UNLOCKED P_1 FINITE ELEMENTS FOR THE MINDLIN–REISSNER THICK PLATE PROBLEM

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Abstract: Minimization problems of Mindlin–Reissner type treated by the finite element method can exhibit a locking phenomenon due to incompatible choice of the discrete spaces involved. Here such an incompatible choice is considered for which an analysis of an origin of the locking is performed. We modify the discrete equations in consequence, and these equations are then proved to be unlocked. Then a convergence proof is given. Numerical simulations of a Mindlin–Reissner type problem are proposed with the only use of P_1-continuous finite elements and give the expected unlocked result. The numerical added cost amounts mainly to invert a diagonal matrix in the case of P_1 finite elements.

1 Introduction

The motivation of this study was to try to use P_1-continuous finite elements: These elements are easy to compute and they are proposed in any finite element codes. Here we shall use the Matlab PDE Toolbox where we do not have the choice of the finite elements: The P_1-continuous finite elements are the only finite elements proposed. These elements are however well-known to yield a locked solution for Mindlin–Reissner type problem if no cure is introduced. In this paper we propose a new modification of the discrete problem to circumvent this problem.

The approach presented differs from more usual approaches that try to find adapted finite elements to a given discrete problem: See for example Arnold [1] for the analysis of the 1-D problem (Timoshenko beam), Brezzi and Fortin [8] [9] and Brezzi, Fortin and Sternberg [10] for the presentation and ideas of cures, Arnold and Brezzi [2] for a new idea that extracts from the start coercivity in both variables, Chenais and Paumier [14] for a choice of unlocked finite elements, Arnold and Falk [3] for analysis of some elements and a point on some new methods, Chapelle and Stenberg [13] for what they call "a slight modification" of the discrete equations, Capatina-Papaghiuc [12] for the use of non conforming finite elements.

We also refer to Brezzi and Bathe [7] for an introduction to the locking phenomenon and its relation to the inf-sup condition, to Babuska and Suri [5] for the computation of the degree of the locking.

The starting point of this paper is different: we start from a choice of (conforming) finite elements that are known to give a locked solution, and it is a modification of the discrete problem (addition of a new term) that is considered. This yields an added numerical cost that however will be negligeable when P_1 finite elements are used since this will introduce a diagonal matrix inverted at (almost) no cost.

In Section 2, Mindlin–Reissner type problems are introduced. Section 3 deals with a generalized problem and we compare the respective coercivity constants of the continuous and discrete problem, and then correct the discrete problem. In Section 4 the proof of the unlocking for the cured equations is performed. In Section 5 computations of errors are proposed. And in the last Section, numerical results are shown.

2 The Mindlin-Reissner equations

2.1 Notation

Let m be an integer ≥ 1 and Ω be a bounded regular domain in $\mathbb{R}^m$ which boundary is denoted Γ.

We equip $\mathbb{R}^m$ with its canonical scalar product $(\cdot,\cdot)_{\mathbb{R}^m}$. Thus for any differentiable function $f : \Omega \to \mathbb{R}$ we shall represent its differential $df(\underline{x})$ by its gradient $\text{grad}f(\underline{x})$ defined by $df(\underline{x})(\underline{v}) = (\text{grad}f(\underline{x}), \underline{v})_{\mathbb{R}^m}$ for all $\underline{v} \in \mathbb{R}^m$.

Then we consider the canonical basis or $\mathbb{R}^m$, and any vecteur $\underline{v} \in \mathbb{R}^m$ will be given by its components v_i in this basis, for $i =$

$1,...,n$. In particular, grad$f(\underline{x})$ is represented by the transpose of the line matrix $(\frac{\partial f}{\partial x_1}(\underline{x}),...,\frac{\partial f}{\partial x_n}(\underline{x}))$, for any $\underline{x}\in\Omega$.

We denote by $L^2(\Omega)$ the space of measurable functions that are square integrables on Ω, then by $H^1(\Omega)=\{v\in L^2(\Omega):$ grad$v\in L^2(\Omega)^m\}$, then by $H_0^1(\Omega)=\{v\in H^1(\Omega):v_{|\Gamma}=0\}$. The scalar product in $L^2(\Omega)$ will be denoted $(u,v)_{L^2}=\int_\Omega u(x)v(x)\,dx$. In $L^2(\Omega)^m$, we shall use the scalar product:

$$(\underline{u},\underline{v})_{L^2}=\int_\Omega(\underline{u}(x),\underline{v}(x))_{\mathbb{R}^m}\,dx=\int_\Omega\sum_{i=1}^m u_i(x)v_i(x)\,dx$$

where $\underline{u}=(u_i)_{i=1,...,m}$ in the canonical basis. And in $L^2(\Omega)^{m^2}$, we shall use the scalar product:

$$(\underline{\underline{A}},\underline{\underline{B}})_{L^2}=\int_\Omega\sum_{i,j=1}^m A_{ij}(x)B_{ij}(x)\,dx$$

with the generic matrix notation $\underline{\underline{A}}=(A_{ij})_{i,j=1,...,m}$: This last notation is used for "(grad$\underline{u}$, grad$\underline{v})_{L^2}$" where grad$\underline{u}$ is represented by the matrix $[\frac{\partial u_i}{\partial x_j}]_{i,j=1,...,m}$.

The associated norm in $L^2(\Omega)$ will be denoted $||v||_{L^2}=(v,v)_{L^2}^{\frac{1}{2}}$ for any $v\in L^2(\Omega)$ or $v\in L^2(\Omega)^m$ or $v\in L^2(\Omega)^{m^2}$. The norm used in $H_0^1(\Omega)$ or in $H_0^1(\Omega)^m$ is $||v||_{H_0^1}=||\text{grad}v||_{L^2}$, where v is a scalar or vector function.

In $H_0^1(\Omega)$, we shall use the Poincaré's inequality (since Ω is bounded):

$$\exists\beta>0,\quad\forall u\in H_0^1(\Omega),\quad||\text{grad}u||_{L^2}^2\geq\beta||u||_{L^2}^2. \tag{2.1}$$

For the numerical computation, we shall consider a finite element triangulation of Ω and denote by P_k the finite elements for which the associated functions are continuous over Ω and polynomial of degree k on each triangle. And for a vector function $\underline{v}=(v_i)_{i=1,...,m}$ such that $v_i\in P_k$ for any $i=1,...,m$, we also refer to $\underline{v}$ as being P_k.

And we consider conforming approximations: if the unknown is looked for in a space V, its computed approximation is looked for in a finite dimensional subspace $V_h\subset V$.

2.2 The Mindlin-Reissner problem

The usual Mindlin–Reissner plate model with Dirichlet boundary conditions reads: Find $(\underline{u},p)\in(H_0^1(\Omega))^2\times H_0^1(\Omega)$ such that:

$$M(\underline{u},p)=\inf_{(\underline{v},q)\in(H_0^1(\Omega))^2\times H_0^1(\Omega)}M(\underline{v},q), \tag{2.2}$$

where:

$$M(\underline{v},q) = \frac{1}{2}a(\underline{v},\underline{v}) + \frac{\lambda}{2}||\underline{v} - \text{grad}q||^2_{L^2} - (f,q)_{L^2}, \qquad (2.3)$$

and:

$$a(\underline{v},\underline{v}) = \frac{E}{12(1-\nu^2)}\int_\Omega \Big[(\frac{\partial v_1}{\partial x} + \nu\frac{\partial v_2}{\partial y})\frac{\partial v_1}{\partial x} + (\nu\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y})\frac{\partial v_2}{\partial y} + \frac{(1-\nu)}{2}(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x})^2 \Big]\, dx\, dy.$$

E is the Young modulus, ν the Poisson coefficient. In (2.2) and (2.3) $\underline{u}$ is the rotation vector, p the transverse displacement, f the tranverse load, $\lambda = \frac{kE}{2(1+\nu)t^2}$ where k is a shear correction factor and t is the thickness of the plate. And as the thickness vanishes, λ increases to infinity. The value of E and ν are such that $a(\cdot,\cdot)$ is a scalar product on $H^1_0(\Omega)^2$ equivalent to the $H^1_0(\Omega)^2$ usual $(u,v)_{H^1_0} = (\text{grad}\underline{u}, \text{grad}\underline{v})_{L^2}$ scalar product.

The interpretation of this Mindlin–Reissner plate model is classical and can be found for example in Brezzi and Fortin [9] and references therein.

In this paper we consider in (2.3) any $a(\cdot,\cdot)$ that is a scalar product on $H^1_0(\Omega)^2$ equivalent to the $H^1_0(\Omega)^2$ usual $(\cdot,\cdot)_{H^1_0}$ scalar product. And for the numerical computation (to test the locking or the unlocking) we choose $a(\cdot,\cdot) = (\cdot,\cdot)_{H^1_0}$. In that case, the Euler associated problem to (2.2) reads: Find $(\underline{u},p) \in (H^1_0(\Omega))^2 \times H^1_0(\Omega)$ such that for all $(\underline{v},q) \in (H^1_0(\Omega))^2 \times H^1_0(\Omega)$:

$$\begin{cases} (\text{grad}\underline{u}, \text{grad}\underline{v})_{L^2} + \lambda(\underline{u} - \text{grad}p, \underline{v})_{L^2} = (\underline{g},\underline{v})_{L^2}, \\ \lambda(\underline{u} - \text{grad}p, -\text{grad}q)_{L^2} = (f,q)_{L^2}, \end{cases} \qquad (2.4)$$

i.e.:

$$\Phi((\underline{u},p),(\underline{v},q)) = (\underline{g},\underline{v})_{L^2} + (f,q)_{L^2},$$

where:

$$\Phi((\underline{u},p),(\underline{v},q)) = (\text{grad}\underline{u}, \text{grad}\underline{v})_{L^2} + \lambda(\underline{u} - \text{grad}p, \underline{v} - \text{grad}q)_{L^2}.$$

3 General setting and interpretation of the problem

3.1 The setting

We consider a Hilbert space H and denote $(\cdot,\cdot) = (\cdot,\cdot)_H$ and $||.|| = ||.||_H$ the scalar product and associated norms in H. If k is any integer, we also denote $(\cdot,\cdot) = (\cdot,\cdot)_H$ and $||.|| = ||.||_H$ the scalar product and associated norms in the Cartesian product $H^k = H \times H ... \times H$ (k times). Then we consider two Hilbert spaces V and Q such that $V \subset H^m$ and $Q \subset H^n$, m and n integers, and we suppose that V is dense in H^m.

