e}_i. \tag{A.9}$$

Proof. If $\sum_{i=1}^n \lambda_i \pi_{ei} = 0$, then $0 = (\sum_{i=1}^n \lambda_i \pi_{ei})(\vec{e}_j) = \sum_{i=1}^n \lambda_i \pi_{ei}(\vec{e}_j) = \sum_{i=1}^n \lambda_i \delta_{ij} = \lambda_j$ for all j, thus $(\pi_{ei})_{i=1,...,n}$ is a family of n independent vectors in E^*. Then let $\ell \in E^*$ and $m = \sum_i (\ell.\vec{e}_i)\pi_{ei}$. Thus $m \in E^*$ (since E^* is a vector space), and $m(\vec{e}_j) = \sum_i (\ell.\vec{e}_i)(\pi_{ei}.\vec{e}_j) = \sum_i (\ell.\vec{e}_i)\delta_{ij} = \ell.\vec{e}_j$, for all j, thus $m = \ell$, thus $\ell = \sum_i (\ell.\vec{e}_i)\pi_{ei}$, thus $\text{Vect}\{(\pi_{ei})_{i=1,...,n}\}$ span E^*; Thus $(\pi_{ei})_{i=1,...,n}$ is a basis in E^*; Thus $\dim E^* = n$. (Use duality notations if you prefer.) ∎

Example A.14 The size of a child is represented on a wall by a bipoint vector $\vec{u}$. An English observer chooses the foot as unit of length and thus makes a vertical bipoint vector "one-foot long" which he names $\vec{a}$. And then defines the linear form $\pi_a : \text{Vect}\{\vec{u}\} \to \mathbb{R}$ by $\pi_a.\vec{a} = 1$. Thus π_a is a measuring instrument, and $\vec{u} = s_a\vec{a}$ where $s_a = \pi_a.\vec{u}$ is the size of the child in foot.

A French observer chooses the metre as unit of length and thus makes a vertical bipoint vector "one-metre long" which he names $\vec{b}$. And then defines the linear form $\pi_b : \text{Vect}\{\vec{u}\} \to \mathbb{R}$ by $\pi_b.\vec{b} = 1$. Thus π_b is a measuring instrument, and $\vec{u} = s_b\vec{b}$ where $s_b = \pi_b.\vec{u}$ is the size of the child in metre. ∎

Exercice A.15 Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases and let (π_{ai}) and (π_{bi}) be the dual bases. Let $\lambda \neq 0$. Prove:

$$\text{If, } \forall i = 1, ..., n, \ \vec{b}_i = \lambda \vec{a}_i, \ \text{ then, } \forall i = 1, ..., n, \ \pi_{bi} = \frac{1}{\lambda}\pi_{ai} \quad (\text{i.e. } b^i = \tfrac{1}{\lambda} a^i). \tag{A.10}$$

Answer. $\pi_{bi}.\vec{b}_j = \delta_{ij} = \pi_{ai}.\vec{a}_j = \pi_{ai}.\frac{\vec{b}_j}{\lambda} = \frac{1}{\lambda}\pi_{ai}.\vec{b}_j$ for all j (since π_{ai} is linear). ∎

A.3.3 Example: aeronautical units

(Fundamental if you fly.) International aeronautical units: Horizontal length = nautical mile (NM), altitude = English foot (ft).

Example A.16 O = the position of the control tower, and a plane p is located thanks to the bipoint vector $\vec{x} = \overrightarrow{\mathcal{O}p}$. A traffic controller chooses $\vec{e}_1$ = the vector of length 1 NM oriented South (first runway), $\vec{e}_2$ = the vector of length 1 NM oriented Southwest (second runway), $\vec{e}_3$ = the vertical vector of length 1 ft: His referential is $\mathcal{R} = (\mathcal{O}, (\vec{e}_1, \vec{e}_2, \vec{e}_3))$. The dual basis is $(\pi_{e1}, \pi_{e2}, \pi_{e3})$ defined by $\pi_{ei}(\vec{e}_j) = \delta_{ij}$ for all i, j, cf. (A.7). He writes $\vec{x} = \sum_{i=1}^n x_i\vec{e}_i \in \vec{\mathbb{R}^n}$, so that $x_1 = \pi_{e1}(\vec{x})$ = the distance to the south in NM, $x_2 = \pi_{e2}(\vec{x})$ = the distance to the southwest in NM, $x_3 = \pi_{e3}(\vec{x})$ = the altitude in ft.

Here the basis $(\vec{e}_i)$ is not a Euclidean basis. This non Euclidean basis $(\vec{e}_i)$ is however vital if you fly: A Euclidean basis is not essential to life... See next remark A.17. ▪

Remark A.17 The metre is the international unit for NASA that launched the Mars Climate Orbiter probe... But for the Mars Climate Orbiter landing procedure, NASA asked Lockheed Martin (who uses the foot) to do the computation. Result? The probe burned in the Martian atmosphere because of $\lambda \sim 3$ times too high a speed during the landing procedure: One metre is $\lambda \sim 3$ times one foot, and someone forgot it... NASA and Lockheed Martin used a Euclidean dot product... But not the same: One based on a metre, and one based on the foot. Objectivity and covariance can be useful! ▪

A.3.4 Matrix representation of a linear form

Let $\ell \in E^*$. Let $(\vec{e}_i)$ be a basis. With the components ℓ_i of ℓ, cf. (A.9),

$$[\ell]_{|\pi_e} = (\,\ell_1 \quad \dots \quad \ell_n\,) \overset{\text{noted}}{=} [\ell]_{|\vec{e}} \quad \text{(row matrix)} \tag{A.11}$$

is called the matrix of ℓ relative to $(\vec{e}_i)$. Thus, if $\vec{x} \in E$ and $\vec{x} =^{\text{clas.}} \sum_{i=1}^n x_i\vec{e}_i =^{\text{dual}} \sum_{i=1}^n x^i\vec{e}_i$, then $\ell.\vec{x} = (\sum_{i=1}^n \ell_i\pi_{ei}).(\sum_{j=1}^n x_j\vec{e}_j) = \sum_{i,j=1}^n \ell_i x_j \pi_{ei}.\vec{e}_j = \sum_{i,j=1}^n \ell_i x_j \delta_{ij} = \sum_{i=1}^n \ell_i x_i = [\ell]_{|\pi_e}.[\vec{x}]_{|\vec{e}}$, so

$$\ell.\vec{x} = [\ell]_{|\pi_e}.[\vec{x}]_{|\vec{e}} \overset{\text{clas.}}{=} \sum_{i=1}^n \ell_i x_i \overset{\text{dual}}{=} \sum_{i=1}^n \ell_i x^i \overset{\text{noted}}{=} [\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}, \tag{A.12}$$

with the usual matrix computation rule: A $1*n$ matrix times a $n*1$ matrix.

In particular for the dual basis $(\pi_{ei}) = (e^i)$ (classical and duality notations),

$$[\pi_{ej}]_{|\pi_e} = [e^j]_{|e} = (0 \;\dots\; 0 \underbrace{\;1\;}_{j\text{th position}} 0 \;\dots\; 0) \overset{\text{noted}}{=} [\pi_{ej}]_{|\vec{e}} = [e^j]_{|\vec{e}} \quad (= \text{row matrix } [\vec{e}_j]_{|\vec{e}}^T). \tag{A.13}$$

Remark A.18 Relative to a basis, a vector is represented by a column matrix, cf. (A.2), and a linear form by a row matrix, cf. (A.11). This enables:

- The use of matrix calculation to compute $\ell.\vec{x} = [\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}$, cf. (A.12), not to be confused with an inner dot product calculation $\vec{x} \bullet \vec{y} := (\vec{x}, \vec{y})_g = [\vec{x}]_{|\vec{e}}^T.[g]_{\pi_e}.[\vec{y}]_{|\vec{e}}$ relative to an inner dot product $(\cdot,\cdot)_g$ in E.
- Not to confuse the "nature of objects": Relative to a basis, a (contravariant) vector is a mathematical object represented by a column matrix, while a linear form (covariant vector) is a mathematical object represented by a row matrix. Cf. remark A.11. ▪

A.3.5 Example: Thermodynamic

Consider the Cartesian space $\vec{\mathbb{R}^2} = \{(T, P) \in \mathbb{R} \times \mathbb{R}\} = \{(\text{temperature,pressure})\}$. There is no meaningful inner dot product in this $\vec{\mathbb{R}^2}$: What would $\sqrt{T^2+P^2}$ mean (Pythagoras: Can you add Kelvin degrees and pressure (kg.m^{-1}·s^{-2})? Thus, in thermodynamics, a (covariant) dual bases is fundamental for calculations.

E.g.: After a choice of temperature and pressure units, consider the basis $(\vec{E}_1{=}(1,0), \vec{E}_2{=}(0,1))$ in $\mathbb{R}\times\mathbb{R} =^{\text{noted}} \vec{\mathbb{R}^2}$; Let $\vec{X} = T\vec{E}_1 + P\vec{E}_2 =^{\text{noted}} (T, P) \in \vec{\mathbb{R}^2}$, and let $(\pi_{E1}, \pi_{E2}) = (E^1, E^2) =^{\text{noted}} (dT, dP)$ be the (covariant) dual basis. The first principle of thermodynamics tells that the density α of internal energy is an exact differential form: $\exists U \in C^1(\vec{\mathbb{R}^2}; \mathbb{R})$ s.t. $\alpha = dU$. So, at any $\vec{X}_0 = (T_0, P_0)$, the linear form $\alpha(\vec{X}_0) = \alpha_1(\vec{X}_0)\,dT + \alpha_2(\vec{X}_0)\,dT \in (\mathbb{R}^2)^*$ is given by $\alpha_1 = \frac{\partial U}{\partial T}$ and $\alpha_2 = \frac{\partial U}{\partial P}$:

$$dU(\vec{X}_0) = \frac{\partial U}{\partial T}(\vec{X}_0)\,dT + \frac{\partial U}{\partial P}(\vec{X}_0)\,dP \quad \text{so} \quad [dU(\vec{X}_0)]_{|\vec{E}} = \begin{pmatrix} \frac{\partial U}{\partial T}(\vec{X}_0) & \frac{\partial U}{\partial P}(\vec{X}_0) \end{pmatrix} \text{ (row matrix).} \tag{A.14}$$

With matrix computation, column matrices for vectors, row matrices for linear forms:

$$[\vec{E}_1]_{|\vec{E}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [\vec{E}_2]_{|\vec{E}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [\vec{X}_0]_{|\vec{E}} = \begin{pmatrix} T_0 \\ P_0 \end{pmatrix}, \quad [\delta\vec{X}]_{|\vec{E}} = \begin{pmatrix} \delta T \\ \delta P \end{pmatrix}, \quad \text{and} \tag{A.15}$$

$$[E^1]_{|\vec{E}} = [dT]_{|\vec{E}} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad [E^2]_{|\vec{E}} = [dP]_{|\vec{E}} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad [dU]_{|\vec{E}} = \begin{pmatrix} \frac{\partial U}{\partial T} & \frac{\partial U}{\partial P} \end{pmatrix} \tag{A.16}$$

give

$$dU(\vec{X}_0).\delta\vec{X} = \begin{pmatrix} \frac{\partial U}{\partial T}(\vec{X}_0) & \frac{\partial U}{\partial P}(\vec{X}_0) \end{pmatrix} . \begin{pmatrix} \delta T \\ \delta P \end{pmatrix} = \frac{\partial U}{\partial T}(\vec{X}_0)\delta T + \frac{\partial U}{\partial P}(\vec{X}_0)\delta P. \tag{A.17}$$

This is a "covariant calculation" (in particular no inner dot product has been used). And we have the first order Taylor expansion in the vicinity of $\vec{X}_0 = (T_0, P_0)$, with $\delta X = (\delta T, \delta P)$:

$$\begin{aligned} U(\vec{X}_0 + \delta\vec{X}) &= U(\vec{X}_0) + dU(\vec{X}_0).\delta\vec{X} + o(\delta\vec{X}) \\ &= U(T_0, P_0) + \delta T \frac{\partial U}{\partial T}(T_0, P_0) + \delta P \frac{\partial U}{\partial T}(T_0, P_0) + o((\delta T, \delta P)). \end{aligned} \tag{A.18}$$

A.4 Einstein convention

A.4.1 Definition

When you work with components (after a choice of a basis), the goal is to visually differentiate a linear form from a vector (to visually differentiate covariance from contravariance).

Framework: a finite dimension vector space E, $\dim E = n$, and duality notations.

Einstein Convention:

1. A basis in E (contravariant) is written with bottom indices: E.g., $(\vec{e}_i)$ is a basis in E.
2. A vector $\vec{x} \in E$ (contravariant) has its components relative to $(\vec{e}_i)$ (quantification) written with top indices: $\vec{x} = \sum_{i=1}^n x^i \vec{e}_i$, and is represented by the column matrix $[\vec{x}]_{|\vec{e}} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$. (Classical notations: $\vec{x} = \sum_{i=1}^n x_i \vec{e}_i$, and column matrix of x_i.)
3. The (covariant) dual basis of $(\vec{e}_i)$ (in $E^* = \mathcal{L}(E;\mathbb{R})$) is written with top indices, so (e^i) is the dual basis of the basis $(\vec{e}_i)$. (Classical notations: (π_{ei}).)
4. A linear form $\ell \in E^*$ (covariant vector) has its components relative to (e^i) (quantification) written with bottom indices: $\ell = \sum_{i=1}^n \ell_i e^i$, and its matrix representation is the row matrix $[\ell]_{|\vec{e}} = \begin{pmatrix} \ell_1 & \dots & \ell_n \end{pmatrix}$.
5. Optional: You can use "the repeated index convention", i.e. omit the sum sign $\sum$ when there are repeated indices at a different position. E.g. $\sum_{i=1}^n x^i \vec{e}_i =^{\text{noted}} x^i \vec{e}_i$, $\sum_{i=1}^n \ell_i e^i =^{\text{noted}} \ell_i e^i$, $\sum_{i=1}^n L^i{}_j \vec{e}_i =^{\text{noted}} L^i_j \vec{e}_i$, $\sum_{i,j=1}^n g_{ij} x^i y^j =^{\text{noted}} g_{ij} x^i y^j$, ... In fact, before computers and word processors, printing $\sum_{i=1}^n$ was not an easy task. With LaTeX it's asy: In this manuscript the sum sign $\sum$ is not omitted (and some confusions are avoided).

A.4.2 Do not mistake yourself

1. Einstein's convention is just meant not to confuse a linear function with a vector.
2. It only deals with quantification relative to a basis.
3. Classical notations are as good as duality notations, even if you are told that classical notations cannot detect obvious errors in component manipulations... But duality notations can be easily (and are often) misused in classical mechanics (cf. the paradigmatic example of the vectorial dual basis treated at § F.8), and mainly adds confusion to the confusion.
4. The convention does not admit shortcuts; E.g. with a metric: $g(\vec{u}, \vec{v}) = \sum_{i,j=1}^n g_{ij} u^i v^j$ shows the observer dependence on a choice of a basis and on the chosen metric (with the g_{ij}); And even if $g_{ij} = \delta_{ij}$ you can***not*** write $g(\vec{u}, \vec{v}) = \sum_{i,j=1}^n u^i v^j$: You have to write $g(\vec{u}, \vec{v}) = \sum_{i,j=1}^n g_{ij} u^i v^j$: Unmissable in physics because you need to see the metric and bases in use.
5. Golden rule: Return to classical notations if in doubt. (Einstein's convention can add confusions, untruths, misinterpretations, absurdities, misuses...)

A.5 Matrix and transposed matrix

The definitions can be found in any elementary books, e.g., Strang [21]. Recall:

• $\mathcal{M}_{mn}$ will be the space of $m*n$ matrices; It is a vector space (with the usual rules).

• Product: If $M=[M_{ij}]_{\substack{i=1,...,m\\j=1,...,n}} \in \mathcal{M}_{mn}$ and $M=[M_{ij}]_{\substack{i=1,...,n\\j=1,...,p}} \in \mathcal{M}_{np}$ then their product is the $m*p$ matrix $M.N=[(M.N)_{ij}]_{\substack{i=1,...,m\\j=1,...,p}} \in \mathcal{M}_{mp}$ where $(M.N)_{ij}=\sum_{k=1}^n M_{ik}N_{kj}$.

• Transposed : If $M=[M_{ij}]_{\substack{i=1,...,m\\j=1,...,n}} \in \mathcal{M}_{mn}$ then its transposed is the matrix $M^T=[(M^T)_{ij}]_{\substack{i=1,...,n\\j=1,...,m}} \in \mathcal{M}_{nm}$ defined by

$$(M^T)_{ij} := M_{ji} \tag{A.19}$$

(swapping rows and columns). E.g., $M=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ gives $M^T=\begin{pmatrix}1&3\\2&4\end{pmatrix}$, and $(M^T)_{12}{=}M_{21}{=}3$.

• M is symmetric iff $M^T=M$ (requires $m{=}n$).

• $(M.N)^T=N^T.M^T$ (because $\sum_k M_{jk}N_{ki}=\sum_k (N^T)_{ik}(M^T)_{kj}$).

• $M\in\mathcal{M}_{nn}$ is invertible iff $\exists N\in\mathcal{M}_{nn}$ s.t. $M.N=I$, and then $N=^{\text{noted}} M^{-1}$.

Exercice A.19 Prove: If M is an $n*n$ invertible matrix then M^T is invertible and $(M^T)^{-1}=(M^{-1})^T$ ($=^{\text{noted}} M^{-T}$); And if M is symmetric, then M^{-1} is symmetric.

Answer. $M.M^{-1}=I$ gives $(M^{-1})^T.M^T=I^T=I$, thus M^T is invertible and $(M^T)^{-1}=(M^{-1})^T$. Thus if $M=M^T$ then $M^{-1}=(M^{-1})^T$. ▪

A.6 Change of basis formulas

E is a finite dimension vector space, $\dim E=n$, $(\vec{e}_{old,i})$ and $(\vec{e}_{new,i})$ are two bases in E, $(\pi_{old,i})$ and $(\pi_{new,i})$ are the associated dual bases in E^*, written (e^i_{old}) and (e^i_{new}) with duality notations.

A.6.1 Change of basis endomorphism and transition matrix

Definition A.20 The change of basis endomorphism $\mathcal{P}\in\mathcal{L}(E;E)$ from $(\vec{e}_{old,i})$ to $(\vec{e}_{new,i})$ is the endomorphism (= the linear map $E\to E$) defined by, for all $j\in[1,n]_{\mathbb{N}}$,

$$\boxed{\mathcal{P}.\vec{e}_{old,j}=\vec{e}_{new,j}}. \tag{A.20}$$

Let

$$\vec{e}_{new,j} \overset{\text{clas.}}{=} \sum_{i=1}^n P_{ij}\,\vec{e}_{old,i} \overset{\text{dual}}{=} \sum_{i=1}^n P^i{}_j\,\vec{e}_{old,i},\quad \text{i.e.}\quad [\vec{e}_{new,j}]_{|\vec{e}_{old}} = \begin{pmatrix}P_{1j}\\ \vdots\\ P_{nj}\end{pmatrix} = \begin{pmatrix}P^1{}_j\\ \vdots\\ P^n{}_j\end{pmatrix}, \tag{A.21}$$

i.e. the $P_{ij}=P^i{}_j$ are the components of $\vec{e}_{new,j}$ in $(\vec{e}_{old,i})$. And (A.20) gives $\mathcal{P}.\vec{e}_{old,j}=\sum_{i=1}^n P_{ij}\,\vec{e}_{old,i}$, so $[\mathcal{P}]_{|\vec{e}_{old}} =^{\text{clas.}} [P_{ij}] =^{\text{dual}} [P^i{}_j]$ is the matrix of the endomorphism $\mathcal{P}$ relative to the basis $(\vec{e}_{old,i})$.

Definition A.21 The matrix $P =^{\text{clas.}} [P_{ij}] =^{\text{dual}} [P^i{}_j]$ is the transition matrix from $(\vec{e}_{old,i})$ to $(\vec{e}_{new,i})$.

You may find other "component type" notations:

$$\vec{e}_{new,j} \overset{\text{clas.}}{=} \sum_{i=1}^n (P_j)_i\,\vec{e}_{old,i} \overset{\text{dual}}{=} \sum_{i=1}^n (P_j)^i\,\vec{e}_{old,i},\quad \text{i.e.}\quad [\vec{e}_{new,j}]_{|\vec{e}_{old}} = \begin{pmatrix}(P_j)_1\\ \vdots\\ (P_j)_n\end{pmatrix} = \begin{pmatrix}(P_j)^1\\ \vdots\\ (P_j)^n\end{pmatrix}. \tag{A.22}$$

So $P_{ij}=P^i{}_j=(P_j)_i=(P_j)^i$ are four notations for the i-th component of $\vec{e}_{new,j}=\mathcal{P}.\vec{e}_{old,j}$ in $(\vec{e}_{old,i})$.

A.6.2 Inverse of the transition matrix

The inverse endomorphism $\mathcal{Q}:=\mathcal{P}^{-1}\in\mathcal{L}(E;E)$ of $\mathcal{P}$ in (A.20) is given by, for all $j\in[1,n]_{\mathbb{N}}$,

$$\vec{e}_{old,j}=\mathcal{Q}.\vec{e}_{new,j}\quad (=\mathcal{P}^{-1}.\vec{e}_{new,j}). \tag{A.23}$$

So $\mathcal{Q}$ is change of basis endomorphism from $(\vec{e}_{new,i})$ to $(\vec{e}_{old,i})$, and $Q:=[\mathcal{Q}]_{|\vec{e}_{new}}=[Q_{ij}]$ is the transition matrix from $(\vec{e}_{new,i})$ to $(\vec{e}_{old,i})$:

$$\vec{e}_{old,j}=\sum_{i=1}^n Q_{ij}\vec{e}_{new,i},\quad [\vec{e}_{old,j}]_{|\vec{e}_{new}} = \begin{pmatrix}Q_{1j}\\ \vdots\\ Q_{nj}\end{pmatrix}. \tag{A.24}$$

Use other notation if you prefer: $Q_{ij}=(Q_j)_i=Q^i{}_j=(Q_j)^i$

Proposition A.22

$$Q = P^{-1}. \tag{A.25}$$

Proof. $\vec{e}_{new,j} = \mathcal{P}.\vec{e}_{old,j} = \sum_{i=1}^n P_{ij}\vec{e}_{old,i} = \sum_{i=1}^n P_{ij}(\sum_{k=1}^n Q_{ki}\vec{e}_{new,k}) = \sum_{k=1}^n(\sum_{i=1}^n Q_{ki}P_{ij})\vec{e}_{new,k} = \sum_{k=1}^n (Q.P)_{kj}\vec{e}_{new,k}$ for all j, thus $(Q.P)_{kj} = \delta_{kj}$ for all j,k. Hence $Q.P = I$, i.e. (A.25). ∎

Exercice A.23 Prove $\left\{\begin{array}{l} [\mathcal{P}]_{|\vec{e}_{old}} = [\mathcal{P}]_{|\vec{e}_{new}} = P, \\ [\mathcal{Q}]_{|\vec{e}_{new}} = [\mathcal{Q}]_{|\vec{e}_{old}} = Q, \end{array}\right\}$, i.e.

$$\left\{\begin{array}{ll} \mathcal{P}.\vec{e}_{new,j} = \sum_{i,j=1}^n P_{ij}\vec{e}_{new,i} & (= \sum_{i,j=1}^n P^i{}_j\vec{e}_{new,i} = \sum_{i,j=1}^n (P_j)^i\vec{e}_{new,i}), \\ \mathcal{Q}.\vec{e}_{old,j} = \sum_{i,j=1}^n Q_{ij}\vec{e}_{old,i} & (= \sum_{i,j=1}^n Q^i{}_j\vec{e}_{old,i} = \sum_{i,j=1}^n (Q_j)^i\vec{e}_{old,i}). \end{array}\right. \tag{A.26}$$

Answer. $Z = [Z_{ij}] = [\mathcal{P}]_{|\vec{e}_{new}}$ means $\mathcal{P}.\vec{e}_{new,j} = \sum_i Z_{ij}\vec{e}_{new,i}$, i.e. $\vec{e}_{new,j} = \mathcal{Q}.(\sum_{i=1}^n Z_{ij}\vec{e}_{new,i}) = \sum_{i=1}^n Z_{ij}\mathcal{Q}.\vec{e}_{new,i} = \sum_{i=1}^n Z_{ij}(\sum_{k=1}^n Q_{ki}\vec{e}_{new,k}) = \sum_{k=1}^n(\sum_{i=1}^n Q_{ki}Z_{ij})\vec{e}_{new,k} = \sum_{k=1}^n (Q.Z)_{kj}\vec{e}_{new,k}$ for all j, thus $(Q.Z)_{kj} = \delta_{kj}$ for all j,k, thus $Q.Z = I$, thus $Z = P$. Idem for Q, thus (A.26). ∎

Remark A.24 $P^T \neq P^{-1}$ in general. E.g., $(\vec{e}_{old,i}) = (\vec{a}_i)$ is a foot-built Euclidean basis, $(\vec{e}_{new,i}) = (\vec{b}_i)$ is a metre-built Euclidean basis, and $\vec{b}_i = \lambda\vec{a}_i$ for all i (the basis are "aligned"), so $P = \lambda I$; Thus $P^T = \lambda I$ and $P^{-1} = \frac{1}{\lambda}I \neq P^T$, since $\lambda = \frac{1}{0.3048} \neq 1$. Thus it is essential not to confuse P^T and P^{-1}, cf. e.g. the Mars Climate Orbiter probe crash (remark A.17). ∎

A.6.3 Change of dual basis

Proposition A.25 *$(\pi_{new,i}) = (e^i_{new})$ and $(\pi_{old,i}) = (e^i_{old})$ being the dual bases of $(\vec{e}_{new,i})$ and $(\vec{e}_{old,i})$, for all $i \in [1,n]_{\mathbb{N}}$,*

$$\pi_{new,i} \overset{clas.}{=} \sum_{j=1}^n Q_{ij}\pi_{old,j} \quad = \quad e^i_{new} \overset{dual}{=} \sum_{j=1}^n Q^i{}_j e^j_{old}, \tag{A.27}$$

and

$$[\pi_{new,i}]_{|\vec{e}_{old}} = (\,Q_{i1} \quad \ldots \quad Q_{in}\,) = [e^i_{new}]_{|\vec{e}_{old}} = (\,Q^i{}_1 \quad \ldots \quad Q^i{}_n\,) \quad \textit{(the i-th row of Q)}. \tag{A.28}$$

Proof. $\pi_{new,i}(\vec{e}_{old,k}) =^{(A.24)} \pi_{new,i}(\sum_j Q_{jk}\vec{e}_{new,j}) = \sum_j Q_{jk}\,\pi_{new,i}(\vec{e}_{new,j}) = \sum_j Q_{jk}\,\delta_{ij} = Q_{ik}$, and $\sum_j Q_{ij}\pi_{old,j}(\vec{e}_{old,k}) = \sum_j Q_{ij}\delta_{jk} = Q_{ik}$, true for all i,k, thus $\pi_{new,i} = \sum_j Q_{ij}$, i.e. (A.27) ∎

A.6.4 Change of coordinate system for vectors and linear forms

Proposition A.26 *Let $\vec{x} \in E$ and $\ell \in E^*$. Then*

$$\begin{array}{ll} \bullet\ [\vec{x}]_{|\vec{e}_{new}} = P^{-1}.[\vec{x}]_{|\vec{e}_{old}} & \textit{(contravariance formula for vectors: between column matrices)}, \\ \bullet\ [\ell]_{|\vec{e}_{new}} = [\ell]_{|\vec{e}_{old}}.P & \textit{(covariance formula for linear forms: between row matrices)}. \end{array} \tag{A.29}$$

And the scalar value $\ell.\vec{x}$ is computed indifferently with one or the other basis (objective result):

$$\ell.\vec{x} = [\ell]_{|\vec{e}_{old}}.[\vec{x}]_{|\vec{e}_{old}} = [\ell]_{|\vec{e}_{new}}.[\vec{x}]_{|\vec{e}_{new}}. \tag{A.30}$$

Proof. Let $\vec{x} = \sum_j x_j\vec{e}_{old,j} = \sum_i y_i\vec{e}_{new,i}$. We have $\vec{x} = \sum_j x_j\vec{e}_{old,j} = \sum_j x_j(\sum_{i=1}^n Q_{ij}\vec{e}_{new,i}) = \sum_{ij} Q_{ij}x_j\vec{e}_{new,i}$, thus $y_i = \sum_j Q_{ij}x_j$ for all i, thus (A.29)$_1$.

And $\ell = \sum_j m_j\pi_{new,j} = \sum_i \ell_i\pi_{old,i} =^{(A.27)} \sum_{ij}\ell_i P_{ij}\pi_{new,j}$ gives $m_j = \sum_i \ell_i P_{ij}$ for all j, thus (A.29)$_2$.

Thus $[\ell]_{|\vec{e}_{new}}.[\vec{x}]_{|\vec{e}_{new}} = ([\ell]_{|\vec{e}_{old}}.P).(P^{-1}.[\vec{x}]_{|\vec{e}_{old}}) = [\ell]_{|\vec{e}_{old}}.[\vec{x}]_{|\vec{e}_{old}}$, hence (A.30).

Use duality notations if you prefer. ∎

Notation: Let $\vec{x} \in E$, $\vec{x} = \sum_j x_j\vec{e}_{old,j} = \sum_i y_i\vec{e}_{new,i}$. Hence (A.29) give $y_i = \sum_{j=1}^n Q_{ij}x_j$, which tells: y_i is the function defined by $y_i(x_1,...,x_n) = \sum_{j=1}^n Q_{ij}x_j$, thus $Q_{ij} = \frac{\partial y_i}{\partial x_j}(x_1,...,x_n)$; Similarly with P_{ij}; Which is written

$$Q_{ij} = \frac{\partial y_i}{\partial x_j}, \quad \text{and} \quad P_{ij} = \frac{\partial x_i}{\partial y_j}. \tag{A.31}$$

(Use duality notations if you prefer: $Q^i{}_j = \frac{\partial y^i}{\partial x^j}$ and $P^i{}_j = \frac{\partial x^i}{\partial y^j}$.)

Exercice A.27 Check that (A.29) applies to $\vec{e}_{new,j}$ and $\pi_{new,i}$.

Answer. Let $(\vec{E}_i)$ be the canonical basis in $\mathcal{M}_{n1}$ the space of $n*1$ matrices. Thus $[\vec{e}_{new,j}]_{|\vec{e}_{new}} = \vec{E}_j$ and $P.[\vec{e}_{new,j}]_{|\vec{e}_{new}} =^{(A.29)} [\vec{e}_{new,j}]_{|\vec{e}_{old}}$ reads $P.\vec{E}_j = [\vec{e}_{new,j}]_{|\vec{e}_{old}}$ = column j of P : True.

$[\pi_{new,i}]_{|\vec{e}_{old}} = \vec{E}_i^T$, thus $[\pi_{new,i}]_{|\vec{e}_{new}}.Q =^{(A.29)} [\pi_{new,i}]_{|\vec{e}_{old}}$ reads $\vec{E}_i^T.Q = [\pi_{new,i}]_{|\vec{e}_{old}}$ = row i of Q : True. ▪

A.7 Bidual basis (and contravariance)

Definition A.28 The dual of E^* is $E^{**} := (E^*)^* = \mathcal{L}(E^*;\mathbb{R})$ and is named the bidual of E. E^{**} is also called the space of contravariant vectors. (the space of directional derivatives see § T.1).

Then let $(\vec{e}_i)$ be a basis in E, let (π_{ei}) be its dual basis (basis in E^*). The dual basis (∂_i) of (π_{ei}) is called the bidual basis of $(\vec{e}_i)$. Duality notations: (∂_i) is the dual basis of (e^i).

Thus, the linear forms $\partial_i \in E^{**} = \mathcal{L}(E^*;\mathbb{R})$ are characterized by, for all j,

$$\partial_i.\pi_{ej} = \delta_{ij} \quad (= \pi_{ej}.\vec{e}_i), \quad \text{so:} \quad \ell = \sum_{i=1}^n \ell_i \pi_{ei} \quad \text{iff} \quad \ell_i = \partial_i.\ell \quad (= \ell.\vec{e}_i). \tag{A.32}$$

Indeed, $\partial_i(\ell) = \partial_i(\sum_{j=1}^n \ell_j \pi_{ej}) = \sum_{j=1}^n \ell_j \partial_i(\pi_{ej}) = \sum_{j=1}^n \ell_j \delta_{ij} = \ell_i$.

Duality notation: $\partial_i.e^j = \delta_i^j = e^j.\vec{e}_i$ and $\ell = \sum_{i=1}^n \ell_i e^i$.

Remark A.29 The notation ∂_i refers to the derivation in the direction $\vec{e}_i$ because $\partial_i(df(\vec{x})) = df(\vec{x}).\vec{e}_i$; And $\partial_i =^{\text{noted}} \vec{e}_i$ in differential geometry. Indeed, with the natural canonical isomorphism $\mathcal{J}$: $\left\{\begin{matrix} E & \to E^{**} \\ \vec{u} & \to \mathcal{J}(\vec{u}) \end{matrix}\right\}$ given by $\mathcal{J}(\vec{u}).\ell := \ell.\vec{u}$ for all $\ell \in E^*$, see (U.9), we can identify $\vec{u}$ and $\mathcal{J}(\vec{u})$ (observer independent identification), thus $\partial_i = \mathcal{J}(\vec{e}_i) =^{\text{noted}} \vec{e}_i$; And (A.32) reads $\vec{e}_i.\pi_{ej} = \delta_{ij}$ and $\ell_i = \vec{e}_i.\ell$. ▪

A.8 Bilinear forms

A.8.1 Definition

Let E and F be vector spaces.

Definition A.30 • A bilinear form is a function $\beta(\cdot,\cdot) : \left\{\begin{matrix} E \times F & \to \mathbb{R} \\ (\vec{u},\vec{w}) & \to \beta(\vec{u},\vec{w}) \end{matrix}\right\}$ satisfying:

$\beta(\vec{u}_1+\lambda\vec{u}_2,\vec{w}) = \beta(\vec{u}_1,\vec{w})+\lambda\beta(\vec{u}_2,\vec{w})$ (linearity for the first variable) and $\beta(\vec{u},\vec{w}_1+\lambda\vec{w}_2) = \beta(\vec{u},\vec{w}_1)+\lambda\beta(\vec{u},\vec{w}_2)$ (linearity for the second variable) for all $\vec{u},\vec{u}_1,\vec{u}_2 \in E$, $\vec{w},\vec{w}_1,\vec{w}_2 \in F$, $\lambda \in \mathbb{R}$.

• $\mathcal{L}(E,F;\mathbb{R})$ is the set of bilinear forms $E \times F \to \mathbb{R}$.

• If $(\ell,m) \in E^* \times F^*$, then the bilinear form $\ell \otimes m \in \mathcal{L}(E,F;\mathbb{R})$ defined by

$$(\ell \otimes m)(\vec{u},\vec{w}) = \ell(\vec{u})m(\vec{w}) \quad (= (\ell.\vec{u})(m.\vec{w})), \tag{A.33}$$

for all $(\vec{u},\vec{w}) \in E \times F$, is called an elementary bilinear form.

A.8.2 The transposed of a bilinear form (objective)

(Warning: Not to be confused with the subjective definition of a transposed of a linear map which depends on choices of inner dot products, see e.g. (A.52).)

Definition A.31 If $\beta \in \mathcal{L}(E,F;\mathbb{R})$ then its transposed is the bilinear form $\beta^T \in \mathcal{L}(F,E;\mathbb{R})$ defined by, for all $(\vec{w},\vec{u}) \in F \times E$,

$$\beta^T(\vec{w},\vec{u}) = \beta(\vec{u},\vec{w}). \tag{A.34}$$

(This definition is observer independent, i.e. same definition for all observers; In particular the definition of β^T doesn't require a basis or an inner dot product.)

A.8.3 Inner dot products, and metrics

Definition A.32 Here $F=E$ and $\beta\in\mathcal{L}(E,E;\mathbb{R})$.
- β is (semi-)positive iff, for all $\vec{u}\in E$, $\beta(\vec{u},\vec{u})\geq 0$.
- β is definite positive iff, for all $\vec{u}\neq\vec{0}$, $\beta(\vec{u},\vec{u})>0$.
- β is symmetric iff $\beta^T=\beta$, i.e., for all $\vec{u},\vec{v}\in E$, $\beta(\vec{u},\vec{v})=\beta(\vec{v},\vec{u})$.

Definition A.33 • An "inner dot product" (or "scalar dot product", or "scalar inner dot product", or "inner scalar product", or "inner product") in a vector space E is a bilinear form $g\in\mathcal{L}(E,E;\mathbb{R})$,

$$g \overset{\text{noted}}{=} g(\cdot,\cdot) \overset{\text{noted}}{=} (\cdot,\cdot)_g \overset{\text{noted}}{=} \cdot \bullet_g \cdot, \quad \text{i.e.}\quad g(\vec{u},\vec{w})=(\vec{u},\vec{w})_g \overset{\text{noted}}{=} \vec{u}\bullet_g\vec{w},\ \ \forall\vec{u},\vec{w}\in E, \tag{A.35}$$

which is symmetric and definite positive: $g(\vec{u},\vec{w})=g(\vec{w},\vec{v})$ for all $\vec{u},\vec{w}$, and $g(\vec{u},\vec{u})>0$ for all $\vec{u}\neq\vec{0}$.
- A "semi-inner dot product" is a symmetric and semi-positive bilinear form.

Definition A.34 Let $(\cdot,\cdot)_g$ be an inner dot product in E.
- Two vectors $\vec{u},\vec{w}\in E$ are $(\cdot,\cdot)_g$-orthogonal iff $(\vec{u},\vec{w})_g=0$.
- The associated norm with $(\cdot,\cdot)_g$ is the function $||.||_g : E\to\mathbb{R}_+$ defined by, for all $\vec{u}\in E$,

$$||\vec{u}||_g=\sqrt{(\vec{u},\vec{u})_g}. \tag{A.36}$$

It is called a semi-norm iff $(\cdot,\cdot)_g$ is a symmetric and semi-positive bilinear form.

Proposition A.35 *(Cauchy–Schwarz inequality.) $(\cdot,\cdot)_g$ being an inner dot product in E,*

$$\forall\vec{u},\vec{w}\in E,\quad |(\vec{u},\vec{w})_g|\leq||\vec{u}||_g||\vec{w}||_g. \tag{A.37}$$

And $|(\vec{u},\vec{w})_g|=||\vec{u}||_g||\vec{w}||_g$ iff $\vec{u}$ and $\vec{w}$ are parallel. And $||.||_g$ is indeed a norm.

Proof. Let $p(\lambda)=||\vec{u}{+}\lambda\vec{w}||_g^2=(\vec{u}{+}\lambda\vec{w},\vec{u}{+}\lambda\vec{w})_g$, so $p(\lambda)=a\lambda^2+b\lambda+c$ where $a=||\vec{w}||_g^2$, $b=2(\vec{u},\vec{w})_g$ and $c=||\vec{u}||_g^2$. With $p(\lambda)\geq 0$ (since$(\cdot,\cdot)_g$ is positive), we get $b^2-4ac\geq 0$, thus (A.37); And $p(\lambda)=0$ iff $\vec{u}{+}\lambda\vec{w}=\vec{0}$. Then $||u||_g=0$ iff $(\vec{u},\vec{u})_g=0$ iff $\vec{u}=0$ since $(\cdot,\cdot)_g$ is definite positive, and $||\vec{u}||_g=\sqrt{(\vec{u},\vec{u})_g}\geq 0$, and $||\lambda\vec{u}||_g=\sqrt{(\lambda\vec{u},\lambda\vec{u})_g}=\sqrt{\lambda^2(\vec{u},\vec{u})_g}=|\lambda|\,||\vec{u}||_g$, and $||\vec{u}+\vec{w}||_g^2=(\vec{u}+\vec{w},\vec{u}+\vec{w})_g=||\vec{u}||_g^2+2(\vec{u},\vec{w})_g+||\vec{w}||_g^2\leq||\vec{u}||_g^2+2||\vec{u}||_g\,||\vec{w}||_g+||\vec{w}||_g^2=(||\vec{u}||_g+||\vec{w}||_g)^2$ thanks to Cauchy–Schwarz inequality, thus $||\vec{u}+\vec{w}||_g\leq||\vec{u}||_g+||\vec{w}||_g$; Thus $||.||_g$ is a norm. ∎

Definition A.36 (Metric.) 1- In $\mathbb{R}^n$ our usual affine geometric space, $n=1$, 2 or 3, with $\vec{\mathbb{R}^n}$ = the usual associated vector space made of bipoint vectors. Let $\Omega\subset\mathbb{R}^n$ be open in $\mathbb{R}^n$. A metric in Ω is a C^∞ function $g:\left\{\begin{array}{ll}\Omega & \to\mathcal{L}(\vec{\mathbb{R}^n},\vec{\mathbb{R}^n};\mathbb{R})\\ p & \to g(p)\overset{\text{noted}}{=} g_p\end{array}\right\}$ such that g_p is an inner dot product in $\vec{\mathbb{R}^n}$ at each $p\in\Omega$. Particular Case: If the g_p is independent of p then a metric is simply called a inner dot product (e.g. a Euclidean metric is called a Euclidean dot product).

2- In a differentiable manifold Ω, a metric is a C^∞ $\binom{0}{2}$ tensor g s.t. $g(p)$ is an inner dot product in the tangent plane $T_p\Omega$ at each $p\in\Omega$. A Riemannian metric is a metric s.t. $g(p)$ is a Euclidean dot product in $T_p\Omega$ at each $p\in\Omega$.

A.8.4 Quantification: Matrice $[\beta_{ij}]$ and tensorial representation

$\dim E=n$, $\dim F=m$, $\beta\in\mathcal{L}(E,F;\mathbb{R})$, $(\vec{a}_i)$ is a basis in E which dual basis is (π_{ai}), $(\vec{b}_i)$ is a basis in F which dual basis is (π_{bi}). (With duality notations, $(\pi_{ai})=(a^i)$ and $(\pi_{bi})=(b^i)$.)

Definition A.37 The components of $\beta\in\mathcal{L}(E,F;\mathbb{R})$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ are the nm reals

$$\beta_{ij}:=\beta(\vec{a}_i,\vec{b}_j),\quad\text{and}\quad[\beta]_{|\vec{a},\vec{b}}=[\beta_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}\overset{\text{noted}}{=}[\beta_{ij}] \tag{A.38}$$

is the matrix of β relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$. And if $F=E$ and $(\vec{b}_i)=(\vec{a}_i)$ then

$$[\beta]_{|\vec{a},\vec{a}}\overset{\text{noted}}{=}[\beta]_{|\vec{a}}. \tag{A.39}$$

Proposition A.38 *A bilinear form $\beta \in \mathcal{L}(E,F;\mathbb{R})$ is known as soon as the nm scalars $\beta_{ij} = \beta(\vec{a}_i,\vec{b}_j)$ are known:*

$$\beta = \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_{ij}\pi_{ai}\otimes\pi_{bj}, \quad \text{and} \quad \beta(\vec{u},\vec{w}) = [\vec{u}]_{|\vec{a}}{}^T.[\beta]_{|\vec{a},\vec{b}}.[\vec{w}]_{|\vec{b}} = \sum_{i,j=1}^{n}\beta_{ij}u_iw_j \tag{A.40}$$

for all $(\vec{u},\vec{w}) \in E\times F$ with $\vec{u} = \sum_i u_i\vec{a}_i$ and $\vec{w} = \sum_i w_i\vec{b}_i$.

And a basis in $\mathcal{L}(E,F;\mathbb{R})$ is made of the nm functions $\pi_{ai}\otimes\pi_{bj}$, and $\dim\mathcal{L}(E,F;\mathbb{R}) = nm$.

(Duality notations: $\beta = \sum_{i=1}^n\sum_{j=1}^m\beta_{ij}a^i\otimes b^j$ and $\beta(\vec{u},\vec{w}) = \sum_{i,j=1}^n\beta_{ij}u^iw^j$.)

Proof. β being bilinear, $\vec{u} = \sum_{i=1}^n u_i\vec{a}_i$ and $\vec{w} = \sum_{j=1}^n w_j\vec{b}_j$ give $\beta(\vec{u},\vec{w}) = \sum_{i,j=1}^n u_iw_j\beta(\vec{a}_i,\vec{b}_j) = \sum_{i,j=1}^n u_i\beta_{ij}w_j = ([\vec{u}]_{|\vec{a}})^T.[\beta]_{|\vec{a},\vec{b}}.[\vec{w}]_{|\vec{b}}$. Thus if the β_{ij} are known then β is known.

And $(\pi_{ai}\otimes\pi_{bj})(\vec{a}_k,\vec{b}_\ell) \stackrel{(A.33)}{=} (\pi_{ai}.\vec{a}_k)(\pi_{bj}.\vec{b}_\ell) = \delta_{ik}\delta_{j\ell}$ (all the elements of the matrix $[\pi_{ai}\otimes\pi_{bj}]_{|\vec{a},\vec{b}}$ are zero except the element at the intersection of row i and column j which is equal to 1).

Thus $\sum_{i,j=1}^n\beta_{ij}(\pi_{ai}\otimes\pi_{bj})(\vec{u},\vec{w}) = \sum_{i,j=1}^n\beta_{ij}u_iw_j = \beta(\vec{u},\vec{v})$, for all $\vec{u},\vec{w}$, thus $\beta := \sum_{i,j=1}^n\beta_{ij}(\pi_{ai}\otimes\pi_{bj})$, thus the $\pi_{ai}\otimes\pi_{bj}$ span $\mathcal{L}(E,F;\mathbb{R})$. And $\sum_{ij}\lambda_{ij}(\pi_{ai}\otimes\pi_{bj}) = 0$ implies $0 = (\sum_{ij}\lambda_{ij}(\pi_{ai}\otimes\pi_{bj}))(\vec{a}_k,\vec{b}_\ell) = \sum_{ij}\lambda_{ij}(\pi_{ai}\otimes\pi_{bj})(\vec{a}_k,\vec{b}_\ell) = \sum_{ij}\lambda_{ij}\delta_{ik}\delta_{j\ell} = \lambda_{k\ell} = 0$ for all k,ℓ; Thus the $\pi_{ai}\otimes\pi_{bj}$ are independent. Thus $(\pi_{ai}\otimes\pi_{bj})$ is a basis in $\mathcal{L}(E,F;\mathbb{R})$ and $\dim(\mathcal{L}(E,F;\mathbb{R})) = nm$. ∎

Example A.39 $\dim E = \dim F = 2$. $[\beta]_{|\vec{a},\vec{b}} = \begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}$ means $\beta(\vec{a}_1,\vec{b}_1) = \beta_{11} = 1$, $\beta(\vec{a}_1,\vec{b}_2) = \beta_{12} = 2$, $\beta(\vec{a}_2,\vec{b}_1) = \beta_{21} = 0$, $\beta(\vec{a}_2,\vec{b}_2) = \beta_{22} = 3$. And $\beta_{12} = [\vec{a}_1]_{|\vec{a}}^T.[\beta]_{|\vec{a},\vec{b}}.[\vec{b}_2]_{|\vec{b}} = (\,1 \quad 0\,).\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}.\begin{pmatrix}0\\ 1\end{pmatrix} = 2$. ∎

Exercice A.40 Let $\beta \in \mathcal{L}(E,E;\mathbb{R})$, let $(\vec{a}_i)$ and $(\vec{b}_i)$ be two bases in A, and let $\lambda \in \mathbb{R}^*$. Prove:

$$\text{if,}\ \ \forall i \in [1,n]_{\mathbb{N}},\ \ \vec{b}_i = \lambda\vec{a}_i, \quad \text{then} \quad [\beta]_{|\vec{b}} = \lambda^2[\beta]_{|\vec{a}}. \tag{A.41}$$

(A change of unit, e.g. from foot to metre, has a true influence on the matrix of a bilinear form.)

Answer. $\vec{b}_i = \lambda\vec{a}_i$ give $\beta(\vec{b}_i,\vec{b}_j) = \beta(\lambda\vec{a}_i,\lambda\vec{a}_j) = \lambda^2\beta(\vec{a}_i,\vec{a}_j)$ (bilinearity), thus $[\beta]_{|\vec{b}} = \lambda^2[\beta]_{|\vec{a}}$. ∎

Exercice A.41 Prove

$$[\beta^T]_{\vec{b},\vec{a}} = ([\beta]_{\vec{a},\vec{b}})^T, \quad \text{written} \quad [\beta^T] = [\beta]^T. \tag{A.42}$$

Answer. Let$[\beta]_{\vec{a},\vec{b}} = [\beta_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$ and $[\beta^T]_{\vec{b},\vec{a}} = [\gamma_{ij}]_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$. We have $\gamma_{ij} = \beta^T(\vec{b}_i,\vec{a}_j) = \beta(\vec{a}_j,\vec{b}_i) = \beta_{ji}$, qed. ∎

A.9 Linear maps

A.9.1 Definition

Let E and F be vector spaces.

Definition A.42 • A function $L : E \to F$ is linear iff $L(\vec{u}_1 + \lambda\vec{u}_2) = L(\vec{u}_1) + \lambda L(\vec{u}_2)$ for all $\vec{u}_1,\vec{u}_2 \in E$ and all $\lambda \in \mathbb{R}$ (distributivity rule). And (distributivity notation):

$$L(\vec{u}) \stackrel{\text{noted}}{=} L.\vec{u}, \quad \text{so} \quad L(\vec{u}_1 + \lambda\vec{u}_2) = L.(\vec{u}_1 + \lambda\vec{u}_2) = L.\vec{u}_1 + \lambda L.\vec{u}_2. \tag{A.43}$$

NB: This dot notation $L(\vec{u}) =^{\text{noted}} L.\vec{u}$ is a linearity notation (distributivity type notation);

- It is an "outer" dot product between a (linear) function and a vector;
- It is ***not*** an "inner" dot product since L and $\vec{u}$ don't belong to a same space.
- It is ***not*** a matrix product (no quantification with bases has been done yet).

Definition A.43 $\mathcal{L}(E;F)$ is the set of linear maps $E \to F$ (vector space, subspace of $(\mathcal{F}(E;F),+,.)$).

If $F = E$ then a linear map $L \in \mathcal{L}(E;E)$ is called an endomorphism in E.

If $F = \mathbb{R}$ then a linear map $E \to \mathbb{R}$ is called a linear form, and $E^* := \mathcal{L}(E;\mathbb{R})$ is the dual of E.

$L_i(E;F)$ is the space of invertible linear maps $E \to F$, i.e. $L \in L_i(E;F)$ iff $\exists M \in L_i(F;E)$ s.t. $L\circ M = I_F$ and $M\circ L = I_E$ where I_E and I_F are the identity maps in E and F.

Vocabulary: If E is a finite dimension vector space, $\dim E = n$, then, in algebra, the set $(L_i(E;E),\circ)$ of invertible endomorphisms equipped with the composition rule is called $GL_n(E)$ = "the linear group" (it is indeed a group, easy check). Particular case: The "linear group" of $n*n$ invertible matrices is $GL_n(\mathcal{M}_n) = (L_i(\mathcal{M}_n;\mathcal{M}_n),.)$ = the set of invertible matrices equipped with the matrix product.

Exercice A.44 (Math exercise.) $E = (E,||.||_E)$ and $F = (F,||.||_F)$ are Banach spaces, and $L_{ic}(E;F)$ is the space of invertible linear continuous maps $E \to F$ with its usual norm $||L|| = \sup_{||\vec{x}||_E=1} ||L.\vec{x}||_F$. Let $Z : \left\{ \begin{array}{ll} L_{ic}(E;F) & \to L_{ic}(E;F) \\ L & \to L^{-1} \end{array} \right\}$. Prove: $dZ(L).M = -L^{-1}\circ M \circ L^{-1}$, for all $M \in L_{ic}(E;F)$ (and Z is differentiable in any direction).

Answer. Consider $\lim_{h\to0} \frac{Z(L+hM)-Z(L)}{h} = \lim_{h\to0} \frac{(L+hM)^{-1}-L^{-1}}{h}$ ($=^{\text{noted}}$ $dZ(L).M$ if the limit exists). With $N = L^{-1}.M$ we have $L+hM = L(I+hN)$, and $(I+hN)$ is invertible as soon as $||hN|| < 1$, i.e. $h < \frac{1}{||N||} = \frac{1}{||L^{-1}.M||}$, its inverse being $I - hN + h^2N - ...$ (Neumann series); Thus $I+hN = I-hN+o(h)$, and $(L+hM)^{-1} = (I+hN)^{-1}.L^{-1} = (I-hN+o(h)).L^{-1} = L^{-1} - hN.L^{-1} + o(h)$. Thus $\frac{(L+hM)^{-1}-L^{-1}}{h} = \frac{L^{-1}-hN.L^{-1}+o(h)-L^{-1}}{h} = -N.L^{-1} + o(1) \longrightarrow_{h\to0} -N.L^{-1}$. ▪

A.9.2 Quantification: Matrices $[L_{ij}] = [L^i{}_j]$

$\dim E = n$, $\dim F = m$, $L \in \mathcal{L}(E;F)$, $(\vec{a}_i)$ is a basis in E and $(\vec{b}_i)$ is a basis in F.

Definition A.45 The components of a linear map $L \in \mathcal{L}(E;F)$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ are the nm reals named L_{ij} (classical notation) $= L^i{}_j$ (duality notation), which are the components of the vectors $L.\vec{a}_j$ relative to the basis $(\vec{b}_i)$. That is:

$$L.\vec{a}_j \overset{\text{clas.}}{=} \sum_{i=1}^m L_{ij}\vec{b}_i \overset{\text{dual}}{=} = \sum_{i=1}^m L^i{}_j\vec{b}_i, \quad \text{i.e.} \quad [L.\vec{a}_j]_{|\vec{b}} \overset{\text{clas.}}{=} \begin{pmatrix} L_{1j} \\ \vdots \\ L_{mj} \end{pmatrix} \overset{\text{dual}}{=} \begin{pmatrix} L^1{}_j \\ \vdots \\ L^m{}_j \end{pmatrix}. \tag{A.44}$$

And

$$[L]_{|\vec{a},\vec{b}} \overset{\text{clas.}}{=} [L_{ij}]_{\substack{i=1,...,m \\ j=1,...,n}} \overset{\text{dual}}{=} [L^i{}_j]_{\substack{i=1,...,m \\ j=1,...,n}} \overset{\text{noted}}{=} [L_{ij}] \overset{\text{noted}}{=} [L^i{}_j] \tag{A.45}$$

is the matrix of L relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$; So $[L.\vec{a}_j]_{|\vec{b}}$ is the j-th column of $[L]_{|\vec{a},\vec{b}}$.

Particular case: If $E = F$, i.e. if L is an endomorphism in E, and if $(\vec{b}_i) = (\vec{a}_i)$ then

$$[L]_{|\vec{a},\vec{a}} \overset{\text{noted}}{=} [L]_{|\vec{a}}. \tag{A.46}$$

Example A.46 $n = m = 2$. $[L]_{|\vec{a},\vec{b}} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ means $L.\vec{a}_1 = \vec{b}_1$ and $L.\vec{a}_2 = 2\vec{b}_1 + 3\vec{b}_2$ (column reading). Here L_{11}=1, L_{12}=2, L_{21}=0, L_{22}=3 (duality notations: $L^1{}_1$=1, $L^1{}_2$=2, $L^2{}_1$=0, $L^2{}_2$=3). ▪

Let $L \in \mathcal{L}(E;F)$. For all $\vec{u} \in E$, $\vec{u} = \sum_{j=1}^n u_j\vec{a}_j = \sum_{j=1}^n u^j\vec{a}_j$, we get, thanks to linearity,

$$L.\vec{u} \overset{\text{clas.}}{=} \sum_{i=1}^m\sum_{j=1}^n L_{ij}u_j\vec{b}_i \overset{\text{dual}}{=} \sum_{i=1}^m\sum_{j=1}^n L^i{}_j u^j\vec{b}_i, \quad \text{i.e.} \quad \boxed{[L.\vec{u}]_{|\vec{b}} = [L]_{|\vec{a},\vec{b}}.[\vec{u}]_{|\vec{a}}}. \tag{A.47}$$

Shortened notation: $[L.\vec{u}] = [L].[\vec{u}]$ when the bases are implicit.

Proposition A.47 *A linear map $L \in \mathcal{L}(E;F)$ is known as soon as the n vectors $L.\vec{a}_1$, ..., $L.\vec{a}_n$ are known. And, for $i,k = 1,...,n$ and $j = 1,...,m$, the linear maps $\mathcal{L}_{ij} \in \mathcal{L}(E;F)$ defined by $\mathcal{L}_{ij}.\vec{a}_k = \delta_{jk}\vec{b}_i$ (all the elements of the matrix $[\mathcal{L}_{ij}]_{|\vec{a},\vec{b}}$ vanish except the element at the intersection of row i and column j which is equal to 1), constitute a basis $\in \mathcal{L}(E;F)$. So, $\dim(\mathcal{L}(E;F)) = nm$.*

(Duality notations: $\mathcal{L}_{ij} =^{noted} \mathcal{L}_i{}^j$, and $\mathcal{L}_i{}^j.\vec{a}_k = \delta^j_k\vec{b}_i$.)

Proof. $\vec{u} \in E$ and $\vec{u} = \sum_k u_j\vec{a}_j$ give $L.\vec{u} = \sum_j u_jL.\vec{a}_j$, since L is linear. Thus L is known iff the n vectors $L.\vec{a}_j$ are known for all $j = 1,...,n$; And $\sum_{ij} L_{ij}\mathcal{L}_{ij}.\vec{a}_k = \sum_{ij} L_{ij}\delta_{jk}\vec{b}_i = \sum_i L_{ik}\vec{b}_i = L.\vec{a}_k$, for all k, thus $\sum_{ij} L_{ij}\mathcal{L}_{ij} = L$, i.e. $L = \sum_{ij} L_{ij}\mathcal{L}_{ij}$, thus the $\mathcal{L}_{ij}$ span $\mathcal{L}(E;F)$. And $\sum_{i=1}^m\sum_{j=1}^n\lambda_{ij}\mathcal{L}_{ij} = 0$ implies $\sum_{i=1}^m\sum_{j=1}^n\lambda_{ij}\mathcal{L}_{ij}.\vec{a}_k = \sum_{i=1}^m\sum_{j=1}^n\lambda_{ij}\delta_{jk}\vec{b}_i = \sum_{i=1}^m\lambda_{ik}\vec{b}_i = \vec{0}$ for all k, thus $\lambda_{ik} = 0$ for all i,k (because $(\vec{b}_i)$ is a basis). Thus the $\mathcal{L}_{ij}$ are independent. Thus $(\mathcal{L}_{ij})_{\substack{i=1,...,n \\ j=1,...,m}}$ is a basis in $\mathcal{L}(E;F)$. ▪

A.10 Trace of an endomorphism

The trace of a $n*n$ matrix $[L_{ij}]$ is $\mathrm{Tr}([L_{ij}]) = \sum_{i=1}^n L_{ii}$ = sum of its diagonal elements.

E is a vector space, $\dim E = n$, $(\vec{a}_i)$ is a basis in E.

Definition A.48 The trace of an endomorphism $L \in \mathcal{L}(E;E)$, with $L.\vec{a}_j = \sum_{i=1}^n L_{ij}\vec{a}_i = \sum_{i=1}^n L^i{}_j\vec{a}_i$, is the real

$$\mathrm{Tr}(L) = \sum_{i=1}^n L_{ii} = \mathrm{Tr}([L]_{|\vec{a}}) \quad (= \sum_{i=1}^n L^i{}_i). \tag{A.48}$$

And the trace operator is the linear map $\mathrm{Tr} : \begin{Bmatrix} \mathcal{L}(E;E) & \to \mathbb{R} \\ L & \to \mathrm{Tr}(L) \end{Bmatrix}$.

Proposition A.49 *The real* $\mathrm{Tr}(L)$ *is independent of any basis in* E*: If* $(\vec{a}_i)$ *and* $(\vec{b}_i)$ *are bases in* E*, then*

$$\mathrm{Tr}([L]_{|\vec{a}}) = \mathrm{Tr}([L]_{|\vec{b}}) = \mathrm{Tr}(L). \tag{A.49}$$

If $L, M \in \mathcal{L}(E;E)$ *then*

$$\mathrm{Tr}(L \circ M) = \mathrm{Tr}(M \circ L) = \sum_{i,j=1}^n L_{ij}M_{ji} = \mathrm{Tr}([L]_{|\vec{a}}.[M]_{|\vec{a}}). \tag{A.50}$$

Proof. $L.\vec{a}_j = \sum_i L_{ij}\vec{a}_i$ and $M.\vec{a}_j = \sum_i M_{ij}\vec{a}_i$ give $(L \circ M).\vec{a}_j = L.(M.\vec{a}_j) = \sum_k M_{kj}L.\vec{a}_k = \sum_{ik} M_{kj}L_{ik}\vec{a}_i = \sum_i(\sum_k L_{ik}M_{kj})\vec{a}_i$. Thus $\mathrm{Tr}(L \circ M) = \sum_i(\sum_k L_{ik}M_{ki}) = \sum_{ij} L_{ij}M_{ji} = \mathrm{Tr}([L]_{|\vec{a}}.[M]_{|\vec{a}}) = \sum_{ij} L_{ji}M_{ij} = \mathrm{Tr}(M \circ L)$. And $[L]_{|\vec{b}} = P^{-1}.[L]_{|\vec{a}}.P$ where P is the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$ (change of basis formula for endomorphisms see (A.103)), thus $\mathrm{Tr}([L]_{|\vec{b}}) = \mathrm{Tr}(P^{-1}.[L]_{|\vec{a}}.P) = \mathrm{Tr}((P^{-1}.[L]_{|\vec{a}}).P) = \mathrm{Tr}(P.(P^{-1}.[L]_{|\vec{a}})) = \mathrm{Tr}((P.P^{-1}).[L]_{|\vec{a}}) = \mathrm{Tr}([L]_{|\vec{a}})$. ∴

Example A.50 If $\vec{b}_i = \lambda\vec{a}_i$ for all i (change of unit of measurement), $\mathrm{Tr}(L) = \sum_i M_{ii} = \sum_i N_{ii}$. Trivial check here: $L.\vec{b}_j = \sum_i N_{ij}\vec{b}_i$ gives $L.(\lambda\vec{a}_j) = \sum_i N_{ij}(\lambda\vec{a}_i)$, thus $L.\vec{a}_j = \sum_i N_{ij}\vec{a}_i$, thus $N = M$. ∴

Exercice A.51 For $L := \vec{w} \otimes \ell$ (defined by $(\vec{w} \otimes \ell).\vec{u} = (\ell.\vec{u})\vec{w}$ for all $\vec{u}$), check:

$$\mathrm{Tr}(\vec{w} \otimes \ell) = \ell.\vec{w}. \tag{A.51}$$

Answer. $\vec{w} = \sum_i w_i\vec{a}_i$ and $\ell = \sum_i \ell_i\pi_{ai}$ give $[\vec{w} \otimes \ell) = [w_i\ell_j]$, thus $\mathrm{Tr}(\vec{w} \otimes \ell) = \sum_i w_i\ell_i = \sum_i \ell_i w_i = \ell.\vec{w}$. ∴

Remark A.52 The "trace" of a bilinear form $g : E \times E \to \mathbb{R}$ (e.g. an inner dot product) defined by $T_a(g) = \sum_i g_{ii}$, where $(\vec{a}_i)$ is a basis and $g(\vec{a}_i, \vec{a}_j) = g_{ij}$, is useless (not used) because it depends on the choice of the basis $(\vec{a}_i)$: E.g. if $\vec{b}_i = \lambda\vec{a}_i$ then $T_b(g) = \lambda^2 T_a(g) \neq T_a(g)$ when $\lambda \neq \pm 1$. ∴

A.11 A transposed endomorphism: Depends on a chosen inner dot product

Not to be confused with the transposed of a matrix, cf. (A.19), and not to be confused with the transposed of a bilinear form which is observer independent, cf. (A.34).

A.11.1 Definition (requires an inner dot product: Not objective)

E is a finite dimensional vector space, $g(\cdot,\cdot) = (\cdot,\cdot)_g = .\bullet_g.$ is an inner dot product in E.

Definition A.53 Let $L \in \mathcal{L}(E;E)$ (endomorphism). Its transposed relative to $(\cdot,\cdot)_g$, also called the $(\cdot,\cdot)_g$-transposed, is the endomorphism $L_g^T \in \mathcal{L}(E;E)$ defined by

$$\forall \vec{u}, \vec{w} \in E, \quad (L_g^T.\vec{w}, \vec{u})_g = (\vec{w}, L.\vec{u})_g, \quad \text{i.e.} \quad (L_g^T.\vec{w}) \bullet_g \vec{u} = \vec{w} \bullet_g (L.\vec{u}). \tag{A.52}$$

(It depends on $(\cdot,\cdot)_g$, so L has an infinite number of transposed, see e.g. exercise A.56.)

Isometric framework, so $(\cdot,\cdot)_g$ is an imposed Euclidean dot product (English, French,...); Then $L_g^T =^{\text{noted}} L^T$ and (A.52) is written $(L^T.\vec{w}) \bullet \vec{u} = \vec{w} \bullet (L.\vec{u})$.

Exercice A.54 Prove: If $(E,(\cdot,\cdot)_g)$ is an Hilbert space and if $L\in\mathcal{L}(E;E)$ is continuous, then L^T_g exists, is unique, and is continuous (apply the Riesz representation theorem F.1).

(If E is finite dimensional then see next § for a direct computation.)

Answer. Let $\vec{w}\in E$, then let $\ell_{\vec{w}g}:\vec{u}\in E\to\ell_{\vec{w}g}(\vec{u}):=(\vec{w},L.\vec{u})_g\in\mathbb{R}$. $\ell_{\vec{w}g}$ is linear (trivial since L is linear and $(\cdot,\cdot)_g$ is bilinear) and continuous: $|\ell_{\vec{w}g}.\vec{u}|\le||\vec{w}||_g||L.\vec{u}||_g\le||\vec{w}||_g||L||\,||\vec{u}||_g$ gives $||\ell_{\vec{w}g}||_{E^*}\le||L||\,||\vec{w}||_g<\infty$. Let $\vec{\ell}_{\vec{w}g}\in E$ be the $(\cdot,\cdot)_g$-Riesz representation of $\ell_{\vec{w}g}\in E^*$: So $\ell_{\vec{w}g}.\vec{u}=(\vec{\ell}_{\vec{w}g},\vec{u})_g$ for all $\vec{u}$ and $||\vec{\ell}_{\vec{w}g}||_g=||\ell_{\vec{w}g}||_{E^*}$. Then define $L^T_g:\vec{w}\in E\to L^T_g(\vec{w}):=\vec{\ell}_{\vec{w}g}\in E$; So $(L^T_g(\vec{w}),\vec{u})_g=(\vec{\ell}_{\vec{w}g},\vec{u})_g=\ell_{\vec{w}g}.\vec{u}=(\vec{w},L.\vec{u})_g$, thus L^T_g is linear (since $(\cdot,\cdot)_g$ is bilinear) and continuous: $||L^T_g.\vec{w}||_g=||\vec{\ell}_{\vec{w}g}||_g=||\ell_{\vec{w}g}||_{E^*}\le||L||\,||\vec{w}||_g$ gives $||L^T_g||\le||L||_{\mathcal{L}(E;E)}<\infty$. Uniqueness: if M^T_g also satisfies $(M^T_g.\vec{w},\vec{u})_g=(\vec{w},L.\vec{u})_g$ then $(M^T_g.\vec{w},\vec{u})_g=(L^T_g.\vec{w},\vec{u})_g$, for all $\vec{u},\vec{w}$, thus $M^T_g=L^T_g$. ▪

A.11.2 Quantification with bases

$(\vec{e}_i)$ is a basis in E, $[g]_{|\vec{e}}=[g_{ij}]=g(\vec{e}_i,\vec{e}_j)$, $[L]_{|\vec{e}}=[L_{ij}]$ and $[L^T_g]_{|\vec{e}}=[(L^T_g)_{ij}]$ (classical notation):

$$L.\vec{e}_j=\sum_{i=1}^n L_{ij}\vec{e}_i,\quad L^T_g.\vec{e}_j=\sum_{i=1}^n (L^T_g)_{ij},\quad\text{i.e.}\quad [L]_{|\vec{e}}=[L_{ij}]\overset{\text{noted}}{=}[L],\quad [L^T_g]_{|\vec{e}}=[(L^T_g)_{ij}]\overset{\text{noted}}{=}[L^T_g].\tag{A.53}$$

(A.52) gives $[\vec{u}]^T_{|\vec{e}}.[g]_{|\vec{e}}.[L^T_g.\vec{w}]_{|\vec{e}}=[L.\vec{u}]^T_{|\vec{e}}.[g]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}$, thus $[\vec{u}]^T_{|\vec{e}}.[g]_{|\vec{e}}.[L^T_g]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}=[\vec{u}]^T_{|\vec{e}}.[L]^T_{|\vec{e}}.[g]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}$, for all $\vec{u},\vec{w}$, thus

$$[g]_{|\vec{e}}.[L^T_g]_{|\vec{e}}=[L]^T_{|\vec{e}}.[g]_{|\vec{e}},\quad\text{i.e.}\quad \sum_{k=1}^n g_{ik}(L^T_g)_{kj}=\sum_{k=1}^n L_{ki}\,g_{kj},\tag{A.54}$$

so $[L^T_g]_{|\vec{e}}=[g]^{-1}_{|\vec{e}}.[L]^T_{|\vec{e}}.[g]_{|\vec{e}}$, written (the basis being implicit)

$$\boxed{[L^T_g]=[g]^{-1}.[L]^T.[g]},\quad\text{i.e.}\quad (L^T_g)_{ij}=\sum_{k,\ell=1}^n([g]^{-1})_{ik}L_{\ell k}g_{\ell j}.\tag{A.55}$$

Duality notations: $L.\vec{e}_j=\sum_{i=1}^n L^i{}_j\vec{e}_i$, $L^T_g.\vec{e}_j=\sum_{i=1}^n(L^T_g)^i{}_j$, $[L]_{|\vec{e}}=[L^i{}_j]$, $[L^T_g]_{|\vec{e}}=[(L^T_g)^i{}_j]$, and

$$\sum_{k=1}^n g_{ik}(L^T_g)^k{}_j=\sum_{k=1}^n L^k{}_i\,g_{kj},\quad\text{i.e.}\quad (L^T_g)^i{}_j=\sum_{k,\ell=1}^n([g]^{-1})_{ik}L^\ell{}_k\,g_{\ell j}.\tag{A.56}$$

Particular case $(\vec{e}_i)$ is $(\cdot,\cdot)_g$-orthonormal: Then $[g]_{|\vec{e}}=[\delta_{ij}]$ and $(L^T_g)_{ij}=L_{ji}$.

Remark A.55 Warning: The last equation $(A.56)_2$ is also written, only because it looks nice (!),

$$(L^T_g)^i{}_j=\sum_{k,\ell=1}^n g^{ik}L^\ell{}_k\,g_{\ell j}\quad\text{when}\quad([g]_{\vec{e}})^{-1}=[g_{ij}]^{-1}\overset{\text{noted}}{=}[g^{ij}].\tag{A.57}$$

But it does ***not*** satisfy Einstein notation because it has nothing to do with covariance-contravariance here. In fact g^{ij} is the short notation for $(g^\sharp)^{ij}$, see (F.34). And $g^\sharp$ has nothing to do here...

So don't be fooled by the notation g^{ij}, defined by $[g^{ij}]:=[g_{ij}]^{-1}$. Use classical notations to avoid misuses and misinterpretations. ▪

Exercice A.56 In $\vec{\mathbb{R}^2}$, let $(\vec{e}_1,\vec{e}_2)$ be a basis. Let $L\in\mathcal{L}(\vec{\mathbb{R}^2};\vec{\mathbb{R}^2})$ be defined by $[L]_{|\vec{e}}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Find two inner dot products $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ in $\vec{\mathbb{R}^2}$ such that $L^T_g\ne L^T_h$ (a transposed endomorphism is not unique, is not intrinsic to L, since it depends on a choice of an inner dot product by an observer).

Answer. Calculations with (A.54):

Choose $(\cdot,\cdot)_g$ given by $[g]_{|\vec{e}}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=[I]$. Thus $[L^T_g]_{|\vec{e}}=[I].[L]_{|\vec{e}}.[I]=\begin{pmatrix}0&1\\1&0\end{pmatrix}$; So $L^T_g=L$.

Choose $(\cdot,\cdot)_h$ given by $[h]_{|\vec{e}}=\begin{pmatrix}1&0\\0&2\end{pmatrix}$. Thus $[L^T_h]_{|\vec{e}}=[h]^{-1}_{|\vec{e}}.[L]_{|\vec{e}}.[h]_{|\vec{e}}=\begin{pmatrix}0&2\\\frac12&0\end{pmatrix}$; So $L^T_h\ne L$.

Thus $L^T_h\ne L^T_g$, e.g., $\vec{e}_2=L^T_g.\vec{e}_1\ne L^T_h.\vec{e}_1=\frac12\vec{e}_2$. ▪

Exercice A.57 Prove: If L is invertible then L^T_g is invertible, and $(L^T_g)^{-1} = (L^{-1})^T_g$ (written L^{-T}_g).

Answer. Suppose: $\exists \vec{w} \in E$, $\vec{w} \neq \vec{0}$, s.t. $L^T_g.\vec{w} = 0$. L being invertible, $\exists! \vec{u} \in E$ s.t. $L.\vec{u} = \vec{w}$, with $\vec{u} \neq \vec{0}$ since $\vec{w} \neq \vec{0}$ and L is linear; And $L^T_g.\vec{w} = 0$ gives $L^T_g.L.\vec{u} = 0$, thus $(L^T_g.L.\vec{u}, \vec{u})_g = 0$, thus $||L.\vec{u}||^2_g = 0$, thus $L.\vec{u} = 0$, thus $\vec{u} = 0$ since L is linear bijective; Absurd. Thus $\mathrm{Ker}(L^T_g) = \{\vec{0}\}$, thus L^T_g is invertible since it is an endomorphism. And $(L^T_g.(L^{-1})^T_g.\vec{u}, \vec{w})_g \stackrel{(A.52)}{=} ((L^{-1})^T_g.\vec{u}, L.\vec{w})_g \stackrel{(A.52)}{=} (\vec{u}, (L^{-1}).L.\vec{w})_g = (\vec{u}, \vec{w})_g = (L^T_g.(L^T_g)^{-1}.\vec{u}, \vec{w})_g$, true $\forall \vec{u}, \vec{w}$, thus $L^T_g.(L^{-1})^T_g = L^T_g.(L^T_g)^{-1}$, thus $(L^{-1})^T_g = (L^T_g)^{-1}$ since L^T_g is invertible. ▪

Exercice A.58 Special case of proportional inner dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$: $\exists \lambda > 0$ s.t. $(\cdot,\cdot)_a = \lambda^2 (\cdot,\cdot)_b$. Prove: $L^T_a = L^T_b$: Two proportional inner dot products give the same transposed endomorphism.

Answer. $(L^T_b.\vec{w}, \vec{u})_b = (\vec{w}, L.\vec{u})_b = \lambda^2 (\vec{w}, L.\vec{u})_a = \lambda^2 (L^T_a.\vec{w}, \vec{u})_a = (L^T_a.\vec{w}, \vec{u})_b$, for all $\vec{u}, \vec{w}$, so $L^T_b = L^T_a$. ▪

Exercice A.59 Prove: $\mathrm{Tr}(L^T_g) = \mathrm{Tr}(L)$ (independent of g).

Answer. $\mathrm{Tr}(L^T_g) = \mathrm{Tr}([L^T_g]_{|\vec{e}}) = \mathrm{Tr}([g]^{-1}_{|\vec{e}}.[L]^T_{|\vec{e}}.[g]_{|\vec{e}}) = \mathrm{Tr}([g]_{|\vec{e}}.[g]^{-1}_{|\vec{e}}.[L]^T_{|\vec{e}}) = \mathrm{Tr}([L]^T_{|\vec{e}}) = \mathrm{Tr}([L]_{|\vec{e}}) = \mathrm{Tr}(L)$. ▪

A.11.3 Dangerous tensorial notation for endomorphisms

Recall: The transposed β^T of a bilinear form β is objective, cf. (A.34): We don't need any tool like an inner dot product to define β^T.

Not to be confused with: The transposed $L^T_g =^{\text{noted}} L^T$ of a linear map L is subjective: It depends on a choice of an inner dot products $(\cdot,\cdot)_g$ by an observer.

E.g., a bilinear form $\beta \in \mathcal{L}(E,E;\mathbb{R})$ satisfies $[\beta^T]_{|\vec{e}} = [\beta]_{|\vec{e}}{}^T$. But a linear endomorphism $L \in \mathcal{L}(E;E)$ satisfies $[L^T_g]_{|\vec{e}} \neq [L]_{|\vec{e}}{}^T$ in general: E.g. take $[L]_{|\vec{e}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $[g]_{|\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and use (A.55).

Hence it is ***dangerous*** to represent an endomorphism in a basis with its "bilinear tensorial representation" when dealing with the transposed: $L \in \mathcal{L}(E;E)$ is naturally canonically represented by the bilinear form $\beta_{(L)} \in \mathcal{L}(E^*,E;\mathbb{R})$ (and $\beta_{(L)} \notin \mathcal{L}(E,E;\mathbb{R})$): With (a^i) the dual basis of $(\vec{a}_i)$,

$$L.\vec{a}_j = \sum_{i=1}^{n} L^i{}_j \vec{a}_i \quad \text{gives} \quad \beta_{(L)} = \sum_{i,j=1}^{n} L^i{}_j \vec{a}_i \otimes a^j, \quad \text{thus} \quad \beta_{(L)}{}^T \stackrel{(A.34)}{=} \sum_{i,j=1}^{n} L^j{}_i a^i \otimes \vec{a}_j. \tag{A.58}$$

And, a $(\cdot,\cdot)_g$ being chosen, $L^T_g \in \mathcal{L}(F;E)$ is represented by the bilinear form $\beta_{(L^T_g)} \in \mathcal{L}(E^*,E;\mathbb{R})$:

$$L^T_g.\vec{a}_j = \sum_{i=1}^{n} (L^T_g)^i{}_j \vec{a}_i \quad \text{gives} \quad \beta_{(L^T_g)} = \sum_{i,j=1}^{n} (L^T_g)^i{}_j \vec{a}_i \otimes a^j; \quad \text{Thus} \quad \boxed{\beta_{(L^T_g)} \neq \beta_{(L)}{}^T} \tag{A.59}$$

because: 1- $\vec{a}_i \otimes a^j \neq a^i \otimes \vec{a}_j$ (!), and

2- $(\beta_L)^T$ is independent of any inner dot product, while L^T_g depends on a chosen inner dot product, see (A.55): $(L^T_g)^i{}_j = \sum_{k,\ell=1}^{n} ([g]^{-1})_{ik} L^\ell{}_k \, g_{\ell j} \neq L^j{}_i$ in general, while $(\beta_{(L)}{}^T)^i{}_j = (\beta_{(L)})^j{}_i$ (always).

3- $(\beta_L)^T \in \mathcal{L}(E^*,E;\mathbb{R})$ is the tensorial representation of the adjoint $L^* \in \mathcal{L}(E^*;E^*)$ of L, i.e. $(\beta_L)^T = \beta_{(L^*)}$, see (A.81).

So in continuum mechanics it is strongly advised ***not to use the tensorial notation*** for linear maps when dealing with transposed.

A.11.4 Symmetric endomorphism (depends on a $(\cdot,\cdot)_g$)

Definition A.60 An endomorphism $L \in \mathcal{L}(E;E)$ is $(\cdot,\cdot)_g$-symmetric iff $L^T_g = L$:

$$L \ (\cdot,\cdot)_g\text{-symmetric} \iff L^T_g = L \iff (L.\vec{u}, \vec{w})_g = (\vec{u}, L.\vec{w})_g, \quad \forall \vec{u}, \vec{w} \in E. \tag{A.60}$$

Remark A.61 The symmetric character of an endomorphism L is not intrinsic to the endomorphism: It depends on $(\cdot,\cdot)_g$; See exercise A.56 where L is $(\cdot,\cdot)_g$-symmetric while it is ***not*** $(\cdot,\cdot)_h$-symmetric. ▪

A.11.5 The general flat $^\flat$ notation for an endomorphism (depends on a $(\cdot,\cdot)_g$)

Definition A.62 Let $(\cdot,\cdot)_g$ be an inner dot product in a vector space E, and let $L \in \mathcal{L}(E;E)$. Its associated bilinear form $L^\flat_g \in \mathcal{L}(E,E;\mathbb{R})$ is defined by, for all $\vec{u},\vec{w} \in E$,

$$L^\flat_g(\vec{u},\vec{w}) := (\vec{u}, L.\vec{w})_g. \tag{A.61}$$

(The bilinearity of $L^\flat_g$ is trivial$L^\flat_g$.) (The bilinear form $L^\flat_g$ is continuous as soon as L is: $|L^\flat_g(\vec{u},\vec{v})| \le ||g||\,||L.\vec{u}||\,||\vec{v}|| \le (||g||\,||L||)\,||\vec{u}||\,||\vec{v}||$.) We have thus defined the $(\cdot,\cdot)_g$-dependent operator:

$$(.)^\flat_g = \mathcal{J}_g(.) : \begin{cases} \mathcal{L}(E;E) & \to \mathcal{L}(E,E;\mathbb{R}) \\ \qquad L & \to \mathcal{J}_g(L) := L^\flat_g, \end{cases} \tag{A.62}$$

This operator transforms a contravariance into a covariance: Indeed, with the natural canonical isomorphism $\mathcal{L}(E;E) \simeq \mathcal{L}(E^*,E;\mathbb{R})$, L is represented by a bilinear form $\widetilde{L} \in \mathcal{L}(E^*,E;\mathbb{R})$ (a $\binom{1}{1}$ tensor) which is transformed by $(.)^\flat_g$ into a bilinear form $L^\flat_g \in \mathcal{L}(E,E;\mathbb{R})$ (a $\binom{0}{2}$ tensor).

Quantification: Let $(\vec{e}_i)$ be a basis in E, and $[g]_{|\vec{e}} = [g_{ij}]$, $[L]_{|\vec{e}} = [L^i{}_j]$ and $[L^\flat_g]_{|\vec{e}} = [(L^\flat_g)_{ij}]$, i.e.

$$g = \sum_{i,j=1}^n g_{ij} e^i \otimes e^j, \quad L\vec{e}_j = \sum_{i=1}^n L^i{}_j \vec{e}_i, \quad L^\flat_g = \sum_{i,j=1}^n (L^\flat_g)_{ij} e^i \otimes e^j. \tag{A.63}$$

Then

$$\boxed{[L^\flat_g] = [g].[L]}. \tag{A.64}$$

Indeed: $(L^\flat_g)_{ij} = L^\flat_g(\vec{e}_i,\vec{e}_j) \overset{(A.61)}{=} (\vec{e}_i, L.\vec{e}_j)_g = (\vec{e}_i, \sum_k L^k{}_j \vec{e}_k)_g = \sum_k L^k{}_j g_{ik} = ([g].[L])_{ij}$.

Exercice A.63 With the natural canonical isomorphism $L \in \mathcal{L}(E;E) \simeq T_L \in \mathcal{L}(E^*,E;\mathbb{R})$ given by $T_L(\ell,\vec{w}) = \ell.L.\vec{w}$, prove:

$$L^\flat_g = g.T_L \in \mathcal{L}(E,E;\mathbb{R}) \simeq \mathcal{L}(E^*;E). \tag{A.65}$$

(A change of variance, here from the $\binom{1}{1}$ tensor $T_L \simeq L$ to the $\binom{0}{2}$ tensor $L^\flat_g$, is necessarily observer dependent: There is no natural canonical isomorphism between a vector space E and its dual E^*, see § U.2.)

Answer. If $L.\vec{e}_j = \sum_i L^i{}_j \vec{e}_i$ then $T_L = \sum_{ij} L^i{}_j \vec{e}_i \otimes e^j$, thus $g = \sum_{ij} g_{ij} e^i \otimes e^j$ gives $g.T_L = \sum_{ijk} g_{ik} L^k{}_j \vec{e}_i \otimes e^j$. And $L^\flat_g(\vec{e}_i,\vec{e}_j) =^{(A.61)} (\vec{e}_i, L.\vec{e}_j)_g = \sum_k L^k{}_j (\vec{e}_i,\vec{e}_k)_g = \sum_k L^k{}_j g_{ik}$, thus $L^\flat_g = \sum_{ijk} g_{ik} L^k{}_j \vec{e}_i \otimes e^j = g.T_L$. ▪

A.12 A transposed of a linear map: depends on chosen inner dot products

This paragraph is needed to define the transposed of a deformation gradient.

A.12.1 Definition (depends on two inner dot products)

$(E,(\cdot,\cdot)_g)$ and $(F,(\cdot,\cdot)_h)$ are Hilbert spaces, and $L \in \mathcal{L}(E;F)$ (supposed continuous when E and F are infinite dimensional). E.g., $E = \vec{\mathbb{R}}^n_{t_0}$, $F = \vec{\mathbb{R}}^n_t$, deformation gradient $L = d\Phi^{t_0}_t(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$, cf. (4.1), $(\cdot,\cdot)_g$ is the foot built Euclidean dot product chosen by the observer at t_0 (measurements at t_0), $(\cdot,\cdot)_h$ is the metre built Euclidean dot product chosen by the observer at t (measurements at t).

Definition A.64 The transposed of $L \in \mathcal{L}(E;F)$ relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ is $L^T_{gh} \in \mathcal{L}(F;E)$ defined by, for all $(\vec{u},\vec{w}) \in E \times F$,

$$(L^T_{gh}.\vec{w},\vec{u})_g = (\vec{w}, L.\vec{u})_h, \tag{A.66}$$

where we used the dot notation $L^T_{gh}(\vec{w}) =^{\text{noted}} L^T_{gh}.\vec{w}$ since L^T_{gh} is linear. This defines the map

$$(.)^T_{gh} : \begin{cases} \mathcal{L}(E;F) & \to \mathcal{L}(F;E) \\ \qquad L & \to (.)^T_{gh}(L) := L^T_{gh} \end{cases} \tag{A.67}$$

(So a linear map has an infinite number of transposed (it depends on inner dot products.)

And if $F = E$ and $(\cdot,\cdot)_h = (\cdot,\cdot)_g$ then $L^T_{gh} = L^T_g$, see § A.11 (transposed of an endomorphism).

A.12.2 Quantification with bases

Let $(\vec{a}_i)_{i=1,...,n}$ and $(\vec{b}_i)_{i=1,...,m}$ be bases in E and F, let $[g]_{|\vec{a}} = [g_{ij}] = [g(\vec{a}_i, \vec{a}_j)]$, $[h]_{|\vec{b}} = [h_{ij}] = [h(\vec{b}_i, \vec{b}_j)]$, and let (classical notation)

$$\begin{aligned} L.\vec{a}_j &= \sum_{i=1}^{m} L_{ij}\vec{b}_i, \quad \text{i.e.} \quad [L]_{|\vec{a},\vec{b}} = [L_{ij}] \overset{\text{noted}}{=} [L], \\ L^T_{gh}.\vec{b}_j &= \sum_{i=1}^{n} (L^T_{gh})_{ij}\vec{a}_i, \quad \text{i.e.} \quad [L^T_{gh}]_{|\vec{b},\vec{a}} = [(L^T_{gh})_{ij}] \overset{\text{noted}}{=} [L^T_{gh}]. \end{aligned} \tag{A.68}$$

(A.66) gives $[\vec{u}]^T_{|\vec{a}}.[g]_{|\vec{a}}.[L^T_{gh}.\vec{w}]_{|\vec{w}} = ([L.\vec{u}]_{|\vec{b}})^T.[h]_{|\vec{b}}.[\vec{w}]_{|\vec{b}}$ for all $\vec{u}, \vec{w}$, thus, $[g]_{|\vec{a}}.[L^T_{gh}]_{|\vec{b},\vec{a}} = ([L]_{|\vec{a},\vec{b}})^T.[h]_{|\vec{b}}$ and $[L^T_{gh}]_{|\vec{b},\vec{a}} = [g]^{-1}_{|\vec{a}}.([L]_{|\vec{a},\vec{b}})^T.[h]_{|\vec{b}}$. Shortened notation with implicit bases:

$$[g].[L^T] = [L]^T.[h], \quad \text{i.e.} \quad \sum_{k=1}^{n} g_{ik}(L^T_{gh})_{kj} = \sum_{k=1}^{m} L_{ki}\, h_{kj}, \tag{A.69}$$

i.e.

$$\boxed{[L^T] = [g]^{-1}.[L]^T.[h]}, \quad \text{i.e.} \quad (L^T_{gh})_{ij} = \sum_{k=1}^{n}\sum_{\ell=1}^{m} ([g]^{-1})_{ik} L_{\ell k} h_{\ell j}. \tag{A.70}$$

Duality notations: $L.\vec{e}_j = \sum_{i=1}^n L^i{}_j \vec{e}_i$, $[L]_{|\vec{e}} = [L^i{}_j]$, $L^T_{gh}.\vec{e}_j = \sum_{i=1}^n (L^T_{gh})^i{}_j$, $[L^T_{gh}]_{|\vec{e}} = [(L^T_{gh})^i{}_j]$, and

$$\sum_{k=1}^{n} g_{ik}(L^T_{gh})^k{}_j = \sum_{k=1}^{n} L^k{}_i\, h_{kj}, \quad \text{i.e.} \quad (L^T_{gh})^i{}_j = \sum_{k,\ell=1}^{n} ([g]^{-1})_{ik} L^\ell{}_k\, h_{\ell j} \quad \Big(\overset{\text{noted}}{=} \sum_{k,\ell=1}^{n} (g^{ik} L^\ell{}_k\, h_{\ell j}\Big). \tag{A.71}$$

(Be careful with the notation $([g]^{-1})_{ik} =^{\text{noted}} g^{ij}$, see remark A.55.)

Exercice A.65 Prove: If L is invertible then $(L^T_{gh})^{-1} = (L^{-1})^T_{hg}$.

Answer. $(L^T_{gh}.(L^{-1})^T_{hg}.\vec{u}, \vec{w})_g = ((L^{-1})^T_{hg}.\vec{u}, L.\vec{w})_h = (\vec{u}, L^{-1}.L.\vec{w})_g = (\vec{u}, \vec{w})_g = (L^T_{gh}.(L^T_{gh})^{-1}.\vec{u}, \vec{w})_g$, true $\forall \vec{u}, \vec{w}$. ∎

A.12.3 Deformation gradient symmetric: Absurd

The symmetry of a linear map $L \in \mathcal{L}(E;F)$ is a nonsense if $E \neq F$.

E.g.: The gradient of deformation $F^{t_0}_t(p_{t_0}) = d\Phi^{t_0}_t(p_{t_0}) =^{\text{noted}} F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ cannot be symmetric since $F^T \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0})$. Idem for the first Piola–Kirchhoff tensor $\mathit{I\!\!K}^{t_0}_t$, which motivates the introduction of the symmetric second Piola–Kirchhoff tensor $\mathit{S\!\!K}^{t_0}_t$, see Marsden–Hughes [14] or § O.2.4.

A.12.4 Isometry

Definition A.66 A linear map $L \in \mathcal{L}(E;F)$ is an isometry relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ iff

$$\forall \vec{u}, \vec{w} \in E, \quad (L.\vec{u}, L.\vec{w})_h = (\vec{u}, \vec{w})_g, \quad \text{i.e.} \quad L^T_{gh} \circ L = I_E \text{ (identity in } E). \tag{A.72}$$

Thus, if $L \in \mathcal{L}(E;F)$ is an isometry and $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis, then $(L.\vec{e}_i)$ is a $(\cdot,\cdot)_h$-orthonormal basis, since $(L.\vec{e}_i, L.\vec{e}_j)_h = (\vec{e}_i, \vec{e}_j)_g = \delta_{ij}$ for all i, j.

In particular, an endomorphism $L \in \mathcal{L}(E;E)$ is a $(\cdot,\cdot)_g$-isometry iff

$$\forall \vec{u}, \vec{w} \in E, \quad (L.\vec{u}, L.\vec{w})_g = (\vec{u}, \vec{w})_g, \quad \text{i.e.} \quad L^T_g \circ L = I_E. \tag{A.73}$$

Exercice A.67 Let $\vec{f} : E \to F$. Prove:

$$\text{if, } \forall \vec{u}, \vec{w}, \ (\vec{f}(\vec{u}), \vec{f}(\vec{w}))_h = (\vec{u}, \vec{w})_g \ \text{ then } \vec{f} \text{ is linear.} \tag{A.74}$$

Answer. Let $(\vec{e}_i)$ be a $(\cdot,\cdot)_g$-orthonormal basis; Thus $(\vec{f}(\vec{e}_i))$ is a $(\cdot,\cdot)_h$-orthonormal basis (since $\vec{f}$ is an isometry). Thus, if $\vec{u} = \sum_{i=1}^n x_i\vec{e}_i$ then $\vec{f}(\vec{u}) \overset{b.o.n.}{=} \sum_{i=1}^{n} (\vec{f}(\vec{u}), \vec{f}(\vec{e}_i))_h \vec{f}(\vec{e}_i) \overset{hyp.}{=} \sum_{i=1}^{n} (\vec{u}, \vec{e}_i)_g \vec{f}(\vec{e}_i) \overset{b.o.n.}{=} \sum_{i=1}^{n} x_i \vec{f}(\vec{e}_i)$, thus $\vec{f}(\vec{u}+\lambda\vec{w}) = \sum_{i=1}^{n} (x_i + \lambda y_i)\vec{f}(\vec{e}_i) = \sum_{i=1}^{n} x_i\vec{f}(\vec{e}_i) + \lambda\sum_{i=1}^{n} y_i\vec{f}(\vec{e}_i) = \vec{f}(\vec{u}) + \lambda\vec{f}(\vec{w})$, thus $\vec{f}$ is linear. ∎

Exercice A.68 $\mathbb{R}^n$ is an affine space, $\vec{\mathbb{R}^n}$ is the usual associated vector space, and $(\cdot,\cdot)_g$ is an inner dot product in $\vec{\mathbb{R}^n}$. Definition: A distance-preserving function $f : p \in \mathbb{R}^n \to f(p) \in \mathbb{R}^n$ is a function s.t.

$$||\overrightarrow{f(p)f(q)}||_g = ||\overrightarrow{pq}||_g, \quad \forall p,q \in \mathbb{R}^n. \tag{A.75}$$

Prove: If f is a distance-preserving function, then f is affine.

Answer. Let $O \in \mathbb{R}^n$ (an origin) and $\vec{f} : \vec{x} = \overrightarrow{Op} \in \vec{\mathbb{R}^n} \to \vec{f}(\vec{x}) := \overrightarrow{f(O)f(p)}$ (vectorial associated function). Let $\vec{x} = \overrightarrow{Op}$ and $\vec{y} = \overrightarrow{Oq}$. Then the remarkable identity $2(\vec{f}(\vec{x}),\vec{f}(\vec{y}))_g = ||\vec{f}(\vec{x})||_g^2 + ||\vec{f}(\vec{y})||_g^2 - ||\vec{f}(\vec{x}) - \vec{f}(\vec{y})||_g^2$ gives $2(\vec{f}(\vec{x}),\vec{f}(\vec{y}))_g = ||\vec{f}(\vec{x})||_g^2 + ||\vec{f}(\vec{y})||_g^2 - ||\overrightarrow{f(q)f(p)}||_g^2 = ||\vec{f}(\vec{x})||_g^2 + ||\vec{f}(\vec{y})||_g^2 - ||\overrightarrow{qp}||_g^2 = ||\vec{x}||_g^2 + ||\vec{y}||_g^2 - ||\vec{x} - \vec{y}||_g^2 = 2(\vec{x},\vec{y})_g$, thus $\vec{f}$ is an isometry, thus $\vec{f}$ is linear cf. (A.74), thus f is affine since $f(p) = f(O) + \vec{f}(\overrightarrow{Op})$. ▪

A.13 The adjoint of a linear map (objective)

(May produce misunderstandings, misuses, problematic mechanical interpretations, if not understood.)

A linear map $L \in \mathcal{L}(E;F)$ has one and only one adjoint L^* (intrinsic to L); Must not be confused with the many transposed $L^T := L^T_{gh}$ which depend on inner dot products.

A.13.1 Definition

E and F are vector spaces, and $E^* = \mathcal{L}(E;\mathbb{R})$ and $F^* = \mathcal{L}(F;\mathbb{R})$ are the dual spaces (of linear forms).

Definition A.69 Let $L \in \mathcal{L}(E;F)$; Its adjoint is the linear map $L^* \in \mathcal{L}(F^*;E^*)$ canonically defined by

$$L^* : \begin{cases} F^* & \to E^* \\ m & \to L^*(m) := m \circ L, \quad \text{written} \quad L^*.m = m.L \end{cases} \tag{A.76}$$

thanks to the linearity of m, L and L^*, i.e., for all $(\vec{u},m) \in E \times F^*$,

$$L^*(m)(\vec{u}) := m(L(\vec{u})), \quad \text{written} \quad (L^*.m).\vec{u} = m.L.\vec{u} \tag{A.77}$$

thanks to the linearity of m, L and L^*.

(The linearity of L^* is trivial. And $||L^*.m||_{E^*} = ||m.L||_{E^*} \le ||m||_{F^*}||L||_{\mathcal{L}(E;F)}$ gives $||L^*||_{\mathcal{L}(F^*;E^*)} \le ||L||_{\mathcal{L}(E;F)} < \infty$, thus L^* is continuous when L is.)

A.13.2 Quantification

E and F are finite dimensional, $\dim E = n$, $\dim F = m$, and $(\vec{a}_i)$ and $(\vec{b}_i)$ are bases in E and F and (π_{ai}) and (π_{bi}) are the (covariant) dual bases. Let $[L]_{|\vec{a},\vec{b}} \overset{\text{noted}}{=} [L]$, $[L^*]_{|\pi_b,\pi_a} \overset{\text{noted}}{=} [L^*]$, $[m]_{|b} \overset{\text{noted}}{=} [m]$ and $[\vec{u}]_{|\vec{a}} \overset{\text{noted}}{=} [\vec{u}]$ (the matrices relative to the chosen bases). (A.77) gives

$$\forall (m,\vec{u}) \in F^* \times E, \;\; [L^*].[m].[\vec{u}] = [m].[L].[\vec{u}], \quad \text{thus} \quad \forall m \in F^*, \;\; [L^*].[m]^T = ([L]^T.[m]^T \tag{A.78}$$

(recall: $m \in F^*$, thus $[m]$ is a row matrix). Thus

$$\boxed{[L^*] = [L]^T} \quad \text{(transposed matrix)}. \tag{A.79}$$

(Full notation: $[L^*]_{|\pi_b,\pi_a} = ([L]_{|\vec{a},\vec{b}})^T$: There is no inner dot products here.)

Details: $L.\vec{a}_j = \sum_{i=1}^m L_{ij}\vec{b}_i$, i.e. $[L]_{|\vec{a},\vec{b}} = [L_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$, and $L^*.\pi_{bj} = \sum_{i=1}^n (L^*)_{ij}\pi_{ai}$, i.e. $[L^*]_{|\pi_b,\pi_a} = [(L^*)_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$, (A.77) gives, for all $(i,j) \in [1,n]_\mathbb{N} \times [1,m]_\mathbb{N}$,

$$(L^*.\pi_{bj}).\vec{a}_i = \pi_{bj}.(L.\vec{a}_i), \quad \text{thus} \quad \boxed{(L^*)_{ij} = L_{ji}} \quad \text{gives} \quad [L^*] = [L]^T. \tag{A.80}$$

Duality notations: $L.\vec{a}_j = \sum_{i=1}^m L^i{}_j\vec{b}_i$, i.e. $[L]_{|\vec{a},\vec{b}} = [L^i{}_j]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$, and $L^*.b^j = \sum_{i=1}^n (L^*)_i{}^j a^i$, i.e. $[L^*]_{|b,a} = [((L^*)_i{}^j]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$, thus, for all $(i,j) \in [1,n]_\mathbb{N} \times [1,m]_\mathbb{N}$,

$$(L^*.b^j).\vec{a}_i = b^j.(L.\vec{a}_i), \quad \text{thus} \quad (L^*)_i{}^j = L^j{}_i \quad \text{and} \quad [L^*] = [L]^T. \tag{A.81}$$

(Recall: If in doubt then don't use the duality notations! Use classical notations.)

NB: Reminder: The transposed b^T of a bilinear b form is intrinsic to b, and the adjoint L^* of a linear map L is intrinsic to L; But a transposed L^T of a linear form L is ***not*** intrinsic to the linear form (it depends on chosen inner dot products):

Watch out for the (unfortunate) vocabulary "transposed"!

A.13.3 Relation with the transposed when inner dot products are introduced

let $L \in \mathcal{L}(E;F)$. We need inner dot products $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ in E and F to define $L^T = L^T_{gh}$. To have a functional relation between L^* and L^T_{gh}, we use the $(\cdot,\cdot)_g$-Riesz representation mapping $\vec{R}_g$: $\left\{\begin{array}{ll} E^* & \to E \\ \ell & \to \vec{R}_g(\ell) = \vec{\ell}_g \end{array}\right\}$ defined by $\ell.\vec{u} = (\vec{\ell}_g, \vec{u})_g$ for all $\vec{u} \in E$, see (F.3); Idem in F.

Let $L \in \mathcal{L}(E;F)$ (continuous). For all $\vec{u} \in E$ and all $m \in F^*$ we have

$$(L^*.m).\vec{u} \overset{(A.77)}{=} m.(L.\vec{u}), \quad \text{thus} \quad (\vec{R}_g(L^*.m), \vec{u})_g = (\vec{R}_h(m), L.\vec{u})_h, \tag{A.82}$$

thus $((\vec{R}_g \circ L^*).m), \vec{u})_g = ((L^T_{gh} \circ \vec{R}_h).m, L.\vec{u})_g$. Thus $\vec{R}_g \circ L^* = L^T_{gh} \circ \vec{R}_h$, i.e.

$$\boxed{L^T_{gh} = \vec{R}_g \circ L^* \circ (\vec{R}_h)^{-1}} \quad \text{i.e.} \quad \begin{array}{ccc} E & \overset{L^T_{gh}}{\longleftarrow} & F \\ \vec{R}_g \uparrow & & \uparrow \vec{R}_h \\ E^* & \underset{L^*}{\longleftarrow} & F^* \end{array} \quad \text{is a commutative diagram.} \tag{A.83}$$

Exercice A.70 From (A.83), recover (A.69), i.e. $[L^T_{gh}] = [g]^{-1}.[L]^T.[h]$.

Answer. $[L^T_{gh}] =^{(A.83)} [\vec{R}_g].[L^*].[\vec{R}_h]^{-1} =^{(F.8)} [g]^{-1}.[L]^T.[h]$. ▪

A.14 Tensorial representation of a linear map (dangerous)

Consider the natural canonical isomorphism (between linear maps $E \to F$ and bilinear forms $F^* \times E \to \mathbb{R}$)

$$\widetilde{\mathcal{J}} : \left\{\begin{array}{ll} \mathcal{L}(E;F) & \to \mathcal{L}(F^*, E; \mathbb{R}) \\ L & \to \beta_L = \widetilde{\mathcal{J}}(L) \end{array}\right\} \quad \text{where} \quad \beta_L(m, \vec{u}) := m.(L.\vec{u}), \quad \forall (m, \vec{u}) \in F^* \times E, \tag{A.84}$$

see § U.4. And β_L is also named L for calculations purposes, see (A.87).

Quantification: $(\vec{a}_i)_{i=1,\ldots,n}$ is a basis in E, $(\vec{b}_i)_{i=1,\ldots,m}$ is a basis in F which dual basis is (π_{bi}), $L \in \mathcal{L}(E;F)$. Then (A.84) gives

$$\beta_L(\pi_{bi}, \vec{a}_i) = \pi_{bi}.L.\vec{a}_i. \tag{A.85}$$

Thus, if $L.\vec{a}_j = \sum_{i=1}^m L_{ij} \vec{b}_i$, i.e. $[L]_{\vec{a},\vec{b}} = [L_{ij}]$, then

$$\beta_L = \sum_{i=1}^m \sum_{j=1}^n L_{ij} \vec{b}_i \otimes \pi_{aj}, \quad \text{i.e.} \quad [\beta_L]_{\vec{b},\pi_a} = [L]_{\vec{a},\vec{b}} \overset{\text{noted}}{=} [\beta_L]_{\vec{a},\vec{b}}. \tag{A.86}$$

Indeed, $(\sum_{ij} L_{ij} \vec{b}_i \otimes \pi_{aj})(\pi_{bk}, \vec{a}_\ell) = \sum_{ij} L_{ij} (\vec{b}_i \otimes \pi_{aj})(\pi_{bk}, \vec{a}_\ell) = \sum_{ij} L_{ij} (\vec{b}_i.\pi_{bk})(\pi_{aj}.\vec{a}_\ell) = \sum_{ij} L_{ij} \delta_{ki} \delta_{j\ell} = L_{k\ell} = \pi_{bk}.L.\vec{a}_\ell$, so (A.85) gives (A.86).

Duality notations: $L.\vec{a}_j = \sum_{i=1}^m L^i{}_j \vec{b}_i$ and $\beta_L = \sum_{i=1}^m \sum_{j=1}^n L^i{}_j \vec{b}_i \otimes a^j$.

Contraction rule. If you write $L =^{\text{noted}} \beta_L = \sum_{i=1}^m \sum_{j=1}^n L_{ij} \vec{b}_i \otimes \pi_{aj}$, then the vector $L.\vec{u} \in F$ is computed thanks to the "contraction rule":

$$L.\vec{u} = \beta_L.\vec{u} = (\sum_{i=1}^m \sum_{j=1}^n L_{ij} \vec{b}_i \otimes \underbrace{\pi_{aj}).\vec{u}}_{\text{contraction}} := \sum_{i=1}^m \sum_{j=1}^n L_{ij} \vec{b}_i (\pi_{aj}.\vec{u}) = \sum_{i=1}^m \sum_{j=1}^n L_{ij} u_j \vec{b}_i, \tag{A.87}$$

which is the expected result.

Duality notations: $L.\vec{u} = (\sum_{i=1}^m \sum_{j=1}^n L^i{}_j \vec{b}_i \otimes \underbrace{a^j).\vec{u}}_{\text{contraction}} = \sum_{i=1}^m \sum_{j=1}^n L^i{}_j \vec{b}_i (a^j.\vec{u}) = \sum_{i=1}^m \sum_{j=1}^n L^i{}_j u^j \vec{b}_i.$

Remark A.71 Warning: The bilinear form β_L should not be confused with the linear map L: The domain of definition of β_L is $F^* \times E$, and β_L acts on the two objects ℓ (linear form) and $\vec{u}$ (vector) to get a ***scalar*** result; While the domain of definition of L is E, and L acts one object $\vec{u}$ to get a ***vector*** result. You can use the tensorial notation for L... only to calculate $L.\vec{u}$ as in (A.87) (contraction rule). ▪

A.15 Change of basis formulas for bilinear forms and linear maps

A.15.1 Notations

Let A and B be finite dimension vector spaces, $\dim A = n$, $\dim B = m$. (E.g. application to the change of basis formula for the deformation gradient $A{=}\vec{\mathbb{R}}^n_{t_0} \to B{=}\vec{\mathbb{R}}^n_t$.)

Let $(\vec{a}_{\text{old},i})$ and $(\vec{a}_{\text{new},i})$ be two bases in A, and $(\vec{b}_{\text{old},i})$ and $(\vec{b}_{\text{new},i})$ be two bases in B. Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the change of basis endomorphisms from old to new bases, and $P_A := [\mathcal{P}_A]_{|\vec{a}_{\text{old}}} = [P_{Aij}]$ and $P_B := [\mathcal{P}_B]_{|\vec{b}_{\text{old}}} = [P_{Bij}]$ be the associated transition matrices, and $Q_A = P_A^{-1}$ and $Q_B = P_B^{-1}$. So:

$$\begin{aligned} \vec{a}_{\text{new},j} &= \mathcal{P}_A.\vec{a}_{\text{old},i} = \sum_{i,j=1}^n P_{Aij}\vec{a}_{\text{old},i}, \quad \pi_{anew,j} = \sum_{i=1}^n Q_{Aij}\pi_{aold,i}, \\ \vec{b}_{\text{new},j} &= \mathcal{P}_B.\vec{b}_{\text{old},i} = \sum_{i,j=1}^m P_{Bij}\vec{b}_{\text{old},i}, \quad \pi_{bnew,j} = \sum_{i,j=1}^n Q_{Bij}\pi_{bold,i}. \end{aligned} \tag{A.88}$$

Dual not.: $\vec{a}_{\text{new},j} = \sum_{i=1}^n P_A{}^i{}_j\vec{a}_{\text{old},i}$, $a^i_{\text{new}} = \sum_{j=1}^n Q_A{}^i{}_j a^j_{\text{old}}$, $\vec{b}_{\text{new},j} = \sum_{i=1}^n P_B{}^i{}_j\vec{b}_{\text{old},i}$, $b^i_{\text{new}} = \sum_{j=1}^n Q_B{}^i{}_j b^j_{\text{old}}$.

A.15.2 Change of coordinate system for bilinear forms $\in \mathcal{L}(A,B;\mathbb{R})$

Let $g \in \mathcal{L}(A,B;\mathbb{R})$, and, for all $(i,j) \in [1,n]_{\mathbb{N}} \times [1,m]_{\mathbb{N}}$,

$$g(\vec{a}_{\text{old},i},\vec{b}_{\text{old},j}) = M_{ij}, \quad g(\vec{a}_{\text{new},i},\vec{b}_{\text{new},j}) = N_{ij}, \quad \text{i.e.} \quad \begin{cases} [g]_{|olds} = M = [M_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \\ [g]_{|news} = N = [N_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}. \end{cases} \tag{A.89}$$

Proposition A.72 *Change of basis formula:*

$$\boxed{[g]_{|news} = P_A{}^T.[g]_{|olds}.P_B}, \quad \text{i.e.} \quad N = P_A{}^T.M.P_B. \tag{A.90}$$

In particular, if $A = B$ and $(\vec{a}_{\text{old},i}) = (\vec{b}_{\text{old},i})$ and $(\vec{a}_{\text{new},i}) = (\vec{b}_{\text{new},i})$, then $P_A = P_B =^{noted} P$, and

$$\boxed{[g]_{|new} = P^T.[g]_{old}.P}, \quad \text{i.e.} \quad N = P^T.M.P. \tag{A.91}$$

Proof. $N_{ij} = g(\vec{a}_{\text{new},i},\vec{b}_{\text{new},j}) = \sum_{k\ell} P_A{}^k{}_i P_B{}^\ell{}_j g(\vec{a}_{\text{old},k},\vec{b}_{\text{old},\ell}) = \sum_{k\ell} P_A{}^k{}_i M_{k\ell} P_B{}^\ell{}_j = \sum_{k\ell}(P_A{}^T)^i{}_k M_{k\ell} P_B{}^\ell{}_j$. ∴

Exercice A.73 Prove (objective result):

$$g(\vec{u},\vec{w}) = [\vec{u}]^T_{|\vec{a}_{\text{new}}}.[g]_{|news}.[\vec{w}]_{|\vec{b}_{\text{new}}} = [\vec{u}]^T_{|\vec{a}_{\text{old}}}.[g]_{|olds}.[\vec{w}]_{|\vec{b}_{\text{old}}}. \tag{A.92}$$

Answer. $[\vec{u}]^T_{|\vec{a}_{\text{new}}}.[g]_{|news}.[\vec{w}]_{|\vec{b}_{\text{new}}} = (P_A{}^{-1}.[\vec{u}]_{|\vec{a}_{\text{old}}})^T.(P_A{}^T.[g]_{|olds}.P_B).(P_B{}^{-1}.[\vec{w}]_{|\vec{b}_{\text{old}}})$. ∴

A.15.3 Change of coordinate system for bilinear forms $\in \mathcal{L}(A^*,B^*;\mathbb{R})$

Let $z \in \mathcal{L}(A^*,B^*;\mathbb{R})$, and, for all $(i,j) \in [1,n]_{\mathbb{N}} \times [1,m]_{\mathbb{N}}$,

$$z(a^i_{\text{old}},b^j_{\text{old}}) = M^{ij}, \quad z(a^i_{\text{new}},b^j_{\text{new}}) = N^{ij}, \quad \text{i.e.} \quad \begin{cases} [z]_{|olds} = M = [M^{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \\ [z]_{|news} = N = [N^{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}. \end{cases} \tag{A.93}$$

Proposition A.74 *Change of basis formula:*

$$[z]_{|news} = P_A{}^{-T}.[z]_{|olds}.P_B{}^{-1}, \quad \text{i.e.} \quad N = P_A{}^{-T}.M.P_B{}^{-1}. \tag{A.94}$$

In particular, if $A = B$ and $(\vec{a}_{\text{old},i}) = (\vec{b}_{\text{old},i})$ and $(\vec{a}_{\text{new},i}) = (\vec{b}_{\text{new},i})$, then $P_A = P_B =^{noted} P$, and

$$[z]_{|new} = P^{-T}.[z]_{old}.P^{-1}, \quad \text{i.e.} \quad N = P^{-T}.M.P^{-1}. \tag{A.95}$$

Proof. $N_{ij} = z(a^i_{\text{new}},b^j_{\text{new}}) = \sum_{k\ell} Q_A{}^k{}_i Q_B{}^\ell_j z(a^k_{\text{old}},b^\ell_{\text{old}}) = \sum_{k\ell} Q_A{}^k{}_i M^{k\ell} Q_B{}^\ell{}_j = \sum_{k\ell}(Q_A{}^T)^i{}_k M^{k\ell} Q_B{}^\ell{}_j$. ∴

A.15.4 Change of coordinate system for bilinear forms $\in \mathcal{L}(B^*, A; \mathbb{R})$

(Toward linear maps $L \in \mathcal{L}(A;B) \simeq \mathcal{L}(B^*, A; \mathbb{R})$ thanks to the natural canonical isomorphism.)

Let $T \in \mathcal{L}(B^*, A; \mathbb{R})$, and, for all $(i,j) \in [1,n]_{\mathbb{N}} \times [1,m]_{\mathbb{N}}$,

$$T(b^i_{\text{old}}, \vec{a}_{\text{old},j}) = M^i{}_j, \quad T(b^i_{\text{new}}, \vec{a}_{\text{new},j}) = N^i{}_j, \quad \text{i.e.} \quad \begin{cases} [T]_{|olds} = M = [M^i{}_j]_{\substack{i=1,...,n \\ j=1,...,m}}, \\ [T]_{|news} = N = [N^i{}_j]_{\substack{i=1,...,n \\ j=1,...,m}}. \end{cases} \tag{A.96}$$

Proposition A.75 *Change of basis formula:*

$$\boxed{[T]_{|news} = P_B^{-1}.[T]_{|olds}.P_A}, \quad \text{i.e.} \quad N = Q_A.M.P_B. \tag{A.97}$$

In particular, if $A = B$ and $(\vec{a}_{\text{old},i}) = (\vec{b}_{\text{old},i})$ and $(\vec{a}_{\text{new},i}) = (\vec{b}_{\text{new},i})$, then $P_A = P_B =^{noted} P$, and

$$\boxed{[T]_{|new} = P^{-1}.[T]_{old}.P}, \quad \text{i.e.} \quad N = P^{-1}.M.P. \tag{A.98}$$

Proof. $N^i{}_j = T(b^i_{\text{new}}, \vec{a}_{\text{new},j}) = \sum_{k\ell} Q_B{}^i{}_k P_A{}^\ell{}_j T(b^i_{\text{old}}, \vec{a}_{\text{old},j}) = \sum_{k\ell} Q_B{}^i{}_k M^i{}_j P_A{}^\ell{}_j$ ∎

A.15.5 Change of coordinate system for tri-linear forms $\in \mathcal{L}(A^*, A, A; \mathbb{R})$

(Toward $d^2\vec{u}$: For a vector field $\vec{u} \in \Gamma(U) \simeq T^1_0(U)$, $\vec{u}(p) \in \vec{\mathbb{R}^n}$, its differential satisfies $d\vec{u}(p) \in \mathcal{L}(\vec{\mathbb{R}^n}; \vec{\mathbb{R}^n}) \simeq \mathcal{L}(\mathbb{R}^{n*}, \vec{\mathbb{R}^n}; \mathbb{R})$, and $d^2\vec{u}(p) \in \mathcal{L}(\vec{\mathbb{R}^n}; \mathcal{L}(\vec{\mathbb{R}^n}; \vec{\mathbb{R}^n})) \simeq \mathcal{L}(\mathbb{R}^{n*}, \vec{\mathbb{R}^n}, \vec{\mathbb{R}^n}; \mathbb{R})$, see § T.1.3.)

Consider a tri-linear form $T \in \mathcal{L}(A^*, A, A; \mathbb{R})$, and $[T]_{|\vec{a}_{\text{old}}} = [M^i_{jk}]$ and $[T]_{|\vec{a}_{\text{new}}} = [N^i_{jk}]$, so where

$$M^i_{jk} = T(a^i_{\text{old}}, \vec{a}_{\text{old},j}, \vec{a}_{\text{old},k}), \quad N^i_{jk} = T(a^i_{\text{new}}, \vec{a}_{\text{new},j}, \vec{a}_{\text{new},k}). \tag{A.99}$$

Then

$$N^i_{jk} = \sum_{\lambda,\mu,\nu=1}^{n} Q^i_\lambda P^\mu_j P^\nu_k M^\lambda_{\mu\nu}. \tag{A.100}$$

Indeed $\sum_{\lambda\mu\nu} M^\lambda_{\mu\nu} \vec{a}_{old,\lambda} \otimes a^\mu_{old} \otimes a^\nu_{old} = \sum_{\lambda\mu\nu ijk} M^\lambda_{\mu\nu} Q^i_\lambda P^\mu_j P^\nu_k \vec{a}_{\text{new},i} \otimes a^j_{\text{new}} \otimes a^k_{\text{new}}$. ∎

A.15.6 Change of coordinate system for linear maps $\in \mathcal{L}(A;B)$

Notation of § A.15.1. Let $L \in \mathcal{L}(A;B)$ be a linear map, and let, for all $j = 1, ..., n$,

$$\begin{cases} L.\vec{a}_{\text{old},j} = \sum_{i=1}^{m} M_{ij} \vec{b}_{\text{old},i} = \sum_{i=1}^{m} M^i{}_j \vec{b}_{\text{old},i} \quad \text{i.e.} \quad [L]_{|olds} = M = [M_{ij}] = [M^i{}_j]_{\substack{i=1,...,m \\ j=1,...,n}}, \\ L.\vec{a}_{\text{new},j} = \sum_{i=1}^{m} N_{ij} \vec{b}_{\text{new},i} = \sum_{i=1}^{m} N^i{}_j \vec{b}_{\text{new},i} \quad \text{i.e.} \quad [L]_{|news} = N = [N_{ij}] = [N^i{}_j]_{\substack{i=1,...,m \\ j=1,...,n}}, \end{cases} \tag{A.101}$$

with classical and duality notations.

Proposition A.76 *Change of bases formula:*

$$\boxed{[L]_{|news} = P_B^{-1}.[L]_{|olds}.P_A}, \quad \text{i.e.} \quad N = P_B^{-1}.M.P_A. \tag{A.102}$$

Particular case L endomorphism: $A = B$, $(\vec{a}_{\text{old},i}) = (\vec{b}_{\text{old},i})$, $(\vec{a}_{\text{new},i}) = (\vec{b}_{\text{new},i})$, $P_A = P_B =^{noted} P$ and

$$\boxed{[L]_{|new} = P^{-1}.[L]_{|old}.P}, \quad \text{i.e.} \quad N = P^{-1}.M.P. \tag{A.103}$$

Proof. $L.\vec{a}_{\text{new},j} = \sum_i N^i{}_j \vec{b}_{\text{new},i} = \sum_{ik} N^i{}_j P_B{}^k{}_i \vec{b}_{\text{old},k} = \sum_k (P_B.N)^k{}_j \vec{b}_{\text{old},k}$ and $L.\vec{a}_{\text{new},j} = L.(\sum_i P_A{}^i{}_j \vec{a}_{\text{old},i}) = \sum_i P_A{}^i{}_j \sum_k M^k{}_i \vec{b}_{\text{old},k} = \sum_k (M.P_A)^k{}_j \vec{b}_{\text{old},k}$, for all j, thus $P_B.N = M.P_A$. ∎

Exercice A.77 Prove:

$$\ell.L.\vec{u} = [\ell]_{|\vec{b}_{new}}.[L]_{|news}.[\vec{u}]_{|\vec{a}_{new}} = [\ell]_{|\vec{b}_{old}}.[L]_{|olds}.[\vec{u}]_{|\vec{a}_{old}} \quad \text{(objective result).} \tag{A.104}$$

Answer. $[\ell]_{|\vec{b}_{new}}.[L]_{|news}.[\vec{u}]_{|\vec{a}_{new}} = ([\ell]_{|\vec{b}_{old}}.P_B).(P_B^{-1}.[L]_{|olds}.P_A).(P_A^{-1}.[\vec{u}]_{|\vec{a}_{old}})$. ▪

Remark A.78 Bilinear forms in $\mathcal{L}(A,A;\mathbb{R})$ and endomorphisms in $\mathcal{L}(A;A)$ behave differently: The formulas (A.91) and (A.103) should not be confused since $P^{-1} \neq P^T$ in general. E.g., if an English observer uses a Euclidean (old) basis $(\vec{a}_i) = (\vec{a}_{old,i})$ in foot, if a French observer uses a Euclidean (new) basis $(\vec{b}_i) = (\vec{a}_{new,i})$ in metre, and if (simple case) $\vec{b}_i = \lambda\vec{a}_i$ for all i (change of unit), then

$$[L]_{|new} = [L]_{|old}, \quad \text{while} \quad [g]_{|new} = \underbrace{\lambda^2}_{>10}[g]_{|old}. \tag{A.105}$$

Quite different results! I.e. $P^{-1}.[L]_{|old}.P \neq P^T.[L]_{|old}.P$ for a general change of basis. See the Mars Climate Orbiter crash, remark A.17, where someone forgot that 1 foot $\neq$ 1 metre. ▪

B Euclidean Frameworks

Time and space are decoupled (classical mechanics). $\mathbb{R}^n$ is the geometric affine space, $n = 1,2,3$, and $\vec{\mathbb{R}^n}$ is the associated usual vector space made of "bi-point vectors".

B.1 Euclidean basis

Manufacturing of a Euclidean basis.

An observer chooses a unit of measure (foot, metre, a unit of length used by Euclid, the diameter a of pipe...) and makes a "unit rod" of length 1 in this unit.

Postulate: The length of the rod does not depend on its direction in space.

• Space dimension $n = 1$: This rod models a vector $\vec{e}_1$ which makes a basis $(\vec{e}_1)$ called the Euclidean basis relative to the chosen unit of measure.

• Space dimension $n \geq 2$:
- The observers makes three rods of length 3, 4 and 5, to build a triangle (A,B,C) with A, B and C are the vertices and A not on the side on length 5.
- Pythagoras: $3^2 + 4^2 = 5^2$ gives: The triangle (A,B,C) is said to have a right angle at A.
- Two vectors $\vec{u}$ and $\vec{w}$ in $\vec{\mathbb{R}^n}$ are orthogonal iff the triangle (A,B,C) can be positioned such that $\vec{AB}$ and $\vec{AC}$ are parallel to $\vec{u}$ and $\vec{w}$.
- A basis $(\vec{e}_i)_{i=1,\dots,n}$ is Euclidean relative to the chosen unit of measurement iff the $\vec{e}_i$ are two to two orthogonal and their length is 1 (relative to the chosen unit).

Example B.1 An English observer defines a Euclidean basis $(\vec{a}_i)$ using the foot. A French observer defines a Euclidean basis $(\vec{b}_i)$ using the metre. We have

$$1\,\text{foot} = \mu\,\text{metre}, \quad \mu = 0.3048, \quad \text{and} \quad 1\,\text{metre} = \lambda\,\text{foot}, \quad \lambda = \frac{1}{\mu} \simeq 3.28. \tag{B.1}$$

($\mu = 0,3048$ is the official length in metre for the English foot.) E.g., the bases are "aligned" iff, for all i,

$$\vec{b}_i = \lambda\vec{a}_i \quad \text{(change of measurement unit),} \tag{B.2}$$

thus the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$ is $P = \lambda I$, thus $P^T = P$, $P^{-1} = \frac{1}{\lambda}I$ and $P^T.P = \lambda^2 I$. ▪

Remark B.2 The bases used in practice are not all Euclidean. E.g., see example A.16 if you fly. ▪

B.2 Euclidean dot product

Definition B.3 An observer has built his Euclidean basis $(\vec{e}_i)$. The associated Euclidean dot product is the bilinear form $g(\cdot,\cdot) = (\cdot,\cdot)_g \in \mathcal{L}(\vec{\mathbb{R}^n}, \vec{\mathbb{R}^n}; \mathbb{R})$ defined by

$$g(\vec{e}_i, \vec{e}_j) \overset{\text{noted}}{=} g_{ij} = \delta_{ij}, \quad \forall i,j, \quad \text{i.e.} \quad [g]_{|\vec{e}} = I. \tag{B.3}$$

I.e., with $(\pi_{ei}) = (e^i)$ the dual basis of $(\vec{e}_i)$ (with classical and duality notations),

$$(\cdot,\cdot)_g := \sum_{i=1}^n \pi_{ei} \otimes \pi_{ei} = \sum_{i=1}^n e^i \otimes e^i. \tag{B.4}$$

With Einstein's convention, $(\cdot,\cdot)_g := \sum_{i,j=1}^n g_{ij} e^i \otimes e^j$: You have to write g_{ij} (although $= \delta_{ij}$ here). E.g. with the repeated index convention: $(\cdot,\cdot)_g := g_{ij} e^i \otimes e^j$.

Thus, for all $\vec{x}, \vec{y} \in \vec{\mathbb{R}^n}$, with $\vec{x} = \sum_{i=1}^n x_i \vec{e}_i$ and $\vec{y} = \sum_{i=1}^n y_i \vec{e}_i$ (classical notations),

$$(\vec{x}, \vec{y})_g = \sum_{i=1}^n x_i y_i = [\vec{x}]^T_{|\vec{e}} . [\vec{y}]_{|\vec{e}} \overset{\text{noted}}{=} \vec{x} \bullet_g \vec{y}. \tag{B.5}$$

Duality notations: $\vec{x} = \sum_{i=1}^n x^i \vec{e}_i$, $\vec{y} = \sum_{i=1}^n y^i \vec{e}_i$ and $(\vec{x}, \vec{y})_g = \sum_{i=1}^n x^i y^i$.
With Einstein's convention: $(\vec{x}, \vec{y})_g := \sum_{i,j=1}^n g_{ij} x^i y^j$.

Definition B.4 The associated norm is $||.||_g := \sqrt{(\cdot,\cdot)_g}$, and the length of a vector $\vec{x}$ relative to the chosen Euclidean unit of measurement is $||\vec{x}||_g := \sqrt{(\vec{x}, \vec{x})_g} = \sqrt{\vec{x} \bullet_g \vec{x}}$.

Thus with a Euclidean basis $(\vec{e}_i)$ used to build $(\cdot,\cdot)_g$, if $\vec{x} = \sum_{i=1}^n x_i \vec{e}_i$, then $||\vec{x}||_g = \sqrt{\sum_{i=1}^n x_i^2}$ is the length of $\vec{x}$ relative to the chosen Euclidean unit of measure (Pythagoras).

Duality notations: $||\vec{x}||_g = \sqrt{\sum_{i=1}^n (x^i)^2}$. Einstein convention: $||\vec{x}||_g = \sqrt{\sum_{i,j=1}^n g_{ij} x^i x^j}$.

Definition B.5 The angle $\theta(\vec{x}, \vec{y})$ between two vectors $\vec{x}, \vec{y} \in \vec{\mathbb{R}^n} - \{\vec{0}\}$ is defined by

$$\cos(\theta(\vec{x}, \vec{y})) = (\frac{\vec{x}}{||\vec{x}||_g}, \frac{\vec{y}}{||\vec{y}||_g})_g. \tag{B.6}$$

(With a computer, this formula gives $\theta(\vec{x}, \vec{y}) = \arccos((\frac{\vec{x}}{||\vec{x}||_g}, \frac{\vec{y}}{||\vec{y}||_g})_g)$ in $[0, \pi]$.)

B.3 Two Euclidean dot products are proportional

Consider two Euclidean bases in $\vec{\mathbb{R}^n}$: $(\vec{a}_i)$, e.g. built with the foot, and $(\vec{b}_i)$, e.g. built with the metre; And let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be the associated Euclidean dot products.

Proposition B.6 *If* $\lambda = ||\vec{b}_1||_g$, *then* $||\vec{b}_i||_g = \lambda$ *for all* $i = 1, ..., n$ *and*

$$(\cdot,\cdot)_g = \lambda^2 (\cdot,\cdot)_h, \quad \textit{and} \quad ||.||_g = \lambda ||.||_h. \tag{B.7}$$

Proof. By definition of a Euclidean basis, the length of the rod that enabled to define $(\vec{b}_i)$ is independent of i, cf. § B.1, thus $||\vec{b}_i||_g = ||\vec{b}_1||_g$ for all i, and here $||\vec{b}_i||_g =^{\text{noted}} \lambda$. Thus $||\vec{b}_i||_g^2 = \lambda^2 = \lambda^2 ||\vec{b}_i||_h^2$ for all i, since $||\vec{b}_i||_h^2 = 1$. And if $i \neq j$ then $(\vec{b}_i, \vec{b}_j)_g = 0 = (\vec{b}_i, \vec{b}_j)_h$ since $\vec{b}_i$ and $\vec{b}_j$ form a right angle (Pythagoras), cf. (B.4). Hence $(\vec{b}_i, \vec{b}_j)_g = \lambda^2 (\vec{b}_i, \vec{b}_j)_h$ for all i, j, thus $(\vec{x}, \vec{y})_g = \lambda^2 (\vec{x}, \vec{y})_h$ for all $\vec{x}, \vec{y}$ (bilinearity of inner dot products, thus (B.7). ∎

Example B.7 Continuation of example B.1: $(\cdot,\cdot)_a = \sum_{i=1}^n a^i \otimes a^i$ is the English Euclidean dot product (foot), and $(\cdot,\cdot)_b = \sum_{i=1}^n b^i \otimes b^i$ is the French Euclidean dot product (metre). (B.7) and (B.1) give:

$$(\cdot,\cdot)_a = \lambda^2 (\cdot,\cdot)_b \quad \text{and} \quad ||.||_a = \lambda ||.||_b, \quad \text{with} \quad \lambda \simeq 3.28 \quad \text{and} \quad \lambda^2 \simeq 10.76. \tag{B.8}$$

In particular, if $\vec{w}$ is s.t. $||\vec{w}||_b = 1$ (its length is 1 metre), then $||\vec{w}||_a = \lambda$ (its length is $\lambda \simeq 3.28$ foot). ∎

B.4 Counterexample: Non existence of a Euclidean dot product)

1- Thermodynamic: Let T be the temperature and P the pressure, and consider the Cartesian vector space $\{(T,P)\} = \{(\text{temperature,pressure})\} = \mathbb{R}\times\mathbb{R}$. There is no associated Euclidean dot product: An associated norm would give $||(T,P)|| = \sqrt{T^2+P^2} \in \mathbb{R}$ which is meaningless (incompatible dimensions). See § A.3.5.

2- Polar coordinate system $\vec{q} = (r,\theta) \in \mathbb{R}\times\mathbb{R}$: There is no Euclidean norm $\sqrt{r^2+\theta^2}$ for $\vec{q}$ that is physically meaningful (incompatible dimensions), see example 6.11.

B.5 Euclidean transposed of a deformation gradient

Let $n\in\{1,2,3\}$ and consider a linear map $L\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ (e.g., $L = d\Phi^{t_0}_t(P) = F^{t_0}_t(P)$).

Let $(\cdot,\cdot)_G$ be a Euclidean dot product in $\vec{\mathbb{R}}^n_{t_0}$ (used in the past by someone), and let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be Euclidean dot products in $\vec{\mathbb{R}}^n_t$ (the actual space where the results are obtained by two observers, e.g., $(\cdot,\cdot)_g$ built with a foot and $(\cdot,\cdot)_h$ built with a metre). Let L^T_{Gg} and L^T_{Gh} be the transposed of L relative to the dot products: L^T_{Gg} and L^T_{Gh} in $\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0})$ are characterized by, cf. (A.66), for all $(\vec{X},\vec{y})\in\vec{\mathbb{R}}^n_{t_0}\times\vec{\mathbb{R}}^n_t$,

$$(L^T_{Gg}.\vec{y},\vec{X})_G = (L.\vec{X},\vec{y})_g \quad\text{and}\quad (L^T_{Gh}.\vec{y},\vec{X})_G = (L.\vec{X},\vec{y})_h. \tag{B.9}$$

Corollary B.8

$$\textit{If}\quad (\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h \quad\textit{then}\quad L^T_{Gg} = \lambda^2 L^T_{Gh}. \tag{B.10}$$

NB: Do not forget λ^2 (e.g. $\lambda^2\simeq 10$ if an English man works with a French man).

Proof. $(L^T_{Gg}.\vec{y},\vec{X})_G \overset{(B.9)}{=} (L.\vec{X},\vec{y})_g \overset{hyp.}{=} \lambda^2(L.\vec{X},\vec{y})_h \overset{(B.9)}{=} \lambda^2(L^T_{Gh}.\vec{y},\vec{X})_G$ for all $\vec{X}\in\vec{\mathbb{R}}^n_{t_0}$ and all $\vec{y}\in\vec{\mathbb{R}}^n_t$, thus $L^T_{Gg}.\vec{y} = \lambda^2 L^T_{Gh}.\vec{y}$ for all $\vec{y}\in\vec{\mathbb{R}}^n_t$, thus $L^T_{Gg} = \lambda^2 L^T_{Gh}$. ∴

B.6 The Euclidean transposed for endomorphisms

Let $n\in\{1,2,3\}$ and consider an endomorphism $L\in\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$; E.g. $L = d\vec{v}_t(p)$ the differential of the Eulerian velocity. Let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be dot products in $\vec{\mathbb{R}^n}$. Let L^T_g and L^T_h be the transposed of L relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$: L^T_g and L^T_h in $\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ are characterized by, cf. (A.52), for all $\vec{x},\vec{y}\in\vec{\mathbb{R}}^n_t$,

$$(L^T_g.\vec{y},\vec{x})_g = (L.\vec{x},\vec{y})_g, \quad\text{and}\quad (L^T_h.\vec{y},\vec{x})_h = (L.\vec{x},\vec{y})_h. \tag{B.11}$$

Corollary B.9

$$\textit{If}\quad (\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h \quad\textit{then}\quad L^T_g = L^T_h \overset{\text{noted}}{=} L^T \tag{B.12}$$

(an endomorphism type relation). Thus we can speak of "the Euclidean transposed of an endomorphism".

Proof. $(L^T_g.\vec{y},\vec{x})_g \overset{(B.11)}{=} (L.\vec{x},\vec{y})_g \overset{hyp.}{=} \lambda^2(L.\vec{x},\vec{y})_h \overset{(B.11)}{=} \lambda^2(L^T_h.\vec{y},\vec{x})_h \overset{hyp}{=} (L^T_h.\vec{y},\vec{x})_g$ for all $\vec{x},\vec{y}\in\vec{\mathbb{R}^n}$, thus $L^T_g.\vec{y} = L^T_h.\vec{y}$ for all $\vec{y}\in\vec{\mathbb{R}^n}$, thus $L^T_g = L^T_h$. ∴

B.7 Unit normal vector, unit normal form

The results in this § are not objective: We need a Euclidean dot product (need a unit of length: Foot? Meter?) to get Euclidean orthogonality and a unit normal vector.

Framework: $n = 2$ or 3, $(\cdot,\cdot)_g$ is a Euclidean dot product in $\vec{\mathbb{R}^n}$ and, for all $\vec{u},\vec{w}\in\vec{\mathbb{R}^n}$,

$$(\vec{u},\vec{w})_g \overset{\text{noted}}{=} \vec{u}\bullet_g\vec{w} \tag{B.13}$$

(or $=^{\text{noted}}$ $\vec{u}\bullet\vec{w}$ when one chosen Euclidean dot product is imposed to all).

Ω is a regular open bounded set in $\mathbb{R}^n$, and $\Gamma := \partial\Omega$ is its regular surface. If $p\in\Gamma$ then $T_p\Gamma$ is the tangent plane at p to Γ. Let $(\vec{\beta}_1(p),...,\vec{\beta}_{n-1}(p))$ be a basis in $T_p\Gamma$ (e.g. obtained thanks to a coordinate system describing Γ).

B.7.1 Unit normal vector

Call $\vec{n}_g(p)$ the unit outward normal vector at $p \in \Gamma$ at $T_p\Gamma$ relative to $(\cdot,\cdot)_g$; So $\vec{n}_g(p) \bullet_g \vec{\beta}_i(p) = 0$ for all $i = 1,...,n{-}1$, and $||\vec{n}_g(p)||_g = 1$, i.e. $\vec{n}_g$ is defined on Γ by (up to its sign)

$$\forall i = 1,...,n{-}1,\quad \vec{\beta}_i \bullet_g \vec{n}_g = 0, \quad \text{and} \quad \vec{n}_g \bullet_g \vec{n}_g = 1 \quad (= ||\vec{n}_g||_g^2), \tag{B.14}$$

i.e., at any $p \in \Gamma$, $\vec{n}_g(p)$ is orthogonal to the hyperplane $\mathrm{Vect}\{\vec{\beta}_1(p),...,\vec{\beta}_{n-1}(p)\}$ and $\vec{n}_g(p)$ is unitary.

So $(\vec{\beta}_1(p),...,\vec{\beta}_{n-1}(p),\vec{n}_g(p))$ is a basis at p in $\vec{\mathbb{R}^n}$, written in short $(\vec{\beta}_1,...,\vec{\beta}_{n-1},\vec{n}_g)$. Drawing.

Thus, if $\vec{w} \in \vec{\mathbb{R}^n}$ is a vector at p, $\vec{w} = \sum_{i=1}^{n-1} w_i\vec{\beta}_i + w_n\vec{n}_g$ (classical notations) then

$$w_n = \vec{w} \bullet_g \vec{n}_g = \text{ the normal component of } \vec{w} \text{ at } p \text{ at } \Gamma. \tag{B.15}$$

(w_n depends on $(\cdot,\cdot)_g$.) (Duality notations: $\vec{w} = \sum_{i=1}^{n-1} w^i\vec{\beta}_i + w^n\vec{n}_g$ and $w^n = (\vec{w},\vec{n}_g)_g$.)

Exercice B.10 Let $(\vec{a}_i)$ be a basis in $\vec{\mathbb{R}^n}$, let $g_{ij} = g(\vec{a}_i,\vec{a}_j)$ for all i,j, and let $\vec{\beta}_j = \sum_{i=1}^n B_{ij}\vec{a}_i$ for $j = 1,...,n{-}1$. Compute the components n_i of $\vec{n}_g = \sum_{i=1}^n n_i\vec{a}_i$. Particular case $(\vec{a}_i)$ is $(\cdot,\cdot)_g$-orthonormal?

Answer. (B.14) gives $[\vec{\beta}_i]_{|\vec{a}}^T.[g]_{|\vec{a}}.[\vec{n}_g]_{|\vec{a}} = 0$ for $i = 1,...,n{-}1$: We get $n{-}1$ linear equations. With one more equation given by $[\vec{n}_g]_{|\vec{a}}^T.[g]_{|\vec{a}}.[\vec{n}_g]_{|\vec{a}} = 1$: We get $\vec{n}_g$ up to its sign.

E.g. if $(\vec{a}_i)$ is $(\cdot,\cdot)_g$-orthonormal, then $\sum_{j=1}^n B_{ij}n_j = 0$ for $j = 1,...,n{-}1$, with $\sum_{i=1}^n n_i^2 = 1$. ∎

Exercice B.11 Let $(\vec{a}_i)$ be a Euclidean basis in foot, $(\vec{b}_i)$ a Euclidean basis in metre, $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ the associated Euclidean dot products, so $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$ with $\lambda \simeq 3.28$, cf. (B.7). Let $\vec{n}_a(p)$ and $\vec{n}_b(p)$ be the corresponding unit outward normal vectors, cf. (B.14). 1- Prove (up to the sign):

$$\vec{n}_b = \lambda\vec{n}_a, \quad \text{and} \quad (\vec{w},\vec{n}_a)_a = \lambda(\vec{w},\vec{n}_b)_b \quad \forall \vec{w} \in \vec{\mathbb{R}^n} \tag{B.16}$$

2- Then let $\vec{n}_a = \sum_{i=1}^m n_{ai}\vec{a}_i$ and $\vec{n}_b = \sum_{i=1}^m n_{bi}\vec{b}_i$; Prove:

$$\text{If, } \forall i = 1,...,n,\ \vec{b}_i = \lambda\vec{a}_i \quad \text{then} \quad \forall i = 1,...,n,\ n_{ai} = n_{bi}. \tag{B.17}$$

So the vectors $\vec{n}_a$ and $\vec{n}_b$ are different ($\lambda > 1$), and their respective components are equal... relative to different bases! And of course $1 = ||\vec{n}_a||_a^2 = \sum_{i=1}^n (n_{ai})^2 = \sum_{i=1}^n (n_{bi})^2 = ||\vec{n}_b||_b^2 = 1$.

Answer. $\vec{n}_a(p) \parallel \vec{n}_b(p)$, since the vectors are Euclidean and orthogonal to $T_p\Gamma$ cf. (B.14). And $||.||_a = \lambda||.||_b$ cf. (B.8), thus $||\vec{n}_b||_b = 1 = ||\vec{n}_a||_a = \lambda||\vec{n}_a||_b = ||\lambda\vec{n}_a||_b$, so $\vec{n}_b = \pm\lambda\vec{n}_a$. And they both are outward vectors, so $\vec{n}_b = +\lambda\vec{n}_a$. Thus $(\vec{w},\vec{n}_a)_a = \lambda^2(\vec{w},\vec{n}_a)_b = \lambda^2(\vec{w},\frac{\vec{n}_b}{\lambda})_b = \lambda(\vec{w},\vec{n}_b)_b$.

And if $\vec{b}_i = \lambda\vec{a}_i$ (B.16) gives $\sum_{i=1}^n n_b^i\vec{b}_i = \lambda\sum_{i=1}^n n_a^i\vec{a}_i = \sum_{i=1}^n n_a^i(\lambda\vec{a}_i) = \sum_{i=1}^n n_a^i\vec{b}_i$, then $n_a^i = n_b^i$. ∎

B.7.2 Unit normal form $n^\flat$ associated to $\vec{n}$

For mathematicians: May produce misunderstandings and lack of mechanical interpretations. Don't forget: $n^\flat$ is obtained only after $\vec{n}$ has been defined (thanks to a chosen inner dot product).

Definition B.12 Let $p \in \Gamma$, let $(\cdot,\cdot)_g$ be an inner dot product, and let $\vec{n}_g(p)$ be the outward unit normal. The unit normal form $n_g^\flat(p) \in \mathbb{R}^{n*}$ is the linear form defined by $n_g^\flat(p).\vec{w} := (\vec{n}_g(p),\vec{w})_g$ for all $\vec{w} \in \vec{\mathbb{R}^n}$ vector at p :

$$n_g^\flat.\vec{w} := (\vec{n}_g,\vec{w})_g. \tag{B.18}$$

($=^{\text{noted}}$ $\vec{n} \bullet \vec{w}$ if one chosen Euclidean dot product is imposed to all).

Quantification: Let $(\vec{e}_i)$ be a basis in $\vec{\mathbb{R}^n}$; Then (B.18) gives $[n_g^\flat]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} = [\vec{n}_g]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}$ simply written $[n^\flat].[\vec{w}] = [\vec{n}]^T.[g].[\vec{w}]$ if the basis $(\vec{e}_i)$ is imposed. Hence, with the dual basis (e^i) in $\mathbb{R}^{n*}$,

$$\text{if } \vec{n} = \sum_{i=1}^n n^i\vec{e}_i \text{ and } n^\flat = \sum_{i=1}^n n_ie^i \quad \text{then} \quad n_i = \sum_{j=1}^n g_{ij}n^j, \tag{B.19}$$

where we used the duality notation to justify the $^\flat$ notation: The "top i" gives the "bottom i".

Particular case $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-Euclidean basis, then $n_i = n^i$. As usual the apparent contradiction in the position of the index i in the equation $n_i = n^i$ is due to the implicit use of an inner dot product. Use the Einstein convention to avoid this apparent contradiction: Write $n_i = \sum_{j=1}^n g_{ij}n^j$ even if $g_{ij} = \delta_{ij}$.

Classical notations: Dual basis (π_{ei}), then $n^\flat = \sum_{i=1}^n (n^\flat)_i\pi_{ei}$ and $(n^\flat)_i = \sum_{j=1}^n g_{ij}n_j$.

B.8 Integration by parts (Green–Gauss–Ostrogradsky)

Ω is a regular bounded open set in $\mathbb{R}^n$, $\Gamma = \partial\Omega$, $\varphi \in C^1(\overline{\Omega};\mathbb{R})$, $(\vec{e}_i)$ is a Euclidean basis and $(\cdot,\cdot)_g$ its associated Euclidean dot product, $\frac{\partial\varphi}{\partial x_i}(p) := d\varphi(p).\vec{e}_i$ (usual notation), $\vec{n}_g(p) = \vec{n}(p) = \sum_{i=1}^n n_i(p)\vec{e}_i$ (classical notations) is the $(\cdot,\cdot)_g$-outward normal unit vector at $p \in \Gamma$. Then (Green), for $i = 1,...,n$,

$$\int_{p\in\Omega} \frac{\partial\varphi}{\partial x_i}(p)\, d\Omega = \int_{p\in\Gamma} \varphi(p)n_i(p)\, d\Gamma, \quad \text{in short} \quad \int_\Omega \frac{\partial\varphi}{\partial x_i}\, d\Omega = \int_\Gamma \varphi n_i\, d\Gamma. \tag{B.20}$$

Thus, for any $v \in C^1(\overline{\Omega};\mathbb{R})$, with φv instead of φ in (B.20), we get the integration by parts formula (Green formula):

$$\int_\Omega \frac{\partial\varphi}{\partial x_i} v\, d\Omega = -\int_\Omega \varphi\, \frac{\partial v}{\partial x_i}\, d\Omega + \int_\Gamma \varphi v n_i\, d\Gamma. \tag{B.21}$$

Thus, for any $\vec{v} \in C^1(\overline{\Omega};\vec{\mathbb{R}^n})$ (vector field), with $\vec{v}(p) = \sum_{i=1}^n v_i(p)\vec{e}_i$ we get

$$\int_\Omega \frac{\partial\varphi}{\partial x_i} v_i\, d\Omega = -\int_\Omega \varphi\, \frac{\partial v_i}{\partial x_i}\, d\Omega + \int_\Gamma \varphi v_i n_i\, d\Gamma. \tag{B.22}$$

Thus, with the gradient vector $\vec{\text{grad}}\varphi(p) = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}\vec{e}_i$ and with $\text{div}\vec{v} = \sum_{i=1}^n \frac{\partial v_i}{\partial x_i}$, we get the Gauss–Ostrogradsky formula:

$$\Big(\int_\Omega d\varphi.\vec{v}\, d\Omega =\Big) \quad \int_\Omega \vec{\text{grad}}\varphi \bullet \vec{v}\, d\Omega = -\int_\Omega \varphi\, \text{div}\vec{v}\, d\Omega + \int_\Gamma \varphi\vec{v} \bullet \vec{n}\, d\Gamma. \tag{B.23}$$

(And $\int_\Gamma \varphi\vec{v} \bullet \vec{n}\, d\Gamma$ is the flux through Γ.)

Exercice B.13 Use the differential $d\varphi$ instead of the gradient $\vec{\text{grad}}\varphi$ (which is the $(\cdot,\cdot)_g$-Riesz representation vector of $d\varphi$) to express (B.22). Is the use of $n^\flat$, cf (B.18), useful in that case?

Answer. $d(\varphi\vec{v}) = d\varphi.\vec{v} + \varphi\,\text{div}\vec{v}$, thus $\int_\Omega d\varphi.\vec{v}\, d\Omega = -\int_\Omega \varphi\,\text{div}\vec{v}\, d\Omega + \int_\Gamma \varphi\vec{v} \bullet_g \vec{n}_g\, d\Gamma$. And $\vec{v} \bullet_g \vec{n}_g = n_g^\flat.\vec{v}$, so $\int_\Omega d\varphi.\vec{v}\, d\Omega = -\int_\Omega \varphi\,\text{div}\vec{v}\, d\Omega + \int_\Gamma \varphi n_g^\flat.\vec{v}\, d\Gamma$. But $n_g^\flat$ depends on $\vec{n}_g$ (definition), so there is no reason that justifies the use of $n_g^\flat$ (unless you want to look erudite). ▪

B.9 Stokes theorem

B.9.1 The classic Stokes theorem

Consider a regular oriented 2-D surface $\Sigma \subset \mathbb{R}^3$ parametrized with $\vec{r} : (u,v) \in [a,b] \times [c,d] \to \vec{x} = \vec{r}(u,v) \in \vec{\mathbb{R}^3}$; The unit oriented normal is $\vec{n}(\vec{x}) := \frac{\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}}{||\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}||}(u,v)$ defined at $\vec{x} \in \Sigma = \text{Im}(\vec{r})$. And Σ has a boundary Γ positively parametrized with $\vec{q} : t \in [t_1,t_2] \to \vec{q}(t) \in \vec{\mathbb{R}^3}$: At any $\vec{x} \in \Gamma$ the vector $\vec{n} \times \vec{q}'$ points towards the surface.

Theorem B.14 *If $\vec{f} \in C^1(\vec{\mathbb{R}^3};\vec{\mathbb{R}^3})$ then*

$$\int_\Gamma \vec{f} \bullet d\vec{\ell} = \int_\Sigma \vec{\text{curl}}\vec{f} \bullet d\vec{\Sigma} \quad \Big(= \int_\Sigma \vec{\text{curl}}\vec{f} \bullet \vec{n}\, d\Sigma\Big), \tag{B.24}$$

i.e. $\int_{t=t_1}^{t_2} \vec{f}(\vec{q}(t)) \bullet \vec{q}'(t)\, dt = \int_{u=a}^b \int_{v=c}^d \vec{\text{curl}}\vec{f}(\vec{r}(u,v)) \bullet (\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v})(u,v)\, dudv$.

Proof. See any elementary course, e.g. https://perso.isima.fr/ leborgne//Isimath1ereannee/cousur.pdf. ▪

B.9.2 Generalized Stokes theorem

The curl operator $\vec{\text{curl}}$ is a differential operator which acts on vectors to give vectors. From a covariant point of view, it would be nice to first define a "curl operator" curl as a (linear) function acting on vectors (and eventually representing it with $\vec{\text{curl}}$); Moreover this curl function should "kill the gradient", i.e. should satisfy $\text{curl} \circ d = 0$ (in place of $\vec{\text{curl}} \circ \vec{\text{grad}} = 0$). To do so Cartan developed the "exterior differential" d_{ext} which acts on k-forms (= skew-symmetric covariant tensors), see [5]) and e.g. Marsden–Hughes [14]:

1. The set of $C^\infty(\mathbb{R}^n;\mathbb{R})$ functions is called Ω^0 (the set of $\binom{0}{0}$ tensors = functions); Then define $d_{ext}f := df$ for all $f \in \Omega^0$, i.e. $d_{ext} := d$ (so d_{ext} is the usual differential operator on Ω^0).

2. The set of $C^\infty(\mathbb{R}^n;\mathbb{R}^{n*})$ 1-forms is called Ω^1 (the set of $\binom{0}{1}$ tensors = differential forms); In particular if $f\in\Omega^0$ then the exact differential form $d_{ext}f = df$ is in Ω^1.
3. Definition: A 2-form is a bilinear skew-symmetric $\binom{0}{2}$ tensor (order two covariant), and the set of 2-forms is called Ω^2; So $\beta\in\Omega^2$ iff β is bilinear and $\beta(\vec{u},\vec{w}) = -\beta(\vec{w},\vec{u})$ for all $\vec{u},\vec{v}\in\mathbb{R}^n$ (a 1-form is meant to "measure a length" and a 2-form is meant to "measure a surface"). And the wedge product $\alpha\wedge\beta$ of two 1-forms $\alpha,\beta\in\Omega^1$ is the 2-form $\alpha\wedge\beta\in\Omega^2$ defined by $\alpha\wedge\beta = \alpha\otimes\beta-\beta\otimes\alpha$ (and $\wedge$ is an exterior product defined on Ω^1 to give elements in Ω^2: from "lengths" you get a "surface").
4. Define the exterior differential $d_{ext}:\Omega^1\to\Omega^2$ s.t. $d_{ext}(df) = 0$ for all $f\in\Omega^0$, and $d_{ext}(\alpha\wedge\beta) = d_{ext}\alpha\wedge\beta-\alpha\wedge d_{ext}\beta$ for any $\alpha,\beta\in\Omega^1$.
5. (Generalization.) For $k\geq 2$ define a k-form (also called a differential k-form) to be a skew-symmetric $\binom{0}{k}$ tensor (order k covariant), the set of k-forms being called Ω^k (so $\alpha\in\Omega^k$ satisfies $\alpha(\vec{u}_{\pi(1)},...,\vec{u}_{\pi(k)}) = \mathrm{sgn}(\pi)\alpha(\vec{u}_1,...,\vec{u}_n)$ for all $\vec{u}_1,...,\vec{u}_n\in\mathbb{R}^n$ and all permutations π). On $\Omega^k\times\Omega^\ell$ define the exterior wedge product $\alpha\wedge\beta\in\Omega^{k+\ell}$ by $\alpha\wedge\beta(w_1,...,w_k,w_{k+1},...,w_{k+\ell}) := \frac{1}{k!\ell!}\sum_{\pi\in\sigma}\mathrm{sgn}(\pi)\alpha(w_{\pi_1},...,w_{\pi_k})\beta(w_{\pi_{k+1}},...,w_{\pi_{k+\ell}})$ where σ is the set of permutations. Then define the exterior differential $d_{ext}:\Omega^k\to\Omega^{k+1}$ s.t. $d_{ext}(d_{ext}\gamma) = 0$ for all $\gamma\in\Omega^{k-1}$, and $d_{ext}(\alpha\wedge\beta) = d_{ext}\alpha\wedge\beta+(-1)^k\alpha\wedge d_{ext}\beta$ for any $\alpha\in\Omega^k$ and $\beta\in\Omega^\ell$.
6. $d_{ext} =^{\text{noted}} d$ (creates confusions outside Cartan's framework and for non-mathematicians).

The generalized Stokes theorem (see e.g. Abraham–Marsden [1]) is:

Theorem B.15 *If Σ is n dimensional, if Γ is positively oriented and if $\alpha\in\Omega^{n-1}$ then*

$$\int_\Sigma d_{ext}\alpha = \int_\Gamma \alpha, \tag{B.25}$$

written $\int_\Sigma d\alpha = \int_\Gamma\alpha$.

C Rate of deformation tensor and spin tensor

Let $\widetilde{\Phi}:[t_1,t_2]\times Obj\to\mathbb{R}^n$ be a regular motion, cf. (1.5), and let $\vec{v}:\mathcal{C}\to\vec{\mathbb{R}^n}$ be the Eulerian velocity field, cf. (2.4), that is, $\vec{v}(t,p) = \frac{\partial\Phi}{\partial t}(t,P_{Obj})$ when $p = \widetilde{\Phi}(t,P_{Obj})$.

At t, choose a unit of measurement (foot, metre...) and build the associated Euclidean dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$, cf. § B.2. (We loose the objectivity here). And the same $(\cdot,\cdot)_g$ is used at all t.

C.1 The symmetric and antisymmetric parts of $d\vec{v}$

With the imposed chosen Euclidean dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$, we can consider the transposed endomorphism $d\vec{v}_t(p)^T_g =^{\text{noted}} d\vec{v}_t(p)^T\in\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$, which is defined by, for all $\vec{w}_1,\vec{w}_2\in\vec{\mathbb{R}}^n_t$ vectors at p,

$$(d\vec{v}_t(p)^T.\vec{w}_1,\vec{w}_2)_g = (\vec{w}_1, d\vec{v}_t(p).\vec{w}_2)_g \tag{C.1}$$

cf. § A.12. We have thus defined

$$d\vec{v}_t^T : \begin{cases}\Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)\\ p & \to d\vec{v}_t^T(p) := d\vec{v}_t(p)^T\end{cases} \tag{C.2}$$

Other usual notations (definitions): $d\vec{v}_t(p)^T =^{\text{noted}} d\vec{v}(t,p)^T =^{\text{noted}} d\vec{v}^T(t,p)$.

Definition C.1 The (Eulerian) rate of deformation tensor, or stretching tensor, is the $(\cdot,\cdot)_g$-symmetric part of $d\vec{v}$:

$$\mathcal{D} = \frac{d\vec{v}+d\vec{v}^T}{2},\quad\text{i.e.,}\quad \forall(t,p)\in\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t),\ \ \mathcal{D}(t,p) = \frac{d\vec{v}(t,p)+d\vec{v}(t,p)^T}{2}. \tag{C.3}$$

The (Eulerian) spin tensor is the $(\cdot,\cdot)_g$-antisymmetric part of $d\vec{v}$:

$$\Omega = \frac{d\vec{v}-d\vec{v}^T}{2},\quad\text{i.e.,}\quad \forall(t,p)\in\bigcup_{t\in\mathbb{R}}(\{t\}\times\Omega_t),\ \ \Omega(t,p) = \frac{d\vec{v}(t,p)-d\vec{v}(t,p)^T}{2}. \tag{C.4}$$

(So $d\vec{v} = \mathcal{D}+\Omega$.)

NB: The same notation is used for the set of points $\Omega_t = \Phi^{t_0}_t(\Omega_{t_0})\subset\mathbb{R}^n$ and for the function "the spin tensor" $\Omega_t = \frac{d\vec{v}_t-d\vec{v}_t^T}{2}$: The context removes ambiguities.

C.2 Quantification with a basis

With a basis $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_t$, (C.1) gives

$$[g]_{|\vec{e}}.[d\vec{v}^T]_{|\vec{e}} = [d\vec{v}]^T_{|\vec{e}}.[g]_{|\vec{e}}, \quad \text{and} \quad [d\vec{v}^T]_{|\vec{e}} = [g]^{-1}_{|\vec{e}}.[d\vec{v}]^T_{|\vec{e}}.[g]_{|\vec{e}}. \tag{C.5}$$

In particular, if $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis, then $[d\vec{v}^T]_{|\vec{e}} = [d\vec{v}]^T_{|\vec{e}}$ (orthonormal basis case). Thus for the endomorphisms $\mathcal{D}$ and Ω, with a Euclidean orthonormal basis, with $\mathcal{D}.\vec{e}_j = \sum_{i=1}^n \mathcal{D}_{ij}\vec{e}_i$ and $\Omega.\vec{e}_j = \sum_{i=1}^n \Omega_{ij}\vec{e}_i$ then $\mathcal{D}_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$ and $\Omega_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i})$, that is,

$$[\mathcal{D}]_{|\vec{e}} = \frac{[d\vec{v}]_{|\vec{e}} + [d\vec{v}]^T_{|\vec{e}}}{2} \quad \text{and} \quad [\Omega]_{|\vec{e}} = \frac{[d\vec{v}]_{|\vec{e}} - [d\vec{v}]^T_{|\vec{e}}}{2} \quad \text{(Euclidean framework)}. \tag{C.6}$$

Duality notations: $\mathcal{D}.\vec{e}_j = \sum_{i=1}^n \mathcal{D}^i_j \vec{e}_i$, $\mathcal{D}^i_j = \frac{1}{2}(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i})$ and $\Omega.\vec{e}_j = \sum_{i=1}^n \Omega^i{}_j \vec{e}_i$, $\Omega^i{}_j = \frac{1}{2}(\frac{\partial v^i}{\partial x^j} - \frac{\partial v^j}{\partial x^i})$, where $\mathcal{D}^i_j = \mathcal{D}^j_i$ and $\Omega^i{}_j = -\Omega^j{}_i$.

D Interpretation of the rate of deformation tensor

We are interested in the evolution of the deformation gradient $F(t) := F^{t_0}_{p_{t_0}}(t)$ along the trajectory of a particle P_{Obj} which was at p_{t_0} at t_0. So let $\vec{A} = \vec{a}(t_0, p_{t_0})$ and $\vec{B} = \vec{b}(t_0, p_{t_0})$ be vectors at t_0 at $p_{t_0} \in \Omega_{t_0}$, and consider their push-forwards by the flow $\Phi^{t_0}_t$ (the transported vectors), i.e. the vectors defined at t at $p(t) = \Phi^{t_0}_{p_{t_0}}(t)$ by

$$\vec{a}(t, p(t)) := F(t).\vec{A} \quad \text{and} \quad \vec{b}(t, p(t)) := F(t).\vec{B}. \tag{D.1}$$

see (4.3) and figure 4.1. Then consider the function

$$(\vec{a}, \vec{b})_g : \begin{cases} \mathcal{C} & \to \mathbb{R} \\ (t, p_t) & \to (\vec{a}, \vec{b})_g(t, p_t) := (\vec{a}(t, p_t), \vec{b}(t, p_t))_g. \end{cases} \tag{D.2}$$

Proposition D.1 *The rate of deformation tensor* $\mathcal{D} = \frac{d\vec{v} + d\vec{v}^T}{2}$ *gives half the evolution rate between two vectors deformed by the flow, that is, along trajectories,*

$$\frac{D(\vec{a}, \vec{b})_g}{Dt} = 2(\mathcal{D}.\vec{a}, \vec{b})_g. \tag{D.3}$$

Proof. $f(t) := (\vec{a}(t, p(t)), \vec{b}(t, p(t)))_g = (F(t).\vec{A}, F(t).\vec{B})_g$ gives (with $(\cdot,\cdot)_g$ independent of t)

$$f'(t) = (F'(t).\vec{A}, F(t).\vec{B})_g + (F(t).\vec{A}, F'(t).\vec{B})_g. \tag{D.4}$$

Thus with $F'(t) =^{(3.33)} d\vec{v}(t, p(t)).F(t)$ and $\vec{a}(t, p(t)) = F(t).\vec{A}$ and $\vec{b}(t, p(t)) = F(t).\vec{B}$,

$$\begin{aligned} f'(t) &= (d\vec{v}(t, p(t)).\vec{a}(t, p(t)), \vec{b}(t, p(t)))_g + (\vec{a}(t, p(t)), d\vec{v}(t, p(t)).\vec{b}(t, p(t)))_g \\ &= ((d\vec{v}(t, p(t)) + d\vec{v}(t, p(t))^T).\vec{a}(t, p(t)), \vec{b}(t, p(t)))_g, \end{aligned} \tag{D.5}$$

i.e. (D.3), since $f(t) = (\vec{a}, \vec{b})_g(t, p(t))$ gives $f'(t) = \frac{D(\vec{a}, \vec{b})_g}{Dt}(t, p(t))$. ∎

E Rigid body motions and the spin tensor

Choose a Euclidean dot product $(\cdot,\cdot)_g$ (required to characterize a rigid body motion).

Simple definition: A rigid body motion is a motion whose Eulerian velocity satisfies $d\vec{v} + d\vec{v}^T = 0$, i.e., $\mathcal{D} = 0$ (Eulerian approach independent of any initial time t_0 chosen by some observer).

But the usual classical introduction to rigid body motion relies on some initial time t_0 (Lagrangian approach). So, we start with the Lagrangian approach: Consider a motion $\widetilde{\Phi}$, fix a $t_0 \in \mathbb{R}$, consider the associated Lagrangian motion Φ^{t_0}, and for a fixed t the associated motion $\Phi^{t_0}_t$. The first order Taylor expansion of $\Phi^{t_0}_t$ in the vicinity of a $p_{t_0} \in \Omega_{t_0}$ is, with $d\Phi^{t_0}_t(p_{t_0}) =^{\text{noted}} F^{t_0}_t(p_{t_0})$,

$$\Phi^{t_0}_t(q_{t_0}) = \Phi^{t_0}_t(p_{t_0}) + F^{t_0}_t(p_{t_0}).\overrightarrow{p_{t_0}q_{t_0}} + o(\overrightarrow{p_{t_0}q_{t_0}}). \tag{E.1}$$

E.1 Affine motions and rigid body motions

E.1.1 Affine motions ...

Definition E.1 Φ^{t_0} is an affine motion (understood "affine motion in space") iff $\Phi_t^{t_0}$ is an "affine motion", i.e. iff $\Phi_t^{t_0}$ is a C^1 diffeomorphism (in space), and (E.1) reads, for all $p_{t_0}, q_{t_0} \in \Omega_{t_0}$ and all $t \in [t_1, t_2]$,

$$\Phi_t^{t_0}(q_{t_0}) = \Phi_t^{t_0}(p_{t_0}) + F_t^{t_0}(p_{t_0}).\overrightarrow{p_{t_0}q_{t_0}}. \tag{E.2}$$

Marsden–Hughes notations: $\Phi(Q) = \Phi(P) + F(P).\overrightarrow{PQ}$.

Proposition E.2 and definition. *If Φ^{t_0} is an affine motion, then $F_t^{t_0}(p_{t_0})$ is independent of p_{t_0}, i.e., for all $t \in]t_1, t_2[$ and $p_{t_0}, q_{t_0} \in \Omega_{t_0}$,*

$$F_t^{t_0}(p_{t_0}) = F_t^{t_0}(q_{t_0}) \overset{noted}{=} F_t^{t_0}. \tag{E.3}$$

And then $dF_t^{t_0}(p_{t_0}) = 0$, i.e. $d^2\Phi_t^{t_0}(p_{t_0}) = 0$. And for all $t \in]t_1, t_2[$, Φ^t is an affine motion, i.e. for all $\tau \in]t_1, t_2[$ and all $p_t, q_t \in \Omega_t$,

$$\Phi_\tau^t(q_t) = \Phi_\tau^t(p_t) + F_\tau^t.\overrightarrow{p_tq_t}. \tag{E.4}$$

And $\widetilde{\Phi}$ is said to be an affine motion.

Proof. $q_{t_0} = p_{t_0} + \overrightarrow{p_{t_0}q_{t_0}}$ gives $\Phi_t^{t_0}(q_{t_0}) = \Phi_t^{t_0}(p_{t_0} + \overrightarrow{p_{t_0}q_{t_0}}) = \Phi_t^{t_0}(p_{t_0}) + d\Phi_t^{t_0}(p_{t_0}).\overrightarrow{p_{t_0}q_{t_0}}$, and, similarly, $\Phi_t^{t_0}(p_{t_0}) = \Phi_t^{t_0}(q_{t_0} + \overrightarrow{q_{t_0}p_{t_0}}) = \Phi_t^{t_0}(q_{t_0}) + d\Phi_t^{t_0}(q_{t_0}).\overrightarrow{q_{t_0}p_{t_0}}$. Thus (addition) $\Phi_t^{t_0}(q_{t_0}) + \Phi_t^{t_0}(p_{t_0}) = \Phi_t^{t_0}(p_{t_0}) + \Phi_t^{t_0}(q_{t_0}) + (d\Phi_t^{t_0}(p_{t_0}) - d\Phi_t^{t_0}(q_{t_0})).\overrightarrow{p_{t_0}q_{t_0}}$, thus $(d\Phi_t^{t_0}(p_{t_0}) - d\Phi_t^{t_0}(q_{t_0})).\overrightarrow{p_{t_0}q_{t_0}} = 0$, true for all p_{t_0}, q_{t_0}, thus $d\Phi_t^{t_0}(p_{t_0}) - d\Phi_t^{t_0}(q_{t_0}) = 0$, i.e. (E.3).

Thus $d^2\Phi_t^{t_0}(p_{t_0}).\vec{u}_{t_0} = \lim_{h\to 0} \frac{d\Phi_t^{t_0}(p_{t_0}+h\vec{u}_{t_0}) - d\Phi_t^{t_0}(p_{t_0})}{h} = \lim_{h\to 0} \frac{d\Phi_t^{t_0} - d\Phi_t^{t_0}}{h} = 0$ for all p_{t_0} and all $\vec{u}_{t_0}$, thus $d^2\Phi_t^{t_0}(p_{t_0}) = 0$ for all p_{t_0}, thus $d^2\Phi_t^{t_0} = 0$.

And (5.17) gives $(\Phi_\tau^t \circ \Phi_t^{t_0})(p_{t_0}) = \Phi_\tau^{t_0}(p_{t_0})$, thus, with $p_t = \Phi_t^{t_0}(p_{t_0})$, we get $d\Phi_\tau^t(p_t).d\Phi_t^{t_0}(p_{t_0}) = d\Phi_\tau^{t_0}(p_{t_0})$, thus $d\Phi_\tau^t(p_t) = d\Phi_\tau^{t_0}(p_{t_0}).d\Phi_t^{t_0}(p_{t_0})^{-1}$, and (E.2) gives

$$d\Phi_\tau^t(p_t) = d\Phi_\tau^{t_0}.d\Phi_t^{t_0\,-1} \overset{noted}{=} d\Phi_\tau^t \quad \text{(independent of } p_t\text{)}, \tag{E.5}$$

thus (E.4). ∎

Corollary E.3 *With $\vec{v}$ the Eulerian velocity and $\vec{V}^{t_0}$ the Lagrangian velocity: If $\widetilde{\Phi}$ is affine then, $\vec{v}_t$ is affine for all t, and $\vec{V}_t^{t_0}$ is affine for all t_0, t, i.e., $d\vec{v}_t(p_t) = d\vec{v}_t$ for all $p_t \in \Omega_t$ (independent of p_t), and $d\vec{V}_t^{t_0}(p_{t_0}) =^{noted} d\vec{V}_t^{t_0}$ for all $p_{t_0} \in \Omega_{t_0}$ (independent of p_{t_0}). So, for all $p_t, q_t \in \Omega_t$ and $p_{t_0}, q_{t_0} \in \Omega_{t_0}$,*

$$\begin{cases} \bullet \;\; \vec{v}_t(q_t) = \vec{v}_t(p_t) + d\vec{v}_t.\overrightarrow{p_tq_t}, \\ \bullet \;\; \vec{V}_t^{t_0}(q_{t_0}) = \vec{V}_t^{t_0}(p_{t_0}) + d\vec{V}_t^{t_0}.\overrightarrow{p_{t_0}q_{t_0}}. \end{cases} \tag{E.6}$$

Proof. (E.2) gives $\Phi^{t_0}(t, q_{t_0}) = \Phi^{t_0}(t, p_{t_0}) + F^{t_0}(t).\overrightarrow{p_{t_0}q_{t_0}}$, and the derivation in time gives $(E.6)_2$, hence $(E.6)_1$ thanks to $d\vec{V}_t^{t_0}(p_{t_0}) = d\vec{v}_t(p_t).F_t^{t_0}$, cf. (3.27), and $\overrightarrow{p_{t_0}q_{t_0}} = (F_t^{t_0})^{-1}.\overrightarrow{p_tq_t}$, cf. (E.2). ∎

Example E.4 In $\mathbb{R}^2$, with a basis $(\vec{E}_1, \vec{E}_2)$ in $\vec{\mathbb{R}}_{t_0}^n$ and a basis $(\vec{e}_1, \vec{e}_2) \in \vec{\mathbb{R}}_t^n$, then $F_t^{t_0}$ given by $[F_t^{t_0}]_{|\vec{E},\vec{e}} = \begin{pmatrix} 1+t & 2t^2 \\ 3t^3 & e^t \end{pmatrix}$ derives from the affine motion $[\overrightarrow{\Phi_t^{t_0}(p_{t_0})\Phi_t^{t_0}(q_{t_0})}]_{|\vec{e}} = \begin{pmatrix} 1+t & 2t^2 \\ 3t^3 & e^t \end{pmatrix}.[\overrightarrow{p_{t_0}q_{t_0}}]_{|\vec{E}}$. ∎

E.1.2 ... and rigid body motion

Let $\Phi := \Phi_t^{t_0}$ and $F := F_t^{t_0}$ if non ambiguous. Recall: If $P \in \Omega_{t_0}$ and $p = \Phi(P)$ $(\in \Omega_t)$ then the transposed of the linear map $F(P) \in \mathcal{L}(\vec{\mathbb{R}}_{t_0}^n; \vec{\mathbb{R}}_t^n)$ relative to $(\cdot,\cdot)_g$ is the linear map $F^T(p) \in \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_{t_0}^n)$ defined by

$$F^T(p) := F(P)^T : \begin{cases} \vec{\mathbb{R}}_t^n & \to \vec{\mathbb{R}}_{t_0}^n \\ \vec{w}_p & \to F^T(p).\vec{w}_p \quad \text{s.t.} \quad (F^T(p).\vec{w}_p, \vec{U}_P)_g = (\vec{w}_p, F(P).\vec{U}_P)_g, \quad \forall \vec{U}_P \in \vec{\mathbb{R}}_{t_0}^n. \end{cases} \tag{E.7}$$

Which defines the function $F^T : \Omega_t \to \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_{t_0}^n)$.

Particular case: For an affine motion F is independent of P, hence F^T is independent of p.

Definition E.5 A rigid body motion is an affine motion $\widetilde{\Phi}$ such that angles and lengths are unchanged by Φ: For all $t_0, t \in \mathbb{R}$, $P \in \Omega_{t_0}$, $\vec{U}_P, \vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$, and with $p = \Phi(P)$,

$$(F.\vec{U}_P, F.\vec{W}_P)_g = (\vec{U}_P, \vec{W}_P)_g, \quad \text{i.e.} \quad (F^T.F.\vec{U}_P, \vec{W}_P)_g = (\vec{U}_P, \vec{W}_P)_g, \quad \text{i.e.} \quad \boxed{F^T.F = I}. \tag{E.8}$$

In other words, with the Cauchy strain tensor $C \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ defined by $C = F^T.F$, the motion is rigid iff it is affine and

$$\boxed{C = I}, \quad \text{i.e.} \quad \boxed{F^{-1} = F^T}. \tag{E.9}$$

Proposition E.6 *If Φ^{t_0} is a rigid body motion, if $(\vec{A}_i)$ is a $(\cdot,\cdot)_g$-Euclidean basis in $\vec{\mathbb{R}}^n_{t_0}$, if $\vec{a}_{it}(p) = F^{t_0}_t(P).\vec{A}_i$ for all i when $p = \Phi^{t_0}_t(P)$, then $\vec{a}_{it}(p) =^{noted} \vec{a}_{it}$ is independent of p, and $(\vec{a}_{it})$ is a $(\cdot,\cdot)_g$-Euclidean basis with the same orientation than $(\vec{A}_i)$, for all t.*

Proof. $\Phi^{t_0}_t$ is affine, thus, for all t, P, $F^{t_0}_t(P) = F^{t_0}_t$ (independent of P), thus $\vec{a}_{i,t}(p) = F^{t_0}_t.\vec{A}_i \in \vec{\mathbb{R}}^n_t$ is independent of p, for all t. And $(\vec{a}_{it}, \vec{a}_{jt})_g = (F^{t_0}_t.\vec{A}_i, F^{t_0}_t.\vec{A}_j)_g = ({F^{t_0}_t}^T.F^{t_0}_t.\vec{A}_i, \vec{A}_j)_g =^{solid} (I.\vec{A}_i, \vec{A}_j)_g = (\vec{A}_i, \vec{A}_j)_g = \delta_{ij}$ for all i,j, thus $(\vec{a}_{it})$ is $(\cdot,\cdot)_g$-orthonormal basis. And $\det(\vec{a}_{1t}, ..., \vec{a}_{nt}) = \det(F^{t_0}_t.\vec{A}_1, ..., F^{t_0}_t.\vec{A}_n) = \det(F^{t_0}_t)\det(\vec{A}_1, ..., \vec{A}_n) = \det(F^{t_0}_t)$ since $(\vec{A}_i)$ is a $(\cdot,\cdot)_g$- orthonormal basis. And, Φ^{t_0} being regular, $t \to \det(F^{t_0}_t)$ is continuous, does not vanish, with $\det(F^{t_0}_{t_0}) = \det(I) = 1 > 0$; Thus $\det(F^{t_0}_t) > 0$ for all t, thus $\det(\vec{a}_1, ..., \vec{a}_n) > 0$: The bases have the same orientation. ∎

Example E.7 In $\mathbb{R}^2$, a rigid body motion is given by $F^{t_0}_t = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix}$ with θ a regular function s.t. $\theta(t_0) = 0$. ∎

Exercice E.8 Let $\widetilde{\Phi}$ be a rigid body motion. Prove

$$(F^T)'(t) = (F'(t))^T, \quad \text{and} \quad F^T.F' \text{ is antisymmetric: } (F')^T.F + F^T.F' = 0. \tag{E.10}$$

Answer. Let $t \in \mathbb{R}$, $F(t) := F^{t_0}_P(t)$, $p(t) = \Phi^{t_0}_P(t)$, $\vec{U}, \vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ and $\vec{w}(t, p(t)) = F(t).\vec{W}$. Recall: F^T is defined by $F^T(t) := (F(t))^T$, so $(F^T(t).\vec{w}(t,p(t)), \vec{U})_g = (\vec{w}(t,p(t)), F(t).\vec{U})_g$. Thus $((F^T)'(t).\vec{w}(t,p(t)) + F^T(t).\frac{D\vec{w}}{Dt}(t,p(t)), \vec{U})_g = (\frac{D\vec{w}}{Dt}(t,p(t)), F(t).\vec{U})_g + (\vec{w}(t,p(t)), F'(t).\vec{U})_g$, which simplifies into $((F^T)'(t).\vec{w}(t,p(t)), \vec{U})_g = (\vec{w}(t,p(t)), F'(t).\vec{U})_g = ((F'(t))^T.\vec{w}(t,p(t)), \vec{U})_g$, thus $(F^T)'(t) = (F'(t))^T$, for all t.

And (E.8) reads $F^T(t).F(t) = I_{t_0}$, thus $(F^T)'(t).F(t) + F^T(t).F'(t) = 0$, thus $(F')^T(t).F(t) + F^T(t).F'(t) = 0$, thus $F^T(t).F'(t)$ is antisymmetric, for all t. ∎

E.1.3 Alternative definition of a rigid body motion: $d\vec{v} + d\vec{v}^T = 0$

The stretching tensor $\mathcal{D}_t = \frac{d\vec{v}_t + d\vec{v}_t^T}{2}$ and the spin tensor $\Omega_t = \frac{d\vec{v}_t - d\vec{v}_t^T}{2}$ have been defined in (C.3)-(C.4).

Proposition E.9 *(Here no initial time is required: Eulerian approach.) If $\widetilde{\Phi}$ is a rigid body motion, cf. (E.8), then the endomorphism $d\vec{v}_t \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$ is antisymmetric at all t:*

$$d\vec{v}_t + d\vec{v}_t^T = 0, \quad \text{i.e.} \quad d\vec{v}_t = \Omega_t, \quad \text{i.e.} \quad \mathcal{D}_t = 0. \tag{E.11}$$

Conversely, if $d\vec{v}_t + d\vec{v}_t^T = 0$ at all t, then $\widetilde{\Phi}$ is a rigid body motion.

So the relation « $d\vec{v}_t + d\vec{v}_t^T = 0$ for all t » gives an equivalent definition to the definition E.5.

Proof. Let $F(t) := F^{t_0}_P(t)$ and $V(t) := \vec{V}^{t_0}_P(t) = (\Phi^{t_0}_P)'(t) = \vec{v}(t, p_t)$. (E.8) gives $(F.F^T)'(t) = 0 = F'(t).F^T(t) + F(t).(F^T)'(t) \stackrel{(E.10)}{=} F'(t).F^T(t) + (F'(t).F^T(t))^T = dV(t).F(t)^{-1} + (dV(t).F(t)^{-1})^T \stackrel{(3.27)}{=} d\vec{v}(t, p_t) + d\vec{v}(t, p_t)^T$. Thus (E.11).

Conversely, suppose $d\vec{v} + d\vec{v}^T = 0$. Then (D.3) gives $\frac{D(\vec{a}, \vec{b})_g}{Dt} = 0$, thus $(\vec{a}, \vec{b})_g(t, p(t)) = (\vec{a}, \vec{b})_g(t_0, P)$ when $p(t) = \Phi^{t_0}_t(P)$, i.e. $(F^{t_0}_t(P).\vec{A}, F^{t_0}_t(P).\vec{B})_g = (\vec{A}, \vec{B})_g$, for all t, t_0, P, $\vec{A}$, $\vec{B}$: Thus $\widetilde{\Phi}$ is a rigid body motion, cf (E.8). ∎

E.2 Vector and pseudo-vector representations of a spin tensor Ω

We are dealing here with concepts that are sometimes misunderstood.

Framework: $\mathbb{R}^n = \mathbb{R}^3$ with a Euclidean dot product $(\cdot,\cdot)_g$ (so the following is not objective).

E.2.1 Reminder

• The determinant $\det_{|\vec{e}}$ associated with a basis $(\vec{e}_i)$ in $\mathbb{R}^3$ is the alternating multilinear form defined by $\det_{|\vec{e}}(\vec{e}_1,\vec{e}_2,\vec{e}_3) = 1$; The algebraic volume (or signed volume) limited by three vectors $\vec{u}_1,\vec{u}_2,\vec{u}_3$ is $\det_{|\vec{e}}(\vec{u}_1,\vec{u}_2,\vec{u}_3)$; And the (positive) volume is $|\det_{|\vec{e}}(\vec{u}_1,\vec{u}_2,\vec{u}_3)|$, see § L.

• Let A and B be two observers (e.g. A=English and B=French), let $(\vec{a}_i)$ be a Euclidean basis chosen by A (e.g. based on the foot), let $(\vec{b}_i)$ be a Euclidean basis chosen by B (e.g. based on the metre), see § B.1. Let $\lambda = ||\vec{b}_1||_a > 0$ (change of unit of length coefficient). The relation between the determinants is:

$$\det_{|\vec{a}} = \pm\lambda^3 \det_{|\vec{b}} \quad \text{with} \quad \begin{cases} + \text{ if } \det_{|\vec{a}}(\vec{b}_1,\vec{b}_2,\vec{b}_3) > 0 & \text{(i.e. if the bases have the same orientation)}, \\ - \text{ if } \det_{|\vec{a}}(\vec{b}_1,\vec{b}_2,\vec{b}_3) < 0 & \text{(i.e. if the bases have opposite orientation)}. \end{cases} \tag{E.12}$$

In particular, if A and B use the same unit of length, then $\lambda = 1$ and $\det_{|\vec{a}} = \pm\det_{|\vec{b}}$.