We consider a scalar product $(u,v)_V$ on V and denote $||v||_V$ the associate norm, and we suppose that the natural injection from V to H^m is continuous:

$$\exists c_V > 0, \quad \forall v \in V, \quad ||v||_H \le c_V ||v||_V. \tag{3.1}$$

We consider a differential operator $G : Q \to H^m$ such that $(Gp, Gq)_H \overset{\text{denoted}}{=} (p,q)_Q$ defines a scalar product on Q. We denote $||q||_Q = ||Gq||_H$ the associated norm, and we suppose that the natural injection from Q to H^n is continuous:

$$\exists c_Q > 0, \quad \forall q \in Q, \quad ||q||_H \le c_Q ||q||_Q. \tag{3.2}$$

And we equip $V \times Q$ with the norm $(||v||_V^2 + ||q||_Q^2)^{\frac{1}{2}}$.

We then consider finite dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$ and denote by Π_{V_h} and Π_{Q_h} the orthogonal projection operator from H^m to V_h or on Q_h relative to the $(\cdot,\cdot)_H$ scalar product. I.e., for $x \in H^m$ and $y \in H^n$, $\Pi_{V_h} x \in V_h$ and $\Pi_{Q_h} y \in Q_h$ are characterized by:

$$(\Pi_{V_h} x, v_h)_H = (x, v_h)_H, \quad \forall v_h \in V_h,$$
$$(\Pi_{Q_h} y, q_h)_H = (y, q_h)_H, \quad \forall q_h \in Q_h.$$

And we have Pythagorean equality "$||\Pi_{V_h} x||_H^2 + ||x - \Pi_{V_h} x||_H^2 = ||x||_H^2$" as well as the identity "$(x - \Pi_{V_h} x, \tilde{x} - \Pi_{V_h} \tilde{x}) = (x - \Pi_{V_h} x, \tilde{x})$" (idem with $\Pi_{Q_h} y$).

Then we consider a bilinear, continuous and coercive form $a(\cdot,\cdot)$ on V, and denote $||a||$ the continuity constant and $\alpha > 0$ and $\beta > 0$ the constants satisfying:

$$a(v,v) \ge \alpha ||v||_V^2 \quad \text{and} \quad a(v,v) \ge \beta ||v||_H^2, \quad \forall v \in V, \tag{3.3}$$

where β exists since we have supposed that the natural injection from V to H^m is continuous. In particular, when $a(\cdot,\cdot)$ is symmetric, $a(\cdot,\cdot)$ is a scalar product on V equivalent to the $(\cdot,\cdot)_V$ scalar product.

3.2 The continuous problem

For some given $g \in H^m$ and $f \in H^n$ we look at the problem: Find $(u,p) \in V \times Q$ such that for all $(v,q) \in V \times Q$:

$$\Phi((u,p),(v,q)) = (g,v) + (f,q), \tag{3.4}$$

where for any $\lambda > 0$:

$$\Phi((u,p),(v,q)) = a(u,v) + \lambda(u - Gp, v - Gq). \tag{3.5}$$

And we will be interested in the case λ 'large'. Problem (3.4) also reads: Find $(u,p) \in V \times Q$ such that for all $(v,q) \in V \times Q$:

$$\begin{cases} a(u,v) + \lambda(u - Gp, v) = (g, v), \\ \lambda(u - Gp, -Gq) = (f, q). \end{cases} \tag{3.6}$$

In case $a(\cdot,\cdot)$ is symmetric, problem (3.4) also reads: Find $(u,p) \in V \times Q$ such that:

$$M(u,p) = \inf_{(v,q)\in V\times Q} M(v,q), \tag{3.7}$$

where:

$$M(v,q) = \frac{1}{2}a(v,v) + \frac{\lambda}{2}||v - Gq||^2 - (g,v) - (f,q). \tag{3.8}$$

Proposition 3.1 *For all (v,q) in $V \times Q$:*

$$\Phi((v,q),(v,q)) \geq \alpha ||v||_V^2,$$

and:

$$\Phi((v,q),(v,q)) \geq \frac{\beta\lambda}{\lambda+\beta}||q||_Q^2, \tag{3.9}$$

where $\frac{\beta\lambda}{\lambda+\beta}$ (coercivity constant on Q) is optimal and of order 0 in λ.

And $\Phi(\cdot,\cdot)$ is continuous on $V \times Q$: it exists $c_\Phi > 0$ such that for all (u,p) and (v,q) in $V \times Q$:

$$\Phi((u,p),(v,q)) \leq c_\Phi \sqrt{||u||_V^2 + ||p||_Q^2}\sqrt{||v||_V^2 + ||q||_Q^2}, \tag{3.10}$$

where $c_\Phi = O(\lambda)$ (continuity constant $= ||\Phi||$) as λ increases to infinity. Then problem (3.4) is well-posed in $V \times Q$.

Proof. It is straightforward that $\Phi((v,q),(v,q)) \geq \alpha||v||_V^2$ (with α given in (3.3)), for any $(v,q) \in V \times Q$.

With β given in (3.3) we have, for any $(v,q) \in V \times Q$:

$$\Phi((v,q),(v,q)) \geq \beta||v||_H^2 + \lambda||v - Gq||_H^2,$$

i.e.:

$$\Phi((v,q),(v,q)) \geq (\beta + \lambda)||v||_H^2 + 2\lambda||v||_H\,||Gq||_H + \lambda||Gq||_H^2.$$

And the max of the constants c that satisfy "$(\beta+\lambda)x^2+2\lambda xy+\lambda y^2 \geq cy^2$" is $c = \frac{\beta\lambda}{\lambda+\beta}$. Indeed we want that the second order polynomial "$P(X) = (\beta+\lambda)X^2 + 2\lambda X + (\lambda-c)$" to have its discriminant is ≤ 0 (no solution for the equation $P(X) = 0$), i.e such that $\lambda^2 - (\beta+\lambda)(\lambda-c) \leq 0$, i.e. such that $c \leq \frac{\beta\lambda}{\lambda+\beta}$. And then (3.9).

Since v and q are independent variables, and V is dense in H, we can choose v as close as wished to Gq (in the $||.||_H$-norm), so that $c = \frac{\beta\lambda}{\lambda+\beta}$ is indeed the best constant. And as $\lambda \to \infty$, we have $\frac{\beta\lambda}{\lambda+\beta} \to \beta = \beta\lambda^0$: $\frac{\beta\lambda}{\lambda+\beta}$ is of order 0 in λ as λ increases to infinity.

Then with $\alpha_\Phi = \min(\alpha, \frac{\beta\lambda}{\lambda+\beta})$, we get the coercivity of Φ on $V \times Q$.

Then we have, for any $(u,p), (v,q) \in V \times Q$:

$$\begin{aligned}
&\Phi((u,p),(v,q)) \\
&\leq ||a||\,||u||_V||v||_V + \lambda c_V^2||u||_V||v||_V \\
&\qquad + \lambda c_V(||u||_V||q||_Q + ||v||_V||p||_Q) + \lambda||p||_Q||q||_Q
\end{aligned}$$

from which we get (3.10).

Then by Lax–Milgram Theorem the problem is well-posed: There exists a unique solution $(u,p) \in V \times Q$ that satisfies $\Phi((u,p),(u,p)) \leq ||f||_H||p||_H + ||g||_H||u||_H$ and then:

$$(||u||_V^2 + ||p||_Q^2)^{\frac{1}{2}} \leq \frac{\alpha_\Phi}{c_\Phi}(c_V^2||g||_H^2 + c_Q^2||f||_H^2)^{\frac{1}{2}},$$

where c_Q and c_V are the continuity constants defined in (3.1) and (3.2). ∎

Remark 3.2 The difference of order between the coercivity constant $\alpha_\Phi = O(1)$ and the continuity constant $||\Phi|| = O(\lambda)$ is expected to yield difficulties for the discrete associated equations. Indeed the conditioning of the associated matrix to invert will be of order $\frac{||\Phi||}{\alpha_\Phi} = 0(\lambda)$ so that numerics will explode with λ. This is expected.

However, what is not expected is that the discrete solution degrades much faster than expected from the matrix conditioning number: The results for problem (2.4) are already inacceptable for relatively small values of λ on a reasonnable mesh, for example $\lambda = 100$ with an usual mesh. ∎

3.3 The discrete associated problem

The discrete problem associated to (3.4) reads: Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$:

$$\Phi((u_h,p_h),(v_h,q_h)) = (g, v_h) + (f, q_h). \tag{3.11}$$

And for this discrete problem we get (to compare to (3.9)):

Proposition 3.3 *For any* $(v_h,q_h)\in V_h\times Q_h$ *we have:*

$$\Phi((v_h,q_h),(v_h,q_h))\geq\frac{\beta\lambda}{\lambda+\beta}||q_h||_Q^2+\frac{\lambda^2}{\lambda+\beta}||Gq_h-\Pi_{V_h}Gq_h||_H^2.\quad(3.12)$$

Proof. We have:

$$\begin{aligned}(u_h-Gp_h,-Gq_h)&=(u_h-\Pi_{V_h}Gp_h,-Gq_h)+(Gp_h-\Pi_{V_h}Gp_h,Gq_h)\\&=(u_h-\Pi_{V_h}Gp_h,-\Pi_{V_h}Gq_h)+(Gp_h-\Pi_{V_h}Gp_h,Gq_h-\Pi_{V_h}Gq_h),\end{aligned}$$

so that the functional $\Phi(\cdot,\cdot)$ (given in (3.5)) also reads in $V_h\times Q_h$:

$$\begin{aligned}\Phi((u_h,p_h),(v_h,q_h))=\;&a(u_h,v_h)+\lambda(u_h-\Pi_{V_h}Gp_h,v_h-\Pi_{V_h}Gq_h)\\&+\lambda(Gp_h-\Pi_{V_h}Gp_h,Gq_h-\Pi_{V_h}Gq_h).\end{aligned}$$

Then, for any $(v_h,q_h)\in V_h\times Q_h$, with the a computation similar to the one of the previous proof:

$$\Phi((v_h,q_h),(v_h,q_h))\geq\frac{\beta\lambda}{\lambda+\beta}||\Pi_{V_h}Gq_h||^2+\lambda||Gq_h-\Pi_{V_h}Gq_h||^2.$$

Then with $||\Pi_{V_h}Gq_h||^2=||Gq_h||^2-||Gq_h-\Pi_{V_h}Gq_h||^2$ (Pythagorean relation) and $-\frac{\beta\lambda}{\lambda+\beta}+\lambda=\frac{\lambda^2}{\lambda+\beta}$ we get (3.12). ∎

3.4 An interpretation of the locking

We consider the case $H=L^2(\Omega)$, $V=H_0^1(\Omega)^m$ and $Q=H_0^1(\Omega)$, together with $G=\text{grad}$.

For the continuous problem (3.4), the coercivity constant is of order 0 in λ, see (3.9), and $||p||_{H_0^1}=O(||f||_{L^2})$ as λ increases to infinity, i.e. $||p||_{H_0^1}$ is of the same order as $||f||_{L^2}$.

But, for the discrete counterpart (3.11): Suppose V_h is "small" compared to Q_h, for example small such that there exists some direction p_h verifying $\text{grad}p_h\neq 0$ and $\Pi_{V_h}\text{grad}p_h=0$. Then such p_h yield a coercivity constant "$=\beta\frac{\lambda}{\lambda+\beta}+\lambda\frac{\lambda}{\lambda+\beta}=\lambda$ in (3.12) which is of order 1 in λ. Then if such p_h is solution we could have $||p_h||_{H_0^1}=\frac{1}{\lambda}O(||f||_{L^2})$ and p_h vanishes as λ increases to infinity (locking phenomenon).

Such p_h directions (or at least directions p_h such that $\Pi_{V_h}\text{grad}p_h$ is one order of magnitude smaller than $\text{grad}p_h$) are usual in finite element computations: Just take the 1-D case $\Omega=]-1,1[$, its partition $\Omega=]-1,0[\bigcup]0,1[$, the space $H_0^1(\Omega)$, its discrete $V_h=P_1$ approximation which is here of dimension 1 and generated by the hat P_1 basis fonction φ given by $\varphi(-1) = \varphi(1) = 0$ and $\varphi(0) = 1$, that is $\varphi(x) = x{+}1$ on $]-1,0[$ and $\varphi(x) = -x{+}1$ on $]0,1[$. We have $\text{grad}\varphi = \varphi'$ that equals $+1$ on $]-1,0[$ and -1 on $]0,1[$ (discontinuous), whereas its (continuous) projection $\Pi_{V_h}\text{grad}\varphi$ satisfied $\Pi_{V_h}\text{grad}\varphi = 0$ (trivial computation).

3.5 The discrete corrected problem

As a corollary of Proposition 3.3, we define on $V \times Q$:

$$\begin{aligned}&\Phi_{1h}((u,p),(v,q)) \\ &= \Phi((u,p),(v,q)) - \frac{\lambda^2}{\lambda{+}\beta}(Gp{-}\Pi_{V_h}Gp, Gq{-}\Pi_{V_h}Gq)\end{aligned} \tag{3.13}$$

where $\Phi(\cdot,\cdot)$ has been defined in (3.5).

Proposition 3.4 *For any $(v_h,q_h) \in V_h \times Q_h$:*

$$\Phi_{1h}((v_h,q_h),(v_h,q_h)) \geq \frac{\lambda\beta}{\lambda{+}\beta}||Gq_h||^2_{L^2}, \tag{3.14}$$

and $\frac{\lambda\beta}{\lambda+\beta}$ is the maximum coercivity in the variable q_h.

Proof. Φ_{1h} has been built for that, see (3.12). ∎

In many applications, we don't know the value of β: We can only estimate it. And we cannot take an estimate of β larger than β if we don't want to destroy the coercivity in V or V_h. We then define:

$$\beta_c = \frac{\beta}{c}, \quad c \geq 1. \tag{3.15}$$

Remark 3.5 An estimation of the Rayleigh quotient gives an approximation of β. Then for the numerical computations we shall choose $\beta_c \simeq \beta_2 = \frac{\beta}{2}$.

And we will not be able to take $\beta_c = 0$, i.e make $c \to \infty$: This would lead to the vanishing of some required coercivity in p_h. ∎

And we define on $V \times Q$ (and then on $V_h \times Q_h$):

$$\begin{aligned}&\Phi_h((u,p),(v,q)) \\ &= \Phi((u,p),(v,q)) - \frac{\lambda^2}{\lambda{+}\beta_c}(Gp{-}\Pi_{V_h}Gp, Gq{-}\Pi_{V_h}Gq)\end{aligned} \tag{3.16}$$

And the discrete corrected problem now reads: Find $(u_h,p_h) \in V_h \times Q_h$ such that for all $(v_h,q_h) \in V_h \times Q_h$:

$$\Phi_h((u_h,p_h),(v_h,q_h)) = (g,v_h) + (f,q_h), \tag{3.17}$$

i.e.:

$$\begin{cases} a(u_h,v_h)+\lambda(u_h-Gp_h,v_h)=(g,v_h),\\ \lambda(u_h-Gp_h,-Gq_h)-\dfrac{\lambda^2}{\lambda+\beta_c}(Gp_h-\Pi_{V_h}Gp_h,Gq_h)=(f,q_h). \end{cases} \tag{3.18}$$

Proposition 3.6 *We have, for any $(v_h,q_h)\in V_h\times Q_h$:*

$$\Phi_h((v_h,q_h),(v_h,q_h))\geq\alpha||v_h||_V^2,$$

and:

$$\Phi_h((v_h,q_h),(v_h,q_h))\geq\gamma_h||q_h||_Q^2, \tag{3.19}$$

where $\gamma_h=\frac{\lambda^2(\beta-\beta_c)}{(\lambda+\beta)(\lambda+\beta_c)}$ is the maximun coercivity constant on Q_h, and $\gamma_h\in[\frac{\beta\lambda}{\lambda+\beta},2\frac{\beta\lambda}{\lambda+\beta}]$ is of order 0 in λ.

And $\Phi_h(\cdot,\cdot)$ is continuous on $V\times Q$ and on $V_h\times Q_h$ with a continuity constant of order 1 in λ. And problem (3.17) is well-posed.

Proof. The only really new result to prove is (3.19).

We have $\frac{\lambda^2}{\lambda+\beta_c}=\frac{\lambda^2}{\lambda+\beta}+(\frac{\lambda^2}{\lambda+\beta_c}-\frac{\lambda^2}{\lambda+\beta})=\frac{\lambda^2}{\lambda+\beta}+\frac{\lambda^2(\beta-\beta_c)}{(\lambda+\beta)(\lambda+\beta_c)}$, and then:

$$\begin{aligned}&\Phi_h((\underline{v}_h,q_h),(\underline{v}_h,q_h))\\ &=\Phi_{1h}((\underline{v}_h,q_h),(\underline{v}_h,q_h))+\frac{\lambda^2(\beta-\beta_c)}{(\lambda+\beta)(\lambda+\beta_c)}||Gq_h-\Pi_{V_h}Gq_h||_{L^2}^2.\end{aligned}$$

And $\beta-\beta_c=\beta\frac{c-1}{c}<\beta$, then $\frac{\lambda^2(\beta-\beta_c)}{(\lambda+\beta)(\lambda+\beta_c)}=\frac{\lambda(\beta-\beta_c)}{(\lambda+\beta)}\frac{\lambda}{(\lambda+\beta_c)}\in[0,\frac{\lambda\beta}{(\lambda+\beta)}]$, and then the result. ∎

We also have:

$$\begin{aligned}\Phi_h((u_h,p_h),(v_h,q_h))&=a(u_h,v_h)+\lambda(u_h-\Pi_{V_h}Gp_h,v_h-\Pi_{V_h}Gq_h)\\ &\quad+\beta_c\frac{\lambda}{\lambda+\beta_c}(Gp_h-\Pi_{V_h}Gp_h,Gq_h-\Pi_{V_h}Gq_h),\end{aligned}$$

and the practical computation will be done as (with $w_h=\Pi_{V_h}Gp_h$): Find $(u_h,p_h,w_h)\in V_h\times Q_h\times V_h$ such that for all $(v_h,q_h,w_h')\in V_h\times Q_h\times V_h$:

$$\begin{cases} a(u_h,v_h)+\lambda(u_h,v_h)-\lambda(Gp_h,v_h)=(g,v_h),\\ -\lambda(u_h,Gq_h)+\dfrac{\beta_c\lambda}{\lambda+\beta_c}(Gp_h,Gq_h)+\dfrac{\lambda^2}{\lambda+\beta_c}(w_h,Gq_h)=(f,q_h),\\ \dfrac{\lambda^2}{\lambda+\beta_c}(Gp_h,w_h')-\dfrac{\lambda^2}{\lambda+\beta_c}(w_h,w_h')=0. \end{cases} \tag{3.20}$$

And in the case $V_h = P_1$ finite elements, the last equation can be solved very cheaply with the mass lumping technique: The mass matrix resulting from the $(\underline{w}_h, \underline{w}'_h)_{L^2}$ term is made diagonal and its inverse is thus computed at (almost) no cost. And we then solve the two equation problem in two unknowns (3.18) instead of the above three equation problem (3.20).