• With an imposed Euclidean dot product $(\cdot,\cdot)_g$: An endomorphism L is $(\cdot,\cdot)_g$-antisymmetric iff

$$\forall \vec{u},\vec{v},\quad (L.\vec{u},\vec{v})_g + (\vec{u},L.\vec{v})_g = 0, \quad \text{i.e.} \quad L^T = -L. \tag{E.13}$$

E.2.2 Definition of the vector product (cross product)

Let $(\vec{e}_i)$ be a $(\cdot,\cdot)_g$-orthonormal basis, let $\vec{u},\vec{v} \in \vec{\mathbb{R}^3}$, and let $\ell_{\vec{e},\vec{u},\vec{v}} \in \mathcal{L}(\vec{\mathbb{R}^3},\mathbb{R})$ be the linear form defined by

$$\ell_{\vec{e},\vec{u},\vec{v}} : \begin{cases} \vec{\mathbb{R}^3} \to \mathbb{R} \\ \vec{z} \to \ell_{\vec{e},\vec{u},\vec{v}}(\vec{z}) := \det_{|\vec{e}}(\vec{u},\vec{v},\vec{z}) \end{cases} \tag{E.14}$$

(the algebraic volume of the parallelepiped limited by $\vec{u},\vec{v},\vec{z}$ in the Euclidean chosen unit).

Definition E.10 The vector product, or cross product, $\vec{u} \times_{eg} \vec{v}$ (written $\vec{u} \wedge_{eg} \vec{v}$ in french) of two vectors $\vec{u}$ and $\vec{v}$ is the $(\cdot,\cdot)_g$-Riesz representation vector $\vec{u} \times_{eg} \vec{v} \in \vec{\mathbb{R}^3}$ of the linear form $\ell_{\vec{e},\vec{u},\vec{v}}$: It is given by, cf. (F.2):

$$\forall \vec{z} \in \vec{\mathbb{R}^3}, \quad \boxed{(\vec{u} \times_{eg} \vec{v}, \vec{z})_g = \det_{|\vec{e}}(\vec{u},\vec{v},\vec{z})}. \tag{E.15}$$

NB: $\vec{u} \times_{eg} \vec{v}$ depends on $(\cdot,\cdot)_g$ and on the orientation of $(\vec{e}_i)$.

We have thus defined the bilinear cross product operator

$$\times_{eg} : \begin{cases} \vec{\mathbb{R}^3} \times \vec{\mathbb{R}^3} \to \vec{\mathbb{R}^3} \\ (\vec{u},\vec{v}) \to \times_{eg}(\vec{u},\vec{v}) := \vec{u} \times_{eg} \vec{v}. \end{cases} \tag{E.16}$$

(The bilinearity is trivial thanks to the multilinearity of the determinant.)

And if a chosen $(\cdot,\cdot)_g$ is imposed to all, then $\vec{u} \times_{eg} \vec{v} =^{\text{noted}} \vec{u} \times_e \vec{v}$.

Moreover if an orthonormal basis $(\vec{e}_i)$ is imposed to all observers then $\vec{u} \times_e \vec{v} =^{\text{noted}} \vec{u} \times \vec{v}$.

NB: The cross product is not an objective operator! It depends on a chosen Euclidean dot product and on a chosen Euclidean basis (its orientation).

Notation: Isometric framework + imposed Euclidean basis (orientation imposed): (E.15) is written

$$\forall \vec{z} \in \vec{\mathbb{R}^3}, \quad (\vec{u} \times \vec{v}) \bullet \vec{z} = \det(\vec{u},\vec{v},\vec{z}). \tag{E.17}$$

E.2.3 Calculation of the vector product

$\vec{u} = \sum_{i=1}^3 u_i\vec{e}_i$, $\vec{v} = \sum_{i=1}^3 v_i\vec{e}_i$ and (E.15) give

$$(\vec{u} \times_{eg} \vec{v}, \vec{e}_1)_g = \det_{|\vec{e}}(\vec{u}, \vec{v}, \vec{e}_1) = \det\begin{pmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 0 \\ u_3 & v_3 & 0 \end{pmatrix} = \det\begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} = u_2v_3 - u_3v_2. \tag{E.18}$$

Similar calculation: $(\vec{u} \times_{eg} \vec{v}, \vec{e}_2)_e = u_3v_1 - u_1v_3$ and $(\vec{u} \times_{eg} \vec{v}, \vec{e}_3)_e = u_1v_2 - u_2v_1$, thus

$$[\vec{u} \times_{eg} \vec{v}]_{|\vec{e}} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}, \quad \text{i.e.} \quad \vec{u} \times_{eg} \vec{v} = \sum_{i=1}^{3}(u_{i+1}v_{i+2} - u_{i+2}v_{i+1})\,\vec{e}_i \tag{E.19}$$

with the generic notation $w_4 := w_1$ and $w_5 = w_2$ (indices modulo 3): In particular $\vec{e}_i \times_{eg} \vec{e}_{i+1} = \vec{e}_{i+2}$.

Proposition E.11 *1- $\vec{u} \times_{eg} \vec{v} = -\vec{v} \times_{eg} \vec{u}$.*

2- $\vec{u} \parallel \vec{v}$ iff $\vec{u} \times_{eg} \vec{v} = 0$.

3- $\vec{u} \times_{eg} \vec{v}$ is orthogonal to $\mathrm{Vect}\{\vec{u}, \vec{v}\}$ *the linear space generated by $\vec{u}$ and $\vec{v}$.*

4- $\vec{u} \times_{eg} \vec{v}$ depends on the unit of measurement and on the orientation of the $(\cdot,\cdot)_g$- orthonormal basis $(\vec{e}_i)$: Consider two Euclidean dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$, so $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$ for a $\lambda > 0$; Choose a $(\cdot,\cdot)_a$-orthonormal basis $(\vec{a}_i)$ and a $(\cdot,\cdot)_b$-orthonormal basis $(\vec{b}_i)$; Then

$$\vec{u} \times_{aa} \vec{v} = \pm\lambda\vec{u} \times_{bb} \vec{v}, \tag{E.20}$$

with the $+$ sign iff $(\vec{a}_i)$ and $(\vec{b}_i)$ have the same orientation.

Proof. 1- $(\vec{u} \times_{eg} \vec{v}, \vec{z})_g = \det_{|\vec{e}}(\vec{u}, \vec{v}, \vec{z}) = -\det_{|\vec{e}}(\vec{v}, \vec{u}, \vec{z}) = -(\vec{v} \times_{eg} \vec{u}, \vec{z})_g$, for all $\vec{z}$.

2- If $\vec{u} \parallel \vec{v}$ then $\det_{|\vec{e}}(\vec{u}, \vec{v}, \vec{z}) = 0 = (\vec{u} \times_{eg} \vec{v}, \vec{z})_e$, so $\vec{u} \times_{eg} \vec{v} \perp_g \vec{z}$, for all $\vec{z}$. And if $\vec{u} \times_{eg} \vec{v} = 0$ then (E.19) gives $\vec{u} \parallel \vec{v}$.

3- If $\vec{z} \in \mathrm{Vect}\{\vec{u}, \vec{v}\}$ then $\det_{|\vec{e}}(\vec{u}, \vec{v}, \vec{z}) = 0$, thus $(\vec{u} \times_{eg} \vec{v}, \vec{z})_g = 0$ thus $\vec{u} \times_{eg} \vec{v} \perp_g \vec{z}$.

4- $(\vec{u} \times_{aa} \vec{v}, \vec{z})_a \overset{(E.15)}{=} \det_{|\vec{a}}(\vec{u}, \vec{v}, \vec{z}) \overset{(E.12)}{=} \pm\lambda^3 \det_{|\vec{b}}(\vec{u}, \vec{v}, \vec{z}) \overset{(E.15)}{=} \pm\lambda^3(\vec{u} \times_{bb} \vec{v}, \vec{z})_b = \pm\lambda^3 \frac{1}{\lambda^2}(\vec{u} \times_{bb} \vec{v}, \vec{z})_a$, true for all $\vec{z}$, thus (E.20). ∎

Exercice E.12 Prove that $\vec{u} \times_{eg} \vec{v}$ is a contravariant vector.

Answer. It is a vector (Riesz representation vector) in $\vec{\mathbb{R}^3}$, so it is contravariant; Or calculation: It satisfies the contravariance change of basis formula, see (F.18). ∎

E.2.4 Antisymmetric endomorphism represented by a vector

Proposition E.13 *Let $(\vec{e}_i)$ be a $(\cdot,\cdot)_g$-Euclidean basis. If an endomorphism $\Omega \in \mathcal{L}(\vec{\mathbb{R}^3}; \vec{\mathbb{R}^3})$ is $(\cdot,\cdot)_g$-antisymmetric then there exists a unique vector $\vec{\omega}_{eg} \in \vec{\mathbb{R}^3}$ s.t., for all $\vec{y}, \vec{z} \in \vec{\mathbb{R}^3}$,*

$$(\Omega.\vec{y}, \vec{z})_g = \det_{|\vec{e}}(\vec{\omega}_{eg}, \vec{y}, \vec{z}), \tag{E.21}$$

i.e., there exists a unique vector $\vec{\omega}_{eg} \in \vec{\mathbb{R}^3}$ s.t., for all $\vec{y}, \vec{z} \in \vec{\mathbb{R}^3}$,

$$\boxed{\Omega.\vec{y} = \vec{\omega}_{eg} \times_{eg} \vec{y}}, \tag{E.22}$$

And

$$[\Omega]_{|\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \quad \text{iff} \quad [\vec{\omega}_{eg}]_{|\vec{e}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{E.23}$$

In particular $\Omega.\vec{\omega}_{eg} = \vec{0}$ $(= \vec{\omega}_{eg} \times_{eg} \vec{\omega}_{eg})$, i.e. $\vec{\omega}_{eg}$ is an eigenvector of Ω associated with the eigenvalue 0.

Proof. Ω is antisymmetric, thus $[\Omega]_{|\vec{e}}$ is given as in (E.23). In particular $[\Omega.\vec{e}_1]_{|\vec{e}} = [\Omega]_{|\vec{e}}.[\vec{e}_1]_{|\vec{e}} = \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}$.

Calculation of the components of $\vec{\omega}_{eg}$ if it exists: Let $\vec{\omega} = \omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3$; thus $[\vec{\omega} \times_{eg} \vec{e}_1]_{|\vec{e}} = \begin{pmatrix} 0 \\ \omega_3 \\ -\omega_2 \end{pmatrix}$, cf. (E.19), thus $\omega_3 = c$ and $\omega_2 = b$; Idem with $\vec{e}_2$ so that $\omega_1 = a$. Thus if it exists $\vec{\omega}$ is unique. And $\vec{\omega}_{eg}$ given in (E.23) satisfies (E.22): It exists. ∎

Proposition E.14 *Let $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ be two Euclidean dot products (e.g. in foot and metre), let $(\vec{a}_i)$ and $(\vec{b}_i)$ be Euclidean associated bases, let $||\vec{b}_1||_a = \lambda$ (change of unit coefficient), so $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b)$. And $\vec{\omega}_{aa} =^{noted} \vec{\omega}_a$ and $\vec{\omega}_{bb} =^{noted} \vec{\omega}_b$. Suppose $[\Omega]_{|\vec{a}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$, thus $[\vec{\omega}_a]_{|\vec{a}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, cf. (E.23). Then (change of representation vector for Ω):*

- *If $(\vec{b}_i)$ and $(\vec{a}_i)$ have the same orientation, then $\vec{\omega}_b = \lambda\vec{\omega}_a$,*
- *If $(\vec{b}_i)$ and $(\vec{a}_i)$ have opposite orientation, then $\vec{\omega}_b = -\lambda\vec{\omega}_a$,* (E.24)

E.g., if $\vec{b}_i = \lambda\vec{a}_i$ for all i (change of unit, same orientation) then $\vec{\omega}_b = \lambda\vec{\omega}_a$, and if $\vec{b}_1 = -\lambda\vec{a}_1$, $\vec{b}_2 = \lambda\vec{a}_2$, $\vec{b}_3 = \lambda\vec{a}_3$ (change of unit, opposite orientation) then $\vec{\omega}_b = -\lambda\vec{\omega}_a$.

NB: *The formula $\vec{\omega}_b = \pm\lambda\vec{\omega}_a$ is a change of vector formula,* ***not*** *a change of basis formula.*

Proof. Apply (E.20). ∎

Notation: If $(\cdot,\cdot)_g$ is imposed, then $\vec{\omega}_{eg} =^{\text{noted}} \vec{\omega}_e$.

Interpretation of $\vec{\omega}_e$: Suppose $[\Omega]_{|\vec{e}} = \alpha \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So Ω is the rotation with angle $\frac{\pi}{2}$ in the horizontal plane composed with the dilation with ratio α. And $[\vec{\omega}_e]_{|\vec{e}} = \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, thus $\vec{\omega}_e = \alpha\,\vec{e}_3$ is orthogonal to the horizontal plane, hence $\vec{\omega}_e \times_e$ is a rotation around the z-axis composed with a dilation which coefficient is α.

Exercice E.15 Let Ω s.t. $[\Omega]_{|\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ (see (E.23)). Find a direct (relative to $(\vec{e}_i)$) orthonormal basis $(\vec{b}_i)$ s.t. $[\Omega]_{|\vec{b}} = \sqrt{a^2+b^2+c^2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Answer. Let $\vec{b}_3 = \frac{\vec{\omega}_e}{||\vec{\omega}_e||_e}$, so $[\vec{b}_3]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Then let $\vec{b}_1$ be given by $[\vec{b}_1]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$, so $\vec{b}_1 \perp \vec{b}_3$. Then let $\vec{b}_2 = \vec{b}_3 \times_e \vec{b}_1$, that is, $[\vec{b}_2]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2}} \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix}$. Thus $(\vec{b}_i)$ is a direct orthonormal basis, and the transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$ is $P = \begin{pmatrix} [\vec{b}_1]_{|\vec{e}} & [\vec{b}_2]_{|\vec{e}} & [\vec{b}_3]_{|\vec{e}} \end{pmatrix}$. With $[\Omega]_{|\vec{b}} = P^{-1}.[\Omega]_{|\vec{e}}.P$ (change of basis formula), where $P^{-1} = P^T$ (change of orthonormal basis).

With $[\Omega]_{|\vec{e}}.[\vec{b}_1]_{|\vec{e}} = \frac{1}{\sqrt{b^2+c^2}} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} . \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix} = \frac{1}{\sqrt{b^2+c^2}} \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix} = \sqrt{a^2+b^2+c^2}[\vec{b}_2]_{|\vec{e}}$ (expected), $[\Omega]_{|\vec{e}}.[\vec{b}_2]_{|\vec{e}} = \frac{1}{\sqrt{b^2+c^2}} \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} . \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix} = \frac{1}{\sqrt{b^2+c^2}} \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} bc^2 + b(a^2+b^2) \\ -ac^2 - a(a^2+b^2) \\ abc - abc \end{pmatrix} = -\sqrt{a^2+b^2+c^2}[\vec{b}_1]_{|\vec{e}}$ (expected), and $[\Omega]_{|\vec{e}}.[\vec{b}_3]_{|\vec{e}} = [\vec{0}]$ (expected since $\vec{b}_3 \parallel \vec{\omega}_e$). Thus $[\Omega]_{|\vec{e}}.P = \sqrt{a^2+b^2+c^2} \begin{pmatrix} [\vec{b}_2]_{|\vec{e}} & -[\vec{b}_1]_{|\vec{e}} & [\vec{0}]_{|\vec{e}} \end{pmatrix}$. And $(P^{-1}.[\Omega]_{|\vec{e}}.P)_{ij} = (P^T.[\Omega]_{|\vec{e}}.P)_{ij} = [\vec{b}_i]_{|\vec{e}}^T.[\Omega]_{|\vec{e}}.[\vec{b}_j]_{|\vec{e}}$ gives the result. ∎

E.2.5 Curl

Definition E.16 If $\vec{v}$ is a C^1 vector field, if $(\vec{e}_i)$ is a Euclidean basis in $\vec{\mathbb{R}^3}$, and if $\vec{v} = \sum_{i=1}^3 v^i \vec{e}_i$, then the curl (or rotational) of $\vec{v}$ relative to $(\vec{e}_i)$ is the vector field $\vec{\text{curl}}_e \vec{v} = \vec{\text{curl}}_e \vec{v}$ given by

$$\vec{\text{curl}}_e \vec{v} = \sum_{i=1}^{3} (\frac{\partial v_{i+2}}{\partial x_{i+1}} - \frac{\partial v_{i+1}}{\partial x_{i+2}})\, \vec{e}_i, \quad \text{i.e.} \quad [\vec{\text{curl}}_e \vec{v}]_{|\vec{e}} = \begin{pmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{pmatrix}. \tag{E.25}$$

And $\vec{\text{curl}}_e \vec{v} =^{\text{noted}} \vec{\nabla} \times_e \vec{v}$ (notation due to the matrix product $\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$).

Proposition E.17 *Let* $\Omega(t, p_t) = \frac{d\vec{v}(t,p_t) - d\vec{v}(t,p_t)^T}{2}$ *and* $\vec{\omega}_e(t, p_t)$ *be its associated vector relative to the Euclidean basis* $(\vec{e}_i)$*, cf. (E.22). Then*

$$\vec{\omega}_e = \frac{1}{2} \vec{\text{curl}}_e \vec{v}. \tag{E.26}$$

Proof. (C.6) gives $[\Omega]_{|\vec{e}} = \frac{1}{2} \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ . & 0 & \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \\ . & . & 0 \end{pmatrix}$, with $[\Omega]_{|\vec{e}}$ antisymmetric. Thus (E.23), (E.19) and (E.25) gives (E.26). ∎

E.3 Pseudo-vector, and pseudo-cross product

Framework: $\mathcal{M}_{31}$ the space of $3 * 1$ matrices: We leave the framework of the vectors in $\vec{\mathbb{R}^3}$ to enter the matrix world.

E.3.1 Definition

Definition E.18 A column matrix is also called a pseudo-vector or a column vector.

Definition E.19 The pseudo-cross product $\overset{\circlearrowleft}{\times} : \mathcal{M}_{31} \times \mathcal{M}_{31} \to \mathcal{M}_{31}$ is defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{\circlearrowleft}{\times} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix} \overset{\text{noted}}{=} [\vec{x}] \overset{\circlearrowleft}{\times} [\vec{y}], \tag{E.27}$$

notation used when $[\vec{x}] := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $[\vec{y}] := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. So the pseudo-cross product of two pseudo-vectors is a pseudo-vector (is a matrix).

E.3.2 Antisymmetric matrix represented by a pseudo-vector

Recall. An antisymmetric matrix $A = [A_{ij}] \in \mathcal{M}_{nn}$ is s.t. $A_{ji} = -A_{ij}$ for all i, j.

Definition E.20 Consider an antisymmetric matrix $A = [A_{ij}] = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \mathcal{M}_{33}$. The pseudo-vecteur $\overset{\circlearrowleft}{\omega}$ associated to A is the column matrix $\overset{\circlearrowleft}{\omega} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathcal{M}_{31}$.

So, with (E.27),

$$\boxed{A.[\vec{y}] = \overset{\circlearrowleft}{\omega} \overset{\circlearrowleft}{\times} [\vec{y}]}, \quad \text{i.e.} \quad A.\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \overset{\circlearrowleft}{\omega} \overset{\circlearrowleft}{\times} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad \text{for all matrix } [\vec{y}] = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{E.28}$$

E.3.3 Pseudo-vector representations of an antisymmetric endomorphism

Let $\mathbb{R}^3$ be our usual affine space, $\vec{\mathbb{R}^3}$ its associated vector space, $(\cdot,\cdot)_g$ a Euclidean dot product, and $(\vec{e}_i)$ a $(\cdot,\cdot)_g$-Euclidean associated basis. Let Ω be a $(\cdot,\cdot)_g$-antisymmetric endomorphism, so $\Omega^T = -\Omega$, cf. (E.13). Thus $[\Omega]_{|\vec{e}}$ is an antisymmetric matrix. Call $\overset{\circlearrowleft}{\omega}$ the associated pseudo-vector, i.e., cf. (E.28), for all $\vec{y} \in \vec{\mathbb{R}^3}$,

$$[\Omega]_{|\vec{e}}.[\vec{y}]_{|\vec{e}} = \overset{\circlearrowleft}{\omega} \overset{\circlearrowleft}{\times} [\vec{y}]_{|\vec{e}}. \tag{E.29}$$

This formula is widely used in mechanics, and unfortunately sometimes noted $\Omega.\vec{y} = \vec{\omega} \times \vec{y}$:

Be careful: (E.29) is ***not*** a vectorial formula; This is just a formula for matrix calculations which gives false result if a change of basis is considered; E.g., with $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$ be a $(\cdot,\cdot)_g$-Euclidean basis, and $(\vec{b}_1, \vec{b}_2, \vec{b}_3) = (-\vec{a}_1, \vec{a}_2, \vec{a}_3)$. So $(\vec{b}_i)$ is also a $(\cdot,\cdot)_g$-Euclidean basis, but with a different orientation.

1- Vector approach: Let P be the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, so $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $[\Omega]_{|\vec{a}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$. Thus, Ω being an endomorphism, the change of basis formula gives

$$[\Omega]_{|\vec{b}} = P^{-1}.[\Omega]_{|\vec{a}}.P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} . \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & c & -b \\ -c & 0 & -a \\ b & a & 0 \end{pmatrix}. \tag{E.30}$$

Thus the vectors $\vec{\omega}_a$ and $\vec{\omega}_b$ are given by (E.23):

$$[\vec{\omega}_a]_{|\vec{a}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad [\vec{\omega}_b]_{|\vec{b}} = \begin{pmatrix} a \\ -b \\ -c \end{pmatrix}, \quad \text{i.e.} \quad \left\{ \begin{matrix} \vec{\omega}_a = a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3, \\ \vec{\omega}_b = a\vec{b}_1 - b\vec{b}_2 - c\vec{b}_3, \end{matrix} \right\} \quad \text{thus} \quad \boxed{\vec{\omega}_b = -\vec{\omega}_a}. \tag{E.31}$$

Or simply apply (E.24).

2- Matrix approach (E.28) gives $[\Omega]_{|\vec{a}}.[\vec{y}] = \overset{\circlearrowleft}{\omega}_a \overset{\circlearrowleft}{\times} [\vec{y}]$ and $[\Omega]_{|\vec{b}}.[\vec{y}] = \overset{\circlearrowleft}{\omega}_b \overset{\circlearrowleft}{\times} [\vec{y}]$, with

$$\overset{\circlearrowleft}{\omega}_a = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \overset{\circlearrowleft}{\omega}_b = \begin{pmatrix} a \\ -b \\ -c \end{pmatrix}, \quad \text{so} \quad \boxed{\overset{\circlearrowleft}{\omega}_a \neq -\overset{\circlearrowleft}{\omega}_b}. \tag{E.32}$$

And $\overset{\circlearrowleft}{\omega}$ does not represent a single vector either, since it does not satisfy the vector change of basis formula $\overset{\circlearrowleft}{\omega}_b \neq P^{-1}.\overset{\circlearrowleft}{\omega}_a$. Thus $\overset{\circlearrowleft}{\omega}$ is not a vector (is not tensorial): It is just a matrix (called a "pseudo-vector").

E.4 Examples

E.4.1 Rectilinear motion

Let $\widetilde{\Phi} : [t_1, t_2] \times \mathit{Obj} \to \mathbb{R}^n$ be a C^1 motion. Let $t_0 \in]t_1, t_2[$ and $P_{Obj} \in \mathit{Obj}$.

Definition E.21 The motion of P_{Obj} is rectilinear iff, for all $t_0, t \in [t_1, t_2]$,

$$\frac{\widetilde{\Phi}_{P_{Obj}}(t) - \widetilde{\Phi}_{P_{Obj}}(t_0)}{t - t_0} \parallel \widetilde{\Phi}_{P_{Obj}}{}'(t_0), \tag{E.33}$$

i.e. $\forall t_0, t \in \mathbb{R}$, $\exists \alpha_{t_0,t} \in \mathbb{R}$, $\frac{\widetilde{\Phi}_{P_{Obj}}(t) - \widetilde{\Phi}_{P_{Obj}}(t_0)}{t - t_0} = \alpha_{t_0,t} \widetilde{\Phi}_{P_{Obj}}{}'(t_0)$. (E.g., $\widetilde{\Phi}_{P_{Obj}}(t) = O + (t - t_0)^2 \vec{e}_1$.)

The motion of P_{Obj} is rectilinear uniform iff, for all $t_0, t \in [t_1, t_2]$, with $p(t) = \widetilde{\Phi}(t, P_{Obj})$,

$$\widetilde{\Phi}_{P_{Obj}}(t) = \widetilde{\Phi}_{P_{Obj}}(t_0) + (t - t_0)\, \widetilde{\Phi}_{P_{Obj}}{}'(t_0), \quad \text{i.e.} \quad p(t) = p(t_0) + (t - t_0)\, \vec{V}^{t_0}(t_0, p(t_0)) \tag{E.34}$$

(the trajectory is traveled at constant velocity).

E.4.2 Circular motion

$\widetilde{\Phi} : [t_0,T] \times Obj \to \mathbb{R}^2$ is a motion, Φ^{t_0} is the associated motion, $P = \widetilde{\Phi}(t_0, P_{Obj})$. Let $(\vec{E}_1, \vec{E}_2)$ be a Euclidean basis. The motion $\Phi^{t_0}_P$ is a circular motion iff, for all t, $\overrightarrow{\mathcal{O}\Phi^{t_0}_P(t)} = x(t)\vec{E}_1 + y(t)\vec{E}_2$ with

$$\begin{cases} x(t) = a + R\cos(\theta(t)) \\ y(t) = b + R\sin(\theta(t)) \end{cases} \tag{E.35}$$

for some $R > 0$ (called the radius), some $a, b \in \mathbb{R}$, and some function $\theta : \mathbb{R} \to \mathbb{R}$. And $\begin{pmatrix} a \\ b \end{pmatrix} = \mathcal{O}_C \in \mathbb{R}^2$ is the center of the circle and $\theta(t)$ is the angle at t. And the particle P_{Obj} (s.t. $\widetilde{\Phi}(t_0, P_{Obj}) = P$) stays on the circle with center $\mathcal{O}_C$ and radius R.

The circular motion is uniforme iff, for all t, $\theta''(t) = 0$, that is, $\exists \omega_0 \in \mathbb{R}$, $\forall t \in [t_1, t_2]$, $\theta(t) = \omega_0 t$.

Notation: $\vec{\varphi}^{t_0}_P(t) = \overrightarrow{\mathcal{O}_C\Phi^{t_0}_P(t)}$, i.e.

$$\vec{\varphi}^{t_0}_P(t) = R\cos(\theta(t)\vec{E}_1 + R\sin(\theta(t))\vec{E}_2, \quad \text{so} \quad [\vec{\varphi}^{t_0}_P(t)]_{|\vec{E}} = \begin{pmatrix} R\cos(\theta(t)) \\ R\sin(\theta(t)) \end{pmatrix}. \tag{E.36}$$

Thus the Lagrangian velocity of a circular motion is

$$\vec{V}^{t_0}_P(t) = (\Phi^{t_0}_P)'(t) = (\vec{\varphi}^{t_0}_P)'(t), \quad \text{so} \quad [\vec{V}^{t_0}_P(t)]_{|\vec{E}} = R\theta'(t)\begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix} \tag{E.37}$$

(orthogonal to the radius vector $\vec{V}^{t_0}_P(t)$ is to $\vec{\varphi}^{t_0}_P(t)$). And the Lagrangian acceleration $\vec{\Gamma}^{t_0}_P(t)$ is given by

$$[\vec{\Gamma}^{t_0}_P(t)]_{|\vec{E}} = R\theta''(t)\begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix} + R(\theta'(t))^2\begin{pmatrix} -\cos(\theta(t)) \\ -\sin(\theta(t)) \end{pmatrix}. \tag{E.38}$$

Then consider the orthonormal basis $(\vec{e}_r(t), \vec{e}_\theta(t))$ given by

$$[\vec{e}_r(t)]_{|\vec{E}} = [\frac{\vec{\varphi}^{t_0}_P(t)}{||\vec{\varphi}^{t_0}_P(t)||}]_{|\vec{E}} = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}, \quad \text{and} \quad [\vec{e}_\theta(t)]_{|\vec{E}} = \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix}. \tag{E.39}$$

We get

$$\vec{V}^{t_0}_P = R\theta'\,\vec{e}_\theta \quad \text{and} \quad \vec{\Gamma}^{t_0}_P = R\big(\theta''\,\vec{e}_\theta - (\theta')^2\,\vec{e}_r\big). \tag{E.40}$$

Immersed in $\mathbb{R}^3$, the vertical line being given by $\vec{E}_3$:

$$\vec{V}^{t_0}_P(t) = \vec{\omega}(t) \times \vec{\varphi}^{t_0}_P(t), \quad \text{where} \quad \vec{\omega}(t) = \omega(t)\vec{e}_3 \quad \text{and} \quad \omega(t) = \theta'(t). \tag{E.41}$$

So

$$\vec{\Gamma}^{t_0} = \frac{d\vec{\omega}}{dt} \times \vec{\varphi}^{t_0}_P + \vec{\omega} \times \vec{V}^{t_0}_P = R\big(\frac{d\omega}{dt}\,\vec{e}_\theta - \omega^2\,\vec{e}_r\big). \tag{E.42}$$

E.4.3 Motion of a planet (centripetal acceleration)

Illustration: *Obj* is e.g. a planet from the solar system. $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is a Euclidean basis (e.g. fixed relative to stars an $(\vec{e}_1, \vec{e}_2)$ define the ecliptic plane), $(\cdot,\cdot)_g$ is the associated Euclidean dot product, $||.||$ the Euclidean associated norm, $\mathcal{O}$ an origin in $\mathbb{R}^3$ (e.g. the center of the Sun), $\mathcal{R} = (\mathcal{O}, (\vec{e}_i))$, $\widetilde{\Phi} : [t_0,T] \times Obj \to \mathbb{R}^3$ is a motion in $\mathcal{R}$, cf. (1.5), $\Phi^{t_0} =^{\text{noted}} \Phi$ and $\vec{\varphi}^{t_0} := \overrightarrow{\mathcal{O}\Phi_P} =^{\text{noted}} \vec{\varphi}$ are the associated motions, cf. (3.1)-(3.4). So the Lagrangian velocities and accelerations are given by

$$\vec{V}_P(t) = \frac{d\Phi_P}{dt}(t) = \frac{d\vec{\varphi}_P}{dt}(t), \quad \text{and} \quad \vec{A}_P(t) = \frac{d^2\Phi_P}{dt^2}(t) = \frac{d^2\vec{\varphi}_P}{dt^2}(t). \tag{E.43}$$

Definition E.22 The motion of a particle P_{Obj} is a centripetal acceleration motion iff the particle is not static and, at all time, its acceleration vector $\vec{A}(t)$ points to a fixed point F (focus).

We choose a focus F to be the origin of the referential: $\mathcal{O} := F$. So, for all t, $\overrightarrow{\mathcal{O}\Phi_P(t)} \parallel \vec{A}_P(t)$:

$$\overrightarrow{\mathcal{O}\Phi_P(t)} \times \vec{A}_P(t) = \vec{0}, \quad \text{i.e.} \quad \vec{\varphi}_P(t) \times \vec{A}_P(t) = \vec{0}. \tag{E.44}$$

Remark E.23 A rectilinear motion is a centripetal acceleration motion, but such a motion is usually excluded in the definition E.22. ∴

Example E.24 The motion of a planet from the solar system is a centripetal acceleration motion: An elliptical motion with one focus is at the center of the Sun. ∴

Example E.25 The second Newton's law of motion $\sum \vec{f} = m\vec{\gamma}$ (Galilean referential) gives: If at all time $\sum \vec{f}$ is directed to a unique point F, then the motion is a centripetal acceleration motion. ∴

Definition E.26 The areolar velocity at t is the vector

$$\vec{Z}(t) = \frac{1}{2}\vec{\varphi}_P(t) \times \vec{V}_P(t). \tag{E.45}$$

Proposition E.27 *If Φ is a centripetal acceleration motion, then the areolar velocity is contant, that is, $\frac{d\vec{Z}}{dt}(t) = \vec{0}$ pour tout t, so*

$$\vec{Z}(t) = \vec{Z}(t_0), \quad \forall t. \tag{E.46}$$

That is, the position vectors sweep equal areas in equal times. And $\vec{Z}(t_0) = \vec{0}$ iff Φ is a rectilinear motion.
If $\vec{Z}(t_0) \neq \vec{0}$ then :
- $\vec{\varphi}_P(t)$ and $\vec{V}_P(t)$ are orthogonal to $\vec{Z}(t_0)$ at all time t,
- The motion of the particle P_{Obj} takes place in the affine plane orthogonal to $\vec{Z}(t_0)$ passing through O.
- $\vec{V}_P(t)$ never vanishes.

Proof. (E.45) and (E.44) give $2\frac{d\vec{Z}}{dt}(t) = \frac{d\vec{\varphi}_P}{dt}(t) \times \vec{V}_P(t) + \vec{\varphi}(t) \times \frac{d\vec{V}_P}{dt}(t) = \vec{V}_P(t) \times \vec{V}_P(t) + \vec{\varphi}(t) \times \vec{A}_P(t) = \vec{0} + \vec{0}$. Thus $\vec{Z}$ is constant, $\vec{Z}(t) = \vec{Z}(t_0)$ for all t. In particular, if $\vec{Z}(t_0) \neq \vec{0}$ then $\vec{Z}(t) \neq \vec{0}$ pour tout t, and

• $\vec{Z}(t) = \frac{1}{2}\vec{\varphi}_P(t) \times \vec{V}_P(t)$ gives that $\vec{\varphi}_P(t)$ et $\vec{V}_P(t)$ are orthogonal to $\vec{Z}(t_0)$ for all t, thus $\vec{A}_P(t)$ is orthogonal to $\vec{Z}(t_0)$, cf. (E.44).

• The Taylor expansion reads $\vec{\varphi}_P(t) = \vec{\varphi}_P(t_0) + \vec{V}_P(t_0)(t-t_0) + \int_{\tau=t_0}^t \vec{A}_P(\tau)(t-\tau)^2\,d\tau$, with $\vec{V}_P(t_0) \perp \vec{Z}(t_0)$ and $\vec{A}_P(\tau) \perp \vec{Z}(t_0)$ for all τ, thus $\vec{\varphi}_P(t) - \vec{\varphi}_P(t_0) \perp \vec{Z}(t_0)$ for all τ, that is $\overrightarrow{Op(t)} - \overrightarrow{OP} = \overrightarrow{Pp(t)} \perp \vec{Z}(t_0)$ for all τ, Thus $p(t)$ belongs to the affine plane containing P orthogonal to $\vec{Z}(t_0)$, for all t. And $\overrightarrow{OP} = \vec{\varphi}_P(t_0) \perp \vec{Z}(t_0)$, thus O belong to the same plane.

• $\vec{Z}(t) = \vec{Z}(t_0) \neq \vec{0}$ implies $\vec{V}_P(t) \neq \vec{0}$ for all t, and (E.45) gives: $(\vec{\varphi}_P(t), \vec{V}_P(t), \vec{Z}(t_0))$ is a positively-oriented basis. Since $\vec{\varphi}_P$ and $\vec{V}_P$ are continuous and do not vanish, since $\vec{Z}(t_0) \neq \vec{0}$, we get: P_{Obj} "turns around $\vec{Z}(t_0)$" and its velocity never vanishes.

If $\vec{Z}(t) = \vec{0}$ then $\vec{\varphi}_P(t) \parallel \vec{V}_P(t)$ for all t, cf. (E.45), so $\vec{V}_P(t) = f(t)\vec{\varphi}_P(t)$ where f is some scalar function. And $\vec{V}_P(t) = \vec{\varphi}_P{}'(t)$ gives $\vec{\varphi}_P{}'(t) = f(t)\vec{\varphi}_P(t)$, thus $\vec{\varphi}_P(t) = \vec{\varphi}_P(t_0)e^{F(t)}$ where F is a primitive of f s.t. $F(t_0) = 0$, thus $\vec{\varphi}_P(t) \parallel \vec{\varphi}_P(t_0)$, so $\overrightarrow{O\Phi_P(t)} \parallel \overrightarrow{O\Phi_P(t_0)}$, for all t: The motion is rectilinear. ∴

Interpretation. (Non rectilinear motion.) The area swept by $\vec{\varphi}_P(t)$ is, at first order, the area of the triangle whose sides are $\vec{\varphi}_P(t)$ and $\vec{\varphi}_P(t+\tau)$ ("anglular sector"). So, with τ close to 0, let

$$\vec{S}_t(\tau) = \frac{1}{2}\vec{\varphi}_P(t) \times \vec{\varphi}_P(t+\tau), \quad \text{and} \quad S_t(\tau) = ||\vec{S}_t(\tau)||, \tag{E.47}$$

the vectorial and scalar areas. With $\vec{\varphi}_P(t+\tau) = \vec{\varphi}_P(t) + \vec{V}_P(t)\tau + o(\tau)$ (Taylor) we get

$$\vec{S}_t(\tau) = \frac{1}{2}\vec{\varphi}_P(t) \times (\vec{V}_P(t)\tau + o(\tau)), \tag{E.48}$$

Since $\vec{S}_t(0) = 0$ we get $\frac{\vec{S}_t(\tau) - \vec{S}(0)}{\tau} = \frac{1}{2}\vec{\varphi}_P(t) \times \vec{V}_P(t) + o(1)$, then

$$\frac{d\vec{S}_t}{d\tau}(0) = \frac{1}{2}\vec{\varphi}_P(t) \times \vec{V}_P(t) = \vec{Z}(t) = \vec{Z}(t_0), \tag{E.49}$$

thanks to (E.46), thus

$$\frac{d\vec{S}_t}{d\tau}(0) = \frac{d\vec{S}_{t_0}}{d\tau}(0), \qquad \forall t \in [t_0, T], \tag{E.50}$$

that is, the rate of variation of $\vec{S}_t$ is constant. And with $||\vec{S}_t(\Delta\tau)||^2 = (\vec{S}_t(\Delta\tau), \vec{S}_t(\Delta\tau))$ we get

$$\frac{d||\vec{S}_t||^2}{d\tau}(\Delta\tau) = 2(\frac{d\vec{S}_t}{d\tau}(\Delta\tau), \vec{S}_t(\Delta\tau)), \tag{E.51}$$

so, since $\vec{S}_t(0) = 0$,

$$\frac{d||\vec{S}_t||^2}{d\tau}(0) = 0. \tag{E.52}$$

So the function $t \to ||\vec{S}_t(0)||^2 = S_t(0)^2$ is constant, thus $t \to S_t(0)$ est constant, and $\frac{dS_t}{d\tau}(0)$ is constant.

Exercice E.28 Give a parametrization of the swept area, and redo the calculations.

Answer. Let

$$r(t) = ||\vec{\varphi}_P(t)||, \quad \theta(t) = \widehat{p(t)OP} \quad \text{(angle)}, \tag{E.53}$$

then

$$\vec{\varphi}_P(t) = \begin{pmatrix} r(t)\cos(\theta(t)) \\ r(t)\sin(\theta(t)) \\ 0 \end{pmatrix}. \tag{E.54}$$

Thus

$$\vec{V}_P(t) = \begin{pmatrix} r'(t)\cos(\theta(t) - r(t))\theta'(t)\sin(\theta(t)) \\ r'(t)\sin(\theta(t) + r(t))\theta'(t)\cos(\theta(t)) \\ 0 \end{pmatrix}. \tag{E.55}$$

With (E.45) we get

$$\vec{Z}(t) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ r^2(t)\theta'(t) \end{pmatrix}, \quad \text{with} \quad r^2(t)\theta'(t) = r^2(t_0)\theta'(t_0) \quad \text{(constant)}, \tag{E.56}$$

cf. (E.46). A parametrization of the swept area is then

$$\vec{\mathcal{A}} : \left\{ \begin{array}{ll} [0,1] \times [t_0, T] & \to \mathbb{R}^3 \\ (\rho, t) & \to \vec{\mathcal{A}}(\rho, t) \end{array} \right\}, \qquad \vec{\mathcal{A}}(\rho, t) = \begin{pmatrix} \rho\, r(t)\cos(\theta(t)) \\ \rho\, r(t)\sin(\theta(t)) \\ 0 \end{pmatrix}. \tag{E.57}$$

Therefore, the tangent associated vectors are

$$\frac{\partial\vec{\mathcal{A}}}{\partial\rho}(\rho, t) = \begin{pmatrix} r(t)\cos(\theta(t)) \\ r(t)\sin(\theta(t)) \\ 0 \end{pmatrix}, \qquad \frac{\partial\vec{\mathcal{A}}}{\partial t}(\rho, t) = \begin{pmatrix} \rho r'(t)\cos(\theta(t) - \rho r(t))\theta'(t)\sin(\theta(t)) \\ \rho r'(t)\sin(\theta(t) + \rho r(t))\theta'(t)\cos(\theta(t)) \\ 0 \end{pmatrix}, \tag{E.58}$$

hence the vectorial and scalare element areas are

$$d\vec{\sigma} = (\frac{\partial\vec{\mathcal{A}}}{\partial\rho} \times \frac{\partial\vec{\mathcal{A}}}{\partial t})d\rho dt = \begin{pmatrix} 0 \\ 0 \\ \rho r^2\theta'\, d\rho dt \end{pmatrix}, \qquad d\sigma = \rho r^2\theta'\, d\rho d\theta. \tag{E.59}$$

Therefore the area between t_0 and t is

$$\mathcal{A}(t) = \mathcal{A}(t_0) + \int_{\rho=0}^{1}\int_{\tau=t_0}^{t} \rho r^2(\tau)\theta'(\tau)\, d\rho d\tau = \frac{1}{2}\int_{\tau=t_0}^{t} r(\tau)^2\theta'(\tau)\, d\tau. \tag{E.60}$$

Hence

$$\mathcal{A}'(t) = r(t)^2\theta'(t) = r(t_0)^2\theta'(t_0) \quad (= \text{constant} = ||\vec{Z}(t_0)||), \tag{E.61}$$

cf. (E.56).

Exercice E.29 Prove the Binet formulas (non rectilinear central motion):

$$V_P(t)^2 = Z_0^2\Big(\frac{1}{r^2} + (\frac{d\frac{1}{r}}{d\theta})^2\Big)(t), \qquad \vec{\Gamma}_P(t) = -\frac{Z_0^2}{r^2}\Big(\frac{1}{r} + \frac{d^2\frac{1}{r}}{d\theta^2}\Big)(t)\,\vec{e}_r(t), \tag{E.62}$$

for the energy and the acceleration.

Answer. Proposition E.27 tells that Φ is a planar motion. With (E.53) and $\vec{e}_r(t) = \begin{pmatrix}\cos(\theta(t))\\ \sin(\theta(t))\end{pmatrix}$ we have $\vec{\varphi}(t) = r(t)\vec{e}_r(t)$ (in the plane). Let $\vec{e}_\theta(t) = \begin{pmatrix}-\sin(\theta(t))\\ \cos(\theta(t))\end{pmatrix}$, thus

$$\vec{V}(t) = \frac{dr}{dt}(t)\vec{e}_r(t) + r(t)\frac{d\vec{e}_r}{dt}(t) = r'(t)\vec{e}_r(t) + r(t)\theta'(t)\vec{e}_\theta(t).$$

And $\vec{e}_r(t) \perp \vec{e}_\theta(t)$ gives

$$V^2(t) = (r'(t))^2 + (r(t)\theta'(t))^2.$$

Since $\theta'(t) \neq 0$ for all t (non rectilinear central motion) Let $s(\theta(t)) = r(t)$. Let us suppose that θ is C^1, thus $\theta' > 0$ or $\theta' < 0$, and $\theta : t \to \theta(t)$ defines a change of variable. And

$$r'(t) = s'(\theta(t))\theta'(t).$$

And (E.61) and $\theta'(t) = \frac{Z_0}{r^2(t)}$ give

$$V^2(t(\theta)) = (s'(\theta))^2\frac{Z_0^2}{r^4(t)} + r^2(t)\frac{Z_0^2}{r^4(t)} = Z_0^2(\frac{(s'(\theta))^2}{s^4(\theta)} + \frac{1}{s^2(\theta)}) = Z_0^2[\Big(\frac{d\frac{1}{s}}{d\theta}(\theta)\Big)^2 + \frac{1}{s^2(\theta)}].$$

Thus $r(t) = s(\theta)$ and $\frac{dr}{d\theta} := \frac{ds}{d\theta}$ give the first Binet formula. Then

$$\vec{\Gamma}(t) = r''(t)\vec{e}_r(t) + r'(t)\frac{d\vec{e}_r}{dt}(t) + (r'(t)\theta'(t) + r(t)\theta''(t))\vec{e}_\theta(t) + r(t)\theta'(t)\frac{d\vec{e}_\theta}{dt}(t),$$

with $\frac{d\vec{e}_r}{dt} \parallel \vec{e}_\theta$, and $\frac{d\vec{e}_\theta}{dt}(t) = -\theta'(t)\vec{e}_r(t)$, and $\vec{e}_\theta \perp \vec{\Gamma}$ (central motion), we get

$$\vec{\Gamma}(t) = (r''(t) - r(t)(\theta'(t))^2)\vec{e}_r(t).$$

And

$$r'(t) = s'(\theta)\theta'(t) = s'(\theta)\frac{Z_0}{r^2(t)} = Z_0\frac{s'(\theta)}{s^2(\theta)} = -Z_0\frac{d\frac{1}{s}}{d\theta}(\theta),$$

thus

$$r''(t) = -Z_0\frac{d^2\frac{1}{s}}{d\theta^2}(\theta)\,\theta'(t) = -\frac{Z_0^2}{r^2(t)}\frac{d^2\frac{1}{s}}{d\theta^2}(\theta),$$

which is the second Binet formula. ▪

E.5 Screw theory (= torsors, distributors)

See https://perso.isima.fr/leborgne/IsimathMeca/torseur.pdf

F Riesz representation theorem

F.1 The Riesz representation theorem

Framework: $(E, (\cdot,\cdot)_g)$ is Hilbert space (a vector space with an inner dot product $(\cdot,\cdot)_g$) such that, with the associated norm defined by$||\vec{x}||_g := \sqrt{(\vec{x},\vec{x})_g}$, $(E, ||.||_g)$ is a complete space (a Banach space). E.g., $E = \vec{\mathbb{R}^n}$ with a Euclidean dot product, $L^2(\Omega)$ with its inner dot product $(f,g)_{L^2} = \int_\Omega fg\,d\Omega$...

And $E^* = \mathcal{L}(E;\mathbb{R})$ is the space of linear and continuous forms on E (the space of linear "measuring tools") equipped with its usual norm $||\ell||_{E^*} := \sup_{||\vec{x}||_g=1} |\ell.\vec{x}|$.

• We have the easy statement:

$$\forall \vec{v} \in E \text{ (vector)},\ \exists! v_g \in E^* \text{ (linear continuous form) s.t. } v_g.\vec{x} = (\vec{v},\vec{x})_g,\ \ \forall \vec{x} \in E, \tag{F.1}$$

moreover $||v_g||_{E^*} = ||\vec{v}||_g$.

Indeed: Define $v_g : E \to \mathbb{R}$ by $v_g(\vec{x}) = (\vec{v},\vec{x})_g$ for all $\vec{x} \in E$; The definition domain of v_g is E and v_g is trivially linear; And the Cauchy–Schwarz inequality gives $|v_g(\vec{x})| = |(\vec{v},\vec{x})_g| \le ||\vec{v}||_g\,||\vec{x}||_g$ for all $\vec{x} \in E$, thus $||v_g||_{E^*} \le ||\vec{v}||_g < \infty$, thus v_g is continuous; And $|v_g(\vec{v})| = |(\vec{v},\vec{v})_g| = ||\vec{v}||_g\,||\vec{v}||_g$, thus $||v_g||_{E^*} \ge ||\vec{v}||_g$, thus $||v_g||_{E^*} = ||\vec{v}||_g$. And uniqueness: Another w_g satisfying $w_g.\vec{x} = (\vec{v},\vec{x})_g$ gives $(w_g - v_g).\vec{x} = 0$ for all $\vec{x}$, thus $w_g - v_g = 0$.

• The Riesz representation theorem concerns the converse: A "measuring tool" $\ell \in E^*$ can represent with he help of $(\cdot,\cdot)_g$ by a vector $\vec{\ell}_g \in E$:

Theorem F.1 (Riesz representation theorem, and definition) *$(E,(\cdot,\cdot)_g)$ being a Hilbert space,*

$$\forall \ell \in E^* \text{ (linear continuous form)},\ \exists! \vec{\ell}_g \in E \text{ (vector) s.t. } \ell.\vec{x} = (\vec{\ell}_g, \vec{x})_g,\ \ \forall \vec{x} \in E, \tag{F.2}$$

and moreover $||\vec{\ell}_g||_g = ||\ell||_{E^*}$. And $\vec{\ell}_g$ is called the $(\cdot,\cdot)_g$-Riesz representation vector of ℓ.

(Usual notation in finite dimension: $v_g.\vec{x} = \vec{v} \bullet_g \vec{x}$, or simply $v.\vec{x} = \vec{v} \bullet \vec{x}$ if a chosen $(\cdot,\cdot)_g$ is imposed to all observers: Isometric framework.)

Proof. Easy in finite dimension: With a basis $(\vec{e}_i)$, if $[\ell]_{|\vec{e}} = (\,\ell_1 \ \ \dots \ \ \ell_n\,)$ (row matrix since ℓ is a linear form) then (F.2) gives $[\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}} = [\vec{\ell}_g]^T_{\vec{e}}.[g]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}$, thus $[\vec{\ell}_g]_{\vec{e}} = [g]^{-1}_{|\vec{e}}.[\ell]^T_{|\vec{e}}$ (column matrix), thus $\vec{\ell}_g$. Then $|\ell.\vec{x}| = |(\vec{\ell}_g,\vec{x})_g| \le ||\vec{\ell}_g||_g||\vec{x}||_g$, with $|\ell.\vec{\ell}_g| = |(\vec{\ell}_g,\vec{\ell}_g)_g| = ||\vec{\ell}_g||_g||\vec{\ell}_g||_g$, thus $||\ell||_{E^*} = ||\vec{\ell}_g||_g$.

General case, infinite dimension (e.g. $E = L^2(\Omega)$ and the finite element method). Let $\ell \in E^*$. ℓ being linear and continuous, its kernel $\text{Ker}\ell = \ell^{-1}(\{0\})$ is a closed sub-vector space in E. If $\ell = 0$ then $\vec{\ell}_g = \vec{0}$ (trivial). Suppose $\ell \neq 0$, thus $\text{Ker}\ell \subsetneq E$. Thus if $\vec{z} \notin \text{Ker}\ell$ then $\exists! \vec{z}_0 \in \text{Ker}\ell$ (called the $(\cdot,\cdot)_g$-orthogonal projection of $\vec{z}$ on $\text{Ker}\ell$), given by: $\forall \vec{y}_0 \in \text{Ker}\ell$, $(\vec{z}-\vec{z}_0, \vec{y}_0)_g = 0$, so $\vec{z}-\vec{z}_0 \perp_g \text{Ker}\ell$. Then let $\vec{n} := \frac{\vec{z}-\vec{z}_0}{||\vec{z}-\vec{z}_0||_g}$, so $\vec{n} \in (\text{Ker}\ell)^\perp$ (and unitary); Moreover $\dim(\text{Ker}\ell)^\perp = 1$ ($= \dim\mathbb{R}$ = dimension of the codomain of ℓ, see next exercise F.2), so $(\text{Ker}\ell)^\perp = \text{Vect}\{\vec{n}\}$. And $E = \text{Ker}\ell \oplus (\text{Ker}\ell)^\perp$ since both vector spaces are closed (an orthogonal is always closed in a Hilbert space), thus any $\vec{x} \in E$ satisfies $\vec{x} = \vec{x}_0 + (\vec{x}-\vec{x}_0) = \vec{x}_0 + \lambda\vec{n} \in \text{Ker}\ell \oplus (\text{Ker}\ell)^\perp$ where $(\vec{x},\vec{n})_g = 0 + \lambda 1$ and $\ell(\vec{x}) = 0 + \lambda\ell(\vec{n})$, thus $\ell(\vec{x}) = (\vec{x},\vec{n})_g\ell(\vec{n}) = (\vec{x},\ell(\vec{n})\vec{n})_g$ (bilinearity of $(\cdot,\cdot)_g$); Thus $\vec{\ell}_g := \ell(\vec{n})\vec{n}$ satisfies (F.2). And if $\vec{\ell}_{g1}$ and $\vec{\ell}_{g2}$ satisfy (F.2) then $(\vec{\ell}_{g1}-\vec{\ell}_{g2},\vec{x})_g = 0$ for all $\vec{x} \in E$, thus $\vec{\ell}_{g1}-\vec{\ell}_{g2} = 0$. Thus $\vec{\ell}_g$ is unique. And $||\ell||_{E^*} := \sup_{||\vec{x}||_g=1}|\ell(\vec{x})| = \sup_{||\vec{x}||_g=1}|(\vec{\ell}_g,\vec{x})_g| =^{Cauchy}_{Schwarz} ||\vec{\ell}_g||_g$. ∎

Exercice F.2 Prove: If $\ell \in E^*-\{0\}$ then $\dim(\text{Ker}\ell)^\perp = 1$ ($= \dim(\text{Im}(\ell)) = \dim\mathbb{R}$).

Answer. Consider the restriction $\ell_{|\text{Ker}\ell^\perp} : \begin{Bmatrix} (\text{Ker}\ell)^\perp & \to \mathbb{R} \\ \vec{x} & \to \ell_{|\text{Ker}\ell^\perp}.\vec{x} := \ell.\vec{x} \end{Bmatrix}$. It is linear (since ℓ is), it is onto since ℓ is linear and $\ell \neq 0$. And it is one to one since $\ell_{|\text{Ker}\ell^\perp}(\vec{x}) = 0 = \ell(\vec{x})$ gives $\vec{x} \in (\text{Ker}\ell)^\perp \bigcap \text{Ker}\ell = \{\vec{0}\}$ thus $\vec{x} = 0$; Thus $\ell_{|\text{Ker}\ell^\perp}$ is (linear) bijective, thus $\dim(\text{Ker}\ell)^\perp = \dim(\mathbb{R}) = 1$. ∎

F.2 The $(\cdot,\cdot)_g$-Riesz representation operator

The Riesz representation theorem F.1 gives the $(\cdot,\cdot)_g$-Riesz representation operator

$$\vec{R}_g : \begin{Bmatrix} E^* & \to E \\ \ell & \to \vec{R}_g(\ell) := \vec{\ell}_g, \end{Bmatrix} \quad \text{where} \quad (\underbrace{\vec{R}_g(\ell)}_{\vec{\ell}_g}, \vec{v})_g = \ell.\vec{v},\ \ \forall \vec{v} \in E. \tag{F.3}$$

Proposition F.3 *$\vec{R}_g$ is an isomorphism between Banach spaces. And $\vec{R}_g$ is a change of variance tool:*

$$\vec{R}_g \text{ transforms a « covariant } \ell \text{ » into a « contravariant } \vec{\ell}_g \text{ » thanks to the tool } (\cdot,\cdot)_g. \tag{F.4}$$

Proof. Linearity: $(\vec{R}_g(\ell+\lambda m),\vec{x})_g = (\ell+\lambda m).\vec{x} = \ell.\vec{x} + \lambda m.\vec{x} = (\vec{R}_g(\ell),\vec{x})_g + \lambda(\vec{R}_g(m),\vec{x})_g = (\vec{R}_g(\ell) + \lambda\vec{R}_g(m),\vec{x})_g$, for all $\vec{x}$, gives $\vec{R}_g(\ell+\lambda m) = \vec{R}_g(\ell) + \lambda\vec{R}_g(m)$. Bijectivity thanks to (F.1) and (F.2), and $||\vec{\ell}_g||_g = ||\ell||_{E^*}$ thanks to the Riesz representation theorem: Isomorphism between Banach spaces. ∎

NB (fundamental): $\vec{R}_g$ is ***not*** objective since it requires a man made tool (an inner dot product e.g. English or French) to be defined. In fact, an isomorphism $E \leftrightarrow E^*$ can ***never*** be objective, see § U.2.

With $\mathcal{G}$ the set of inner dot products in E, we have thus defined the Riesz representation mapping

$$\vec{R} : \begin{cases} \mathcal{G} \times E^* & \to E \\ (g,\ell) & \to \vec{R}(g,\ell) := \vec{\ell}_g = \vec{R}_g(\ell) = \vec{\ell}(g). \end{cases} \tag{F.5}$$

So $\vec{R}$ has two inputs: A choice $(\cdot,\cdot)_g$ by an observer for the first slot, a linear form for the second slot.

F.3 Quantification with a basis

Here E is finite dimensional, $\dim E = n$, $\ell \in E^*$ (a linear form), $(\cdot,\cdot)_g$ is an inner dot product, $(\vec{e}_i)$ is a basis, (e^i) is the dual basis (duality notations). Let

$$g_{ij} = g(\vec{e}_i, \vec{e}_j), \quad \ell = \sum_{j=1}^n \ell_j e^j, \quad \vec{\ell}_g = \sum_{i=1}^n (\vec{\ell}_g)^i \vec{e}_i, \quad \vec{R}_g.e^j = \sum_{i=1}^n R_g^{ij} \vec{e}_i, \quad \text{i.e.,}$$
$$[g]_{|\vec{e}} = [g_{ij}], \quad [\ell]_{|e} = (\, \ell_1 \;\; ... \;\; \ell_n \,) \text{ (row)}, \quad [\vec{\ell}_g]_{|\vec{e}} = \begin{pmatrix} (\vec{\ell}_g)^1 \\ \vdots \\ (\vec{\ell}_g)^n \end{pmatrix} \text{ (column)}, \quad [\vec{R}]_{|e,\vec{e}} = [R^{ij}]. \tag{F.6}$$

Then (F.2) gives $[\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}} = [\vec{\ell}_g]^T_{|\vec{e}}.[g]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}$ for all $\vec{x}$, thus $[\ell]_{|\vec{e}} = [\vec{\ell}_g]^T_{|\vec{e}}.[g]_{|\vec{e}}$, thus $[\ell]^T_{|\vec{e}} = [g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}}$ (since $[g]_{|\vec{e}} = [g]^T_{|\vec{e}}$), thus

$$\boxed{[\vec{\ell}_g] = [g]^{-1}.[\ell]^T}, \quad \text{i.e.} \quad (\vec{\ell}_g)^i = \sum_{j=1}^n ([g]^{-1})_{ij} \ell_j, \; \forall i. \tag{F.7}$$

And $\vec{R}_g.\ell =^{(F.3)} \vec{\ell}_g$ gives $[\vec{R}_g]_{|e,\vec{e}}.[\ell]^T_{|\vec{e}} = [\vec{\ell}_g]_{\vec{e}}$, thus

$$\boxed{[\vec{R}_g] = [g]^{-1}}, \quad \text{i.e.} \quad R_g^{ij} = ([g]^{-1})_{ij}, \; \forall i,j, \quad \text{thus} \quad \boxed{(\vec{\ell}_g)^i = \sum_{j=1}^n R^{ij} \ell_j}, \; \forall i. \tag{F.8}$$

Remark F.4 If a chosen inner dot product $(\cdot,\cdot)_g$ is imposed (e.g. Euclidean foot based) and if duality notations are used, then a usual notation for $\vec{\ell}_g$ is $\ell^\sharp$, because the bottom index i in ℓ_i has been raised by $\vec{R}_g$ to give $\ell_g^i =^{\text{noted}} \ell^i$. Then (F.2) and (F.8) read, with $\ell^\sharp := \vec{\ell}_g = \sum_i \ell^i \vec{e}_i$,

$$\ell.\vec{x} = \ell^\sharp \bullet \vec{x} \quad \text{and} \quad \begin{pmatrix} \ell^1 \\ \vdots \\ \ell^n \end{pmatrix} = [\vec{R}_g].\begin{pmatrix} \ell_1 \\ \vdots \\ \ell_n \end{pmatrix}, \tag{F.9}$$

We won't use this $\ell^\sharp$ notation (we deal with objectivity: No isometric framework imposed). ∎

F.4 Change of Riesz representation vector, and Euclidean case

Let $\ell \in E^*$, let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be two inner dot products, let $\vec{\ell}_g := \vec{R}_g(\ell)$ and $\vec{\ell}_h := \vec{R}_h(\ell)$, so, $\forall \vec{x} \in E$,

$$(\vec{\ell}_g, \vec{x})_g = \ell.\vec{x} = (\vec{\ell}_h, \vec{x})_h. \tag{F.10}$$

Proposition F.5 *For any basis $(\vec{e}_i)$ in E, we have the change of Riesz representation vector formula:*

$$[h].[\vec{\ell}_h] = [g].[\vec{\ell}_g], \quad \textit{i.e.} \quad [\vec{\ell}_h] = [h]^{-1}.[g].[\vec{\ell}_g], \tag{F.11}$$

short notation for $[h]_{|\vec{e}}.[\vec{\ell}_h]_{|\vec{e}} = [g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}}$, i.e. $[\vec{\ell}_h]_{|\vec{e}} = [h]^{-1}_{|\vec{e}}.[g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}}$.

NB: (F.11) is a "change of vector" formula: one basis, two vectors; It is not a "change of basis" formula (one vector and two sets of components). In particular (for the Euclidean case):

$$\textit{If} \;\; (\cdot,\cdot)_g = \lambda^2 (\cdot,\cdot)_h \quad \textit{then} \quad \vec{\ell}_h = \lambda^2 \vec{\ell}_g. \tag{F.12}$$

Conversely, if $\vec{\ell}_h = \lambda^2 \vec{\ell}_g$ for all linear forms $\ell \in E^*$, then $(\cdot,\cdot)_g = \lambda^2 (\cdot,\cdot)_h$.

*So, a linear form ℓ can**not** be identified with a Riesz representation vector (which one: $\vec{\ell}_g$? $\vec{\ell}_h$?).*

Proof. (F.10) gives $[\vec{x}]^T_{|\vec{e}}.[g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}} = [\vec{x}]^T_{|\vec{e}}.[h]_{|\vec{e}}.[\vec{\ell}_h]_{|\vec{e}}$ for all $\vec{x}$, hence $[g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}} = [h]_{|\vec{e}}.[\vec{\ell}_h]_{|\vec{e}}$, i.e. (F.11).

In particular $\lambda^2(\cdot,\cdot)_h = (\cdot,\cdot)_g$ give $\lambda^2(\vec{\ell}_g, \vec{x})_h = (\vec{\ell}_g, \vec{x})_g =^{(F.10)} (\vec{\ell}_h, \vec{x})_h$ for all $\vec{x}$, hence $\lambda^2 \vec{\ell}_g = \vec{\ell}_h$.

Converse: $\lambda^2 \vec{\ell}_g = \vec{\ell}_h$ for all ℓ gives $\lambda^2(\vec{\ell}_g, \vec{x})_h = (\vec{\ell}_h, \vec{x})_h \overset{(F.10)}{=} (\vec{\ell}_g, \vec{x})_g$, for all $\vec{x}$ and for all $\vec{\ell}_g$ (because $\vec{R}_g$ is an isomorphism cf. prop. (F.3)), thus $\lambda^2(\cdot,\cdot)_h = (\cdot,\cdot)_g$. ∎

Example F.6 If $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are the Euclidean dot products made with the foot and the metre then, with (F.10),

$$(\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h \quad\Longrightarrow\quad \vec{\ell}_h = \lambda^2\vec{\ell}_g, \quad \text{with} \quad \lambda^2 > 10: \tag{F.13}$$

A linear form ℓ is represented by quite different vectors by an English observer ($\vec{\ell}_g$ "small") and by a French observer ($\vec{\ell}_h$ "big")! So a Riesz representation vector is (very) subjective, and certainly not "canonical" (a word that you may find in books where... nothing is defined... nor justified...). ▪

Example F.7 Aviation: If you do want to use a Riesz representation vector to represent a $\ell \in \mathbb{R}^{n*}$, it is vital to know which Euclidean dot product is in use, cf. the Mars Climate Orbiter probe crash (remark A.17). Recall: The foot is the international unit of altitude for aviation. ▪

F.5 Riesz representation vector and gradients

If $f \in C^1(\mathbb{R}^n;\mathbb{R})$ and $p \in \mathbb{R}^n$, the differential of f at p is the linear form $df(p) \in \mathbb{R}^{n*}$ defined by, for all $\vec{w} \in \vec{\mathbb{R}^n}$,

$$df(p).\vec{w} := \lim_{h\to 0} \frac{f(p+h\vec{w}) - f(p)}{h}. \tag{F.14}$$

See (T.6) (definition independent of any inner dot product or basis).

If you choose an inner dot product $(\cdot,\cdot)_g$ then you can define the gradient $\vec{\mathrm{grad}}_g f(p)$: It is the $(\cdot,\cdot)_g$-Riesz representation vector of $df(p)$:

$$\vec{\mathrm{grad}}_g f(p) := \vec{R}_g(df(p)), \quad \text{i.e.} \quad df(p).\vec{w} = (\vec{\mathrm{grad}}_g f(p), \vec{w})_g, \quad \forall \vec{w} \in \vec{\mathbb{R}^n}. \tag{F.15}$$

E.g. (F.13) gives

$$\vec{\mathrm{grad}}_h f(p) = \lambda^2 \vec{\mathrm{grad}}_g f(p) \quad \text{with} \quad \lambda^2 > 10 \ \text{(English vs French)}: \tag{F.16}$$

The gradient is very dependent on the observer (a gradient is subjective, the differential is objective).

Remark F.8 We already had this observer dependence in the 1-D case $f : x \in \mathbb{R} \to f(x) \in \mathbb{R}$:

Question: What does $f'(x) = 3$ mean? Answer:

11- For one observer, it means $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$... where in the departure space the observer has chosen a basis vector $\vec{a}$ of length 1 for him (e.g. 1 foot, 1 Fahrenheit...) which he calls $\vec{a} = 1$; So, with no abusive notations, his derivative $f'(x)$ is in fact $f'_a(x) := df(x).\vec{a} = \lim_{h\to 0} \frac{f(x+h\vec{a})-f(x)}{h}$.

12- For another observer, it means $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$... where in the departure space the observer has chosen a basis vector $\vec{b}$ of length 1 for him (e.g. 1 metre, 1 Celsius...), and he write $\vec{b} = 1$; So, with no abusive notations, his derivative $f'(x)$ is in fact $f'_b(x) := df(x).\vec{b} = \lim_{h\to 0} \frac{f(x+h\vec{b})-f(x)}{h}$.

13- If $\vec{b} = \lambda\vec{a}$, then

$$\lim_{h\to 0} \frac{f(x+h\vec{b}) - f(x)}{h} = \lim_{h\to 0} \frac{f(x+h\lambda\vec{a}) - f(x)}{h} = \lambda \lim_{h\to 0} \frac{f(x+h\lambda\vec{a}) - f(x)}{h\lambda} = \lambda \lim_{k\to 0} \frac{f(x+k\vec{a}) - f(x)}{k}.$$

E.g. with foot and metre,

$$f'_b(x) = \lambda f'_a(x), \quad \text{with} \quad \lambda \simeq 3.28, \quad \text{so} \quad f'_b(x) \neq f'_a(x). \tag{F.17}$$

In other words, $f'(x) = \dfrac{\text{opposite side}}{\text{adjacent side}}$ depends on the length of the adjacent side: In foot? metre?... ▪

Exercice F.9 We have $f'_b(x) =^{(F.17)} \lambda f'_a(x)$ and $\vec{\mathrm{grad}}_b f(x) =^{(F.16)} \lambda^2 \vec{\mathrm{grad}}_a f(x)$. Why?

Answer. Because (F.17) does not use the Riesz representation theorem. Details: $(\vec{a})$ and $(\vec{b})$ are two bases in $\vec{\mathbb{R}}$, associated inner dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$, and $\vec{b} = \lambda\vec{a}$; thus $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$. And $f'_b(x) = \lambda f'_a(x)$ gives $(\vec{\mathrm{grad}} f_b(x), \vec{b})_b =^{(F.15)} df(x).\vec{b} = f'_b(x) = \lambda f'_a(x) = \lambda df(x).\vec{a} =^{(F.15)} \lambda(\vec{\mathrm{grad}} f_a(x), \vec{a})_a = (\vec{\mathrm{grad}} f_b(x), \lambda\vec{a})_b = \lambda^2(\vec{\mathrm{grad}} f_a(x), \vec{a})_b$, so $\vec{\mathrm{grad}} f_b(x) = \lambda^2 \vec{\mathrm{grad}} f_a(x)$ as expected. ▪

Exercice F.10 With $||.||_g = \lambda||.||_h$ we have $||\vec{\ell}_h||_g = \lambda||\vec{\ell}_h||_h$. Does it contradict the Riesz representation theorem which gives $||\ell|| = ||\vec{\ell}_g||$?

Answer. No, because $||\ell|| := \sup_{\vec{x}} \frac{|\ell.\vec{x}|}{||\vec{x}||_{\mathbb{R}^n}}$ depends on the norm $||.||_{\mathbb{R}^n}$ chosen; Here $||.||_{\mathbb{R}^n}$ is either $||.||_g$ or $||.||_h$. And if $||\ell||_g := \sup_{\vec{x}} \frac{|\ell.\vec{x}|}{||\vec{x}||_g}$ (you have chosen the $||.||_{\mathbb{R}^n} := ||.||_g$), then $||\ell||_h = \sup_{\vec{v}\in\vec{\mathbb{R}^n}} \frac{|\ell.\vec{v}|}{||\vec{v}||_h} = \sup_{\vec{v}\in\vec{\mathbb{R}^n}} \frac{|\ell.\vec{v}|}{\frac{1}{\lambda}||\vec{v}||_g} = \lambda \sup_{\vec{v}\in\vec{\mathbb{R}^n}} \frac{|\ell.\vec{v}|}{||\vec{v}||_g} = \lambda||\ell||_g$. Don't forget: $||\ell|| = \sup(...)$ depends on the choice of a norm: $||.||_g$? $||.||_h$? ▪

F.6 A Riesz representation vector is contravariant

$\vec{\ell}_g$ is a vector in E, cf. (F.2), so it is contravariant. To be convinced:

Exercice F.11 Check:

$$[\vec{\ell}_g]_{|new} = P^{-1}.[\vec{\ell}_g]_{|old} \quad \text{(contravariance formula).} \tag{F.18}$$

Answer. Consider two bases $(\vec{e}_{old,i})$ and $(\vec{e}_{new,i})$ in E. With the change of basis formulas $[\vec{x}]_{|new} = P^{-1}.[\vec{x}]_{|old}$ and $[g]_{|new} = P^T.[g]_{|old}.P$, (F.2) gives, for all $\vec{x}$,

$$\begin{aligned}[\vec{x}]^T_{|old}.[g]_{|old}.[\vec{\ell}_g]_{|old} = \ell.\vec{x} &= [\vec{x}]^T_{|new}.[g]_{|new}.[\vec{\ell}_g]_{|new} \\ &= ([\vec{x}]^T_{|old}.P^{-T}).(P^T.[g]_{|old}.P).[\vec{\ell}_g]_{|new} = [\vec{x}]^T_{|old}.[g]_{|old}.(P.[\vec{\ell}_g]_{|new}),\end{aligned} \tag{F.19}$$

thus $[\vec{\ell}_g]_{|old} = P.[\vec{\ell}_g]_{|new}$ since $[g]$ is invertible (an inner dot product is positive definite), thus (F.18). ∎

Remark F.12 • Dont forget: A representation vector $\vec{\ell}_g$ is not intrinsic to the linear form ℓ because it depends on a $(\cdot,\cdot)_g$ (depends on a observer: foot? metre?). More generally, there is no natural canonical isomorphism between E and E^*, see § U.2: It is impossible to identify a linear form with a vector.
• $\vec{\ell}_g$ is ***not*** compatible with the use of push-forwards and pull-backs, cf. § 7.2.
• $\vec{\ell}_g$ is ***not*** compatible with the use of Lie derivatives, cf. (9.54). ∎

F.7 What is a vector versus a $(\cdot,\cdot)_g$-vector?

1- Originally, a vector was a bipoint vector $\vec{x} = \overrightarrow{AB}$ in $\vec{\mathbb{R}}^3$ used to represent a "material object". E.g. the height of a child is represented on a wall by a vertical bipoint vector $\vec{x}$ starting from A the ground up to B a pencil line. The vector $\vec{x}$ is objective: Any observer uses this same vector to get the height of the child... And then they use "their subjective unit" (foot, metre...) to give a value.

2- Then (mid 19th century), the concept of vector space was introduced: It is a quadruplet $(E, +, K, .)$ where $+$ is an inner law, $(E, +)$ is a group, K is a field, . is a external law on E (called a scalar multiplication) compatible with $+$ (see any math book).

3- And the definition of scalar inner dot product $(\cdot,\cdot)_g$ (in a vector space) was introduced.

4- We can then get non "material" vectors ("subjectively built vectors"). E.g.: start with our usual vector space $\vec{\mathbb{R}}^n$ of bi-point vectors, and consider its dual $\mathbb{R}^{n*} := \mathcal{L}(\vec{\mathbb{R}}^n;\mathbb{R})$. For a given $\ell \in \mathbb{R}^{n*}$ (a given measuring device), consider two observers: An English observer with his foot built Euclidean dot product $(\cdot,\cdot)_g$, and a French observer with with his metre built Euclidean dot product $(\cdot,\cdot)_h$. These observers build their own artificial (man made) Riesz representation vectors $\vec{\ell}_g = \vec{R}_g(\ell)$ and $\vec{\ell}_h = \vec{R}_h(\ell)$, cf (F.13); They remark that $\vec{\ell}_g \neq \vec{\ell}_h$: Their man made vectors are different (subjective).

5- Then, with differential geometry, a vector $\vec{v}$ has been redefined: It is a "tangent vector", which means that there exists a C^1 curve $c : s \in [a,b] \to c(s) \in E$ such that $\vec{v}$ is defined at a $p = c(s) \in \mathrm{Im}(c)$ by $\vec{v}(p) := \vec{c}\,'(s)$. Advantage: This definition of a tangent vector is applicable to "tangent vectors to a surface" i.e. tangent vectors to a manifold, see e.g. § 9.1.1,2-. Then it is shown that $\vec{v}$ is equivalent to $\frac{\partial}{\partial \vec{v}}$ = the directional derivative in the direction $\vec{v}$ (natural canonical isomorphism $E \simeq E^{**}$ see § U.3).
For other equivalent definitions of vectors, see e.g. Abraham–Marsden [1].

F.8 The "$(\cdot,\cdot)_g$-dual vectorial bases" of one basis (and warnings)

Framework: E is a finite dimensional vector space, $\dim E = n$ (e.g. $E = \vec{\mathbb{R}}^3$). An observer chooses an inner dot product $(\cdot,\cdot)_g$ (e.g., in $\vec{\mathbb{R}}^3$, a foot-built Euclidean dot product, hence the results will be subjective). And $(\vec{e}_i)$ is some basis in E.

F.8.1 A basis and its many associated "dual vectorial basis"

Definition F.13 Let $(\vec{e}_i)$ be a basis in E. Its $(\cdot,\cdot)_g$-dual vectorial basis (or $(\cdot,\cdot)_g$-vectorial dual basis, or $(\cdot,\cdot)_g$-dual basis) is the basis $(\vec{e}_{ig})$ in E defined by

$$\forall j = 1,...,n, \quad (\vec{e}_{ig}, \vec{e}_j)_g = \delta_{ij}, \quad \text{i.e.} \quad \boxed{\vec{e}_{ig} \bullet_g \vec{e}_j = \delta_{ij}}. \tag{F.20}$$

NB: A vectorial dual basis is not unique: It depends on the chosen inner dot product, see e.g. (F.24).

Definition F.14 (Equivalent definition.) Let (π_{ei}) be the (covariant) dual basis of the basis $(\vec{e}_i)$: The π_{ei} are the linear forms defined by $\pi_{ei}.\vec{e}_j = \delta_{ij}$ for all j, cf. (A.7). (The $\pi_{ei} \in E^*$ are objective, i.e. the same for all observers). The $(\cdot,\cdot)_g$-dual vectorial basis of the basis $(\vec{e}_i)$ is the basis $(\vec{e}_{ig})$ in E made of the $(\cdot,\cdot)_g$-Riesz representative vectors of the π_{ei}:

$$\boxed{\vec{e}_{ig} := \vec{R}_g(\pi_{ei})}, \quad \text{so defined by} \quad \vec{e}_{ig} \bullet_g \vec{v} = \pi_{ei}.\vec{v}, \quad \forall \vec{v} \in E. \tag{F.21}$$

where $\vec{R}_g$ is the $(\cdot,\cdot)_g$-Riesz operator, see (F.3).

Duality notations: with (e^i) the dual basis,

$$\vec{e}_{ig} := \vec{R}_g(e^i), \quad \text{i.e.} \quad (\vec{e}_{ig}, \vec{v})_g = e^i.\vec{v}, \ \forall \vec{v} \in E. \tag{F.22}$$

The position of the index i is down on the left and up on the right, because $\vec{R}_g$ changes the variance type.

NB: Pay attention to the notations: $\vec{e}_{ig}$ is a contravariant vector: $\vec{e}_{ig} \in E$. So if you use the Einstein convention then the index i in $\vec{e}_{ig}$ must be a bottom index.

Exercice F.15 Prove that the vectors $\vec{e}_{ig}$ satisfy the contravariant change of basis formula

$$[\vec{e}_{ig}]_{|new} = P^{-1}.[\vec{e}_{ig}]_{|old} \quad \text{(the } \vec{e}_{jg} \text{ are “contravariant vectors”).} \tag{F.23}$$

Answer. • First answer: $\vec{e}_{ig}$ is a vector in E, thus it is contravariant.

• Second answer: Apply (F.18) since $\vec{e}_{ig}$ is a Riesz-representation vector.

• Third answer = direct computation: Consider two bases $(\vec{a}_i)$ and $(\vec{b}_i)$, and the transition matrix P from $(\vec{a}_i)$ to $(\vec{b}_i)$. (F.20) and the change of basis formulas give $[\vec{e}_j]_{|\vec{a}}^T.[g]_{|\vec{a}}.[\vec{e}_{ig}]_{|\vec{a}} = (\vec{e}_{ig}, \vec{e}_j)_g = [\vec{e}_j]_{|\vec{b}}^T.[g]_{|\vec{b}}.[\vec{e}_{ig}]_{|\vec{b}} = (P^{-1}.[\vec{e}_j]_{|\vec{a}})^T.(P^T.[g]_{|\vec{a}}.P).[\vec{e}_{ig}]_{|\vec{b}} = [\vec{e}_j]_{|\vec{a}}^T.[g]_{|\vec{a}}.P.[\vec{e}_{ig}]_{|\vec{b}}$, for all i,j, thus $[\vec{e}_{ig}]_{|\vec{a}} = P.[\vec{e}_{ig}]_{|\vec{b}}$, for all i, i.e. (F.23). ▪

Exercice F.16 Choose one basis $(\vec{e}_i)$ in E. Consider two inner dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ (e.g., a foot and a metre built Euclidean dot products). Call $(\vec{e}_{ia})$ and $(\vec{e}_{ib})$ the $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$-dual vectorial bases of the basis $(\vec{e}_i)$. Prove:

$$(\cdot,\cdot)_a = \lambda^2 (\cdot,\cdot)_b \implies \vec{e}_{ib} = \lambda^2 \vec{e}_{ia}, \ \forall i. \tag{F.24}$$

E.g., $\lambda^2 > 10$ with foot and metre built Euclidean bases: $\vec{e}_{ib}$ is much bigger than $\vec{e}_{ia}$: A vectorial dual basis is ***not*** intrinsic to $(\vec{e}_i)$ (***not*** objective).

Answer. (F.20) gives $(\vec{e}_{ib}, \vec{e}_j)_b = \delta_{ij} = (\vec{e}_{ia}, \vec{e}_j)_a = \lambda^2 (\vec{e}_{ia}, \vec{e}_j)_b$, thus $(\vec{e}_{ib} - \lambda^2 \vec{e}_{ia}, \vec{e}_j)_b = \delta_{ij}$, for all i,j. ▪

Example F.17 If $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis we trivially get $\vec{e}_{ig} = \vec{e}_i$ for all i, i.e., $(\vec{e}_{ig}) = (\vec{e}_i)$.This particular case is not compatible with joint work by an English (foot) and a French (metre) observer. ▪

F.8.2 Components of $\vec{e}_{jg}$ in the basis $(\vec{e}_i)$

Proposition F.18 *The components of $\vec{e}_{jg}$ in the basis $(\vec{e}_i)$ are the R_g^{ij}: for any $j \in [1,n]_\mathbb{N}$,*

$$\vec{e}_{jg} = \sum_{i=1}^n R_g^{ij} \vec{e}_i, \quad \text{i.e.} \quad \vec{e}_{jg} = \sum_{i=1}^n P^i_j \vec{e}_i \ \text{ where } P^i_j = R_g^{ij}, \quad \text{i.e.} \quad [\vec{e}_{jg}]_{|\vec{e}} = [\vec{R}_g]_{|\vec{e}}.[\vec{e}_j]_{|\vec{e}} \tag{F.25}$$

(the j-th column of $[g]_{|\vec{e}}{}^{-1} = [\vec{R}_g]_{|\vec{e}}$). And $[P] = [P^i_j] = [R_g^{ij}]$ is the transition matrix form $(\vec{e}_i)$ to $(\vec{e}_{ig})$. (Recall $\vec{e}_{jg} = \vec{R}_g(e^i)$: Change of variance, thus the position of the index.)

Use classic notations if you prefer: $\vec{e}_{jg} = \sum_i P_{ij} \vec{e}_i = \sum_i R_{g,ij} \vec{e}_i$.

Thus the matrix of $g(\cdot,\cdot)$ in the basis $(\vec{e}_{ig})$ is the inverse of the matrix of $g(\cdot,\cdot)$ in the basis $(\vec{e}_i)$:

$$[g(\vec{e}_{ig}, \vec{e}_{jg})] = \quad [g]_{|\vec{e}_{ig}} = [g]_{|\vec{e}_i}{}^{-1} \quad = ([g(\vec{e}_i, \vec{e}_j)])^{-1}. \tag{F.26}$$

Proof. First proof of (F.25) (straight forward calculation): (F.20) gives

$$\forall i,j, \ \ [\vec{e}_j]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{e}_{ig}]_{|\vec{e}} = \delta_{ij} = [\vec{e}_j]_{|\vec{e}}^T.[\vec{e}_i]_{|\vec{e}}, \quad \text{thus} \quad [g]_{|\vec{e}}.[\vec{e}_{ig}]_{|\vec{e}} = [\vec{e}_i]_{|\vec{e}}, \ \forall i. \tag{F.27}$$

Second proof of (F.25): Apply (F.8) (generic Riesz representation result) to get (F.25).

Then, with $[g]_{|\vec{e}}$ symmetric, $g(\vec{e}_{ig}, \vec{e}_{jg}) = [\vec{e}_{ig}]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{e}_{jg}]_{|\vec{e}} = [\vec{e}_i]_{|\vec{e}}^T.[g]_{|\vec{e}}{}^{-1}.[g]_{|\vec{e}}.[g]_{|\vec{e}}{}^{-1}.[\vec{e}_j]_{|\vec{e}} = [\vec{e}_i]_{|\vec{e}}^T.[g]_{|\vec{e}}{}^{-1}.[\vec{e}_j]_{|\vec{e}} = ([g]_{|\vec{e}}{}^{-1})_{ij}$, thus (F.26). ▪

Example F.19 $\vec{\mathbb{R}^2}$, $[g]_{|\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, thus $[g]_{|\vec{e}}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Thus $\vec{e}_{1g} = \vec{e}_1$ and $\vec{e}_{2g} = \frac{1}{2}\vec{e}_2$. ▪

Remark F.20 Warning, cf remark A.55: When $([g]_{|\vec{e}}^{-1})_{ij} =^{\text{noted}} g^{ij}$ (instead of R^{ij}) then (F.25) reads

$$\vec{e}_{jg} = \sum_{i=1}^{n} g^{ij} \vec{e}_i, \tag{F.28}$$

where the Einstein convention is ***not*** satisfied. The Einstein convention is satisfied with $\vec{e}_{jg} = \sum_{i=1}^{n} (P_j)^i \vec{e}_i$. (And $P = [(P_j)^i] = [P^i{}_j]$ is the transition matrix from $(\vec{e}_i)$ to $(\vec{e}_{ig})$). So in (F.28) g^{ij} is also another name for $(P_j)^i = P^i{}_j$:

$$g^{ij} := (P_j)^i = P^i{}_j. \tag{F.29}$$

We insist: $M = [g]_{|\vec{e}} = [M_{ij}]$ is a matrix, and its inverse is the matrix $M^{-1} = [M_{ij}]^{-1} = [N_{ij}]$: A matrix is just a collection of scalars, it is not tensorial (has nothing to do with the Einstein convention), and its inverse is also a collection of scalars, and you don't change this fact by calling $M^{-1} = [M^{ij}]$.

And because $(P_j)^i$ equals $([g]_{|\vec{e}}^{-1})_{ij} =^{\text{noted}} g^{ij}$, some people rename $\vec{e}_{jg}$ as $\vec{e}^j$... to get $\vec{e}^j = \sum_{i=1}^{n} g^{ij} \vec{e}_i$... to have the illusion to satisfy Einstein's convention, which is false: They confuse covariance and contravariance... and add confusion to the confusion...

NB: Recall: If in trouble with a notation which comes as a surprise (the notation g^{ij} here), use classic notations: Then no misuse of Einstein's convention and no possible misinterpretation. ▪

F.8.3 Multiple admissible notations for the components of $\vec{e}_{jg}$

Let $\mathcal{P} \in \mathcal{L}(E;E)$ be the change of basis endomorphism from $(\vec{e}_i)$ to $(\vec{e}_{ig})$: defined by $\mathcal{P}.\vec{e}_j = \vec{e}_{jg}$. And let $P = [\mathcal{P}]_{|\vec{e}}$ = the transition matrix from $(\vec{e}_i)$ to $(\vec{e}_{ig})$. We have multiple admissible (non confusing) notations for the components of $\vec{e}_{jg}$ relative to the basis $(\vec{e}_i)$:

$$\vec{e}_{jg} = \mathcal{P}.\vec{e}_j = \underbrace{\sum_{j=1}^{n} P_{ij} \vec{e}_i = \sum_{j=1}^{n} (P_j)_i \vec{e}_i}_{\text{clas.}} = \underbrace{\sum_{j=1}^{n} (P_j)^i \vec{e}_i = \sum_{j=1}^{n} P^i{}_j \vec{e}_i}_{\text{dual}}, \tag{F.30}$$

i.e. the i-th component of the vector $\vec{e}_{jg}$ has the names $P_{ij} = (P_j)_i = (P_j)^i = P^i{}_j$ or P^i_j, i.e. $P = [\mathcal{P}]_{|\vec{e}} = [P_{ij}] = [(P_j)_i] = [(P_j)^i] = [P^i{}_j]$ (four different notations for the same matrix), i.e.

$$\forall j, \quad [\vec{e}_{jg}]_{|\vec{e}} = P.[\vec{e}_j]_{|\vec{e}} = \begin{pmatrix} P_{1j} \\ \vdots \\ P_{nj} \end{pmatrix} = \begin{pmatrix} (P_j)_1 \\ \vdots \\ (P_j)_n \end{pmatrix} = \begin{pmatrix} P^1{}_j \\ \vdots \\ P^n{}_j \end{pmatrix} = \begin{pmatrix} (P_j)^1 \\ \vdots \\ (P_j)^n \end{pmatrix} \tag{F.31}$$

= the j-th column of P. You can choose any notation, depending on your current need or mood...

F.8.4 (Huge) differences between "the (covariant) dual basis" and "a dual vectorial basis"

1. A basis $(\vec{e}_i)$ has an infinite number of vectorial dual bases $(\vec{e}_{ig})$, as many as the number of inner dot products $(\cdot,\cdot)_g$ (observer dependents), see (F.25). And two observers with two different inner dot product get two different dual vectorial bases.

2. While a basis $(\vec{e}_i)$ has a unique intrinsic (covariant) dual basis $(\pi_{ei}) \overset{\text{noted}}{=} (e^i)$, cf. (A.7): Two observers who consider the same basis $(\vec{e}_i)$ have the same (covariant) dual basis.

3. If you fly, it is vital to use the dual basis $(\pi_{ei}) = (e^i)$: It is possibly fatal if you confuse foot and metre at takeoff and at landing (if you survived takeoff) because of the choice of different Euclidean dot product $(\cdot,\cdot)_g$ or $(\cdot,\cdot)_h$; See e.g. the Mars Climate Orbiter crash, remark A.17.

4. Einstein's convention can help... only if it is properly applied.

F.8.5 About the notation g^{ij} = shorthand notation for $(g^\sharp)^{ij}$

Definition F.21 $g(\cdot,\cdot) = (\cdot,\cdot)_g$ being an inner dot product in E, the Riesz associated inner dot product $g^\sharp(\cdot,\cdot) = (\cdot,\cdot)_{g^\sharp}$ in E^* is the bilinear form in $\mathcal{L}(E^*,E^*;\mathbb{R})$ defined by, for all $\ell, m \in E^*$,

$$(\ell, m)_{g^\sharp} := (\vec{\ell}_g, \vec{m}_g)_g, \quad \text{where} \quad \vec{\ell}_g = \vec{R}_g(\ell) \quad \text{and} \quad \vec{m}_g = \vec{R}_g(m) \tag{F.32}$$

(the $(\cdot,\cdot)_g$-Riesz representation vectors). ($g^\sharp(\cdot,\cdot)$ is indeed an inner dot product in E^*: easy check.)

Quantification: With $(\vec{e}_i)$ a basis in E and (e^i) its dual basis (duality notations). (F.32) gives:

$$(g^\sharp)^{ij} := g^\sharp(e^i, e^j) \overset{(F.32)}{=} g(\vec{e}_{ig}, \vec{e}_{jg}), \quad \text{thus} \quad [g^\sharp]_{|e} \overset{(F.25)}{=} = [g]^{-1}_{|\vec{e}}, \quad \text{i.e.} \quad \boxed{[(g^\sharp)^{ij}] = [g_{ij}]^{-1}}. \tag{F.33}$$

And

$$\text{shorthand notation:} \quad \boxed{[(g^\sharp)^{ij}] \overset{\text{noted}}{=} [g^{ij}]}. \tag{F.34}$$

Classical notations: $[g^\sharp]_{|e} = [(g^\sharp)_{ij}] = [g^\sharp(\pi_{ei}, \pi_{ej})] = [g(\vec{e}_{ig}, \vec{e}_{jg})] = [g_{ij}]^{-1} = ([g]_{|\vec{e}})^{-1}$.

Exercice F.22 How do we compute $g^\sharp(\ell, m)$ with matrix computations?