Remark 3.7 If $a(\cdot,\cdot)$ is symmetric, problem (3.17) also reads: Find $(u_h, p_h) \in V_h \times Q_h$ such that:

$$M_h(u_h, p_h) = \inf_{(v_h,q_h)\in V_h\times Q_h} M_h(v_h, q_h), \tag{3.21}$$

where

$$\begin{aligned} M_h(v_h, q_h) = \frac{1}{2}a(v_h, v_h) + \frac{\lambda}{2}||v_h - Gq_h||^2 \\ - \frac{\lambda^2}{2(\lambda+\beta_c)}||Gq_h - \Pi_{V_h}Gq_h||^2 - (f, q_h) - (g, v_h), \end{aligned}$$

(to compare to (3.8)). ∎

3.6 Error computation

We consider the solution (u, p) of (3.4) and the solution (u_h, p_h) of (3.17) (discrete corrected problem). We have the convergence result:

Proposition 3.8 *For any $(u_i, p_i) \in V_h \times Q_h$, it exists $d_h > 0$ such that:*

$$\begin{aligned} &(||u-u_h||_V^2 + ||p-p_h||_Q^2)^{\frac{1}{2}} \\ &\le d_h(\lambda)[(||u-u_i||_V^2 + ||p-p_i||_Q^2)^{\frac{1}{2}} + ||Gp - \Pi_{V_h}Gp||_H], \end{aligned} \tag{3.22}$$

and $d_h = d_h(\lambda) = O(\lambda)$.

Proof. We use the finite element technique: We have, for any $(v_h, q_h) \in V_h \times Q_h$:

$$\Phi_h((u-u_h, p-p_h), (v_h, q_h)) = \frac{\lambda^2}{\lambda+\beta_c}(Gp - \Pi_{V_h}Gp, Gq_h).$$

And for any $(v_i, p_i) \in V_h \times Q_h$ (interpolants), we get:

$$\begin{aligned} \Phi_h((u_i-u_h, p_i-p_h), (v_h, q_h)) = \Phi_h((u_i-u, p_i-p), (v_h, q_h)) \\ + \frac{\lambda^2}{\lambda+\beta_c}(Gp - \Pi_{V_h}Gp, Gq_h). \end{aligned}$$

With Proposition 3.6 we get, with $v_h = u_i - u_h$, $q_h = p_i - p_h$ and $0 \le \frac{\lambda}{\lambda+\beta} \le 1$:

$$\begin{aligned}
&\alpha_{\Phi_h}(||u_i - u_h||_V^2 + ||p_i - p_h||_Q^2) \\
&\le ||\Phi_h||(||u - u_i||_V^2 + ||p - p_i||_Q^2)^{\frac{1}{2}}(||u_i - u_h||_V^2 + ||p_i - p_h||_Q^2)^{\frac{1}{2}} \\
&\qquad + \lambda ||Gp - \Pi_{V_h} Gp||_H ||G(p_i - p_h)||_H,
\end{aligned}$$

where $\alpha_{\Phi_h} = \min(\alpha, \frac{\beta\lambda}{\lambda+\beta})$. And with $||G(p_i - p_h)||_H = ||p_i - p_h||_Q \le (||u_i - u_h||_Q^2 + ||p_i - p_h||_Q^2)^{\frac{1}{2}}$ we get:

$$\begin{aligned}
&(||u_i - u_h||_V^2 + ||p_i - p_h||_Q^2)^{\frac{1}{2}} \\
&\le \frac{||\Phi_h||}{\alpha_{\Phi_h}}(||u - u_i||_V^2 + ||p - p_i||_Q^2)^{\frac{1}{2}} + \frac{\lambda}{\alpha_{\Phi_h}} ||Gp - \Pi_{V_h} Gp||_H.
\end{aligned}$$

Then, with $||u - u_h||^2 \le 2||u - u_i||^2 + 2||u_i - u_h||^2$ and $||p - p_h||^2 \le 2||p - p_i||^2 + 2||p_i - p_h||^2$, we get (3.22). ∎

Remark 3.9 We cannot expect a bound independent of λ since the continuity constant $||\Phi||$ for the continuous problem is of order $O(\lambda)$, whereas the coercivity constant is of order $O(1)$.

Then the above error computation shows that the added correction term does not destroy the error: We have the usual expected result. ∎

3.7 The variable $s = (\lambda+\beta)u - \lambda Gp$

We extract β_c-coercivity out of $a(\cdot,\cdot)$ (procedure used to compute the coercivity in the variable q_h). Rewrite (3.6) as: Find $(u, p) \in V \times Q$ such that for all $(v, q) \in V \times Q$:

$$\begin{cases}
(a(u, v) - \beta_c(u, v)) + (\lambda+\beta_c)(u, v) - \lambda(Gp, v) = (g, v), \\
\dfrac{\lambda}{\lambda+\beta_c}[(\lambda+\beta_c)u - \lambda Gp, -Gq) + \beta_c(Gp, Gq)] = (f, q).
\end{cases}$$

Then we introduce the variable $s = (\lambda+\beta_c)u - \lambda Gp \in H^m$, and the problem reads: Find $(u, p, s) \in V \times Q \times H^m$ such that for all $(v, q, s') \in V \times Q \times H^m$:

$$\begin{cases}
(a(u, v) - \beta_c(u, v)) + (s, v) = (g, v), \\
\dfrac{\beta_c\lambda}{\lambda+\beta_c}(Gp, Gq) - \dfrac{\lambda}{\lambda+\beta_c}(s, Gq) = (f, q), \\
(u, s') - \dfrac{\lambda}{\lambda+\beta_c}(Gp, s') - \dfrac{1}{\lambda+\beta_c}(s, s') = 0.
\end{cases} \tag{3.23}$$

This problem is well-posed since (3.6) is.

Now consider the discrete problem associated to (3.23): Find $(u_h, p_h, s_h) \in V_h \times Q_h \times V_h$ such that for all $(v_h, q_h, s'_h) \in V_h \times Q_h \times V_h$:

$$\begin{cases} (a(u_h, v_h) - \beta_c(u_h, v_h)) + (s_h, v_h) = (g, v_h), \\ \dfrac{\beta_c \lambda}{\lambda+\beta_c}(Gp_h, Gq_h) - \dfrac{\lambda}{\lambda+\beta_c}(s_h, Gq_h) = (f, q_h), \\ (u_h, s'_h) - \dfrac{\lambda}{\lambda+\beta_c}(Gp_h, s'_h) - \dfrac{1}{\lambda+\beta_c}(s_h, s'_h) = 0, \end{cases} \tag{3.24}$$

where the new variable s_h in looked for in V_h. Since $V_h \subset V$ and $V \subset H$, we have $V_h \subset H$ and we still deal with conforming finite elements.

Then $(3.24)_3$ gives:

$$s_h = (\lambda+\beta_c)u_h - \lambda \Pi_{V_h} G p_h \in V_h,$$

and then by elimination of s_h in $(3.24)_{1,2}$, problem (3.24) reads: Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in V_h \times Q_h$:

$$\begin{cases} a(u_h, v_h) + \lambda(u_h - Gp_h, v_h)_{L^2} = (g, v_h)_{L^2}, \\ (-\lambda u_h + \dfrac{\lambda^2}{\lambda+\beta_c}\Pi_{V_h} G p_h + \dfrac{\lambda\beta_c}{\lambda+\beta_c} G p_h, G q_h)_{L^2} = (f, q_h)_{L^2}. \end{cases}$$

And this is the discrete corrected problem (3.17) since

$$\frac{\lambda^2}{\lambda+\beta_c}\Pi_{V_h} G p_h + \frac{\lambda\beta_c}{\lambda+\beta_c} G p_h = \lambda G p_h - \frac{\lambda^2}{\lambda+\beta_c}(Gp_h - \Pi_{V_h} G p_h).$$

Here we have just rewritten the discrete counterpart (3.24) of the continuous problem (3.23) without the addition of any term.

Then problem (3.23) could be considered as the one to be discretized to avoid any locking, instead of problem (3.4) which requires a stabilization term (when P_1 finite elements are used).

Case of a substitution problem. Take $a(\cdot,\cdot) = (\cdot,\cdot)_H$ and $f = 0$, and then consider problem (3.6) that now reads: Find $(u, p) \in H^m \times Q$ such that for all $(v, q) \in H^m \times Q$:

$$\begin{cases} (u, v) + \lambda(u - Gp, v) = (g, v), \\ (u - Gp, -Gq) = 0. \end{cases} \tag{3.25}$$

This is a substitution problem: $(3.25)_1$ gives $(\lambda+1)u = \lambda G p + g$ and then $(3.25)_2$ gives:

$$(Gp, Gq) = (g, Gq).$$

But the discrete counterpart of (3.25) gives $(\lambda+1)u_h = \lambda\Pi_{V_h}Gp_h + \Pi_{V_h}g$ and then:

$$(\lambda+1)(Gp_h, Gq_h) - \lambda(\Pi_{V_h}Gp_h, Gq_h) = (\Pi_{V_h}g, Gq_h).$$

This is not anymore a substitution problem in the discrete case (unless V_h is large enough so that $\Pi_{V_h}Gp_h = Gp_h$). And we could get locking.

Now, if we consider equations (3.23), for which we take $\beta_c = \beta = 1$ (the coercivity constant $\beta = 1$ is known), we get: Find $(u, p, s) \in H^m \times Q \times H^m$ such that for all $(v, q, s') \in H^m \times Q \times H^m$:

$$\begin{cases} (s, v) = (g, v), \\ \dfrac{\lambda}{\lambda+1}(Gp, Gq) - \dfrac{\lambda}{\lambda+1}(s, Gq) = 0, \\ (u - \dfrac{\lambda}{\lambda+1}Gp, s') - \dfrac{1}{\lambda+1}(s, s') = 0, \end{cases} \tag{3.26}$$

and the discrete counterpart immediatly gives:

$$(Gp_h, Gq_h) = (\Pi_{V_h}g, Gq_h) \tag{3.27}$$

for any $q_h \in Q_h$. This is the expected discrete result in that case of a plain substitution problem.

Thus, to formulate problem (3.6) in terms of problem (3.23) seems to yield a reasonnable direct discrete problem (3.24) to avoid the locking phenomenon.

4 Proof of the unlocking

For the continuous problem, following Brezzi and Fortin [9], we suppose that $g = 0$ together with $a(\cdot, \cdot)$ symmetric and we are interested in a lower bound:

$$c(f) \le \Phi((u, p), (u, p))$$

where $c(f) > 0$ is a constant independent of λ and > 0 as soon as $f \neq 0$.

The idea is that (u, p) realizes the minimum of M defined in (3.7) and that at (u, p) we have $M(u, p)$ is < 0. And to prove that this minimum is < 0 it is sufficient to prove that it is already < 0 in some subspace.

This has been done by Brezzi and Fortin [9] in the subspace $\{(w, z) \in V \times Q : w = Gz\}$.

For the discrete corrected associated equations (3.17):

Proposition 4.1 *With $g = 0$ the discrete corrected problem (3.17) has a unique solution that satisfies:*

$$c_0(f) \le \Phi_h((u_h, p_h), (u_h, p_h)). \tag{4.1}$$

where $c_0(f) > 0$ is independent of λ.

Proof. The proof follows the steps of the continuous case, see Brezzi and Fortin [9] and is done in the case $\beta_c = \beta$ for simplicity (similar proof when $0 < \beta_c < \beta$):

1- The solution (u_h, p_h) realizes the minimum of the functional M_h defined in (3.21). We suppose that $\lambda > \beta$ (we are interested in the limit case $\lambda \to \infty$) so that $\frac{\beta}{2} \le \frac{\beta\lambda}{\lambda+\beta} \le \beta$. And we look for a minimum of this functional in the subspace $\{(v_h, q_h) \in V_h \times Q_h : v_h = \Pi_{V_h} G q_h\}$. We then consider the functional defined on Q_h by:

$$\begin{aligned} J(z_h) &= M_h(\Pi_{V_h} G z_h, z_h) \\ &= \frac{1}{2} a(\Pi_{V_h} G z_h, \Pi_{V_h} G z_h) + \frac{\beta\lambda}{2(\lambda+\beta)} ||G z_h - \Pi_{V_h} G z_h||^2 - (f, z_h). \end{aligned} \tag{4.2}$$

This functional is α-convexe with $\alpha = \frac{1}{4}$ as soon as $\lambda \ge \beta$ since, for any q_h, z_h in Q_h:

$$\begin{aligned} J''(q_h)(z_h, z_h) &\ge \frac{\beta}{2} ||\Pi_{V_h} G z_h||_H^2 + \frac{\beta\lambda}{2(\lambda+\beta)} ||G z_h - \Pi_{V_h} G z_h||_H^2, \\ &\ge \frac{1}{2} \frac{\beta\lambda}{\lambda+\beta} ||G z_h||_H^2 \ge \frac{1}{4} ||z_h||_Q^2. \end{aligned}$$

We are in finite dimensional spaces and then a minimum exists in Q_h and is unique: We denote it q_h^λ. And $q_h^\lambda \ne 0$ unless $f = 0$, or more precisely $q_h^\lambda \ne 0$ as soon as there exists one $z_h \in Q_h$ such that $(f, z_h) \ne 0$, which is assumed in any finite element method (the space Q_h is 'dense in the limit' in Q).

Since q_h^λ realizes the minimum of J, we have (Euler associated equation to J), for all $z_h \in Q_h$:

$$a(\Pi_{V_h} G q_h^\lambda, \Pi_{V_h} G z_h) + \frac{\beta\lambda}{\lambda+\beta} (G q_h^\lambda - \Pi_{V_h} G q_h^\lambda, G z_h - \Pi_{V_h} G z_h) = (f, z_h). \tag{4.3}$$

Then for such a minimum we have, as soon as $\lambda \ge \beta$, replacing in (4.2) (f, z_h) by the left-hand side of (4.3):

$$\begin{aligned}
&M_h(\Pi_{V_h} G q_h^\lambda, q_h^\lambda) \\
&= -\frac{1}{2} a(\Pi_{V_h} G q_h^\lambda, \Pi_{V_h} G q_h^\lambda) - \frac{\beta\lambda}{2(\lambda+\beta)} ||G q_h^\lambda - \Pi_{V_h} G q_h^\lambda||_H^2, \\
&\leq -\frac{\beta}{2} ||\Pi_{V_h} G q_h^\lambda||_H^2 - \frac{\beta}{4} ||G q_h^\lambda - \Pi_{V_h} G q_h^\lambda||_H^2 \leq -\frac{\beta}{4} ||G q_h^\lambda||_H^2, \\
&\leq -\frac{\beta}{4} ||q_h^\lambda||_Q^2 < 0,
\end{aligned}$$

which is strictly negative as soon as $f \neq 0$. Then the Euler equation (3.17) (with $g{=}0$) gives $(f, p_h) = \Phi_h((u_h, p_h), (u_h, p_h))$ and then $M_h(u_h, p_h) = -\frac{1}{2}\Phi_h((u_h, p_h), (u_h, p_h))$. And we get, together with $M_h(u_h, p_h) \leq M_h(\Pi_{V_h} G q_h^\lambda, q_h^\lambda)$:

$$\frac{1}{2}\Phi_h((u_h, p_h), (u_h, p_h)) \geq \frac{\beta}{4} ||q_h^\lambda|_Q^2 > 0.$$

2- Now we prove that q_h^λ stays away from 0 as λ increases to infinity. Consider the limit case $\lambda = \infty$. Problem (4.3) then reads: For all $z_h \in Q_h$:

$$a(\Pi_{V_h} G q_h^\infty, \Pi_{V_h} G z_h) + \beta(G q_h^\infty - \Pi_{V_h} G q_h^\infty, G z_h - \Pi_{V_h} G z_h) = (f, z_h),$$

where q_h^∞ exists and $q_h^\infty \neq 0$: We have supposed $f \neq 0$ and this problem also reads as a minimum problem of an elliptic functional since:

$$\begin{aligned}
&a(\Pi_{V_h} G q_h^\infty, \Pi_{V_h} G q_h) + \beta(G q_h^\infty - \Pi_{V_h} G q_h^\infty, G q_h - \Pi_{V_h} G q_h) \\
&\geq \beta ||\Pi_{V_h} G q_h^\infty||_H^2 + \beta ||G q_h^\infty||_H^2 - \beta ||\Pi_{V_h} G q_h^\infty||_H^2 = \beta ||G q_h^\infty||_H^2.
\end{aligned}$$

Then denoting $e_q = q_h^\infty - q_h^\lambda$, with (4.3) we get, forall $z_h \in Q_h$, together with $\beta = \frac{\beta\lambda}{\lambda+\beta} + \frac{\beta^2}{\lambda+\beta}$:

$$\begin{aligned}
&a(\Pi_{V_h} G e_q, \Pi_{V_h} G z_h) + \frac{\beta\lambda}{\lambda+\beta}(G e_q - \Pi_{V_h} G e_q, G z_h - \Pi_{V_h} G z_h) \\
&= -\frac{\beta^2}{\lambda+\beta}(G q_h^\infty - \Pi_{V_h} G q_h^\infty, G z_h - \Pi_{V_h} G z_h).
\end{aligned}$$

Now with $z_h = e_q$ we get:

$$\begin{aligned}
&\beta ||\Pi_{V_h} G q_h^\infty||_H^2 + \frac{\beta\lambda}{\lambda+\beta} ||G e_q - \Pi_{V_h} G e_q||^2 \\
&\leq \frac{\beta^2}{\lambda+\beta} ||G q_h^\infty - \Pi_{V_h} G q_h^\infty||_H ||G e_q - \Pi_{V_h} G e_q||_H, \\
&\leq \frac{\beta^2}{\lambda+\beta}\Big(\frac{\beta}{2\lambda} ||G q_h^\infty - \Pi_{V_h} G q_h^\infty||_H^2 + \frac{\lambda}{2\beta} ||G e_q - \Pi_{V_h} G e_q||_H^2\Big).
\end{aligned}$$

so that, since $\beta \geq \frac{\beta\lambda}{2(\lambda+\beta)}$:

$$\frac{\beta\lambda}{2(\lambda+\beta)}||Ge_q||_H^2 \leq \frac{\beta^3}{2\lambda(\lambda+\beta)}||Gq_h^\infty - \Pi_{V_h} Gq_h^\infty||_H^2 \underset{\lambda\to\infty}{\longrightarrow} 0.$$

And with $\frac{\beta\lambda}{2(\lambda+\beta)} = O(\frac{\beta}{2})$ we get that $q_h^\lambda \to q_h^\infty$ in Q when $\lambda \to \infty$. And with $q_h^\infty \neq 0$ (we have suppose $f \neq 0$) we have (4.1) and the discrete corrected problem (3.17) is unlocked. ∎

5 Error computation with $s = (\lambda+\beta)u - \lambda Gp$

We want to prove that $||u - u_h||_V$ and $||p - p_h||_Q$ converges to 0 with h independently of λ, conversely to the result of Proposition 3.8.

5.1 Continuous equations

5.1.1 The problem

We consider problem (3.23) that we rewrite: Find $(u, p, s) \in V \times Q \times H^m$ such that for all $(v, q, s') \in V \times Q \times H^m$:

$$\begin{cases} [(a(u,v) - \beta_c(u,v)) + \beta_c \dfrac{\lambda}{\lambda+\beta_c}(Gp, Gq)] \\ \qquad\qquad + (s, v - \dfrac{\lambda}{\lambda+\beta_c} Gq) = (g, v) + (f, q), \\ (u - \dfrac{\lambda}{\lambda+\beta_c} Gp, s') - \dfrac{1}{\lambda+\beta_c}(s, s') = 0. \end{cases} \tag{5.1}$$

This problem reads as the usual constraint type problem: Find $(u, p, s) \in V \times Q \times H^m$ such that for all $(v, q, s') \in V \times Q \times H^m$:

$$\begin{cases} \tilde{a}((u,p),(v,q)) + \tilde{b}((v,q), s) = (g, v) + (f, q), \\ \tilde{b}((u,p), s') - \varepsilon \tilde{c}(s, s') = 0, \end{cases} \tag{5.2}$$

where $\varepsilon = \frac{1}{\lambda+\beta_c}$ and:

$$\begin{cases} \tilde{a}((u,p),(v,q)) = a(u,v) - \beta_c(u,v) + \beta_c \dfrac{\lambda}{\lambda+\beta_c}(Gp, Gq), \\ \tilde{b}((u,p), s') = (u - \dfrac{\lambda}{\lambda+\beta_c} Gp, s'), \\ \tilde{c}(s, s') = (s, s')_H. \end{cases}$$

Then we consider the functional defined on $V \times Q \times H^m$:

$$\begin{aligned}&\psi((u,p,s),(v,q,s'))\\&=\tilde{a}((u,p),(v,q))+\tilde{b}((v,q),s)-\tilde{b}((u,p),s')+\varepsilon\tilde{c}(s,s')\end{aligned} \tag{5.3}$$

and the problem (5.1) reads: Find $(u,p,s) \in V \times Q \times H^m$ such that for all $(v,q,s') \in V \times Q \times H^m$:

$$\psi((u,p,s),(v,q,s')) = (g,v) + (f,q). \tag{5.4}$$

This problem is well-posed since (3.23) is.