Answer. $\ell = \sum_{i=1}^n \ell_i e^i$ and $m = \sum_{j=1}^n m_j e^j$ give $g^\sharp(\ell,m) = \sum_{i,j=1}^n \ell_i m_j g^\sharp(e^i,e^j) = \sum_{i,j=1}^n \ell_i (g^\sharp)^{ij} m_j = [\ell]_{|\vec{e}}.[g^\sharp]_{|\vec{e}}.[m]^T_{|\vec{e}} = [\ell]_{|\vec{e}}.[g]^{-1}_{|\vec{e}}.[m]^T_{|\vec{e}}$ (a linear form is represented by a row matrix,). ▪

Exercice F.23 (F.32) tells that the $\binom{2}{0}$ tensor $g^\sharp \in \mathcal{L}(E^*,E^*;\mathbb{R})$ was created from the $\binom{0}{2}$ tensor $g = (\cdot,\cdot)_g \in \mathcal{L}(E,E;\mathbb{R})$ using twice the $(\cdot,\cdot)_g$-Riesz representation theorem.

1- Show that if you use the $(\cdot,\cdot)_g$-Riesz representation theorem just once you get the $\binom{1}{1}$ tensor $g^\natural \in \mathcal{L}(E^*,E;\mathbb{R}) \simeq \mathcal{L}(E;E)$ which is the identity endomorphism:

$$g^\natural = I. \tag{F.35}$$

2- Reciprocal: What is the $\binom{0}{2}$ tensor $g^\flat \in \mathcal{L}(E,E;\mathbb{R})$ that you create from the identity $I \in \mathcal{L}(E;E)$ when using the $(\cdot,\cdot)_g$-Riesz representation theorem once?

3- Summary: $\widetilde{I} = g^\natural$ gives $(\widetilde{I})^\flat = g^\flat = g$ and $(\widetilde{I})^\sharp = g^\sharp$

Answer. 1- $g^\natural \in \mathcal{L}(E^*,E;\mathbb{R})$ is defined by $g^\natural(\ell,\vec{w}) = (\vec{\ell}_g, \vec{w})_g$ for all $(\ell,\vec{w}) \in E^* \times E$, where $\vec{\ell}_g$ is the $(\cdot,\cdot)_g$-Riesz representation vector of ℓ. Thus $g^\natural(\ell,\vec{w}) = \ell.\vec{w} = \ell.I.\vec{w}$, for all $(\ell,\vec{w}) \in E^* \times E$, hence $g^\natural \in \mathcal{L}(E^*,E;\mathbb{R})$ is naturally canonically associated with the identity $I \in \mathcal{L}(E;E)$.

2- The identity operator $I \in \mathcal{L}(E;E)$ (observer independent) is naturally canonically associated with the $\binom{1}{1}$ tensor $\widetilde{I} \in \mathcal{L}(E^*,E;\mathbb{R})$ defined by $\widetilde{I}(\ell,\vec{w}) = \ell.I.\vec{w} = \ell.\vec{w}$ for all $(\ell,\vec{w}) \in E^* \times E$, thus $\widetilde{I} = g^\natural$. ▪

G Cauchy–Green deformation tensor $C = F^T.F$

Framework: $\widetilde{\Phi} : \left\{ \begin{array}{ll} [t_0,T] \times Obj & \to \mathbb{R}^n \\ (t, P_{Obj}) & \to \widetilde{\Phi}(t, P_{Obj}) \end{array} \right\}$ is a motion of Obj, $\Omega_\tau = \widetilde{\Phi}(\tau, P_{Obj})$ is the configuration of Obj at any τ. Then $\Phi^{t_0}(t, p_{t_0}) := \widetilde{\Phi}(t, P_{Obj})$ when $p_{t_0} = \widetilde{\Phi}(t_0, p_{t_0})$, and if t is fixed then $\Phi_t^{t_0}(p_{t_0}) := \Phi^{t_0}(t, p_{t_0})$ and $\Phi := \Phi_t^{t_0} : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} & \to p_t = \Phi(p_{t_0}) \end{array} \right\}$. And $F(P) := d\Phi(P) : \left\{ \begin{array}{ll} \vec{\mathbb{R}}^n_{t_0} & \to \vec{\mathbb{R}}^n_t \\ \vec{W} & \to \vec{w} = F(p_{t_0}).\vec{W} := \lim_{h\to 0} \dfrac{\Phi(p_{t_0}+h\vec{W}) - \Phi(p_{t_0})}{h} \end{array} \right\}$ (deformation gradient at p_{t_0} between t_0 and t).

G.1 Goal

Construction of C (summary of Cauchy's approach):

1- At t_0, consider two vectors $\vec{W}_1$ and $\vec{W}_2$ at a point $P \in \Omega_{t_0}$.

2- At t, they have been distorted by the motion to become the vectors $F.\vec{W}_1$ and $F.\vec{W}_2$ at $p = \Phi(P)$.

3- Then choose a Euclidean dot product $(\cdot,\cdot)_g =^{\text{noted}} \cdot \bullet \cdot$, the same at all t (to simplify);

4- Then, by definition of the transposed, $(F.\vec{W}_1) \bullet (F.\vec{W}_2) = (F^T.F.\vec{W}_1) \bullet \vec{W}_2$: You have got the Cauchy strain tensor $C := F^T.F$;

5- Then $(F.\vec{W}_1) \bullet (F.\vec{W}_2) - \vec{W}_1 \bullet \vec{W}_2 = ((C-I).\vec{W}_1) \bullet \vec{W}_2$ gives a measure of the deformation with $\vec{W}_2$ as a reference, value used to build a first order constitutive law for the stress (Cauchy).

G.2 Transposed F^T: Inner dot products required

We first recall the functional definition of F^T; Then we get the usual matrix representation of F^T relative to observers (quantification).

G.2.1 Definition of the function F^T

At t_0, a past observer chose an inner dot product $(\cdot,\cdot)_G$ in $\vec{\mathbb{R}}^n_{t_0}$, and at t the present observer chooses an inner dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$. Let $P \in \Omega_{t_0}$ and $p = \Phi(P)$ $(\in \Omega_t)$. The transposed of the linear map $F(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ relative to $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ is the linear map $F(P)^T_{Gg} \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0})$ defined by, for all $\vec{U}_P \in \vec{\mathbb{R}}^n_{t_0}$ vector at P and $\vec{w}_p \in \vec{\mathbb{R}}^n_t$ vector at p,

$$(F(P)^T_{Gg}.\vec{w}_p, \vec{U}_P)_G = (F(P).\vec{U}_P, \vec{w}_p)_g, \quad \text{written} \quad \boxed{(F^T.\vec{w}) \bullet_G \vec{U} = \vec{w} \bullet_g (F.\vec{U})}, \tag{G.1}$$

see (A.66). Don't forget that $F^T := F(P)^T_{Gg}$ depends on $(\cdot,\cdot)_G$, $(\cdot,\cdot)_g$, a $P \in \Omega_{t_0}$, t_0 and t. Recall: F stands for $F^{t_0}_t$, so $F(P)^T_{Gg}$ stands for $F^{t_0}_t(P)^T_{Gg}$.

So $F(P)^T_{Gg} : \vec{\mathbb{R}}^n_t \to \vec{\mathbb{R}}^n_{t_0}$ acts on vectors defined at p, which defines

$$F^T_{Gg} : \begin{cases} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0}) \\ p & \to \boxed{F^T_{Gg}(p) := F(P)^T_{Gg}} \quad \text{where} \quad P = \Phi^{-1}(p). \end{cases} \tag{G.2}$$

Hence (G.1) reads $(F^T_{Gg}(p).\vec{w}_p, \vec{U}_P)_G = (F(P).\vec{U}_P, \vec{w}_p)_g$, written in short $(F^T.\vec{w}) \bullet_G \vec{U} = \vec{w} \bullet_g (F.\vec{U})$. Don't forget that $F^T := F^T_{Gg}(p)$ depends on $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ and $p \in \Omega_t$.

Recall: F stands for $F^{t_0}_t$, so $F^T_{Gg}(p)$ stands for $(F^{t_0}_t)^T_{Gg}(p)$ $(= F^{t_0}_t(P)^T_{Gg})$.

Exercice G.1 1. With the ambiguous notation $F^T.\vec{z}.\vec{W} = \vec{z}.F.\vec{W} = F.\vec{W}.\vec{z} = \vec{W}.F^T.\vec{z}$, which dots are inner dot products?

2. With ambiguous notations, what does $F.\vec{W}_1.F.\vec{W}_2 = \vec{W}_1.F^T.F.\vec{W}_2$ mean?

Answer. 1. No choice: $(\vec{W},\vec{z}) \in \vec{\mathbb{R}}^n_{t_0} \times \vec{\mathbb{R}}^n_t$, and meaning $(F^T.\vec{z}) \bullet_G \vec{W} = \vec{z} \bullet_g (F.\vec{W}) = (F.\vec{W}) \bullet_g \vec{z} = \vec{W} \bullet_G (F^T.\vec{z})$.

2. No choice: $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$, and meaning $(F.\vec{W}_1) \bullet_g (F.\vec{W}_2) = \vec{W}_1 \bullet_G (F^T.F.\vec{W}_2)$. ▪

Remark G.2 On a surface Ω (a manifold), (G.1) is defined for all $(\vec{U}_P, \vec{w}_p) \in T_P\Omega_{t_0} \times T_p\Omega_t$. ▪

G.2.2 Quantification with bases (matrix representation)

Classical notations: $(\vec{a}_i)$ is a basis in $\vec{\mathbb{R}}^n_{t_0}$, and $(\vec{b}_i)$ is a basis in $\vec{\mathbb{R}}^n_t$. Marsden–Hughes duality notations: $(\vec{E}_I)$ is a basis in $\vec{\mathbb{R}}^n_{t_0}$ and $(\vec{e}_i)$ is a basis in $\vec{\mathbb{R}}^n_t$. And the reference to the points P and p is omitted to lighten the writings (use the full notation of § G.2.1 if in doubt). Let

$$G_{ij} = (\vec{a}_i, \vec{a}_j)_G,\ g_{ij} = (\vec{b}_i, \vec{b}_j)_g,\ F.\vec{a}_j = \sum_{i=1}^n F_{ij}\vec{b}_i,\ F^T.\vec{b}_j = \sum_{i=1}^n (F^T)_{ij}\vec{a}_j, \tag{G.3}$$

and $[G]_{|\vec{a}} := [G_{ij}] \overset{\text{not.}}{=} [G]$, $[g]_{|\vec{b}} := [g_{ij}] \overset{\text{not.}}{=} [g]$, $[F]_{|\vec{a},\vec{b}} = [F_{ij}] \overset{\text{not.}}{=} [F]$, $[F^T]_{|\vec{b},\vec{a}} = [(F^T)_{ij}] \overset{\text{not.}}{=} [F^T]$.

(G.1) gives $[\vec{U}]^T.[G].[F^T.\vec{w}] = [F.\vec{U}]^T.[G].[\vec{w}]$, thus $[\vec{U}]^T.[G].[F^T].[\vec{w}] = [\vec{U}]^T.[F]^T.[g].[\vec{w}]$, for all $\vec{U}, \vec{w}$, thus

$$[G].[F^T] = [F]^T.[g], \quad \text{i.e.} \quad \boxed{[F^T] = [G]^{-1}.[F]^T.[g]}. \tag{G.4}$$

(More precisely: $[G]_{|\vec{a}}.[F^T]_{|\vec{b},\vec{a}} = [F]_{|\vec{a},\vec{b}}{}^T.[g]_{\vec{b}}$, i.e. $[F^T]_{|\vec{b},\vec{a}} = [G]_{|\vec{a}}{}^{-1}.([F]_{|\vec{a},\vec{b}})^T.[g]_{\vec{b}}$.) So:

$$\sum_{k=1}^n G_{ik}(F^T)_{kj} = \sum_{k=1}^n F_{ki}g_{kj}, \quad \text{i.e.} \quad (F^T)_{ij} = \sum_{k,\ell=1}^n ([G]^{-1})_{ik}F_{\ell k}g_{\ell j} \tag{G.5}$$

Remark G.3 If $(\vec{a}_i)$ and $(\vec{b}_i)$ are $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$-orthonormal bases, then $[G] = I = [g]$, thus $[C] = [F]^T.[F]$. But recall: If work with coordinate systems then the bases are usually the coordinate system bases which are ***not*** orthonormal in general, i.e. $[G]^{-1} \neq I$ and $[g]^{-1} \neq I$ in general. ▪

Exercice G.4 Detail the obtaining of (G.5) (classical notation), then use Marsden duality notations to express (G.5).

Answer. Classical notations: $(F^T.\vec{b}_j, \vec{a}_i)_G = (\vec{b}_j, F.\vec{a}_i)_g$ and (G.3) gives $(\sum_{k=1}^n (F^T)_{kj}\vec{a}_k, \vec{a}_i)_G = (\vec{b}_j, \sum_{k=1}^n F_{ki}\vec{b}_k)_g$, thus $\sum_{k=1}^n (F^T)_{kj}(\vec{a}_k, \vec{a}_i)_G = \sum_{k=1}^n F_{ki}(\vec{b}_j, \vec{b}_k)_g$ with $F_{ki} = ([F]^T)_{ik}$, thus (G.5).

Marsden duality notations: Basis $(\vec{E}_I)$ at P at t_0, basis $(\vec{e}_i)$ at p at t, $G_{IJ} = G(\vec{E}_I, \vec{E}_j)$, $g_{ij} = g(\vec{e}_i, \vec{e}_j)$, $F.\vec{E}_J = \sum_{i=1}^n F^i{}_J \vec{e}_i$, $F^T.\vec{e}_j = \sum_{I=1}^n (F^T)^I{}_j \vec{E}_I$, thus:

$$\sum_{K=1}^n G_{IK}(F^T)^K{}_j = \sum_{k=1}^n F^k{}_I g_{kj}, \quad \text{i.e.} \quad (F^T)^I{}_j = \sum_{K,k=1}^n G^{IK} F^k{}_K g_{kj} \quad \text{where} \quad [G^{IJ}] := [G_{IJ}]^{-1}.$$ ▪

G.2.3 Remark: F^*

(F^* doesn't seem to be very useful in mechanics, apart from making simple things difficult... or playing with components and pseudo-duality notations...).

For mathematicians (no "magic tricks"):

Definition G.5 The adjoint of the linear map $F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ (acting on vectors) is the linear map $F^* \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t; \vec{\mathbb{R}}^{n*}_{t_0})$ (acting on functions) canonically defined by, for all $m \in \vec{\mathbb{R}}^{n*}_t$,

$$F^*(m) := m \circ F, \quad \text{written} \quad F^*.m = m.F \ \ (\in \vec{\mathbb{R}}^{n*}_{t_0}). \tag{G.6}$$

So, for all $(m, \vec{W}) \in \vec{\mathbb{R}}^{n*}_t \times \vec{\mathbb{R}}^n_{t_0}$,

$$(F^*.m).\vec{W} = m.F.\vec{W} \ \ (\in \mathbb{R}). \tag{G.7}$$

Quantification (matrix representation): We use (G.3), and (π_{ai}) and (π_{bi}) the (covariant) dual bases of $(\vec{a}_i)$ and $(\vec{b}_i)$. Let $(F^*)_{ij}$ be the components of F^* relative to these dual bases:

$$F^*.\pi_{bj} = \sum_{I=1}^n (F^*)_{ij}\pi_{ai}, \quad \text{i.e.} \quad [F^*]_{|\pi_b,\pi_a} = [(F^*)_{ij}]. \tag{G.8}$$

(G.7) gives $(F^*.\pi_{bj}).\vec{a}_i = \pi_{bj}.F.\vec{a}_i$, thus

$$\forall i,j, \ \boxed{(F^*)_{ij} = F_{ji}}, \quad \text{i.e.} \quad [F^*]_{|\pi_b,\pi_a} = ([F]_{|\vec{a},\vec{b}})^T, \quad \text{in short} \quad [F^*] = [F]^T. \tag{G.9}$$

Marsden duality notations: $F^*.e^j = \sum_{I=1}^n (F^*)_I{}^j E^I$ gives $(F^*)_I{}^j = F^j{}_I$ for all I, j.

Interpretation of F^*. As usual in classical mechanics, we use Euclidean dot products, here $(\cdot,\cdot)_G$ in $\vec{\mathbb{R}}^n_{t_0}$ and $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$. Then we use the $(\cdot,\cdot)_G$-Riesz representation vector $\vec{R}_G(F^*.m) \in \vec{\mathbb{R}}^n_{t_0}$ of $F^*.m \in \vec{\mathbb{R}}^{n*}_{t_0}$, and the $(\cdot,\cdot)_g$-Riesz representation vector $\vec{R}_g(m) \in \vec{\mathbb{R}}^n_t$ of $m \in \vec{\mathbb{R}}^{n*}_t$; So, for all $m \in \vec{\mathbb{R}}^{n*}_t$ and $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$,

$$(F^*.m).\vec{W} = \vec{R}_G(F^*.m) \bullet_G \vec{W}, \quad \text{and} \quad m.(F.\vec{W}) = \vec{R}_g(m) \bullet_g F.\vec{W} = (F^T.\vec{R}_g(m)) \bullet_G \vec{W}. \tag{G.10}$$

Thus (G.7) gives $\vec{R}_G(F^*.m) = F^T.\vec{R}_g(m)$, thus

$$\vec{R}_G.F^* = F^T.\vec{R}_g, \quad \text{i.e.} \quad F^* = \vec{R}_G{}^{-1}.F^T.\vec{R}_g. \tag{G.11}$$

Remark G.6 The definition of F^* is intrinsic to F (objective), while the definition of F^T is ***not*** intrinsic to F (***not*** objective) since it needs inner dot products (observer choices) to be defined. ▪

G.3 Cauchy–Green deformation tensor C

G.3.1 Definition of C

Consider vectors $\vec{W}_i \in \vec{\mathbb{R}}^n_{t_0}$ at P, $i = 1, 2$, and their push forwards $\vec{w}_i$ at $p = \Phi(P)$, i.e.

$$\vec{w}_i = F.\vec{W}_i, \tag{G.12}$$

short notation for $\vec{w}_i(p) = F(P).\vec{W}_i(P)$. With chosen inner dot products $(\cdot,\cdot)_G$ in $\vec{\mathbb{R}}^n_{t_0}$ and $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$, we get

$$(\vec{w}_1, \vec{w}_2)_g = (F.\vec{W}_1, F.\vec{W}_2)_g = (\underbrace{F^T.F}_{C}.\vec{W}_1, \vec{W}_2)_G. \tag{G.13}$$

More precisely: $(\vec{w}_{1p}, \vec{w}_{2p})_g = (F(P).\vec{W}_{1P}, F(P).\vec{W}_{2P})_g =^{(G.2)} (F^T_{Gg}(p).F(P).\vec{W}_{1P}, \vec{W}_{2P})_G$.

Definition G.7 The (right) Cauchy–Green deformation tensor at $P \in \Omega_{t_0}$ relative to $(\cdot,\cdot)_G$, $(\cdot,\cdot)_g$, t_0 and t is the endomorphism $C^{t_0}_{t,Gg}(P) =^{\text{noted}} C_{Gg}(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ defined by

$$C_{Gg}(P) := F^T_{Gg}(p) \circ F(P), \quad \text{in short} \quad \boxed{C := F^T.F}. \tag{G.14}$$

(More precisely: $C^{t_0}_{t,Gg}(P) := F^{t_0}_t(P)^T_{Gg} \circ F^{t_0}_t(P)$.)

So

$$\underbrace{F^T \circ F}_{=^{\text{not.}}\ C} : \vec{W} \in \vec{\mathbb{R}}^n_{t_0} \ \overset{F}{\longrightarrow}\ F(\vec{W}) \in \vec{\mathbb{R}}^n_t \ \overset{F^T}{\longrightarrow}\ \underbrace{F^T(F(\vec{W}))}_{=^{\text{not.}}\ C(\vec{W})} \in \vec{\mathbb{R}}^n_{t_0}. \tag{G.15}$$

(Recall: F and F^T are linear, thus $C = F^T \circ F$ is linear and written $C = F^T.F$.)

And (G.13) tells that C is characterized by, for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$,

$$\vec{w}_1 \bullet_g \vec{w}_2 = \boxed{(C.\vec{W}_1) \bullet_G \vec{W}_2 = (F.\vec{W}_1) \bullet_g (F.\vec{W}_2)}. \tag{G.16}$$

Moreover C is a $(\cdot,\cdot)_G$-symmetric endomorphism in $\vec{\mathbb{R}}^n_{t_0}$, i.e., for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$,

$$(C.\vec{W}_1, \vec{W}_2)_G = (\vec{W}_1, C.\vec{W}_2)_G, \quad \text{i.e.} \quad (C.\vec{W}_1) \bullet_G \vec{W}_2 = \vec{W}_1 \bullet_G (C.\vec{W}_2). \tag{G.17}$$

Indeed: $(C.\vec{W}_1, \vec{W}_2)_G = (F^T.F.\vec{W}_1, \vec{W}_2)_G = (F.\vec{W}_1, F.\vec{W}_2)_g = (\vec{W}_1, F^T.F.\vec{W}_2)_G = (\vec{W}_1, C.\vec{W}_2)_G$.

G.3.2 Quantification

(G.14) gives $[C] = [F^T].[F]$, with $[F^T] =^{(G.4)} [G]^{-1}.[F]^T.[g]$, thus

$$\boxed{[C] = [G]^{-1}.[F]^T.[g].[F]}, \tag{G.18}$$

short notation for $[C_{Gg}]_{|\vec{a}} = [G]^{-1}_{|\vec{a}}.([F]_{|\vec{a},\vec{b}})^T.[g]_{|\vec{b}}.[F]_{|\vec{a},\vec{b}}$.

Exercice G.8 Use classical notation, then duality notations, to express (G.18) with components.

Answer. Classical notations:

$$F.\vec{a}_j = \sum_{i=1}^n F_{ij}\vec{b}_i \quad \text{and} \quad C.\vec{a}_j = \sum_{i=1}^n C_{ij}\vec{a}_i, \quad \text{i.e.} \quad [F]_{|\vec{a},\vec{b}} = [F_{ij}] \quad \text{and} \quad [C]_{|\vec{a}} = [C_{ij}]. \tag{G.19}$$

Hence $(\vec{a}_i, C.\vec{a}_j)_G = (F.\vec{a}_i, F.\vec{a}_j)_g$, thus $(\vec{a}_i, \sum_k C_{kj}\vec{a}_k)_G = (\sum_k F_{ki}\vec{b}_k, \sum_\ell F_{\ell j}\vec{b}_\ell)_g$, thus $\sum_k C_{kj}(\vec{a}_i,\vec{a}_k)_G = \sum_{k\ell} F_{ki}(\vec{b}_k,\vec{b}_\ell)_g F_{\ell j}$, i.e.

$$\sum_{k=1}^n G_{ik}C_{kj} = \sum_{k,\ell=1}^n F_{ki}\, g_{k\ell} F_{\ell j} = \sum_{k,\ell=1}^n ([F]^T)_{ik}\, g_{k\ell} F_{\ell j}, \quad \text{so} \quad [G].[C] = [F]^T.[g].[F], \tag{G.20}$$

so $C_{ij} = \sum_{k,\ell,m=1}^n ([G]^{-1})_{im} F_{km}\, g_{k\ell} F_{\ell j} = \sum_{k,\ell,m=1}^n ([G]^{-1})_{im} ([F]^T)_{mk}\, g_{k\ell} F_{\ell j}$. Duality notations:

$$\begin{aligned} &F.\vec{E}_J = \sum_{i=1}^n F^i{}_J \vec{e}_i \quad \text{and} \quad C.\vec{E}_J = \sum_{I=1}^n C^I{}_J \vec{E}_I, \quad \text{i.e.} \quad [F]_{|\vec{E},\vec{e}} = [F^i{}_J] \quad \text{and} \quad [C]_{|\vec{E}} = [C^I{}_J], \quad \text{and} \\ &\sum_{K=1}^n G_{IK}C^K{}_J = \sum_{k,\ell=1}^n F^k{}_I\, g_{k\ell} F^\ell{}_J, \quad \text{and} \quad C^I{}_J = \sum_{k,\ell,M=1}^n G^{IM} F^k{}_M\, g_{k\ell} F^\ell{}_J \quad \text{when} \quad [G^{IJ}] := [G_{IJ}]^{-1}. \end{aligned} \tag{G.21}$$

▪

Exercice G.9 $(\cdot,\cdot)_G$ is a Euclidean dot product in foot, $(\cdot,\cdot)_g$ is a Euclidean dot product in metre, so $(\cdot,\cdot)_g = \mu^2(\cdot,\cdot)_G$ with $\mu = 0.3048$; And $(\vec{a}_i)$ is a $(\cdot,\cdot)_G$-orthonormal basis, and $(\vec{b}_i) := (\vec{a}_i)$. Prove:

$$[C] = \mu^2[F]^T.[F]. \tag{G.22}$$

Answer. $[C]_{|\vec{a}} =^{(G.18)} [G]^{-1}_{|\vec{a}}.[F]^T_{|\vec{a},\vec{a}}.[g]_{|\vec{a}}.[F]_{|\vec{a},\vec{a}}$ gives $[C]_{|\vec{a}} = I.[F]^T_{|\vec{a},\vec{a}}.\mu^2 I.[F]_{|\vec{a},\vec{a}}$. Shorten notation = (G.22). ▪

G.4 Time Taylor expansion of C

Here we use a unique inner dot product $(\cdot,\cdot)_G = (\cdot,\cdot)_g$ at all time (to compare "comparable values" in the vicinity of t_0). And we use an orthonormal basis $(\vec{a}_i)$ to lighten the notations, thus $[G]_{|\vec{a}} = I = [g]_{|\vec{a}}$ and (G.18) gives $[C]_{|\vec{a}} = [F]^T_{|\vec{a}}.[F]_{|\vec{a}}$, written $[C] = [F]^T.[F]$.

A time Taylor expansion implicitly imposes "along a trajectory of a fixed particle".

So P is fixed, $F^{t_0}(t,P) := F^{t_0}_t(P)$ and $F^{t_0}_P(t) := F^{t_0}(t,P)$, and $F^{t_0}_P(t) =^{\text{noted}} F(t)$.

And $C^{t_0}(t,P) := C^{t_0}_t(P)$ and $C^{t_0}_P(t) := C^{t_0}(t,P)$, and $C^{t_0}_P(t) =^{\text{noted}} C(t) = F(t)^T.F(t)$.

And here $[C(t)] = [F(t)]^T.[F(t)]$.

And $\vec{V}^{t_0}_t(P) =^{\text{noted}} \vec{V}(t)$ and $\vec{A}^{t_0}_t(P) =^{\text{noted}} \vec{A}(t)$ are the Lagrangian velocities and accelerations.

We have $\Phi(t{+}h) = \Phi(t) + h\,\vec{V}(t) + \frac{h^2}{2}\,\vec{A}(t) + o(h^2)$, thus $F(t{+}h) = F(t) + h\,d\vec{V}(t) + \frac{h^2}{2}\,d\vec{A}(t) + o(h^2)$, thus

$$\begin{aligned}
[C(t{+}h)] &= [F(t{+}h)]^T.[F(t{+}h)] = [F]^T(t{+}h).[F(t{+}h)]\\
&= \Big([F]^T + h\,d[\vec{V}]^T + \frac{h^2}{2}\,d[\vec{A}]^T + o(h^2)]\Big)\Big([F + h\,d[\vec{V}] + \frac{h^2}{2}\,d[\vec{A}] + o(h^2)]\Big)(t)\\
&= \Big([C] + h\,([F^T].[d\vec{V}]{+}[d\vec{V}]^T.[F]) + \frac{h^2}{2}\,([F]^T.[d\vec{A}]{+}2[d\vec{V}]^T.[d\vec{V}]{+}[d\vec{A}]^T.[F]) + o(h^2)\Big)(t).
\end{aligned}\tag{G.23}$$

Together with

$$[C(t{+}h)] = [C(t)] + h\,[C'(t)] + \frac{h^2}{2}\,[C''(t)]o(h^2). \tag{G.24}$$

thus

$$[C'] = [F^T].[d\vec{V}]{+}[d\vec{V}]^T.[F] \quad \text{and} \quad [C''] = [F]^T.[d\vec{A}]{+}2[d\vec{V}]^T.[d\vec{V}]{+}[d\vec{A}]^T.[F]. \tag{G.25}$$

In particular $[C'(t_0)] = [d\vec{V}(t_0)]{+}[d\vec{V}(t_0)]^T$, so

$$[C(t_0{+}h)] = I + h\,([d\vec{V}]{+}[d\vec{V}]^T)(t_0) + \frac{h^2}{2}\,([d\vec{A}]{+}2[d\vec{V}]^T.[d\vec{V}]{+}[d\vec{A}]^T)(t_0) + o(h^2). \tag{G.26}$$

Abusively written $C(t_0{+}h) = I + (d\vec{V} + d\vec{V}^T)(t_0) + \frac{h^2}{2}\,(d\vec{A} + 2d\vec{V}^T.d\vec{V} + d\vec{A}^T)(t_0) + o(h^2)$, but don't forget it is a matrix meaning.

With Eulerian variables and $\vec{v}(t,p)$ and $\vec{\gamma}(t,p)$ the Eulerian velocities and accelerations at t at $p = \Phi^{t_0}_t(t,P)$: We have $d\vec{V}^{t_0}(t,P) = d\vec{v}(t,p(t)).F(t)$ and $d\vec{A}^{t_0}(t,P) = d\vec{\gamma}(t,p(t)).F(t)$, thus

$$\begin{aligned}
C^{t_0}_P(t{+}h) = C^{t_0}_P(t) &+ h\,(F^T(t).(d\vec{v} + d\vec{v}^T)(t,p(t)).F(t))\\
&+ \frac{h^2}{2}\,(F^T(t).(d\vec{\gamma} + 2d\vec{v}^T.d\vec{v} + d\vec{\gamma}^T)(t,p(t)).F(t)) + o(h^2).
\end{aligned}\tag{G.27}$$

abusive notation of $[C^{t_0}_P(t{+}h)] = ...$ (matrices).

Remark G.10 $F'' = d\vec{A}$ is easy to interpret, but $C'' = F^T.d\vec{A} + 2d\vec{V}^T.d\vec{V} + d\vec{A}^T.F = (F^T.d\vec{A} + d\vec{V}^T.d\vec{V}) + (F^T.d\vec{A} + d\vec{V}^T.d\vec{V})^T$ is not that easy to interpret (and in not linear in $\vec{V}$).

We already had a problem with the composition of flows: The formula $F^{t_0}_{t_2} = F^{t_1}_{t_2}.F^{t_0}_{t_1}$ is simple (determinism), but the formula $C^{t_0}_{t_2} = (F^{t_0}_{t_2})^T.F^{t_0}_{t_2} = (F^{t_0}_{t_1})^T.(F^{t_1}_{t_2})^T.F^{t_1}_{t_2}.F^{t_0}_{t_1} = (F^{t_0}_{t_1})^T.C^{t_1}_{t_2}.F^{t_0}_{t_1}$ is "not that simple" ($\neq C^{t_1}_{t_2}.C^{t_0}_{t_1}$). (Indeed, to consider C instead of F amounts to consider the "motion squared", cf. $(C.\vec{W},\vec{W})_g = ||F.\vec{W}||^2_g$.)

Since $C'(t_0) = d\vec{V}(t_0) + d\vec{V}(t_0)^T$ this may have little consequences for linear approximation near t_0, but ultimately not small consequences for second-order approximations (and large deformations) if C'' is used to make constitutive laws. The consideration of Lie derivatives may be an interesting alternative. ▪

G.5 Remark: $C^\flat$

For mathematicians: May produce errors, misuses, covariance-contravariance confusion, see next § G.5.2. For the general $^\flat$ notation see § A.11.5.

G.5.1 Definition of $C^\flat$...

Definition G.11 At $P \in \Omega_{t_0}$, the bilinear form $C^\flat_{Gg}(P) =^{\text{noted}} C^\flat \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}, \vec{\mathbb{R}}^n_{t_0}; \mathbb{R})$ associated with the linear map $C_{Gg}(P) =^{\text{noted}} C(P) =^{\text{noted}} C \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ is defined by, for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$ vectors at P,

$$C^\flat(\vec{W}_1, \vec{W}_2) := (\vec{W}_1, C.\vec{W}_2)_G \quad (= (F.\vec{W}_1, F.\vec{W}_2)_g). \tag{G.28}$$

NB: using $(\cdot,\cdot)_G$ we have changed the variance of C (a $\binom{1}{1}$ tensor) to build $C^\flat$ (a $\binom{0}{2}$ tensor).

$C^\flat$ is a bilinear symmetric form (trivial) and is a metric in $\vec{\mathbb{R}}^n_{t_0}$ (trivial F being a diffeomorphism), but not a Euclidean one (it is if $C = I$ i.e. for rigid body motions).

Quantification: (G.28) gives $[\vec{W}_2]^T.[C^\flat].[\vec{W}_1] = [\vec{W}_2]^T.[G].[C].[\vec{W}_1]$ for all $\vec{W}_1, \vec{W}_2$ since $C^\flat$ and $(\cdot,\cdot)_G$ are symmetric, thus

$$[C^\flat] = [G].[C] \quad (= [F]^T.[g].[F]). \tag{G.29}$$

More precisely: $[C^\flat]_{|\vec{E}} = [G]_{|\vec{E}}.[C]_{|\vec{E}} = ([F]_{|\vec{E},\vec{e}})^T.[g]_{|\vec{e}}.[F]_{|\vec{E},\vec{e}}$.
Classical notations: $C^\flat = \sum_{ij} C_{ij} \pi_{ai} \otimes \pi_{aj}$ and $C.\vec{a}_j = \sum_i C_{ij} \vec{a}_i$ and $G = \sum_{ij} G_{ij} \pi_{ai} \otimes \pi_{aj}$ give

$$(C^\flat)_{ij} = \sum_k G_{ik} C_{kj} \quad (= \sum_{k\ell} F_{ki} g_{k\ell} F_{\ell j}). \tag{G.30}$$

Duality notations: $C^\flat = \sum_{IJ} C_{IJ} E^I \otimes E^J$ and $C.\vec{E}_j = \sum_I C^I_J \vec{E}_i$ and $G = \sum_{IJ} G_{IJ} E^I \otimes E^J$ give

$$C_{IJ} = \sum_K C^K{}_J G_{KI} \quad (= \sum_{k\ell} F^k{}_I g_{k\ell} F^\ell{}_J), \tag{G.31}$$

which justifies the flat notation: The top index I in $[C] = [C^I{}_J]$ has been transformed into a bottom index in $[C^\flat] = [C_{IJ}]$ (the use of an inner dot product changes the variance).

G.5.2 ... and remarks about $C^\flat$... and Jaumann

$C^\flat$ can also be defined only with $(\cdot,\cdot)_g$ by, for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$,

$$C^\flat_g(\vec{W}_1, \vec{W}_2) := (F.\vec{W}_1, F.\vec{W}_2)_g, \tag{G.32}$$

i.e., $C^\flat := C^\flat_g := g^*$ the pull-back of the metric $(\cdot,\cdot)_g$ by Φ, see (8.9).

- However $C^\flat = C^\flat_g$ is useless in itself: $C^\flat$ is ***not*** a Euclidean dot product (it is a metric defined at each P by $C^\flat_g(P)(\vec{W}_1, \vec{W}_2) := (F(P).\vec{W}_1, F(P).\vec{W}_2)_g$ for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$ vectors at P). $C^\flat$ is only useful to characterize a deformation if the value $C^\flat(\vec{W}_1, \vec{W}_2)$ can be compared with the initial value $(\vec{W}_1, \vec{W}_2)_G$, i.e. if a Euclidean dot product $(\cdot,\cdot)_G$ was introduced in $\vec{\mathbb{R}}^n_{t_0}$: This is why $C^\flat$ is classically defined from C, cf. (G.28).
- There is no objective "trace" for a $\binom{0}{2}$ tensor like $C^\flat$, while $\mathrm{Tr}(C)$ is objective (endomorphism).
- The Lie derivatives of a second order tensor depends on the type of the tensor, and the Lie derivative of the $\binom{1}{1}$ tensor like C gives the Jaumann derivative, which is usually preferred to the Lie derivative of the $\binom{0}{2}$ tensor like $C^\flat$ which is the lower convected Lie derivative, see next remark G.12.
- So the introduction and use of $C^\flat$ in mechanics mostly complicate things unnecessarily, and interferes with basic understandings like the distinction between covariance and contravariance.

Remark G.12 Interpretation issue with Jaumann (and the use of $C^\flat$ should be avoided in mechanics).

With $\frac{D(d\vec{v})}{Dt} =^{(2.26)} d(\frac{D\vec{v}}{Dt}) - d\vec{v}.d\vec{v} = d\vec{\gamma} - d\vec{v}.d\vec{v}$ and with orthonormal bases, $2\mathcal{D} = \frac{D(d\vec{v})}{Dt} + \frac{D(d\vec{v})^T}{Dt} = d\vec{\gamma} + d\vec{\gamma}^T - d\vec{v}.d\vec{v} - d\vec{v}^T.d\vec{v}^T$ (matrix meaning), thus, with (G.27) (matrix meaning),

$$\begin{aligned} C''(t) &= F(t)^T.(2\frac{D\mathcal{D}}{Dt} + d\vec{v}.d\vec{v} + d\vec{v}^T.d\vec{v}^T + 2d\vec{v}^T.d\vec{v})(t, p(t)).F(t) \\ &= 2F(t)^T.(\frac{D\mathcal{D}}{Dt} + \mathcal{D}.d\vec{v} + d\vec{v}^T.\mathcal{D})(t, p(t)).F(t). \end{aligned} \tag{G.33}$$

The $\frac{D\mathcal{D}}{Dt} + \mathcal{D}.d\vec{v} + d\vec{v}^T.\mathcal{D}$ term looks like a lower-convected Lie derivative, but with $d\vec{v}^T$ instead of $d\vec{v}^*$, cf. (9.61); So you may find (G.33) abusively written: $C'' = 2F^T.\mathcal{L}_{\vec{v}}\mathcal{D}.F$, or $(C^\flat)'' = 2F^T.\mathcal{L}_{\vec{v}}\mathcal{D}^\flat_g.F$ where $\mathcal{D}^\flat_g := \frac{d\vec{v}^\flat_g + (d\vec{v}^\flat_g)^T}{2}$. But you get disappointing results (values) using the lower convected Lie derivative (Jaumann is usually preferred). ▪

G.6 Stretch ratio and deformed angle

Here $(\cdot,\cdot)_g = (\cdot,\cdot)_G$, i.e. at t_0 and t we use the same Euclidean dot product, to be able to compare the lengths relative to the same unit of measurement. (If $(\cdot,\cdot)_g \neq (\cdot,\cdot)_G$ then use $(\cdot,\cdot)_g = \mu^2(\cdot,\cdot)_G$.)

G.6.1 Stretch ratio

The stretch ratio at $P \in \vec{\mathbb{R}}^n_{t_0}$ between t_0 and t for a $\vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$ is defined by

$$\lambda(\vec{W}_P) := \frac{||\vec{w}_p||_G}{||\vec{W}_P||_G} = \frac{||F_P.\vec{W}_P||_G}{||\vec{W}_P||_G} \quad (= ||F_P.(\frac{\vec{W}_P}{||\vec{W}_P||_G})||_G) \tag{G.34}$$

where $\vec{w}_p = F_P.\vec{W}_P$ is the deformed vector by the motion at $p = \Phi(P)$. I.e., in short

$$\forall \vec{W} \in \vec{\mathbb{R}}^n_{t_0} \text{ s.t. } ||\vec{W}|| = 1, \quad \lambda(\vec{W}) := ||F.\vec{W}||. \tag{G.35}$$

(You may find: $\lambda(d\vec{X}) = ||F.d\vec{X}||$ with $d\vec{X}$ a unit vector(!); This notation should be avoided, see § 4.3.)

G.6.2 Deformed angle

Recall: The angle $\theta_{t_0} = (\widehat{\vec{W}_1, \vec{W}_2})$ between two vectors $\vec{W}_1$ and $\vec{W}_2$ in $\vec{\mathbb{R}}^n_{t_0} - \{\vec{0}\}$ at $P \in \Omega_{t_0}$ is defined by

$$\cos(\theta_{t_0}) = \frac{\vec{W}_1}{||\vec{W}_1||_G} \bullet_G \frac{\vec{W}_2}{||\vec{W}_2||_G} \quad (= (\frac{\vec{W}_1}{||\vec{W}_1||_G}, \frac{\vec{W}_2}{||\vec{W}_2||_G})_G). \tag{G.36}$$

Andhe deformed angle θ_t between the deformed vectors $\vec{w}_i = F.\vec{W}_i$ at $p = \Phi^{t_0}_t(P)$, with $(\cdot,\cdot)_g = (\cdot,\cdot)_G$,

$$\cos(\theta_t) := (\widehat{\vec{w}_1, \vec{w}_2}) = \frac{\vec{w}_1}{||\vec{w}_1||_G} \bullet_G \frac{\vec{w}_2}{||\vec{w}_2||_G} = \frac{(C.\vec{W}_1) \bullet_G \vec{W}_2}{||\vec{w}_1||_G \, ||\vec{w}_2||_G}. \tag{G.37}$$

G.7 Decompositions of C

G.7.1 Spherical and deviatoric tensors

Definition G.13 The deformation spheric tensor is

$$C_{sph} = \frac{1}{n}\text{Tr}(C)\, I, \tag{G.38}$$

with $\text{Tr}(C)$ = the trace of the endomorphism C (there is no "trace" for the $\binom{0}{2}$ tensor $C^\flat$).

Definition G.14 The deviatoric tensor is

$$C_{dev} = C - C_{sph}. \tag{G.39}$$

So $\text{Tr}(C_{dev}) = 0$ and $C = C_{sph} + C_{dev}$.

G.7.2 Rigid motion

The deformation is rigid iff, for all t_0, t,

$$(F^{t_0}_t)^T.F^{t_0}_t = I, \quad \text{i.e.} \quad C^{t_0}_t = I, \quad \text{written} \quad C = I = F^T.F. \tag{G.40}$$

After a rigid body motion, lengths and angles are left unchanged.

G.7.3 Diagonalization of C

Proposition G.15 *$C = F^T.F$ being symmetric positive, C is diagonalizable, its eigenvalues are positive, and $\vec{\mathbb{R}}^n_{t_0}$ has an orthonormal basis made of eigenvectors of C.*

Proof. $(C(P).\vec{W}_1, \vec{W}_2)_G = (F(P).\vec{W}_1, F(P).\vec{W}_2)_g = (\vec{W}_1, C(P).\vec{W}_2)_G$, thus C is $(\cdot,\cdot)_G$-symmetric.

$(C.\vec{W}_1, \vec{W}_1)_G = (F.\vec{W}_1, F.\vec{W}_1)_g = ||F.\vec{W}_1||^2_g > 0$ when $\vec{W}_1 \neq \vec{0}$, since F invertible ($\Phi^{t_0}_t$ is supposed to be a diffeomorphism). Thus C est $(\cdot,\cdot)_G$-symmetric definite positive real endomorphism. ∎

Definition G.16 Let λ_i be the eigenvalues of C. Then the $\sqrt{\lambda_i}$ are called the principal stretches. And the associated eigenvectors give the principal directions.

G.7.4 Mohr circle

This § deals with general properties of $3*3$ symmetric positive endomorphism, like $C_t^{t_0}(P)$.

Consider $\vec{\mathbb{R}^3}$ with a Euclidean dot product $(\cdot,\cdot)_{\mathbb{R}^3}$ and a $(\cdot,\cdot)_{\mathbb{R}^3}$-orthonormal basis $(\vec{a}_i)$.

Let $\mathcal{M} : \vec{\mathbb{R}^3} \to \vec{\mathbb{R}^3}$ be a symmetric positive endomorphism. Thus $\mathcal{M}$ is diagonalizable in a $(\cdot,\cdot)_{\mathbb{R}^3}$-orthonormal basis $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$, that is, $\exists\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}$, $\exists\vec{e}_1,\vec{e}_2,\vec{e}_3 \in \vec{\mathbb{R}^3}$ s.t.

$$\mathcal{M}.\vec{e}_i = \lambda_i\vec{e}_i \quad\text{and}\quad (\vec{e}_i,\vec{e}_j)_{\mathbb{R}^3} = \delta_{ij}, \quad\text{so}\quad [\mathcal{M}]_{|\vec{e}} = \text{diag}(\lambda_1,\lambda_2,\lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{G.41}$$

And the orthonormal basis $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ is ordered s.t. $\lambda_1 \ge \lambda_2 \ge \lambda_3$ (>0).

Let S be the unit sphere in $\mathbb{R}^3$, that is the set $\{(x,y,z) : x^2+y^2+z^2=1\}$. Its image $\mathcal{M}(S)$ by $\mathcal{M}$ is the ellipsoid $\{(x,y,z) : \frac{x^2}{\lambda_1^2}+\frac{y^2}{\lambda_2^2}+\frac{z^2}{\lambda_3^2}=1\}$. Then consider $\vec{n}=\sum_i n_i\vec{e}_i$ s.t. $||\vec{n}||_{\mathbb{R}^3}=1$:

$$[\vec{n}]_{|\vec{e}} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \quad\text{with}\quad n_1^2+n_2^2+n_3^2=1. \tag{G.42}$$

Thus its image $\vec{A}=\mathcal{M}.\vec{n} \in \mathcal{M}(S)$ satisfies

$$\vec{A}=\mathcal{M}.\vec{n}, \quad [\vec{A}]_{|\vec{e}} = \begin{pmatrix} \lambda_1 n_1 \\ \lambda_2 n_2 \\ \lambda_3 n_3 \end{pmatrix}. \tag{G.43}$$

Then define

$$A_n = (\vec{A},\vec{n})_{\mathbb{R}^3}, \quad \vec{A}_\perp = \vec{A} - A_n\vec{n}, \quad A_\perp := ||\vec{A}_\perp||. \tag{G.44}$$

So $\vec{A} = A_n\vec{n} + \vec{A}_\perp \in \text{Vect}\{\vec{n}\} \otimes \text{Vect}\{\vec{n}\}^\perp$. (Remark: $\vec{A}_\perp$ is not orthonormal to the ellipsoid $\mathcal{M}(S)$, but is orthonormal to the initial sphere S.)

Mohr Circle purpose: To find a relation:

$$A_\perp = f(A_n), \tag{G.45}$$

relation between "the normal force A_n" (to the initial sphere) and the "tangent force$A_\perp$" (to the initial sphere).

(G.42), (G.43) and $A_n = (\mathcal{M}.\vec{n},\vec{n})_{\mathbb{R}^3}$ give

$$\begin{cases} n_1^2+n_2^2+n_3^2=1, \\ \lambda_1 n_1^2+\lambda_2 n_2^2+\lambda_3 n_3^2 = A_n \\ \lambda_1^2 n_1^2+\lambda_2^2 n_2^2+\lambda_3^2 n_3^2 = ||\vec{A}||^2 = A_n^2+A_\perp^2. \end{cases} \tag{G.46}$$

This is linear system with the unknowns n_1^2, n_2^2, n_3^2. The solution is

$$\begin{cases} n_1^2 = \dfrac{A_\perp^2+(A_n-\lambda_2)(A_n-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \\ n_2^2 = \dfrac{A_\perp^2+(A_n-\lambda_3)(A_n-\lambda_1)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}, \\ n_3^2 = \dfrac{A_\perp^2+(A_n-\lambda_1)(A_n-\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}. \end{cases} \tag{G.47}$$

The n_i^2 being non negative, and with $\lambda_1 > \lambda_2 > \lambda_3 \ge 0$, we get

$$\begin{cases} A_\perp^2+(A_n-\lambda_2)(A_n-\lambda_3) \ge 0, \\ A_\perp^2+(A_n-\lambda_3)(A_n-\lambda_1) \le 0, \\ A_\perp^2+(A_n-\lambda_1)(A_n-\lambda_2) \ge 0. \end{cases} \tag{G.48}$$

Then let $x=A_n$ and $y=A_\perp$, and consider, for some $a,b\in\mathbb{R}$, the equation

$$y^2+(x-a)(x-b)=0, \quad\text{so}\quad (x-\frac{a+b}{2})^2+y^2 = \frac{(a-b)^2}{4}.$$

This is the equation of a circle centered at $(\frac{a+b}{2},0)$ with radius $\frac{|a-b|}{2}$.

Thus (G.48)$_2$ tells that A_n and $A_\perp$ are inside the circle centered at $(\frac{\lambda_1+\lambda_3}{2},0)$ with radius $\frac{\lambda_1-\lambda_3}{2}$, and (G.48)$_{1,3}$ tell that A_n and $A_\perp$ are outside the other circles (adjacent and included in the first, drawing).

Exercice G.17 What happens if $\lambda_1=\lambda_2=\lambda_3>0$?

Answer. Then $\left\{\begin{array}{l} n_1^2+n_2^2+n_3^2=1, \\ n_1^2+n_2^2+n_3^2=\dfrac{A_n}{\lambda_1}, \\ n_1^2+n_2^2+n_3^2=\dfrac{A_n^2+A_\perp^2}{\lambda_1^2}. \end{array}\right\}$ Thus $A_n=\lambda_1$ and $A_n^2+A_\perp^2=\lambda_1^2$, thus $A_\perp=0$. Here $C=\lambda_1 I$, and we deal with a dilation: $A_\perp=0$. ∎

Exercice G.18 What happens if $\lambda_1=\lambda_2>\lambda_3>0$?

Answer. Then $\left\{\begin{array}{l} n_1^2+n_2^2+n_3^2=1, \\ \lambda_1(1-n_3^2)+\lambda_3 n_3^2=A_n, \\ \lambda_1^2(1-n_3^2)+\lambda_3^2 n_3^2=A_n^2+A_\perp^2. \end{array}\right\}$ Thus $A_n=\lambda_1-(\lambda_1-\lambda_3)n_3^2\in[\lambda_3,\lambda_1]$, and $A_\perp=\pm(\lambda_1^2-(\lambda_1^2-\lambda_3^2)n_3^2-A_n^2)^{\frac12}$, with $A_n^2+A_\perp^2$ a point on the circle with radius $\lambda_1^2(1-n_3^2)+\lambda_3^2 n_3^2$. ∎

G.8 Green–Lagrange deformation tensor E

(G.13) gives $(\vec w_1,\vec w_2)_g=(F.\vec W_1,F.\vec W_2)_g=(C.\vec W,\vec W)_G$ at $p=\Phi(P)$, thus

$$(\vec w_1,\vec w_2)_g-(\vec W_1,\vec W_2)_G=((C-I).\vec W_1,\vec W_2)_G. \tag{G.49}$$

Definition G.19 The Green–Lagrange tensor (or Green–Saint Venant tensor) at P relative to t_0 and t is the endomorphism $E_t^{t_0}(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ defined by

$$E_t^{t_0}(P):=\frac{C_t^{t_0}(P)-I_{t_0}}{2},\quad \text{in short}\quad \boxed{E=\frac{C-I}{2}}\quad (=\frac{F^T.F-I}{2}). \tag{G.50}$$

(In particular $E=0$ for rigid body motions.) And $E_t^{t_0}:\Omega_{t_0}\to\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ is the Green–Lagrange tensor relative to t_0 and t.

The $\frac12$ is introduced because $(C.,.)=(F.,F.)$ corresponds to the “motion squared”, see the following linearization.

And we get the time Taylor expansion of $E_P^{t_0}(t)=\frac12(C_P^{t_0}(t)-I_{t_0})$ with $p(t)=\Phi_P^{t_0}(t)$ and (G.27):

$$\begin{aligned} E_P^{t_0}(t+h)&=F_P^{t_0}(t)^T.\Big(h\,\frac{d\vec v+d\vec v^T}{2}+\frac{h^2}{2}\,(\frac{d\vec\gamma+d\vec\gamma^T}{2}+d\vec v^T.d\vec v)\Big)(t,p(t)).F_P^{t_0}(t)+o(h^2)\\ &=F_P^{t_0}(t)^T.\Big(h\,\mathcal{D}+h^2\,(\frac{D\mathcal{D}}{Dt}+\mathcal{D}.d\vec v+d\vec v^T.\mathcal{D})\Big)(t,p(t)).F_P^{t_0}(t)+o(h^2). \end{aligned} \tag{G.51}$$

G.9 Small deformations (linearization): The infinitesimal strain tensor $\underline{\underline{\varepsilon}}$

G.9.1 Landau notations big-O and little-o

Reminder. Let $f,g:\mathbb{R}\to\mathbb{R}$ and $x_0\in\mathbb{R}$.

- $$f=O(g)\text{ near }x_0\iff \exists C>0,\ \exists\eta>0,\ \forall x\ s.t.\ |x-x_0|<\eta,\ |f(x)|<C|g(x)|. \tag{G.52}$$

and f is said to be “comparable with g” near x_0. If $|g|>0$ then it reads $\frac{|f(x)|}{|g(x)|}<C$.

And $\frac{|f(x)|}{|x^n|}<C$ near x=0 means $f=O(x^n)$ near x_0=0.

- $$f=o(g)\text{ near }x_0\iff \forall\varepsilon>0,\ \exists\eta>0,\ \forall x\ s.t.\ |x-x_0|<\eta,\ |f(x)|<\varepsilon|g(x).\tag{G.53}$$

and f is said to be “negligible compared with g near x_0”. If $|g|>0$ then it reads $\frac{|f(x)|}{|g(x)|}\longrightarrow_{x\to x_0}0$.

And $\frac{|f(x)|}{|x^n|}\longrightarrow_{x\to 0}0$ means $f=o(x^n)$ near x_0=0.

G.9.2 Definition of the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$

The motion is supposed to be C^2. Along a trajectory, with $F_P^{t_0}(t_0) = I$ we have, near t_0,

$$F_P^{t_0}(t_0{+}h) = I + O(h), \tag{G.54}$$

thus $F_P^{t_0}(t_0{+}h).\vec{W} = \vec{W} + O(h)$ for all $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, i.e., near t_0, with $(\cdot,\cdot)_g = (\cdot,\cdot)_G$,

$$||\vec{w} - \vec{W}|| = O(h) \quad \text{when} \quad \vec{w} = F_P^{t_0}(t_0{+}h).\vec{W}. \tag{G.55}$$

Full notation: $||F_P^{t_0}(t).\vec{W}_P - \vec{W}_P||_g = O(t{-}t_0)$ near t_0. (More precisely $||F_P^{t_0}(t).\vec{W}_P - S_t^{t_0}.\vec{W}_P||_g = O(t{-}t_0)$ with Marsden shifter $S_t^{t_0}$, to avoid using any ubiquity gift.)

Definition G.20 With the same inner dot product $(\cdot,\cdot)_g$ used at all time: If $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis, the same at all time, then the infinitesimal strain tensor at P is the matrix defined by

$$[\underline{\underline{\varepsilon}}(P)]_{|\vec{e}} = \frac{[F(P)]_{|\vec{e}} + [F(P)]_{|\vec{e}}^T}{2} - [I], \tag{G.56}$$

written

$$\underline{\underline{\varepsilon}} := \frac{F + F^T}{2} - I \quad \text{(matrix meaning)}. \tag{G.57}$$

(And more precisely, at $P \in \Omega_{t_0}$ and between t_0 and t, $[\underline{\underline{\varepsilon}}_t^{t_0}(P)]_{|\vec{e}} = \frac{[F_t^{t_0}(P)]_{|\vec{e}} + [F_t^{t_0}(P)]_{|\vec{e}}^T}{2} - [I]$.)

So $\underline{\underline{\varepsilon}}.\vec{W} = \frac{F.\vec{W} + F^T.\vec{W}}{2} - .\vec{W}$ means $[\underline{\underline{\varepsilon}}]_{|\vec{e}}.[\vec{W}]_{|\vec{e}} = \frac{[F]_{|\vec{e}}.[\vec{W}]_{|\vec{e}} + [F]_{|\vec{e}}^T.[\vec{W}]_{|\vec{e}}}{2} - [\vec{W}]_{|\vec{e}}$.

Remark G.21 $\underline{\underline{\varepsilon}}$ in (G.57) can***not*** be a tensor (cannot be a function) since $F_t^{t_0}(P) : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ and $F_t^{t_0}(P)^T : \vec{\mathbb{R}}^n_t \to \vec{\mathbb{R}}^n_{t_0}$ and $I_{t_0} : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_{t_0}$ don't have the same definition domain.

So $\underline{\underline{\varepsilon}}$ is not a function, is not a tensor: It is a matrix... But is called "the infinitesimal strain tensor"... ∎

Proposition G.22 *The Green–Lagrange tensor* $E = \frac{F^T.F-I}{2} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ *satisfies near* t_0*:*

$$E = \underline{\underline{\varepsilon}} + o(t{-}t_0) \quad \left(= \frac{F + F^T}{2} - I + o(t{-}t_0)\right) \quad \textit{(matrix meaning)}, \tag{G.58}$$

which means $[E] = [\underline{\underline{\varepsilon}}] + o(t{-}t_0) = \frac{[F]+[F^T]}{2} - [I] + o(t{-}t_0))$.

And "for small deformations" we write $E \simeq \underline{\underline{\varepsilon}}$, *i.e.* $E \simeq \frac{F+F^T}{2} - I$.

Interpretation: (G.58) is a linearization of E, *since we keep the linear part of the "quadratic"* $E = \frac{1}{2}(F^T.F - I)$ *given by* $(E.\vec{W},\vec{U})_g = \frac{1}{2}\big((F.\vec{W},F.\vec{U})_g - (\vec{W},\vec{U})_g\big)$ *for all* $\vec{U},\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ *("motion squared" cf. the* $(F\cdot,F\cdot)_g$ *term).*

Proof. A $(\cdot,\cdot)_g$-orthonormal basis being chosen, $[F^T] =^{(G.4)} [F]^T$, thus $[C] = [F]^T.[F]$, thus

$$2[E] = [C] - [I] = [F]^T.[F] - [I] = ([F]^T - [I]).([F] - [I]) + [F]^T + [F] - 2[I]. \tag{G.59}$$

Then, near t_0 and with $h = t{-}t_0$, (G.54) gives $([F]^T - [I]).([F] - I]) = O(h)O(h) = O(h^2)$, thus $2[E] = [F]^T + [F] - 2[I] + O(h)$, thus (G.58). ∎

G.9.3 The classic approach is weird

The classic approach is weird: It applies the small displacement hypothesis to the Green–Lagrange tensor $E = \frac{F^T.F-I}{2}$ which is then linearized to get $\underline{\underline{\varepsilon}} = \frac{F+F^T}{2} - I$, that is, cf. (G.58):

Starting with F, the classical approach "squares the motion" to get E, then...
linearizes E ... to get back to F... with a spurious F^T...(!)

H Finger tensor $F.F^T$ (left Cauchy–Green tensor)

Finger's approach is consistent with the foundations of relativity (Galileo classical relativity or Einstein general relativity): We can only do measurements at the current time t, and we can refer to the past.

There is a lot of misunderstandings, as was the case for the Cauchy–Green deformation tensor C, due to the lack of precise definitions: Definition domain? Value domain? Points at stake (p or P)? Euclidean dot product (English? French?)? Covariance? Contravariance?...

H.1 Definition

Let $\widetilde{\Phi}$ be motion, $t_0 \in \mathbb{R}$, Φ^{t_0} the associated motion, $P \in \Omega_{t_0}$, $t \in \mathbb{R}$, and $F_t^{t_0}(P) := d\Phi_t^{t_0}(P) \in \mathcal{L}(\vec{\mathbb{R}^n_{t_0}}; \vec{\mathbb{R}^n_t})$. And let $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ be Euclidean dot products in $\vec{\mathbb{R}^n_{t_0}}$ and $\vec{\mathbb{R}^n_t}$.

Definition H.1 The Finger tensor $\underline{\underline{b}}_t^{t_0}(p_t)$, or left Cauchy–Green deformation tensor, at t at p_t relative to t_0 is the endomorphism $\in \mathcal{L}(\vec{\mathbb{R}^n_t}; \vec{\mathbb{R}^n_t})$ defined by, with $P = {\Phi_t^{t_0}}^{-1}(p_t)$,

$$\underline{\underline{b}}_t^{t_0}(p_t) := F_t^{t_0}(P).(F_t^{t_0})^T_{Gg}(p_t) \quad \text{written in short} \quad \boxed{b = F.F^T}, \tag{H.1}$$

i.e. is defined by $(\underline{\underline{b}}_t^{t_0}(p_t).\vec{w}_1, \vec{w}_2)_g = (F_t^{t_0}(P)^T.\vec{w}_1, F_t^{t_0}(P)^T.\vec{w}_2)_G = ((F_t^{t_0})^T(p_t).\vec{w}_1, (F_t^{t_0})^T(p_t).\vec{w}_2)_G$, for all $\vec{w}_1, \vec{w}_2$ vectors at $p_t \in \Omega_t$, written in short

$$(\underline{\underline{b}}.\vec{w}_1, \vec{w}_2)_g = (F^T.\vec{w}_1, F^T.\vec{w}_2)_G. \tag{H.2}$$

(To compare with $C = F^T.F$ and $(C.\vec{W}_1, \vec{W}_2)_G = (F.\vec{W}_1, F.\vec{W}_2)_g$.)

And the Finger tensor relative to t_0 is

$$\underline{\underline{b}}^{t_0} : \begin{cases} \mathcal{C} = \bigcup_t (\{t\} \times \Omega_t) & \to \mathcal{L}(\vec{\mathbb{R}^n_t}; \vec{\mathbb{R}^n_t}) \\ \qquad (t, p_t) & \to \underline{\underline{b}}^{t_0}(t, p_t) := \underline{\underline{b}}_t^{t_0}(p_t). \end{cases} \tag{H.3}$$

NB: $\underline{\underline{b}}^{t_0}$ looks like a Eulerian function, but isn't, since it depends on a t_0.

Other definition found:

$$B_t^{t_0} := \underline{\underline{b}}_t^{t_0} \circ (\Phi_t^{t_0})^{-1}, \quad \text{i.e.} \quad B_t^{t_0}(P) := \underline{\underline{b}}_t^{t_0}(p_t) = F_t^{t_0}(P).F_t^{t_0}(P)^T, \quad \text{written} \quad B = F.F^T. \tag{H.4}$$

Pay attention: $B_t^{t_0}(P) \in \mathcal{L}(\vec{\mathbb{R}^n_t}; \vec{\mathbb{R}^n_t})$ is an endomorphism at t at p_t, not at t_0 at P: E.g., $B_t^{t_0}(P).\vec{w}_t(p_t) = \underline{\underline{b}}_t^{t_0}(p_t).\vec{w}_t(p_t)$ is meaningful, while $B_t^{t_0}(P).\vec{W}_{t_0}(P)$ is absurd.

Remark H.2 For mathematicians. The push-forward by $\Phi := \Phi_t^{t_0}$ of the Cauchy–Green deformation tensor $C = F^T.F$ is $\Phi_*(C) = F.C.F^{-1} = F.F^T = \underline{\underline{b}}$, cf. (8.15): It is the Finger tensor. So the endomorphism C in $\vec{\mathbb{R}^n_{t_0}}$ is the pull-back of the endomorphism $\underline{\underline{b}}$ in $\vec{\mathbb{R}^n_t}$. (However a push-forward and a pull-back don't depend on any inner dot product while the transposed F^T does...). ∴

H.2 $\underline{\underline{b}}^{-1}$

With pull-backs (towards the virtual power principle at t). With $p_t = \Phi_t^{t_0}(P)$ and $\vec{W}_i(P) = (F_t^{t_0}(P))^{-1}.\vec{w}_i(p_t)$:

$$(\vec{W}_1, \vec{W}_2)_G = (F^{-1}.\vec{w}_1, F^{-1}.\vec{w}_2)_G = (F^{-T}.F^{-1}.\vec{w}_1, \vec{w}_2)_g = (\underline{\underline{b}}^{-1}.\vec{w}_1, \vec{w}_2)_g. \tag{H.5}$$

So $\underline{\underline{b}}^{-1} := (\underline{\underline{b}}_t^{t_0})^{-1}$ is useful:

$$(\underline{\underline{b}}_t^{t_0})^{-1} : \begin{cases} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}^n_t}; \vec{\mathbb{R}^n_t}) \\ p_t & \to (\underline{\underline{b}}_t^{t_0})^{-1}(p_t) = F_t^{t_0}(P)^{-T}.F_t^{t_0}(P)^{-1} = H_t^{t_0}(p_t)^T.H_t^{t_0}(p_t) \end{cases} \tag{H.6}$$

with $p_t = \Phi_t^{t_0}(P)$ and $H_t^{t_0}(p_t) = (F_t^{t_0}(P))^{-1}$ cf. (4.43). Thus we can define

$$(\underline{\underline{b}}^{t_0})^{-1} : \begin{cases} \bigcup_t (\{t\} \times \Omega_t) & \to \mathcal{L}(\vec{\mathbb{R}^n_t}; \vec{\mathbb{R}^n_t}) \\ \qquad (t, p_t) & \to (\underline{\underline{b}}^{t_0})^{-1}(t, p_t) := (\underline{\underline{b}}_t^{t_0})^{-1}(p_t). \end{cases} \tag{H.7}$$

Remark: $(\underline{\underline{b}}^{t_0})^{-1}$ looks like a Eulerian function, but isn't, since it depends on t_0.

In short:

$$\underline{\underline{b}}^{-1} = H^T.H, \quad \text{to compare with} \quad C = F^T.F, \tag{H.8}$$

and with $\vec{w} = F.\vec{W}$,

$$\underline{\underline{b}}^{-1}.\vec{w} = H^T.\vec{W}, \quad \text{to compare with} \quad C.\vec{W} = F^T.\vec{w}, \tag{H.9}$$

and with $\vec{W}_i = F^{-1}.\vec{w}_i$, i.e. $\vec{w}_i = F.\vec{W}_i$,

$$(\underline{\underline{b}}^{-1}.\vec{w}_1, \vec{w}_2)_g = (\vec{W}_1, \vec{W}_2)_G, \quad \text{to compare with} \quad (C.\vec{W}_1, \vec{W}_2)_G = (\vec{w}_1, \vec{w}_2)_g. \tag{H.10}$$

Remark H.3 For mathematicians. $p_t = \Phi_t^{t_0}(P)$, $b(p_t) = F(P).F(P)^T$ and $C(P) = F(P)^T.F(P)$ give

$$\underline{\underline{b}}(p_t).F(P) = F(P).C(P), \tag{H.11}$$

written $\underline{\underline{b}} = F.C.F^{-1}$. Thus $\underline{\underline{b}}^{-1} = F.C^{-1}.F^{-1}$, so

$$\Phi_t^{t_0*}\underline{\underline{b}}^{-1} = F^{-1}.\underline{\underline{b}}^{-1}.F = F^{-1}.F^{-T} = (F^T.F)^{-1} = C^{-1}, \tag{H.12}$$

i.e. the pull-back of $\underline{\underline{b}}^{-1}$ is C^{-1}, i.e. $\underline{\underline{b}}^{-1}$ is the push-forward of C^{-1}. ▪

H.3 Time derivatives of $\underline{\underline{b}}^{-1}$

With (H.7) let $(\underline{\underline{b}}^{t_0})^{-1} =^{\text{noted}} \underline{\underline{b}}^{-1} = H^T.H$. Thus, along a trajectory, and with (4.47), we get

$$\begin{aligned}\frac{D\underline{\underline{b}}^{-1}}{Dt} &= \frac{DH^T}{Dt}.H + H^T.\frac{DH}{Dt} = -d\vec{v}^T.H^T.H - H^T.H.d\vec{v} \\ &= -\underline{\underline{b}}^{-1}.d\vec{v} - d\vec{v}^T.\underline{\underline{b}}^{-1}.\end{aligned} \tag{H.13}$$

Exercice H.4 Prove (H.13) with (H.10).

Answer. (H.10) gives $\frac{D}{Dt}(\underline{\underline{b}}^{-1}.\vec{w}_1, \vec{w}_2)_g = 0 = (\frac{D\underline{\underline{b}}^{-1}}{Dt}.\vec{w}_1, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.\frac{D\vec{w}_1}{Dt}, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.\vec{w}_1, \frac{D\vec{w}_2}{Dt})_g$, and $\vec{w}_i(t,p(t)) = F^{t_0}(t,P).\vec{W}t_0(P)$ gives $\frac{D\vec{w}_i}{Dt} = d\vec{v}.\vec{w}_i$, thus $(\frac{D\underline{\underline{b}}^{-1}}{Dt}.\vec{w}_1, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.d\vec{v}.\vec{w}_1, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.\vec{w}_1, d\vec{v}.\vec{w}_2)_g = 0$, thus (H.13). ▪

Exercice H.5 Prove (H.13) with $F^T.\underline{\underline{b}}^{-1}.F = I_{t_0}$.

Answer. $\underline{\underline{b}}^{-1} = (F.F^T)^{-1} = F^{-T}.F^{-1}$ gives $F^T.\underline{\underline{b}}^{-1}.F = I_{t_0}$, thus $(F^T)'.\underline{\underline{b}}^{-1}.F + F^T.\frac{D\underline{\underline{b}}^{-1}}{Dt}.F + F^T.\underline{\underline{b}}^{-1}.F' = 0$, thus $F^T.d\vec{v}^T.\underline{\underline{b}}^{-1}.F + F^T.\frac{D\underline{\underline{b}}^{-1}}{Dt}.F + F^T.\underline{\underline{b}}^{-1}.d\vec{v}.F = 0$, thus (H.13). ▪

H.4 Euler–Almansi tensor $\underline{\underline{a}}$

Euler–Almansi approach is consistent with the foundations of relativity (Galileo relativity or Einstein general relativity): We can only do measurements at the current time t, and we can refer to the past.

At t in Ω_t, consider the Finger tensor $\underline{\underline{b}} = F.F^T$ and its inverse $\underline{\underline{b}}^{-1} = F^{-T}.F^T = H^T.H$ cf. (H.8).

Definition H.6 Euler–Almansi tenor at $p_t \in \Omega_t$ is the endomorphism $\underline{\underline{a}}_t^{t_0}(p_t) \in \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_t^n)$ defined by

$$\underline{\underline{a}}_t^{t_0}(p_t) = \frac{1}{2}(I_t - \underline{\underline{b}}_t^{t_0}(p_t)^{-1}) = \frac{1}{2}(I_t - H(p_t)^T.H(p_t)), \tag{H.14}$$

written

$$\underline{\underline{a}} = \frac{1}{2}(I - \underline{\underline{b}}^{-1}) = \frac{1}{2}(I - H^T.H), \tag{H.15}$$

to compare with the Green–Lagrange tensor $E = \frac{1}{2}(C - I) = \frac{1}{2}(F^T.F - I) \in \mathcal{L}(\vec{\mathbb{R}}_{t_0}^n; \vec{\mathbb{R}}_{t_0}^n)$.

Remark: $\underline{\underline{a}}^{t_0}$ looks like a Eulerian function, but isn't, since it depends on t_0.

(H.10) gives ($\vec{w}_i = F.\vec{W}_i$)

$$(\vec{w}_1, \vec{w}_2)_g - (\vec{W}_1, \vec{W}_2)_G = 2(\underline{\underline{a}}.\vec{w}_1, \vec{w}_2)_g, \tag{H.16}$$

to compare with $(\vec{w}_1, \vec{w}_2)_g - (\vec{W}_1, \vec{W}_2)_G = 2(E.\vec{W}_1, \vec{W}_2)_G$. (This also gives $(\underline{\underline{a}}.\vec{w}_1, \vec{w}_2)_g = (E.\vec{W}_1, \vec{W}_2)_G$.) And (H.15) gives

$$F^T.\underline{\underline{a}}.F = E, \quad \text{i.e.} \quad \underline{\underline{a}} = F^{-T}.E.F^{-1}, \tag{H.17}$$

standing for $F_t^{t_0}(P)^T.\underline{\underline{a}}_t^{t_0}(p).F_t^{t_0}(P) = E_t^{t_0}(P)$ when $p = \Phi_t^{t_0}(P)$.

Remark H.7 $\underline{\underline{a}}_t^{t_0}$ is not the push-forward of $E_t^{t_0}$ by $\Phi_t^{t_0}$ (the push-forward is $F.E.F^{-1}$). ▪

H.5 Time Taylor expansion for $\underline{\underline{a}}$

(H.13) gives

$$\frac{D\underline{\underline{a}}}{Dt} = \frac{\underline{\underline{b}}^{-1}.d\vec{v} + d\vec{v}^T.\underline{\underline{b}}^{-1}}{2}. \tag{H.18}$$

H.6 Almansi modified Infinitesimal strain tensor $\widetilde{\underline{\underline{\varepsilon}}}$

Same Euclidean framework as in § G.9.2, and matrix meaning again.

We have $I - \underline{\underline{b}}^{-1} = I - H^T.H = -(I - H^T).(I - H) + 2I - H^T - H$ where H stands for $H_t^{t_0}(p_t)$. Thus, for small displacement we get $I - \underline{\underline{b}}^{-1} = 2I - H^T - H + O(h)$, so

$$\underline{\underline{a}}(t, p(t)) = \widetilde{\underline{\underline{\varepsilon}}}(t, p(t)) + O(h) \quad \text{where} \quad \widetilde{\underline{\underline{\varepsilon}}} := I - \frac{H + H^T}{2}. \tag{H.19}$$

And, with $t = t_0 + h$ we have $F^{t_0}(t, P) = I + (t - t_0)\, d\vec{v}(t, P) + o(t - t_0)$, cf. (4.37), thus we have $H^{t_0}(t, p(t)) = F^{t_0}(t, P)^{-1} = I - (t - t_0)\, d\vec{v}(t, P) + o(t - t_0)$ when $p(t) = \Phi^{t_0}(t, P)$. Thus

$$F^{t_0}(t, P) - I = I - H^{t_0}(t, p(t)) + O(t - t_0). \tag{H.20}$$

Therefore, for small displacements ($|t - t_0| << 1$):

$$\underline{\underline{a}}(t, p(t)) \simeq \widetilde{\underline{\underline{\varepsilon}}}(t, p(t)) \simeq \underline{\underline{\varepsilon}}^{t_0}(t, P) \quad \text{(matrix meaning)}. \tag{H.21}$$

I Polar decompositions of F ("isometric objectivity")

Regular motion $\widetilde{\Phi} : (t, P_{Obj}) \in [t_0, T] \times Obj \to p_t = \widetilde{\Phi}(t, P_{Obj}) \in \mathbb{R}^n$, $\Omega_t = \widetilde{\Phi}(t, Obj)$, associated Lagrangian motion $\Phi_t^{t_0} : (t, p_{t_0}) \in [t_0, T] \times \Omega_{t_0} \to p_t = \Phi^{t_0}(t, p_{t_0}) := \widetilde{\Phi}(t, P_{Obj}) \in \mathbb{R}^n$ when $p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj})$, deformation gradient $F_t^{t_0}(p_{t_0}) := d\Phi_t^{t_0}(p_{t_0}) =^{\text{noted}} F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$.

The covariant objectivity is abandoned here, due to the need for inner dot products $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$ to define $F^T \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0})$ and build $C = F^T.F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$.

Recall: $(F_t^{t_0})^T_{Gg}(p_t) =^{\text{noted}} F^T$ is defined by $(F^T.\vec{w}, \vec{U})_G = (F.\vec{U}, \vec{w})_g$ for all $(\vec{U}, \vec{w}) \in \vec{\mathbb{R}}^n_{t_0} \times \vec{\mathbb{R}}^n_t$, and $C^{t_0}_{t,Gg}(p_{t_0}) := (F_t^{t_0})^T_{Gg}(p_t) \circ F_t^{t_0}(p_{t_0}) =^{\text{noted}} C = F^T.F$ is a $(\cdot,\cdot)_G$-symmetric endomorphism in $\vec{\mathbb{R}}^n_{t_0}$ since $(C.\vec{X}, \vec{Y})_G = (F^T.F.\vec{X}, \vec{Y})_G = (F.\vec{X}, F.\vec{Y})_g = (\vec{X}, F^T.F.\vec{Y})_G = (\vec{X}, C.\vec{Y})_G$ for all $\vec{X}, \vec{Y} \in \vec{\mathbb{R}}^n_{t_0}$.

I.1 $F = R.U$ (right polar decomposition)

C being $(\cdot,\cdot)_G$-symmetric, $\exists \alpha_1, ..., \alpha_n \in \mathbb{R}$ (the eigenvalues), $\exists \vec{c}_1, ..., \vec{c}_n \in \vec{\mathbb{R}}^n_{t_0}$ (associated eigenvectors), s.t.

$$\forall i \in [1, n]_{\mathbb{N}}, \;\; C.\vec{c}_i = \alpha_i \vec{c}_i, \quad \text{and} \quad (\vec{c}_i) \text{ is a } (\cdot,\cdot)_G\text{-orthonormal basis in } \vec{\mathbb{R}}^n_{t_0}, \tag{I.1}$$

i.e. $(\vec{c}_i, \vec{c}_j)_G = \delta_{ij}$ for all $i, j \in [1, n]_{\mathbb{N}}$.

So, with $(\vec{E}_i)$ a $(\cdot,\cdot)_G$-orthonormal basis in $\vec{\mathbb{R}}^n_{t_0}$, $[C]_{\vec{c}} = D := \text{diag}(\alpha_1, ..., \alpha_n)$ is the diagonal matrix of eigenvalues, and with $P = [P_{ij}]$ the transition matrix from $(\vec{E}_i)$ to $(\vec{c}_i)$ (i.e. $\vec{c}_j = \sum_i P_{ij} \vec{E}_i$ for all j), (I.1) reads

$$[C]_{\vec{E}}.P = P.D \quad \text{and} \quad P^T.P = I, \quad \text{so} \quad D = P^{-1}.[C]_{\vec{E}}.P \quad \text{and} \quad P^{-1} = P^T. \tag{I.2}$$

And F being regular, $0 < ||F.\vec{c}_i||_g^2 = (F.\vec{c}_i, F.\vec{c}_i)_g = (C.\vec{c}_i, \vec{c}_i)_G = \alpha_i ||\vec{c}_i||_G^2$, thus $\alpha_i > 0$, for all i.

Definition I.1 With (I.1), the right stretch tensor $U \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ is the endomorphism defined by

$$\forall i \in [1, n]_{\mathbb{N}}, \;\; U.\vec{c}_i = \sqrt{\alpha_i}\, \vec{c}_i, \tag{I.3}$$

the $\sqrt{\alpha_i}$ being called the principal stretches. (Full notation: $U := U^{t_0}_{t,Gg}(p_{t_0})$.)

So $[U]_{|\vec{c}} = \text{diag}(\sqrt{\alpha_1}, ..., \sqrt{\alpha_n}) = \sqrt{D}$, and (I.3) reads

$$[U]_{|\vec{E}}.P = P.\sqrt{D}, \quad \text{so} \quad \sqrt{D} = P.[U]_{|\vec{E}}.P^{-1}. \tag{I.4}$$

And U is $(\cdot,\cdot)_G$-symmetric since $(U^T.\vec{c}_i, \vec{c}_j)_G = (\vec{c}_i, U.\vec{c}_j)_G = (\vec{c}_i, \sqrt{\alpha_j}\vec{c}_j)_G = \sqrt{\alpha_j}\delta_{ij} = \sqrt{\alpha_i}\delta_{ij} = (\sqrt{\alpha_i}\vec{c}_i, \vec{c}_j)_G = (U.\vec{c}_i, \vec{c}_j)_G$ for all i, j. And $(U \circ U).\vec{c}_j = U(U.\vec{c}_j) = U(\sqrt{\alpha_j}\vec{c}_j) = \sqrt{\alpha_j} U(\vec{c}_j) = \sqrt{\alpha_j}\sqrt{\alpha_j}\vec{c}_i = \alpha_j.\vec{c}_j = C.\vec{c}_j$ for all j, hence

$$C = U \circ U \overset{\text{noted}}{=} U.U \overset{\text{noted}}{=} U^2, \quad \text{and} \quad U \overset{\text{noted}}{=} \sqrt{C}. \tag{I.5}$$

Definition I.2 The orthogonal transformation $R \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ is the linear map defined by

$$R := F \circ U^{-1} \overset{\text{noted}}{=} F.U^{-1}. \tag{I.6}$$

(Full notation: $R^{t_0}_{t,Gg}(p_{t_0}) = F^{t_0}_t(p_{t_0}) \circ (U^{t_0}_t(p_{t_0})_{Gg})^{-1}$.) And

$$\boxed{F = R \circ U} \overset{\text{noted}}{=} R.U \quad \text{is called the right polar decomposition of } F. \tag{I.7}$$

Proposition I.3 *1-*

$$R^T \circ R = I, \quad i.e. \quad R^{-1} = R^T, \tag{I.8}$$

written $R^T.R = I$, i.e. R sends a $(\cdot,\cdot)_G$-orthonormal basis in $\vec{\mathbb{R}}^n_{t_0}$ to a $(\cdot,\cdot)_g$-orthonormal basis in $\vec{\mathbb{R}}^n_t$.

2- The right polar decomposition $F = R \circ U$ is unique: If $F = R_2 \circ U_2$ with $U_2 \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ symmetric definite positive and $R_2 \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ s.t. $R_2^{-1} = R_2^T$, then $U_2 = U$ and $R_2 = R$.

Proof. 1- $R^T \circ R \overset{(I.6)}{=} U^{-T} \circ F^T \circ F \circ U^{-1} = U^{-1} \circ C \circ U^{-1} = U^{-1} \circ (U \circ U) \circ U^{-1} = I$ identity in $\vec{\mathbb{R}}^n_{t_0}$. Thus $(R.\vec{E}_i, R.\vec{E}_j)_g = (R^T.R.\vec{E}_i, \vec{E}_j)_G = (\vec{E}_i, \vec{E}_j)_G = \delta_{ij}$ for all i, j: $(R.\vec{E}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis.

2- U_2 being symmetric definite positive, call $\sqrt{\beta_i}$ its eigenvalues (all positive) and $(\vec{d}_i)$ a $(\cdot,\cdot)_G$-orthonormal basis made of associated eigenvectors. We have $C = (U_2^T.R_2^T).(R_2.U_2) = U_2.(R_2^T.R_2).U_2 = U_2.I.U_2 = U_2^2$, thus $C.\vec{d}_j = U_2^2.\vec{d}_i = \beta_i \vec{d}_i$, thus the β_i are eigenvalues of C and the $\vec{d}_i$ are associated eigenvectors. Thus, even if it means reordering (β_i), $\beta_i = \alpha_i$ and $\vec{d}_i \in \text{Ker}(C - \alpha_i I)$, for all i, and $U.\vec{d}_i \overset{(I.3)}{=} \sqrt{\alpha_i}\vec{d}_i = U_2.\vec{d}_i$ for all i, thus $U_2 = U$. Thus $R_2 = F.U_2^{-1} = F.U^{-1} = R$. ∎

I.2 $F = S.R_0.U$ (shifted right polar decomposition)

We need to be more precise if the gift of ubiquity is prohibited: Because we work with the affine space $\mathbb{R}^n$, we can consider the Marsden's shifter, with $p_t = \Phi^{t_0}_t(p_{t_0})$,

$$S := S^{t_0}_t(p_{t_0}) : \begin{cases} T_{p_{t_0}}(\Omega_{t_0}) & \to T_{p_t}(\Omega_t) \\ (p_{t_0}, \vec{w}_{t_0,p_{t_0}}) & \to (p_t, \vec{w}_{t,p_t}) \quad \text{where} \quad \vec{w}_{t,p_t} := \vec{w}_{t_0,p_{t_0}}. \end{cases} \tag{I.9}$$

Shorten (misleading) notation:

$$S := S^{t_0}_t(p_{t_0}) : \begin{cases} \vec{\mathbb{R}}^n_{t_0} & \to \vec{\mathbb{R}}^n_t \\ \vec{W} & \to \vec{w} = S.\vec{W} = \vec{W}. \end{cases} \tag{I.10}$$

NB: 1- S is not "the identity" unless you have time and space ubiquity gift, since $\vec{w}_{t_0,p_{t_0}}$ is defined at t_0 at p_{t_0} while $\vec{w}_{t,p_t} = S.\vec{w}_{t_0,p_{t_0}}$ is defined at t at p_t, and $t \neq t_0$ and $p_t \neq p_{t_0}$ in general;

2- S is not a topological identity since it changes the norms in general: You consider $||\vec{w}_{t_0,p_{t_0}}||_G$ in $\vec{\mathbb{R}}^n_{t_0}$ and $||S.\vec{w}_{t_0,p_{t_0}}(t,p_t)||_g = ||\vec{w}_{t_0,p_{t_0}}||_g$ in $\vec{\mathbb{R}}^n_t$.

Notation: Let $R_0 \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ be the endomorphism defined by

$$R_0 := S^{-1} \circ R \overset{\text{noted}}{=} S^{-1}.R, \quad \text{so} \quad R = S.R_0 \quad (= S \circ R_0). \tag{I.11}$$

(Full notations: $(R_0)^{t_0}_{t,Gg}(p_{t_0}) := (S^{t_0}_t(p_{t_0}))^{-1}(R^{t_0}_{t,Gg}(p_{t_0})) \in \mathcal{L}(T_{p_{t_0}}(\Omega_{t_0}); T_{p_{t_0}}(\Omega_{t_0}))$.) Thus

$$F = S \circ R_0 \circ U \quad \text{written} \quad \boxed{F = S.R_0.U}. \tag{I.12}$$

Proposition I.4 *If $(\cdot,\cdot)_g = (\cdot,\cdot)_G$ (same inner dot product in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$) then*

$$S^T.S = I, \quad i.e. \quad S^{-1} = S^T. \tag{I.13}$$

And the endomorphism $R_0 = S^{-1}.R \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ is a change of $(\cdot,\cdot)_G$-orthonormal basis:

$$R_0^T.R_0 = I, \quad i.e. \quad R_0^{-1} = R_0^T. \tag{I.14}$$

Proof. $(S^T.S.\vec{U},\vec{W})_G \overset{\text{transposed}}{=} (S.\vec{U},S.\vec{W})_g \overset{(I.9)}{=} (\vec{U},\vec{W})_g = (\vec{U},\vec{W})_G$ (here $(\cdot,\cdot)_g = (\cdot,\cdot)_G$), for all $\vec{U},\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, thus $S^T.S = I$, thus $S^{-1} = S^T$.

Thus $I = S.S^T$ and $R_0 = S^T.R$, thus $(R_0^T.R_0.\vec{U},\vec{W})_G = (R_0.\vec{U},R_0.\vec{W})_G = (S^T.R.\vec{U},S^T.R.\vec{W})_G = (S.S^T.R.\vec{U},R.\vec{W})_g = (R.\vec{U},R.\vec{W})_g =^{(I.8)} (\vec{U},\vec{W})_G$, for all $\vec{U},\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, thus $R_0^T.R_0 = I$. ∎

Interpretation of (I.12): F is composed of: The pure deformation U (endomorphism in $\vec{\mathbb{R}}^n_{t_0}$), the change of orthonormal basis with R_0 (endomorphism in $\vec{\mathbb{R}}^n_{t_0}$), and the shift operator $S : T_{p_{t_0}}(\Omega_{t_0}) \to T_{p_t}(\Omega_t)$ (from past to present time and position).

I.3 $F = V.R$ (left polar decomposition)

Same steps than for the right polar decomposition.

Let $\underline{\underline{b}}^{t_0}_t(p_t) := F^{t_0}_t(p_{t_0}) \circ (F^{t_0}_t)^T(p_t) \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ (the Finger tensor), written $\underline{\underline{b}} = F.F^T$. The endomorphism $\underline{\underline{b}}$ being symmetric definite positive: $\exists \beta_1,...,\beta_n \in \mathbb{R}^*_+$ (the eigenvalues) and $\exists \vec{z}_1,...,\vec{z}_n \in \vec{\mathbb{R}}^n_t$ (associated eigenvectors) s.t.

$$\forall i \in [1,n]_{\mathbb{N}},\ \ \underline{\underline{b}}.\vec{z}_i = \beta_i \vec{z}_i, \quad \text{and} \quad (\vec{z}_i) \text{ is a } (\cdot,\cdot)_g\text{-orthonormal basis in } \vec{\mathbb{R}}^n_t. \tag{I.15}$$

The left stretch tensor $V \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ is the endomorphism defined by,

$$\forall i \in [1,n]_{\mathbb{N}},\ \ V.\vec{z}_i = \sqrt{\beta_i}\vec{z}_i, \quad \text{and} \quad V \overset{\text{noted}}{=} \sqrt{\underline{\underline{b}}}. \tag{I.16}$$

(Full notation: $V^{t_0}_{t,Gg}(p_t) = \sqrt{\underline{\underline{b}}^{t_0}_t(p_t)_{Gg}}$.) Then define the linear map $R_\ell \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ by

$$R_\ell := V^{-1}.F, \tag{I.17}$$

so that

$$\boxed{F = V.R_\ell}, \quad \text{called the left polar decomposition of } F. \tag{I.18}$$

Proposition I.5 *1- $\underline{\underline{b}} = V.V =^{noted} V^2$, V is symmetric definite positive, $R_\ell^{-1} = R_\ell^T$. And the left polar decomposition $F = V.R_\ell$ is unique.*

2- $R_\ell = R$ and $V = R.U.R^{-1}$ (so U and V are similar), thus U and V have the same eigenvalues (square root of those of C): $\alpha_i = \beta_i$ and, with (I.1), $\vec{z}_i = R.\vec{c}_i$ is an associated eigenvector of $\underline{\underline{b}}$, for all i.

Proof. 1- "Copy" the proof of prop. I.3 with F^{-1} and $\underline{\underline{b}}^{-1} = (F^{-1})^T.(F^{-1})$ instead of F and $C = F^T.F$.