5.1.2 The norms

We define on $V \times Q \times H^m$ the following norm:

$$|||(v,q,s')||| = (||v||_V^2 + ||q||_Q^2 + ||s'||_{V'}^2 + ||G^t s'||_{Q'}^2 + \varepsilon||s'||_H^2)^{\frac{1}{2}}. \tag{5.5}$$

Proposition 5.1 *ψ is continuous on $(V \times Q \times H^m)^2$: Exists $c_\psi > 0$, for any (u,p,s) and (v,q,s') in $V \times Q \times H^m$:*

$$|\psi((u,p,s),(v,q,s'))| \le c_\psi\, |||(u,p,s)|||\ |||(v,q,s')|||, \tag{5.6}$$

where $c_\psi = O(1)$ as $\lambda \to \infty$.

Proof. We have:

$$\begin{aligned}&|\psi((u,p,s),(v,q,s'))|\\&\le (\alpha+\beta_c c_V^2)||u||_V||v||_V + \frac{\beta_c\lambda}{\lambda+\beta_c}||p||_Q||q||_Q\\&\quad + ||v||_V||s||_{V'} + ||u||_V||s'||_{V'}\\&\quad + \frac{\lambda}{\lambda+\beta_c}(||q||_Q||G^t s||_{Q'} + ||p||_Q||G^t s'||_{Q'}) + \varepsilon||s||_H||s'||_H,\end{aligned}$$

and then (5.6). ∎

We also define on $V_h \times Q_h \times V_h$ the following norm:

$$|||(v_h,q_h,s'_h)|||_h = (||v_h||_V^2 + ||q_h||_Q^2 + ||s'_h||_{V'}^2 + \varepsilon||s'_h||_H^2)^{\frac{1}{2}}. \tag{5.7}$$

(The $||G^t s||_{Q'}$ term is absent by comparison to (5.5).)

Proposition 5.2 *ψ is continuous on $(V \times Q \times H^m) \times (V_h \times Q_h \times V_h)$ in the following sense: It exists $c_{\psi_1} > 0$, for any $(u,p,s) \in V \times Q \times H^m$ and $(v_h,q_h,s'_h) \in V_h \times Q_h \times V_h$:*

$$\begin{aligned}&|\psi((u,p,s),(v_h,q_h,s'_h))|\\&\le c_{\psi_1}(|||(u,p,s)|||^2 + ||\Pi_{V_h}Gp||_V^2)^{\frac{1}{2}}\ |||(v_h,q_h,s'_h)|||_h,\end{aligned} \tag{5.8}$$

where $c_\psi = O(1)$ as $\lambda \to \infty$.

Proof. We have:

$$
\begin{aligned}
&|\psi((u,p,s),(v_h,q_h,s'_h))|\\
&\le (\alpha{+}\beta_c c_V^2)||u||_V||v_h||_V+\frac{\beta_c\lambda}{\lambda+\beta}||p||_Q||q_h||_Q\\
&\quad+||v_h||_V||s||_{V'}+||u||_V||s'_h||_{V'}\\
&\quad+\frac{\lambda}{\lambda+\beta_c}(||q_h||_Q||G^ts||_{Q'}+||\Pi_{V_h}Gp||_V||s'_h||_{V'})+\varepsilon||s||_H||s'_h||_H,
\end{aligned}
$$

and then (5.8). ∎

5.1.3 The inf-sup conditions

And we have the two inf-sup conditions:

$$
\forall s\in H^m,\quad \left\{\begin{array}{l}\sup\limits_{v\in V}\dfrac{\langle s,v\rangle}{||v||_V}\ge ||s||_{V'},\\[2mm] \sup\limits_{q\in Q}\dfrac{\langle s,Gq\rangle}{||q||_Q}\ge ||G^ts||_{Q'}.\end{array}\right. \tag{5.9}
$$

These inequalities are in fact equalities by definition of the dual norms $||.||_{V'}$ and $||.||_{Q'}$.

Proposition 5.3 *The solution of (3.23) satisfies*

$$
(||u||_V^2+||p||_Q^2+||s||_{V'}^2+||G^ts||_{Q'}^2+\frac{1}{\lambda+\beta}||s||_H^2)^{\frac12}\le c(||f||_{Q'}+||g||_{V'}), \tag{5.10}
$$

where c is a constant independent of λ as soon as for example $\lambda\ge\max(\beta_c,1)$.

Proof. Choose any $(u,p,s)\in V\times Q\times H^m$.

1- Case $c{>}1$, i.e. $\beta_c{<}\beta$. We have, as soon as $\lambda\ge\beta_c$:

$$
\begin{aligned}
a(u,u)-\beta_c||u||_H^2 &\ge \frac{\alpha}{c}||u||_V^2-\frac{\beta}{c}||u||_H^2+\frac{\alpha(c-1)}{c}||u||_V^2\\
&\ge \frac{\alpha(c-1)}{c}||u||_V^2
\end{aligned}
$$

so that, since $\lambda\ge\beta_c$ and then $\beta_c\frac{\lambda}{\lambda+\beta_c}\ge\frac{\beta_c}{2}$, for any given (u,p,s):

$$
\psi_h((u,p,s),(u,p,s))\ge\frac{\alpha(c-1)}{c}||u||_V^2+\frac{\beta_c}{2}||p||_Q^2+\varepsilon||s||_H^2.
$$

Then with $(5.9)_1$ we have the existence of $v_s \in V$ such that:

$$\psi((u,p,s),(v_s,0,0)) \geq -(||a||+\beta_c c_V^2)||u||_V||v_s||_V + ||s||_{V'}||v_s||_V,$$

where we can choose v_s such that $||v_s||_V = ||s||_{V'}$. Then, with
$(||a||+\beta_c c_V^2)||u||_V||v_s||_V \leq \frac{1}{2}(||a||+\beta_c c_V^2)^2||u||_V^2 + \frac{1}{2}||v_s||_V^2$,
we get:

$$\psi((u,p,s),(v_s,0,0)) \geq -\frac{(||a||+\beta_c c_V^2)^2}{2}||u||_V^2 + \frac{1}{2}||s||_{V'}^2.$$

Then with $(5.9)_2$ we have the existence of $q_s \in Q$ such that, with $\frac{\lambda}{\lambda+\beta_c} \geq \frac{1}{2}$ (we have supposed $\lambda \geq \beta_c$):

$$\psi((u,p,s),(0,q_s,0)) \geq -\beta_c||p||_Q||q_s||_Q + \frac{1}{2}||G^t s||_{Q'}||q_s||_Q,$$

where we can choose q_s such that $||q_s||_Q = ||G^t s||_{Q'}$. Then, with
$\beta_c||p||_Q||q_s||_Q \leq {\beta_c}^2||p||_Q^2 + \frac{1}{4}||q_s||_Q^2$,
we get:

$$\psi((u,p,s),(0,q_s,0)) \geq -{\beta_c}^2||p||_Q^2 + \frac{1}{4}||G^t s||_{Q'}^2.$$

Then we get, for any $\kappa_1, \kappa_2 > 0$:

$$\begin{aligned}&\psi((u,p,s),(u,p,s)+\kappa_1(v_s,0,0)+\kappa_2(0,q_s,0))\\ &\quad\geq \Big(\frac{\alpha(c-1)}{c}-\kappa_1\frac{(||a||+\beta_c c_V^2)^2}{2}\Big)||u||_V^2+\Big(\frac{\beta_c}{2}-\kappa_2{\beta_c}^2\Big)||p||_Q^2 \\ &\qquad+\varepsilon||s||_H^2+\frac{\kappa_1}{2}||s||_{V'}^2+\frac{\kappa_2}{4}||G^t s||_{Q'}^2.\end{aligned} \tag{5.11}$$

Then choose $\kappa_1{=}\frac{1}{2}\frac{2\alpha(c-1)}{c(||a||+\beta_c c_V^2)^2}$ and $\kappa_2{=}\frac{1}{4\beta_c}$ to get, with $(v,q,s') = (u{+}\kappa_1 v_s, p{+}\kappa_2 q_s, s)$:

$$\begin{aligned}&\psi((u,p,s),(v,q,s'))\\ &\geq \frac{\alpha(c-1)}{2c}||u||_V^2 + \frac{\beta_c}{4}||p||_Q^2 + \varepsilon||s||_H^2 + \frac{\kappa_1}{2}||s||_{V'}^2 + \frac{\kappa_2}{4}||G^t s||_{Q'}^2.\end{aligned}$$

Then with $|||(v,q,s')||| = O(|||(u,p,s)|||)$ (easy to check) we get:

$$\psi((u,p,s),(v,q,s')) \geq c_0|||(u,p,s)|||\ |||(v,q,s')|||, \tag{5.12}$$

with $c_0 > 0$ independent of ε (i.e. of λ).