2- $F = V.R_\ell = R_\ell.(R_\ell^{-1}.V.R_\ell)$ with $R_\ell^{-1}.V.R_\ell$ symmetric (since $(R_\ell^{-1}.V.R_\ell)^T = R_\ell^T.V^T.R_\ell^{-T} = R_\ell^{-1}.V.R_\ell$) and definite positive (since $(R_\ell^{-1}.V.R_\ell.\vec{y}_i,\vec{y}_j)_g = (R_\ell^{-1}.V.R_\ell.\vec{y}_i,R_\ell^{-T}.\vec{y}_j)_g = (V.R_\ell.\vec{y}_i,R.\vec{y}_j)_g = (V.\vec{z}_i,\vec{z}_j)_g = \beta_i$ where the $\vec{y}_i := R_\ell^{-1}\vec{z}_i$ make a basis). Thus $F = R.U = R_\ell.(R_\ell^{-1}.V.R_\ell)$ gives $R = R_\ell$ (uniqueness of the right polar decomposition). Hence $R.U = V.R$ (so V and U are similar), hence V and U have the same eigenvalues and if $\vec{c}_i$ is an eigenvector of U then $R.\vec{c}_i$ is an eigenvector of V: Indeed $V.(R.\vec{c}_i) = R.U.\vec{c}_i = R.(\alpha_i\vec{c}_i) = \alpha_i(R.\vec{c}_i)$ for all i. ∎

J Linear elasticity: A Classical "tensorial" approach

J.1 Definition of elasticity

(See Ciarlet [8].) Motion $\widetilde{\Phi} : [t_1,t_2] \times Obj \to \mathbb{R}^n$, $\Omega_t := \widetilde{\Phi}(t,Obj) \subset \mathbb{R}^n$ for all $t \in [t_1,t_2]$, $t_0 \in [t_1,t_2]$, associated motion $\Phi^{t_0} : (t,P) \in [t_1,t_2] \times \Omega_{t_0} \to \Phi^{t_0}(t,P) \in \mathbb{R}^n$, $\Phi^{t_0}_t(P) := \Phi^{t_0}(t,P)$, $F^{t_0}_t := d\Phi^{t_0}_t$ (deformation gradient), and an imposed Euclidean dot product $(\cdot,\cdot)_g =^{\text{noted}} .\bullet..$

Definition J.1 A material is elastic iff, at any $t \in [t_1,t_2]$ and $p \in \Omega_t$, the Cauchy stress vector $\vec{T}_t(p)$ at t and p only depends on the deformation gradient $F^{t_0}_t(P) := d\Phi^{t_0}_t(P)$ when $p = \Phi^{t_0}_t(P)$ for any $t_0 \in [t_1,t_2]$ and $P \in \Omega_{t_0}$. I.e. there exists a mapping $\widehat{T} : \Omega_{t_0} \times \mathcal{M}_{nn} \to \vec{\mathbb{R}}^n$ (constitutive equation) s.t.

$$\vec{T}_t(p) = \widehat{T}(P,F^{t_0}_t(P)) \quad \text{when} \quad p = \Phi^{t_0}_t(P). \tag{J.1}$$

J.2 Classical approach ("isometric objectivity"), and an issue

Recall: With $F(P) := F_t^{t_0}(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$, the transposed $F(P)_g^T =^{\text{noted}} F_g^T(p) =^{\text{noted}} F^T \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0})$ relative to $(\cdot,\cdot)_g$ is defined by $(F^T.\vec{w},\vec{U})_g := (\vec{w},F.\vec{U})_g$ for all $(\vec{U},\vec{w}) \in \vec{\mathbb{R}}^n_{t_0} \times \vec{\mathbb{R}}^n_t$. And the infinitesimal strain "tensor" (which is not a tensor but a matrix) is defined relative to a $(\cdot,\cdot)_g$-Euclidean basis $(\vec{e}_i)$ (the same at all time) by

$$[\underline{\underline{\varepsilon}}]_{|\vec{e}} = \frac{[F]_{|\vec{e}} + [F]^T_{|\vec{e}}}{2} - I, \quad \text{written} \quad \underline{\underline{\varepsilon}} = \frac{F + F^T}{2} - I. \tag{J.2}$$

And then the homogeneous isotropic elasticity constitutive law reads with λ, μ the Lamé coefficients and $\underline{\underline{\sigma}}$ the Cauchy stress "tensor":

$$\underline{\underline{\sigma}} = \lambda \text{Tr}(\underline{\underline{\varepsilon}})I + 2\mu\underline{\underline{\varepsilon}} = \big(\lambda\text{Tr}(F) - (\lambda+2\mu)\big)I + \mu(F + F^T), \tag{J.3}$$

matrix equation which stands for

$$\underline{\underline{\sigma}} = \lambda \text{Tr}([\underline{\underline{\varepsilon}}]_{|\vec{e}})I + 2\mu[\underline{\underline{\varepsilon}}]_{|\vec{e}} = \big(\lambda\text{Tr}([F]_{|\vec{e}}) - (\lambda+2\mu)\big)I + \mu([F]_{|\vec{e}} + [F]^T_{|\vec{e}}). \tag{J.4}$$

(Recall: F is not an endomorphism, so $\text{Tr}(F)$ is meaningless: It is $\text{Tr}([F]_{|\vec{e}})$ which is meant in (J.3)).

Remark J.2 We can also first start with the matrix expression $\underline{\underline{\sigma}}_{init} = \lambda\text{Tr}(F) + \mu(F - 2I)$ where we see the expected dependence on $F = d\Phi$ (meaning: $\underline{\underline{\sigma}}_{init} = \lambda\text{Tr}([F]_{|\vec{e}}) + \mu([F]_{|\vec{e}} - 2I)$; Then in a Galilean Euclidean framework the stress "tensor" is symmetric, and we write $\underline{\underline{\sigma}} = \frac{\underline{\underline{\sigma}}_{init} + \underline{\underline{\sigma}}^T_{init}}{2}$ to get (J.3). ▪

Remark J.3 Issue (recall): Adding F and F^T (and I) to make $2\underline{\underline{\varepsilon}}$ (in (J.2)) is a mathematical nonsense since they don't have the same domain or codomain: $F : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ while $F^T : \vec{\mathbb{R}}^n_t \to \vec{\mathbb{R}}^n_{t_0}$ (and I is some identity operator so codomain = domain). Thus $\underline{\underline{\varepsilon}}$ can't be a function: It is the matrix in (J.3) (obtained with some Euclidean basis). So $\text{Tr}(\underline{\underline{\varepsilon}}) := \text{Tr}([\underline{\underline{\varepsilon}}]_{|\vec{e}}) = \frac{\text{Tr}([F]_{|\vec{e}}) + \text{Tr}([F^T]_{|\vec{e}})}{2} - n = \text{Tr}([F]_{|\vec{e}}) - n$ (trace of a matrix). Idem

$$\underline{\underline{\sigma}}.\vec{n} = \lambda\text{Tr}(\underline{\underline{\varepsilon}})\vec{n} + 2\mu\underline{\underline{\varepsilon}}.\vec{n} \quad \text{means} \quad \underline{\underline{\sigma}}.[\vec{n}]_{|\vec{e}} = \lambda\text{Tr}([\underline{\underline{\varepsilon}}]_{|\vec{e}})[\vec{n}]_{|\vec{e}} + 2\mu[\underline{\underline{\varepsilon}}]_{|\vec{e}}.[\vec{n}]_{|\vec{e}} \tag{J.5}$$

with $\vec{n}$ the $(\cdot,\cdot)_g$-normal unit out of Ω_t (not out of Ω_{t_0}...). So, despite the eventual claims, neither $\underline{\underline{\varepsilon}}$ nor $\underline{\underline{\sigma}}$ are tensors (they don't have any functional meaning). ▪

Remark J.4 You may read: "For small displacements the Eulerian variable $p = p_t$ and the Lagrangian variable $P = p_{t_0}$ can be confused": $p_t \simeq p_{t_0}$ (so Ω_{t_0} and Ω_t are "almost equal"). Which means that you use the zero-th order Taylor expansions $p_t = \Phi^{t_0}_{p_{t_0}}(t) = p_{t_0} + o(1)$. But you can***not*** then use the first order Taylor expansion (in time) in following calculations (you cannot use velocities)... ▪

J.3 A functional formulation ("isometric objectivity")

Can the constitutive law (J.3) be modified into a functional expression? Yes:

1. Consider the "right polar decomposition" $F = R.U$ where $U \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$, cf. (I.6). The Green Lagrange tensor $E = \frac{C-I}{2}$ (endomorphism in $\vec{\mathbb{R}}^n_{t_0}$) then reads, with (I.8),

$$E = \frac{U^2 - I_{t_0}}{2} = \frac{(U - I_{t_0})^2 + 2(U - I_{t_0})}{2}. \tag{J.6}$$

Then, with $U - I_{t_0} = O(h)$ (small deformation approximation), we get the modified infinitesimal strain tensor at $p_{t_0} \in \Omega_{t_0}$

$$\boxed{\underline{\underline{\tilde{\varepsilon}}} = U - I_{t_0}} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0}), \tag{J.7}$$

endomorphism in $\vec{\mathbb{R}}^n_{t_0}$. (Full notation $\underline{\underline{\tilde{\varepsilon}}}^{t_0}_{t,Gg}(p_{t_0}) = U^{t_0}_{t,Gg}(p_{t_0}) - I_{t_0}(p_{t_0})$.) Thus, for all $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$,

$$\boxed{\underline{\underline{\tilde{\varepsilon}}}.\vec{W} = U.\vec{W} - \vec{W} = R^{-1}.\vec{w} - \vec{W}} \in \vec{\mathbb{R}}^n_{t_0}, \quad \text{when} \quad \vec{w} = F.\vec{W} \text{ (push-forward)}. \tag{J.8}$$

Interpretation: From $\vec{w} = F.\vec{W} = R.U.\vec{W} \in \vec{\mathbb{R}}^n_t$ (the deformed by the motion), first remove the "shifted rotation" to get $R^{-1}.\vec{w} = U.\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, then remove the initial $\vec{W}$ to obtain $R^{-1}.\vec{w} - \vec{W} = \underline{\underline{\tilde{\varepsilon}}}.\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$. In particular $||\underline{\underline{\tilde{\varepsilon}}}.\vec{W}||_G = ||(U - I_{t_0}).\vec{W}||_G$ measures the relative elongation undergone by $\vec{W}$.

2. Then you get a constitutive law with the stress "tensor" $\widetilde{\Sigma}(\Phi) =^{\text{noted}} \widetilde{\Sigma} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ functionally well defined:

$$\boxed{\widetilde{\Sigma} = \lambda \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) I_{t_0} + 2\mu \underline{\underline{\widetilde{\varepsilon}}}} = \lambda \text{Tr}(U - I_{t_0}) I_{t_0} + 2\mu(U - I_{t_0}). \tag{J.9}$$

(The trace $\text{Tr}(\underline{\underline{\widetilde{\varepsilon}}})$ is well defined since $\underline{\underline{\widetilde{\varepsilon}}}$ is an endomorphism.) And, at $p_{t_0} \in \Omega_{t_0}$, for any $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$,

$$\widetilde{\Sigma}.\vec{W} = \lambda \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) \vec{W} + 2\mu \underline{\underline{\widetilde{\varepsilon}}}.\vec{W} = \lambda \text{Tr}(U - I_{t_0}) \vec{W} + 2\mu(U.\vec{W} - \vec{W}) \quad \in \vec{\mathbb{R}}^n_{t_0}. \tag{J.10}$$

3. Then "rotate and shift" with R to get into $\vec{\mathbb{R}}^n_t$ at p_t,

$$\begin{aligned} R.\widetilde{\Sigma}.\vec{W} &= \lambda \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) R.\vec{W} + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} = \lambda \text{Tr}(U - I_{t_0}) R.\vec{W} + 2\mu R.(U - I_{t_0}).\vec{W} \\ &= \lambda \text{Tr}(U - I_{t_0}) R.\vec{W} + 2\mu(F - R).\vec{W}, \\ &= \lambda \text{Tr}(U - I_{t_0}) R.\vec{W} + 2\mu(\vec{w} - R.\vec{W}), \quad \text{where} \quad \vec{w} = F.\vec{W} = R.U.\vec{W}. \end{aligned} \tag{J.11}$$

You have defined the two point "tensor" (functionally well defined)

$$R.\widetilde{\Sigma} = \lambda \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) R + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}} \ \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t). \tag{J.12}$$

4. You get the constitutive law for the stress "tensor" (well defined symmetric endomorphism) in $\vec{\mathbb{R}}^n_t$:

$$(\underline{\underline{\widetilde{\sigma}}}(\Phi) =) \quad \boxed{\underline{\underline{\widetilde{\sigma}}} = R \circ \widetilde{\Sigma} \circ R^{-1}} \overset{\text{noted}}{=} R.\widetilde{\Sigma}.R^{-1} \ \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t). \tag{J.13}$$

So, for any $\vec{w} \in \Omega_t$,

$$\underline{\underline{\widetilde{\sigma}}}.\vec{w} = R.\widetilde{\Sigma}.R^{-1}.\vec{w} \ \in \vec{\mathbb{R}}^n_t. \tag{J.14}$$

Interpretation of (J.13)-(J.14): Shift and rigid rotate backward by applying R^{-1}, apply the elastic stress law with $\widetilde{\Sigma}$ which corresponds to a rotation free motion, then shift and rigid rotate forward by applying R.

Detailed expression for (J.13)-(J.14): With $\text{Tr}(R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1}) = \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}})$ (see exercise J.6), we get, at (t, p_t),

$$\begin{aligned} \underline{\underline{\widetilde{\sigma}}} = \lambda \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}})\, I_t + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1} &= \lambda \text{Tr}(U - I_{t_0})\, I_t + 2\mu R.(U - I_{t_0}).R^{-1} \\ &= \lambda \text{Tr}(U - I_{t_0})\, I_t + 2\mu(F.R^{-1} - I_t). \end{aligned} \tag{J.15}$$

And for any $\vec{w} \in \vec{\mathbb{R}}^n_t$, and with $\vec{w} = R.\vec{W}$, you get

$$\begin{aligned} \underline{\underline{\widetilde{\sigma}}}.\vec{w} = \lambda \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}})\, \vec{w} + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} &= \lambda \text{Tr}(U - I_{t_0})\, \vec{w} + 2\mu R.(U - I_{t_0}).\vec{W} \\ &= \lambda \text{Tr}(U - I_{t_0})\, \vec{w} + 2\mu(R.U.R^{-1}.\vec{w} - \vec{w}). \end{aligned} \tag{J.16}$$

To compare with the classical "functionally meaningless" (J.5).

Remark J.5 Doing so, you avoid the use of the Piola–Kirchhoff tensors. ▪

Exercice J.6 Prove: $\text{Tr}(R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1}) = \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) = \sum_i (\alpha_i - 1)$. (NB: $\underline{\underline{\widetilde{\varepsilon}}}$ is an endomorphism in $\vec{\mathbb{R}}^n_{t_0}$ while $R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1}$ is an endomorphism in $\vec{\mathbb{R}}^n_t$.)

Answer. $\det_{|\vec{e}}(R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1} - \lambda I_t) = \det_{|\vec{e}}(R.(\underline{\underline{\widetilde{\varepsilon}}} - \lambda I_{t_0}).R^{-1}) = \det_{|\vec{E},\vec{e}}(R). \det_{|\vec{E}}(\underline{\underline{\widetilde{\varepsilon}}} - \lambda I). \det_{|\vec{e},\vec{E}}(R^{-1}) = \det_{|\vec{E}}(\underline{\underline{\widetilde{\varepsilon}}} - \lambda I)$ for all Euclidean bases $(\vec{E}_i)$ and $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$. (With $L = \underline{\underline{\widetilde{\varepsilon}}}$ and components, $\text{Tr}(R.L.R^{-1}) = \sum_i (R.L.R^{-1})^i_i = \sum_{ijk} R^i_j L^j_k (R^{-1})^k_i = \sum_{jk} (R^{-1}.R)^k_j L^j_k = \sum_{jk} \delta^k_j L^j_k = \sum_j L^j_j = \text{Tr}(L)$.) ▪

Exercice J.7 Elongation in $\mathbb{R}^2$ along the first axis : origin O, same Euclidean basis $(\vec{E}_1, \vec{E}_2)$ and Euclidean dot product at all time, $\xi > 0$, $t \geq t_0$, $L, H > 0$, $P \in [0, L] \times [0, H]$, $[\overrightarrow{OP}]_{|\vec{E}} = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$, and $[\overrightarrow{O\Phi^{t_0}_t(P)}]_{|\vec{E}} = \begin{pmatrix} X_0 + \xi(t - t_0) X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} X_0(\kappa + 1) \\ Y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = [\overrightarrow{Op}]_{|\vec{E}}$, where $\kappa = \xi(t - t_0) > 0$ for $t > t_0$.

1- Give F, C, $U = \sqrt{C}$ and $R = F.U^{-1}$. Relation with the classical expression ?

2- Spring $\overrightarrow{OP} = \overrightarrow{Oc_{t_0}}(s) = X_0 \vec{E}_1 + Y_0 \vec{E}_2 + s\vec{W}$, i.e. $[\overrightarrow{OP}]_{|\vec{E}} = [\overrightarrow{Oc_{t_0}}]_{|\vec{E}} = \begin{pmatrix} X_0 + sW_1 \\ Y_0 + sW_2 \end{pmatrix}_{|\vec{E}}$ with $s \in [0, L]$ and $\vec{W} = W_1 \vec{E}_1 + W_2 \vec{E}_2$. Give the deformed spring, i.e. give $p = c_t(s) = \Phi^{t_0}_t(c_{t_0}(s))$, and $\vec{c}_t{}'$, and the stretch ratio.

Answer. 1- $[F]=[d\Phi]=\begin{pmatrix}\kappa{+}1 & 0\\ 0 & 1\end{pmatrix}$, same Euclidean dot product and basis at all time, thus $[F^T]=[F]^T=[F]$, then $[C]=[F^T].[F]=[F]^2=\begin{pmatrix}(\kappa{+}1)^2 & 0\\ 0 & 1\end{pmatrix}$, thus $[U]=[F]=\begin{pmatrix}\kappa{+}1 & 0\\ 0 & 1\end{pmatrix}$, thus $[R]=[I]$. All the matrices are given relative to the basis $(\vec E_i)$, thus F,C,U,R (e.g., $C.\vec E_1=(\kappa{+}1)^2\vec E_1$ and $C.\vec E_2=\vec E_2$).

Since $R=I$ and $[\underline{\underline{\varepsilon}}]=[\underline{\underline{\tilde\varepsilon}}]$, (J.15) gives the usual result $[\underline{\underline{\sigma}}]=\lambda\mathrm{Tr}([\underline{\underline{\varepsilon}}])I+2\mu[\underline{\underline{\varepsilon}}]$, cf (J.3) (matrix meaning).

2- $\overrightarrow{Oc_t(s)}=\overrightarrow{O\Phi_t^{t_0}(c_{t_0}(s))}=\begin{pmatrix}(X_0{+}sW_1)(\kappa{+}1)\\ Y_0{+}sW_2\end{pmatrix}_{|\vec E}$, thus $\vec c_t{}'(s)=\begin{pmatrix}W_1(\kappa{+}1)\\ W_2\end{pmatrix}_{|\vec E}$, stretch ration $\frac{W_1^2(\kappa+1)^2+W_2^2}{W_1^2+W_2^2}$ at (t,p_t). ▪

Exercice J.8 Simple shear in $\mathbb{R}^2$: $[\overrightarrow{O\Phi_t^{t_0}(P)}]_{|\vec E}=\begin{pmatrix}X+\xi(t{-}t_0)Y\\ Y\end{pmatrix}=^{\text{noted}}\begin{pmatrix}X+\kappa Y\\ Y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}=[\overrightarrow{Op}]_{|\vec E}$. Same questions, and moreover give the eigenvalues of C.

Answer. 1- $[F]=\begin{pmatrix}1 & \kappa\\ 0 & 1\end{pmatrix}$, $[C]=\begin{pmatrix}1 & 0\\ \kappa & 1\end{pmatrix}.\begin{pmatrix}1 & \kappa\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & \kappa\\ \kappa & \kappa^2{+}1\end{pmatrix}$. Eigenvalues: $\det(C-\lambda I)=\lambda^2-(2{+}\kappa^2)\lambda+1$. Discriminant $\Delta=(2{+}\kappa^2)^2-4=\kappa^2(\kappa^2{+}4)$. Eigenvalues $\alpha_\pm=\frac12(2{+}\kappa^2\pm\kappa\sqrt{\kappa^2{+}4})$. (We check that $\alpha_\pm>0$.) Eigenvectors $\vec v_\pm$(main directions of deformations) given by $(1{-}\alpha_\pm)x+\kappa y=0$, i.e., $y=\frac12(\kappa\pm\sqrt{\kappa^2{+}4})x$, thus, e.g., $\vec v_\pm=\begin{pmatrix}2\\ \kappa\pm\sqrt{\kappa^2{+}4}\end{pmatrix}$. (We check that $\vec v_+\perp\vec v_-$.) With P the transition matrix from $(\vec E_1,\vec E_2)$ to $(\frac{\vec v_+}{||\vec v_+||},\frac{\vec v_-}{||\vec v_-||})$ and $D=\mathrm{diag}(\alpha_+,\alpha_-)$ we get $C=P.D.P^{-1}$ (with $P^{-1}=P^T$ since here $(\frac{\vec v_+}{||\vec v_+||},\frac{\vec v_-}{||\vec v_-||})$ is an orthonormal basis), thus $U=P.\sqrt{D}.P^{-1}$ (we check that $U^T=U$ and $U^2=C$). And $R=F.U^{-1}$.

2- $\overrightarrow{Oc_t(s)}=\overrightarrow{O\Phi_t^{t_0}(c_{t_0}(s))}=\begin{pmatrix}(X_0{+}sW_1)+\kappa(Y_0{+}sW_2)\\ Y_0{+}sW_2\end{pmatrix}$, thus $[\vec c_t{}'(s)]=\begin{pmatrix}W_1+\kappa W_2\\ W_2\end{pmatrix}$. Stretch ratio $\frac{(W_1+\kappa W_2)^2+W_2^2}{W_1^2+W_2^2}$ at (t,p_t). ▪

J.4 Second functional formulation: With the Finger tensor

The above approach uses the push-forward, i.e. uses F (you arrive with your memory). You may prefer to use the pull-back, i.e. use F^{-1} (you remember the past which is Cauchy's point of view): Then you use $F^{-1}=R^{-1}.V^{-1}$ the right polar decomposition of F^{-1}, and you consider the "tensor"

$$\underline{\underline{\tilde{\tilde\varepsilon}}}_t=V^{-1}-I_t\ \in\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t),\tag{J.17}$$

and

$$\underline{\underline{\sigma}}_t=\lambda\mathrm{Tr}(\underline{\underline{\tilde{\tilde\varepsilon}}}_t)I_t+2\mu\underline{\underline{\tilde{\tilde\varepsilon}}}_t,\quad\text{and}\quad\underline{\underline{\sigma}}_t.\vec n_t=\lambda\mathrm{Tr}(\underline{\underline{\tilde{\tilde\varepsilon}}}_t)\,\vec n_t+2\mu\underline{\underline{\tilde{\tilde\varepsilon}}}_t.\vec n_t.\tag{J.18}$$

(Quantities functionally well defined).

K Displacement

K.1 The displacement vector $\vec{\mathcal U}$

In $\mathbb{R}^n$, let $p_t=\Phi_t^{t_0}(p_{t_0})$. Then the bi-point vector

$$\vec{\mathcal U}_t^{t_0}(p_{t_0})=\Phi_t^{t_0}(p_{t_0})-I_{t_0}(p_{t_0})=p_t-p_{t_0}=\overrightarrow{p_{t_0}p_t}\tag{K.1}$$

is called the displacement vector at p_{t_0} relative to t_0 and t. This defines the map

$$\vec{\mathcal U}_t^{t_0}:\begin{cases}\Omega_{t_0}\to\vec{\mathbb{R}}^n\\ p_{t_0}\to\vec{\mathcal U}_t^{t_0}(p_{t_0}):=p_t-p_{t_0}=\overrightarrow{p_{t_0}p_t}\quad\text{when}\quad p_t=\Phi_t^{t_0}(p_{t_0}).\end{cases}\tag{K.2}$$

Thus we have defined

$$\vec{\mathcal U}^{t_0}:\begin{cases}[t_0,T]\times\Omega_{t_0}\to\vec{\mathbb{R}}^n\\ (t,p_{t_0})\to\vec{\mathcal U}^{t_0}(t,p_{t_0}):=\vec{\mathcal U}_t^{t_0}(p_{t_0}),\end{cases}\quad\text{and}\quad\vec{\mathcal U}_{p_{t_0}}^{t_0}:\begin{cases}[t_0,T]\to\vec{\mathbb{R}}^n\\ t\to\vec{\mathcal U}_{p_{t_0}}^{t_0}(t):=\vec{\mathcal U}_t^{t_0}(p_{t_0}).\end{cases}\tag{K.3}$$

Remark K.1 $\vec{\mathcal U}_t^{t_0}(p_{t_0})$ doesn't define a vector field (it is not tensorial), because $\vec{\mathcal U}_t^{t_0}(p_{t_0})=p_t-p_{t_0}=\overrightarrow{p_{t_0}p_t}$ is a bi-point vector which is neither in $\vec{\mathbb{R}}^n_{t_0}$ nor in $\vec{\mathbb{R}}^n_t$ since $p_{t_0}\in\Omega_{t_0}$ and $p_t\in\Omega_t$ (it requires time and space ubiquity gift). In particular, it makes no sense on a non-plane surface (manifold). More at § K.5. ▪

Remark K.2 For elastic solids in $\mathbb{R}^n$, the function $\vec{\mathcal{U}}^{t_0}$ is often considered to be the unknown; But the "real" unknown is the motion Φ^{t_0}. And it is sometimes confused with the extension of a 1-D spring. But see figure 4.1: $||\vec{w}_{t_0}(p_{t_0})||$ represents the initial length and $||\vec{w}_{t_0*}(t,p_t)||$ represents the current length of the spring, and the difference $||\vec{w}_{t_0*}(t,p_t)|| - ||\vec{w}_{t_0}(p_{t_0})||$ can be very small ($\ll 1$) while the length of the displacement vector $||\vec{\mathcal{U}}_t^{t_0}|| = p_t - p_{t_0}$ can be very long ($\gg 1$).

K.2 The differential of the displacement vector

The differential of $\vec{\mathcal{U}}_t^{t_0}$ at p_{t_0} is (matrix meaning)

$$d\vec{\mathcal{U}}_t^{t_0}(p_{t_0}) = d\Phi_t^{t_0}(p_{t_0}) - I_{t_0} = F_t^{t_0}(p_{t_0}) - I_{t_0}, \quad \text{written} \quad d\vec{\mathcal{U}} = F - I, \tag{K.4}$$

which means $[d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})] = [d\Phi_t^{t_0}(p_{t_0})] - [I_{t_0}]$ relative to some basis. It doesn't defined a function, because $F_t^{t_0}(p_{t_0}) : \vec{\mathbb{R}}_{t_0}^n \to \vec{\mathbb{R}}_t^n$ while $I_{t_0} : \vec{\mathbb{R}}_{t_0}^n \to \vec{\mathbb{R}}_{t_0}^n$. Idem, with $\vec{W} \in \vec{\mathbb{R}}_{t_0}^n$, matrix meaning

$$d\vec{\mathcal{U}}.\vec{W} = F.\vec{W} - \vec{W} : \quad \text{means} \quad [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})].[\vec{W}] = [F_t^{t_0}(p_{t_0})].[\vec{W}] - [\vec{W}]. \tag{K.5}$$

K.3 Deformation "tensor" $\underline{\underline{\varepsilon}}$ (matrix), bis

(K.4) gives (matrix meaning)

$$F_t^{t_0}(p_{t_0}) = I_{t_0} + d\vec{\mathcal{U}}_t^{t_0}(p_{t_0}), \quad \text{written} \quad F = I + d\vec{\mathcal{U}}. \tag{K.6}$$

Therefore, Cauchy–Green deformation tensor $C = F^T.F$ reads, in short, (matrix meaning)

$$C = I + d\vec{\mathcal{U}} + d\vec{\mathcal{U}}^T + d\vec{\mathcal{U}}^T.d\vec{\mathcal{U}} \quad \text{(matrix meaning)}, \tag{K.7}$$

i.e. $[C_t^{t_0}(p_{t_0})] = [I_{t_0}] + [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})] + [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})]^T + [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})]^T.[d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})]$.

Thus the Green–Lagrange deformation tensor $E = \frac{C-I}{2}$, cf. (G.50), reads, in short, (matrix meaning)

$$E = \frac{d\vec{\mathcal{U}} + d\vec{\mathcal{U}}^T}{2} + \frac{1}{2} d\vec{\mathcal{U}}^T.d\vec{\mathcal{U}} \quad \text{(matrix meaning)}. \tag{K.8}$$

Thus the deformation tensor $\underline{\underline{\varepsilon}}$, cf. (G.57), reads (matrix meaning)

$$\underline{\underline{\varepsilon}} = E - \frac{1}{2}(d\vec{\mathcal{U}})^T.d\vec{\mathcal{U}}, \tag{K.9}$$

with $\underline{\underline{\varepsilon}}$ the "linear part" of E (small displacements: we only used the first order derivative $d\Phi_t^{t_0}$).

K.4 Small displacement hypothesis, bis

(Usual introduction.) Let $p_t = \Phi_t^{t_0}(p_{t_0})$, $i = 1, 2$, $\vec{W}_i \in \vec{\mathbb{R}}_{t_0}^n$, $\vec{w}_i(p_t) = F_t^{t_0}(p_{t_0}).\vec{W}_i(p_{t_0}) \in \vec{\mathbb{R}}_t^n$ (the push-forwards), written $\vec{w}_i = F.\vec{W}_i$. Then define (matrix meaning)

$$\vec{\Delta}_i := \vec{w}_i - \vec{W}_i = d\mathcal{U}.\vec{W}_i, \quad \text{and} \quad ||\vec{\Delta}||_\infty = \max(||\vec{\Delta}_1||_{\mathbb{R}^n}, ||\vec{\Delta}_2||_{\mathbb{R}^n}). \tag{K.10}$$

Then the small displacement hypothesis reads (matrix meaning):

$$||\vec{\Delta}||_\infty = o(||\vec{W}||_\infty). \tag{K.11}$$

Thus $\vec{w}_i = \vec{W}_i + \vec{\Delta}_i$ (with $\vec{\Delta}_i$ "small") and the hypothesis $(\cdot,\cdot)_g = (\cdot,\cdot)_G$ (same inner dot product at t_0 and t) give

$$(\vec{w}_1, \vec{w}_2)_G - (\vec{W}_1, \vec{W}_2)_G = (\vec{\Delta}_1, \vec{W}_2)_G + (\vec{\Delta}_2, \vec{W}_1)_G + (\vec{\Delta}_1, \vec{\Delta}_2)_G.$$

So (K.9) gives $2(E.\vec{W}_1, \vec{W}_2)_G = 2(\underline{\underline{\varepsilon}}.\vec{W}_1, \vec{W}_2)_G + (d\vec{\mathcal{U}}^T.d\vec{\mathcal{U}}.\vec{W}_1, \vec{W}_2)_G$, And (K.11) gives

$$(E.\vec{W}_1, \vec{W}_2)_G = (\underline{\underline{\varepsilon}}.\vec{W}_1, \vec{W}_2)_G + O(||\vec{\Delta}||_\infty^2), \tag{K.12}$$

so $E_t^{t_0}$ is approximated by $\underline{\underline{\varepsilon}}_t^{t_0}$, that is, $E_t^{t_0} \simeq \underline{\underline{\varepsilon}}_t^{t_0} = \frac{F+F^T}{2} - I = \frac{d\vec{\mathcal{U}}+d\vec{\mathcal{U}}^T}{2}$ (matrix meaning).

K.5 Displacement vector with differential geometry

K.5.1 The shifter

We give the steps, see Marsden–Hughes [14].

• **Affine case** $\mathbb{R}^n$ (continuum mechanics). Recall (I.9): With $p = \Phi_t^{t_0}(P)$, the shifter is:

$$\widetilde{S_t^{t_0}} : \begin{cases} \Omega_{t_0} \times \vec{\mathbb{R}}^n_{t_0} & \to \Omega_t \times \vec{\mathbb{R}}^n_t \\ (P, \vec{Z}_P) & \to \widetilde{S_t^{t_0}}(P, \vec{Z}_P) = (p, S_t^{t_0}(\vec{Z}_P)) \quad \text{with} \quad S_t^{t_0}(\vec{Z}_P) = \vec{Z}_P. \end{cases} \tag{K.13}$$

(The vector is unchanged but the time and the application point have changed: A real observer has no ubiquity gift). So:

$$S_t^{t_0} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \quad \text{and} \quad [S_t^{t_0}]_{|\vec{e}} = I \text{ identity matrix,} \tag{K.14}$$

the matrix equality being possible after the choice of a unique basis at t_0 and at t. And (simplified notation) $\widetilde{S_t^{t_0}}(P, \vec{Z}_P) =^{\text{noted}} S_t^{t_0}(\vec{Z}_P)$. Then the deformation tensor $\underline{\underline{\varepsilon}}$ at P can be defined by

$$\underline{\underline{\varepsilon}}_t^{t_0}(P).\vec{Z}(P) = \frac{(S_t^{t_0})^{-1}(F_t^{t_0}(P).\vec{Z}(P)) + F_t^{t_0}(P)^T.(S_t^{t_0}(P).\vec{Z}(P))}{2} - \vec{Z}(P), \tag{K.15}$$

in short: $\underline{\underline{\varepsilon}}.\vec{Z} = \frac{(S_t^{t_0})^{-1}(F.\vec{Z}) + F^T.(S_t^{t_0}.\vec{Z})}{2} - \vec{Z})$.

• **In a manifold:** Ω is a manifold (like a surface in $\mathbb{R}^3$ from which we cannot take off). Let $T_P\Omega_{t_0}$ be the tangent space à P (the fiber at P), and $T_p\Omega_t$ be the tangent space à p (the fiber at p). In general $T_P\Omega_{t_0} \neq T_p\Omega_t$ (e.g. on the sphere "the Earth"). The bundle (the union of fibers) at t_0 is $T\Omega_{t_0} = \bigcup_{P \in \Omega_{t_0}}(\{P\} \times T_P\Omega_{t_0})$, and the bundle at t is $T\Omega_t = \bigcup_{p \in \Omega_t}(\{p\} \times T_p\Omega_t)$. Then the shifter

$$\widetilde{S_t^{t_0}} : \begin{cases} T\Omega_{t_0} & \to T\Omega_t \\ (P, \vec{Z}_P) & \to \widetilde{S_t^{t_0}}(P, \vec{Z}_P) = (p, S_t^{t_0}(\vec{Z}_P)), \end{cases} \tag{K.16}$$

where $S_t^{t_0}(\vec{Z}_P)$ is defined such that it distorts $\vec{Z}_P$ "as little as possible" along geodesics.

E.g., on a sphere along a path which is a geodesic, if θ_{t_0} is the angle between $\vec{Z}_P$ and the tangent vector to the geodesic at P, then θ_{t_0} is also the angle between $S_t^{t_0}(\vec{Z}_P)$ and the tangent vector to the geodesic at p, and $S_t^{t_0}(\vec{Z}_P)$ has the same length than $\vec{Z}_P$ (at constant speed in a car you think the geodesic is a straight line, although $S_t^{t_0}(\vec{Z}_P) \neq \vec{Z}_P$: the Earth is not flat).

K.5.2 The displacement vector

(Affine space framework, Ω_{t_0} open set in $\mathbb{R}^n$.) Let $P \in \Omega_{t_0}$, $\vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$, $p = \Phi_t^{t_0}(P) \in \Omega_t$, and $d\Phi_t^{t_0} = F_t^{t_0} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$. Define

$$\widetilde{\delta\vec{\mathcal{U}}_t^{t_0}} : \begin{cases} \Omega_{t_0} \times \vec{\mathbb{R}}^n_{t_0} & \to \Omega_t \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \\ (P, \vec{Z}_P) & \to \widetilde{\delta\vec{\mathcal{U}}_t^{t_0}}(P, \vec{Z}_P) = (p, \delta\vec{\mathcal{U}}_t^{t_0}(\vec{Z}_P)) \quad \text{with} \quad \delta\vec{\mathcal{U}}_t^{t_0}(\vec{Z}_P) = (F_t^{t_0} - S_t^{t_0}).\vec{Z}_P. \end{cases} \tag{K.17}$$

Then $\widetilde{\delta\vec{\mathcal{U}}_t^{t_0}} = F_t^{t_0} - S_t^{t_0} : P \in \Omega_{t_0} \to \widetilde{\delta\vec{\mathcal{U}}_t^{t_0}}(P) = F_t^{t_0}(P) - S_t^{t_0}(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ is a two-point tensor. And

$$\begin{aligned} C_t^{t_0} &= (F_t^{t_0})^T.F_t^{t_0} = (\delta\mathcal{U}_t^{t_0} + S_t^{t_0})^T.(\delta\mathcal{U}_t^{t_0} + S_t^{t_0}) \\ &= I + (S_t^{t_0})^T.\delta\mathcal{U}_t^{t_0} + (\delta\mathcal{U}_t^{t_0})^T.S_t^{t_0} + (\delta\mathcal{U}_t^{t_0})^T.\delta\mathcal{U}_t^{t_0}, \end{aligned} \tag{K.18}$$

since $(S_t^{t_0})^T.S_t^{t_0} = I$ identity in $T\Omega_{t_0}$: Indeed, $((S_t^{t_0})^T.S_t^{t_0}.\vec{A}, \vec{B})_{\mathbb{R}^n} = (S_t^{t_0}.\vec{A}, S_t^{t_0}.\vec{B})_{\mathbb{R}^n} = (\vec{A}, \vec{B})_{\mathbb{R}^n}$, cf. (K.13), for all $\vec{A}, \vec{B}$. Then the Green–Lagrange tensor is defined on Ω_{t_0} by

$$E_t^{t_0} = \frac{1}{2}(C_t^{t_0} - I_{t_0}) = \frac{(S_t^{t_0})^T.\delta\mathcal{U}_t^{t_0} + S_t^{t_0}.(\delta\mathcal{U}_t^{t_0})^T}{2} + \frac{1}{2}(\delta\mathcal{U}_t^{t_0})^T.\delta\mathcal{U}_t^{t_0}, \tag{K.19}$$

to compare with (G.50).

L Determinants

L.1 Alternating multilinear form

Let E be a vector space, and let $\mathcal{L}(E,...,E;\mathbb{R}) =^{\text{noted}} \mathcal{L}(E^n;\mathbb{R})$ be the set of multilinear forms, i.e. $m \in \mathcal{L}(E^n;\mathbb{R})$ iff

$$m(...,\vec{x}+\lambda\vec{y},...) = m(...,\vec{x},...) + \lambda m(...,\vec{y},...) \tag{L.1}$$

for all $\vec{x},\vec{y} \in E$, all $\lambda \in \mathbb{R}$, and any "slot".

E.g., $m(\lambda_1\vec{x}_1,...,\lambda_n\vec{x}_n) = (\prod_{i=1,...,n}\lambda_i)\, m(\vec{x}_1,...,\vec{x}_n)$, for all $\lambda_1,...,\lambda_n \in \mathbb{R}$ and all $\vec{x}_1,...,\vec{x}_n \in E$.

In particular a 1-alternating multilinear function is a linear form, also called a 1-form. And the set of 1-forms is $\Omega^1(E) = E^*$. Suppose $n \geq 2$.

Definition L.1 $\mathcal{A}\ell : \left\{ \begin{array}{ll} E^n & \to \mathbb{R} \\ (\vec{v}_1,...,\vec{v}_n) & \to \mathcal{A}\ell(\vec{v}_1,...,\vec{v}_n) \end{array} \right\} \in \mathcal{L}(E^n;\mathbb{R})$ is a n-alternating multilinear form iff, for all $\vec{u},\vec{v} \in E$,

$$\mathcal{A}\ell(...,\vec{u},...,\vec{v},...) = -\mathcal{A}\ell(...,\vec{v},...,\vec{u},...), \tag{L.2}$$

the other elements being unchanged. The set of n-alternating multilinear forms is

$$\Omega^n(E) = \{\mathcal{A}\ell \in \mathcal{L}(E^n;\mathbb{R}) : \mathcal{A}\ell \text{ is alternating}\}. \tag{L.3}$$

If $\mathcal{A}\ell, \mathcal{B}\ell \in \Omega^n(E)$ and $\lambda \in \mathbb{R}$ then $\mathcal{A}\ell + \lambda\mathcal{B}\ell \in \Omega^n(E)$ thanks to the linearity for each variable. Thus $\Omega^n(E)$ is a vector space, sub-space of $\mathcal{L}(E^n;\mathbb{R})$.

L.2 Leibniz formula

Particular case $\dim E{=}n$. Let $\mathcal{A}\ell \in \Omega^n(E)$ (a n-alternating multilinear form). Recall (see e.g. Cartan [5]):

1- A permutation $\sigma : [1,n]_\mathbb{N} \to [1,n]_\mathbb{N}$ is a bijective map (i.e. one-to-one and onto); Let S_n be the set of permutations of $[1,n]_\mathbb{N}$.

2- A transposition $\tau : [1,n]_\mathbb{N} \to [1,n]_\mathbb{N}$ is a permutation that exchanges two elements, that is, $\exists i,j$ s.t. $\tau(...,i,...,j,...) = (...,j,...,i,...)$, the other elements being unchanged.

3- A permutation is a composition of transpositions (theorem left as an exercise, see Cartan). And a permutation is even iff the number of transpositions is even, and a permutation is odd iff the number of transpositions is odd. The parity (even or odd character) of a permutation is an invariant.

4- The signature $\varepsilon(\sigma) = \pm 1$ of a permutation σ is $+1$ if σ is even, and is -1 if σ is odd.

Proposition L.2 (Leibniz formula) *Let $\mathcal{A}\ell \in \Omega^n(E)$. Let $(\vec{e}_i)_{i=1,...,n} =^{noted} (\vec{e}_i)$ be a basis in E. For all vectors $\vec{v}_1,...,\vec{v}_n \in E$, with $\vec{v}_j = \sum_{i=1}^n v_j^i \vec{e}_i$ for all j,*

$$\mathcal{A}\ell(\vec{v}_1,...,\vec{v}_n) = c \sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{j=1}^n v_j^{\sigma(j)} = c \sum_{\tau\in S_n} \varepsilon(\tau) \prod_{i=1}^n v_{\tau(i)}^i \quad \textit{(with } c := \mathcal{A}\ell(\vec{e}_1,...,\vec{e}_n)\textit{)}. \tag{L.4}$$

Thus if $c = \mathcal{A}\ell(\vec{e}_1,...,\vec{e}_n)$ is known, then $\mathcal{A}\ell$ is known. Thus

$$\dim(\Omega^n(E)) = 1. \tag{L.5}$$

(Classic not.: $\vec{v}_j = \sum_{i=1}^n v_{ij}\vec{e}_i$, $\mathcal{A}\ell(\vec{v}_1,...,\vec{v}_n) = c\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{i=1}^n v_{\sigma(i),i} = c\sum_{\tau\in S_n}\varepsilon(\tau)\prod_{i=1}^n v_{i,\tau(i)}$.)

Proof. Let $F := \mathcal{F}([1,n]_\mathbb{N};[1,n]_\mathbb{N}) =^{\text{noted}} [1,n]_\mathbb{N}^{[1,n]_\mathbb{N}}$ be the set of functions $i : \left\{ \begin{array}{ll} [1,n]_\mathbb{N} & \to [1,n]_\mathbb{N} \\ k & \to i_k = i(k) \end{array} \right\}$. $\mathcal{A}\ell$ being multilinear, $\mathcal{A}\ell(\vec{v}_1,...,\vec{v}_n) = \sum_{j_1=1}^n v_1^{j_1} \mathcal{A}\ell(\vec{e}_{j_1},\vec{v}_2,...,\vec{v}_n)$ ("the first column" development). By recurrence we get $\mathcal{A}\ell(\vec{v}_1,...,\vec{v}_n) = \sum_{j_1,...,j_n=1}^n v_1^{j_1}...v_n^{j_n} \mathcal{A}\ell(\vec{e}_{j_1},...,\vec{e}_{j_n}) = \sum_{j\in F}\prod_{k=1}^n v_k^{j(k)} \mathcal{A}\ell(\vec{e}_{j(1)},...,\vec{e}_{j(n)})$.

And $\mathcal{A}\ell(\vec{e}_{i_1},...,\vec{e}_{i_n}) \neq 0$ iff $i : k \in \{1,...,n\} \to i(k) = i_k \in \{1,...,n\}$ is one-to-one (thus bijective). Thus $\mathcal{A}\ell(\vec{v}_1,...,\vec{v}_n) = \sum_{\sigma\in S_n}\prod_{i=1}^n v_i^{\sigma(i)} \mathcal{A}\ell(\vec{e}_{\sigma(1)},...,\vec{e}_{\sigma(n)}) = \sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{i=1}^n v_i^{\sigma(i)} \mathcal{A}\ell(\vec{e}_1,...,\vec{e}_n)$, which is the first equality in (L.4). Then $\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{i=1}^n v_i^{\sigma(i)} = \sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{i=1}^n v_{\sigma^{-1}(i)}^{\sigma(\sigma^{-1}(i))}$ since σ is bijectif, thus $\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{i=1}^n v_i^{\sigma(i)} = \sum_{\tau\in S_n}\varepsilon(\tau^{-1})\prod_{i=1}^n v_{\tau(i)}^i$, thus the second equality in (L.4) since $\varepsilon(\tau)^{-1} = \varepsilon(\tau)$. (See Cartan [5].) ∎

L.3 Determinant of vectors

Definition L.3 $(\vec{e}_i)_{i=1,...,n}$ being a basis in E, the determinant relative to $(\vec{e}_i)$ is the alternating multilinear form $\det_{|\vec{e}} \in \Omega^n(E)$ defined by

$$\det_{|\vec{e}}(\vec{e}_1, ..., \vec{e}_n) = 1. \tag{L.6}$$

And the determinant relative to $(\vec{e}_i)$ of n vectors $\vec{v}_i$ is , with $\vec{v}_j = \sum_{i=1}^n v_j^i \vec{e}_i$ for all j, and with prop. L.2 (here $c = 1$),

$$\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{j=1}^n v_j^{\sigma(j)} = \sum_{\tau \in S_n} \varepsilon(\tau) \prod_{i=1}^n v_{\tau(i)}^i \tag{L.7}$$

In particular,

$$\Omega^n(E) = \text{Vect}\{\det_{|\vec{e}}\} \quad \text{(the 1-D vector space spanned by } \det_{|\vec{e}}). \tag{L.8}$$

And any $\mathscr{A} \in \Omega^n(E)$ reads

$$\mathscr{A} = \mathscr{A}(\vec{e}_1, ..., \vec{e}_n) \det_{|\vec{e}}. \tag{L.9}$$

And if $(\vec{b}_i)$ is another basis then

$$\det_{|\vec{b}} = c \det_{|\vec{e}} \quad \text{where} \quad c = \det_{|\vec{b}}(\vec{e}_1, ..., \vec{e}_n). \tag{L.10}$$

Exercice L.4 Change of measuring unit: If $(\vec{a}_i)$ is a basis and $\vec{b}_j = \lambda \vec{a}_j$ for all j, prove

$$\forall j = 1, ..., n, \quad \vec{b}_j = \lambda \vec{a}_j \quad \Longrightarrow \quad \det_{|\vec{a}} = \lambda^n \det_{|\vec{b}} \tag{L.11}$$

(gives the relation between volumes relative to a change of measuring unit in the Euclidean case).

Answer. $\det_{|\vec{a}}(\vec{b}_1, ..., \vec{b}_n) = \det_{|\vec{a}}(\lambda\vec{a}_1, ..., \lambda\vec{a}_n) \overset{multi}{\underset{linear}{=}} \lambda^n \det_{|\vec{a}}(\vec{a}_1, ..., \vec{a}_n) \overset{(L.6)}{=} \lambda^n \overset{(L.6)}{=} \lambda^n \det_{|\vec{b}}(\vec{b}_1, ..., \vec{b}_n)$. ▪

Proposition L.5 $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0$ *iff* $(\vec{v}_1, ..., \vec{v}_n)$ *is a basis; Or equivalently,* $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0$ *iff* $\vec{v}_1, ..., \vec{v}_n$ *are linearly dependent.*

Proof. If $\sum_{i=1}^n c_i \vec{v}_i = 0$ and one of the $c_i \neq 0$ and then a $\vec{v}_i$ is a linear combination of the others thus $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0$ (since $\det_{|\vec{e}}$ is alternate); Thus $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0 \Rightarrow$ the $\vec{v}_i$ are independent. And if the $\vec{v}_i$ are independent then $(\vec{v}_1, ..., \vec{v}_n)$ is a basis, thus $\det_{|\vec{v}}(\vec{v}_1, ..., \vec{v}_n) = 1 \neq 0$, with $\det_{|\vec{v}} = c \det_{|\vec{e}}$, thus $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0$. ▪

Exercice L.6 In $\mathbb{R}^2$. Let $\vec{v}_1 = \sum_{i=1}^2 v_1^i \vec{e}_i$ and $\vec{v}_2 = \sum_{j=1}^2 v_2^j \vec{e}_j$ (duality notations). Prove:

$$\det_{|\vec{e}}(\vec{v}_1, \vec{v}_2) = v_1^1 v_2^2 - v_1^2 v_2^1. \tag{L.12}$$

Answer. Development relative to the first column: $\det_{|\vec{e}}(\vec{v}_1, \vec{v}_2) = \det_{|\vec{e}}(v_1^1 \vec{e}_1 + v_1^2 \vec{e}_2, \vec{v}_2) = v_1^1 \det_{|\vec{e}}(\vec{e}_1, \vec{v}_2) + v_1^2 \det_{|\vec{e}}(\vec{e}_2, \vec{v}_2)$. Then $\det_{|\vec{e}}(\vec{v}_1, \vec{v}_2) = 0 + v_1^1 v_2^2 \det(\vec{e}_1, \vec{e}_2) + v_1^2 v_2^1 \det(\vec{e}_2, \vec{e}_1) + 0 = v_1^1 v_2^2 - v_1^2 v_2^1$. ▪

Exercice L.7 In $\mathbb{R}^3$, with $\vec{v}_j = \sum_{i=1}^3 v_j^i \vec{e}_i$, prove:

$$\det(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \sum_{i,j,k=1}^3 \varepsilon_{ijk} v_1^i v_2^j v_3^k, \tag{L.13}$$

where $\varepsilon_{ijk} = \frac{1}{2}(j-i)(k-j)(k-i)$, i.e. $\varepsilon_{ijk} = 1$ if $(i,j,k) = (1,2,3)$, $(3,1,2)$ or $(2,3,1)$ (even signature), $\varepsilon_{ijk} = -1$ if $(i,j,k) = (3,2,1)$, $(1,3,2)$ and $(2,1,3)$ (odd signature), and $\varepsilon_{ijk} = 0$ otherwise.

Answer. Development relative to the first then second then third column (as in exercise L.6).
Result $= v_1^1 v_2^2 v_3^3 + v_2^1 v_3^2 v_1^3 + v_3^1 v_1^2 v_2^3 - v_1^3 v_2^2 v_3^1 - v_2^3 v_3^2 v_1^1 - v_3^3 v_1^2 v_2^1$. ▪

L.4 Determinant of a matrix

Let $M = [M_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,n}} \in \mathcal{M}_{nn}$(a $n*n$ real matrix). Let $(\vec{E}_i)$ be the canonical basis in $\mathcal{M}_{n1}$ (space of $n*1$ matrices). Let $\vec{v}_j \in \mathcal{M}_{n1}$ and call M_{ij} its components: $\vec{v}_j = \sum_{i=1}^n M_{ij}\vec{E}_i$. So $[\vec{v}_1]_{|\vec{E}} = M.[\vec{E}_1]_{|\vec{E}}$. And $[\vec{v}_1]_{|\vec{E}} =^{\text{noted}} \vec{v}_1$ because the canonical basis will be systematically used in $\mathcal{M}_{n1}$.

So $M = (\,\vec{v}_1, ..., \vec{v}_n\,) = \big(\,M.\vec{E}_1, ..., M.\vec{E}_n\,\big) = [M_{ij}]$.

Definition L.8 The determinant of the matrix $M = (\,\vec{v}_1, ..., \vec{v}_n\,) = [M_{ij}]$ is

$$\det(M) := \det_{|\vec{E}}(\vec{v}_1, ..., \vec{v}_n) \quad (= \det_{|\vec{E}}\big(\,M.\vec{E}_1, ..., M.\vec{E}_n\,\big) = \det([M_{ij}])). \tag{L.14}$$

Proposition L.9 • $\det(I) = 1$.
• *If* $M, N \in \mathcal{M}_{nn}$ *then* $\det(M.N) = (\det M)(\det N)$.
• *If* $M \in \mathcal{M}_{nn}$ *then* $\det(M^T) = \det(M)$.

Proof. • $\det(I) := \det_{|\vec{E}}(\vec{E}_1, ..., \vec{E}_n) = 1$.

• Define $a, b : E^n \to \mathbb{R}$ by $a(\vec{v}_1, ..., \vec{v}_n) := \det_{|\vec{E}}(M.\vec{v}_1, ..., M.\vec{v}_n)$ and $b(\vec{v}_1, ..., \vec{v}_n) := \det_{|\vec{E}}(M.N.\vec{v}_1, ..., M.N.\vec{v}_n)$. They are alternated forms (since the matrix product is linear) in $\Omega^1(E)$. Thus $b = \lambda a$ for a $\lambda \in \mathbb{R}$ since $\dim(\Omega^n(E)) = 1$. Thus $\det(M.N) = b(\vec{E}_1, ..., \vec{E}_n) = \lambda a(\vec{E}_1, ..., \vec{E}_n) = \lambda \det(M)$, and in particular $\det(N) = \det(I.N) = \lambda \det(I) = \lambda$.

• $\det[M_{ij}] = \det_{|\vec{E}}(\vec{v}_1, ..., \vec{v}_n) \overset{(L.7)}{=} \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^n v_i^{\sigma(i)} = \sum_{\tau \in S_n} \varepsilon(\tau) \prod_{i=1}^n v_{\tau(i)}^i = \det[M_{ji}]$. ∎

Exercice L.10 Let $g(\cdot, \cdot)$ be an inner dot product, $(\vec{e}_i)$ be a basis, $g_{ij} = g(\vec{e}_i, \vec{e}_j)$. Prove $\det([g_{ij}]) > 0$.

Answer. $[g]_{|\vec{e}}$ is symmetric def. > 0, $[g]_{|\vec{e}} = P^T.D.P$, $\det([g]_{|\vec{e}}) = \det(P)^2 \prod_{i=1}^n (\lambda_i) > 0$. ∎

L.5 Volume

Definition L.11 *Let* $(\vec{e}_i)$ *be a Euclidean basis. Consider a parallelepiped in* $\mathbb{R}^n$ *which sides are given by the vectors* $\vec{v}_1, ..., \vec{v}_n$*; Its algebraic volume and its volume relative to* $(\vec{e}_i)$ *are*

$$\textit{algebraic volume} = \det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n), \quad \textit{and} \quad \textit{volume} = \big|\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n)\big|. \tag{L.15}$$

If $n = 2$ *then volume is also called an area. If* $n = 1$ *then volume is also called a length.*

Notation. Let $(\vec{e}_i)$ be a Cartesian basis and $(e^i) = (dx^i)$ be the dual basis. Then, cf. Cartan [6],

$$\det_{|\vec{e}} \overset{\text{noted}}{=} e^1 \times ... \times e^n = dx^1 \times ... \times dx^n. \tag{L.16}$$

And, for integration, the volume element (non negative) uses a Euclidean basis $(\vec{e}_i)$ and is

$$d\Omega(\vec{x}) = |\det_{|\vec{e}}| = |dx^1 \times ... \times dx^n| \overset{\text{noted}}{=} dx^1...dx^n. \tag{L.17}$$

And the volume of a regular domain Ω is

$$0 \le |\Omega| := \int_\Omega d\Omega = \int_{\vec{x} \in \Omega} dx^1...dx^n. \tag{L.18}$$

(cf. Riemann approach: any regular volume Ω can be approximated with cubes as small as wished.)

Exercice L.12 Let $\Psi : \left\{ \begin{array}{rl} [a_1, b_1] \times ... \times [a_n, b_n] & \to \Omega \\ \vec{q} = (q_1, ..., q_n) & \to \vec{x} = (x_1 = \Psi_1(\vec{q}), ..., x_n = \Psi_n(\vec{q})) \end{array} \right\}$ be a parametric description of a domain Ω. Prove

$$d\Omega(\vec{x}) = |J_\Psi(\vec{q})|\, dq^1...dq^n, \quad \text{and} \quad |\Omega| = \int_{\vec{q}} |J_\Psi(\vec{q})|\, dq^1...dq^n, \tag{L.19}$$

where $J_\Psi(\vec{q}) = \det_{|\vec{e}}[d\Psi(\vec{q})]_{|\vec{e}} = \det[\frac{\partial \Psi}{\partial q_i}(\vec{q})] = \det_{|\vec{e}}(\vec{p}_1, ...\vec{p}_n)$ is the Jacobian matrix of Ψ at $\vec{q}$ = the volume at $\vec{x} = \Psi(\vec{q})$ relative to $\vec{e}_i$) limited by the tangent vectors $\vec{p}_i(\vec{x}) = \frac{\partial \Psi}{\partial q_i}(\vec{q})$.

Answer. Polar coordinates for illustration purpose (immediate generalization): Consider the disk Ω parametrized with the polar coordinate system $\Psi : \left\{ \begin{array}{ll}]0,R] \times [0,2\pi] & \to \mathbb{R}^2 \\ \vec{q} = (\rho,\theta) & \to \vec{x} = (x = \rho\cos\theta, y = \rho\sin\theta) \end{array} \right\}$ where a Euclidean basis $(\vec{e}_1, \vec{e}_2)$ is used in $\mathbb{R}^2$ (so $\vec{x} = \rho\cos\theta\vec{e}_1 + \rho\sin\theta\vec{e}_2$). The associated polar basis at $\vec{x} = \Psi(\vec{q})$ is $(\vec{p}_1(\vec{x}) = \frac{\partial\Psi}{\partial\rho}(\rho,\theta), \vec{p}_2(\vec{x}) = \frac{\partial\Psi}{\partial\theta}(\rho,\theta))$, so $[\vec{p}_1(\vec{x})]_{|\vec{e}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $[\vec{p}_2(\vec{x})]_{|\vec{e}} = \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \end{pmatrix}$. Thus $\det_{|\vec{e}}(\vec{p}_1(\vec{x}), \vec{p}_2(\vec{x})) = \rho$ (> 0 here), thus $d\Omega = |\rho|\, d\rho d\theta = \rho\, d\rho d\theta$. Thus the volume is $|\Omega| = \int_{\vec{x}\in\Omega} d\Omega = \int_{\rho=0}^{R}\int_{\theta=0}^{2\pi} \rho\, d\rho d\theta = \pi R^2$. ∎

Exercice L.13 What is the "volume element" on a regular surface Σ in $\mathbb{R}^3$, called the "surface element"?

Answer. Let $\Psi : \left\{ \begin{array}{ll} [a_1,b_2] \times [a_2,b_2] & \to \mathbb{R}^3 \\ (u,v) & \to \vec{x} = \Psi(u,v) = x_1(u,v)\vec{e}_1 + ... + x_3(u,v)\vec{e}_3 \end{array} \right\}$ be a regular parametrization of the geometric surface $\Sigma = \mathrm{Im}(\Psi)$, where $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is a Euclidean basis in $\mathbb{R}^3$. Thus $\vec{t}_1(\vec{x}) = \frac{\partial\Psi}{\partial u}(u,v)$ and $\vec{t}_2(\vec{x}) = \frac{\partial\Psi}{\partial v}(u,v)$ are the tangent vectors at Σ at $\vec{x} = \Psi(u,v)$. Hence a normal unit vector is $\vec{n}(\vec{x}) = \frac{\vec{t}_1(\vec{x})\times\vec{t}_2(\vec{x})}{||\vec{t}_1(\vec{x})\times\vec{t}_2(\vec{x})||}$, thus $\det_{|\vec{e}}(\vec{t}_1, \vec{t}_2, \vec{n}) = ||\vec{t}_1(\vec{x})\times\vec{t}_2(\vec{x})||$ is the area of the parallelogram which sides are given by $\vec{t}_1$ and $\vec{t}_2$ (volume with height 1). Thus the surface element at $\vec{x} = \Psi(u,v)$ is $d\Sigma(\vec{x}) = ||\vec{t}_1(\vec{x}) \times \vec{t}_2(\vec{x})||\, dudv = ||\frac{\partial\Psi}{\partial u}(u,v) \times \frac{\partial\Psi}{\partial v}(u,v)||\, dudv$. Thus $|\Sigma| = \int_{\vec{x}\in\Sigma} d\Sigma(\vec{x}) = \int_{u=a_1}^{b_1}\int_{v=a_2}^{b_2} ||\frac{\partial\Psi}{\partial u}(u,v) \times \frac{\partial\Psi}{\partial v}(u,v)||\, dudv$. ∎

L.6 Determinant of an endomorphism

L.6.1 Definition and basic properties

Definition L.14 The determinant of an endomorphism $L \in \mathcal{L}(E;E)$ relative to a basis $(\vec{e}_i)$ in E is

$$\widetilde{\det_{|\vec{e}}}(L) := \det_{|\vec{e}}(L.\vec{e}_1, ..., L.\vec{e}_n) \overset{\text{noted}}{=} \det_{|\vec{e}}(L.\vec{e}_1, ..., L.\vec{e}_n). \tag{L.20}$$

the last notation if the context is not ambiguous. This define $\widetilde{\det}_{|\vec{e}} : \mathcal{L}(E;E) \to \mathbb{R}$.

Proposition L.15 *Let $L \in \mathcal{L}(E;E)$.*
1- If $L = I$ the identity, then $\widetilde{\det}_{|\vec{e}}(I) = 1$, for all basis $(\vec{e}_i)$.
2- For all $\vec{v}_1, ..., \vec{v}_n \in E$,

$$\det_{|\vec{e}}(L.\vec{v}_1, ..., L.\vec{v}_n) = \widetilde{\det_{|\vec{e}}}(L) \det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n). \tag{L.21}$$

3- If $L.\vec{e}_j = \sum_{i=1}^n L_{ij}\vec{e}_i$, then

$$\widetilde{\det_{|\vec{e}}}(L) = \det([L]_{|\vec{e}}) \quad (= \det([L_{ij}])). \tag{L.22}$$

4- For all $M \in \mathcal{L}(E;E)$, and with $M \circ L =^{noted} M.L$ (thanks to linearity),

$$\widetilde{\det_{|\vec{e}}}(M.L) = \widetilde{\det_{|\vec{e}}}(M)\, \widetilde{\det_{|\vec{e}}}(L) = \widetilde{\det_{|\vec{e}}}(L.M). \tag{L.23}$$

5- L is invertible iff $\widetilde{\det}_{|\vec{e}}(L) \neq 0$.
6- If L is invertible then

$$\widetilde{\det_{|\vec{e}}}(L^{-1}) = \frac{1}{\widetilde{\det}_{|\vec{e}}(L)}. \tag{L.24}$$

7- If $(\cdot,\cdot)_g$ is an inner dot product in E and L_g^T is the $(\cdot,\cdot)_g$ transposed of L (i.e., $(L_g^T\vec{w}, \vec{u})_g = (\vec{w}, L.\vec{u})_g$ for all $\vec{u}, \vec{w} \in E$) then

$$\widetilde{\det_{|\vec{e}}}(L_g^T) = \widetilde{\det_{|\vec{e}}}(L). \tag{L.25}$$

8- If $(\vec{e}_i)$ and $(\vec{b}_i)$ are two $(\cdot,\cdot)_g$-orthonormal bases in $\vec{\mathbb{R}}_t^n$, then $\det_{|\vec{b}} = \pm \det_{|\vec{e}}$.

Proof. 1- $\widetilde{\det}_{|\vec{e}}(I) =^{(L.20)} \det_{|\vec{e}}(I.\vec{e}_1, ..., I.\vec{e}_n) =^{(L.6)} \det_{|\vec{e}}(\vec{e}_1, ..., \vec{e}_n) = 1$, true for all basis.

2- Let $m : (\vec{v}_1, ..., \vec{v}_n) \to m(\vec{v}_1, ..., \vec{v}_n) := \det_{|\vec{e}}(L.\vec{v}_1, ..., L.\vec{v}_n)$: It is a multilinear alternated form, since L is linear; Thus $m =^{(L.9)} m(\vec{e}_1, ..., \vec{e}_n) \det_{|\vec{e}}$; With $m(\vec{e}_1, ..., \vec{e}_n) =^{(L.20)} \widetilde{\det}_{|\vec{e}}(L)$, thus (L.21).

3- Apply (L.14) with $M = [L]_{|\vec{e}}$ to get (L.22).

4- $\det_{|\vec{e}}((M.L).\vec{e}_1, ..., (M.L).\vec{e}_n) = \det_{|\vec{e}}(M.(L.\vec{e}_1), ..., M.(L.\vec{e}_n)) =^{(L.21)} \widetilde{\det}_{|\vec{e}}(M) \det_{|\vec{e}}(L.\vec{e}_1, ..., L.\vec{e}_n)$.

5- If L is invertible, then $1 = \widetilde{\det}_{|\vec{e}}(I) = \widetilde{\det}_{|\vec{e}}(L.L^{-1}) = \widetilde{\det}_{|\vec{e}}(L)\,\widetilde{\det}_{|\vec{e}}(L^{-1})$, thus $\widetilde{\det}_{|\vec{e}}(L) \neq 0$.
If $\widetilde{\det}_{|\vec{e}}(L) \neq 0$ then $\det_{|\vec{e}}(L.\vec{e}_1, ..., L.\vec{e}_n) \neq 0$, thus $(L.\vec{e}_1, ..., L.\vec{e}_n)$ is a basis, thus L is invertible.
6- (L.23) gives $1 = \widetilde{\det}_{|\vec{e}}(I) = \widetilde{\det}_{|\vec{e}}(L^{-1}.L) = \widetilde{\det}_{|\vec{e}}(L).\,\widetilde{\det}_{|\vec{e}}(L^{-1})$, thus (L.24).
7- $[g]_{|\vec{e}}.[L_g^T]_{|\vec{e}} = ([L]_{|\vec{e}})^T.[g]_{|\vec{e}}$ gives $\det([g]_{|\vec{e}})\det([L_g^T]_{|\vec{e}}) = \det(([L]_{|\vec{e}})^T)\det([g]_{|\vec{e}})$,

8- Let $\mathcal{P}$ be the change of basis endomorphism from $(\vec{e}_i)$ to $(\vec{b}_i)$, and $P = [\mathcal{P}]_{|\vec{e}}$ (the transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$). Both basis being $(\cdot,\cdot)_g$-orthonormal, $P^T.P = I$, thus $\det(P)^2 = 1$, thus $\det(P) = \pm 1 = \widetilde{\det}_{|\vec{e}}(\mathcal{P})$. And $\det_{|\vec{e}}(\vec{b}_1, ..., \vec{b}_n) = \det_{|\vec{e}}(\mathcal{P}.\vec{e}_1, ..., \mathcal{P}.\vec{e}_n) = \widetilde{\det}_{|\vec{e}}(\mathcal{P})\det_{|\vec{e}}(\vec{e}_1, ..., \vec{e}_n) = \widetilde{\det}_{|\vec{e}}(\mathcal{P})\det_{|\vec{b}}(\vec{b}_1, ..., \vec{b}_n)$, thus $\det_{|\vec{e}} = \widetilde{\det}_{|\vec{e}}(\mathcal{P})\det_{|\vec{b}} = \pm\det_{|\vec{b}}$. ∎

Definition L.16 Two $(\cdot,\cdot)_g$-orthonormal bases $(\vec{e}_i)$ and $(\vec{b}_i)$ have the same orientation iff $\det_{|\vec{b}} = +\det_{|\vec{e}}$.

Exercice L.17 Prove $\widetilde{\det}_{|\vec{e}}(\lambda L) = \lambda^n\,\widetilde{\det}_{|\vec{e}}(L)$.

Answer. $\widetilde{\det}_{|\vec{e}}(\lambda L) = \det_{|\vec{e}}(\lambda L.\vec{e}_1, ..., \lambda L.\vec{e}_n) = \lambda^n \det_{|\vec{e}}(L.\vec{e}_1, ..., L.\vec{e}_n) = \lambda^n\,\widetilde{\det}_{|\vec{e}}(L)$. ∎

L.6.2 The determinant of an endomorphism is objective

Proposition L.18 *Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases in E. The determinant of an endomorphism $L \in \mathcal{L}(E;E)$ is objective (observer independent, here basis independent):*

$$(\det([L]_{|\vec{a}}) =) \quad \widetilde{\det}_{|\vec{a}}(L) = \widetilde{\det}_{|\vec{b}}(L) \quad (= \det([L]_{|\vec{b}})). \tag{L.26}$$

NB: *But the determinant of n vectors is* **not** *objective, cf. (L.10) (compare the change of basis formula for vectors $[\vec{w}]_{|\vec{b}} = P^{-1}.[\vec{w}]_{|\vec{a}}$ with the change of basis formula for endomorphisms $[L]_{|\vec{b}} = P^{-1}.[L]_{|\vec{a}}.P$).*

Proof. Let P be the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$. Hence $[L]_{|\vec{b}} = P^{-1}.[L]_{|\vec{a}}.P$ and (L.23)-(L.24) give $\det([L]_{|\vec{b}}) = \det(P^{-1})\det([L]_{|\vec{a}})\det(P) = \det([L]_{|\vec{a}})$. ∎

Exercice L.19 Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases in E, and define $\mathcal{P} \in \mathcal{L}(E;E)$ by $\mathcal{P}.\vec{a}_j = \vec{b}_j$ for all j (the change of basis endomorphism). Prove

$$\det_{|\vec{a}}(\vec{b}_1, ..., \vec{b}_n) = \widetilde{\det}_{|\vec{a}}(\mathcal{P}), \quad \text{thus} \quad \det_{|\vec{a}} = \widetilde{\det}_{|\vec{a}}(\mathcal{P})\det_{|\vec{b}}, \quad \text{i.e.} \quad \det_{|\vec{b}} = \frac{\det_{|\vec{a}}}{\widetilde{\det}_{|\vec{a}}(\mathcal{P})}, \tag{L.27}$$

Answer. $\det_{|\vec{a}}(\vec{b}_1, ..., \vec{b}_n) = \det_{|\vec{a}}(\mathcal{P}.\vec{a}_1, ..., \mathcal{P}.\vec{a}_n) \overset{(L.21)}{=} \widetilde{\det}_{|\vec{a}}(\mathcal{P})\det_{|\vec{a}}(\vec{a}_1, ..., \vec{a}_n) = \widetilde{\det}_{|\vec{a}}(\mathcal{P}) = \widetilde{\det}_{|\vec{a}}(\mathcal{P})\det_{|\vec{b}}(\vec{b}_1, ..., \vec{b}_n)$, thus $\det_{|\vec{a}} = \widetilde{\det}_{|\vec{a}}(\mathcal{P})\det_{|\vec{b}}$. ∎

L.7 Determinant of a linear map

(Needed for the deformation gradient $F_t^{t_0}(P) = d\Phi_t^{t_0}(P) : \vec{\mathbb{R}}_{t_0}^n \to \vec{\mathbb{R}}_t^n$.)

Let A and B be vector spaces, $\dim A = \dim B = n$, and $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases in A and B.

L.7.1 Definition and first properties

Definition L.20 The determinant of a linear map $L \in \mathcal{L}(A;B)$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ is

$$\widetilde{\det}_{|\vec{a},\vec{b}}(L) := \det_{|\vec{b}}(L.\vec{a}_1, ..., L.\vec{a}_n). \tag{L.28}$$

(And $\widetilde{\det}_{|\vec{a},\vec{b}}(L) =^{\text{noted}} \det(L)$ if the bases are implicit.)

Thus, with $L.\vec{a}_j = \sum_{i=1}^n L_{ij}\vec{b}_i$, i.e. $[L]_{|\vec{a},\vec{b}} = [L_{ij}]$, and with (L.14):

$$\widetilde{\det}_{|\vec{a},\vec{b}}(L) = \det([L_{ij}]). \tag{L.29}$$

Proposition L.21 *Let $\vec u_1,...,\vec u_n \in A$. Then*

$$\det_{|\vec b}(L.\vec u_1,...,L.\vec u_n) = \widetilde{\det}_{|\vec a,\vec b}(L)\ \det_{|\vec a}(\vec u_1,...,\vec u_n). \tag{L.30}$$

Proof. $m : (\vec u_1,...,\vec u_n) \in A^n \to m(\vec u_1,...,\vec u_n) := \det_{|\vec b}(L.\vec u_1,...,L.\vec u_n) \in \mathbb{R}$ is a multilinear alternated form since L is linear; And $m(\vec a_1,...,\vec a_n) = \det_{|\vec b}(L.\vec a_1,...,L.\vec a_n) =^{(L.28)} \widetilde{\det}_{|\vec a,\vec b}(L) = \widetilde{\det}_{|\vec a,\vec b}(L)\det_{|\vec a}(\vec a_1,...,\vec a_n)$. Thus $m = \widetilde{\det}_{|\vec a,\vec b}(L)\det_{|\vec a}$, cf. (L.10), thus (L.30). ∎

Corollary L.22 *Let A,B,C be vector spaces such that* $\dim A = \dim B = \dim C = n$. *Let $(\vec a_i)$, $(\vec b_i)$, $(\vec c_i)$ be bases in A,B,C. Let $L : A \to B$ and $M : B \to C$ be linear. Then, with $M \circ L =^{noted} M.L$ (thanks to linearity),*

$$\widetilde{\det}_{|\vec a,\vec c}(M.L) = \widetilde{\det}_{|\vec a,\vec b}(L)\ \widetilde{\det}_{|\vec b,\vec c}(M). \tag{L.31}$$

Proof. $\widetilde{\det}_{|\vec a,\vec c}(M.L) = \det_{|\vec c}(M.L.\vec a_1),...,M.L.\vec a_n)) = \widetilde{\det}_{|\vec b,\vec c}(M)\det_{|\vec b}(L.\vec a_1,...,L.\vec a_n) = \widetilde{\det}_{|\vec b,\vec c}(M)\ \widetilde{\det}_{|\vec a,\vec b}(L)$. ∎

L.7.2 Jacobian of a motion, and dilatation

$F := F_t^{t_0}(p_{t_0}) := d\Phi_t^{t_0}(p_{t_0}) : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ is the deformation gradient at $p_{t_0} \in \Omega_{t_0}$ relative to t_0 and t, cf. (4.1). Let $(\vec E_i)$ be a Euclidean basis in $\vec{\mathbb{R}}^n_{t_0}$ and $(\vec e_i)$ be a Euclidean basis in $\vec{\mathbb{R}}^n_t$ for all $t \ge t_0$. Let F_{ij} be the components of F relative to these bases, so $F.\vec E_j = \sum_{i=1}^n F_{ij}\vec e_i$ for all j and $[F]_{|\vec E,\vec e} = [F_{ij}]$.

Definition L.23 The "volume dilatation rate" at p_{t_0} relative to the Euclidean bases $(\vec E_i)$ and $(\vec e_i)$ is

$$J_{|\vec E,\vec e}(\Phi_t^{t_0})(p_{t_0}) := \widetilde{\det}_{|\vec E,\vec e}(F) \quad (= \det_{|\vec e}(F.\vec E_1,...,F.\vec E_n) = \det([F_{ij}])), \tag{L.32}$$

often written $J_{|\vec E,\vec e} := \det([F]_{|\vec E,\vec e})$ (or simply $J = \det(F)$ when everything is implicit).

So, at t_0 at p_{t_0}, $(\vec E_1,...,\vec E_n)$ is a unit parallelepiped which volume is 1 (relative to the unit of measurement chosen in $\vec{\mathbb{R}}^n_{t_0}$), and, at t at $p_t = \Phi_t^{t_0}(p_{t_0})$, $J_{|\vec E,\vec e}(\Phi_t^{t_0})(p_{t_0}) = \det_{|\vec e}(F.\vec E_1,...,F.\vec E_n)$ is the volume of the parallelepiped $(p_t, F.\vec E_1,...,F.\vec E_n)$ at $p_t = \Phi_t^{t_0}(p_{t_0})$ (relative to the unit of measurement chosen in $\vec{\mathbb{R}}^n_t$).

Interpretation: With $t_2 > t_1 \ge t_0$, and $[\vec e_i)$ is the basis at t_1 and t_2:

- Dilatation if $J_{|\vec E,\vec e}(\Phi_{t_2}^{t_0})(p_{t_0}) > J_{|\vec E,\vec e}(\Phi_{t_1}^{t_0})(p_{t_0})$ (volume increase),
- contraction if $J_{|\vec E,\vec e}(\Phi_{t_2}^{t_0})(p_{t_0}) < J_{|\vec E,\vec e}(\Phi_{t_1}^{t_0})(p_{t_0})$ (volume decrease), and
- incompressibility if $J_{|\vec E,\vec e}(\Phi_{t_2}^{t_0})(p_{t_0}) = J_{|\vec E,\vec e}(\Phi_{t_1}^{t_0})(p_{t_0})$ for all t (volume conservation).

In particular, if $(\vec e_i) = (\vec E_i)$ then $J_{|\vec e,\vec e}(\Phi_{t_0}^{t_0})(p_{t_0}) = 1$, and if $t > t_0$, then

- Dilatation if $J_{|\vec e,\vec e}(\Phi_t^{t_0})(p_{t_0}) > 1$ (volume increase),
- contraction if $J_{|\vec e,\vec e}(\Phi_t^{t_0})(p_{t_0}) < 1$ (volume decrease), and
- incompressibility if $J_{|\vec e,\vec e}(\Phi_t^{t_0})(p_{t_0}) = 1$ for all t (volume conservation).

Exercice L.24 Let $(\vec E_i)$ be a Euclidean basis in $\vec{\mathbb{R}}^n_{t_0}$, and let $(\vec a_i)$ and $(\vec b_i)$ be two Euclidean bases in $\vec{\mathbb{R}}^n_t$ for the same Euclidean dot product $(\cdot,\cdot)_g$. Prove:

$$J_{|\vec E,\vec a}(\Phi_t^{t_0}(P)) = \pm J_{|\vec E,\vec b}(\Phi_t^{t_0}(P)). \tag{L.33}$$

Answer. P being the transition matrix from $(\vec a_i)$ to $(\vec b_i)$, $\det(P) = \pm 1$ here. And (4.26) gives $[F]_{|\vec E,\vec a} = P.[F]_{|\vec E,\vec b}$, thus $\det([F]_{|\vec E,\vec a}) = \pm\det([F]_{|\vec E,\vec b})$, thus $\det_{|\vec a}(F.\vec E_1,...,F.\vec E_n) = \pm\det_{|\vec b}(F.\vec E_1,...,F.\vec E_n)$. ∎

L.7.3 Determinant of the transposed

Let $(A,(\cdot,\cdot)_g)$ and $(B,(\cdot,\cdot)_h)$ be finite dimensional Hilbert spaces. Let $L\in\mathcal{L}(A;B)$ (a linear map). Recall: The transposed $L^T_{gh}\in\mathcal{L}(B;A)$ is defined by, for all $\vec u\in A$ and all $\vec w\in B$, cf. (A.66)

$$(L^T_{gh}.\vec w,\vec u)_g := (\vec w, L.\vec u)_h. \tag{L.34}$$

Let $(\vec a_i)$ be a basis in A and $(\vec b_i)$ be a basis in B. Then

$$\widetilde{\det}([L^T_{gh}]_{|\vec b,\vec a}) = \det([L]_{|\vec a,\vec b})\frac{\det([(\cdot,\cdot)_g]_{|\vec a})}{\det([(\cdot,\cdot)_h]_{|\vec b})}. \tag{L.35}$$

Indeed, (L.34) gives $[(\cdot,\cdot)_g]_{|\vec a}.[L^T_{gh}]_{|\vec b,\vec a} = ([L]_{|\vec a,\vec b})^T.[(\cdot,\cdot)_h]_{|\vec b}$.

L.8 Dilatation rate

A unique Euclidean basis $(\vec e_i)$ at all time is chosen, and $(\cdot,\cdot)_g$ is the associated inner dot product.

L.8.1 $\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0}) = J^{t_0}(t,p_{t_0})\,\mathrm{div}\vec v(t,p_t)$

The Eulerian velocity is $\vec v(t,p_t) = \frac{\partial\widetilde\Phi}{\partial t}(t,P_{Obj})$ at $p_t=\widetilde\Phi(t,P_{Obj})$. The Lagrangian velocity is $\vec V(t,p_{t_0}) = \frac{\partial\Phi^{t_0}}{\partial t}(t,p_{t_0})$ at $p_t=\widetilde\Phi(t,P_{Obj})$ (so with $p_t=\Phi^{t_0}(t,p_{t_0})$). The deformation gradient along the motion of a particle is $F^{t_0}(t,p_{t_0}) := d\Phi^{t_0}_t(t,p_{t_0}) = F^{t_0}_{p_{t_0}}(t)$. The Jacobian "along the motion of a particle" is

$$J^{t_0}_{p_{t_0}}(t) := J^{t_0}(t,p_{t_0}) := J^{t_0}_t(p_{t_0}) = \det_{|\vec e}(F^{t_0}_t(p_{t_0})), \tag{L.36}$$

Lemma L.25 *$\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0})$ satisfies, with $p_t=\Phi^{t_0}_t(p_{t_0})$,*

$$\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0}) = J^{t_0}(t,p_{t_0})\,\mathrm{div}\vec v(t,p_t) \tag{L.37}$$

(value to be considered at t at p_t). In particular, $\widetilde\Phi$ is incompressible iff $\mathrm{div}\vec v(t,p_t)=0$.

Proof. Let $\mathcal{O}$ be a origin in $\mathbb{R}^n$. Let $\overrightarrow{\mathcal{O}\Phi^{t_0}} = \sum_{i=1}^n\Phi^i\vec e_i$, $\vec V^{t_0} = \sum_{i=1}^n V^i\vec e_i$, $\vec v=\sum_{i=1}^n v^i\vec e_i$, $F^{t_0}.\vec E_j = d\Phi^{t_0}.\vec E_j = \sum_{i=1}^n\frac{\partial\Phi^i}{\partial X^j}\vec e_i$. Let $[F^{t_0}]_{|\vec E,\vec e} =^{\text{noted}} F$, $J^{t_0} =^{\text{noted}} J$ and $[d\Phi^i]_{|\vec E} = \begin{pmatrix}\frac{\partial\Phi^i}{\partial X^1} & \dots & \frac{\partial\Phi^i}{\partial X^n}\end{pmatrix} =^{\text{noted}} d\Phi^i$ (row matrix). Thus $J=\det F=\det\begin{pmatrix}d\Phi^1\\ \vdots\\ d\Phi^n\end{pmatrix}$, thus (a determinant is multilinear)

$$\frac{\partial J}{\partial t} = \det\begin{pmatrix}\frac{\partial(d\Phi^1)}{\partial t}\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} + \dots + \det\begin{pmatrix}d\Phi^1\\ \vdots\\ d\Phi^{n-1}\\ \frac{\partial(d\Phi^n)}{\partial t})\end{pmatrix}.$$

With Φ^{t_0} C^2, thus $\frac{\partial(d\Phi^i)}{\partial t}(t,p_{t_0}) \overset{\text{Swhartz}}{=} d(\frac{\partial\Phi^i}{\partial t})(t,p_{t_0}) = dV^i(t,p_{t_0}) = dv^i(t,p_t).F(t,p_{t_0})$, cf. (3.27). Thus

$$\det\begin{pmatrix}\frac{\partial(d\Phi^1)}{\partial t}\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} = \det\begin{pmatrix}\sum_{i=1}^n\frac{\partial v^1}{\partial x^i}d\Phi^i\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} \underset{\text{alternating}}{\overset{\text{det is}}{=}} \det\begin{pmatrix}\frac{\partial v^1}{\partial x^1}d\Phi^1\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} = \frac{\partial v^1}{\partial x^1}\det\begin{pmatrix}d\Phi^1\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} = \frac{\partial v^1}{\partial x^1}J$$

Idem for the other terms, thus

$$\frac{\partial J}{\partial t}(t,p_{t_0}) = \frac{\partial v^1}{\partial x^1}(t,p_t)\;J(t,p_{t_0}) + \dots + \frac{\partial v^n}{\partial x^n}(t,p_t)\;J(t,p_{t_0}) = \mathrm{div}\vec v(t,p_t)\;J(t,p_{t_0}),$$

i.e. (L.37). ∎

Definition L.26 $\mathrm{div}\vec v(t,p_t)$ is the dilatation rate.

L.8.2 Leibniz formula

Proposition L.27 (Leibniz formula) *Under regularity assumptions (e.g. hypotheses of the Lebesgue theorem to be able to differentiate under $\int$) we have*

$$\begin{aligned}\frac{d}{dt}\Big(\int_{p_t\in\Omega_t} f(t,p_t)\,d\Omega_t\Big) &= \int_{p_t\in\Omega_t} \big(\frac{Df}{Dt} + f\,\mathrm{div}\vec v\big)(t,p_t)\,d\Omega_t\\ &= \int_{p_t\in\Omega_t} \big(\frac{\partial f}{\partial t} + df.\vec v + f\,\mathrm{div}(\vec v)\big)(t,p_t)\,d\Omega_t\\ &= \int_{p_t\in\Omega_t} \big(\frac{\partial f}{\partial t} + \mathrm{div}(f\vec v)\big)(t,p_t)\,d\Omega_t.\end{aligned} \tag{L.38}$$

Proof. Let

$$Z(t) := \int_{p\in\Omega_t} f(t,p)\,d\Omega_t = \int_{P\in\Omega_{t_0}} f(t,\Phi^{t_0}(t,P))\,J^{t_0}(t,P)\,d\Omega_{t_0}.$$

(The Jacobian is positive for a regular motion.) Then (derivation under $\int$)

$$\begin{aligned}Z'(t) &= \int_{P\in\Omega_{t_0}} \frac{Df}{Dt}(t,p_t)\,J^{t_0}(t,P) + f(t,p_t)\frac{\partial J^{t_0}}{\partial t}(t,P)\,d\Omega_{t_0}\\ &= \int_{P\in\Omega_{t_0}} \big(\frac{Df}{Dt}(t,p_t)\ + f(t,p_t)\,\mathrm{div}\vec v(t,p_t)\big)J^{t_0}(t,P)\,d\Omega_{t_0},\end{aligned}$$

thanks to (L.37). And $\mathrm{div}(f\vec v) = df.\vec v + f\,\mathrm{div}\vec v$ gives (L.38). ∎

Corollary L.28 *With $(\vec u,\vec w)_g =^{noted} \vec u \bullet \vec w$ (in the given Euclidean framework),*

$$\frac{d}{dt}\int_{\Omega_t} f(t,p_t)\,d\Omega_t = \int_{\Omega_t} \frac{\partial f}{\partial t}(t,p_t)\,d\Omega_t + \int_{\partial\Omega_t} (f\vec v \bullet \vec n)(t,p_t)\,d\Gamma_t, \tag{L.39}$$

sum of the temporal variation within Ω_t and the flux through the surface $\partial\Omega_t$.

Proof. Apply $(\mathrm{L.38})_3$. ∎

L.9 $\partial J/\partial F = J\,F^{-T}$

L.9.1 Meaning of $\frac{\partial \det}{\partial M_{ij}}$?

Let $\mathcal{M}_{nn} = \{M = [M_{ij}] \in \mathbb{R}^{n^2}\}$ be the set of $n*n$ matrices, and consider the function

$$Z := \det : \left\{\begin{array}{rl} \mathcal{M}_{nn} & \to \mathbb{R}\\ M = [M_{ij}] & \to Z(M) := \det(M) = \det([M_{ij}]).\end{array}\right. \tag{L.40}$$

Question: What does $\frac{\partial Z}{\partial M_{ij}}(M)$ mean?

Answer: It is the "standard meaning" of a directional derivative $\frac{\partial f}{\partial x_i}(\vec x) = df(\vec x).\vec e_i$... where here $f = Z$, thus $\vec x =^{\text{noted}} M$ is a matrix (a vector in $\mathcal{M}_{nn}$), and $(\vec e_i)$ is the canonical basis (m_{ij}) in $\mathcal{M}_{nn}$ (all the elements of the matrix m_{ij} vanish but the element at intersection of line i and column j which equals 1). So:

$$\frac{\partial Z}{\partial M_{ij}}(M) := dZ(M).m_{ij} = \lim_{h\to 0}\frac{Z(M+hm_{ij}) - Z(M)}{h} \quad (\in \mathbb{R}). \tag{L.41}$$

L.9.2 Calculation of $\frac{\partial \det}{\partial M_{ij}}$

Proposition L.29

$$\forall i,j,\quad \frac{\partial Z}{\partial M_{ij}}(M) = Z(M)\,(M^{-T})_{ij}, \quad \textit{written} \quad \frac{\partial Z}{\partial M} = Z\,M^{-T}. \tag{L.42}$$

Proof. $\frac{\partial Z}{\partial M_{ij}}(M) := \lim_{h\to0}\frac{\det(M+hm_{ij})-\det(M)}{h}$; The development of the determinant $\det(M+hm_{ij})$ relative to the column j gives

$$\det(M+h[m_{ij}]) = \det(M) + h\,c_{ij} \tag{L.43}$$

where c_{ij} is the (i,j)-th cofactor of M; Thus $\frac{\partial Z}{\partial M_{ij}}(M) = \lim_{h\to0}\frac{Z(M+hm_{ij})-Z(M)}{h} = c_{ij}$; And since $M^{-1} = \frac{1}{\det(M)}[c_{ij}]^T$, i.e. $[c_{ij}] = \det(M)M^{-T}$, we get $\frac{\partial Z}{\partial M_{ij}}(M) = \det(M)(M^{-T})_{ij}$, i.e. (L.42). ∎

L.9.3 $\partial J/\partial F = J\,F^{-T}$ usually written $[\frac{\partial J}{\partial F_{ij}}] = J\,F^{-T}$

Setting of § L.8: With $F := d\Phi(p_{t_0})$ we have $F.\vec{E}_j = \sum_{i=1}^n F_{ij}\vec{e}_i$ where $F_{ij} = \frac{\partial\Phi_i}{\partial X_j}(p_{t_0})$, and

$$\widehat{J}_{|\vec{E},\vec{e}} \overset{\text{noted}}{=} \widehat{J} : \begin{cases} \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t) & \to \mathbb{R} \\ \qquad F & \to \widehat{J}(F) := \det([F_{ij}]) \quad (= \det([\dfrac{\partial\Phi_i}{\partial X_j}(p_{t_0})] = \underset{|\vec{E},\vec{e}}{\widetilde{\det}}(d\Phi(p_{t_0}))), \end{cases} \tag{L.44}$$

so, $\widehat{J}(F) = J(\Phi)$ is the Jacobian of Φ at p_{t_0} relative to $(\vec{E}_i)$ and $(\vec{e}_i)$. Thus (L.42) gives:

Corollary L.30

$$\forall i,j,\quad \frac{\partial\widehat{J}}{\partial F_{ij}}(F) = \widehat{J}(F)\,([F]^{-T})_{ij}, \quad \textit{written} \quad \frac{\partial J}{\partial F} = J\,F^{-T}. \tag{L.45}$$

L.9.4 Interpretation of $\frac{\partial J}{\partial F_{ij}}$?

The first derivations into play are along the directions $\vec{E}_j$ at t_0 because $F_{ij} = \frac{\partial\Phi_i}{\partial X_j} := d\Phi_i.\vec{E}_j$, when $\Phi = \sum_i \Phi_i\vec{e}_i$, so $F.\vec{E}_j = F_{ij}\vec{e}_i$.

Question: What does $\frac{\partial J}{\partial F_{ij}}$ mean ? That is, derivative of J in which direction ?

Answer: 1- "Identify" $F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ with the tensor $\widetilde{F} \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t,\vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ given by $\widetilde{F}(\ell,\vec{U}) = \ell.(F.\vec{U})$; So, if $F.\vec{E}_j = \sum_{i=1}^n F_{ij}\vec{e}_i$ then $\widetilde{F} = \sum_{i,j=1}^n F_{ij}\vec{e}_i\otimes\pi_{Ej}$, where $\vec{e}_i$ is a basis in $\vec{\mathbb{R}}^n_t$ and (π_{Ei}) is the covariant dual basis of $(\vec{E}_i)$ basis in $\vec{\mathbb{R}}^n_{t_0}$.

2- Define the function Jac : $\begin{Bmatrix} \mathcal{L}(\vec{\mathbb{R}}^{n*}_t,\vec{\mathbb{R}}^n_{t_0};\mathbb{R}) & \to \mathbb{R} \\ \qquad \widetilde{F} & \to \text{Jac}(\widetilde{F}) := J(F) = \underset{\vec{E},\vec{e}}{\det}(F) \end{Bmatrix}$.

3- Then it is meaningful to differentiate Jac along the direction $\vec{e}_i\otimes\pi_{Ej} \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t,\vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ to get

$$\frac{\partial\text{Jac}}{\partial F_{ij}}(\widetilde{F}) = \lim_{h\to0}\frac{\text{Jac}(\widetilde{F}+h\vec{e}_i\otimes\pi_{Ej}) - \text{Jac}(\widetilde{F})}{h} \overset{\text{noted}}{=} \frac{\partial J}{\partial F_{ij}}(F). \tag{L.46}$$

(Duality notation: $\frac{\partial\text{Jac}}{\partial F^i_J}(\widetilde{F}) = \lim_{h\to0}\frac{\text{Jac}(\widetilde{F}+h\vec{e}_i\otimes E^J)-\text{Jac}(\widetilde{F})}{h}$.)

Question: This is a derivation in both directions $\vec{e}_i$ in $\vec{\mathbb{R}}^n_t$ (present at p_t) and π_{Ej} in $\vec{\mathbb{R}}^n_{t_0}$ (past at p_{t_0} and dual basis vector); So, what does this derivative mean?

Answer: ?

M Transport of volumes and areas

Here $\mathbb{R}^n = \mathbb{R}^3$ the usual affine space, $t_0, t \in \mathbb{R}$, $\Phi := \Phi^{t_0}_t : \mathbb{R}\times\Omega_{t_0} \to \Omega_t$ is a regular motion, and $F_P = d\Phi(P)$. We need a $(\cdot,\cdot)_g$ be a Euclidean dot product in $\vec{\mathbb{R}^n}$, the same at all time. And $(\vec{E}_i)$ and $(\vec{e}_i)$ are $(\cdot,\cdot)_g$-Euclidean bases in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$.

M.1 Transport of volumes

M.1.1 Transformed parallelepiped

The Jacobian of Φ at P relative to the chosen Euclidean bases is

$$J_P = J(P) := \det_{|\vec{E},\vec{e}}(F_t^{t_0}(P)) \quad (= \det_{|\vec{e}}(F_t^{t_0}(P).\vec{E}_1, ..., F_t^{t_0}(P).\vec{E}_n))), \tag{M.1}$$

cf. (L.32); The motion being regular, $J_P > 0$. And if $(\vec{U}_{1P}, ..., \vec{U}_{nP})$ is a parallelepiped at t_0 at P, if $\vec{u}_{ip} = F_P.\vec{U}_{iP}$, then $(\vec{u}_{1p}, ..., \vec{u}_{np})$ is a parallelepiped at t at $p = \Phi(P)$ which algebraic volume is

$$\det_{|\vec{e}}(\vec{u}_{1p}, ..., \vec{u}_{np}) = J_P \det_{|\vec{E}}(\vec{U}_{1P}, ..., \vec{U}_{nP}). \tag{M.2}$$

M.1.2 Transformed volumes

Riemann integrals and (M.2) give the change of variable formula: For any regular function $f : \Omega_t \to \mathbb{R}$,

$$\int_{p\in\Omega_t} f(p)\, d\Omega_t = \int_{P\in\Omega_{t_0}} f(\Phi(P))\, |J(P)|\, d\Omega_{t_0}. \tag{M.3}$$

Here $J_P > 0$ (regular motion), hence

$$\int_{p\in\Omega_t} f(p)\, d\Omega_t = \int_{P\in\Omega_{t_0}} f(\Phi(P))\, J(P)\, d\Omega_{t_0}. \tag{M.4}$$

In particular, $|\Omega_t| = \int_{p\in\Omega_t} d\Omega_t = \int_{P\in\Omega_{t_0}} J(P)\, d\Omega_{t_0}$.

M.2 Transformed surface

M.2.1 Transformed parallelogram and its area

Consider vectors $\vec{U}_{1P}, \vec{U}_{2P} \in \vec{\mathbb{R}}^n_{t_0}$ at t_0 at P, and, Φ being a diffeomorphism, the two independent vectors $\vec{u}_{1p} = F_P.\vec{U}_{1P}$ and $\vec{u}_{2p} = F_P.\vec{U}_{2P}$ at t at $p = \Phi(P)$. The areas of the associated quadrilaterals are $||\vec{U}_{1P} \times \vec{U}_{2P}||_g$ and $||\vec{u}_{1p} \times \vec{u}_{2p})||_g$, and the unit normal vectors to the quadrilaterals are (up to the sign)

$$\vec{N}_P = \frac{\vec{U}_{1P} \times \vec{U}_{2P}}{||\vec{U}_{1P} \times \vec{U}_{2P}||_g}, \quad \text{and} \quad \vec{n}_p = \frac{\vec{u}_{1p} \times \vec{u}_{2p}}{||\vec{u}_{1p} \times \vec{u}_{2p}||_g}. \tag{M.5}$$

Proposition M.1

$$\vec{u}_{1p} \times \vec{u}_{2p} = J_P\, F_P^{-T}.\big(\vec{U}_{1P} \times \vec{U}_{2P}\big), \quad \textit{in short} \quad \vec{u}_1 \times \vec{u}_2 = J\, F^{-T}.\big(\vec{U}_1 \times \vec{U}_2\big), \tag{M.6}$$

and

$$\vec{n}_p = \frac{F_P^{-T}.\vec{N}_P}{||F_P^{-T}.\vec{N}_P||_g} \quad (\neq F_P.\vec{N}_P \textit{ in general}), \quad \textit{in short} \quad \vec{n} = \frac{F^{-T}.\vec{N}}{||F^{-T}.\vec{N}||_g}. \tag{M.7}$$

Proof. Let $\vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$, and $\vec{w}_p = F_P.\vec{W}_P$. The volume of the parallelepiped $(\vec{u}_{1p}, \vec{u}_{2p}, \vec{w}_p)$ is

$$\begin{aligned}(\vec{u}_{1p} \times \vec{u}_{2p}, \vec{w}_p)_g &= \det_{|\vec{e}}(\vec{u}_{1p}, \vec{u}_{2p}, \vec{w}_p) = J_P \det_{|\vec{E}}(\vec{U}_{1P}, \vec{U}_{2P}, \vec{W}_P) = J_P\, (\vec{U}_{1P} \times \vec{U}_{2P}, \vec{W}_P)_g \\ &= J_P\, (\vec{U}_{1P} \times \vec{U}_{2P}, F_P^{-1}.\vec{w}_p)_g = J_P\, (F_P^{-T}.(\vec{U}_{1P} \times \vec{U}_{2P}), \vec{w}_p)_g,\end{aligned}$$

for all $\vec{w}_p$, thus (M.6), thus $\frac{\vec{u}_{1p}\times\vec{u}_{2p}}{||\vec{u}_{1p}\times\vec{u}_{2p}||_g} = \frac{J_P\, F_P^{-T}.(\vec{U}_{1P}\times\vec{U}_{2P})}{J_P||F_P^{-T}.(\vec{U}_{1P}\times\vec{U}_{2P})||_g}$ (here $J_P > 0$), thus (M.7). ∎

M.2.2 Deformation of a surface

A parametrized surface Ψ_{t_0} in Ω_{t_0} and the associated geometric surface S_{t_0} are defined by

$$\Psi_{t_0} : \left\{ \begin{array}{ll} [a,b]\times[c,d] & \to \Omega_{t_0} \\ (u,v) & \to P = \Psi_{t_0}(u,v) \end{array} \right\} \quad \text{and} \quad S_{t_0} = \mathrm{Im}(\Psi_{t_0}) \subset \Omega_{t_0}. \tag{M.8}$$

Consider the basis $(\vec{E}_1 = (1,0), \vec{E}_2 = (0,1))$ in the space $\mathbb{R}\times\mathbb{R} \supset [a,b]\times[c,d] = \{(u,v)\}$ of parameters, and suppose that Ψ_{t_0} is regular. Thus the tangent vectors at $P = \Psi_{t_0}(u,v) \in S_{t_0}$ given by

$$\begin{cases} \vec{T}_{1P} := d\Psi_{t_0}(u,v).\vec{E}_1 \overset{\text{noted}}{=} \dfrac{\partial \Psi_{t_0}}{\partial u}(u,v), \\ \vec{T}_{2P} := d\Psi_{t_0}(u,v).\vec{E}_2 \overset{\text{noted}}{=} \dfrac{\partial \Psi_{t_0}}{\partial v}(u,v), \end{cases} \tag{M.9}$$

are independent: $\vec{T}_{1P} \times \vec{T}_{2P} \neq \vec{0}$.

Call $\Psi_t := \Phi_t^{t_0} \circ \Psi_{t_0} = \Phi \circ \Psi_{t_0}$ and S_t the transformed parametric and geometric surfaces:

$$\Psi_t := \Phi \circ \Psi_{t_0} : \left\{ \begin{array}{ll} [a,b]\times[c,d] & \to \Omega_{t_0} \\ (u,v) & \to p = \Psi_t(u,v) = \Phi(\Psi_{t_0}(u,v)) \ \ (= \Phi(P)) \end{array} \right\} \quad \text{and} \quad S_t = \Phi(S_{t_0}). \tag{M.10}$$

The tangent vectors at S_t at $p = \Phi_t^{t_0}(P)$ at t:

$$\begin{cases} \vec{t}_{1p} := d\Psi_t(u,v).\vec{E}_1 = \dfrac{\partial \Psi_t}{\partial u}(u,v) = d\Phi_t^{t_0}(P).\dfrac{\partial \Psi_{t_0}}{\partial u}(u,v), \quad \text{i.e.} \quad \vec{t}_{1p} = F_P.\vec{T}_{1P}, \\ \vec{t}_{2p} := d\Psi_t(u,v).\vec{E}_2 = \dfrac{\partial \Psi_t}{\partial v}(u,v) = d\Phi_t^{t_0}(P).\dfrac{\partial \Psi_{t_0}}{\partial v}(u,v), \quad \text{i.e.} \quad \vec{t}_{2p} = F_P.\vec{T}_{2P}, \end{cases} \tag{M.11}$$

are independent since $\Phi_t^{t_0}$ is a diffeomorphism and Ψ_{t_0} is regular.

M.2.3 Euclidean dot product and unit normal vectors

Relative to $(\cdot,\cdot)_g$, the scalar area elements $d\Sigma_P$ at P at S_{t_0} relative to Ψ_{t_0}, and $d\sigma_p$ at p at S_t relative to Ψ_t, are

$$\begin{cases} d\Sigma_P := ||\dfrac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \dfrac{\partial \Psi_{t_0}}{\partial v}(u,v)||_g \, du\, dv \quad (= ||\vec{T}_{1P} \times \vec{T}_{2P}||_g \, du\, dv), \\ d\sigma_p := ||\dfrac{\partial \Psi_t}{\partial u}(u,v) \times \dfrac{\partial \Psi_t}{\partial v}(u,v)||_g \, du\, dv \quad (= ||\vec{t}_{1p} \times \vec{t}_{2p}||_g \, du\, dv). \end{cases} \tag{M.12}$$

And the areas of S_{t_0} and S_t are

$$\begin{cases} |S_{t_0}| = \displaystyle\int_{P\in S_{t_0}} d\Sigma_P = \int_{u=a}^{b} \int_{v=c}^{d} ||\frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v)||_g \, du\, dv, \\ |S_t| = \displaystyle\int_{p\in S_t} d\sigma_p = \int_{u=a}^{b} \int_{v=c}^{d} ||\frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v)||_g \, du\, dv. \end{cases} \tag{M.13}$$

And the unit normal vectors $\vec{N}_P$ at S_{t_0} at P at t_0 and $\vec{n}_p$ at S_t at p at t are (up to the sign)

$$\begin{cases} \vec{N}_P = \dfrac{\frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v)}{||\frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v)||_g} \quad (= \dfrac{\vec{T}_{1P} \times \vec{T}_{2P}}{||\vec{T}_{1P} \times \vec{T}_{2P}||_g}) \\ \vec{n}_p = \dfrac{\frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v)}{||\frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v)||_g} \quad (= \dfrac{\vec{t}_{1p} \times \vec{t}_{2p}}{||\vec{t}_{1p} \times \vec{t}_{2p}||_g}). \end{cases} \tag{M.14}$$

And the vectorial area elements $d\vec{\Sigma}_P$ at P at S_{t_0} and $d\vec{\sigma}_p$ at p at S_t are

$$\begin{cases} d\vec{\Sigma}_P := \vec{N}_P \, d\Sigma_P = \dfrac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \dfrac{\partial \Psi_{t_0}}{\partial v}(u,v)\, du\, dv \quad (= \vec{T}_{1P} \times \vec{T}_{2P} \, du\, dv) \\ d\vec{\sigma}_p := \vec{n}_p \, d\sigma_p = \dfrac{\partial \Psi_t}{\partial u}(u,v) \times \dfrac{\partial \Psi_t}{\partial v}(u,v)\, du\, dv \quad (= \vec{t}_{1p} \times \vec{t}_{2p} \, du\, dv). \end{cases} \tag{M.15}$$

(And the flux through a surface is $\int_\Gamma \vec{f} \bullet \vec{n}\, d\sigma =^{\text{noted}} \int_\Gamma \vec{f} \bullet d\vec{\sigma}$.)

M.2.4 Relations between area elements

$\vec{t}_{1p} \times \vec{t}_{2p} = J_P\, F_P^{-T}.(\vec{T}_{1P} \times \vec{T}_{2P})$, cf. (M.6), gives

$$\frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v) = J_P\, F_P^{-T}.(\frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v)). \tag{M.16}$$

And

$$\vec{n}\, d\sigma_p = \boxed{d\vec{\sigma}_p = J_P\, F_P^{-T}.d\vec{\Sigma}_P} = J_P\, F_P^{-T}.\vec{N}_P\, d\Sigma_P, \quad \text{and} \quad d\sigma_p = J_P\, ||F_P^{-T}.\vec{N}_P||_g\, d\Sigma_P. \tag{M.17}$$

(Check with (M.7).)

M.2.5 Piola identity...

Reminder: The divergence (in continuum mechanics) of a $3*3$ matrix function $M = [M^i_j]$ is: $\text{div}M := \begin{pmatrix} \sum_{j=1}^n \frac{\partial M^1_j}{\partial X^j} \\ \sum_{j=1}^n \frac{\partial M^2_j}{\partial X^j} \\ \sum_{j=1}^n \frac{\partial M^3_j}{\partial X^j} \end{pmatrix}$ $(= \begin{pmatrix} \frac{\partial M^1_1}{\partial X^1} + \frac{\partial M^1_2}{\partial X^2} + \frac{\partial M^1_3}{\partial X^3} \\ \frac{\partial M^2_1}{\partial X^1} + \frac{\partial M^2_2}{\partial X^2} + \frac{\partial M^2_3}{\partial X^3} \\ \frac{\partial M^3_1}{\partial X^1} + \frac{\partial M^3_2}{\partial X^2} + \frac{\partial M^3_3}{\partial X^3} \end{pmatrix})$, cf. (T.66). Its matrix of cofactors $\text{Cof}(M)$ is given by $\text{Cof}(M)^i_j = M^{i+1}_{j+1}M^{i+2}_{j+2} - M^{i+1}_{j+2}M^{i+2}_{j+1}$, and $(\det M)M^{-1} = \text{Cof}(M)^T$.

Application: $\det([F(P)]_{|\vec{E},\vec{e}})\,([F(P)]_{|\vec{E},\vec{e}})^{-T} = \text{Cof}([F(P)]_{|\vec{E},\vec{e}})$; Written in short $\det(F(P))\,F(P)^{-T} = \text{Cof}(F(P))$ (matrix meaning); So, in Ω_{t_0},

$$JF^{-T} = \text{Cof}(F) \quad \text{(matrix meaning)}. \tag{M.18}$$

Proposition M.2 (Piola identity)

$$\text{div}(JF^{-T}) = 0, \quad \textit{i.e.} \quad \forall i,\ \forall P,\ \sum_{j=1}^n \frac{\partial \text{Cof}(F)_{ij}}{\partial X_j}(P) = 0 \ \textit{ or } \ \sum_{J=1}^n \frac{\partial \text{Cof}(F)^i_J}{\partial X^J}(P) = 0. \tag{M.19}$$

Also sometimes ambiguously written $\sum_{j=1}^n \frac{\partial}{\partial X_j}(J\frac{\partial X_i}{\partial x_j}) = 0$ *or* $\sum_{J=1}^n \frac{\partial}{\partial X^J}(\text{Jac}(\frac{\partial X^i}{\partial x^J})) = 0...$

Proof. $\text{Cof}(F)^i_j = F^{i+1}_{j+1}F^{i+2}_{j+2} - F^{i+1}_{j+2}F^{i+2}_{j+1} = \frac{\partial \varphi^{i+1}}{\partial X^{j+1}}\frac{\partial \varphi^{i+2}}{\partial X^{j+2}} - \frac{\partial \varphi^{i+1}}{\partial X^{j+2}}\frac{\partial \varphi^{i+2}}{\partial X^{j+1}}$. Thus

$$\frac{\partial \text{Cof}(F)^i_j}{\partial X^j} = \frac{\partial^2 \varphi^{i+1}}{\partial X^j \partial X^{j+1}}\frac{\partial \varphi^{i+2}}{\partial X^{j+2}} + \frac{\partial \varphi^{i+1}}{\partial X^{j+1}}\frac{\partial^2 \varphi^{i+2}}{\partial X^j \partial X^{j+2}} - \frac{\partial^2 \varphi^{i+1}}{\partial X^j \partial X^{j+2}}\frac{\partial \varphi^{i+2}}{\partial X^{j+1}} - \frac{\partial \varphi^{i+1}}{\partial X^{j+2}}\frac{\partial^2 \varphi^{i+2}}{\partial X^j \partial X^{j+1}}.$$

And summation: The terms cancel out two by two. ∎

M.2.6 ... and Piola transformation

Goal: for a $\vec{u} : \Omega_t \to \vec{\mathbb{R}}^n_t$, find $\vec{U}_{\text{Piola}} : \Omega_{t_0} \to \vec{\mathbb{R}}^n_{t_0}$ s.t., for all $\omega_t = \Phi^{t_0}_t(\omega_{t_0})$ (with ω_{t_0} open subset in Ω_{t_0}),

$$\int_{\partial\omega_{t_0}} \vec{U}_{\text{Piola}} \bullet \vec{N}\, d\Sigma = \int_{\partial\omega_t} \vec{u} \bullet \vec{n}\, d\sigma, \tag{M.20}$$

i.e.

$$\int_{\omega_{t_0}} \text{div}\vec{U}_{\text{Piola}}\, d\Omega_{t_0} = \int_{\omega_t} \text{div}\vec{u}\, d\Omega_t, \tag{M.21}$$

i.e., with (M.4), for all $P \in \Omega_{t_0}$,

$$\text{div}\vec{U}_{\text{Piola}}(P) = J(P)\, \text{div}\vec{u}(\Phi(P)). \tag{M.22}$$

Proposition M.3 *With* $p = \Phi(P)$,

$$\vec{U}_{\text{Piola}}(P) = J(P)F(P)^{-1}.\vec{u}(p), \tag{M.23}$$

i.e., $\vec{U}_{\text{Piola}} := J\,\Phi^*(\vec{u})$, *i.e.* $= J$ *times the pull-back of* $\vec{u}$ *by* Φ. *Hence*

$$\int_{p\in\partial\omega_t} \vec{u}(p) \bullet \vec{n}(p)\, d\sigma = \int_{P\in\partial\omega_{t_0}} \big(J(P)F(P)^{-1}.\vec{u}(\Phi(P))\big) \bullet \vec{N}(P)\, d\Sigma. \tag{M.24}$$

Proof. $d(\vec{u}\circ\Phi)(P)=d\vec{u}(p).F(P)$, thus

$$\begin{aligned}\mathrm{div}((JF^{-1}).(\vec{u}\circ\Phi_t^{t_0}))(P) &\overset{(T.62)}{=} \widetilde{\mathrm{div}}_P(JF^{-1})(P).\vec{u}(\Phi_t^{t_0}(P)) + J(P)F(P)^{-1} \odot (d\vec{u}(p).F(P)) \\ &= \mathrm{div}(JF^{-T})(P)\bullet\vec{u}(p) + J(P)(F(P).F(P)^{-1}) \odot d\vec{u}(p) \\ &\overset{(M.19)}{=} 0 + J(P)I_t \odot d\vec{u}(p) = J(P)\mathrm{div}\vec{u}(p),\end{aligned}$$

thus $\vec{U}_{\mathrm{Piola}}(P) := J(P)F(P)^{-1}.\vec{u}(p)$ satisfies (M.22). (Check with components if you prefer.) ∎

Definition M.4 The Piola transform is the map (between vector fields in Ω_t and Ω_{t_0})

$$\begin{cases} T\Omega_t & \to T\Omega_{t_0} \\ \vec{u} & \to \vec{U}_{\mathrm{Piola}}, \quad \vec{U}_{\mathrm{Piola}}(P) := J(P)F(P)^{-1}.\vec{u}(p) \quad \text{when} \quad p=\Phi_t^{t_0}(P). \end{cases} \tag{M.25}$$

N Conservation of mass

Let $\rho(t,p)=\rho_t(p)$ be the (Eulerian) mass density at t at $p\in\Omega_t$, supposed to be >0; The mass $m(\omega_t)$ of a subset $\omega_t\subset\Omega_t$ is

$$m(\omega_t)=\int_{p\in\omega_t}\rho_t(p)\,d\omega_t. \tag{N.1}$$

Conservation of mass principle (no loss nor production of particles): For all $\omega_{t_0}\subset\Omega_{t_0}$ and all t,

$$m(\omega_t)=m(\omega_{t_0}),\quad \text{i.e.}\quad \int_{p\in\omega_t}\rho_t(p)\,d\omega_t=\int_{P\in\omega_{t_0}}\rho_{t_0}(P)\,d\omega_{t_0}. \tag{N.2}$$

Proposition N.1 *If (N.2) then, with $J_t^{t_0}(P)=\det(d\Phi_t^{t_0}(P))$ (positive Jacobian the motion being supposed regular) and $p=\Phi_t^{t_0}(P)$,*

$$\rho_t(p)=\frac{\rho_{t_0}(P)}{J_t^{t_0}(P)}. \tag{N.3}$$

Proof. The change of variable formula gives

$$\int_{p\in\omega_t}\rho_t(p)\,d\omega_t=\int_{P\in\omega_{t_0}}\rho_t(\Phi_t^{t_0}(P))\,J_t^{t_0}(P)\,d\omega_{t_0},$$

thus (N.2) gives $\rho_t(\Phi_t^{t_0}(P))J_t^{t_0}(P)=\rho_{t_0}(P)$. ∎

Proposition N.2 *$\vec{v}=\vec{v}(t,p_t)$ being the Eulerian velocity at $(t,p_t)\in\mathbb{R}\times\Omega_t$, (N.2) gives*

$$\frac{D\rho}{Dt}+\rho\,\mathrm{div}\vec{v}=0,\quad i.e.\quad \frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\,\vec{v})=0. \tag{N.4}$$

Thus, for all $\omega_t\subset\Omega_t$,

$$\int_{\omega_t}\frac{\partial\rho}{\partial t}\,d\omega_t=-\int_{\partial\omega_t}\rho\vec{v}.\vec{n}\,d\sigma_t. \tag{N.5}$$

Proof. (N.2) gives $\frac{d}{dt}(\int_{p(t)\in\omega_t}\rho(t,p(t))\,d\omega_t)=0$, and Leibniz formula (L.38) applied for all ω_t gives (N.4). Then the Green formula $\int_{\Omega_t}\mathrm{div}(\rho\,\vec{v})\,d\Omega_t=\int_{\partial\Omega_t}\rho\vec{v}.\vec{n}\,d\sigma_t$ gives (N.5). ∎

Exercice N.3 Use (N.3) to prove (N.4).

Answer. $J(t,P)\rho(t,\Phi(t,P))=\rho_{t_0}(P)$ give, with $p_t=\Phi(t,P)$,

$$\frac{\partial J}{\partial t}(t,P)\,\rho(t,p_t)+J(t,P)\left(\frac{\partial\rho}{\partial t}(t,p_t)+d\rho(t,p_t).d\Phi(t,P)\right)=0.$$

Thus $\frac{\partial J}{\partial t}(t,P)=J(t,P)\,\mathrm{div}\vec{v}(t,p)$, cf. (L.37), gives (N.4). ∎

O Work and power

O.1 Definitions

O.1.1 Work along a trajectory

Let α be a differential form (unmissable in thermodynamics, e.g. $\alpha = dU$ the internal energy density, $\alpha = \delta W$ the elementary work, $\alpha = \delta Q$ the elementary heat...).

And consider a regular curve $c : t \in [t_0, T] \to c(t) \in \mathbb{R}^n$ and let $\vec{v}(t, c(t)) := \vec{c}\,'(t)$.

Definition O.1 The work of the differential form α along the curve c is

$$\begin{aligned}\int_c \alpha := \int_{t=t_0}^{T} \alpha(t, c(t)).\vec{c}\,'(t)\, dt \overset{\text{noted}}{=} \int_{t=t_0}^{T} \alpha.d\vec{c} \\ = \int_{t=t_0}^{T} \alpha(t, c(t)).\vec{v}(t, c(t))\, dt \overset{\text{noted}}{=} \int_{t=t_0}^{T} \alpha.\vec{v}\, dt.\end{aligned} \tag{O.1}$$

E.g., $W_T^{t_0}(\alpha, c) = \int_c \delta W =$ work along c of the differential form $\alpha = \delta W$.

E.g. (work of a Lie derivative): $\int_c \mathcal{L}_{\vec{w}}\alpha = \int_{t=t_0}^{T} \mathcal{L}_{\vec{w}}\alpha.\vec{v}\, dt = \int_{t=t_0}^{T} (\frac{\partial\alpha}{\partial t} + d\alpha.\vec{w} + \alpha.d\vec{w}).\vec{v}\, dt$.

Remark O.2 : If α is a stationary and exact differential form, i.e. $\exists U \in C^1$ s.t. $\alpha(t,p) = dU(p)$, then

$$\int_c dU = U(c(T)) - U(c(t_0)) \overset{\text{noted}}{=} \Delta U, \tag{O.2}$$

because $\int_c dU = \int_{t=t_0}^{T} dU(c(t)).\vec{c}\,'(t)\, dt = \int_{t=t_0}^{T} \frac{d(U\circ c)}{dt}(t)\, dt = [U \circ c]_{t_0}^{T} = U(c(T)) - U(c(t_0))$; I.e. the work is independent of the trajectory joining $c(t_0)$ and $c(T)$. ▪

Representation with an Euclidean dot product $(\cdot,\cdot)_g =^{\text{noted}} \,.\bullet_g .$: the linear forms $\alpha_t(p) \in \mathbb{R}^{n*}$ can be represented with its $(\cdot,\cdot)_g$-Riesz representation vector $\vec{f}_t(p) \in \vec{\mathbb{R}^n}$ (observer dependent), hence

$$(\int_c \alpha =) \quad \int_{t=t_0}^{T} \alpha.d\vec{c} = \int_{t=t_0}^{T} \vec{f} \bullet_g d\vec{c} \quad (= \int_{t=t_0}^{T} \vec{f} \bullet_g \vec{v}\, dt). \tag{O.3}$$

In particular if $\vec{f} = \vec{\text{grad}}_g \varphi$ (i.e. if $\alpha = d\varphi$, and $\vec{f}$ is said to derive from a potential φ) then $\int_{t=t_0}^{T} d\varphi.\vec{v}\, dt = \int_{t=t_0}^{T} \vec{f} \bullet_g \vec{v}\, dt = \Delta\varphi$ is independent of the trajectory joining $c(t_0)$ and $c(T)$.

O.1.2 Work

Consider an object Obj, a motion $\widetilde{\Phi} : (t, P_{Obj}) \in [t_0, T] \times Obj \to p(t) = \widetilde{\Phi}(t, P_{Obj}) = \widetilde{\Phi}_{P_{Obj}}(t) \in \mathbb{R}^n$, the trajectories $c_{P_{Obj}} = \widetilde{\Phi}_{P_{Obj}} : t \in [t_0, T] \to p(t) = \widetilde{\Phi}_{P_{Obj}}(t) \in \mathbb{R}^n$, the Eulerian velocities $\vec{v}(t, p(t)) = c_{P_{Obj}}{}'(t)$.

Definition O.3 The work of α along $\widetilde{\Phi}$ is the sum of the works of α along all the trajectories: With $p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj})$, $p_t = \widetilde{\Phi}(t, P_{Obj}) = \widetilde{\Phi}_{P_{Obj}}(t) = \Phi_{p_{t_0}}^{t_0}(t)$ and $\Omega_t = \widetilde{\Phi}(t, Obj)$, it is

$$W_T^{t_0}(\widetilde{\Phi}) = \sum_{P_{Obj} \in Obj} \int_{\widetilde{\Phi}_{P_{Obj}}} \alpha := \int_{p_{t_0} \in \Omega_{t_0}} \Big(\int_{t=t_0}^{T} \alpha(t, \Phi_{p_{t_0}}^{t_0}(t)).\vec{v}(t, \Phi_{p_{t_0}}^{t_0}(t))\, dt \Big)\, d\Omega_{t_0}, \tag{O.4}$$

written $= \int_{p_{t_0} \in \Omega_{t_0}} (\int_{t=t_0}^{T} \alpha.\vec{v}\, dt)\, d\Omega_{t_0}$ (with $\sum_{p_{t_0} \in \Omega_{t_0}}$ for a finite number of particles instead of $\int_{p_{t_0} \in \Omega_{t_0}}$).

O.1.3 The associated power density

Definition: The power density of a differential form α relative to a Eulerian velocity field $\vec{v}$ is the Eulerian function

$$\psi := \alpha.\vec{v} : \begin{cases} \mathcal{C} = \bigcup_{t \in [t_0, T]} (\{t\} \times \Omega_t) & \to \mathbb{R} \\ \qquad (t, p) & \to \psi(t, p) = \alpha(t, p).\vec{v}(t, p). \end{cases} \tag{O.5}$$

And the power at t is, with $\psi_t(p) := \psi(t, p)$,

$$\mathcal{P}_t(\vec{v}_t) := \int_{p \in \Omega_t} \psi_t(p)\, d\Omega = \int_{p \in \Omega_t} \alpha_t(p).\vec{v}_t(p)\, d\Omega \overset{\text{noted}}{=} \int_{\Omega_t} \alpha.\vec{v}\, d\Omega. \tag{O.6}$$

E.g. with a differential form $\mathcal{L}_{\vec{w}}\alpha$ (a Lie derivative of a differential form): $\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_t} \mathcal{L}_{\vec{w}}\alpha.\vec{v}\, d\Omega = \int_{\Omega_t} (\frac{\partial\alpha}{\partial t} + d\alpha.\vec{w} + \alpha.d\vec{w}).\vec{v}\, d\Omega = \int_{\Omega_t} (\frac{\partial\alpha}{\partial t}(t,p) + d\alpha_t(p).\vec{w}_t(p) + \alpha_t(p).d\vec{w}_t(p)).\vec{v}_t(p)\, d\Omega$.

Particular case: If α_t is an exact differential form, i.e. $\exists U_t$ s.t. $\alpha_t = dU_t$, then

$$\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_t} dU_t.\vec{v}_t\,d\Omega = -\int_{\Omega_t} U_t\,\mathrm{div}\vec{v}_t\,d\Omega + \int_\Gamma U_t\,\vec{v}_t \bullet \vec{n}_t\,d\Omega. \tag{O.7}$$

With a Euclidean dot product $(\cdot,\cdot)_g$ and with the $(\cdot,\cdot)_g$-Riesz representation vector $\vec{f}_t$ of α_t we get

$$\mathcal{P}_t(\vec{v}_t) = \int_{p\in\Omega_t} \vec{f}_t(p) \bullet \vec{v}_t(p)\,d\Omega, \tag{O.8}$$

and if $\vec{f}_t = \vec{\mathrm{grad}}U_t$ then $\mathcal{P}_t(\vec{v}_t) = -\int_{\Omega_t} U_t\,\mathrm{div}\vec{v}_t\,d\Omega + \int_\Gamma U_t\,\vec{v}_t \bullet \vec{n}_t\,d\Omega$.

O.2 Piola–Kirchhoff tensors

O.2.1 Classical presentation

At all time, a unique Euclidean basis $(\vec{e}_i)$ and associated Euclidean dot product $.\bullet.$ are imposed.

Usual (first order) hypothesis for the internal stress in a material: The power density is of the type

$$\psi = \underline{\underline{\sigma}} : d\vec{v} \quad \text{(subjective quantity)}, \tag{O.9}$$

which means:

$$\text{If } [\underline{\underline{\sigma}}]_{|\vec{e}} = [\sigma_{ij}] \text{ and } \vec{v} = \sum_{i=1}^n v_i\vec{e}_i \quad \text{then} \quad \psi = \sum_{i,j=1}^n \sigma_{ij}\frac{\partial v_i}{\partial x_j}. \tag{O.10}$$

So the power at t is

$$\mathcal{P}_t(\vec{v}_t) = \int_{p\in\Omega_t} \psi(t,p)\,d\Omega_t = \int_{p\in\Omega_t} \underline{\underline{\sigma}}_t(p) : d\vec{v}_t(p)\,d\Omega_t. \tag{O.11}$$

O.2.2 Objective internal power for the stress

Recall: If $\vec{v}$ is a (regular) Eulerian velocity field, then $d\vec{v}_t(p)$ is an endomorphism at each t and p.

First order hypothesis for the internal stress in a material: There exists an endomorphism $\underline{\underline{\tau}}$ s.t. the power density is given by

$$\psi = \underline{\underline{\tau}} \odot d\vec{v} \quad (\text{objective quantity } = \mathrm{Tr}(\underline{\underline{\tau}}.d\vec{v})), \tag{O.12}$$

And the power at t is

$$\mathcal{P}_t(\vec{v}_t) = \int_{p\in\Omega_t} \underline{\underline{\tau}}_t(p) \odot d\vec{v}_t(p)\,d\Omega_t. \tag{O.13}$$

Quantification with a basis $(\vec{e}_i)$ at t: With $[\underline{\underline{\tau}}]_{|\vec{e}} = [\tau_{ij}]$ and $[d\vec{v}]_{|\vec{e}} = [v_{i|j}]$, i.e. $\underline{\underline{\tau}}.\vec{e}_j = \sum_{i=1}^n \tau_{ij}\vec{e}_i$, $\vec{v} = \sum_{i=1}^n v_i\vec{e}_i$ and $d\vec{v}.\vec{e}_j = \sum_{i=1}^n v_{i|j}\vec{e}_i$,

$$\psi = \sum_{i,j=1}^n \tau_{ij}v_{j|i} \quad \text{and} \quad \mathcal{P}_t(\vec{v}_t) = \sum_{i,j=1}^n \int_{p\in\Omega_t} \tau_{ij}(p)v_{j|i}(p)\,d\Omega_t \quad \text{(objective quantity)}. \tag{O.14}$$

(Duality notations : $[\underline{\underline{\tau}}]_{|\vec{e}} = [\tau^i_{|j}]$, $[d\vec{v}]_{|\vec{e}} = [v^i_{|j}]$, and $\psi = \sum_{i,j=1}^n \tau^i_j v^j_{|i}$.)

(Cartesian basis: $v_{i|j} = \frac{\partial v_i}{\partial x_j} = v^i_{|j} = \frac{\partial v^i}{\partial x^j}$.)

(With chosen Euclidean dot product $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_g$-Euclidean basis $(\vec{e}_i)$: $\underline{\underline{\sigma}} := [\underline{\underline{\tau}}]^T_{|\vec{e}}$ gives

$$\psi = \underline{\underline{\sigma}} : [d\vec{v}]_{|\vec{e}} = \sum_{i,j=1}^n \sigma_{ij}v_{i|j}.) \tag{O.15}$$

O.2.3 The first Piola–Kirchhoff tensor

The Piola–Kirchhoff approach consists in transforming Eulerian quantities into Lagrangian quantities to refer to the initial configuration, with the help of a Euclidean dot product. t_0 and t are fixed, $\Phi^{t_0}_t =^{\text{noted}} \Phi$, $d\Phi^{t_0}_t = F^{t_0}_t =^{\text{noted}} F$, $\vec{V}^{t_0}_t =^{\text{noted}} V$ (Lagrangian velocity). Recall: $\vec{V}^{t_0}_t(P) = \vec{v}_t(\Phi^{t_0}_t(P))$

gives $d\vec{V}_t^{t_0}(P) = d\vec{v}_t(p).F(P)$ when $p = \Phi^{t_0}(t,P)$, written $d\vec{V}(P) = d\vec{v}(p).F(P)$. Thus (O.13) gives (the Jacobian $J_t^{t_0}(P) =^{\text{noted}} J(P)$ being positive for a regular motion)

$$\begin{aligned}\mathcal{P}_t(\vec{v}_t) &= \int_{P\in\Omega_{t_0}} \underline{\underline{\tau}}_t(\Phi(P)) \odot \left(d\vec{V}(P).F(P)^{-1}\right) J(P)\, d\Omega_{t_0} \\ &= \int_{P\in\Omega_{t_0}} \left(J(P)\, F(P)^{-1}.\underline{\underline{\tau}}_t(\Phi(P))\right) \odot d\vec{V}(P)\, d\Omega_{t_0} \quad \text{(objective)}.\end{aligned} \tag{O.16}$$

Quantification: Choose a basis and a Euclidean dot product $(\cdot,\cdot)_g$ to get

$$\mathcal{P}_t(\vec{v}_t) = \int_{P\in\Omega_{t_0}} \underbrace{(J(P)\, \underline{\underline{\tau}}_t(p)^T.F(P)^{-T})}_{I\!\!K(P)} : d\vec{V}(P)\, d\Omega_{t_0} \quad \text{(subjective)}. \tag{O.17}$$

Definition O.4 Relative to t_0 and t and a Euclidean dot product, and with $\underline{\underline{\tau}}_t(p)^T =^{\text{noted}} \underline{\underline{\sigma}}_t(p)$, the first Piola–Kirchhoff tensor at $P \in \Omega_{t_0}$ is the linear map $I\!\!K_{t,g}^{t_0}(P) =^{\text{noted}} I\!\!K(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ defined by

$$I\!\!K(P) = J(P)\, \underline{\underline{\sigma}}_t(\Phi(P)).F(P)^{-T}, \quad \text{written} \quad \boxed{I\!\!K = J\, \underline{\underline{\sigma}}.F^{-T}}. \tag{O.18}$$

So

$$\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_{t_0}} I\!\!K(P) : d\vec{V}_t(P)\, d\Omega_{t_0}. \tag{O.19}$$

Remark O.5 Looking at (O.16), we can also define $\Pi_t^{t_0}(P) = J_t^{t_0}(P) F_t^{t_0}(P)^{-1}.\underline{\underline{\tau}}_t(\Phi_t^{t_0}(P))$ (objective) which can be called "the objective Piola–Kirchhoff tensor". And we have $\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_{t_0}} \Pi_t^{t_0}(P) \odot d\vec{V}_t^{t_0}(P)\, d\Omega_{t_0}$ (objective); And then introduce a Euclidean dot product to use the transposed to define $I\!\!K_t^{t_0}(P) = \Pi_t^{t_0}(P)^T$ (subjective). ▲

O.2.4 The second Piola–Kirchhoff tensor

$I\!\!K(p_{t_0})$ is not symmetric: It can't be since $I\!\!K(p_{t_0}) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ is not an endomorphism. To get a symmetric tensor, the second Piola–Kirchhoff tensor is defined:

Definition O.6 The second Piola–Kirchhoff tensor is the endomorphism $S\!\!K(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ defined by, in short,

$$S\!\!K = F^{-1}.I\!\!K = JF^{-1}.\underline{\underline{\sigma}}.F^{-T}. \tag{O.20}$$

In particular, if $\underline{\underline{\sigma}}_t(p) \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$ is $(\cdot,\cdot)_g$-symmetric then $S\!\!K(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ is symmetric.

Thus, with the pull-back of the endomorphism $d\vec{v}_t \in \mathcal{L}(\vec{\mathbb{R}}^n_t, \vec{\mathbb{R}}^n_t)$,

$$(\Phi^* d\vec{v}_t)(P) = F(P)^{-1}.d\vec{v}_t(p_t).F(P), \tag{O.21}$$

and with $d\vec{v}_t(p_t) = d\vec{V}(P).F(P)^{-1}$ and $\underline{\underline{\sigma}}_t(p)$ symmetric (so $S\!\!K$ is symmetric),

$$\begin{aligned}\mathcal{P}_t(\vec{v}_t) &= \int_{\Omega_{t_0}} I\!\!K : d\vec{V}\, d\Omega_{t_0} = \int_{\Omega_{t_0}} (F.S\!\!K) : d\vec{V}\, d\Omega_{t_0} = \int_{\Omega_{t_0}} ([F].[S\!\!K]) : [d\vec{V}]^T\, d\Omega_{t_0} \\ &= \int_{\Omega_{t_0}} S\!\!K : (d\vec{V}^T.F)\, d\Omega_{t_0} = \int_{\Omega_{t_0}} S\!\!K : (\frac{F^T.d\vec{V} + d\vec{V}^T.F}{2})\, d\Omega_{t_0}.\end{aligned} \tag{O.22}$$

Remark O.7 It is a "chosen time derivative" of $S\!\!K(t) = J(t)F(t)^{-1}.\underline{\underline{\sigma}}(t).F(t)^{-T}$ that leads to some kind of Lie derivative as explain in books in continuum mechanics, see footnote page 26. ▲

O.3 Classical hyper-elasticity and the notation $\partial W/\partial F$

O.3.1 Notation $\partial W/\partial F$

A and B are finite dimensional spaces, $\dim A = n$, $\dim B = m$, and $\widehat{W} \in C^1(\mathcal{L}(A;B);\mathbb{R})$, so

$$\widehat{W} : \begin{cases} \mathcal{L}(A;B) & \to \mathbb{R} \\ L & \to \widehat{W}(L) \end{cases}, \quad \text{and} \quad d\widehat{W} : \begin{cases} \mathcal{L}(A;B) & \to \mathcal{L}(\mathcal{L}(A;B);\mathbb{R}) \\ L & \to d\widehat{W}(L) \end{cases} \tag{O.23}$$

is given by $d\widehat{W}(L)(M) \overset{\text{linearity}}{\underset{\text{notation}}{=}} d\widehat{W}(L).M = \lim_{h\to 0} \dfrac{\widehat{W}(L+hM)-\widehat{W}(L)}{h}$ for all $M \in \mathcal{L}(A;B)$. Notation when L is the name of the variable:

$$d\widehat{W}(L) \overset{\text{noted}}{=} \frac{\partial \widehat{W}}{\partial L}(L), \quad \text{so} \quad d\widehat{W}(L).M \overset{\text{noted}}{=} \frac{\partial \widehat{W}}{\partial L}(L).M. \tag{O.24}$$

Example O.8 $A = B = \vec{\mathbb{R}^n}$, and $\widehat{W}(L) := \mathrm{Tr}(L)$ (the trace of an endomorphism $L \in \mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^n})$). Here $d\mathrm{Tr}(L)(M) = \lim_{h\to 0} \frac{\mathrm{Tr}(L+hM)-\mathrm{Tr}(L)}{h} = \mathrm{Tr}(M)$ since the trace is linear: $\frac{\partial \mathrm{Tr}}{\partial L}(L) := d\mathrm{Tr}(L) = \mathrm{Tr}$. ▪

Example O.9 $A = \vec{\mathbb{R}}^n_{t_0}$, $B = \vec{\mathbb{R}}^n_t$, $L = F = d\Phi^{t_0}_t(p_{t_0})$. Then $d\widehat{W}(F).M =^{\text{noted}} \frac{\partial \widehat{W}}{\partial F}(F).M \in \mathbb{R}$ is the derivative of $\widehat{W}$ at $F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ in a direction $M \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$. ▪

O.3.2 Expression with bases (quantification) and the notation $\partial W/\partial L_{ij}$

Let $(\vec{a}_i) \in A^n$ and $(\vec{b}_i) \in B^m$ be bases in A and B, and let $(\pi_{ai}) \in (A^*)^n$ be the dual basis of $(\vec{a}_i)$. Then consider the basis $(\mathcal{L}_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} =^{\text{noted}} (\vec{b}_i \otimes \pi_{aj})$ in $\mathcal{L}(A;B)$ (made of the linear maps $\mathcal{L}_{ij} : A \to B$ defined by $\mathcal{L}_{ij}.\vec{a}_\ell = \delta_{j\ell}\vec{b}_i$ for all $i = 1,\dots,m$ and $j,\ell = 1,\dots,n$).

The derivation of $\widehat{W}$ at a $L \in \mathcal{L}(A;B)$ in the direction of a basis vector $\mathcal{L}_{ij}$ is, cf. (T.14),

$$\partial_{\mathcal{L}_{ij}}(L) = \frac{\partial \widehat{W}}{\partial \mathcal{L}_{ij}}(L) = d\widehat{W}(L).\mathcal{L}_{ij} \overset{\text{noted}}{=} \frac{\partial \widehat{W}}{\partial L_{ij}}(L) \quad (= \lim_{h\to 0} \frac{\widehat{W}(L+h\mathcal{L}_{ij})-\widehat{W}(L)}{h}) \tag{O.25}$$

notation used when the L_{ij} are the components of L, i.e. $L = \sum_{i=1}^m \sum_{j=1}^n L_{ij}\mathcal{L}_{ij}$ (i.e. $L.\vec{a}_j = \sum_{i=1}^m L_{ij}\vec{b}_i$ for all j, i.e. $[L]_{|\vec{a},\vec{b}} = [L_{ij}]$). So, the Jacobian matrix of $\widehat{W}$ at L relative to $(\mathcal{L}_{ij})$ is

$$[d\widehat{W}(L)]_{|\mathcal{L}_{ij}} = [\frac{\partial \widehat{W}}{\partial L_{ij}}(L)]_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \overset{\text{noted}}{=} [d\widehat{W}(L)]_{|\vec{a},\vec{b}} = [d\widehat{W}(L)_{ij}]. \tag{O.26}$$

So, $d\widehat{W}(L)$ being linear, if $M = \sum_{ij} M_{ij}\mathcal{L}_{ij}$ then (linearity)

$$d\widehat{W}(L).M = \sum_{ij} M_{ij}\, d\widehat{W}(L).\mathcal{L}_{ij} = [M]_{|\vec{a},\vec{b}} : [d\widehat{W}(L)]_{|\vec{a},\vec{b}} = \sum_{ij} M_{ij} \frac{\partial \widehat{W}}{\partial L_{ij}}(L) \tag{O.27}$$

($= [d\widehat{W}(L)]_{|\vec{a},\vec{b}} : [M]_{|\vec{a},\vec{b}}$) with the double matrix contraction.

Duality notations: $a^i := \pi_{ai}$, $\mathcal{L}_i{}^j =^{\text{noted}} \vec{b}_i \otimes a^j$ (because $L(E;E) \simeq L(E^*,E;\mathbb{R})$), $[M]_{|\vec{a},\vec{b}} = [M^i{}_j]$, i.e. $M.\vec{a}_j = \sum_j M^i{}_j\vec{b}_i$ for all j, written $M = \sum_{ij} M^i{}_j \vec{b}_i \otimes a^j$; $d\widehat{W}(L).\mathcal{L}_i{}^j =^{\text{noted}} \frac{\partial \widehat{W}}{\partial L^i{}_j}(L)$, so $[d\widehat{W}(L)]_{|\vec{b}_i\otimes a^j} = [\frac{\partial \widehat{W}}{\partial L^i{}_j}(L)]$, and

$$d\widehat{W}(L).M = \sum_{ij} \frac{\partial \widehat{W}}{\partial L^i{}_j}(L) M^i{}_j. \tag{O.28}$$

NB: $d\widehat{W}(L) \in \mathcal{L}(\mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^m});\mathbb{R})$ and $M = \mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^m})$ are different kinds of mathematical objects, hence $[M]_{|\vec{a},\vec{b}} : [d\widehat{W}(L)]_{|\vec{a},\vec{b}}$ is nothing but a "term to term product" called "double matrix contraction".

Example O.10 Continuing example O.8 with $(\vec{b}_i) = (\vec{a}_i)$: Then $\widehat{W}(L) = \mathrm{Tr}(L)$ gives $d\widehat{W}(L).M = \mathrm{Tr}(M) = \sum_i M_{ii}$, thus $\frac{\partial \widehat{W}}{\partial L_{ij}}(L) = \delta_{ij}$ for all i,j, thus $[d\widehat{W}(L)]_{|\vec{e}} = [I] = [\frac{\partial \mathrm{Tr}}{\partial L_{ij}}(L)]$ (identity matrix), and we recover $d\mathrm{Tr}(L)(M) = [\frac{\partial \mathrm{Tr}}{\partial L_{ij}}(L)] : [M] = [I] : [M] = \sum_{i=1}^n M_{ii} = \mathrm{Tr}(M)$. ▪

Remark O.11 Continuing example O.9: The meaning of the derivation $\frac{\partial\widehat{W}}{\partial F_{ij}} = \frac{\partial\widehat{W}}{\partial F^i_J}$ is intriguing: $\frac{\partial\widehat{W}}{\partial F_{ij}}(F) = d\widehat{W}(F).\mathcal{L}_{ij} = d\widehat{W}(F).(\vec{e}_i \otimes \pi_{Ej})$ is a derivation "at the same time" in the directions $\vec{e}_i$ (at (t,p)) and π_{Ej} (dual basis of $(\vec{E}_i)$ at (t_0,P)). Duality notation: $\frac{\partial\widehat{W}}{\partial F^i_J}(F) = d\widehat{W}(F).(\vec{e}_i \otimes E^J)$. ▪

O.3.3 Motions and ω-lemma

Generalization of (O.23): With U_A open subset in a affine space which associated vector space is A, let

$$\widehat{W} : \begin{cases} U_A \times \mathcal{L}(A;B) & \to \mathbb{R} \\ (P,L) & \to \widehat{W}(P,L) \end{cases}, \quad \text{and} \quad \widehat{W}_P(L) := \widehat{W}(P,L) \ \ (\text{at any fixed } P \in U_A). \tag{O.29}$$

And let (usual notation) $d\widehat{W}_P(L) =^{\text{noted}} \partial_2\widehat{W}(P,L) =^{\text{noted}} \frac{\partial\widehat{W}}{\partial L}(P,L)$: So, for all $M \in \mathcal{L}(A;B)$,

$$\frac{\partial\widehat{W}}{\partial L}(P,L).M = \lim_{h\to 0} \frac{\widehat{W}(P,L+hM) - \widehat{W}(P,L)}{h} \quad (= d(\widehat{W}_P)(L).M = \partial_2\widehat{W}(P,L).M). \tag{O.30}$$

Then consider a motion $\Phi := \Phi^{t_0}_t : \Omega_{t_0} \to \Omega_t$, and $F := d\Phi : P \in \Omega_{t_0} \to d\Phi(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$; And define

$$f : \begin{cases} C^1(\overline{\Omega_{t_0}};\overline{\Omega_t}) & \to C^0(\overline{\Omega_{t_0}};\mathbb{R}) \\ \Phi & \to f(\Phi) := \widehat{W}(.,d\Phi(.)) \end{cases}, \quad \text{so} \quad f(\Phi)(P) = \widehat{W}(P,d\Phi(P)) = \widehat{W}_P(d\Phi(P)). \tag{O.31}$$

So f is a function of Φ which only depends on its first (covariant) gradient $F = d\Phi$. (Toward: "The power of a motion Φ at P only depends on the deformation gradient".)

So $df : \begin{cases} C^1(\overline{\Omega_{t_0}};\overline{\Omega_t}) & \to \mathcal{L}(C^1(\overline{\Omega_{t_0}};\overline{\Omega_t});C^0(\overline{\Omega_{t_0}};\mathbb{R}) \\ \Phi & \to df(\Phi) \end{cases}$, and

$$df(\Phi) : \begin{cases} C^1(\overline{\Omega_{t_0}};\overline{\Omega_t}) & \to C^0(\overline{\Omega_{t_0}};\mathbb{R}) \\ \Psi & \to df(\Phi).\Psi \end{cases}, \quad \text{with} \quad (df(\Phi).\Psi)(P) = df(\Phi(P)).\Psi(P). \tag{O.32}$$

Lemma O.12 (ω-lemma) *If f and $\widehat{W}$ are C^1 then, for all $\Phi,\Psi \in C^1(\overline{\Omega_{t_0}};\overline{\Omega_t})$,*

$$\boxed{df(\Phi).\Psi = \frac{\partial\widehat{W}}{\partial F}(.,d\Phi).d\Psi} = \partial_2\widehat{W}(.,d\Phi).d\Psi, \tag{O.33}$$

i.e. $(df(\Phi).\Psi)(P) = \frac{\partial\widehat{W}}{\partial F}(P,d\Phi(P)).d\Psi(P)$ *, for all* $P \in \Omega_{t_0}$.

Proof. $df(\Phi)(\Psi) = \lim_{h\to 0} \frac{f(\Phi+h\Psi)-f(\Phi)}{h} \in C^0(\overline{\Omega_{t_0}};\overline{\Omega_t})$, i.e., for any $P \in \overline{\Omega_{t_0}}$ we have $df(\Phi)(\Psi)(P) = \lim_{h\to 0} \frac{f(\Phi+h\Psi)(P)-f(\Phi)(P)}{h} = \lim_{h\to 0} \frac{\widehat{W}_P(d\Phi(P)+h\,d\Psi(P))-\widehat{W}_P(d\Phi(P))}{h} = d\vec{W}_P(d\Phi(P)).d\Psi(P)$, i.e. (O.33) ▪

Quantification: With bases $(\vec{E}_i)$ and $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$ and $d\Psi.\vec{E}_j = \sum_{i=1}^n \frac{\partial\Psi_i}{\partial X_j}\vec{e}_i$, we get

$$df(\Phi).\Psi = \sum_{i,j=1}^n \frac{\partial\widehat{W}}{\partial F_{ij}}(.,d\Phi)\frac{\partial\Psi_i}{\partial X_j}(.) \overset{\text{noted}}{=} [\frac{\partial\widehat{W}}{\partial F_{ij}}(.,d\Phi)] : [\frac{\partial\Psi_i}{\partial X_j}(.)] \overset{\text{noted}}{=} [\frac{\partial\widehat{W}}{\partial F}] : [d\Psi], \tag{O.34}$$

Marsden duality notations: $df(\Phi).\Psi = \sum_{i,J=1}^n \frac{\partial\widehat{W}}{\partial F^i_J}\frac{\partial\Psi^i}{\partial X^J} = [\frac{\partial\widehat{W}}{\partial F^i_J}] : [\frac{\partial\Psi^i}{\partial X^J}] = [\frac{\partial\widehat{W}}{\partial F}] : [d\Psi]$.

O.3.4 Application to classical hyper-elasticity: $I\!\!K = \partial W/\partial F$

$(\vec{e}_i) = (\vec{E}_I)$ is Euclidean basis and $(\cdot,\cdot)_g$ is its associated Euclidean dot product, the sames at all times t. Let $\underline{\underline{\sigma}}_t(p)$ be the Cauchy stress tensor at t at $p = \Phi(P)$. Let $I\!\!K = I\!\!K(\Phi)$ be the first Piola–Kirchhoff tensor, i.e. $I\!\!K(\Phi)(P) = \det(d\Phi(P))\,\underline{\underline{\sigma}}_t(\Phi(P)).d\Phi(P)^{-T}$, cf. (O.18).

Definition O.13 If there exists a function $\widehat{I\!\!K}$ such that (first order hypothesis)

$$I\!\!K(\Phi)(P) = \widehat{I\!\!K}(P,d\Phi(P)) \tag{O.35}$$

then $\widehat{I\!\!K}$ is called a constitutive function ($I\!\!K$ only depends on $d\Phi = F$ the first order derivative of Φ).

Definition O.14 The material is hyper-elastic iff $\exists\widehat{W}:\left\{\begin{array}{ll}\Omega_{t_0}\times\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t) & \to\mathbb{R}\\ (P,L) & \to\widehat{W}(P,L)\end{array}\right\}$ s.t.

$$(\mathbb{K}(\Phi)=)\qquad\widehat{\mathbb{K}}(.,d\Phi)=\frac{\partial\widehat{W}}{\partial F}(.,d\Phi),\quad\text{written}\quad\mathbb{K}=\frac{\partial W}{\partial F},\tag{O.36}$$

that is, $\widehat{\mathbb{K}}(P,F(P))=\frac{\partial\widehat{W}}{\partial F}(P,F(P))$ for all $P\in\Omega_{t_0}$, where $F=d\Phi$.

Quantification (Marsden notations): (E^I) dual basis of $(\vec{E}_I)$, $F.\vec{E}_J=\sum_{i=1}^n F^i_J\vec{e}_i$, $\mathbb{K}.\vec{E}_J=\sum_{i=1}^n\mathbb{K}^i{}_J\vec{e}_i$, and $[\mathbb{K}(.,\Phi)]=[\mathbb{K}^i_J(.,\Phi)]=[\frac{\partial\widehat{W}}{\partial F^i_J}(.,F)]$: For any (virtual) motion $\Psi:\Omega_{t_0}\to\Omega_t$,

$$\widehat{\mathbb{K}}(.,d\Phi).d\Psi=\sum_{iJ}\frac{\partial\widehat{W}}{\partial F^i_J}(.,F)\frac{\partial\Psi^i}{\partial X^J}=[\widehat{\mathbb{K}}(.,F)]:[d\Psi],\tag{O.37}$$

which means, $\widehat{\mathbb{K}}(d\Phi)(d\Psi)(P)=\sum_{iJ}\frac{\partial\widehat{W}}{\partial F^i_J}(P,F^{t_0}_t(P))\frac{\partial\Psi^i}{\partial X^J}(P)$ for all $P\in\Omega_{t_0}$.

Exercice O.15 With $C=F^T.F=C(F)$, and with $F=\sum_{iK}F^i_K\vec{e}_i\otimes E^K$, prove

$$\frac{\partial C}{\partial F^i_J}(F)=\sum_K F^i_K(\vec{E}_J\otimes E^K+\vec{E}_K\otimes E^J)\quad(=dC(F).(\vec{e}_i\otimes E^J)),\tag{O.38}$$

and

$$\frac{\partial\sqrt{C}}{\partial F}=\frac{1}{2}\big(\sqrt{C}\,\big)^{-1}.\frac{\partial C}{\partial F},\quad\text{i.e.}\quad 2\sqrt{C}.d(\sqrt{C})=dC.\tag{O.39}$$

Answer. Euclidean basis, thus $(\vec{e}_i\otimes E^J)^T=\vec{E}_J\otimes e^j$, and $F^T=\sum_{Ik}F^k_I\vec{E}_I\otimes e^k$. Thus

$$\begin{aligned}C(F+h\vec{e}_i\otimes E^J)&=(F+h\vec{e}_i\otimes E^J)^T.(F+h\vec{e}_i\otimes E^J)=(F^T+h\vec{E}_J\otimes e^i).(F+h\vec{e}_i\otimes E^J)\\&=F^T.F+h\,(\vec{E}_J\otimes e^i).F+h\,F^T.(\vec{e}_i\otimes E^J)+h^2\,(\vec{E}_J\otimes e^i).(\vec{e}_i\otimes E^J)\\&=C(F)+h\,(\sum_K F^i_K\vec{E}_J\otimes E^K+\sum_K F^i_K\vec{E}_K\otimes E^J)+h^2\,\vec{E}_J\otimes E^J.\end{aligned}\tag{O.40}$$

Thus (O.38). And $dC(F)$ is linear, hence $dC(F).L=\sum_{iJ}L^i_J\,dC(F).\vec{e}_i\otimes E^J$.

With $\sqrt{f}:\vec{x}\to\sqrt{f}(\vec{x}):=\sqrt{f(\vec{x})}$ we have $\frac{f(\vec{x}+h\vec{z}_k)-f(\vec{x})}{h}=(\sqrt{f}(\vec{x}+h\vec{z}_k)+\sqrt{f}(\vec{x})).\frac{\sqrt{f}(\vec{x}+h\vec{z}_k)-\sqrt{f}(\vec{x})}{h}$, thus $h\to 0$ gives $df(\vec{x}).\vec{z}_k=2\sqrt{f}(\vec{x}).d\sqrt{f}(\vec{x}).\vec{z}_k$, thus $df(\vec{x})=2\sqrt{f}(\vec{x}).d\sqrt{f}(\vec{x})$.

In particular, $f=C$ and $\vec{x}=F$ give $dC(F)=2\sqrt{C}(F).d\sqrt{C}(F)$, thus (O.39). ∎

O.3.5 Corollary (hyper-elasticity): $\mathbb{S}\mathbb{K}=\partial W/\partial C$

For the second Piola–Kirchhoff tensor $\mathbb{S}\mathbb{K}=F^{-1}.\mathbb{K}$: We get the existence of a function $\widetilde{W}$: $\left\{\begin{array}{ll}\Omega_{t_0}\times\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0}) & \to\mathbb{R}\\ (P,L) & \to\widetilde{W}(P,L)\end{array}\right\}$ s.t. (constitutive function), with $C=F^T.F$,

$$\widehat{\mathbb{S}\mathbb{K}^{t_0}_t}(.,C)=\frac{\partial\widetilde{W}}{\partial C}(.,C).\tag{O.41}$$

See Marsden and Hughes [14] for details and the thermodynamical hypotheses required.

P Balance of momentum

P.1 Introduction: Cauchy's hypothesis

See the introduction of [7]. Summary: formerly expansion-contraction normal forces and bending forces were considered. Cauchy proposed reducing these forces to a single force (not generally perpendicular to the surface on which it is applied) which can be deduced from tensions exerted on three orthogonal planes.

So take 3 orthonormal planes 1,2,3 at one points, with unit normals $\vec{n}_1,\vec{n}_2,\vec{n}_3$ and three forces $\vec{T}_1,\vec{T}_2,\vec{T}_3$ exerted on the planes, and the tension is obtained with a "tensor" $\underline{\underline{\sigma}}$ s.t. $\underline{\underline{\sigma}}.\vec{n}_i=\vec{T}_i$, $i=1,2,3$.

Later Cauchy's hypothesis was transformed into the master balance law (to satisfy newton's principle $\sum\vec{f}=m\vec{\gamma}$) and its consequence called Cauchy's theorem (which is in fact Cauchy's hypothesis).

P.2 Framework

$\widetilde{\Phi} : [t_0, T] \times Obj \to \mathbb{R}^n$ is a regular motion, $\Omega_t = \widetilde{\Phi}(t, Obj)$, $\Gamma_t = \partial\Omega_t$ (the boundary), $\vec{v}$ is the Eulerian velocity field, ω_t is a regular sub domain in Ω_t and $\partial\omega_t$ is its boundary.

An observer chooses a Euclidean basis $(\vec{e}_i)$ (e.g. made with the foot or the metre) and call $(\cdot,\cdot)_g$ the associated Euclidean dot product. And $\vec{n}(t,p) = \vec{n}_t(p)$ is the outer unit normal at t at $p \in \partial\omega_t$.

All the functions are assumed to be regular enough to validate the following calculations.

Let $\rho : \left\{ \begin{array}{ll} \bigcup_{t\in[t_0,T]} (\{t\} \times \Omega_t) & \to \mathbb{R} \\ (t, p_t) & \to \rho(t, p_t) \end{array} \right\}$ (a mass density), let $\vec{f} : \left\{ \begin{array}{ll} \bigcup_{t\in[t_0,T]} (\{t\} \times \Omega_t) & \to \vec{\mathbb{R}^n} \\ (t, p_t) & \to \vec{f}(t, p_t) \end{array} \right\}$ (a body force density), and let $\vec{T} : \left\{ \begin{array}{ll} \bigcup_{t\in[t_0,T]} (\{t\} \times \partial\omega_t \times \vec{\mathbb{R}^n_t}) & \to \vec{\mathbb{R}^n} \\ (t, p_t, \vec{n}(p_t)) & \to \vec{T}(t, p_t, \vec{n}(p_t)) \end{array} \right\}$ (a surface force density) defined for any regular subset $\omega_t \subset \Omega_t$.

P.3 Master balance law

Definition P.1 The balance of momentum is satisfied by ρ, $\vec{f}$ and $\vec{T}$ iff, for all regular open subset ω_t in Ω_t,

$$\frac{d}{dt}\Big(\int_{\omega_t} \rho\vec{v}\, d\Omega_t\Big) = \int_{\omega_t} \vec{f}\, d\Omega_t + \int_{\partial\omega_t} \vec{T}\, d\Gamma_t \quad \text{(master balance law).} \tag{P.1}$$

(It is in fact a linearity hypothesis, see theorem P.2.) Equivalent to, with (L.38),

$$\int_{\omega_t} \frac{D(\rho\vec{v})}{Dt} + \rho\vec{v}\,\mathrm{div}\vec{v}\, d\Omega_t = \int_{\omega_t} \vec{f}\, d\Omega_t + \int_{\partial\omega_t} \vec{T}\, d\Gamma_t. \tag{P.2}$$

With the conservation of mass hypothesis, cf. (N.4), we then have

$$\int_{\omega_t} \rho\frac{D\vec{v}}{Dt}\, d\Omega_t = \int_{\omega_t} \vec{f}\, d\Omega_t + \int_{\partial\omega_t} \vec{T}\, d\Gamma_t, \tag{P.3}$$

with $\frac{D\vec{v}}{Dt} = \vec{\gamma} =$ the Eulerian acceleration.

P.4 Cauchy theorem $\vec{T} = \underline{\underline{\sigma}}.\vec{n}$ (stress tensor $\underline{\underline{\sigma}}$)

Theorem P.2 (Cauchy first law: Cauchy stress tensor) *If the master balance law (P.1) is satisfied, then $\vec{T}$ is linear in $\vec{n}$, that is, there exists a Eulerian tensor $\underline{\underline{\sigma}} \in T^1_1(\Omega_t)$, called the Cauchy stress tensor, s.t. on all $\partial\omega_t$, in short*

$$\vec{T} = \underline{\underline{\sigma}}.\vec{n}, \tag{P.4}$$

where $\vec{n}$ is the unit outward normal to $\partial\omega_t$ (i.e., $\vec{T}(t,p_t) = \underline{\underline{\sigma}}(t,p_t).\vec{n}(t,p_t)$ for all t and $p_t \in \partial\omega_t$).

(Remark: This result is based on Cauchy's hypothesis that a "tension" $\vec{T}$ on a surface depends on the unit normal $\vec{n}$ to the surface, see [7], and thus $\int_\Gamma \vec{T}\, d\Gamma = \int_\Omega \mathrm{div}\underline{\underline{\sigma}}\, d\Omega$. Hence the "tension" $\vec{T}$ is obtained from a tensor $\underline{\underline{\sigma}}$, i.e., with a basis, 3 functions T_1, T_2, T_3 are obtained from 9 functions σ_{ij}.)

The proof is based on:

Lemma P.3 *Let* $\varphi : \left\{ \begin{array}{ll} \overline{\Omega} & \to \mathbb{R} \\ p & \to \varphi(p) \end{array} \right\} \in C^1(\overline{\Omega}; \mathbb{R})$ *and* $\psi : \left\{ \begin{array}{ll} \overline{\Omega} \times \vec{\mathbb{R}^3} & \to \mathbb{R} \\ (p, \vec{w}) & \to \psi(p, \vec{w}) \end{array} \right\} \in C^1(\overline{\Omega}, \vec{\mathbb{R}^3}; \mathbb{R})$. *If*

$$\forall\omega \subset \Omega,\ \omega \text{ open}, \int_{p\in\omega} \varphi(p)\, d\Omega = \int_{p\in\partial\omega} \psi(p, \vec{n}(p))\, d\Gamma, \tag{P.5}$$

(this hypothesis imposes that $\int_\omega \varphi$ only depends on $\int_{\partial\omega}$ and on $\vec{n}$, and not on the curvature or on higher derivatives on Γ) then φ is linear in $\vec{n}$ and is a divergence:

$$\exists \vec{k} \in C^1(\overline{\Omega}; \vec{\mathbb{R}^3}) \text{ s.t. } \psi = (\vec{k}, \vec{n})_g, \quad \text{and} \quad \varphi = \mathrm{div}\vec{k}. \tag{P.6}$$

(Thus under the hypothesis (P.5) the scalar function φ is obtained from a vector function $\vec{k}$, i.e., with a basis, φ is obtained from 3 functions k_1, k_2, k_3.)

Proof. (Lemma P.3.) Standard proof: Let $p \in \Omega \subset \mathbb{R}^3$. Consider the tetrahedral defined by its vertices p, $p+(h_1,0,0)$, $p+(0,h_2,0)$ and $p+(0,0,h_3)$, with $h_i > 0$ for all i. (On each face of a tetrahedron, the unit normal vector is uniform.) Let Σ_1 the side which outer unit normal is $-\vec{E}_1$: It area is $\sigma_1 = \frac{1}{2}h_2h_3$ (square triangle). Idem for Σ_2 and Σ_3. Let Σ be the fourth side: its area is $\sigma = \frac{1}{2}\sqrt{h_2^2h_3^2 + h_3^2h_1^2 + h_1^2h_2^2}$ and its outer unit normal is $\vec{n} = \frac{1}{2\sigma}(h_2h_3, h_3h_1, h_1h_2)$ (see exercise P.5), that is $\vec{n} = (n_1, n_2, n_3)$ with $n_i = \frac{\sigma_i}{\sigma}$ pour $i = 1,2,3$. The volume of the tetrahedral is $\frac{1}{6}h_1h_2h_3 =^{\text{noted}} \ell^3$. Let $M := \sup_{p\in\overline{\Omega}} |\varphi(p)|$; We have $M < \infty$, since φ is continuous in $\overline{\Omega}$. Then (P.5) give

$$M\ell^3 \geq |\int_{\partial\omega_t} \psi(p,\vec{n}(p))\,d\Gamma|, \quad \text{so} \quad \int_{\partial\omega_t} \psi(p,\vec{n}(p))\,d\Gamma = O(\ell^3). \tag{P.7}$$

And ψ being continuous, the mean value theorem applied on Σ_i gives: There exists $p_i \in \Sigma_i$ s.t.

$$\int_{\Sigma_i} \psi(p,\vec{n}(p))\,d\Gamma = \sigma_i\psi(p_i,\vec{n}_i).$$

Thus

$$\int_{\partial\omega_t} \psi(p,\vec{n}(p))\,d\Gamma = \Big(\sigma_1\psi(p_1,-\vec{E}_1) + \sigma_2\psi(t,p_2,-\vec{E}_2) + \sigma_3\psi(p_3,-\vec{E}_3) + \sigma\psi(p_4,\vec{n})\Big).$$

Then, Ψ being continous, (P.7) gives

$$\sigma_1\psi(p_1,-\vec{E}_1) + \sigma_2\psi(p_2,-\vec{E}_2) + \sigma_3\psi(p_3,-\vec{E}_3) + \sigma\psi(p_4,\vec{n}) = O(\ell^3). \tag{P.8}$$

We flatten the tetrahedron on the yz face by taking $h_2 = h_3 =^{\text{noted}} h$ and $h_1 = h^2$; Thus $\sigma_1 = \frac{1}{2}h^2$, $\sigma_2 = o(h^2)$, $\sigma_3 = o(h^2)$, $\sigma \sim \sigma_1$, $\ell^3 = \frac{1}{6}h^4$, with $\vec{n} \sim -\vec{n}_1 = \vec{E}_1$ and $p_i \sim p$; Then

$$\psi(p,-\vec{E}_1) + \psi(p,+\vec{E}_1) = 0. \tag{P.9}$$

Idem with xz and xy. And for a fixed tetrahedron with h_1, h_2, h_3 given, consider the smaller tetrahedron with $\varepsilon h_1, \varepsilon h_2, \varepsilon h_3$. Then as $\varepsilon \to 0$ (P.8) with (P.9) give

$$\psi(p,\vec{n}) = -\frac{\sigma_i}{\sigma}\psi(p,-\vec{E}_1) - \frac{\sigma_2}{\sigma}\psi(p,-\vec{E}_2) - \frac{\sigma_3}{\sigma}\psi(p,-\vec{E}_3) = \sum_{i=1}^{3} n_i\psi(p,\vec{E}_i),$$

since $n_i = \frac{\sigma_i}{\sigma}$ pour $i = 1,2,3$. The same steps can be done for any (inclined) tetrahedron (or apply a change of variable to get back to the above tetrahedron). Thus ψ_p is a linear map in $\vec{n}_p$, that is, there exists a linear form α_p s.t. $\psi_p(\vec{n}_p) = \alpha_p.\vec{n}_p$ for any $p \in \partial\omega$. And the Riesz representation theorem gives: $\exists\vec{k}_p$ s.t. $\alpha_p.\vec{n}_p = (\vec{k}_p,\vec{n}_p)_g =^{\text{noted}} \vec{k}_p \bullet \vec{n}_p$. ∎

Proof. (Theorem.) With $\vec{\varphi} = \rho\frac{D\vec{v}}{Dt} - \vec{f} = \sum_{i=1}^n \varphi^i\vec{e}_i$, apply Lemma P.3 to the φ_i, cf. (P.3). ∎

Corollary P.4 *With* $\text{div}\underline{\underline{\sigma}} := \sum_{i=1}^n(\sum_{j=1}^n \frac{\partial\sigma_{ij}}{\partial x_j})\vec{e}_i$ *(definition of "the matrix divergence" see (T.66)),*

$$\begin{cases} \vec{f} + \text{div}\underline{\underline{\sigma}} = \rho\dfrac{D\vec{v}}{Dt} & \text{in } \Omega_t, \\ \underline{\underline{\sigma}}.\vec{n} = \vec{T} & \text{on } \Gamma_t \end{cases} \tag{P.10}$$

(matrix meaning). (With duality notations, $\text{div}\underline{\underline{\sigma}} := \sum_{i=1}^n(\sum_{j=1}^n \frac{\partial\sigma_j^i}{\partial x^j})\vec{e}_i$.*)*

Proof. Apply the divergence Formula to (P.3). ∎

Exercice P.5 Consider a triangle T in $\mathbb{R}^3$ which vertices are $A = (h_1,0,0)$, $B = (0,h_2,0)$, $C = (0,0,h_3)$. Prove that $\vec{n} = (h_2h_3, h_3h_1, h_1h_2)$ is orthogonal to T and that $\sigma = \frac{1}{2}\sqrt{h_2^2h_3^2 + h_3^2h_1^2 + h_1^2h_2^2}$ is its area.

Answer. Consider the parametric surface $\vec{r}(t,u) = A + t\vec{AB} + u\vec{AC}$ for $t,u \in [0,1]$ describing the triangle. Thus $\vec{n} = \frac{\partial\vec{r}}{\partial t} \times \frac{\partial\vec{r}}{\partial u} = \vec{AB} \times \vec{AC} = \begin{pmatrix} -h_1 \\ h_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -h_1 \\ 0 \\ h_3 \end{pmatrix} = \begin{pmatrix} h_2h_3 \\ h_3h_1 \\ h_1h_2 \end{pmatrix}$ is orthonormal. And $d\sigma = ||\frac{\partial\vec{r}}{\partial t} \times \frac{\partial\vec{r}}{\partial u}||dudt = \sqrt{h_2^2h_3^2 + h_3^2h_1^2 + h_1^2h_2^2}dudt$. Thus $\sigma = \int_{t=0}^1\int_{u=0}^1 d\sigma = \sqrt{h_2^2h_3^2 + h_3^2h_1^2 + h_1^2h_2^2}$ is twice the aera of the triangle. ∎

Q Balance of moment of momentum

Definition Q.1 The balance of moment of momentum is satisfied by ρ, $\vec{f}$ and $\vec{T}$ iff for all regular sub-open set $\omega_t \subset \Omega_t$

$$\frac{d}{dt}\int_{\omega_t} \rho\,\overrightarrow{OM} \times \vec{v}\,d\Omega_t = \int_{\omega_t} \rho\,\overrightarrow{OM} \times \vec{f}\,d\Omega_t + \int_{\partial\omega_t} \overrightarrow{OM} \times \vec{T}\,d\Gamma_t, \tag{Q.1}$$

equality called the master balance of moment of momentum law. (This excludes e.g. Cosserat continua materials.)

Theorem Q.2 *(Cauchy second law.) If the master balance law (so $\vec{T} = \underline{\underline{\sigma}}.\vec{n}$) and the master balance of moment of momentum law are satisfied then $\underline{\underline{\sigma}}$ is symmetric.*

Proof. Standard proof: Let $\vec{x} = \overrightarrow{OM} = \sum_i x_i\vec{E}_i$, and $\vec{T} = \sum_i T_i\vec{E}_i = \underline{\underline{\sigma}}.\vec{n} = \sum_{ij}\sigma_{ij}n_j\vec{E}_i$. Then (first component) $(\vec{x}\times\vec{T})_1 = x_2T_3 - x_3T_2 = x_2(\sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3) - x_3(\sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3) = (x_2\sigma_{31} - x_3\sigma_{21})n_1 + (x_2\sigma_{32} - x_3\sigma_{22})n_2 + (x_2\sigma_{33} - x_3\sigma_{23})n_3$. Thus $\int_{\partial\omega_t}(\vec{x}\times\vec{T})_1\,d\Gamma_t = \int_{\omega_t}\frac{\partial(x_2\sigma_{31}-x_3\sigma_{21})}{\partial x_1} + \frac{\partial(x_2\sigma_{32}-x_3\sigma_{22})}{\partial x_2} + \frac{\partial(x_2\sigma_{33}-x_3\sigma_{23})}{\partial x_3}\,d\Omega_t = \int_{\omega_t} x_2(\text{div}\underline{\underline{\sigma}})_3 + x_3(\text{div}\underline{\underline{\sigma}})_2 + \sigma_{32} - \sigma_{23}\,d\omega_t$.

(P.10) gives $\rho\frac{D\vec{v}}{Dt} - \vec{f} = \text{div}\underline{\underline{\sigma}}$, thus $\vec{x}\times(\rho\vec{\gamma} - \vec{f}) = \vec{x}\times\text{div}\underline{\underline{\sigma}}$, so the first component of $\vec{x}\times(\rho\vec{\gamma}-\vec{f})$ is $x_2(\text{div}\underline{\underline{\sigma}})_3 - x_3(\text{div}\underline{\underline{\sigma}})_2$, cf. (P.10). Thus (Q.1) gives $\int_{\omega_t}\sigma_{32}-\sigma_{23}\,d\omega_t = 0$. True for all ω_t, thus $\sigma_{32}-\sigma_{23} = 0$. Idem for the other components: $\underline{\underline{\sigma}}$ is symmetric. ∎

R Uniform tensors in $\mathcal{L}^r_s(E)$

Uniform tensors enable to define without ambiguity the "objective contraction rules". Uniform tensors are scalar valued multilinear functions acting on both vectors and linear forms.

NB: In classical mechanics courses, what is called a "tensor" generally not a tensor but a matrix. E.g. you may encounter the expression "Euclidean tensor" which means: The matrix representation of "something" with respect to a Euclidean basis (based on the foot, metre,...) chosen by some observer. (An "Euclidean tensor" is a non-sense, e.g. can you define a "Euclidean vector"?)

R.1 Tensorial product and multilinear forms

Let $A_1, ..., A_n$ be n finite dimension vector spaces. And $A_i^* = \mathcal{L}(A_i;\mathbb{R})$ the set of linear forms.

R.1.1 Tensorial product of functions

Let $f_1 : A_1 \to \mathbb{R}$, ..., $f_n : A_n \to \mathbb{R}$ be n functions. Their tensorial product is the function $f_1 \otimes ... \otimes f_n : A_1 \times ... \times A_n \to \mathbb{R}$ defined by (separate variable function)

$$(f_1 \otimes ... \otimes f_n)(\vec{x}_1, ..., \vec{x}_n) = f_1(\vec{x}_1)...f_n(\vec{x}_n). \tag{R.1}$$

(E.g., $n = 2$ and $A_1 = A_2 = \mathbb{R}$ and $(\cos\otimes\sin)(x,y) = \cos(x)\sin(y)$.)

R.1.2 Tensorial product of linear forms: multilinear forms

Let $\mathcal{L}(A_1, ..., A_n;\mathbb{R})$ be the set of $\mathbb{R}$-multilinear forms on the Cartesian product $A_1 \times ... \times A_n$, that is, the set of the functions $M : A_1 \times ... \times A_n \to \mathbb{R}$ s.t., for all $i = 1, ..., n$, all $\vec{x}_i, \vec{y}_i \in A_i$ and all $\lambda \in \mathbb{R}$,

$$M(..., \vec{x}_i + \lambda\vec{y}_i, ...) = M(..., \vec{x}_i, ...) + \lambda\, M(..., \vec{y}_i, ...), \tag{R.2}$$

the other variables being unchanged.

Definition: An elementary tensor is multilinear form $M = \ell_1 \otimes .... \otimes \ell_n$, with $\ell_i \in A_i^*$ for all i; So

$$\forall(\vec{x}_i)_{i\in\mathbb{N}^*} \in \prod_{i=1}^n A_i, \quad (\ell_1 \otimes ... \otimes \ell_n)(\vec{x}_1, ..., \vec{x}_n) = (\ell_1.\vec{x}_1)...(\ell_n.\vec{x}_n) \ \in \mathbb{R}. \tag{R.3}$$

(The dot in $\ell_i.\vec{x}_i$ is **not** an inner dot product: It is the duality "outer product" $\ell_i.\vec{x}_i := \ell_i(\vec{x}_i)$, cf. (A.43).)

R.2 Uniform tensors in $\mathcal{L}_s^0(E)$

Let E be a real vector space, with $\dim(E) = n \in \mathbb{N}^*$. In this section we consider the first overlay on E made of multilinear forms M on E, called the uniform tensors of type 0 s or of type $\binom{0}{s}$.

E.g., $M \in \mathcal{L}_1^0(E)$ a linear form, $M \in \mathcal{L}_2^0(E)$ an inner dot product, $M \in \mathcal{L}_n^0(E)$ a determinant...

Notations for quantification purposes: $(\vec{e}_i)$ is a basis in E, (π_{ei}) is its (covariant) dual basis (basis in $E^* = \mathcal{L}(E;\mathbb{R})$), (∂_i) is its bidual basis (basis in $E^{**} = \mathcal{L}(E^*;\mathbb{R})$).

R.2.1 Definition of type $\binom{0}{s}$ uniform tensors

$\mathcal{L}_0^0(E) := \mathbb{R}$, and if $s \in \mathbb{N}^*$ then

$$\mathcal{L}_s^0(E) := \mathcal{L}(\underbrace{E \times ... \times E}_{s \text{ times}};\mathbb{R}) \tag{R.4}$$

is called the set of uniform tensors of type $\binom{0}{s}$ on E.

R.2.2 Example: Type $\binom{0}{1}$ uniform tensor = linear forms

A type $\binom{0}{1}$ uniform tensor is an element of $\mathcal{L}_1^0(E) = \mathcal{L}(E;\mathbb{R}) = E^*$: It is a linear form $\ell \in \mathcal{L}_1^0(E) = E^*$.

Quantification: With $\ell_i := \ell(\vec{e}_i)$ we have, cf. (A.11),

$$\ell = \sum_{i=1}^n \ell_i \pi_{ei}, \quad \text{and} \quad [\ell]_{|\pi_e} = (\, \ell_1 \quad ... \quad \ell_n \,) \overset{\text{noted}}{=} [\ell]_{|\vec{e}} \tag{R.5}$$

(row matrix for a linear form). Duality notations: (e^i) is the covariant dual basis and $\ell = \sum_{i=1}^n \ell_i e^i$.

Thus, if $\vec{v} \in E$, $\vec{v} = \sum_{i=1}^n v_i \vec{e}_i$, then $\vec{v}$ is represented by $[\vec{v}]_{|\vec{e}} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ (column matrix for a vector), and the matrix calculation rules give

$$\ell(\vec{v}) = [\ell]_{|\vec{e}}.[\vec{v}]_{|\vec{e}} = (\, \ell_1 \quad ... \quad \ell_n \,).\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^n \ell_i v_i \overset{\text{noted}}{=} \ell.\vec{v}. \tag{R.6}$$

Duality notations: $\vec{v} = \sum_{i=1}^n v^i \vec{e}_i$ and $\ell(\vec{v}) = \sum_{i=1}^n \ell_i v^i$, and Einstein's convention is satisfied.

R.2.3 Example: Type $\binom{0}{2}$ uniform tensor

A type $\binom{0}{2}$ uniform tensor is an element of $\mathcal{L}_2^0(E) = \mathcal{L}(E,E;\mathbb{R})$: It is a bilinear form $T \in \mathcal{L}(E,E;\mathbb{R})$.

Quantification: Let $T_{ij} := T(\vec{e}_i, \vec{e}_j)$. Then, with $\vec{v} = \sum_{i=1}^n v_i \vec{e}_i$ and $\vec{w} = \sum_{i=1}^n w_i \vec{e}_i$,

$$T(\vec{v},\vec{w}) = \sum_{i,j=1}^n T_{ij} v_i w_j = [\vec{v}]_{|\vec{e}}^T.[T]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}, \quad \text{i.e.} \quad T = \sum_{i,j=1}^n T_{ij} \pi_{ei} \otimes \pi_{ej}. \tag{R.7}$$

Duality notations: $T(\vec{v},\vec{w}) = \sum_{i,j=1}^n T_{ij} v^i w^j$, and Einstein's convention is satisfied.

An elementary uniform tensor in $\mathcal{L}_2^0(E)$ is a tensor $T = \ell \otimes m$, where $\ell, m \in E^*$. And so, for all $\vec{v}, \vec{w} \in E$,

$$(\ell \otimes m)(\vec{v},\vec{w}) = (\ell.\vec{v})(m.\vec{w}). \tag{R.8}$$

R.2.4 Example: Determinant

The determinant is a alternating $\binom{0}{n}$ uniform tensor, cf. (L.2).

R.3 Uniform tensors in $\mathcal{L}_s^r(E)$

In this section we consider an over-overlay on E: The multilinear forms acting on both vectors ($\in E$) and functions $\in E^*$ (linear forms).

R.3.1 Definition of type $\binom{r}{s}$ uniform tensors

Let $r, s \in \mathbb{N}$ s.t. $r+s \geq 1$. The set of multilinear forms

$$\mathcal{L}^r_s(E) := \mathcal{L}(\underbrace{E^* \times ... \times E^*}_{r \text{ times}}, \underbrace{E \times ... \times E}_{s \text{ times}}; \mathbb{R}) \tag{R.9}$$

is called the set of uniform tensors of type $\binom{r}{s}$ on E.

The case $r = 0$ has been considered at § R.2.

When $r \geq 1$, a tensor $T \in \mathcal{L}^r_s(E)$ is a functional: Its domain of definition contains a set of functions (the set $E^* = \mathcal{L}(E;\mathbb{R})$).

R.3.2 Example: Type $\binom{1}{0}$ uniform tensor: Identified with a vector

A uniform $\binom{1}{0}$ tensor is a element $T \in \mathcal{L}^1_0(E) = \mathcal{L}(E^*;\mathbb{R}) = \mathcal{L}(\mathcal{L}(E;\mathbb{R});\mathbb{R}) = E^{**}$. With the natural canonical isomorphism

$$\mathcal{J} : \begin{cases} E & \to E^{**} = \mathcal{L}^1_0(E) \\ \vec{w} & \to \mathcal{J}(\vec{w}) = w, \quad \text{defined by} \quad w(\ell) := \ell(\vec{w}), \quad \forall \ell \in E^*, \end{cases} \tag{R.10}$$

cf. (U.9) and prop. U.5,

$$w \overset{\text{noted}}{=} \vec{w}, \quad \text{so} \quad w.\ell \overset{\text{noted}}{=} \vec{w}.\ell \quad (= \ell.\vec{w}). \tag{R.11}$$

So a $\binom{1}{0}$ type uniform tensor w is identified (natural canonical) to the vector $\vec{w} = \mathcal{J}^{-1}(w)$.

Interpretation: E^{**} is the set of directional derivatives. Indeed, if $\mathcal{E}$ is an affine space, if E is the associated vector space, if $p \in \mathcal{E}$, and if f is a differentiable function at p, then $w.df(p) =^{(R.10)} df(p).\vec{w}$ is the directional derivative along $\vec{w}$.

Remark: In differential geometry, $w.df$ is written $\vec{w}(f)$, so $\vec{w}(f)(p) := df(p).\vec{w}$, the definition of a vector being a directional derivative.

Quantification: For all i, j,

$$\partial_i.\pi_{ej} = \delta_{ij} = \pi_{ej}.\vec{e}_i, \quad \text{thus} \quad \partial_i = \mathcal{J}(\vec{e}_i) \overset{\text{noted}}{=} \vec{e}_i. \tag{R.12}$$

Duality notations: $\partial_i.e^j = \delta^j_i = e^j.\vec{e}_i$. E.g., if f is a C^1 function then $df(p) = \sum_{i=1}^n f_{|i}(p)\,\pi_{ei}$ $(= \sum_{i=1}^n f_{|i}(p)\,e^i)$ and

$$\partial_i(df(p)) = df(p).\vec{e}_i = f_{|i}(p) \overset{\text{noted}}{=} \partial_i(f)(p) \overset{\text{noted}}{=} \vec{e}_i(f)(p). \tag{R.13}$$

R.3.3 Example: Type $\binom{1}{1}$ uniform tensor

An elementary uniform tensor in $\mathcal{L}^1_1(E)$ is a tensor $T = u \otimes \beta$, where $u \in E^{**}$ and $\beta \in E^*$. And, with $\vec{u} = J^{-1}(u) \in E$, cf. (R.10), we also write $T = \vec{u} \otimes \beta$. Thus, for all $\ell \in E^*$ and $\vec{w} \in E$

$$(u \otimes \beta)(\ell, \vec{w}) = u(\ell)\beta(\vec{w}) = \ell(\vec{u})\beta(\vec{w}) \overset{\text{noted}}{=} \vec{u}(\ell)\beta(\vec{w}) \overset{\text{noted}}{=} (\vec{u} \otimes \beta)(\ell, \vec{w}). \tag{R.14}$$

Quantification: Let $T(\pi_{ei}, \vec{e}_j)$. So

$$T = \sum_{i,j=1}^n T_{ij}\ \vec{e}_i \otimes \pi_{ej}, \quad \text{and} \quad [T]_{|\vec{e}} = [T_{ij}], \tag{R.15}$$

$[T]_{|\vec{e}} = [T_{ij}]$ being the matrix of T relative to the basis $(\vec{e}_i)$. Duality notations: $T(e^i, \vec{e}_j) = T^i{}_j$, $[T]_{|\vec{e}} = [T^i{}_j]$, $T = \sum_{i,j=1}^n T^i{}_j \vec{e}_i \otimes e^j$, and Einstein's convention is satisfied.

Thus with $\ell \in E^*$, $\ell = \sum_{i=1}^n \ell_i e^i \in E^*$, and $\vec{w} \in E$, $\vec{w} = \sum_{i=1}^n w^i \vec{e}_i \in E$, (R.15) gives

$$T(\ell, \vec{w}) = \sum_{i,j=1}^n T_{ij}\vec{e}_i(\ell)\pi_{ej}(\vec{w}) = \sum_{i,j=1}^n T_{ij}\ell_i w_j = [\ell]_{|\vec{e}}.[T]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} \tag{R.16}$$

($[\ell]_{|\vec{e}}$ is a row matrix). Duality notations: $T(\ell, \vec{w}) = \sum_{i,j=1}^n T^i{}_j \ell_i w^j$ and Einstein convention is satisfied.