2- Case $c{=}1$, i.e. $\beta_c{=}\beta$. Then we have:

$$\begin{aligned}a(u,u){-}\beta||u||_H^2 &= a(u,u){-}\frac{\beta}{\gamma}||u||_H^2{-}\frac{\beta(\gamma-1)}{\gamma}||u||_H^2\\ &\geq \frac{\alpha(\gamma-1)}{\gamma}||u||_V^2{-}\frac{\beta(\gamma-1)}{\gamma}||u||_H^2,\end{aligned}$$

for any $\gamma>1$. Then we recover coercivity in $||u||_H$ with:

$$\psi((u,p,s),(0,0,-u)) \geq ||u||_H^2 - ||u||_H||Gp||_H - \varepsilon||s||_H||u||_H, \\ \geq \frac{1}{2}||u||_H^2 - ||p||_Q^2 - \varepsilon^2||s||_H^2,$$

so that, for any $\kappa_1,\kappa_2,\kappa_3 > 0$ (with the computation to get (5.11) where γ reads in place of c):

$$\psi((u,p,s),(u,p,s)+\kappa_1(v_s,0,0)+\kappa_2(0,q_s,0)+\kappa_3(0,0,-u)) \\ \geq (\frac{\alpha(\gamma-1)}{\gamma} - \kappa_1\frac{(||a||+\beta c_V^2)^2}{2})||u||_V^2 + (\frac{\beta}{2} - \kappa_2\beta^2 - \kappa_3)||p||_Q^2 \\ +(\frac{\kappa_3}{2} - \frac{\beta(\gamma-1)}{\gamma})||u||_H^2 + \varepsilon(1-\varepsilon\kappa_3)||s||_H^2 + \frac{\kappa_1}{2}||s||_{V'}^2 + \frac{\kappa_2}{4}||G^ts||_{Q'}^2.$$

Then we choose $\kappa_1=\frac{1}{2}\frac{2\alpha(\gamma-1)}{\gamma(||a||+\beta c_V^2)^2}$, $\kappa_2=\frac{1}{8\beta}$ and $\kappa_3=\frac{\beta}{8}$, so that we also now choose γ such that $\frac{\beta(\gamma-1)}{\gamma}=\frac{1}{2}\frac{\kappa_3}{2}$, i.e. $\gamma=1+\frac{1}{31}=\frac{32}{31}$ to get, with $(v,q,s') = (u+\kappa_1 v_s, p+\kappa_2 q_s, s-\kappa_3 u)$:

$$\psi((u,p,s),(v,q,s)) \\ \geq \frac{\alpha(\gamma-1)}{2\gamma}||u||_V^2 + \frac{\kappa_3}{4}||u||_H^2 + \frac{\beta}{4}||p||_Q^2 + \varepsilon(1-\varepsilon\kappa_3)||s||_H^2 \\ + \frac{\kappa_1}{2}||s||_{V'}^2 + \frac{\kappa_2}{4}||G^ts||_{Q'}^2.$$

Here $\frac{\alpha(\gamma-1)}{2\gamma}=\frac{\alpha}{64}$, $\frac{\kappa_3}{4}=\frac{\beta}{32}$, $\varepsilon(1-\varepsilon\kappa_3)=\varepsilon\frac{8\lambda+7\beta}{8(\lambda+\beta)}$, $\frac{\kappa_1}{2}=\frac{\alpha}{64(||a||+\beta c_V^2)^2}$ and $\frac{\kappa_2}{4}=\frac{1}{32\beta}$.

Then with $|||(v,q,s')||| = O(|||(u,p,s)|||)$ (easy to check) we get:

$$\psi((u,p,s),(v,q,s')) \geq c_0|||(u,p,s)|||\ |||(v,q,s')|||,$$

with $c_0 > 0$ independent of ε (i.e. of λ).

Then with Proposition 5.1, we get (5.10). ∎

5.2 Discrete associated equations – case 1

5.2.1 The discrete problem

The discrete associated problem is the discrete counterpart of (5.4): Find $(u_h,p_h,s_h) \in V_h \times Q_h \times V_h$ such that for all $(v_h,q_h,s_h') \in V_h \times Q_h \times V_h$:

$$\psi((u_h,p_h,s_h),(v_h,q_h,s_h')) = (g,v_h) + (f,q_h). \tag{5.13}$$

5.2.2 The first discrete inf-sup condition

We have:

$$\forall s_h \in V_h, \quad \sup_{v_h \in V_h} \frac{\langle s_h, v_h \rangle}{||v_h||_V} \geq ||s_h||_{V_h'}. \tag{5.14}$$

where $||s_h||_{V_h'}$ means the dual norm relative to the restriction on V_h: $||s_h||_{V_h'} = \sup_{v_h \in V_h} \frac{\langle s_h, v_h \rangle}{||v_h||_V}$.

Lemma 5.4 *If the interpolation estimate $||\Pi_{V_h} v||_V \leq c||v||_V$ holds with $c > 0$ independent of h, then the norms $||.||_{V'}$ and $||.||_{V_h'}$ are equivalents on V_h. In particular:*

$$\exists c_1 > 0, \quad \forall s_h \in V_h, \quad \sup_{v_h \in V_h} \frac{\langle s_h, v_h \rangle}{||v_h||_V} \geq c_1 ||s_h||_{V'}, \tag{5.15}$$

where c_1 is independent of h.

Proof. We have, for any $s_h \in V_h$:

$$\begin{aligned} ||s_h||_{V'} &= \sup_{v \in V} \frac{(s_h, v)_H}{||v||_V} = \sup_{v \in V} \frac{(s_h, \Pi_{V_h} v)_H}{||\Pi_{V_h} v||_V} \frac{||\Pi_{V_h} v||_V}{||v||_V} \\ &\leq c \sup_{v_h \in V_h} \frac{\langle s_h, v_h \rangle_{V',V}}{||v_h||_V} = c||s_h||_{V_h'}, \end{aligned}$$

thanks to the interpolation inequality, with c independent of h. Then with $(5.14)_1$ and $\frac{1}{||s_h||_{V'}} \geq \frac{1}{c||s_h||_{V_h'}}$ we deduce (5.15) with $c_1 = \frac{1}{c}$ independent of h. Together with the trivial relation $||.||_{V_h'} \leq ||.||_{V'}$, we conclude that the norms $||.||_{V'}$ and $||.||_{V_h'}$ are equivalents on V_h. ∎

5.2.3 First error computation

Lemma 5.5 *Suppose the interpolation estimate $||\Pi_{V_h} v||_V \leq c||v||_V$ holds with $c > 0$ independent of h. Then there exists a constant $c_0 > 0$ such that for any $(u_h, p_h, s_h) \in V_h \times Q_h \times V_h$ there exists $(v_h, q_h, s_h') \in V_h \times Q_h \times V_h$ verifying:*

$$\psi((u_h, p_h, s_h), (v_h, q_h, s_h')) \geq c_0 |||(u_h, p_h, s_h)|||_h |||(v_h, q_h, s_h')||_h, \tag{5.16}$$

where $c_0 = O(1)$ as $\lambda \to \infty$ (with $|||.|||_h$ defined in (5.7)).

Proof. 1- Case $c{>}1$, i.e. $\beta_c{<}\beta$. We have, as soon as $\lambda \geq \beta_c$:

$$\begin{aligned} a(u_h, u_h) - \beta_c ||u_h||_H^2 &\geq \frac{\alpha}{c} ||u_h||_V^2 - \frac{\beta}{c} ||u_h||_H^2 + \frac{\alpha(c-1)}{c} ||u_h||_V^2 \\ &\geq \frac{\alpha(c-1)}{c} ||u_h||_V^2 \end{aligned}$$

so that, since $\lambda \geq \beta_c$ and then $\beta_c \frac{\lambda}{\lambda+\beta_c} \geq \frac{\beta_c}{2}$, for any given (u_h, p_h, s_h):

$$\psi((u_h,p_h,s_h),(u_h,p_h,s_h)) \geq \frac{\alpha(c-1)}{c}||u_h||_V^2 + \frac{\beta_c}{2}||p_h||_Q^2 + \varepsilon||s_h||_H^2.$$

Then with (5.15) we have the existence of $v_s \in V$ such that:

$$\begin{aligned}&\psi((u_h,p_h,s_h),(v_s,0,0))\\&\geq -(||a||+\beta_c c_V^2)||u_h||_V||v_s||_V + c_1||s_h||_{V'}||v_s||_V,\end{aligned}$$

where we can choose v_s such that $||v_s||_V = ||s_h||_{V'}$. Then, using $(||a||+\beta_c c_V^2)||u_h||_V||v_s||_V \leq \frac{1}{2c_1}(||a||+\beta_c c_V^2)^2||u_h||_V^2 + \frac{c_1}{2}||v_s||_V^2$, we get:

$$\psi((u_h,p_h,s_h),(v_s,0,0)) \geq -\frac{(||a||+\beta_c c_V^2)^2}{2c_1}||u_h||_V^2 + \frac{c_1}{2}||s_h||_{V'}^2.$$

Then we get, for any $\kappa_1 > 0$:

$$\begin{aligned}&\psi((u_h,p_h,s_h),(u_h,p_h,s_h)+(\kappa_1 v_s,0,0))\\&\geq \left(\frac{\alpha(c-1)}{c} - \frac{\kappa_1(||a||+\beta_c c_V^2)^2}{2c_1}\right)||u_h||_V^2 + \frac{\beta_c}{2}||p_h||_Q^2\\&\quad + \varepsilon||s_h||_H^2 + \frac{\kappa_1 c_1}{2}||s_h||_{V'}^2.\end{aligned} \tag{5.17}$$

Choose $\kappa_1 = \frac{1}{2}\frac{\alpha(c-1)2c_1}{c(||a||+\beta_c c_V^2)^2}$ and $(v_h, q_h, s_h') = (u_h+\kappa_1 v_s, p_h, s_h)$ to get:

$$\begin{aligned}&\psi((u_h,p_h,s_h),(v_h,q_h,s_h'))\\&\geq \frac{\alpha(c-1)}{2c}||u_h||_V^2 + \frac{\beta_c}{4}||p_h||_Q^2 + \varepsilon||s_h||_H^2 + \frac{\kappa_1 c_1}{2}||s_h||_{V'}^2\end{aligned}$$

Then with $|||(v_h,q_h,s_h')|||_h = O(||||(u_h,p_h,s_h)||||)_h$ (easy to check) we get (5.16).