R.3.4 Example: Type $\binom{1}{2}$ uniform tensor

The same steps are applied to any tensor. E.g., if $T \in \mathcal{L}^1_2(E)$, then with duality notations, $T^i{}_{jk} = T(e^i, \vec{e}_j, \vec{e}_k)$ and

$$T = \sum_{i,j,k=1}^{n} T^i{}_{jk} \vec{e}_i \otimes e^j \otimes e^k, \quad \text{and} \quad T(\ell, \vec{u}, \vec{w}) = \sum_{i,j,k=1}^{n} T^i{}_{jk} \ell_i u^j w^k. \tag{R.17}$$

R.4 Exterior tensorial products

Let $T_1 \in \mathcal{L}^{r_1}_{s_1}(E)$ and $T_2 \in \mathcal{L}^{r_2}_{s_2}(E)$. Their tensorial product is the tensor $T_1 \otimes T_2 \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}(E)$ defined by

$$(T_1 \otimes T_2)(\ell_{1,1}, ..., \ell_{2,1}, ..., \vec{u}_{1,1}, ..., \vec{u}_{2,1}, ...) := T_1(\ell_{1,1}, ..., \vec{u}_{1,1}, ...)T_2(\ell_{2,1}, ..., \vec{u}_{2,1}, ...). \tag{R.18}$$

Particular case: with $\lambda \in \mathcal{L}^0_0(E) = \mathbb{R}$ and $T \in \mathcal{L}^r_s(E)$,

$$\lambda \otimes T = T \otimes \lambda := \lambda T \in \mathcal{L}^r_s(E). \tag{R.19}$$

Example R.1 let $T_1, T_2 \in \mathcal{L}^1_1(E)$. Quantification: Let $T_1 = \sum_{i,j=1}^n (T_1)^i_j \vec{e}_i \otimes e^j$ and let $T_2 = \sum_{k,m=1}^n (T_2)^k_m \vec{e}_k \otimes e^m$; Then $T_1 \otimes T_2 = \sum_{i,j,k,m=1}^n (T_1)^i_k (T_2)^j_m \vec{e}_i \otimes \vec{e}_j \otimes e^k \otimes e^m \in \mathcal{L}^2_2(E)$. ∴

Remark R.2 Alternative definition: $T_1 \widetilde{\otimes} T_2 := \sum_{i,j,k,m=1}^n (T_1)^i_j (T_2)^k_m \vec{e}_i \otimes e^j \otimes \vec{e}_k \otimes e^m \in \mathcal{L}(E^*, E, E^*, E; \mathbb{R})$. And we get back to the previous definition thanks to the natural canonical isomorphism $\widetilde{J} : \mathcal{L}(E^*, E, E^*, E; \mathbb{R}) \to \mathcal{L}(E^*, E^*, E, E; \mathbb{R}) = \mathcal{L}^2_2(E)$ defined by $\widetilde{J}(\widetilde{T}) = T$ where $T(\ell, m, \vec{v}, \vec{w}) = \widetilde{T}(\ell, \vec{v}, m, \vec{w})$. ∴

R.5 Contractions

R.5.1 Contraction of a linear form with a vector

Let $\ell \in \mathcal{L}^0_1(E) = E^*$ and $\vec{w} \in E$. Their contraction is the value

$$\ell(\vec{w}) \overset{\text{linearity}}{=} \ell.\vec{w} \overset{\text{noted}}{=} \vec{w}.\ell. \tag{R.20}$$

And with a basis $(\vec{e}_i)$ and its dual basis (π_{ei}), $\ell = \sum_{i=1}^n \ell_i \pi_{ei}$ and $\vec{w} = \sum_{i=1}^n w_i \vec{e}_i$ give

$$\ell.\vec{w} = \sum_{i=1}^{n} \ell_i w_i = [\ell]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} = \sum_{i=1}^{n} w_i \ell_i = \vec{w}.\ell = \text{Tr}(\vec{w} \otimes \ell), \tag{R.21}$$

where Tr is the objective trace operator $\text{Tr} : \mathcal{L}(E;E) \simeq \mathcal{L}^1_1(E) \to \mathbb{R}$ (defined by $\text{Tr}(\vec{e}_i \otimes \pi_{ej}) = \delta^i_j$). Duality notations: $\ell.\vec{w} = \sum_{i=1}^n \ell_i w^i$, and Einstein convention is satisfied.

Exercice R.3 Use the change of coordinate formulas to prove that the computation $\ell.\vec{w}$ in (R.21) gives a result independent of the basis.

Answer. Let P be the change of basis matrix. So $[\vec{w}]_{new} = P^{-1}.[\vec{w}]_{old}$ and $[\ell]_{new} = [\ell]_{old}.P$, cf. (A.29), thus $[\ell]_{new}.[\vec{w}]_{new} = ([\ell]_{old}.P).(P^{-1}.[\vec{w}]_{old}) = [\ell]_{old}.(P.P^{-1}).[\vec{w}]_{old} = [\ell]_{old}[\vec{w}]_{old}$ $(= \ell.\vec{w})$. ∴

R.5.2 Contraction of a $\binom{1}{1}$ tensor and a vector

Let $\ell \in E^*$ and $\vec{u} \in E$. The contraction of the elementary tensor $\vec{w} \otimes \ell \in \mathcal{L}^1_1(E)$ with $\vec{u}$ is defined by:

$$(\vec{w} \otimes \underbrace{\ell).\vec{u}}_{\text{contraction}} = (\ell.\vec{u})\vec{w}. \tag{R.22}$$

Thus, if $(\vec{e}_i)$ is a basis in E and (π_{ei}) is the dual basis, and $T = \sum_{i,j=1}^n T_{ij} \vec{e}_i \otimes \pi_{ej} \in \mathcal{L}^1_1(E)$ and $\vec{u} = \sum_{j=1}^n u_j \vec{e}_j \in E$, then

$$T = \sum_{i,j=1}^{n} T_{ij} \vec{e}_i \otimes e^j \quad \Longrightarrow \quad T.\vec{u} = \sum_{i,j=1}^{n} T_{ij} u^j_j \vec{e}_i \tag{R.23}$$

because $\pi_{ej}(\vec{u}) = u_j$. Duality notations: $T.\vec{u} = \sum_{i,j=1}^n T^i_j w^j \vec{e}_i$.

Then, with the natural canonical isomorphism ($\mathcal{L}^1_1(E)$ =) $\mathcal{L}(E,E^*;\mathbb{R}) \simeq \mathcal{L}(E;E)$, see (U.7), any endomorphism $L \in \mathcal{L}(E;E)$ defined by $L.\vec{e}_j = \sum_{i=1}^n L_{ij}\vec{e}_i$ can be written, for calculation purpose,

$$\widetilde{L} = \sum_{i,j=1}^n L_{ij}\vec{e}_i \otimes \pi_{ej} \overset{\text{noted}}{=} L, \quad \text{which means} \quad L.\vec{u} \overset{(R.22)}{=} \sum_{i=1}^n L_{ij}u_j\vec{e}_i \tag{R.24}$$

when $\vec{u} = \sum_i u_j\vec{e}_j$, since $\pi_{ej}(\vec{u}) = u_j$. Duality notations: $L = \sum_{i,j=1}^n L^i{}_j\vec{e}_i \otimes e^j$.

R.5.3 Contractions of uniform tensors

More generally, the contraction of two tensors, if meaningful, is defined thanks to (R.20): Let $T_1 \in \mathcal{L}^{r_1}_{s_1}(E)$, $T_2 \in \mathcal{L}^{r_2}_{s_2}(E)$, $\ell \in E^*$ and $\vec{u} \in E$.

Definition R.4 The objective contraction of $T_1 \otimes \ell \in \mathcal{L}^{r_2}_{s_2+1}(E)$ and $\vec{u} \otimes T_2 \in \mathcal{L}^{r_2+1}_{s_2}(E)$ is the tensor $(T_1 \otimes \ell).(\vec{u} \otimes T_2) \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}$ given by

$$(T_1 \otimes \underbrace{\ell).(\vec{u}}_{\text{contraction}} \otimes T_2) := (\ell.\vec{u})\, T_1 \otimes T_2. \tag{R.25}$$

In particular $(T_1 \otimes \ell).\vec{u} = (\ell.\vec{u})\,T_1$ (as in (R.22)), and $\ell.(\vec{u} \otimes T_2) = (\ell.\vec{u})\,T_2$.

And the objective contraction of $T_1 \otimes \vec{u} \in \mathcal{L}^{r_2+1}_{s_2}(E)$ and $\ell \otimes T_2 \in \mathcal{L}^{r_2}_{s_2+1}(E)$ is the tensor $(T_1 \otimes \vec{u}).(\ell \otimes T_2) \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}$ given by

$$(T_1 \otimes \vec{u}).(\ell \otimes T_2) = (\vec{u}.\ell)\, T_1 \otimes T_2 \quad (= (\ell.\vec{u})\, T_1 \otimes T_2). \tag{R.26}$$

Quantification with a basis $(\vec{e}_i)$, examples to avoid cumbersome notations:

Example R.5 Let $T \in \mathcal{L}^1_1(E) = \mathcal{L}^1_{0+1}(E)$, $T = \sum_{i,j=1}^n T^i_j\vec{e}_i \otimes e^j$. With $\vec{w} \in E \sim E^{**} = \mathcal{L}^1_0(E)$, $\vec{w} = \sum_{j=1}^n w^j\vec{e}_j$, (R.25) gives $T.\vec{w} \in \mathcal{L}^1_0(E) \sim E$ and

$$T.\vec{w} = \sum_{i,j=1}^n T^i_j w^j \vec{e}_i, \quad \text{i.e.} \quad [T.\vec{w}]_{|\vec{e}} = [T]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} \quad \text{(column matrix)}. \tag{R.27}$$

(Einstein's convention is satisfied.) Indeed, $T.\vec{w} = \sum_{i,j,k=1}^n T^i_j w^k(\vec{e}_i \otimes e^j).\vec{e}_k = \sum_{i,j,k=1}^n T^i_j w^k \vec{e}_i(e^j.\vec{e}_k) = \sum_{i,j,k=1}^n T^i_j w^k \vec{e}_i(\delta^j_k) = \sum_{i,j=1}^n T^i_j w^j \vec{e}_i$. With $\ell \in E^* = \mathcal{L}^0_1(E)$, $\ell = \sum_{i=1}^n \ell_i e^i$, (R.25) gives $\ell.T \in \mathcal{L}^0_1(E) = E^*$ and

$$\ell.T = \sum_{i,j=1}^n \ell_i T^i_j e^j, \quad \text{i.e.} \quad [\ell.T]_{|\vec{e}} = [\ell]_{|\vec{e}}.[T]_{|\vec{e}} \quad \text{(row matrix)}. \tag{R.28}$$

(Einstein's convention is satisfied.) Indeed $\ell.T = (\sum_{i=1}^n \ell_i e^i).(\sum_{j,k=1}^n T^k_j \vec{e}_k \otimes e^j) = \sum_{i,j,k=1}^n \ell_i T^k_j (e^i.\vec{e}_k)e^j = \sum_{i,j=1}^n \ell_i T^i_j e^j$. ∴

Example R.6 Let $S,T \in \mathcal{L}^1_1(E)$, $S = \sum_{i,k=1}^n S^i_k \vec{e}_i \otimes e^k$ and $T = \sum_{j,k=1}^n T^k_j \vec{e}_k \otimes e^j$. Then

$$S.T = \sum_{i,j,k=1}^n S^i_k T^k_j \vec{e}_i \otimes e^j, \quad \text{i.e.} \quad [S.T]_{|\vec{e}} = [S]_{|\vec{e}}.[T]_{|\vec{e}} \tag{R.29}$$

(Einstein's convention is satisfied.) Indeed $S.T = (\sum_{i,k=1}^n S^i_k \vec{e}_i \otimes e^k).(\sum_{j,m=1}^n T^m_j \vec{e}_m \otimes e^j) = \sum_{i,j,k,m=1}^n S^i_k T^m_j \vec{e}_i(e^k.\vec{e}_m) \otimes e^j = \sum_{i,j,k=1}^n S^i_k T^k_j \vec{e}_i \otimes e^j$. ∴

Example R.7 Let $T \in \mathcal{L}^1_2(E)$, $T = \sum_{i,j,k=1}^n T^i_{jk} \vec{e}_i \otimes e^j \otimes e^k$, and $\vec{u},\vec{w} \in E \sim \mathcal{L}^1_0(E)$, $\vec{w} = \sum_{i=1}^n w^i \vec{e}_i$ and $\vec{u} = \sum_{i=1}^n u^i \vec{e}_i$. Then

$$T.\vec{w} = \sum_{i,j,k=1}^n T^i_{jk} w^k \vec{e}_i \otimes e^j \in \mathcal{L}^1_1(E), \quad \text{and} \quad (T.\vec{w}).\vec{u} = \sum_{i,j,k=1}^n T^i_{jk} w^k u^j \vec{e}_i \overset{\text{noted}}{=} T(\vec{u},\vec{w}). \tag{R.30}$$

(Einstein's convention is satisfied.) So $[T.\vec{w}]_{|\vec{e}} = [\sum_{k=1}^n T^i_{jk} w^k]_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$. And with $\ell \in E^*$, $\ell = \sum_{i=1}^n \ell_i e^i$,

$$((T.\vec{w}).\vec{u}).\ell = \sum_{i,j,k=1}^n T^i_{jk} w^k u^j \ell_i = T(\ell,\vec{u},\vec{w}) = \ell.T(\vec{u},\vec{w}) = \ell.(T.\vec{w}).\vec{u}. \tag{R.31}$$

∴

R.5.4 Objective double contractions of uniform tensors

Definition R.8 Let $S,T \in \mathcal{L}^1_1(E)$. And let $(\vec{e}_i)$ be a basis in E, (e^i) its dual basis, $S = \sum_{i,j=1}^n S^i_j \vec{e}_i \otimes e^j$ and $T = \sum_{i,j=1}^n T^i_j \vec{e}_i \otimes e^j$. The double objective contraction $S \mathbin{\textcircled{:}} T$ of S and T is defined by

$$S \mathbin{\textcircled{:}} T = \mathrm{Tr}(S.T) = \sum_{i,j=1}^n S^i_j T^j_i \quad (= \sum_{i,j=1}^n T^i_j S^j_i = T \mathbin{\textcircled{:}} S). \tag{R.32}$$

(Einstein convention is satisfied.)

Proposition R.9 *$S \mathbin{\textcircled{:}} T$ defined in (R.32) is an invariant: It is the trace $\mathrm{Tr}(L_S \circ L_T)$ of the endomorphisms $L_S, L_T \in \mathcal{L}(E;E)$ naturally canonically associated to S and T (given by $\ell.L_S.\vec{u} := S(\ell,\vec{u})$ and $\ell.L_T.\vec{u} := T(\ell,\vec{u})$ for all $(\vec{u},\ell) \in E \times E^*$). So the real value $\sum_{i,j=1}^n S^i_j T^j_i$ has the same real value regardless of the chosen basis $(\vec{e}_i)$. (Which is not the case of the term to term matrix multiplication $S : T = \sum_{i,j=1}^n S^i_j T^i_j$, see next § R.5.5 and example R.13.)*

Proof. Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be two bases and $P = [P^i_j]$ be the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, i.e., $\vec{b}_j = \sum_{i=1}^n P^i_j \vec{a}_i$ for all j. Let $Q = [Q^i_j] := P^{-1}$. Then $b^i = \sum_{i=1}^n Q^i_j a^i$. Let $S = \sum_{ij} (S_a)^i_j \vec{a}_i \otimes a^j = \sum_{ij} (S_b)^i_j \vec{b}_i \otimes b^j$. So $[(S_b)^i_j] = P^{-1}.[(S_a)^i_j].P$ (change of basis formula for $\binom{1}{1}$ tensors identified with endomorphisms), i.e. $(S_b)^i_j = \sum_{km} Q^i_k (S_a)^k_m P^m_j$ for all i,j. Idem with T. Thus $\sum_{i,j} (S_b)^i_j (T_b)^j_i = \sum_{i,j,k,m,\alpha,\beta} Q^i_k (S_a)^k_m P^m_j Q^j_\alpha (T_a)^\alpha_\beta P^\beta_i = \sum_{i,j,k,m,\alpha,\beta} (S_a)^k_m (T_a)^\alpha_\beta P^\beta_i Q^i_k P^m_j Q^j_\alpha = \sum_{k,m,\alpha,\beta} (S_a)^k_m (T_a)^\alpha_\beta \delta^\beta_k \delta^m_\alpha = \sum_{k,m} (S_a)^k_m (T_a)^m_k$. ∎

Definition R.10 More generally, the objective double contractions $S \mathbin{\textcircled{:}} T$ of uniform tensors, is obtained by applying the objective simple contraction twice consecutively, when applicable.

E.g., $T_1 \otimes \ell_{1,1} \otimes \ell_{1,2}$ and $\vec{u}_{2,1} \otimes \vec{u}_{2,2} \otimes T_2$ give

$$\begin{aligned}(T_1 \otimes \ell_{1,1} \otimes \underbrace{\ell_{1,2}).(\vec{u}_{2,1}}_{\text{first}} \otimes \vec{u}_{2,2} \otimes T_2) &= (\ell_{1,2}.\vec{u}_{2,1})(T_1 \otimes \underbrace{\ell_{1,1}) \otimes (\vec{u}_{2,2}}_{\text{second}} \otimes T_2) \\ &= (\ell_{1,2}.\vec{u}_{2,1})(\ell_{1,1}.\vec{u}_{2,2})\, T_1 \otimes T_2.\end{aligned} \tag{R.33}$$

Example R.11 Let $S \in \mathcal{L}^1_2(E)$, $T \in \mathcal{L}^2_1(E)$, $S = \sum_{i,j,k=1}^n S^i_{jk} \vec{e}_i \otimes e^j \otimes e^j$, $T = \sum_{\alpha,\beta,\gamma=1}^n T^{\alpha\beta}_\gamma \vec{e}_\alpha \otimes \vec{e}_\beta \otimes e^\gamma$. Then

$$S.T = \sum_{i,j,k,\beta,\gamma=1}^n S^i_{jk} T^{k\beta}_\gamma \vec{e}_i \otimes e^j \otimes \vec{e}_\beta \otimes e^\gamma, \quad \text{and} \quad S \mathbin{\textcircled{:}} T = \sum_{i,j,k,\gamma=1}^n S^i_{jk} T^{kj}_\gamma \vec{e}_i \otimes e^\gamma. \tag{R.34}$$

(Einstein's convention is satisfied.) ∎

Exercice R.12 If $S \in \mathcal{L}(E,F;\mathbb{R})$, $T \in \mathcal{L}(F,G;\mathbb{R})$ and $U \in \mathcal{L}(G,E;\mathbb{R})$ then prove

$$S \mathbin{\textcircled{:}} (T.U) = (S.T) \mathbin{\textcircled{:}} U = (U.S) \mathbin{\textcircled{:}} T \quad \text{(circular permutation)}. \tag{R.35}$$

Answer. If $S = \sum S^i_j \vec{a}_i \otimes b^j$, $T = \sum T^i_j \vec{b}_i \otimes c^j$ and $U = \sum U^i_j \vec{c}_i \otimes a^j$, then $T.U = \sum T^i_k U^k_j \vec{b}_i \otimes a^j$, thus $S \mathbin{\textcircled{:}} (T.U) = \sum S^i_m T^m_k U^k_i$, and $S.T = \sum S^i_k T^k_j \vec{a}_i \otimes c^j$, so $(S.T) \mathbin{\textcircled{:}} U = \sum S^i_k T^k_m U^m_i$. And the second equality thanks to the symmetry of $\mathbin{\textcircled{:}}$, i.e. $(S.T) \mathbin{\textcircled{:}} U = U \mathbin{\textcircled{:}} (S.T) = (U.S) \mathbin{\textcircled{:}} T$ with the previous calculation. ∎

We define in the same way the triple objective contraction (apply the simple contraction three times consecutively). E.g., with (R.34) we get

$$S \mathbin{\textcircled{$\vdots$}} T = \sum_{i,j,k=1}^n S^i{}_{jk} T^{kj}{}_i. \tag{R.36}$$

(Einstein's convention is satisfied.)

R.5.5 Non objective double contraction: Double matrix contraction

The engineers often use the double matrix contraction of second order tensors defined by (term to term multiplication): If $S=[S_{ij}]=[S^i_j]$ and $T=[T_{ij}]=[T^i_j]$ then

$$S:T:=\sum_{i,j=1}^{n}S_{ij}T_{ij}=\sum_{i,j=1}^{n}S^i_jT^i_j\overset{\text{noted}}{=}\mathrm{Tr}(S.T^T). \tag{R.37}$$

Einstein's convention is ***not*** satisfied, and the result is observer dependent for associated endomorphism:

Example R.13 Let $(\vec{e}_i)$ be a basis, let $S\in\mathcal{L}(E;E)$ given by $[S]_{\vec{e}}=\begin{pmatrix}0&4\\2&0\end{pmatrix}$ (so $S.\vec{e}_1=2\vec{e}_2$ and $S.\vec{e}_2=4\vec{e}_1$). Then the double matrix contraction (R.37) gives

$$S:S=[S]_{\vec{e}}:[S]_{\vec{e}}=4*4+2*2=20. \tag{R.38}$$

Change of basis: let $\vec{b}_1=\vec{e}_1$ and $\vec{b}_2=2\vec{e}_2$. The transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$ is $P=\begin{pmatrix}1&0\\0&2\end{pmatrix}$. Thus $[S]_{\vec{b}}=P^{-1}.[S]_{\vec{e}}.P=\begin{pmatrix}1&0\\0&\frac{1}{2}\end{pmatrix}.\begin{pmatrix}0&8\\2&0\end{pmatrix}=\begin{pmatrix}0&8\\1&0\end{pmatrix}$. Thus

$$S:S=[S]_{\vec{b}}:[S]_{\vec{b}}=8*8+1*1=65\neq 20. \tag{R.39}$$

To be compared with the double objective contraction: $[S]_{\vec{e}}\mathbin{\textcircled{:}}[S]_{\vec{e}}=4*2+2*4=16=[S]_{\vec{b}}\mathbin{\textcircled{:}}[S]_{\vec{b}}=S\mathbin{\textcircled{:}}S$ (observer independent result = objective result).

So it is absurd to use $S:S$ (double matrix contraction) if you need objectivity: Recall that the foot is the international vertical unit in aviation, and thus the use of the double **objective** contraction is vital, while the use of the double matrix contraction can be fatal (really). Also see the Mars climate orbiter probe crash. ▪

Exercice R.14 Let $S\in\mathcal{L}^0_2(E)$ (e.g. a metric), let $(\vec{a}_i)$ be a Euclidean basis in foot, and let $(\vec{b}_i)=(\lambda\vec{a}_i)$ be the related euclidean basis in metre (change of unit). Give $[S]_{|\vec{a}}:[S]_{|\vec{a}}$ and $[S]_{|\vec{b}}:[S]_{|\vec{b}}$ and compare. (The simple and double objective contractions are impossible here since S and T are not compatible.)

Answer. Let $S=\sum_{i,j=1}^nS_{a,ij}a^i\otimes a^j=\sum_{i,j=1}^nS_{b,ij}b^i\otimes b^j$. Since $(\vec{b}_i)=(\lambda\vec{a}_i)$ we have $b^i=\frac{1}{\lambda}a^i$. Thus $\sum_{i,j=1}^nS_{a,ij}a^i\otimes a^j=\sum_{i,j=1}^nS_{a,ij}\lambda^2b^i\otimes b^j$, thus $\lambda^2S_{a,ij}=S_{b,ij}$. Thus

$$[S]_{|\vec{b}}:[S]_{|\vec{b}}=\sum_{i,j=1}^n(S_{b,ij})^2=\lambda^4\sum_{i,j=1}^n(S_{a,ij})^2=\lambda^4[S]_{|\vec{a}}:[S]_{|\vec{a}}, \tag{R.40}$$

with $\lambda^4\geq 100$: Quite a difference isn't it? ▪

R.6 Kronecker (contraction) tensor, trace

Definition R.15 The Kronecker tensor is the $\binom{1}{1}$ uniform tensor $\underline{\underline{\delta}}\in\mathcal{L}^1_1(E)$ defined by

$$\forall(\ell,\vec{u})\in E^*\times E,\quad \underline{\underline{\delta}}(\ell,\vec{u}):=\ell.\vec{u}. \tag{R.41}$$

And the Kronecker symbols relative to a basis $(\vec{e}_i)$ are the reals defined by, calling (π_{ei}) the dual basis,

$$\delta_{ij}:=\delta(\pi_{ei},\vec{e}_j)=\left\{\begin{matrix}1\text{ if }i=j,\\0\text{ if }i\neq j,\end{matrix}\right\}\quad\text{i.e.}\quad \underline{\underline{\delta}}:=\sum_{i=1}^n\pi_{ei}\otimes e^i,\quad [\underline{\underline{\delta}}]=[\delta_j]=[I] \tag{R.42}$$

(identity matrix whatever the basis). Duality notations: $\delta^i_j:=\delta(e^i,\vec{e}_j)$, $\underline{\underline{\delta}}:=\sum_{i=1}^n\vec{e}_i\otimes e^i$ and $[\underline{\underline{\delta}}]=[\delta^i_j]$.

Definition R.16 The trace of a $\binom{1}{1}$ uniform tensor $T\in\mathcal{L}^1_1(E)$ is

$$\widetilde{\mathrm{Tr}}(T)=\underline{\underline{\delta}}\mathbin{\textcircled{:}}T\quad(=\mathrm{Tr}(L_T)) \tag{R.43}$$

(with the natural canonical isomorphism $T\in\mathcal{L}^1_1(E)\simeq L_T\in\mathcal{L}(E;E)$ given by $T(\ell,\vec{v}):=\ell.L_T.\vec{v}$).

Thus $\widetilde{\mathrm{Tr}}(T)=\sum_{i=1}^nT^i{}_i$.

In particular $\widetilde{\mathrm{Tr}}(\underline{\underline{\delta}})=n$, and $\widetilde{\mathrm{Tr}}(\vec{v}\otimes\ell)=\sum_iv^i\ell_i=\ell.\vec{v}$ when $\vec{v}=\sum_iv^i\vec{e}_i$ and $\ell=\sum_j\ell_je^j$.

S Tensors in $T^r_s(U)$

S.1 Fundamental counter-example (derivation), and modules

Let A and B be any sets, and let $\mathcal{F}(A;B)$ be the set of functions $A \to B$. The "plus" inner operation and the "dot" outer operation are defined by, for all $f,g \in \mathcal{F}(A;B)$, all $\lambda \in \mathbb{R}$ and all $p \in A$,

$$\begin{cases} (f+g)(p) := f(p)+g(p), \quad \text{and} \\ (\lambda.f)(p) := \lambda\, f(p), \quad \lambda.f \overset{\text{noted}}{=} \lambda f. \end{cases} \tag{S.1}$$

$(\mathcal{F}(A;B),+,.,\mathbb{R})$ is thus a vector space on the field $\mathbb{R}$ (see any elementary course) called $\mathcal{F}(A;B)$.

But the field $\mathbb{R}$ is "too small" to define a tensor which can be seen as "a linear tool that satisfies the change of coordinate system rules":

Example S.1 Fundamental counter-example: Derivation. Let U be an open set in $\mathbb{R}^n$. The derivation $d : \vec{w} \in C^1(U;\vec{\mathbb{R}^n}) \to d\vec{w} \in C^0(U;\mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^n}))$ is $\mathbb{R}$-linear: In particular $d(\lambda\vec{w}) = \lambda(d\vec{w})$ for all $\lambda \in \mathbb{R}$...

...but d doesn't satisfy the change of coordinate system rules, see (T.36).

So a derivation it ***not*** a tensor (it is a "spray", see Abraham–Marsden [1]).

In fact, one requirement for T to be a tensor is, e.g. with $T = \vec{w}$ a vector field: For all $\varphi \in C^\infty(U;\mathbb{R})$, and all $\vec{w} \in \Gamma(U)$ (C^∞-vector field),

$$T(\varphi\vec{w}) = \varphi\, T(\vec{w}). \tag{S.2}$$

While

$$d(\varphi\vec{w}) \neq \varphi\, d(\vec{w}), \quad \text{because} \quad d(\varphi\vec{w}) = \varphi\, d\vec{w} + d\varphi.\vec{w}. \tag{S.3}$$

Thus the elementary $\mathbb{R}$-linearity requirement "$T.(\lambda\vec{w}) = \lambda(T.\vec{w})$ for all $\lambda \in \mathbb{R}$ is not sufficient to characterize a tensor: The $\mathbb{R}$-linearity has to be replaced by the $C^\infty(U;\mathbb{R})$-linearity, cf. (S.2).

Thus we will have to replace a real vector space $(V,+,.,\mathbb{R})$ over the field $\mathbb{R}$ with the "module" $(V,+,.,C^\infty(U;\mathbb{R}))$ over the ring $C^\infty(U;\mathbb{R})$, which mainly amounts to consider (S.1) for all $\lambda = \varphi \in C^\infty(U;\mathbb{R})$. Remark: The use of a module is very similar to the use of a vector space, but for the use of the inverse: all real $\lambda \neq 0$ has a multiplicative inverse in $\mathbb{R}$ (namely $\frac{1}{\lambda}$), but a function $f \in C^\infty(U;\mathbb{R})$ s.t. "$f \neq 0$ and f vanishes at one point" doesn't have a multiplicative inverse in $C^\infty(U;\mathbb{R})$. ▪

S.2 Field of functions and vector fields

U is an open set in the affine space , its associated space being E which is $\vec{\mathbb{R}}$, $\vec{\mathbb{R}^2}$ or $\vec{\mathbb{R}^3}$. The definition of tensors is done at a fixed time t (concerns the space variables in classical mechanics). The approach is first qualitative, then quantitative with a basis $(\vec{e}_i(p))$ and its dual basis $(\pi_{ei}(p)) = (e^i(p))$, at any $p \in \mathcal{E}$.

S.2.1 Framework of classical mechanics

$\mathcal{E}$ is the affine space $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$ made of points p, and $E = \vec{\mathbb{R}^n}$ is the usual associated vector space $\vec{\mathbb{R}}$, $\vec{\mathbb{R}^2}$ or $\vec{\mathbb{R}^3}$ made of bipoint vectors $\vec{w} = \vec{pq} =^{\text{noted}} q-p$, and we then write $q = p+\vec{w}$, which means: If $O \in \mathcal{E}$ (an origin) then $\overrightarrow{Oq} = \overrightarrow{Op} + \vec{w}$ (which is Chasles' relation $\vec{pq} = \overrightarrow{pO} + \overrightarrow{Oq}$), relation independent of the choice of O, and hence the vectors $\vec{w}$ in E are called "free vectors" (congruence relation: $\vec{u}\mathcal{R}\vec{w}$ iff $\vec{u} = \vec{w}$, i.e. $\overrightarrow{p_1q_1}\mathcal{R}\overrightarrow{p_2q_2}$ iff $\overrightarrow{p_1q_1} = \overrightarrow{p_2q_2}$, i.e. $\overrightarrow{p_1q_1}\mathcal{R}\overrightarrow{p_2q_2}$ iff $\overrightarrow{p_1O} + \overrightarrow{Oq_1} = \overrightarrow{p_2O} + \overrightarrow{Oq_2}$, i.e. $\overrightarrow{p_1q_1}\mathcal{R}\overrightarrow{p_2q_2}$ iff $\overrightarrow{p_1p_2} = \overrightarrow{q_1q_2}$).

S.2.2 Vector fields

Let $\vec{w} : \left\{\begin{matrix} U & \to E \\ p & \to \vec{w}(p) \end{matrix}\right\}$ be a vector valued function. The associated field is

$$\widetilde{\vec{w}} : \begin{cases} U & \to U \times E \\ p & \to \widetilde{\vec{w}}(p) = (p;\vec{w}(p)). \end{cases} \tag{S.4}$$

So $\text{Im}\widetilde{\vec{w}} = \{(p;\vec{w}(p)) : p \in U\}$ is the graph of $\vec{w}$, and the definition of $\widetilde{\vec{w}}$ tells that the vector $\vec{w}(p)$ has to be drawn at p called the base point (first component of $\widetilde{\vec{w}}(p)$); and $\widetilde{\vec{w}}(p)$ is called a vector at p. Usual rules:

$$\widetilde{\vec{u}}(p) + \widetilde{\vec{w}}(p) = (p, \vec{u}(p) + \vec{w}(p)), \quad \text{and} \quad \lambda\widetilde{\vec{u}}(p) = (p, \lambda\vec{u}(p)) \tag{S.5}$$

(usual rules for "vectors at p"). To lighten the notations, $\widetilde{\vec{w}}(p) =^{\text{noted}} \vec{w}(p)$ (but don't forget it is a pointed vector). Notations:

$$\Gamma(U) = T_0^1(U) := \text{ the set of vector fields on } U = \text{the set of } \binom{1}{0} \text{ tensors on } U. \tag{S.6}$$

More precisely, we will use the definition of vector fields (see e.g. Abraham–Marsden [1]): A vector field is built from tangent vectors to curves. It makes sense on non planar surfaces, and more generally on differential manifolds.

Example S.2 Discrete case: n "force vectors" $\vec{f}_i(p_i)$ applied at n points $p_i \in \mathbb{R}^3$ give the discrete vector field $\widetilde{\vec{f}} : p_i \in \{p_1, ..., p_n\} \subset \mathbb{R}^3 \to \widetilde{\vec{f}}(p_i) = (p_i, \vec{f}_i(p_i)) \in \mathbb{R}^3 \times \vec{\mathbb{R}^3}$ where p_i is "the point of application" of $\vec{f}_i(p_i)$, and $\widetilde{\vec{f}}(p_i) = (p_i, \vec{f}_i)$ is a pointed vector. Essential in mechanics. ▪

S.2.3 Field of functions

Let $f : \left\{ \begin{array}{ll} U & \to \mathbb{R} \\ p & \to f(p) \end{array} \right\}$ be a scalar valued function. The associated field is

$$\widetilde{f} : \left\{ \begin{array}{ll} U & \to U \times \mathbb{R} \\ p & \to \widetilde{f}(p) := (p; f(p)), \end{array} \right. \tag{S.7}$$

and the first component p of the couple $\widetilde{f}(p) = (p; f(p))$ is called the base point. So $\text{Im}\widetilde{f} = \{(p; f(p)) : p \in U\}$ is the graph of f. Definition:

$$T_0^0(U) := \{\text{field of functions}\} = \text{the set of } \binom{0}{0} \text{ tensors on } U, \tag{S.8}$$

or the set of tensors of order 0 on U. Abusive short notations (to lighten the writings):

$$\widetilde{f}(p) \overset{\text{noted}}{=} f(p), \quad \text{and} \quad T_0^0(U) \overset{\text{noted}}{=} C^\infty(U; \mathbb{R}), \tag{S.9}$$

but keep the base point in mind (no ubiquity gift).

In $T_0^0(U)$, the internal sum is defined by, for all $\widetilde{f}, \widetilde{g} \in T_0^0(U)$ with $\widetilde{f}(p) = (p; f(p))$ and $\widetilde{g}(p) = (p; g(p))$,

$$(\widetilde{f} + \widetilde{g})(p) := (p; (f+g)(p)) \quad (= (p; f(p) + g(p))), \tag{S.10}$$

and the external multiplication on the ring $C^\infty(U; \mathbb{R})$ is defined by, for all $\varphi \in C^\infty(U; \mathbb{R})$,

$$(\varphi\widetilde{f})(p) := (p; (\varphi f)(p)) \quad (= (p; \varphi(p) f(p))) \tag{S.11}$$

(the base point p remains unchanged). Thus $(T_0^0(U), +, .)$ is a module over the ring $C^\infty(U; \mathbb{R})$.

S.3 Differential forms

The basic concept is that of vector fields. A first over-layer is made of differential forms (which "measure vector fields"):

Definition S.3 Let $\alpha \left\{ \begin{array}{ll} U & \to E^* \\ p & \to \alpha(p) \end{array} \right\}$ (so $\alpha(p)$ is a linear form at p). The associated differential form (also called a 1-form) is "the field of linear forms" defined by

$$\widetilde{\alpha} : \left\{ \begin{array}{ll} U & \to U \times E^* \\ p & \to \widetilde{\alpha}(p) = (p; \alpha(p)) \quad (= \text{"a pointed linear form at } p\text{"}). \end{array} \right. \tag{S.12}$$

And p is called the base point, and $\text{Im}\widetilde{\alpha} = \{(p; \alpha(p)) : p \in U\}$ is the graph of α.

Thus, if $\widetilde{\alpha} \in \Omega^1(U)$ (differential form) and $\widetilde{\vec{w}} \in \Gamma(U)$ (vector field), then $\widetilde{\alpha}.\widetilde{\vec{w}} \in T^0_0(U)$ (field of scalar valued functions) satisfies

$$\widetilde{\alpha}.\widetilde{\vec{w}} : \begin{cases} U & \to U \times \mathbb{R} \\ p & \to (\widetilde{\alpha}.\widetilde{\vec{w}})(p) = (p; (\alpha.\vec{w})(p)) = (p; \alpha(p).\vec{w}(p)) \ \in U \times \mathbb{R}. \end{cases} \tag{S.13}$$

Short notation:

$$\widetilde{\alpha}(p) \overset{\text{noted}}{=} \alpha(p), \quad \text{instead of } \widetilde{\alpha}(p) = (p; \alpha(p)), \tag{S.14}$$

but keep the base point in mind. And

$$\Omega^1(U) = T^0_1(U) := \text{ the set of differential forms on } U = \text{the set of } \binom{0}{1} \text{ tensors on } U. \tag{S.15}$$

S.4 Tensors

A second over-layer is introduced with the tensors with are "functions defined on vector fields and on differential forms" (which "measure vector fields and differential forms").

Let $r, s \in \mathbb{N}$, $r+s \geq 1$, and let $T : \left\{ \begin{matrix} U & \to \mathcal{L}^r_s(E) \\ p & \to T(p) \end{matrix} \right\}$ (so $T(p)$ is a uniform $\binom{r}{s}$ tensor for each p, cf. (R.3.1)). And consider the associated function

$$\widetilde{T} : \begin{cases} U & \to U \times \mathcal{L}^r_s(E) \\ p & \to \widetilde{T}(p) = (p; T(p)) \end{cases} \tag{S.16}$$

Abusive short notation:

$$\widetilde{T}(p) \overset{\text{noted}}{=} T(p) \quad \text{instead of } \widetilde{T}(p) = (p; T(p)), \tag{S.17}$$

but keep the base point in mind.

Definition S.4 (Abraham–Marsden [1].) $\widetilde{T}$ is a tensor of type $\binom{r}{s}$ iff T is $C^\infty(U;\mathbb{R})$-multilinear (not only $\mathbb{R}$-multilinear), i.e., for all $f \in C^\infty(U;\mathbb{R})$, all z_1, z_2 vector field or differentiable form where applicable, and all $p \in U$,

$$\begin{cases} T(p)(..., z_1(p) + z_2(p), ...) = T(p)(..., z_1(p), ...) + T(p)(..., z_2(p), ...), \quad \text{and} \\ T(p)(..., f(p) z_1(p), ...) = f(p)\, T(p)(..., z_1(p), ...), \end{cases} \tag{S.18}$$

written in short

$$\begin{cases} T(..., z_1 + z_2, ...) = T(..., z_1, ...) + T(..., z_2, ...), \quad \text{and} \\ T(..., f z_1, ...) = f\, T(..., z_1, ...). \end{cases} \tag{S.19}$$

And

$$T^r_s(U) := \text{ the set of } \binom{r}{s} \text{ type tensors on } U. \tag{S.20}$$

(Recall: $T^0_0(U) := C^\infty(U;\mathbb{R})$ the set of function fields, cf. (S.7).)

Remark S.5 Definition in differential geometry lessons: A tensor is a section of a certain bundle over a manifold. For classical mechanics, definition S.4 gives an equivalent definition. ▪

S.5 First Examples

S.5.1 Type $\binom{0}{1}$ tensor = differential forms

If $T \in T^0_1(U)$ then $T(p) \in E^*$, so $T = \alpha \in \Omega^1(U)$ is a differential form: $T^0_1(U) \subset \Omega^1(U)$.

Converse: Does a differential form $\alpha \in \Omega^1(U)$ defines a $\binom{0}{1}$ type tensor on U? Yes: We have to check (S.18), which is trivial. So $\alpha \in T^0_1(U)$, so $\Omega^1(U) \subset T^0_1(U)$.

Thus

$$T^0_1(U) = \Omega^1(U). \tag{S.21}$$

S.5.2 Type $\binom{1}{0}$ tensor (identified to a vector field)

Let $T \in T^0_1(U)$, so $T(p) \in \mathcal{L}^1_0(E) = \mathcal{L}(E^*;\mathbb{R}) = E^{**}$ for all $p \in U$. Thus, thanks to the natural canonical isomorphism $E^{**} \simeq E$, $T(p)$ can be identified to a vector, thus $T^0_1(U) \subset \Gamma(U)$.

Converse: Does a vector field $\vec{w} \in \Gamma(U)$ defines a $\binom{1}{0}$ type tensor on U? Yes: We have to check (S.18), which is trivial. So $\Gamma(U) \subset T^1_0(U)$.

Thus

$$T^1_0(U) \simeq \Gamma(U). \tag{S.22}$$

S.5.3 A metric is a $\binom{0}{2}$ tensor

Let $T \in T^0_2(U)$, so $T(p) \in \mathcal{L}^0_2(E)$ for all $p \in U$, and $T(\vec{u},\vec{w}) \in T^0_0(U)$ for all $\vec{u},\vec{w} \in \Gamma(U)$.

Definition S.6 A metric g on U is a $\binom{0}{2}$ type tensor on U such that, for all $p \in E$, $g(p) =^{\text{noted}} g_p$ is an inner dot product on E.

S.6 $\binom{1}{1}$ tensor, identification with fields of endomorphisms

Let $T \in T^1_1(U)$, so $T(p) \in \mathcal{L}^1_1(E)$ for all $p \in U$, and $T(\alpha,\vec{w}) \in T^0_0(U)$ for all $\alpha \in \Omega^1(U)$ and $\vec{w} \in \Gamma(U)$ (so $T(p)(\alpha(p),\vec{w}(p)) \in \mathbb{R}$ for all p).

The associated field of endomorphisms on U is $\widetilde{L}_T : \left\{ \begin{array}{l} U \to U \times \mathcal{L}(E;E) \\ p \to \widetilde{L}_T(p) = (p, L_T(p)) \end{array} \right\}$ where $L_T(p)$ is identified with $T(p)$ thanks to the natural canonical isomorphism $\mathcal{L}(E;E) \simeq \mathcal{L}(E^*,E;\mathbb{R}) = \mathcal{L}^1_1(E)$ given by

$$\forall \ell \in E^*,\ \forall \vec{w} \in E, \quad \ell.(L_T(p).\vec{w}) = T(p)(\ell,\vec{w}). \tag{S.23}$$

S.7 Unstationary tensor

Let $t \in [t_1,t_2] \subset \mathbb{R}$. Let $(T_t)_{t\in[t_1,t_2]}$ be a family of $\binom{r}{s}$ tensors, cf. (S.16). Then $T : t \to T(t) := T_t$ is called an unstationary tensor. And the set of unstationary tensors is also noted $T^r_s(U)$. E.g., a Eulerian velocity field is a $\binom{1}{0}$ unstationary vector field.

T Differential, its eventual gradients, divergences

T.1 Differential

The definition of the differential of a function is observer independent: All observers have the same definition (qualitative: no man made tool required, like a basis or an inner dot product).

T.1.1 Framework

Classical Framework: $\mathcal{E}$ are $\mathcal{F}$ affine spaces associated with vector spaces E and F, and $||.||_E$ and $||.||_F$ are norms in E and F such that $(E,||.||_E)$ and $(F,||.||_F)$ are complete (we need limit "that stay in the space" as $h \to 0$). U is an open set in $\mathcal{E}$, and $\Phi : \left\{ \begin{array}{l} U \to \mathcal{F} \\ p \to p_{\mathcal{F}} = \Phi(p) \end{array} \right\}$ is a function. If applicable, $\mathcal{E}$ and/or $\mathcal{F}$ can be replaced by E and/or F. Reminder:

Definition T.1 At $p \in U$ the function Φ is continuous iff $\Phi(q) \underset{q\to p}{\longrightarrow} \Phi(p)$ relative to the considered norms, i.e., $||\Phi(q)-\Phi(p)||_F \longrightarrow_{||q-p||_E\to 0} 0$, also written (Landau notation): Near p,

$$\Phi(q) = \Phi(p) + o(1), \tag{T.1}$$

called "the zero-th order Taylor expansion of Φ near p". Which means:

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \text{s.t. } \forall q \in \mathcal{E} \text{ satisfying } ||q-p||_E < \eta,\ ||\Phi(q)-\Phi(p)||_F < \varepsilon. \tag{T.2}$$

And $C^0(U;\mathcal{F})$ is the set of functions that are continuous at all $p \in U$.

T.1.2 Directional derivative and differential (observer independent)

Let $p \in U$, $\vec{u} \in E$, and let $f : \mathbb{R} \to \mathcal{F}$ defined by

$$f(h) := \Phi(p + h\vec{u}). \tag{T.3}$$

(In a manifold: $f(h) := \Phi(c(h))$ where c is a C^1 curve s.t. $c(0) = p$ and $c'(0) = \vec{u}$.)

Definition T.2 The function Φ is differentiable at p in the direction $\vec{u}$ iff f is derivable at 0, i.e. iff the limit $f'(0) = \lim_{h\to 0} \frac{\Phi(p+h\vec{u})-\Phi(p)}{h} =^{\text{noted}} d\Phi(p)(\vec{u})$ exists in F, i.e. iff, near p,

$$\Phi(p + h\vec{u}) = \Phi(p) + h\, d\Phi(p)(\vec{u}) + o(h), \tag{T.4}$$

equation called the first order Taylor expansion of Φ at p in the direction $\vec{u}$ (it is the first order Taylor expansion of f near p).

Then $d\Phi(p)(\vec{u})$ is called the directional derivative of Φ at p in the direction $\vec{u}$.

And if, for all $\vec{u} \in E$, $d\Phi(p)(\vec{u})$ exists (in F) then Φ is called Gâteaux differentiable at p.

Exercice T.3 Prove: If Φ is Gâteaux differentiable at p then $d\Phi(p)$ is homogeneous, i.e., $d\Phi(p)(\lambda\vec{u}) = \lambda\, d\Phi(p)(\vec{u})$ for all $\vec{u} \in E$ and all $\lambda \in \mathbb{R}$.

Answer. $\lim_{h\to 0} \frac{\Phi(p+h(\lambda\vec{u}))-\Phi(p)}{h} = \lambda \lim_{h\to 0} \frac{\Phi(p+\lambda h\vec{u})-\Phi(p)}{\lambda h} = \lambda \lim_{k\to 0} \frac{\Phi(p+k\vec{u})-\Phi(p)}{k}$. ∎

Definition T.4 If Φ is Gateaux differentiable and if moreover $d\Phi(p)$ is linear and continuous at p, then Φ is said to be differentiable at p (or Fréchet differentiable at p).

In that case (T.4) reads

$$\Phi(q) = \Phi(p) + h\, d\Phi(p).\vec{pq} + o(||\vec{pq}||_E), \tag{T.5}$$

since $d\Phi(p)(\vec{u}) =^{\text{noted}} d\Phi(p).\vec{u}$ for all $\vec{u} \in E$ (linearity of $\Phi(p)$).

And the affine function $\mathit{aff}_p : q \to \mathit{aff}_p(q) := \Phi(p) + d\Phi(p).\vec{pq}$ is the affine approximation of Φ at p. (So, the graph of aff_p is the tangent plane of Φ at p.)

Definition T.5 $\Phi : U \to \mathcal{F}$ is differentiable in U iff Φ is differentiable at all $p \in U$. Then its differential is the map

$$d\Phi : \begin{cases} U & \to \mathcal{L}(E; F) \\ p & \to d\Phi(p). \end{cases} \tag{T.6}$$

And $C^1(U; \mathcal{F})$ is the set of differentiable functions ψ such that $d\Phi \in C^0(U; \mathcal{L}(E; F))$.

And $C^2(U; \mathcal{F})$ is the set of differentiable functions ψ such that $d\Phi \in C^1(U; \mathcal{L}(E; F))$.

... And $C^k(U; \mathcal{F})$ is the set of differentiable functions ψ such that $d\Phi \in C^{k-1}(U; \mathcal{L}(E; F))$....

Proposition T.6 *The differentiation (or derivation) operator* $d : \left\{ \begin{matrix} C^1(U; \mathcal{F}) & \to C^0(U; \mathcal{L}(E; F)) \\ \Phi & \to d\Phi \end{matrix} \right\}$ *is* $\mathbb{R}$*-linear ("a derivation is linear").*

Proof. $d(\Phi + \lambda\Psi)(p).\vec{u} = \lim_{h\to 0} \frac{(\Phi+\lambda\Psi)(p+h\vec{u})-(\Phi+\lambda\Psi)(p)}{h} = \lim_{h\to 0} \frac{\Phi(p+h\vec{u})-\Phi(p)+\lambda\Psi(p+h\vec{u})-\lambda\Psi(p)}{h} = \lim_{h\to 0} \frac{\Phi(p+h\vec{u})-\Phi(p)}{h} + \lambda \lim_{h\to 0} \frac{\Psi(p+h\vec{u})-\Psi(p)}{h} = d\Phi(p).\vec{u} + \lambda d\Psi(p).\vec{u} = (d\Phi(p) + \lambda d\Psi(p)).\vec{u}$ for all p and $\vec{u}$, thus $d(\Phi + \lambda\Psi) = d\Phi + \lambda d\Psi$ for all $\lambda \in \mathbb{R}$ and $\Phi, \Psi \in C^1(U; \mathcal{F})$. ∎

Exercice T.7 Prove: if $f \in C^1(U; \mathbb{R})$ (scalar values) and $\Phi \in C^1(U; \mathcal{F})$ then, for all $\vec{u} \in E$,

$$d(f\Phi).\vec{u} = (df.\vec{u})\Phi + f(d\Phi.\vec{u}) \tag{T.7}$$

(also written $d(f\Phi) = \Phi \otimes df + f\, d\Phi$ for a use with contraction rules).

Answer.

$$\begin{aligned} d(f\Phi)(p).\vec{u} &= \lim_{h\to 0} \frac{f(p+h\vec{u})\Phi(p+h\vec{u}) - f(p)\Phi(p)}{h} \\ &= \lim_{h\to 0} \frac{f(p+h\vec{u})\Phi(p+h\vec{u}) - f(p)\Phi(p+h\vec{u})}{h} + \frac{f(p)\Phi(p+h\vec{u}) - f(p)\Phi(p)}{h} \\ &= \lim_{h\to 0} \frac{f(p+h\vec{u}) - f(p)}{h}(\Phi(p) + o(1)) + \lim_{h\to 0} f(p)\frac{\Phi(p+h\vec{u}) - \Phi(p)}{h} \\ &= (df(p).\vec{u})\Phi(p) + f(p)(d\Phi(p).\vec{u}). \end{aligned} \tag{T.8}$$

∎

Remark T.8 In differential geometry, the tangent map is

$$T\Phi : \begin{cases} U \times E & \to \mathcal{F} \times F \\ (p, \vec{u}) & \to T\Phi(p, \vec{u}) = (\Phi(p), d\Phi(p).\vec{u}). \end{cases} \tag{T.9}$$

The two points p (input) and $\Phi(p)$ (output) are the base points, and the two vectors $\vec{u}$ (input) and $d\Phi(p).\vec{u}$ (output) are the initial vector and its push-forward by Φ. ∎

T.1.3 Notation for the second order Differential

Let $\Phi \in C^2(U;\mathcal{F})$; Thus $d\Phi \in C^1(U;\mathcal{L}(E;F))$, thus $d(d\Phi) \in C^0(U;\mathcal{L}(E;\mathcal{L}(E;F)))$; So, for $p \in U$ and $\vec{u} \in E$, we have $d(d\Phi)(p).\vec{u} = \lim_{h\to 0} \frac{d\Phi(p+h\vec{u})-d\Phi(p)}{h} \in \mathcal{L}(E;F)$, and, with $\vec{v} \in E$ we have $(d(d\Phi)(p).\vec{u}).\vec{v} \in F$.

The bilinear map $d^2\Phi(p) \in \mathcal{L}(E,E;F)$ is defined by

$$d^2\Phi(p)(\vec{u},\vec{v}) = (d(d\Phi)(p).\vec{u}).\vec{v}, \tag{T.10}$$

thanks to the natural canonical isomorphism $L \in \mathcal{L}(E;\mathcal{L}(E;F)) \leftrightarrow T_L \in \mathcal{L}(E,E;F)$ given by $T_L(\vec{u}_1,\vec{u}_2) := (L.\vec{u}_1).\vec{u}_2$ for all $\vec{u}_1,\vec{u}_2 \in E$; Thus $L =^{\text{noted}} T_L$, thus $d(d\Phi) =^{\text{noted}} d^2\Phi(p) \in \mathcal{L}(E,E;F)$.

This gives the usual second order Taylor expansion of Φ (supposed C^2) near p in the direction $\vec{u}$:

$$\Phi(p+h\vec{u}) = \Phi(p) + h\, d\Phi(p).\vec{u} + \frac{h^2}{2}\, d^2\Phi(p)(\vec{u},\vec{u}) + o(h^2) \tag{T.11}$$

(=the second order Taylor expansion of $f : h \to f(h) = \Phi(p+h\vec{u})$ near $h=0$, cf. (T.3)).

And Schwarz's theorem tells: If Φ is C^2 then $d^2\Phi(p)$ is symmetric, i.e. $d^2\Phi(p)(\vec{u},\vec{v}) = d^2\Phi(p)(\vec{v},\vec{u})$.

T.2 A basis and the j-th partial derivative (subjective)

Definition T.9 Let $\Phi \in C^1(U;\mathcal{F})$, $\vec{u} \in \Gamma(U)$ (a vector field), $p \in U$. The derivative of Φ at p along $\vec{u}$ is defined by

$$\partial_{\vec{u}}\Phi(p) := d\Phi(p).\vec{u}(p) \overset{\text{noted}}{=} \frac{\partial\Phi}{\partial\vec{u}}(p) \quad (= \lim_{h\to 0} \frac{\Phi(p+h\vec{u}(p)) - \Phi(p)}{h} \in F). \tag{T.12}$$

This defines the directional derivative operator along $\vec{u}$:

$$\partial_{\vec{u}} : \begin{cases} C^1(U;\mathcal{F}) & \to C^0(U;F) \\ \Phi & \to \partial_{\vec{u}}(\Phi) := d\Phi.\vec{u}, \quad \text{so} \quad \partial_{\vec{u}}(\Phi)(p) := d\Phi(p).\vec{u}(p). \end{cases} \tag{T.13}$$

(And $\partial_{\vec{u}}(\Phi) =^{\text{noted}} \vec{u}(\Phi)$ in differential geometry thanks to $E \simeq E^{**}$ which gives $\partial_{\vec{u}} \simeq \vec{u}$.)

In particular, if $(\vec{e}_i(p))$ is a basis at p, then the j-th partial derivative of Φ at p is

$$\partial_{\vec{e}_j}\Phi(p) := d\Phi(p).\vec{e}_j(p) = \frac{\partial\Phi}{\partial\vec{e}_j}(p) \overset{\text{noted}}{=} \partial_j\Phi(p) \overset{\text{noted}}{=} \Phi_{|j}(p) \quad (= \lim_{h\to 0} \frac{\Phi(p+h\vec{e}_j(p)) - \Phi(p)}{h}), \tag{T.14}$$

and the j-th directional derivative operator is

$$\partial_{\vec{e}_j} = \partial_j = \frac{\partial}{\partial\vec{e}_j} : \begin{cases} C^1(U;\mathcal{F}) & \to C^0(U;F) \\ \Phi & \to \boxed{\partial_j\Phi := d\Phi.\vec{e}_j} = \partial_{\vec{e}_j}\Phi = \frac{\partial\Phi}{\partial\vec{e}_j} = \Phi_{|j}. \end{cases} \tag{T.15}$$

Moreover if U is an open set in the vector space E, if $p =^{\text{noted}} \vec{x} = \sum_{i=1}^n x_i\vec{e}_i \in E$, if $(\vec{e}_i)$ is a Cartesian basis in E, then for any we have the usual notation:

$$\partial_{\vec{e}_j}\Phi(\vec{x}) \overset{\text{noted}}{=} \frac{\partial\Phi}{\partial x_j}(\vec{x}), \quad \text{i.e.} \quad \partial_{\vec{e}_j}\Phi \overset{\text{noted}}{=} \frac{\partial\Phi}{\partial x_j}. \tag{T.16}$$

Warning: This notation $\frac{\partial}{\partial x_j}$ is ambiguous since it depends on the name of a component.

T.3 Application 1: Scalar valued functions

T.3.1 Differential of a scalar valued function (objective)

Here $\Phi \stackrel{\text{noted}}{=} f : \left\{ \begin{matrix} U & \to \mathbb{R} \\ p & \to f(p) \end{matrix} \right\}$ is a C^1 scalar valued function, so $df \in \Omega^1(U) \cap C^0(U; E^*)$ (a C^0 differential form). So $df(p) \in E^*$ for all $p \in U$, and $df(p).\vec{u} = \lim_{h\to 0} \frac{f(p+h\vec{u})-f(p)}{h} \in \mathbb{R}$ for all $\vec{u} \in E$.

Exercice T.10 Prove: If $f, g \in C^1(U;\mathbb{R})$ then (derivative of a product)

$$d(fg) = (df)g + f(dg), \tag{T.17}$$

i.e., $d(fg).\vec{w} = (df.\vec{w})g + f(dg.\vec{w})$ for all $\vec{w} \in \Gamma(U)$.

Answer. $\lim_{h\to 0} \frac{f(p+h\vec{w})g(p+h\vec{w})-f(p)g(p)}{h} = \lim_{h\to 0} \frac{f(p+h\vec{w})g(p+h\vec{w})-f(p)g(p+h\vec{w})}{h} + \lim_{h\to 0} \frac{f(p)g(p+h\vec{w})-f(p)g(p)}{h} = \lim_{h\to 0} \frac{f(p+h\vec{w})-f(p)}{h}(g(p)+o(1)) + \lim_{h\to 0} f(p)\frac{g(p+h\vec{w})-g(p)}{h}$, calculation that only requires the first order (affine) approximation of f and g: We get the same result as with the affine functions $f(x) = a_0+a_1x$ and $g(x) = b_0+b_1x$, which give $(fg)(x) = a_0b_0 + (a_0b_1+a_1b_0)x + a_1b_1x^2$, and then $(fg)'(x) = a_0b_1+a_1b_0 + 2a_1b_1x$, which is indeed equal to $(f'g + fg')(x) = a_1(b_0+b_1x) + (a_0+a_1x)b_1$. ▪

T.3.2 Quantification ...

If $(\vec{e}_i(p))$ is a basis at p, then

$$df(p).\vec{e}_j(p) \stackrel{(T.15)}{=} \partial_{\vec{e}_j} f(p) \stackrel{\text{noted}}{=} \partial_j f(p) \stackrel{\text{noted}}{=} f_{|j}(p). \tag{T.18}$$

Thus, with $(\pi_{ei}(p))$ the dual basis of the basis $(\vec{e}_i(p))$,

$$df(p) = \sum_{j=1}^{n} f_{|j}(p)\pi_{ej}(p) \quad \text{and} \quad [df(p)]_{|\vec{e}} = (\, f_{|1}(p) \quad \dots \quad f_{|n}(p)\,) \quad \text{(row matrix)}. \tag{T.19}$$

Duality notations: $\pi_{ei} = e^i$, $\vec{u} = \sum_{j=1}^n u^j \vec{e}_j$, $df = \sum_{j=1}^n f_{|j}\, e^j$, $df.\vec{u} = \sum_{j=1}^n f_{|j} u^j$.

Interpretation. In $E = \mathbb{R}^n$, call $c_{pi} : h \in [-\varepsilon, \varepsilon] \to c_{pi}(h) = p + h\vec{e}_i(p) \in \vec{\mathbb{R}^n}$ the i-th coordinate line at p: Hence $c_{pi}'(0) = \vec{e}_i(p)$ is the tangent vector at $p = c_{pi}(0)$ to $\text{Im}(c_{pi})$. Thus $(f \circ c_{pi})'(0) = df(p).\vec{e}_i(p)$ is the tangent vector at p to the image $f \circ c_{pi} =^{\text{noted}} f(c_{pi})$ or the i-th coordinate line at p.

Exercice T.11 Prove: $(fg)_{|j} = f_{|j}\, g + f\, g_{|j}$ when $f, g : U \to \mathbb{R}$ are C^1 scalar valued functions.

Answer. Apply (T.8): here $d(fg) = g\, df + f\, dg$, i.e. $d(fg).\vec{e}_j = (df.\vec{e}_j)\, g + f\,(dg.\vec{e}_j)$ for all j. ▪

T.3.3 ... and the notation $\frac{\partial f}{\partial x_i}$....

O is an origin in $\mathcal{E}$, $(\vec{e}_i)$ is a Cartesian basis in E, $(\pi_{ei}) =^{\text{noted}} (dx_i)$ is the (covariant) dual basis, $p =^{\text{noted}} \overrightarrow{Op} = \vec{x} = \sum_{i=1}^n x_i \vec{e}_i$. Then (unmissable in thermodynamics)

$$\partial_i f \stackrel{\text{noted}}{=} \frac{\partial f}{\partial x_j}, \quad \text{so} \quad df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, dx_j, \tag{T.20}$$

i.e. $df(\vec{x}) = \sum_j \frac{\partial f}{\partial x_j}(\vec{x})\, dx_j$, i.e. $df(\vec{x}).\vec{u} = \sum_j \frac{\partial f}{\partial x_j}(\vec{x}) u_j$ for all $\vec{u} = \sum_j u_j \vec{e}_j \in E$. And $\partial_i f(p) = df(p).\vec{e}_i$ is the derivative along the i-th Cartesian coordinate line at $\vec{x}$.

(Duality notations: $df = \sum_j \frac{\partial f}{\partial x^j}\, dx^j$ and $df(\vec{x}).\vec{u} = \sum_j \frac{\partial f}{\partial x^j}(\vec{x}) u^j$.)

T.3.4 ... is subjective

An English observer chooses a Euclidean basis $(\vec{a}_i)$ made with the foot, writes $\vec{x} = \sum_i x_i \vec{a}_i$ and uses $\frac{\partial f}{\partial x_i}$.
A French observer chooses a Euclidean basis $(\vec{b}_i)$ made with the metre, writes $\vec{x} = \sum_i x_i \vec{b}_i$ and uses $\frac{\partial f}{\partial x_i}$.
But the English $\frac{\partial f}{\partial x_i}$ is not equal to the French $\frac{\partial f}{\partial x_i}$.

Indeed, if $\vec{x} = \sum_i x_{a,i}\vec{a}_i = \sum_i x_{b,i}\vec{b}_i$, then $\frac{\partial f}{\partial x_{a,i}}(p) = df(p).\vec{a}_i$ while $\frac{\partial f}{\partial x_{b,i}}(p) = df(\vec{x}).\vec{b}_i$, and e.g.

$$\text{if } \vec{b}_i = \lambda\vec{a}_i, \ \forall i, \quad \text{then} \quad \boxed{\frac{\partial f}{\partial x_{b,i}} = \lambda\frac{\partial f}{\partial x_{a,i}}} \quad \text{(change of unit formula)}, \tag{T.21}$$

since $df(p).\vec{b}_j = df(p).(\lambda\vec{a}_j) = \lambda\, df(p).\vec{a}_j$ (linearity of $df(\vec{x})$). (Duality notations: $\frac{\partial f}{\partial x_b^j} = \lambda\frac{\partial f}{\partial x_a^j}$.)

More generally, with P the transition matrix from $(\vec{a}_i(p))$ to $(\vec{b}_i(p))$, we have $[df(p)]_{|\vec{b}} = [df(p)]_{|\vec{a}}.P(p)$ (change of basis formula for linear forms):

$$[df]_{|\vec{b}} = [df]_{|\vec{a}}.P, \quad \text{i.e.} \quad \frac{\partial f}{\partial x_{b,j}} = \sum_{i=1}^n \frac{\partial f}{\partial x_{a,i}}P_{ij} \quad \text{written} \quad \frac{\partial f}{\partial x_{b,j}} = \sum_{i=1}^n \frac{\partial f}{\partial x_{a,i}}\frac{\partial x_{a,i}}{\partial x_{b,j}}. \tag{T.22}$$

(Duality notations: $\frac{\partial f}{\partial x_b^j} = \sum_{i=1}^n \frac{\partial f}{\partial x_a^i}P^i_j$, written $\frac{\partial f}{\partial x_b^j} = \sum_{i=1}^n \frac{\partial f}{\partial x_a^i}\frac{\partial x_a^i}{\partial x_b^j}$.)

Remark T.12 Why this last notation $P_{ij} =^{\text{noted}} \frac{\partial x_{a,i}}{\partial x_{b,j}}$?
Answer : $[\vec{x}]_{|\vec{a}} = P.[\vec{x}]_{|\vec{b}}$, tells that $[\vec{x}]_{|\vec{a}}$ is a function of $[\vec{x}]_{|\vec{b}}$, so is written $[\vec{x}]_{|\vec{a}}([\vec{x}]_{|\vec{b}}) = P.[\vec{x}]_{|\vec{b}}$, so

$$\begin{pmatrix} x_a^1(x_b^1,...,x_b^n) \\ \vdots \\ x_a^1(x_b^1,...,x_b^n) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n P^1_j x_b^j \\ \vdots \\ \sum_{j=1}^n P^n_j x_b^j \end{pmatrix}, \quad \text{thus} \quad \frac{\partial x_a^i}{\partial x_b^j}(x_b^1,...,x_b^n) = P^i_j, \ \forall i,j. \tag{T.23}$$

More details: With an origin $O \in \mathcal{E}$ and $\vec{x} = \overrightarrow{Op}$, define $f_a, f_b \in C^1(\mathcal{M}_{n1};\mathbb{R})$ by $f_a([\vec{x}]_{|\vec{a}}) := f(p)$ and $f_b([\vec{x}]_{|\vec{b}}) := f(p)$. Thus $f_b([\vec{x}]_{|\vec{b}}) = f_a([\vec{x}]_{|\vec{a}}) = (f_a \circ [\vec{x}]_{|\vec{a}})([\vec{x}]_{|\vec{b}})$, hence (T.22) should be written (with no abusive notations):

$$\frac{\partial f_b}{\partial x_b^i}([\vec{x}]_{|\vec{b}}) = \sum_{j=1}^n \frac{\partial f_a}{\partial x_a^j}([\vec{x}]_{|\vec{a}})\frac{\partial x_a^j}{\partial x_b^i}([\vec{x}]_{|\vec{b}}). \tag{T.24}$$

Question: Why did we need to introduce f_a and f_b (and not just keep f)? Answer: Because $\vec{x} \in \mathbb{R}^n$ while $[\vec{x}]_{|\vec{a}}, [\vec{x}]_{|\vec{b}} \in \mathcal{M}_{n1}$ and $[\vec{x}]_{|\vec{a}} \neq [\vec{x}]_{|\vec{b}}$: A vector $\vec{x}$ can't be reduced to a matrix of components (which one?). ▪

T.3.5 Gradient (subjective: requires some inner dot product)

Let $f \in C^1(U;\mathbb{R})$ (a C^1 scalar valued function). Choose (subjective) an inner dot product $(\cdot,\cdot)_g$ in E.

Definition T.13 The $(\cdot,\cdot)_g$-conjugate gradient $\vec{\text{grad}}_g f(p) =^{\text{noted}} \vec{\nabla}_g f(p)$ of f at $p \in U$ relative to $(\cdot,\cdot)_g$ is the vector in E defined by

$$\forall \vec{u} \in E, \quad \boxed{df(p).\vec{u} = (\vec{\text{grad}}_g f(p), \vec{u})_g} = \vec{\text{grad}}_g f(p) \bullet_g \vec{u} \overset{\text{noted}}{=} \vec{\nabla}_g f(p) \bullet_g \vec{u}. \tag{T.25}$$

If an inner dot product $(\cdot,\cdot)_g$ is imposed then $\vec{\text{grad}}_g f =^{\text{noted}} \vec{\text{grad}} f = \vec{\nabla} f$ is called the gradient of f.

So $\vec{\text{grad}}_g f(p) \overset{(F.3)}{=} \vec{R}_g(df(p))$ is the $(\cdot,\cdot)_g$-Riesz representation vector in E of the linear form $df(p) \in E^*$.

Fundamental: An English observer with his foot, his Euclidean basis $(\vec{a}_i)$ and associated Euclidean dot product $(\cdot,\cdot)_a$, and a French observer with his metre, his Euclidean basis $(\vec{b}_i)$ and associated Euclidean dot product $(\cdot,\cdot)_b$: They do ***not*** have the same gradient. E.g. if $(\vec{b}_i) = (\lambda\vec{a}_i)$ then

$$\vec{\text{grad}}_b f \overset{(F.13)}{=} \lambda^2 \vec{\text{grad}}_a f \quad \text{with} \quad \lambda^2 > 10. \tag{T.26}$$

$\vec{\text{grad}}_b f$ is quite different from $\vec{\text{grad}}_a f$ isn't it? And to forget this fact leads to accidents like the crash of the Mars Climate Orbiter probe, cf. remark A.17.

Subjective first order Taylor expansion: If an inner dot product $(\cdot,\cdot)_g$ exists and is used, then the first order Taylor expansion (T.4) gives

$$f(p+h\vec{u}) = f(p) + h\,(\vec{\text{grad}}_g f(p), \vec{u})_g + o(h) \qquad (= f(p) + h\,\vec{\text{grad}}_g f(p) \bullet_g \vec{u} + o(h)). \tag{T.27}$$

Fundamental once again (we insist):

• An inner dot product does not always exist (as a meaningful tool), see § B.4 (thermodynamics), thus, for a C^1 function, a gradient does not always exists (contrary to a differential).

• $df(p)$ is a linear form (covariant) while $\vec{\text{grad}}_g f(p)$ is a vector (contravariant). In particular the change of basis formulas differ, cf. (A.29):

$$[df]_{|new} = [df]_{|old}.P, \quad \text{while} \quad [\vec{\text{grad}}_g f]_{|new} = P^{-1}.[\vec{\text{grad}}_g f]_{|old}. \tag{T.28}$$

• df cannot be identified with $\vec{\text{grad}} f$ (with one?) (Recall: there is no natural canonical isomorphims between E and E^*.) Vocabulary: The differential df is also called the "covariant gradient of f", while the vector $\vec{\text{grad}}_g f$ is called the "contravariant gradient of f relative to $(\cdot,\cdot)_g$".

Isometric Euclidean framework: If one Euclidean dot product is imposed to all observers (foot? metre?) then $\vec{\text{grad}}_g f \overset{\text{noted}}{=} \vec{\text{grad}} f = \vec{\nabla} f$ and (T.25) is written $df.\vec{u} = \vec{\text{grad}} f \bullet \vec{u} = \vec{\nabla} f \bullet \vec{u}$.

Exercice T.14 Cartesian basis $(\vec{e}_i)$ and $(\cdot,\cdot)_g$ given by $[g]_{|\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Give $[\vec{\text{grad}}_g f]_{|\vec{e}}$.

Answer. $[df]_{|\vec{e}} = (\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2})$ (row matrix), thus (T.25) gives $[\vec{\text{grad}}_g f]_{|\vec{e}} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{1}{2}\frac{\partial f}{\partial x_2} \end{pmatrix}$ (column matrix $\neq [df]^T$). ▪

T.4 Application 2: Coordinate system basis and Christoffel symbols

(Needed when dealing with covariance.)

T.4.1 Coordinate system, and coordinate system basis

$(\vec{A}_i)$ is the canonical basis of the Cartesian vector space $\mathbb{R} \times ... \times \mathbb{R}$ (n times), $U_{par} =]a_1, b_1[\times...\times]a_n, b_n[$ is an non empty open set called the set of parameters, $\vec{q} = \sum_i q_i \vec{A}_i = (q_1, ..., q_n) \in U_{par}$.

O is an origin in the affine geometric space $\mathbb{R}^n$, $(\vec{a}_i)$ is a Cartesian basis in $\vec{\mathbb{R}^n}$, $\vec{x} = \overrightarrow{Op} \in \vec{\mathbb{R}^n}$ for all $p \in \mathbb{R}^n$, $U = \{\vec{x} \in \vec{\mathbb{R}^n}\}$ is an open set, and $\Psi : \vec{q} \in U_{par} \to \vec{x} \in U$ is a C^2-diffeomorphism called a coordinate system.

Ψ being a diffeomorphism, at any $\vec{x} = \Psi(\vec{q}) \in U$, the basis $(\vec{e}_i(\vec{x}))$ defined by, for all i,

$$\vec{e}_i(\vec{x}) := d\Psi(\vec{q}).\vec{A}_i, \tag{T.29}$$

is called the coordinate system basis at $\vec{x}$. Its dual basis at $\vec{x}$ is made of the linear forms $\pi_{ei}(\vec{x}) \overset{\text{noted}}{=} dq_i(\vec{x}) \in \mathbb{R}^{n*}$ defined by, for all i, j,

$$dq_i(\vec{x}).\vec{e}_j(\vec{x}) = \delta_{ij} \quad (= e^i(\vec{x}).\vec{e}_j(\vec{x})). \tag{T.30}$$

Duality notations: $e^i(\vec{x}) \overset{\text{noted}}{=} dq^i(\vec{x})$, thus $dq^i(\vec{x}).\vec{e}_j(\vec{x}) = \delta^i_j$ for all i, j.

Remark T.15 Pay attention to the notations that could contradict themselves: In U_{par} the dual basis (π_{Ai}) of the Cartesian basis $(\vec{A}_i)$ is a uniform basis (independent of $\vec{q}$)... and is (almost) never written (dq_i), because the notation dq_i is used for the dual basis cf. (T.30). Historical notations...

E.g., cf. the polar coordinate system at § 6.6.2: $\vec{q} = r\vec{A}_1 + \theta\vec{A}_2 \overset{\text{noted}}{=} (r, \theta) = (q_1, q_2)$, and $\vec{e}_i(\vec{x}) := d\psi(\vec{q}).\vec{A}_i$ at $\vec{x} = \Psi(\vec{q})$ is the polar basis, and $(dq_1(\vec{x}), dq_2(\vec{x})) = (dr(\vec{x}), d\theta(\vec{x}))$ is the dual basis at $\vec{x}$. ▪

T.4.2 Parametric expression of a differential

A function $f : \left\{ \begin{array}{ll} U & \to \mathbb{R} \\ \vec{x} & \to f(\vec{x}) \end{array} \right\}$ can be studied with $g := f \circ \Psi : \left\{ \begin{array}{ll} U_{par} & \to \mathbb{R} \\ \vec{q} & \to g(\vec{q}) := f(\vec{x}) \text{ when } \vec{x} = \Psi(\vec{q}) \end{array} \right\}$ thanks to Ψ (diffeomorphism). In particular, if f is C^1,

$$dg(\vec{q}) = df(\vec{x}).d\Psi(\vec{q}) \quad \text{when} \quad \vec{x} = \Psi(\vec{q}) \tag{T.31}$$

so for all j,

$$\frac{\partial(f \circ \Psi)}{\partial q_j}(\vec{q}) = \frac{\partial g}{\partial q_j}(\vec{q}) := dg(\vec{q}).\vec{A}_j = df(\vec{x}).d\Psi(\vec{q}).\vec{A}_j = df(\vec{x}).\vec{e}_j(\vec{x}) \overset{\text{noted}}{=} \frac{\partial f}{\partial q_j}(\vec{x}) \;... \tag{T.32}$$

... **Warning (notations!):** f is a function of $\vec{x}$, not of $\vec{q}$ (!), and the notations $\frac{\partial f}{\partial q_j}(\vec{x})$ means $:= \frac{\partial(f\circ\Psi)}{\partial q_j}(\vec{q})$ when $\vec{x} = \Psi(\vec{q})$, and nothing else. Historical notations...

Thus with $(dq_j(\vec{x}))$ the dual basis of the coordinate basis $(\vec{e}_i(\vec{x})) = (\vec{A}_{i*}(\vec{x}))$ at $\vec{x}$,

$$df(\vec{x}) = \sum_{j=1}^{n} \frac{\partial f}{\partial q_j}(\vec{x})\, dq_j(\vec{x}). \tag{T.33}$$

(Check: $(\sum_{j=1}^{n} \frac{\partial f}{\partial q_j}(\vec{x})\, dq_j(\vec{x})).\vec{e}_i(\vec{x}) = \sum_{j=1}^{n} \frac{\partial f}{\partial q_j}(\vec{x})\, (dq_j(\vec{x}).\vec{e}_i(\vec{x}) = \frac{\partial f}{\partial q_j}(\vec{x}) =^{(T.32)} df(\vec{x}).\vec{e}_j(\vec{x})$.)

Duality notations: $df(\vec{x}) = \sum_j \frac{\partial f}{\partial q^j}(\vec{x})\, dq^j(\vec{x})$.

T.4.3 Christoffel symbols

We use duality notations for readability.

Definition T.16 In a coordinate system basis $(\vec{e}_i(\vec{x}))$ in E, the Christoffel symbol $\gamma_{jk}^i(\vec{x}) \in \mathbb{R}$ are the components of the vector $d\vec{e}_k(p).\vec{e}_j(p)$, i.e. $d\vec{e}_k(\vec{x}).\vec{e}_j(\vec{x}) = \sum_{k=1}^{n} \gamma_{jk}^i(\vec{x})\vec{e}_i(\vec{x})$: In U,

$$\boxed{d\vec{e}_k.\vec{e}_j = \sum_{i=1}^{n} \gamma_{jk}^i \vec{e}_i}, \quad \text{or} \quad d\vec{e}_j.\vec{e}_i = \sum_{k=1}^{n} \gamma_{ij}^k \vec{e}_k. \tag{T.34}$$

So, with $(e^i(\vec{x}))$ the dual basis of $(\vec{e}_i(\vec{x}))$, $\gamma_{jk}^i := e^i.d\vec{e}_k.\vec{e}_j$, and, for calculations with contractions, $d\vec{e}_k =^{\text{noted}} \sum_{ij} \gamma_{jk}^i \vec{e}_i \otimes e^j$ (or $d\vec{e}_j = \sum_{k=1}^{n} \gamma_{ij}^k \vec{e}_k \otimes e^i$).

(The Christoffel symbols vanish if $(\vec{e}_i)$ is Cartesian.)

(Differential geometry in manifolds: The $\gamma_{jk}^i = e^i.\nabla_{\vec{e}_j}\vec{e}_k$ are the component of the connection ∇, the usual connection in a surface in $\mathbb{R}^n$ being the Riemannian connection, in which case $\nabla_{\vec{e}_j}\vec{e}_k$ is the orthogonal projection of $d\vec{e}_k.\vec{e}_j$ on the surface relative to a Euclidean dot product. E.g. in $\mathbb{R}^n$ the usual connection ∇ is nothing but the differential d: It is ***not*** the gradient $\vec{\nabla}$...! In facts, $[\vec{\text{grad}}\varphi]_{|\vec{e}} = [\nabla\varphi]_{|\vec{e}}^T = [d\varphi]_{|\vec{e}}^T$ once an inner dot product and a basis have been chosen (hence confusions in mechanics).

Exercice T.17 Polar coordinate system, see remark 6.12, $d\vec{e}_2.\vec{e}_2 = -r\vec{e}_1$, thus $\gamma_{22}^1 = -r$ and $\gamma_{22}^2 = 0$. ∴

Exercice T.18 Prove: If $(\vec{e}_i(\vec{x}))$ is the coordinate system basis of a C^2 coordinate system, then:

$$\forall i,j,\ d\vec{e}_i.\vec{e}_j = d\vec{e}_j.\vec{e}_i \quad (= \frac{\partial^2\Psi}{\partial q^i \partial q^j}), \quad \text{and} \quad \forall i,j,k,\ \gamma_{ji}^k = \gamma_{ij}^k \quad \text{(symmetry for lower indices)}. \tag{T.35}$$

Answer. $\vec{e}_i(\vec{x}) = (\vec{e}_i \circ \Psi)(\vec{q}) = d\Psi(\vec{q}).\vec{A}_i$ gives $d\vec{e}_i(\vec{x}).\vec{e}_i(\vec{x}) = d\vec{e}_i(\Psi(\vec{q})).d\Psi(\vec{q}).\vec{A}_j = d(d\Psi(\vec{q}).\vec{A}_i).\vec{A}_j = \frac{\partial \frac{\partial\Psi}{\partial q^i}}{\partial q^j} = \frac{\partial \frac{\partial\Psi}{\partial q^j}}{\partial q^i}$ (Schwarz theorem since Ψ is C^2) $= d\vec{e}_j(\vec{x}).\vec{e}_i(\vec{x}) = \frac{\partial^2\Psi}{\partial q^j \partial q^i}(\vec{q})$, thus $\sum_{k=1}^{n} \gamma_{ij}^k \vec{e}_k = \sum_{k=1}^{n} \gamma_{ji}^k \vec{e}_k$. ∴

Exercice T.19 Consider two coordinate system bases $(\vec{a}_i(\vec{x}))$ and $(\vec{b}_i(\vec{x}))$ at $\vec{x}$, $P(\vec{x}) = [P_j^i(\vec{x})]$ the transition matrix from $(\vec{a}_i(\vec{x}))$ to $(\vec{b}_i(\vec{x}))$, and $Q = P^{-1}$. Using the generic notation $d\vec{e}_k.\vec{e}_j = \sum_{i=1}^{n} \gamma_{jk,e}^i \vec{e}_i$, prove the change of basis formula for the Christoffel symbols:

$$\gamma_{jk,b}^i = \sum_{\lambda,\mu,\nu=1}^{n} Q_\lambda^i P_j^\mu P_k^\nu \gamma_{\mu\nu,a}^\lambda + \sum_{\lambda,\mu=1}^{n} Q_\lambda^i P_j^\mu (dP_k^\lambda.\vec{a}_\mu) \quad (= \sum_{\lambda,\mu,\nu=1}^{n} Q_\lambda^i P_j^\mu P_k^\nu \gamma_{\mu\nu,a}^\lambda + \sum_{\lambda=1}^{n} Q_\lambda^i (dP_k^\lambda.\vec{b}_j)). \tag{T.36}$$

(Because of the term $\sum_{\mu\nu} Q_\lambda^i P_j^\mu (dP_k^\lambda.\vec{a}_\mu)$, a derivation is not a tensor.)

Answer. $\vec{b}_k(\vec{x}) = \sum_\nu P_k^\nu(\vec{x})\vec{a}_\nu(\vec{x})$ gives $d\vec{b}_k.\vec{b}_j = \sum_\nu (dP_k^\nu.\vec{b}_j)\vec{a}_\nu + \sum_\nu P_k^\nu (d\vec{a}_\nu.\vec{b}_j) = \sum_{\mu\nu} P_j^\mu (dP_k^\nu.\vec{a}_\mu)\vec{a}_\nu + \sum_{\mu\nu} P_k^\nu P_j^\mu (d\vec{a}_\nu.\vec{a}_\mu)$; And $b^i = \sum_\lambda Q_\lambda^i a^\lambda$, thus

$$\gamma_{jk,b}^i = b^i.d\vec{b}_k.\vec{b}_j = \sum_{\lambda\mu\nu} Q_\lambda^i P_j^\mu (dP_k^\nu.\vec{a}_\mu) a^\lambda.\vec{a}_\nu + \sum_{\lambda\mu\nu} Q_\lambda^i P_j^\mu P_k^\nu a^\lambda.(d\vec{a}_\nu.\vec{a}_\mu) = \sum_{\lambda\mu} Q_\lambda^i P_j^\mu (dP_k^\lambda.\vec{a}_\mu) + \sum_{\lambda\mu\nu} Q_\lambda^i P_j^\mu P_k^\nu \gamma_{\mu\nu,a}^\lambda,$$

thus (T.36). ∴

T.5 Application 3: Differential of a vector field

Here $F = E = \vec{\mathbb{R}^n}$, $\Phi =^{\text{noted}} \vec{w} \in \Gamma(U)$ is a vector field. Thus $d\vec{w}(p) \in \mathcal{L}(E;E)$ and $d\vec{w}.\vec{u}$ is a vector field in E for all $\vec{u} \in \Gamma(U)$, given by $(d\vec{w}.\vec{u})(p) = d\vec{w}(p).\vec{u}(p) = \lim_{h\to 0} \frac{\vec{w}(p+h\vec{u}(p))-\vec{w}(p)}{h} \in E$.

Quantification: $(\vec{e}_i(p))$ is a basis at p in E. Call $w_i(p) \in \mathbb{R}$ the components of $\vec{w}(p)$, i.e. $\vec{w}(p) = \sum_{i=1}^n w_i(p)\vec{e}_i(p)$. And call $w_{i|j}(p)$ the components of $d\vec{w}(p)$ (endomorphism in E):

$$\vec{w} = \sum_{i=1}^n w_i\vec{e}_i, \quad d\vec{w}.\vec{e}_j = \sum_{i=1}^n w_{i|j}\vec{e}_i, \quad [d\vec{w}]_{|\vec{e}} = [w_{i|j}] \quad \text{(Jacobian matrix).} \tag{T.37}$$

And tensorial notations for calculations with contractions: $(\pi_{ei}(p))$ being the dual basis,

$$d\vec{w} = \sum_{i,j=1}^n w_{i|j}\vec{e}_i \otimes \pi_{ej}. \tag{T.38}$$

Duality notations: $\vec{w} = \sum_{i=1}^n w^i\vec{e}_i$, $d\vec{w}.\vec{e}_j = \sum_{i,j=1}^n w^i_{|j}\vec{e}_i$, $[d\vec{w}]_{|\vec{e}} = [w^i_{|j}]$, and $d\vec{w} = \sum_{i,j=1}^n w^i_{|j}\vec{e}_i \otimes e^j$.

In a Cartesian basis: Here $(\vec{e}_i)$ is uniform, so $\vec{w}(p) = \sum_{i=1}^n w_i(p)\vec{e}_i$ gives $d\vec{w}(p).\vec{e}_j = \sum_{i=1}^n (dw_i(p).\vec{e}_j)\vec{e}_i$, thus (T.37) gives

$$w_{i|j} = \frac{\partial w_i}{\partial x_j}(p) \overset{\text{noted}}{=} w_{i,j}, \quad \text{so} \quad [d\vec{w}]_{|\vec{e}} = [\frac{\partial w_i}{\partial x_j}]. \tag{T.39}$$

Duality notations: $w^i_{|j} = \frac{\partial w^i}{\partial x^j}$ and $[d\vec{w}]_{|\vec{e}} = [\frac{\partial w^i}{\partial x^j}]$.

In a coordinate system basis: With the coordinate system described in § T.4 and the duality notations for readability (and usage). $\vec{w}(p) = \sum_{i=1}^n w^i(p)\vec{e}_i(p)$ gives, for all j,

$$d\vec{w}.\vec{e}_j = \sum_{i=1}^n (dw^i.\vec{e}_j)\vec{e}_i + \sum_{i=1}^n w^i(d\vec{e}_i.\vec{e}_j) \quad (= \sum_{i=1}^n w^i_{|j}\vec{e}_i). \tag{T.40}$$

(Tensorial notations to be used with contractions: $d\vec{w} = \sum_i \vec{e}_i \otimes dw^i + \sum_i w^i\, d\vec{e}_i = \sum_{ij} w^i_{|j}\vec{e}_i \otimes e^j$.)

And $\sum_i w^i(d\vec{e}_i.\vec{e}_j) =^{(T.34)} \sum_{ik} w^i\gamma^k_{ji}\vec{e}_k = \sum_{ik} w^k\gamma^i_{jk}\vec{e}_i$, thus, for all i, j,

$$\boxed{w^i_{|j} = \frac{\partial w^i}{\partial q^j} + \sum_{k=1}^n w^k\gamma^i_{jk}} \quad \text{where} \quad \frac{\partial w^i}{\partial q^j} := dw^i.\vec{e}_j. \tag{T.41}$$

($\frac{\partial w^i}{\partial q^j} := dw^i.\vec{e}_j$ is the derivation along the j-th coordinate line of the scalar valued function w^i).

(In particular, if $\vec{w} = \vec{e}_\ell = \sum_i \delta^i_\ell\vec{e}_i$, we recover $d\vec{e}_\ell.\vec{e}_j = \sum_i 0\vec{e}_i + \sum_{ik} \delta^k_\ell\gamma^i_{jk}\vec{e}_i = \sum_i \gamma^i_{j\ell}\vec{e}_i$, cf. (T.34).)

Exercice T.20 $d\vec{w}(p)$ being an endomorphism, with exercise T.19, $\vec{w} = \sum_i u^i\vec{a}_i = \sum_i v^i\vec{b}_i$ and $Q = P^{-1}$, check (calculations):

$$[d\vec{w}]_{|\vec{b}} = P^{-1}.[d\vec{w}]_{|\vec{a}}.P, \quad \text{i.e.} \quad v^i_{|j} = \sum_{k,\ell=1}^n Q^i_k u^k_{|\ell} P^\ell_j. \tag{T.42}$$

Answer. $[\vec{w}]_{|\vec{b}} = Q.[\vec{w}]_{|\vec{a}}$, i.e. $v^i = \sum_k Q^i_k u^k$ for all i, thus $dv^i.\vec{b}_j = \sum_\lambda (dQ^i_\lambda.\vec{b}_j)u^\lambda + \sum_\lambda Q^i_\lambda(du^\lambda.\vec{b}_j)$, thus

$$\begin{aligned} v^i_{|j} &\overset{(T.34)}{=} dv^i.\vec{b}_j + \sum_k v^k\gamma^i_{jk,b} \\ &\overset{(T.36)}{=} \sum_{\lambda\mu} u^\lambda P^\mu_j(dQ^i_\lambda.\vec{a}_\mu) + \sum_{\lambda\mu} Q^i_\lambda P^\mu_j(du^\lambda.\vec{a}_\mu) + \sum_{k\omega\lambda\mu\nu}(Q^k_\omega u^\omega)Q^i_\lambda P^\mu_j P^\nu_k\gamma^\lambda_{\mu\nu,a} + \sum_{k\lambda\mu\nu}(Q^k_\lambda u^\lambda)Q^i_\nu P^\mu_j(dP^\nu_k.\vec{a}_\mu) \end{aligned}$$

And $Q^k_\omega P^\lambda_k = \delta^\lambda_\omega$ gives $(dQ^k_\omega.\vec{a}_\mu)P^\lambda_k + Q^k_\omega(dP^\lambda_k.\vec{a}_\mu) = 0$, thus the fourth term reads

$$\sum_{k\lambda\mu\nu} u^\lambda Q^i_\nu P^\mu_j Q^k_\lambda(dP^\nu_k.\vec{a}_\mu) = -\sum_{k\lambda\mu\nu} u^\lambda Q^i_\nu P^\mu_j P^\nu_k(dQ^k_\lambda.\vec{a}_\mu) = -\sum_{\lambda\mu} u^\lambda P^\mu_j(dQ^i_\lambda.\vec{a}_\mu),$$

which cancels the first term: Thus $v^i_{|j} = \sum_{\lambda\mu} Q^i_\lambda P^\mu_j(du^\lambda.\vec{a}_\mu) + \sum_{\lambda\mu\nu} u^\nu Q^i_\lambda P^\mu_j\gamma^\lambda_{\mu\nu} = \sum_{\lambda\mu} Q^i_\lambda u^i_{|j} P^\mu_j$, i.e. (T.42). ▪

T.6 Application 4: Differential of a differential form

Here $F=\mathbb{R}$, $\Phi =^{\text{noted}} \ell \in \Omega^1(U)$ (differential form) supposed C^1, $p\in U$, so $\ell(p)\in E^*$. Its differential at p in a direction $\vec{u}$ is $d\ell(p).\vec{u} = \lim_{h\to 0}\frac{\ell(p+h\vec{u})-\ell(p)}{h} \in E^*$. And $(d\ell(p).\vec{u}).\vec{v} = \lim_{h\to 0}\frac{\ell(p+h\vec{u}).\vec{v}-\ell(p).\vec{v}}{h} \in \mathbb{R}$ for all $\vec{u},\vec{v}\in E$.

Quantification: $(\pi_{ei}(p)$ its the dual basis.

Call $\ell_i(p)\in\mathbb{R}$ the components of $\ell(p)$, i.e. $\ell(p)=\sum_{i=1}^n \ell_i(p)\pi_{ei}(p)$. And call $\ell_{i|j}(p)$ the components of $d\ell(p)\in\mathcal{L}(E;E^*)$:

$$\ell=\sum_{i=1}^n \ell_i\pi_{ei},\quad d\ell.\vec{e}_j=\sum_{i=1}^n \ell_{i|j}\pi_{ei},\quad [d\ell]_{|\vec{e}}=[\ell_{i|j}]. \tag{T.43}$$

Tensorial notations, to be used with contractions: $d\ell=\sum_{i,j=1}^n \ell_{i|j}\pi_{ei}\otimes\pi_{ej}$.

Duality notations: $\ell=\sum_i \ell_i e^i$, $d\ell.\vec{e}_j=\sum_{i=1}^n \ell_{i|j}e^i$, $[d\ell]_{|\vec{e}}=[\ell_{i|j}]$, and $d\ell=\sum_{i,j=1}^n \ell_{i|j}e^i\otimes e^j$.

In a Cartesian basis: Here $(\vec{e}_i)$ is uniform, so

$$\ell_{i|j}=\frac{\partial \ell_i}{\partial x_j}(p) \overset{\text{noted}}{=} \ell_{i,j},\quad \text{so}\quad [d\ell]_{|\vec{e}}=[\frac{\partial \ell_i}{\partial x_j}]. \tag{T.44}$$

Duality notations: $\ell_{i|j}=d\ell_i.\vec{e}_j=\frac{\partial \ell_i}{\partial x^j}$ and $[d\ell]_{|\vec{e}}=[\frac{\partial \ell_i}{\partial x^j}]$.

In a coordinate system basis: With duality notations and Christoffel symbols:

$$\boxed{de^i.\vec{e}_j=-\sum_{k=1}^n \gamma^i_{jk}e^k}. \tag{T.45}$$

Indeed, $e^i.\vec{e}_k=\delta^i_k$ gives $(de^i.\vec{e}_j).\vec{e}_k+e^i.(d\vec{e}_k.\vec{e}_j)=0$, thus $(de^i.\vec{e}_j).\vec{e}_k=-e^i.\sum_\ell \gamma^\ell_{jk}\vec{e}_\ell=-\gamma^i_{jk}$. Thus

$$\boxed{\ell_{i|j}=\frac{\partial \ell_i}{\partial q^j}-\sum_{k=1}^n \ell_k\gamma^k_{ji}}\quad \text{where}\quad \frac{\partial \ell_i}{\partial q^j}(p):=d\ell_i(p).\vec{e}_i(p). \tag{T.46}$$

Indeed, $\ell=\sum_i \ell_i e^i$ gives $d\ell.\vec{e}_j=\sum_i (d\ell_i.\vec{e}_j)e^i+\sum_i \ell_i(de^i.\vec{e}_j)=\sum_i(d\ell_i.\vec{e}_j)e^i-\sum_{ik}\ell_i\gamma^i_{jk}e^k$.

T.7 Application 5: Differential of a 1 1 tensor

Consider a C^1 $\binom{1}{1}$ tensor $\underline{\underline{\tau}}:\begin{Bmatrix} U & \to \mathcal{L}(E^*,E;\mathbb{R}) \\ p & \to \underline{\underline{\tau}}(p)\end{Bmatrix}$. Its differential $d\underline{\underline{\tau}}:\begin{Bmatrix} U & \to \mathcal{L}(E;\mathcal{L}(E^*,E;\mathbb{R})) \\ p & \to d\underline{\underline{\tau}}(p)\end{Bmatrix}$ is defined by $d\underline{\underline{\tau}}(p).\vec{u}=\lim_{h\to 0}\frac{\underline{\underline{\tau}}(p+h\vec{u})-\underline{\underline{\tau}}(p)}{h}\in\mathcal{L}(E^*,E;\mathbb{R})$, so $(d\underline{\underline{\tau}}(p).\vec{u})(\ell,\vec{v})=\lim_{h\to 0}\frac{\underline{\underline{\tau}}(p+h\vec{u})(\ell,\vec{v})-\underline{\underline{\tau}}(p)(\ell,\vec{v})}{h}$ $(\in\mathbb{R})$, for all $\vec{u},\vec{v}\in E$ and $\ell\in E^*$.

Quantification (duality notations): Basis $(\vec{e}_i(p))$ in E at p, dual basis $(e^i(p))$, call $\tau^i_j(p)$ the components of $\underline{\underline{\tau}}(p)$, call $\tau^i_{j|k}(p)$ the components of $d\underline{\underline{\tau}}(p)$:

$$\underline{\underline{\tau}}=\sum_{ij}\tau_{ij}\vec{e}_i\otimes e^j,\quad \boxed{d\underline{\underline{\tau}}.\vec{e}_k=\sum_{i,j=1}^n \tau^i_{j|k}\vec{e}_i\otimes e^j},\quad \text{or}\quad d\underline{\underline{\tau}}=\sum_{i,j,k=1}^n \tau^i_{j|k}\vec{e}_i\otimes e^j\otimes e^k. \tag{T.47}$$

(Classical notations: $\underline{\underline{\tau}}=\sum_{ij}\tau_{ij}\vec{e}_i\otimes\pi_{ej}$, $d\underline{\underline{\tau}}.\vec{e}_k=\sum_{ij}\tau_{ij|k}\vec{e}_i\otimes\pi_{ej}$, and $d\underline{\underline{\tau}}=\sum_{ijk}\tau_{ij|k}\vec{e}_i\otimes\pi_{ej}\otimes\pi_{ek}$.)

Cartesian basis: $d\underline{\underline{\tau}}(p).\vec{e}_k=\sum_{ij}(d\tau^i_j(p).\vec{e}_k)\vec{e}_i\otimes e^j=\sum_{ijk}\frac{\partial\tau^i_j}{\partial x^k}(p)\vec{e}_i\otimes e^j\otimes e^k$ gives

$$\tau^i_{j|k}=\frac{\partial \tau^i_j}{\partial x^k}\overset{\text{noted}}{=}\tau^i_{j,k}\quad (:=d\tau^i_j.\vec{e}_k). \tag{T.48}$$

Coordinate system basis: $\underline{\underline{\tau}}(p)=\sum_{i,j=1}^n \tau^i_j(p)\vec{e}_i(p)\otimes e^j(p)$ gives, for all k,

$$\begin{aligned} d\underline{\underline{\tau}}.\vec{e}_k &= \textstyle\sum_{ij}(d\tau^i_j.\vec{e}_k)\vec{e}_i\otimes e^j+\sum_{ij}\tau^i_j(d\vec{e}_i.\vec{e}_k)\otimes e^j+\sum_{ij}\tau^i_j\vec{e}_i\otimes(de^j.\vec{e}_k)\\ &= \textstyle\sum_{ij}(d\tau^i_j.\vec{e}_k)\vec{e}_i\otimes e^j+\sum_{ij\ell}\tau^i_j\gamma^\ell_{ki}\vec{e}_\ell\otimes e^j-\sum_{ij\ell}\tau^i_j\gamma^j_{k\ell}\vec{e}_i\otimes e^\ell\\ &= \textstyle\sum_{ij}(d\tau^i_j.\vec{e}_k)\vec{e}_i\otimes e^j+\sum_{ij\ell}\tau^\ell_j\gamma^i_{k\ell}\vec{e}_i\otimes e^j-\sum_{ij\ell}\tau^i_\ell\gamma^\ell_{kj}\vec{e}_i\otimes e^j \end{aligned} \tag{T.49}$$

thus

$$\boxed{\tau^i_{j|k} = \frac{\partial \tau^i_j}{\partial q^k} + \sum_{\ell=1}^n \tau^\ell_j \gamma^i_{k\ell} - \sum_{\ell=1}^n \tau^i_\ell \gamma^\ell_{kj}} \quad \text{where} \quad \frac{\partial \tau^i_j}{\partial q^k} := d\tau^i_j.\vec{e}_k. \tag{T.50}$$

(We have the + sign from vector fields, cf. (T.41), and the − sign from differential forms, cf. (T.46).)

Exercice T.21 If $\vec{u} \in E$, $\ell \in E^*$ then for the elementary $\binom{1}{1}$ tensor $\underline{\underline{\tau}} = \vec{u} \otimes \ell$ prove:

$$d(\vec{u} \otimes \ell).\vec{e}_k = (d\vec{u}.\vec{e}_k) \otimes \ell + \vec{u} \otimes (d\ell.\vec{e}_k), \quad \text{and} \quad (\vec{u} \otimes \ell)^i_{j|k} = u^i_{|k}\ell_j + u^i \ell_{j|k}, \tag{T.51}$$

when $\vec{u} = \sum_i u^i \vec{e}_i$, $\ell = \sum_j \ell_j e^j$, $d\vec{u}.\vec{e}_k = \sum_i u^i_{|k}\vec{e}_i$, $d\ell.\vec{e}_k = \sum_j \ell_{j|k} e^j$.

Answer. $\underline{\underline{\tau}} = \vec{u} \otimes \ell = \sum_{ij} \tau^i_j \vec{e}_i \otimes e^j$. where $\tau^i_j = u^i \ell_j$, and $d\underline{\underline{\tau}}.\vec{e}_k = \sum_{i,j=1}^n \tau^i_{j|k} \vec{e}_i \otimes e^j$ where $\tau^i_{j|k} = (u^i\ell_j)_{|k} = u^i_{|k}\ell_j + u^i\ell_{j|k} = (\vec{u} \otimes \ell)^i_{j|k}$. Thus (similar to the derivation of a product):

$$\begin{aligned} d(\vec{u} \otimes \ell)(p).\vec{e}_k(p) &= \lim_{h\to 0} \frac{(\vec{u} \otimes \ell)(p+h\vec{e}_k(p)) - (\vec{u} \otimes \ell)(p)}{h} = \lim_{h\to 0} \frac{\vec{u}(p+h\vec{e}_k(p)) \otimes \ell(p+h\vec{e}_k(p)) - \vec{u}(p) \otimes \ell(p)}{h} \\ &= \lim_{h\to 0} \frac{\vec{u}(p+h\vec{e}_k(p)) \otimes \ell(p+h\vec{e}_k(p)) - \vec{u}(p+h\vec{e}_k(p)) \otimes \ell(p)}{h} + \lim_{h\to 0} \frac{\vec{u}(p+h\vec{e}_k(p)) \otimes \ell(p) - \vec{u}(p) \otimes \ell(p)}{h} \\ &= \lim_{h\to 0} (\vec{u}(p+h\vec{e}_k(p)) \otimes (\frac{\ell(p+h\vec{e}_k(p)) - \ell(p)}{h}) + \lim_{h\to 0} (\frac{\vec{u}(p+h\vec{e}_k(p)) - \vec{u}(p)}{h}) \otimes \ell(p) \\ &= \vec{u}(p) \otimes (d\ell(p).\vec{e}_k(p)) + (d\vec{u}(p).\vec{e}_k(p)) \otimes \ell(p), \end{aligned}$$

thus $(T.51)_1$. Which gives $d(\vec{u} \otimes \ell).\vec{e}_k = (\sum_i u^i\vec{e}_i) \otimes (\sum_j \ell_{j|k} e^j) + (\sum_i u^i_{|k}\vec{e}_i) \otimes (\sum_j \ell_j e^j)$, thus $(T.51)_2$. ▟

T.8 Divergence of a vector field: Invariant

$\Gamma(U)$ is the set of C^1 vector fields in U, and $\mathrm{Tr} : \mathcal{L}(E;E) \to \mathbb{R}$ is the trace operator.

Definition T.22 The divergence operator is

$$\mathrm{div} := \mathrm{Tr} \circ d : \begin{cases} \Gamma(U) & \to C^0(U;\mathbb{R}) \\ \quad \vec{w} & \to \mathrm{div}\vec{w} := \mathrm{Tr}(d\vec{w}), \end{cases} \tag{T.52}$$

so $\mathrm{div}\vec{w}(p) = \mathrm{Tr}(d\vec{w}(p))$ is the trace of the endomorphism $d\vec{w}(p)$.

Tr and d are linear, hence div = Tr ∘ d is $\mathbb{R}$-linear (composed of two $\mathbb{R}$-linear maps).

Proposition T.23 *The divergence of a vector field is objective (is an invariant): Same value for all observers (objective quantity) intrinsic to* $\vec{w}$.

Proof. The differential and the trace are objective. (Computation: $\vec{w} = \sum_i u^i\vec{a}_i = \sum_i v^i\vec{b}_i$ gives $v^i_{|j} = \sum_{k\ell} Q^i_k u^k_{|\ell} P^\ell_j$, see (T.42), thus $\sum_i v^i_{|i} = \sum_{ik\ell} P^\ell_i Q^i_k u^k_{|\ell} = \sum_{k\ell} \delta^\ell_k u^k_{|\ell} = \sum_k u^k_{|k}$.) ▟

Quantification: $\vec{w} \in \Gamma(U)$, $(\vec{e}_i)$ is a basis, $\vec{w} = \sum_{i=1}^n w_i\vec{e}_i$ with classical notations, and $w_{i|j}(p)$ are the components of the vector $d\vec{w}(p).\vec{e}_j(p)$ in the basis $(\vec{e}_i(p))$. Thus

$$\boxed{\mathrm{div}\vec{w} = \sum_{i=1}^n w_{i|i}}. \tag{T.53}$$

Duality notations: $\vec{w} = \sum_{i=1}^n w^i\vec{e}_i$, $d\vec{w}.\vec{e}_j = \sum_{i=1}^n w^i_{|j}\vec{e}_i$, $[d\vec{w}]_{|\vec{e}} = [w^i_{|j}]$, $\mathrm{div}\vec{w} = \sum_{i=1}^n w^i_{|i}$.

Cartesian basis $(\vec{e}_i)$ (classical notations): $dw_i.\vec{e}_j =^{\text{noted}} \frac{\partial w_i}{\partial x_j}$ and

$$w_{i|j} = \frac{\partial w_i}{\partial x_i}, \quad \text{thus} \quad \mathrm{div}\vec{w} = \sum_{i=1}^n \frac{\partial w_i}{\partial x_i}. \tag{T.54}$$

(Duality notations: $\mathrm{div}\vec{w} = \sum_{i=1}^n \frac{\partial w^i}{\partial x^i}$.)

Coordinate system basis $(\vec{e}_i)$ (duality notations): With the Christoffel symbols, cf. (T.34), (T.41) gives

$$w^i_{|i} = \frac{\partial w^i}{\partial q^i} + \sum_{i=1}^n w^k \gamma^i_{ik}, \quad \text{thus} \quad \mathrm{div}\vec{w} = \sum_{i=1}^n \frac{\partial w^i}{\partial q^i} + \sum_{i,k=1}^n w^k \gamma^i_{ik}. \tag{T.55}$$

Exercice T.24 Prove:

$$\text{div}(f\vec{w}) = df.\vec{w} + f\,\text{div}\vec{w}. \tag{T.56}$$

Answer. $d(f\vec{w}) = \vec{w}\otimes df + f\,d\vec{w}$ gives $\text{Tr}(d(f\vec{w})) = \text{Tr}(\vec{w}\otimes df) + \text{Tr}(f\,d\vec{w}) = df.\vec{w} + f\,\text{Tr}(d\vec{w})$. Use a coordinate system if you prefer. ▪

Remark T.25 If $\alpha = \sum_{i=1}^n \alpha_i e^i$ is a differential form, then $d\alpha = \sum_{i=1}^n \alpha_{i|j} e^i \otimes e^j$ where $\alpha_{i|j} := \vec{e}_i.d\alpha.\vec{e}_j$. Here it is impossible to define an objective trace of $d\alpha$ like $\sum_{i=1}^n \alpha_{i|i}$: The result depends on the choice of the basis (the Einstein convention is not satisfied, and e.g. with a Euclidean basis the result depends on the choice of unit of length: Foot? Meter?). Thus the objective (or intrinsic) divergence of a differential form is a nonsense. E.g. the trace of an inner dot product $(\cdot,\cdot)_g$ is a nonsense. ▪

T.9 Objective divergence for 1 1 tensors

To create an objective divergence for a second order $\binom{1}{1}$ tensor $\underline{\underline{\tau}} \in T_1^1(U)$, in (T.47) we have to contract an admissible index with the "differential index k". So, no choice: Contract i and k to get $\widetilde{\text{div}}\underline{\underline{\tau}} := \sum_{i,j=1}^n \tau^i_{j|i} e^j$. Let us start with:

Definition T.26 Let $\vec{u} \in \Gamma(U)$ and $\ell \in \Omega^1(U)$ be C^1. The objective divergence of the elementary $\binom{1}{1}$ tensor $\vec{u}\otimes\ell \in T_1^1(U)$ is the differential form $\widetilde{\text{div}}(\vec{u}\otimes\ell) \in \Omega^1(U)$ defined by

$$\widetilde{\text{div}}(\vec{u}\otimes\ell) = (\text{div}\vec{u})\ell + d\ell.\vec{u}, \quad \text{meaning} \quad \widetilde{\text{div}}(\vec{u}\otimes\ell).\vec{w} := (\text{div}\vec{u})(\ell.\vec{w}) + (d\ell.\vec{w}).\vec{u} \tag{T.57}$$

for all $\vec{w} \in E$. (No basis and no inner dot product needed.)

And the objective divergence operator $\widetilde{\text{div}} : \left\{ \begin{array}{rl} T_1^1(U) & \to \Omega^1(U) \\ \underline{\underline{\tau}} & \to \widetilde{\text{div}}\underline{\underline{\tau}} \end{array} \right\}$ is the linear map defined on elementary tensors with (T.57).

Quantification: If $(\vec{e}_i)$ is a **Cartesian basis**, (e^i) its dual basis, $\vec{u} = \sum_i u^i \vec{e}_i$, $\vec{w} = \sum_i w^i \vec{e}_i$, $\ell = \sum_j \ell_j e^j$, then $\vec{u}\otimes\ell = \sum_{ij} u^i \ell_j \vec{e}_i \otimes e^j$ and $\text{div}\vec{u} = \sum_i \frac{\partial u^i}{\partial x^i}$ and $d\ell = \sum_{ij} \frac{\partial \ell_j}{\partial x^i} e^i \otimes e^j$, thus $d\ell.\vec{w} = \sum_j \frac{\partial \ell_j}{\partial x^i} w^j e^i$, thus $\widetilde{\text{div}}(\vec{u}\otimes\ell).\vec{w} = (\text{div}\vec{u})(\ell.\vec{w}) + (d\ell.\vec{w}).\vec{u} = \sum_i \frac{\partial u^i}{\partial x^i} \sum_j \ell_j w^j + \sum_{ij} \frac{\partial \ell_j}{\partial x^i} w^j u^i$, thus

$$\widetilde{\text{div}}(\vec{u}\otimes\ell) = \sum_{i,j=1}^n \Big(\frac{\partial u^i}{\partial x^i}\ell_j + \frac{\partial \ell_j}{\partial x^i} u^i\Big) e^j \quad (= (\text{div}\vec{u})\ell + d\ell.\vec{u}). \tag{T.58}$$

Thus for the elementary tensor $\underline{\underline{\tau}} = \vec{u}\otimes\ell = \sum_{ij} u^i \ell_j \vec{e}_i \otimes e^j = \sum_{ij} \tau^i_j \vec{e}_i \otimes e^j$, where $\tau^i_j = u^i \ell_j$, we get $d\underline{\underline{\tau}}.\vec{e}_k = \sum_{ijk} \frac{\partial \tau^i_j}{\partial x^k} \vec{e}_i \otimes e^j$ and $\widetilde{\text{div}}(\underline{\underline{\tau}}) = \sum_{ij} \frac{\partial \tau^i_j}{\partial x^i} e^i$, with $\frac{\partial \tau^i_j}{\partial x^k} = \frac{\partial u^i}{\partial x^k}\ell_j + u^i \frac{\partial \ell_j}{\partial x^k}$, so $\frac{\partial \tau^i_j}{\partial x^i} = \frac{\partial u^i}{\partial x^i}\ell_j + u^i \frac{\partial \ell_j}{\partial x^i}$.

Thus, by linearity of $\widetilde{\text{div}}$, for all tensors $\underline{\underline{\tau}} \in T_1^1(U)$, we have with (T.48):

$$\boxed{\widetilde{\text{div}}\underline{\underline{\tau}} = \sum_{i,j=1}^n \frac{\partial \tau^i_j}{\partial x^i} e^j}, \quad \text{i.e.} \quad [\widetilde{\text{div}}\underline{\underline{\tau}}]_{|\vec{e}} = \begin{pmatrix} \sum_i \frac{\partial \tau^i_1}{\partial x^i} & \dots & \sum_i \frac{\partial \tau^i_n}{\partial x^i} \end{pmatrix} \tag{T.59}$$

(row matrix since $\widetilde{\text{div}}\underline{\underline{\tau}}$ is a differential form). I.e., we have contracted i and k in $d\underline{\underline{\tau}} = \sum_{ijk} \frac{\partial \tau^i_j}{\partial x^k} \vec{e}_i \otimes e^j \otimes e^k$.

And in a coordinate system basis, with (T.47):

$$\widetilde{\text{div}}\underline{\underline{\tau}} = \sum_{i,j=1}^n \tau^i_{j|i} e^j, \quad \text{i.e.} \quad [\widetilde{\text{div}}\underline{\underline{\tau}}]_{|\vec{e}} = \begin{pmatrix} \sum_i \tau^i_{1|i} & \dots & \sum_i \tau^i_{n|i} \end{pmatrix}. \tag{T.60}$$

(Classical notations: $\widetilde{\text{div}}\underline{\underline{\tau}} := \sum_{i,j=1}^n \tau_{ij|i} \pi_{ej}$, i.e. $[\widetilde{\text{div}}\underline{\underline{\tau}}]_{|\vec{e}} = (\sum_i \tau_{i1|i} \quad \dots \sum_i \tau_{in|i})$.)

Exercice T.27 Prove: If $f \in C^1(U;\mathbb{R})$ and $\underline{\underline{\tau}} = \sum_{i,j=1}^n \tau^i_j \vec{e}_i \otimes e^j \in T_1^1(U) \cap C^1$ then

$$\widetilde{\text{div}}(f\underline{\underline{\tau}}) = df.\underline{\underline{\tau}} + f\,\widetilde{\text{div}}\underline{\underline{\tau}}. \tag{T.61}$$

Answer. $f\underline{\underline{\tau}} = \sum_{ij} f\tau^i_j \vec{e}_i \otimes e^j$ gives $d(f\underline{\underline{\tau}}) = \sum_{ijk} (f\tau^i_j)_{|k} \vec{e}_i \otimes e^j \otimes e^k = \sum_{ijk} (f_{|k}\tau^i_j + f\tau^i_{j|k}) \vec{e}_i \otimes e^j \otimes e^k$, thus $\widetilde{\text{div}}(f\underline{\underline{\tau}}) = \sum_{ij} (f_{|i}\tau^i_j + f\tau^i_{j|i}) e^j$; And $df.\underline{\underline{\tau}} + f\,\widetilde{\text{div}}\underline{\underline{\tau}} = \sum_{ij} f_{|i}\tau^i_j e^j + f\sum_{ij} \tau^i_{j|i} e^j$; Thus (T.61). ▪

Exercice T.28 Prove: If $\underline{\underline{\tau}} \in T_1^1(U)$ and $\vec{w} \in \Gamma(U)$ then

$$\boxed{\mathrm{div}(\underline{\underline{\tau}}.\vec{w}) = \widetilde{\mathrm{div}}(\underline{\underline{\tau}}).\vec{w} + \underline{\underline{\tau}} \odot d\vec{w}}. \tag{T.62}$$

Answer. $\underline{\underline{\tau}} = \sum_{ij} \tau^i_j \vec{e}_i \otimes e^j$ and $\vec{w} = \sum_i w^i \vec{e}_i$ give $\underline{\underline{\tau}}.\vec{w} = \sum_{ij} \tau^i_j w^j \vec{e}_i$, thus $\mathrm{div}(\underline{\underline{\tau}}.\vec{w}) = \sum_{ij} \tau^i_{j|i} w^j + \tau^i_j w^j_{|i}$. ∎

Exercice T.29 If $\underline{\underline{\tau}} \in T_1^1(U)$ check with component calculations (since $\widetilde{\mathrm{div}}(\underline{\underline{\tau}}) \in T_1^0(U)$ is objective):

$$[\widetilde{\mathrm{div}}(\underline{\underline{\tau}})]_{|b} = [\widetilde{\mathrm{div}}(\underline{\underline{\tau}})]_{|a}.P \quad \text{(covariance formula)}, \tag{T.63}$$

where P is the transition matrix from a basis $(\vec{a}_i)$ to a basis $(\vec{b}_i)$.

Answer. Let $\underline{\underline{\tau}} = \sum_{ij} \sigma^i_j \vec{a}_i \otimes a^j = \sum_{ij} \tau^i_j \vec{b}_i \otimes b^j$, so $\tau^i_j = \sum_{\lambda\mu} Q^i_\lambda \sigma^\lambda_\mu P^\mu_j$.

1- Cartesian bases: $\sum_i \tau^i_{j|i} = \sum_i d\tau^i_j.\vec{b}_i = \sum_i d(\sum_{\lambda\mu} Q^i_\lambda \sigma^\lambda_\mu P^\mu_j).(\sum_\nu P^\nu_i.\vec{a}_\nu) = \sum_{i\lambda\mu\nu} Q^i_\lambda P^\mu_j P^\nu_i (d\sigma^\lambda_\mu.\vec{a}_\nu) = \sum_{\lambda\mu\nu} \delta^\nu_\lambda P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\nu) = \sum_{\lambda\mu} P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\lambda) = \sum_\mu (\sum_\lambda \sigma^\lambda_{\mu|\lambda}) P^\mu_j$ as desired.

2- Coordinate system bases: $\sum_i \tau^i_{j|i} = \sum_i d\tau^i_j.\vec{e}_i + \sum_{i\ell} \tau^\ell_j \gamma^i_{i\ell,b} - \sum_{i\ell} \tau^i_\ell \gamma^\ell_{ij,b}$ (with j fixed); With

$$\begin{aligned}\sum_i (d\tau^i_j.\vec{b}_i) &= \sum_{i\lambda\mu} Q^i_\lambda\,(d\sigma^\lambda_\mu.\vec{b}_i)\,P^\mu_j + \sum_{i\lambda\mu} (dQ^i_\lambda.\vec{b}_i)\,\sigma^\lambda_\mu\,P^\mu_j + \sum_{i\lambda\mu} Q^i_\lambda\,\sigma^\lambda_\mu\,(dP^\mu_j.\vec{b}_i)\\ &= \sum_{i\lambda\mu\nu} Q^i_\lambda P^\mu_j P^\nu_i (d\sigma^\lambda_\mu.\vec{a}_\nu) + \sum_{i\lambda\mu\nu} \sigma^\lambda_\mu\,P^\mu_j P^\nu_i (dQ^i_\lambda.\vec{a}_\nu) + \sum_{i\lambda\mu\nu} \sigma^\lambda_\mu\,Q^i_\lambda P^\nu_i (dP^\mu_j.\vec{a}_\nu)\\ &= \sum_{\lambda\mu} P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\lambda) - \sum_{i\lambda\mu\nu} \sigma^\lambda_\mu\,P^\mu_j Q^i_\lambda (dP^\nu_i.\vec{a}_\nu) + \sum_{\lambda\mu} \sigma^\lambda_\mu\,(dP^\mu_j.\vec{a}_\lambda)\end{aligned}$$

since $P^\nu_i Q^i_\lambda = \delta^\nu_\lambda$ gives $P^\nu_i (dQ^i_\lambda.\vec{a}_\nu) - Q^i_\lambda (dP^\nu_i.\vec{a}_\nu)$. And, with (T.36),

$$\begin{aligned}\sum_{i\ell} \tau^\ell_j \gamma^i_{i\ell,b} &= \sum_{i\ell} (\sum_{\lambda\mu} Q^\ell_\lambda \sigma^\lambda_\mu P^\mu_j)(\sum_{\alpha\beta\omega} Q^i_\alpha P^\beta_i P^\omega_\ell \gamma^\alpha_{\beta\omega,a} + \sum_{\alpha\beta} Q^i_\alpha P^\beta_i (dP^\alpha_\ell.\vec{a}_\beta))\\ &= \sum_{\lambda\mu\alpha} \sigma^\lambda_\mu P^\mu_j \gamma^\alpha_{\alpha\lambda,a} + \sum_{\ell\lambda\mu\alpha} \sigma^\lambda_\mu Q^\ell_\lambda P^\mu_j (dP^\alpha_\ell.\vec{a}_\alpha),\end{aligned} \tag{T.64}$$

and

$$\begin{aligned}-\sum_{i\ell} \tau^i_\ell \gamma^\ell_{ij,b} &= -\sum_{i\ell} (\sum_{\lambda\mu} Q^i_\lambda \sigma^\lambda_\mu P^\mu_\ell)(\sum_{\alpha\beta\omega} P^\alpha_i P^\beta_j Q^\ell_\omega \gamma^\omega_{\alpha\beta,a} + \sum_{\alpha\omega} P^\alpha_i Q^\ell_\omega (dP^\omega_j.\vec{a}_\alpha))\\ &= -\sum_{\lambda\mu\beta} \sigma^\lambda_\mu P^\beta_j \gamma^\mu_{\lambda\beta,a} - \sum_{\lambda\mu} \sigma^\lambda_\mu (dP^\mu_j.\vec{a}_\lambda).\end{aligned} \tag{T.65}$$

Thus $\sum_i \tau^i_{j|i} = \sum_{\lambda\mu} P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\lambda) + \sum_{\lambda\mu\alpha} \sigma^\lambda_\mu P^\mu_j \gamma^\alpha_{\alpha\lambda,a} - \sum_{\lambda\mu\beta} \sigma^\lambda_\mu P^\beta_j \gamma^\mu_{\lambda\beta,a} = \sum_{\lambda\mu} P^\mu_j \sigma^\lambda_{\mu|\lambda}$ as desired. ∎

T.9.1 Divergence of a 2 0 tensor

Let $\underline{\underline{\tau}} \in T_0^2(U)$ and $\underline{\underline{\tau}} = \sum_{i,j=1}^n \tau^{ij} \vec{e}_i \otimes \vec{e}_j$, thus $d\underline{\underline{\tau}} = \sum_{i,j,k=1}^n \tau^{ij}{}_{|k} \vec{e}_i \otimes \vec{e}_j \otimes e^k$; Then two objective divergences may be defined: by contracting k with i, or k with j. (The Einstein convention is then satisfied.)

T.9.2 Divergence of a 0 2 tensor

Let $\underline{\underline{\tau}} = \sum_{i,j=1}^n \tau_{ij} e^i \otimes e^j \in T_2^0(U)$. Thus $d\underline{\underline{\tau}} = \sum_{i,j,k=1}^n \tau_{ij|k} e^i \otimes e^j \otimes e^k$, and there are no indices to contract to satisfy Einstein convention: There is no objective divergence of 0 2 tensors.

T.10 Euclidean framework and “classic divergence” of a tensor (subjective)

Let $\underline{\underline{\sigma}} \in T_1^1(U)$ be a $\binom{1}{1}$ C^1 tensor (so at any point in U naturally canonically identified with an endomorphism). An observer chooses a Euclidean basis $(\vec{e}_i)$ and call $(\cdot,\cdot)_g$ the associated Euclidean dot product. Let $[\underline{\underline{\sigma}}]_{|\vec{e}} = [\sigma_{ij}]$.

Definition T.30 The usual divergence $\mathrm{div}_e \underline{\underline{\sigma}}$ in continuum mechanics is the column matrix (it is not a vector)

$$\mathrm{div}_e \underline{\underline{\sigma}} := \begin{pmatrix} \sum_{j=1}^n \frac{\partial \sigma_{1j}}{\partial x^j} \\ \vdots \\ \sum_{j=1}^n \frac{\partial \sigma_{nj}}{\partial x^j} \end{pmatrix} = [\widetilde{\mathrm{div}} \underline{\underline{\sigma}}]^T_{\vec{e}} \overset{\text{noted}}{=} \mathrm{div} \underline{\underline{\sigma}}. \tag{T.66}$$

So: Take the divergences of the “row vectors” of $[\underline{\underline{\sigma}}]_{|e} = [\sigma_{ij}]$ to make the “column vector” $[\mathrm{div}_e \underline{\underline{\sigma}}]$.

Proposition T.31 *If $\vec u \in T^1_0(U)$ (a vector field), then*

$$[\mathrm{div}(\underline{\underline{\sigma}}.\vec u)]_{|\vec e} = \mathrm{div}_e\underline{\underline{\sigma}}^T.[\vec u]_{|\vec e} + [\underline{\underline{\sigma}}]^T_{|\vec e} : [d\vec u]_{|\vec e}. \tag{T.67}$$

Proof. (T.62) gives $\mathrm{div}(\underline{\underline{\sigma}}.\vec u) = \widetilde{\mathrm{div}}(\underline{\underline{\sigma}}).\vec u + \underline{\underline{\sigma}} \odot d\vec u$, thus (T.67). Or direct calculation. ∎

More general definition of divergence in classical mechanics : Let $\underline{\underline{\sigma}}$ be a C^1 tensor of order 2 of any kind. Then the divergence $\mathrm{div}_e\underline{\underline{\sigma}}$ of $\underline{\underline{\sigma}}$ relative to the basis $(\vec e_i)$, is the column matrix (it is not a vector)

$$\mathrm{div}_e\underline{\underline{\sigma}} = \begin{pmatrix} \sum_{j=1}^n \frac{\partial \sigma_{1j}}{\partial x^j} \\ \vdots \\ \sum_{j=1}^n \frac{\partial \sigma_{nj}}{\partial x^j} \end{pmatrix}, \quad \text{written} \quad \mathrm{div}_e\underline{\underline{\sigma}} = \sum_{ij} \frac{\partial \sigma_{ij}}{\partial x_j}\vec E_i, \tag{T.68}$$

where $(\vec E_i)$ is the canonical basis in $\mathcal{M}_{n1}$ the space of $n*1$ column vectors.

Exercice T.32 Prove that the "so called vector" $\mathrm{div}\underline{\underline{\sigma}}$ defined by

$$\mathrm{div}\underline{\underline{\sigma}} = \sum_{ij} \frac{\partial \sigma_{ij}}{\partial x_j}\vec e_i \tag{T.69}$$

is not a vector of any kind.

Answer. We have to prove that: If $(\vec a_i)$ and $(\vec b_i)$ are bases, if P is the transition matrix from $(\vec a_i)$ to $(\vec b_i)$, then

$$\text{neither} \quad [\mathrm{div}\underline{\underline{\sigma}}]_{|\vec b} \neq P^{-1}.[\mathrm{div}\underline{\underline{\sigma}}]_{|\vec a} \quad \text{nor} \quad [\mathrm{div}\underline{\underline{\sigma}}]^T_{|\vec b} = [\mathrm{div}\underline{\underline{\sigma}}]^T_{|\vec a}.P, \tag{T.70}$$

i.e. the divergence as defined in (T.69) is neither contravariant nor covariant (does not satisfy any change of basis formula). (Compare with (T.63))

Consider the simple case $\vec b_i = \lambda \vec a_i$, for all i, $\lambda > 1$: Transition matrix $P = \lambda I$, and $P^{-1} = \frac{1}{\lambda}I$.

For a $\binom{1}{1}$ tensor: $\underline{\underline{\sigma}} = \sum_{ij}(\sigma_b)^i_j \vec b_i \otimes b^j = \sum_{ij}(\sigma_a)^i_j \vec a_i \otimes a^j$, $[\underline{\underline{\sigma}}]_{|\vec b} = P^{-1}.[\underline{\underline{\sigma}}]_{|\vec a}.P = \frac{1}{\lambda}.[\underline{\underline{\sigma}}]_{|\vec a}.\lambda = [\underline{\underline{\sigma}}]_{|\vec a}$, i.e. $(\sigma_a)^i_j = (\sigma_b)^i_j$ for all i,j. Thus (T.69) gives $\mathrm{div}_b\underline{\underline{\sigma}} = \sum_{ij}(d(\sigma_b)^i_j.\vec b_j)\vec b_i = \sum_{ij}(d(\sigma_a)^i_j.(\lambda \vec a_j))(\lambda \vec a_i) = \lambda^2 \mathrm{div}_a\underline{\underline{\sigma}}$. Thus $[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec b} \neq P^{-1}.[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec a}$ and $[\mathrm{div}_b\underline{\underline{\sigma}}]^T_{|\vec b} \neq [\mathrm{div}_a\underline{\underline{\sigma}}]^T_{|\vec a}.P$.

For a $\binom{0}{2}$ tensor: $\underline{\underline{\sigma}} = \sum_{ij}\sigma_{b,ij} b^i \otimes b^j = \sum_{ij}\sigma_{a,ij} a^i \otimes a^j$, and $[\underline{\underline{\sigma}}]_{|\vec b} = P^T.[\underline{\underline{\sigma}}]_{|\vec a}.P = \lambda^2[\underline{\underline{\sigma}}]_{|\vec a}$, i.e. $\sigma_{b,ij} = \lambda^2\sigma_{a,ij}$ for all i,j. Thus (T.69) gives $\mathrm{div}_b\underline{\underline{\sigma}} = \sum_{ij}(d\sigma_{b,ij}.\vec b_j)\vec b_i = \lambda^2\sum_{ij}(d\sigma_{a,ij}.(\lambda\vec a_j))(\lambda \vec a_i) = \lambda^4 \mathrm{div}_a\underline{\underline{\sigma}}$. Thus $[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec b} \neq P^{-1}.[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec a}$ and $[\mathrm{div}_b\underline{\underline{\sigma}}]^T_{|\vec b} \neq [\mathrm{div}_a\underline{\underline{\sigma}}]^T_{|\vec a}.P$.

For a $\binom{2}{0}$ tensor: $\underline{\underline{\sigma}} = \sum_{ij}\sigma_b^{ij}\vec b_i \otimes \vec b_j = \sum_{ij}\sigma_a^{ij}\vec a_i \otimes \vec a_j$, and $[\underline{\underline{\sigma}}]_{|\vec b} = P^{-T}.[\underline{\underline{\sigma}}]_{|\vec a}.P^{-1} = \frac{1}{\lambda^2}[\underline{\underline{\sigma}}]_{|\vec a}$, i.e. $\sigma_b^{ij} = \frac{1}{\lambda^2}\sigma_a^{ij}$ for all i,j. Thus (T.69) gives $\mathrm{div}_b\underline{\underline{\sigma}} = \sum_{ij}(d\sigma_b^{ij}.\vec b_j)\vec b_i = \frac{1}{\lambda^2}\sum_{ij}(d\sigma_a^{ij}.(\lambda\vec a_j))(\lambda\vec a_i) = \mathrm{div}_a\underline{\underline{\sigma}}$. Thus $[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec b} \neq P^{-1}.[\mathrm{div}_b\underline{\underline{\sigma}}]_{|\vec a}$ and $[\mathrm{div}_b\underline{\underline{\sigma}}]^T_{|\vec b} \neq [\mathrm{div}_a\underline{\underline{\sigma}}]^T_{|\vec a}.P$. ∎

U Natural canonical isomorphisms

U.1 The adjoint of a linear map

Setting of § A.13: E and F are vector spaces, $E^* = \mathcal{L}(E;\mathbb{R})$ and $F^* = \mathcal{L}(F;\mathbb{R})$ are their dual spaces, and the adjoint of a linear map $\mathcal{P} \in \mathcal{L}(E;F)$ is the linear map $\mathcal{P}^* \in \mathcal{L}(F^*;E^*)$ canonically defined by

$$\forall \ell \in F^*, \quad \mathcal{P}^*(\ell) := \ell \circ \mathcal{P}, \quad \text{written} \quad \mathcal{P}^*.\ell = \ell.\mathcal{P} \tag{U.1}$$

(dot notations $\mathcal{P}^*(\ell) =^{\text{noted}} \mathcal{P}^*.\ell$ and $\ell \circ \mathcal{P} =^{\text{noted}} \ell.\mathcal{P}$ since ℓ and $\mathcal{P}^*$ are linear), i.e., for all $(\ell, \vec u) \in F^* \times E$,

$$\mathcal{P}^*(\ell)(\vec u) = \ell(\mathcal{P}(\vec u)), \quad \text{written} \quad (\mathcal{P}^*.\ell).\vec u = \ell.\mathcal{P}.\vec u. \tag{U.2}$$

Interpretation: If $\mathcal{P}$ is the push-forward of vector fields, then $\mathcal{P}^*$ is the pull-back of differential forms, see remark 7.5. In particular, it will be interpreted with $\mathcal{P} \in \mathcal{L}_i(E;F)$ (linear and invertible = a change of observer).

U.2 An isomorphism $E \simeq E^*$ is never natural (never objective)

Two observers A and B consider a linear map $L \in \mathcal{L}(E;E^*)$; Let $\mathcal{P} \in \mathcal{L}(E;E)$ be the change of observer endomorphism. Willing to work together, A and B ("naturally") consider the diagram

$$\begin{array}{cccl} E & \xrightarrow{L} & E^* & \leftarrow \text{considered by observer } A \\ \mathcal{P}\downarrow & & \uparrow \mathcal{P}^* & \\ E & \xrightarrow[L]{} & E^* & \leftarrow \text{considered by observer } B \end{array} \tag{U.3}$$

Definition U.1 (Spivak [19].) A linear map $L \in \mathcal{L}(E;E^*)$ is natural iff the diagram (U.3) commutes for all $\mathcal{P} \in \mathcal{L}(E;E)$:

$$L \in \mathcal{L}(E;E^*) \text{ is natural } \iff \forall \mathcal{P} \in \mathcal{L}(E;E),\ \ \mathcal{P}^* \circ L \circ \mathcal{P} = L. \tag{U.4}$$

(In that case, if A computes $L.\vec{u}$ with the top line of the diagram, if B computes with the bottom line of the diagram, then they can easily check their results since here $L.\vec{u} = (\mathcal{P}^* \circ L \circ \mathcal{P}).\vec{u}$.)

Question: Does there exist an endomorphism L such that the diagram (U.3) commutes for all change of observers? That is, do we have

$$\exists ? L \in \mathcal{L}(E;E),\ \forall \mathcal{P} \in \mathcal{L}_i(E;E), \quad \mathcal{P}^* \circ L \circ \mathcal{P} = L\ ? \tag{U.5}$$

Answer: Always ***no*** (if $L \neq 0$):

Theorem U.2 *A (non-zero) linear map $L \in \mathcal{L}(E;E^*)$ is not natural: If $L \in \mathcal{L}(E;E^*) - \{0\}$, then*

$$\exists \mathcal{P} \in \mathcal{L}_i(E;E) \quad s.t. \quad L \neq \mathcal{P}^* \circ L \circ \mathcal{P}. \tag{U.6}$$

Proof. (Spivak [19].) It suffices to prove this proposition for $E = \vec{\mathbb{R}}$. Let $L \in \mathcal{L}(\vec{\mathbb{R}};(\vec{\mathbb{R}})^*)$, $L \neq 0$.

Let $(\vec{a}_1)$ be a basis in $\vec{\mathbb{R}}$ (chosen by A). Let $(\vec{b}_1)$ be a basis in $\vec{\mathbb{R}}$ (chosen by B).

Consider $\mathcal{P} \in \mathcal{L}_i(\vec{\mathbb{R}};\vec{\mathbb{R}})$ defined by $\mathcal{P}(\vec{a}_1) = \vec{b}_1$ (change of observer), and let $\lambda \in \mathbb{R}$ s.t. $\vec{b}_1 = \lambda \vec{a}_1$. Then (U.1) gives $\mathcal{P}^*(\ell)(\vec{a}_1) := \ell(\mathcal{P}(\vec{a}_1)) = \ell(\vec{b}_1) = \ell(\lambda\vec{a}_1) = \lambda\ell(\vec{a}_1)$, thus $\mathcal{P}^*(\ell) = \lambda\ell$ for all $\ell \in (\vec{\mathbb{R}})^*$.

Thus $\mathcal{P}^*(L(\mathcal{P}(\vec{a}_1))) = \mathcal{P}^*(L(\lambda\vec{a}_1)) = \lambda\mathcal{P}^*(L(\vec{a}_1)) = \lambda^2 L(\vec{a}_1) \neq L(\vec{a}_1)$ when $\lambda^2 \neq 1$. E.g., $\mathcal{P} = 2I$ gives $L \neq \mathcal{P}^* \circ L \circ \mathcal{P}$ $(= 4L)$, thus (U.6): A (non-zero) linear map $E \to E^*$ cannot be natural.

Example U.3 Consider E s.t. $\dim E = 1$, and consider the linear map $L \in \mathcal{L}(E;E^*)$ which sends a basis $(\vec{a}_1)$ onto its dual basis (π_{a1}), so L is defined by $L.\vec{a}_1 := \pi_{a1}$.

Question: If $(\vec{b}_1)$ is another basis, $\lambda \neq \pm 1$ and $\vec{b}_1 = \lambda\vec{a}_1$ (change of unit of measurement), does $L.\vec{b}_1 = \pi_{b1}$, i.e. does L also sends $(\vec{b}_1)$ onto its dual basis?

Answer: No. Indeed, $\vec{b}_1 = \lambda\vec{a}_1$ gives $\pi_{b1} = \frac{1}{\lambda}\pi_{a1}$, thus $L.\vec{b}_1 = \lambda L.\vec{a}_1 = \lambda\pi_{a1} = \lambda^2\pi_{b1} \neq \pi_{b1}$ since $\lambda^2 \neq 1$. In words: L is not natural, cf. (U.6).

A different presentation: Let L_A and L_B be defined by $L_A.\vec{a}_j = \pi_{aj}$ and $L_B.\vec{b}_j = \pi_{bj}$ for all j. And suppose that $\vec{b}_j = \lambda\vec{a}_j$ for all j. Then, $L_A.\vec{b}_j = \lambda L_A.\vec{a}_j = \lambda\pi_{aj} = \lambda^2\pi_{bj} = \lambda^2 L_B.\vec{b}_j \neq L_B.\vec{b}_j$ when $\lambda^2 \neq 1$, that is, $L_A \neq L_B$ when $\lambda^2 \neq 1$: An operator that sends a basis onto its dual basis is not natural.

Example U.4 Let $(\cdot,\cdot)_g$ be an inner dot product in $E = \vec{\mathbb{R}^n}$. Let $\vec{R}_g \in \mathcal{L}(E^*;E)$ be the Riesz representation map, that is, defined by $\vec{R}_g(\ell) = \vec{\ell}_g$ where $\vec{\ell}_g$ is defined by $(\vec{\ell}_g,\vec{v})_g = \ell.\vec{v}$ for all $\vec{v} \in \vec{\mathbb{R}^n}$, cf (F.3).

Question: Is $\vec{R}_g$ natural?

Answer: No: Consider the diagram $\left(\begin{array}{cccc} & E^* & \xrightarrow{\vec{R}_g} & E & \\ \mathcal{P}^* & \downarrow & & \uparrow & \mathcal{P} \\ & E^* & \xrightarrow[\vec{R}_g]{} & E & \end{array}\right)$ with $\mathcal{P} = \lambda I$, $\lambda \neq \pm 1$. Then $\mathcal{P}^* = \lambda I$, and $\mathcal{P}.\vec{R}_g.\mathcal{P}^*.\ell = \lambda^2\vec{R}_g.\ell \neq \vec{R}_g.\ell$ gives $\mathcal{P}.\vec{R}_g.\mathcal{P}^* \neq \vec{R}_g$: So $\vec{R}_g$ is not natural, cf. (U.6). (You may prefer to consider the diagram (U.3) with $L = \vec{R}_g^{-1}$.)

A different presentation: Consider two distinct Euclidean dot products $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ (e.g., built with a foot and built with a metre). So $(\cdot,\cdot)_h = \lambda^2(\cdot,\cdot)_g$ with $\lambda^2 \neq 1$. Let $\vec{R}_g, \vec{R}_h \in \mathcal{L}(\vec{\mathbb{R}^n}^*;\vec{\mathbb{R}^n})$ be the Riesz operators relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$, that is $\vec{R}_g.\ell = \vec{\ell}_g$ and $\vec{R}_h.\ell = \vec{\ell}_h$ are given by $\ell.\vec{v} = (\vec{\ell}_g,\vec{v})_g = (\vec{\ell}_h,\vec{v})_h$ for all $\vec{v} \in \vec{\mathbb{R}^n}$. We have $\vec{\ell}_h = \lambda^2\vec{\ell}_g$, cf. (F.12), thus $\vec{R}_h = \lambda^2\vec{R}_g \neq \vec{R}_g$ since $\lambda^2 \neq 1$: A Riesz representation operator is not natural (it is observer dependent).

U.3 Natural canonical isomorphism $E \simeq E^{**}$

Two observers A and B consider the same linear map $L \in \mathcal{L}(E; E^{**})$ (where $E^{**} = (E^*)^* = \mathcal{L}(E^*; \mathbb{R})$). Willing to work together, they ("naturally") consider the diagram

$$\begin{array}{ccccc} & E & \xrightarrow{L} & E^{**} & \leftarrow \text{considered by observer } A \\ \mathcal{P}\downarrow & & & \downarrow\mathcal{P}^{**} & \\ & E & \underset{L}{\longrightarrow} & E^{**} & \leftarrow \text{considered by observer } B \end{array} \tag{U.7}$$

where $\mathcal{P} \in \mathcal{L}(E; E)$ is a linear diffeomorphism, $\mathcal{P}^* \in \mathcal{L}(E^*; E^*)$ its adjoint, given by $\mathcal{P}^*(\ell) = \ell \circ \mathcal{P}$ cf. (U.1), and $\mathcal{P}^{**} \in \mathcal{L}_i(E^{**}; E^{**})$ the adjoint of $\mathcal{P}^*$, thus given by $\mathcal{P}^{**}(u) = u \circ \mathcal{P}^*$ for all $u \in E^{**}$ cf. (U.1), i.e. $\mathcal{P}^{**}$ is given by, for all $(\ell, u) \in E^* \times E^{**}$,

$$(\mathcal{P}^{**}(u))(\ell) = u(\ell \circ \mathcal{P}), \quad \text{i.e.} \quad (\mathcal{P}^{**}.u).\ell = u.(\ell.\mathcal{P}). \tag{U.8}$$

Question: Does there exist a linear map $L \in \mathcal{L}(E; E^{**})$ that is natural?
Answer: Yes (particular case of the next proposition):

Proposition U.5 *The canonical isomorphism*

$$\mathcal{J}_E : \begin{cases} E & \to E^{**} \\ \vec{u} & \to u = \mathcal{J}_E(\vec{u}) \quad \textit{defined by} \quad \mathcal{J}_E(\vec{u})(\ell) := \ell.\vec{u}, \quad \forall \ell \in E^*, \end{cases} \tag{U.9}$$

is natural, that is, F being another finite dimensional vector space, the diagram

$$\begin{array}{cccc} & E & \xrightarrow{\mathcal{J}_E} & E^{**} \\ \mathcal{P}\downarrow & & & \downarrow\mathcal{P}^{**} \\ & F & \underset{\mathcal{J}_F}{\longrightarrow} & F^{**} \end{array} \quad \textit{written} \quad \begin{array}{cccc} & E & \xrightarrow{\mathcal{J}} & E^{**} \\ \mathcal{P}\downarrow & & & \downarrow\mathcal{P}^{**} \\ & F & \underset{\mathcal{J}}{\longrightarrow} & F^{**} \end{array} \tag{U.10}$$

commutes for all $\mathcal{P} \in \mathcal{L}(E; F)$, i.e.

$$\forall \mathcal{P} \in \mathcal{L}(E; F), \quad \mathcal{P}^{**} \circ \mathcal{J}_E = \mathcal{J}_F \circ \mathcal{P}, \quad \textit{and we write} \quad E \simeq E^{**}. \tag{U.11}$$

Thus we can use the unambiguous notation (observer independent)

$$\mathcal{J}(\vec{u}) \overset{noted}{=} \vec{u}, \quad \textit{and} \quad \mathcal{J}(\vec{u}).\ell \overset{noted}{=} \vec{u}.\ell \quad (= \ell.\vec{u}). \tag{U.12}$$

(And $u = \mathcal{J}(\vec{u})$ is the derivation operator in the direction $\vec{u}$.)

Proof. (Spivak [19].) It is trivial that $\mathcal{J}_E$ is linear and bijective (E is finite dimensional): It is an isomorphism. Then $(\mathcal{P}^{**} \circ \mathcal{J}_E(\vec{u}))(\ell) \overset{(U.8)}{=} \mathcal{J}_E(\vec{u})(\ell.\mathcal{P}) \overset{(U.9)}{=} (\ell \circ \mathcal{P})(\vec{u}) = \ell(\mathcal{P}(\vec{u})) \overset{(U.9)}{=} \mathcal{J}_F(\mathcal{P}(\vec{u}))(\ell)$, for all $\ell \in F^*$ and all $\vec{u} \in E$, thus $\mathcal{P}^{**} \circ \mathcal{J}_E(\vec{u}) = \mathcal{J}_F(\mathcal{P}(\vec{u}))$, for all $\vec{u} \in E$, thus $\mathcal{P}^{**} \circ \mathcal{J}_E = \mathcal{J}_F \circ \mathcal{P}$. ∎

Proposition U.6 *(Characterization of $\mathcal{J}_E$.) $\mathcal{J}_E$ sends any basis $(\vec{a}_i)$ onto its bidual basis. (Expected, since $\mathcal{J}_E(\vec{u})$ is the directional derivative in the direction $\vec{u}$, whatever $\vec{u}$.)*

Proof. Let $(\vec{a}_i)$ be a basis and (π_{ai}) be its dual basis (defined by $\pi_{ai}.\vec{a}_j = \delta_{ij}$ for all i, j). Then (U.9) gives $\mathcal{J}_E(\vec{a}_j).\pi_{ai} = \pi_{ai}.\vec{a}_j = \delta_{ij}$ for all i, j, thus $(\mathcal{J}_E(\vec{a}_j))$ is the dual basis of (π_{ai}), i.e., is the bidual basis of $(\vec{a}_i)$; True for all basis: $\mathcal{J}_E(\vec{b}_j).\pi_{bi} = \pi_{bi}.\vec{b}_j = \delta_{ij}$ for all i, j. ∎

U.4 Natural canonical isomorphisms $\mathcal{L}(E; F) \simeq \mathcal{L}(F^*, E; \mathbb{R}) \simeq \mathcal{L}(E^*; F^*)$

E, F, A, B are finite dimensional vector spaces. Consider the canonical isomorphism

$$\mathcal{J}_{EF} : \begin{cases} \mathcal{L}(E; F) & \to \mathcal{L}(F^*, E; \mathbb{R}) \\ L & \to \widetilde{L} = \mathcal{J}_{EF}(L) \quad \text{where} \quad \widetilde{L}(\ell, \vec{u}) := \ell.L.\vec{u}, \quad \forall (\ell, \vec{u}) \in F^* \times E. \end{cases} \tag{U.13}$$

Let $\mathcal{P}_1 \in \mathcal{L}_i(E;A)$ and $\mathcal{P}_2 \in \mathcal{L}(F;B)$, and consider the diagram

$$\begin{array}{ccc} \mathcal{L}(E;F) & \xrightarrow{\mathcal{J}_{EF}} & \mathcal{L}(F^*,E;\mathbb{R}) \\ \mathcal{I}_\mathcal{P} \downarrow & & \downarrow \widetilde{\mathcal{I}_\mathcal{P}} \\ \mathcal{L}(A;B) & \underset{\mathcal{J}_{AB}}{\longrightarrow} & \mathcal{L}(B^*,A;\mathbb{R}) \end{array} \tag{U.14}$$

where

$$\mathcal{I}_\mathcal{P}(L) = \mathcal{P}_2.L.\mathcal{P}_1^{-1} \quad \text{and} \quad \widetilde{\mathcal{I}_\mathcal{P}}(\widetilde{L})(b,\vec{a}) = \widetilde{L}(b.\mathcal{P}_2, \mathcal{P}_1^{-1}.\vec{a}) \quad \forall (b,\vec{a}) \in B^* \times A. \tag{U.15}$$

($\mathcal{I}_\mathcal{P}$ and $\widetilde{\mathcal{I}_\mathcal{P}}$ are the push-forwards for linear maps $L \in \mathcal{L}(E;F)$ and for bilinear forms $\widetilde{L} \in \mathcal{L}(F^*,E;\mathbb{R})$.)

Proposition U.7 *The canonical isomorphism $\mathcal{J}_{EF}$ is natural, that is, the diagram (U.14) commutes for all $\mathcal{P}_1 \in \mathcal{L}_i(E,A)$ and all $\mathcal{P}_2 \in \mathcal{L}(F,B)$:*

$$\widetilde{\mathcal{I}_\mathcal{P}} \circ \mathcal{J}_{EF} = \mathcal{J}_{AB} \circ \mathcal{I}_\mathcal{P}, \quad \text{and} \quad \mathcal{L}(E;F) \overset{natural}{\simeq} \mathcal{L}(F^*,E;\mathbb{R}). \tag{U.16}$$

Thus $\mathcal{L}(E^*;F^*) \overset{natural}{\simeq} \mathcal{L}(E;F)$.

Proof. $\mathcal{J}_{AB}(\mathcal{I}_\mathcal{P}(L))(b,\vec{a}) \overset{(U.13)}{=} b.\mathcal{I}_\mathcal{P}(L).\vec{a} \overset{(U.15)}{=} b.(\mathcal{P}_2.L.\mathcal{P}_1^{-1}).\vec{a} = (b.\mathcal{P}_2).L.(\mathcal{P}_1^{-1}.\vec{a}) \overset{(U.13)}{=} \mathcal{J}_{EF}(L)(b.\mathcal{P}_2, \mathcal{P}_1^{-1}.\vec{a})$ $\overset{(U.15)}{=} \widetilde{\mathcal{I}_\mathcal{P}}(\mathcal{J}_{EF}(L))(b,\vec{a})$, true for all $L \in \mathcal{L}(E;F)$, $b \in B^*$, $\vec{a} \in A$, thus (U.16).

Thus $\mathcal{L}(E^*;F^*) \overset{(U.16)}{\simeq} \mathcal{L}((F^*)^*,E^*;\mathbb{R}) \overset{(U.11)}{\simeq} \mathcal{L}(F,E^*;\mathbb{R}) \overset{(U.16)}{\simeq} \mathcal{L}(E^{**};F) \overset{(U.11)}{\simeq} \mathcal{L}(E;F)$. ∴

Consider the canonical isomorphism (defines the transposed of a bilinear map)

$$\mathcal{K}_{EF} : \left\{ \begin{array}{rl} \mathcal{L}(E,F;\mathbb{R}) & \to \mathcal{L}(F,E;\mathbb{R}) \\ T & \to \mathcal{K}_{EF}(T) \end{array} \right\}, \quad \mathcal{K}_{EF}(T)(\vec{u},\vec{v}) := T(\vec{v},\vec{u}), \quad \forall (\vec{u},\vec{v}) \in E \times F, \tag{U.17}$$

and $Z_{AB} \in \mathcal{L}(E,F;\mathbb{R}) \to \mathcal{L}(A,B;\mathbb{R})$ defined by $Z_{AB}(T)(\vec{a},\vec{b}) := T(\mathcal{P}_1^{-1}.\vec{a}, \mathcal{P}_2^{-1}.\vec{b})$ for all $(\vec{a},\vec{b}) \in A \times B$.

Proposition U.8 *The canonical isomorphism $\mathcal{K}_{EF}$ is natural: For all $(\mathcal{P}_1,\mathcal{P}_2) \in \mathcal{L}_i(E;A) \times \mathcal{L}(F;B)$, the diagram* $\begin{array}{ccc} \mathcal{L}(E,F;\mathbb{R}) & \xrightarrow{\mathcal{K}_{EF}} & \mathcal{L}(F,E;\mathbb{R}) \\ Z_{AB} \downarrow & & \downarrow Z_{BA} \\ \mathcal{L}(A,B;\mathbb{R}) & \underset{\mathcal{K}_{AB}}{\longrightarrow} & \mathcal{L}(B,A;\mathbb{R}) \end{array}$ *commutes: $\mathcal{L}(E,F;\mathbb{R}) \overset{natural}{\simeq} \mathcal{L}(F,E;\mathbb{R})$.*

Proof. $\mathcal{K}_{EF}(Z_{AB}(T))(\vec{b},\vec{a}) = Z_{AB}(T)(\vec{a},\vec{b}) = T(\mathcal{P}_2^{-1}.\vec{b}, \mathcal{P}_1^{-1}.\vec{a})$ and $Z_{BA}(\mathcal{K}_{EF}(T))(\vec{a},\vec{b}) = \mathcal{K}_{EF}(T)(\mathcal{P}_1^{-1}.\vec{a}, \mathcal{P}_2^{-1}.\vec{b}) = T(\mathcal{P}_2^{-1}.\vec{b}, \mathcal{P}_1^{-1}.\vec{a})$, thus $\mathcal{K}_{AB} \circ Z_{AB} = Z_{BA} \circ \mathcal{K}_{EF}$. ∴

U.5 Natural canonical isomorphisms $\mathcal{L}(E;\mathcal{L}(E;F)) \simeq \mathcal{L}(E,E;F) \simeq \mathcal{L}(F^*,E,E;\mathbb{R})$

For application to the second order derivative $d(d\vec{u}) \simeq d^2\vec{u}$ and, with $\vec{u} \in T_0^1(U)$, the notation $d\vec{u} \in T_1^1(U)$, then $d^2\vec{u} \in T_2^1(U)$, ..., $d^k\vec{u} \in T_k^1(U)$, ...

Consider the canonical isomorphism

$$\mathcal{J}_{12E} : \left\{ \begin{array}{rl} \mathcal{L}(E;\mathcal{L}(E;F)) & \to \mathcal{L}(E,E;F) \\ T_1 & \to T_2 = \mathcal{J}_{12E}(T_1) \end{array} \right\}, \quad \mathcal{J}_{12E}(T_1)(\vec{u}_1,\vec{u}_2) := T_1(\vec{u}_1).\vec{u}_2 \in F, \quad \forall \vec{u}_1,\vec{u}_2 \in E, \tag{U.18}$$

and the canonical isomorphism

$$\mathcal{J}_{23E} : \left\{ \begin{array}{rl} \mathcal{L}(E,E;F) & \to \mathcal{L}(F^*,E,E;\mathbb{R}) \\ T_2 & \to \mathcal{J}_{23E}(T_2) = T_3 \end{array} \right\}, \quad T_3(\ell,\vec{u},\vec{v}) := \ell.T_2(\vec{u}_1,\vec{u}_2), \quad \forall \vec{u}_1,\vec{u}_2 \in E, \ \forall \ell \in F^*. \tag{U.19}$$

Proposition U.9 *$\mathcal{J}_{12}$ and $\mathcal{J}_{23}$ are natural. Thus $\mathcal{J}_{23} \circ \mathcal{J}_{12}$ is natural.*

Proof. 1- We have to prove that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(E;\mathcal{L}(E;F)) & \stackrel{\mathcal{J}_{12E}}{\longrightarrow} & \mathcal{L}(E,E;F) \\ Z_{AB}\downarrow & & \downarrow Y_{AB} \\ \mathcal{L}(A;\mathcal{L}(A;B)) & \stackrel{\mathcal{J}_{12A}}{\longrightarrow} & \mathcal{L}(A,A;B) \end{array} \quad \text{where} \quad \begin{array}{l} Z_{AB}(T_1)(\vec{a}_1).\vec{a}_2 := T_1(\mathcal{P}_1^{-1}.\vec{a}_1).(\mathcal{P}_1^{-1}.\vec{a}_2), \\ Y_{AB}(T_2)(\vec{a}_1,\vec{a}_2) = T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2), \end{array} \tag{U.20}$$

(the "push-forwards) for all $\vec{a}_1,\vec{a}_2 \in A$ and $L_{AB} \in \mathcal{L}(A;B)$.
Let $T_1 \in \mathcal{L}(E;\mathcal{L}(E;F))$. We have
$\mathcal{J}_{12A}(Z_{AB}(T_1))(\vec{a}_1).\vec{a}_2 = Z_{AB}(T_1)(\vec{a}_1).\vec{a}_2 = T_1(\mathcal{P}_1^{-1}.\vec{a}_1).(\mathcal{P}_1^{-1}.\vec{a}_2)$, and
$Y_{AB}(\mathcal{J}_{12E}(T_1))(\vec{a}_1,\vec{a}_2) = \mathcal{J}_{12E}(T_1)(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2) = T_1(\mathcal{P}_1^{-1}.\vec{a}_1).(\mathcal{P}_1^{-1}.\vec{a}_2)$,
thus $\mathcal{J}_{12A} \circ Z_{AB} = Y_{AB} \circ \mathcal{J}_{12E}$, thus $\mathcal{J}_{12}$ is natural.
2- We have to prove that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(E,E;F) & \stackrel{\mathcal{J}_{23E}}{\longrightarrow} & \mathcal{L}(F^*,E,E;\mathbb{R}) \\ Z_{AB}\downarrow & & \downarrow Y_{AB} \\ \mathcal{L}(A,A;B) & \stackrel{\mathcal{J}_{23A}}{\longrightarrow} & \mathcal{L}(B^*,A,A;\mathbb{R}) \end{array} \quad \text{where} \quad \begin{array}{l} \ell_B.Z_{AB}(T_2)(\vec{a}_1,\vec{a}_2) := (\ell_B.\mathcal{P}_2).T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2), \\ Y_{AB}(T_3)(\ell_B,\vec{a}_1,\vec{a}_2) = T_3(\ell_B.\mathcal{P}_2,\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2), \end{array} \tag{U.21}$$

(the "push-forwards) for all $\vec{a}_1,\vec{a}_2 \in A$ and $\ell_B \in B^*$.
Let $T_2 \in \mathcal{L}(E,E;F)$. We have
$\mathcal{J}_{23A}(\ell_B, Z_{AB}(T_2)(\vec{a}_1,\vec{a}_2)) = \ell_B.Z_{AB}(T_2)(\vec{a}_1,\vec{a}_2) = (\ell_B.\mathcal{P}_2).T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2)$, and
$Y_{AB}(\mathcal{J}_{23A}(T_2))(\ell_B,\vec{a}_1,\vec{a}_2) = \mathcal{J}_{23A}(T_2)(\ell_B.\mathcal{P}_2,\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2) = \ell_B.\mathcal{P}_2.T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2)$
thus $\mathcal{J}_{23A} \circ Z_{AB} = Y_{AB} \circ \mathcal{J}_{23E}$, thus $\mathcal{J}_{23}$ is natural. ∎

V Distribution in brief: A covariant concept

For a full description, see the books of Laurent Schwartz.

V.1 Definitions

Usual notations with Ω an open set in $\mathbb{R}^n$: Let $p \in [1,\infty[$ (e.g. $p = 2$ for finite energy functions), and let

$$L^p(\Omega) := \{f : \Omega \to \mathbb{R} : \int_\Omega |f(x)|^p \, d\Omega < \infty\} \quad \text{and} \quad ||f||_p = (\int_\Omega |f(x)|^p \, d\Omega)^{\frac{1}{p}}, \tag{V.1}$$

the space of functions f such that $|f|^p$ is Lebesgue integrable, with $||.||_p$ its usual norm. $(L^p(\Omega), ||.||_p)$ is a Banach space (a complete normed space). And let

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{R} : \sup_{x\in\Omega}(|f(x)|) < \infty\}, \quad \text{and} \quad ||f||_\infty = \sup_{x\in\Omega}(|f(x)|), \tag{V.2}$$

the space of Lebesgue measurable bounded functions, with $||.||_\infty$ its usual norm. $(L^\infty(\Omega), ||.||_{L^\infty})$ is a Banach space (a complete normed space).

Definition V.1 If $f \in \mathcal{F}(\Omega;\mathbb{R})$, then its support is the set

$$\text{supp}(f) := \overline{\{x \in \Omega : f(x) \neq 0\}} = \text{the closure of } \{x \in \Omega : f(x) \neq 0\}. \tag{V.3}$$

The closure in the definition of supp(f) is required: E.g., if $\Omega =]0,2\pi[$ and $f(x) = \sin x$, then $\{f \neq 0\} := \{x \in \Omega : f(x) \neq 0\} =]0,\pi[\cup]\pi,2\pi[$. Here $\pi \notin \{f \neq 0\}$, but π is a point of interest since sin varies in its vicinity: $f'(\pi) = -1 \neq 0$. So $\{f \neq 0\}$ is "too small", and it is its closure supp(f) $:= \overline{\{f \neq 0\}} = [0,2\pi]$ that is needed: supp(f) = the set where it is interesting to study f.

Schwartz notation:

$$\mathcal{D}(\Omega) := C_c^\infty(\Omega;\mathbb{R}) = \{\varphi \in C^\infty(\Omega;\mathbb{R}) \text{ s.t. supp}(\varphi) \text{ is compact in } \Omega\}. \tag{V.4}$$

E.g., $\Omega = \mathbb{R}$, $\varphi(x) := e^{-\frac{1}{1-x^2}}$ if $x \in]-1,1[$ and $\varphi(x) := 0$ elsewhere: $\varphi \in \mathcal{D}(\mathbb{R})$ with supp(φ) = $[-1,1]$.

Result: $\mathcal{D}(\Omega)$ is a vector space which is dense in $(L^p(\Omega), ||.||_{L^p})$ for any $p \in [1,\infty[$.

Definition V.2 A distribution in Ω is a linear $\mathcal{D}(\Omega)$-continuous[4] function

$$T : \begin{cases} \mathcal{D}(\Omega) & \to \mathbb{R} \\ \varphi & \to T(\varphi) \overset{\text{noted}}{=} \langle T, \varphi \rangle \end{cases} \tag{V.5}$$

The space of distribution in Ω is named $\mathcal{D}'(\Omega)$ (the dual of $\mathcal{D}(\Omega)$).

The notation $\langle T, \varphi \rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle T, \varphi \rangle$ is the "duality bracket" = the "covariance–contravariance bracket" between a continuous linear form $T \in \mathcal{D}'(\Omega)$ and a vector $\varphi \in \mathcal{D}(\Omega)$.

Definition V.3 Let $f \in L^p(\Omega)$. The regular distribution $T_f \in \mathcal{D}'(\Omega)$ associated to f is defined by

$$T_f(\varphi) := \int_\Omega f(x)\varphi(x)\, d\Omega, \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{V.6}$$

So T_f is a measuring instrument with density $dm_f(x) = f(x)\, d\Omega$, i.e. $T_f(\varphi) := \int_\Omega \varphi(x)\, dm_f(x)$.

Definition V.4 Let $x_0 \in \mathbb{R}^n$. The Dirac measure at x_0 is the distribution $T \overset{\text{noted}}{=} \delta_{x_0} \in \mathcal{D}'(\mathbb{R})$ defined by, for all $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\delta_{x_0}(\varphi) = \varphi(x_0), \quad \text{i.e.} \quad \langle \delta_{x_0}, \varphi \rangle = \varphi(x_0). \tag{V.7}$$

And δ_{x_0} is not a regular distribution (δ_{x_0} is not a density measure): There is no integrable function f such that $T_f = \delta_{x_0}$. Interpretation: δ_{x_0} corresponds to an ideal measuring device: The precision is perfect at x_0 (gives the exact value $\varphi(x_0)$ at x_0). In real life δ_{x_0} is the ideal approximation of T_{f_n} where f_n is e.g. given by $f_n(x) = n1_{[x_0, x_0+\frac{1}{n}]}$ (drawing): For all $\varphi \in \mathcal{D}(\Omega)$, $T_{f_n}(\varphi) \longrightarrow_{n\to\infty} \delta_{x_0}(\varphi) = \varphi(x_0)$.

Generalization of the definition: In (V.5) $\mathcal{D}(\Omega) = C_c^\infty(\Omega; \mathbb{R})$ is replaced by $C_c^\infty(\Omega; \vec{\mathbb{R}^n})$. So if you consider a basis $(\vec{e}_i)$ then $\vec{\varphi} \in C_c^\infty(\Omega; \vec{\mathbb{R}^n})$ reads $\vec{\varphi} = \sum_{i=1}^n \varphi^i \vec{e}_i$ with $\varphi^i \in \mathcal{D}(\Omega)$ for all i.

Example V.5 Power: Let $\alpha : \Omega \to T_1^0(\Omega)$ be a differential form. Then the distribution P_α defined by $P_\alpha(\vec{v}) = \int_\Omega \alpha.\vec{v}\, d\Omega$ gives the virtual power associated to α relative to the vector field $\vec{v}$. ∴

V.2 Derivation of a distribution

Let O be a point in $\mathbb{R}^n$ (an origin). If $p \in \mathbb{R}^n$ and if $(\vec{e}_i)$ is a basis in $\vec{\mathbb{R}^n}$, let $\vec{x} = \overrightarrow{Op} = \sum_{i=1}^n x_i \vec{e}_i$.

Definition V.6 The derivative $\frac{\partial T}{\partial x_i}$ of a distribution $T \in \mathcal{D}'(\Omega)$ is the distribution in $\mathcal{D}'(\Omega)$ defined by, for all $\varphi \in \mathcal{D}(\Omega)$,

$$\frac{\partial T}{\partial x_i}(\varphi) := -T(\frac{\partial \varphi}{\partial x_i}), \quad \text{i.e.} \quad \langle \frac{\partial T}{\partial x_i}, \varphi \rangle := -\langle T, \frac{\partial \varphi}{\partial x_i} \rangle. \tag{V.8}$$

($\frac{\partial T}{\partial x_i}$ is indeed a distribution: Easy check.)

Example V.7 If $T = T_f$ is a regular distribution with $f \in C^1(\Omega)$, then $\frac{\partial (T_f)}{\partial x_i} = T_{(\frac{\partial f}{\partial x_i})}$. Indeed, for all $\varphi \in \mathcal{D}(\Omega)$, $\frac{\partial (T_f)}{\partial x_i}(\varphi) = -T_f(\frac{\partial \varphi}{\partial x_i}) = -\int_\Omega f(x) \frac{\partial \varphi}{\partial x_i}\, d\Omega = +\int_\Omega \frac{\partial f}{\partial x_i} \varphi(x)\, d\Omega + \int_\Gamma 0\, d\Gamma$, since φ vanishes on $\Gamma = \partial\Omega$ (the support of φ is compact in Ω), thus $\frac{\partial (T_f)}{\partial x_i}(\varphi) = T_{(\frac{\partial f}{\partial x_i})}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$. ∴

Example V.8 Consider the Heaviside function (the unit step function) $H_0 := 1_{\mathbb{R}_+}$ and the associated distribution $T = T_{H_0}$. Then $\langle (T_{H_0})', \varphi \rangle := -\langle T_{H_0}, \varphi' \rangle = -\int_\Omega H_0(x)\varphi'(x)\, dx = -\int_0^\infty \varphi'(x)\, dx = \varphi(0) = \langle \delta_0, \varphi \rangle$ for any $\varphi \in \mathcal{D}(\mathbb{R})$, thus $(T_{H_0})' = \delta_0$. Written ${H_0}' = \delta_0$ in $\mathcal{D}'(\Omega)$, which is not in a equality between functions, because H_0 is not derivable at 0 as a function, and δ_0 is not a function; It is equality between distributions: The notation ${H_0}'$ can only be used to compute ${H_0}'(\varphi)$ $(= \langle {H_0}', \varphi \rangle := -\langle H_0, \varphi' \rangle)$. ∴

[4] The $\mathcal{D}(\Omega)$-continuity of T is defined by: 1- A sequence $(\varphi_n)_{\mathbb{N}^*}$ in $\mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ towards a function $\varphi \in \mathcal{D}(\Omega)$ iff there exists a compact $K \subset \Omega$ s.t. $\text{supp}(\varphi_n) \subset K$ for all n, and $||\frac{\partial^k \varphi}{\partial x_{i_1} \dots \partial x_{i_k}} - \frac{\partial^k \varphi_n}{\partial x_{i_1} \dots \partial x_{i_k}}||_\infty \longrightarrow_{n\to\infty} 0$ for all $k \in \mathbb{N}$ and all i_j; 2- T is continuous at $\varphi \in \mathcal{D}(\Omega)$ iff $T(\varphi_n) \underset{n\to\infty}{\longrightarrow} T(\varphi)$ for any sequence $(\varphi_n)_{\mathbb{N}} \in \mathcal{D}(\Omega)^{\mathbb{N}} \underset{n\to\infty}{\longrightarrow} \varphi$ in $\mathcal{D}(\Omega)$.

V.3 Hilbert space $H^1(\Omega)$

V.3.1 Motivation

Consider the hat function $\Lambda(x) \left\{ \begin{array}{l} = x+1 \text{ if } x\in[-1,0], \\ = 1-x \text{ if } x\in[0,1], \\ = 0 \text{ otherwise} \end{array} \right\}$ (drawing). When applying the finite element method, it is well-known that, if you use integrals (if you use the virtual power principle which makes you compute average values), then you can consider the derivative of the hat function Λ as if it was the usual derivative, i.e. at the points where the usual computation of Λ' is meaningful, that is,

$$\Lambda'(x) \left\{ \begin{array}{l} = 1 \text{ if } x\in]-1,0[, \\ = -1 \text{ if } x\in]0,1[, \\ = 0 \text{ if } x\in\mathbb{R}-\{-1,0,1\} \end{array} \right. \tag{V.9}$$

(drawing).

Problem: Λ' is not defined at $-1,0,1$ (the function Λ is not derivable at $-1,0,1$);
Question: So does (V.9) and the "usual" computation $I=\int_\mathbb{R}\Lambda'(x)\varphi(x)\,dx$ gives the good result? (This is not a trivial question: E.g., with $H_0=1_{\mathbb{R}_+}$ instead of Λ, we would get the absurd result $H_0'=0$, absurd since $H_0'=\delta_0$.)

Answer: Yes in the distribution meaning, i.e.:
1- Consider T_Λ the regular distribution associated to Λ, cf. (V.6);
2- Then consider $(T_\Lambda)'$, cf. (V.8): We get $\langle (T_\Lambda)',\varphi\rangle \overset{(V.8)}{=} -\langle T_\Lambda,\varphi'\rangle = -\int_\mathbb{R}\Lambda(x)\varphi'(x)\,dx = -\int_{-1}^0\Lambda(x)\varphi'(x)\,dx - \int_0^1\Lambda(x)\varphi'(x)\,dx = +\int_{-1}^0 1_{]-1,0[}(x)\varphi(x)\,dx + \int_0^1 1_{]0,1[}\varphi(x)\,dx$, for any $\varphi\in\mathcal{D}(\mathbb{R})$;
3- Thus $(T_\Lambda)'=T_f$ where $f=1_{]-1,0[}+1_{]0,1[}$, that is $(T_\Lambda)'$ is the regular distribution T_f.
4- Then $T_f=(T_\Lambda)' =^{\text{noted}} \Lambda'$ when used within the distribution framework, i.e. when used with $\varphi\in\mathcal{D}(\mathbb{R})$ and the Lebesgue integral $\int_\Omega\Lambda'(x)\varphi(x)\,dx := -\int_\Omega\Lambda(x)\varphi'(x)\,dx$: Ok for finite element methods.

V.3.2 Definition of $L^2(\Omega)$ and its dual

The space $C^0(\Omega;\mathbb{R})$ is too small in many applications, e.g. to consider step functions; Hence consider $L^2(\Omega) := \{f:\Omega\to\mathbb{R} : \int_\Omega |f(x)|^2\,d\Omega<\infty\}$ (the space of finite energy functions) with its usual inner dot product and norm defined by

$$(u,v)_{L^2} = \int_\Omega u(x)v(x)\,d\Omega \quad\text{and}\quad ||v||_2 = ||v||_{L^2} = \sqrt{(v,v)_{L^2}} = (\int_\Omega v(x)^2\,d\Omega)^{\frac{1}{2}}. \tag{V.10}$$

$(L^2(\Omega),(\cdot,\cdot)_{L^2})$ is a Hilbert space (Riesz-Fisher theorem).
The dual space of $L^2(\Omega)$ is the space

$$L^2(\Omega)' = \mathcal{L}(L^2(\Omega);\mathbb{R}) := \{\ell : L^2(\Omega)\to\mathbb{R} \text{ linear and continuous}\}, \tag{V.11}$$

i.e. the space of linear forms $\ell: L^2(\Omega)\to\mathbb{R}$ s.t. ℓ is linear and $\exists C>0$, $\forall v\in H^1(\Omega)$, $|\ell(v)|\le C||v||_{L^2}$.
$L^2(\Omega)'$ equipped with tbe norm $||\ell||_{L^2(\Omega)'} := \sup\limits_{||v||_{L^2(\Omega)}=1} |\ell(v)|$ is a Banach space.
Duality bracket: If $\ell\in L^2(\Omega)'$ then $\ell(v) =^{\text{noted}} \langle\ell,v\rangle_{L^{2'},L^2}$ for all $v\in L^2(\Omega)$.
And thanks to the $(\cdot,\cdot)_{L^2}$-Riesz representation theorem, a $\ell\in L^2(\Omega)'$ being linear and continuous, $\ell\in L^2(\Omega)'$ can be represented by function $f\in L^2(\Omega)$: $\exists f\in L^2(\Omega)$, $\forall v\in L^2(\Omega)$,

$$(\ell(v)=)\quad \langle\ell,v\rangle = (f,v)_{L^2} \quad (=\int_\Omega f(p)v(p)\,d\Omega). \tag{V.12}$$

NB: $L^2(\Omega)$ is called the "pivot space".
Idem with $\ell\in L^2(\Omega)^{n'}$: $\exists\vec{f}\in L^2(\Omega)^n$, $\forall\vec{v}\in L^2(\Omega)^n$, $\langle\ell,\vec{v}\rangle = (\vec{f},\vec{v})_{L^2} = \int_\Omega \vec{f}(p)\bullet\vec{v}(p)\,d\Omega$, an inner dot product in $\vec{\mathbb{R}^n}$ being a priori given.

V.3.3 Definition of $H^1(\Omega)$ and its dual

The space $C^1(\Omega;\mathbb{R})$ is too small in many applications (e.g., for the Λ function above). We need a larger space where the functions are "derivable is a weaker sense": The distribution sense. Consider a Cartesian basis $(\vec{e}_i)$ in $\vec{\mathbb{R}^n}$.

Definition V.9 The Sobolev space $H^1(\Omega)$ is the subspace of $L^2(\Omega)$ restricted to functions whose generalized derivatives are in $L^2(\Omega)$:

$$H^1(\Omega) = \{v \in L^2(\Omega) : [\vec{\text{grad}}v]_{|\vec{e}} \in L^2(\Omega)^n\} := \{v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega),\ \forall i = 1,...,n\}. \tag{V.13}$$

Usual shortened notation: $H^1(\Omega) = \{v \in L^2(\Omega) : \vec{\text{grad}}v \in L^2(\Omega)^n\}$.

So to check that $v \in H^1(\Omega)$, even if $\frac{\partial v}{\partial x_i}$ does not exists in the classic way (see the above hat function Λ), you have to: 1- Consider its associated regular distribution T_v, 2- Compute $\frac{\partial T_v}{\partial x_i}$ in $\mathcal{D}'(\Omega)$, 3- If, for all i, there exists $f_i \in L^2(\Omega)$ s.t. $\frac{\partial T_v}{\partial x_i} = T_{f_i}$, then $v \in H^1(\Omega)$. 4- Then $T_{f_i} = \frac{\partial T_v}{\partial x_i}$ is noted $\frac{\partial v}{\partial x_i}$ when used with $\varphi \in \mathcal{D}(\Omega)$ and the Lebesgue integral: $\int_\Omega \frac{\partial v}{\partial x_i}(x)\varphi(x)\,dx := \int_\Omega v(x)\frac{\partial \varphi}{\partial x_i}(x)\,dx$.

Then define, for all $u, v \in H^1(\Omega)$,

$$(u,v)_{H^1} = (u,v)_{L^2} + (\vec{\text{grad}}u, \vec{\text{grad}}v)_{L^2}, \quad \text{and} \quad ||v||_{H^1} = (v,v)_{H^1}^{\frac{1}{2}}, \tag{V.14}$$

where $(\vec{\text{grad}}u, \vec{\text{grad}}v)_{L^2} := \sum_{i=1}^n (\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i})_{L^2}$. Thus $(H^1(\Omega), (\cdot,\cdot)_{H^1})$ is a Hilbert space (Riesz–Fisher).

The dual space of $H^1(\Omega)$ is

$$H^1(\Omega)' := \mathcal{L}(H^1(\Omega);\mathbb{R}) := \{\ell : H^1(\Omega) \to \mathbb{R} \text{ linear and continuous}\} \tag{V.15}$$

i.e. the space of linear forms $\ell : H^1(\Omega) \to \mathbb{R}$ s.t. ℓ is linear and $\exists C > 0$, $\forall v \in H^1(\Omega)$, $|\ell(v)| \le C||v||_{H^1}$.

And (duality bracket) if $\ell \in H^1(\Omega)'$ then $\ell(v) =^{\text{noted}} \langle \ell, v\rangle_{H^{1\prime},H^1} =^{\text{noted}} \langle \ell, v\rangle$ for all $v \in H^1(\Omega)$.

Theorem V.10 *$\ell \in H^1(\Omega)'$ iff $\exists (f,\vec{u}) \in L^2(\Omega)\times L^2(\Omega)^n$, $\forall \psi \in H^1(\Omega)$,*

$$\ell(\psi) = (f,\psi)_{L^2} + (\vec{u}, \vec{\text{grad}}\psi)_{L^2}. \tag{V.16}$$

Proof. From Brézis [4] (application of the Riesz representation theorem). The space $Z = L^2(\Omega)\times L^2(\Omega)^3$ with its inner dot product $((f,\vec{u}),(g,\vec{v}))_Z := (f,g)_{L^2} + (\vec{u},\vec{v})_{L^2}$ is a Hilbert space. Let $T : H^1(\Omega) \to Z$ be defined by $T(\psi) = (\psi, \vec{\text{grad}}\psi)$; T is linear and $||T(\psi)||_Z = ||\psi||_{H^1}$, thus $T(\psi) = 0$ imply $\psi = 0$, so T is one-to-one, thus $T^{-1} : \text{Im}T \to H^1(\Omega)$ is well defined. And T^{-1} continuous since $T^{-1}(\psi, \vec{\text{grad}}\psi) = \psi$. (Remark: Im$T$ is not closed in Z.) Let $\ell \in H^1(\Omega)'$, then define $L : \text{Im}(T) \to \mathbb{R}$ by $\langle L, (\psi, \vec{\text{grad}}\psi)\rangle_{Z',Z} = \langle \ell, T^{-1}(\psi, \vec{\text{grad}}\psi)\rangle_{H^{1\prime},H^1}$: so $L = \ell \circ T^{-1}$ is linear continuous since ℓ and T^{-1} are, and $\langle L, (\psi, \vec{\text{grad}}\psi)\rangle_{Z',Z} = \langle \ell, \psi\rangle_{H^{1\prime},H^1}$; With Hahn–Banach theorem, extend $L : \text{Im}(T) \to \mathbb{R}$ to $L_Z : Z \to \mathbb{R}$ linear continuous. Apply Riesz representation theorem: $\exists (f,\vec{u}) \in Z$ s.t. $\langle L_Z, (\psi,\vec{w})\rangle_{Z',Z} = ((f,\vec{u}),(\psi,\vec{w}))_Z = (f,\psi)_{L^2} + (\vec{u},\vec{w})_{L^2}$ for all $(\psi,\vec{w}) \in Z$, in particular for all $(\psi,\vec{w}) \in \text{Im}T$, thus $\langle \ell,\psi\rangle_{H^{1\prime},H^1} = (f,\psi)_{L^2} + (\vec{u}, \vec{\text{grad}}\psi)_{L^2}$ for all $\psi \in H^1(\Omega)$. ∴

NB: For Neumann boundary value problems then (V.16) gives, if $\vec{u} \in H^1(\Omega)$,

$$\langle \ell,\psi\rangle_{(H^1)',H^1} = \int_\Omega f(x)\psi(x)\,dx - \int_\Omega \text{div}\vec{u}(x)\psi(x)\,dx + \int_\Gamma \vec{u}(x) \bullet \vec{n}(x)\,\psi(x)\,dx. \tag{V.17}$$

V.3.4 Subspace $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$

Definition:

$$H_0^1(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1} \quad \text{the closure of } \mathcal{D}(\Omega) \text{ in } H^1(\Omega). \tag{V.18}$$

So $H_0^1(\Omega)$ is closed in $H^1(\Omega)$, hence $(H_0^1(\Omega), (\cdot,\cdot)_{H^1})$ is a Hilbert space. If the boundary $\Gamma = \partial\Omega$ of Ω is bounded and regular then

$$\boxed{H_0^1(\Omega) = \{v \in H^1(\Omega) : v_{|\Gamma} = 0\}}. \tag{V.19}$$

(See Brézis [4].) The dual space of $(H_0^1(\Omega), ||.||_{H^1})$ is the space

$$(H_0^1(\Omega))' := \mathcal{L}(H_0^1(\Omega);\mathbb{R}) := \{\ell : H_0^1(\Omega) \to \mathbb{R} \text{ linear and continuous}\} \overset{\text{noted}}{=} H^{-1}(\Omega), \tag{V.20}$$

i.e. space of linear forms $\ell : H_0^1(\Omega) \to \mathbb{R}$ s.t. $\exists C > 0$, $\forall \psi \in H_0^1(\Omega)$, $|\ell(\psi)| \le C||\psi||_{H^1}$. And then $\ell(\psi) =^{\text{noted}} \langle \ell, \psi\rangle_{H^{-1},H_0^1}$ (duality bracket).

Theorem V.11 *$\ell \in H^{-1}(\Omega) = (H_0^1(\Omega))'$ iff $\exists (f,\vec{g}) \in L^2(\Omega) \times L^2(\Omega)^n$ s.t.*

$$\ell = f - \mathrm{div}\vec{g} \quad (\in \mathcal{D}'(\Omega)), \tag{V.21}$$

i.e., for all $\psi \in H_0^1(\Omega)$,

$$\langle \ell, \psi \rangle_{H^{-1},H_0^1} = \int_\Omega f\psi \, d\Omega + \int_\Omega d\psi.\vec{g} \, d\Omega. \tag{V.22}$$

And if Ω is bounded then we can choose $f = 0$, and moreover if $\vec{g} \in H^1(\Omega)^n$ then

$$\langle \ell, \psi \rangle_{H^{-1},H_0^1} = -\int_\Omega \mathrm{div}\vec{g}(x)\psi(x) \, dx. \tag{V.23}$$

(In fact we only need $\vec{g} \in H_{\mathrm{div}}(\Omega) = \{\vec{g} \in L^2(\Omega)^n : \mathrm{div}\vec{g} \in L^2(\Omega)\}$.)

Proof. Apply (V.16) here with $\psi \in \mathcal{D}(\Omega)$ or $\psi \in H_0^1(\Omega)$, so with $\psi_{|\Gamma} = 0$ (for the integration by parts). ▪

W Basics of thermodynamics

See https://perso.isima.fr/leborgne/IsimathMeca/Thermo.pdf

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