2- Case c=1, i.e. β_c=β. Then we have, for any γ>1:

$$\begin{aligned}a(u_h,u_h) - \beta||u_h||_H^2 &= a(u_h,u_h) - \frac{\beta}{\gamma}||u_h||_H^2 - \frac{\beta(\gamma-1)}{\gamma}||u_h||_H^2\\&\geq \frac{\alpha(\gamma-1)}{\gamma}||u_h||_V^2 - \frac{\beta(\gamma-1)}{\gamma}||u_h||_H^2.\end{aligned}$$

And we have:

$$\begin{aligned}&\psi((u_h,p_h,s_h),(0,0,-u_h))\\&\geq ||u_h||_H^2 - ||u_h||_H||Gp_h||_H - \varepsilon||s_h||_H||u_h||_H\\&\geq \frac{1}{2}||u_h||_H^2 - ||p_h||_Q^2 - \varepsilon^2||s_h||_H^2.\end{aligned} \tag{5.18}$$

Then, for any $\kappa_1, \kappa_3 > 0$ (using (5.17)):

$$\psi((u_h,p_h,s_h),(u_h,p_h,s_h)+(\kappa_1 v_s,0,-\kappa_3 u_h))$$
$$\geq \Big(\frac{\alpha(\gamma-1)}{\gamma}-\frac{\kappa_1(||a||+\beta_\gamma c_V^2)^2}{2c_1}\Big)||u_h||_V^2+\Big(\frac{\beta_\gamma}{2}-\kappa_3\Big)||p_h||_Q^2$$
$$+\Big(\frac{\kappa_3}{2}-\frac{\beta(\gamma-1)}{\gamma}\Big)||u_h||_H^2+\varepsilon(1-\kappa_3\varepsilon)||s_h||_H^2+\frac{\kappa_1 c_1}{2}||s_h||_{V'}^2.$$

Then take γ and κ_3 such that $\frac{\beta}{4\gamma}=\kappa_3$ and $\frac{\kappa_3}{2}=\frac{\beta(\gamma-1)}{\gamma}$, i.e. $\gamma=\frac{9}{8}$ and $\kappa_3=\frac{2\beta}{9}$, so that $(1-\kappa_3\varepsilon)=(1-\frac{2\beta}{9(\lambda+\beta)})\geq\frac{1}{2}$, and with $\kappa_1=\frac{1}{2}\frac{\alpha(\gamma-1)2c_1}{\gamma(||a||+\frac{\beta}{\gamma}c_V^2)^2}$ we get, denoting $(v_h,q_h,s'_h)=(u_h+\kappa_1 v_s,p_h,s_h-\kappa_3 u_h)$:

$$\psi((u_h,p_h,s_h),(v_h,q_h,s'_h))$$
$$\geq \frac{\alpha(\gamma-1)}{2\gamma}||u_h||_V^2+\frac{\beta}{4\gamma}||p_h||_Q^2+\frac{\varepsilon}{2}||s_h||_H^2+\frac{\kappa_1 c_1}{2}||s_h||_{V'}^2.$$

Then with $|||(v_h,q_h,s'_h)|||_h = O(|||(u_h,p_h,s_h)|||)_h$ (easy to check) we get (5.16). ∎

Proposition 5.6 *If $||\Pi_{V_h}v||_V \leq c||v||_V$ holds (interpolation estimate), we have:*

$$\begin{aligned}&(||u-u_h||_V^2+||p-p_h||_Q^2+||s-s_h||_{V'}^2+\varepsilon||s-s_h||_H^2)^{\frac{1}{2}}\\ &\leq c\inf_{(v_i,q_i,s'_i)\in V_h\times Q_h\times V_h}(||u-v_i||_V^2+||p-q_i||_Q^2+||s-s'_i||_{V'}^2\\ &\qquad+\varepsilon||s-s'_i||_H^2+||\Pi_{V_h}(G(p-p_i))||_V^2)^{\frac{1}{2}},\end{aligned}\tag{5.19}$$

where $c=O(1)$ as $\lambda\to\infty$.

Proof. We have:

$$\psi((u-u_h,p-p_h,s-s_h),(v_h,q_h,s'_h))=0,$$

so that for any $(u_i,p_i,s_i)\in V_h\times Q_h\times V_h)$ we get:

$$\psi((u_h-u_i,p_h-p_i,s_h-s_i),(v_h,q_h,s'_h))$$
$$=\psi((u-u_i,p-p_i,s-s_i),(v_h,q_h,s'_h)).$$

We use (5.8) and (5.16) to get:

$$|||(u_h-u_i,p_h-p_i,s_h-s_i)|||_h$$
$$=O\Big((|||(u-u_i,p-p_i,s-s_i)|||^2+||\Pi_{V_h}G(p-p_i)||_V^2)^{\frac{1}{2}}\Big).$$

Then since $|||(u-u_h,p-p_h,s-s_h)|||_h \leq |||(u-u_i,p-p_i,s-s_i)|||_h + |||(u_h-u_i,p_h-p_i,s_h-s_i)|||_h$, we get (5.19). ∎

5.3 Discrete associated equations – case 2

5.3.1 The second discrete inf-sup condition

We have:

$$\forall s_h \in V_h, \quad \sup_{q_h \in Q_h} \frac{\langle s_h, Gq_h \rangle}{||q_h||_Q} \geq ||G^t s_h||_{Q'_h}, \tag{5.20}$$

where $||G^t s_h||_{Q'_h}$ is the dual norm relative to the restriction on V_h: $||G^t s_h||_{Q'_h} = \sup_{q_h \in Q_h} \frac{\langle G^t s_h, q_h \rangle}{||q_h||_Q}$.

Unfortunatly (5.20) is not possible if $||G^t s_h||_{Q'}$ reads in place of $||G^t s_h||_{Q'_h}$: We fall on the usual problem of the satisfaction of a discrete inf-sup condition. We just have, in the general case, a weakened result (degraded inf-sup condition):

Lemma 5.7 *When $G^t(V_h) \subset H^m$ and with the interpolation inequalities $||q - \Pi_{Q_h} q||_H \leq ch^r ||q||_Q$ and $||\Pi_{Q_h} q||_Q \leq c||q||_Q$ where $r \geq 0$ and $c > 0$ is independent of h, we have: Exists $c_2 > 0$ such that for all $s_h \in V_h$:*

$$\sup_{q_h \in Q_h} \frac{\langle s_h, Gq_h \rangle}{||q_h||_Q} \geq c_2 ||G^t s_h||_{Q'} - h^r ||G^t s_h - \Pi_{Q_h} G^t s_h||_H, \tag{5.21}$$

where c_2 is independent of h.

(This Lemma has already been settled in [15], and we recall the proof for sake of completeness.)

Proof. We have, with the notation $q_h = \Pi_{Q_h} q$ when $q \in Q$:

$$\begin{aligned}
||G^t s_h||_{Q'} &= \sup_{q \in Q} \frac{(G^t s_h, q)}{||q||_Q} = \sup_{q \in Q} \frac{(G^t s_h, q_h) + (G^t s_h, q - q_h)}{||q||_Q}, \\
&= \sup_{q \in Q} \Big(\frac{(G^t s_h, q_h)}{||q||_Q} + \frac{(G^t s_h - \Pi_{Q_h} G^t s_h, q - q_h)}{||q||_Q} \Big), \\
&\leq \sup_{q \in Q} \Big(\frac{(G^t s_h, q_h)}{||q_h||_Q} \frac{||q_h||_Q}{||q||_Q} + ||G^t s_h - \Pi_{Q_h} G^t s_h||_H \frac{||q - q_h||_H}{||q||_Q} \Big), \\
&\leq c \, (||G^t s_h||_{Q'_h} + h^r \; ||G^t s_h - \Pi_{Q_h} G^t s_h||_H),
\end{aligned}$$

thanks to the interpolation inequalities. And with $\sup_{Q_h} \frac{\langle s_h, Gq_h \rangle}{||q_h||_Q} = ||G^t s_h||_{Q'_h}$ and with $c_2 = \frac{1}{c}$, we get (5.21). ∎

5.3.2 Stabilized discrete problem

In that case we want control on $||G^t s_h||_{Q'}$. Then we have a problem because of the extra h^r-term in (5.21) that degrades the inf-sup condition. Then consider the equations (3.16) modified (stabilized) by introducing the h^r-term of (5.21): Find $(u_h, p_h, s_h) \in V_h \times Q_h \times V_h$ such that for all $(v_h, q_h, s'_h) \in V_h \times Q_h \times V_h$:

$$\psi_h((u_h, p_h, s_h), (v_h, q_h, s'_h)) = (g, v_h) + (f, q_h), \tag{5.22}$$

where:

$$\begin{aligned} &\psi_h((u_h, p_h, s_h), (v_h, q_h, s'_h)) \\ &= \psi((u_h, p_h, s_h), (v_h, q_h, s'_h)) \\ &\quad + \delta h^{2r} (G^t s_h - \Pi_{Q_h} G^t s_h, G^t s'_h - \Pi_{Q_h} G^t s'_h)_H. \end{aligned} \tag{5.23}$$

$h = h_{\max}$ is the maximum of the diameter of the finite elements and where δ is a stabilization constant given in the next Lemma.

Problem (5.22) will read for the numerical computations

$$\left\{ \begin{aligned} &((\text{grad}\underline{u}_h, \text{grad}\underline{v}_h) - \beta_c(\underline{u}_h, \underline{v}_h)) + \beta_c \frac{\lambda}{\lambda+\beta_c}(\text{grad}p_h, \text{grad}q_h) \\ &\qquad + (\underline{s}_h, \underline{v}_h - \frac{\lambda}{\lambda+\beta_c}\text{grad}q_h) = (g, \underline{v}_h) + (f, q_h), \\ &(\underline{u}_h - \frac{\lambda}{\lambda+\beta_c}\text{grad}p_h, \underline{s}'_h) - \frac{1}{\lambda+\beta_c}(\underline{s}_h, \underline{s}'_h) \\ &\qquad - \delta h^{2r}(\text{div}\underline{s}_h - \Pi_{Q_h}\text{div}\underline{s}_h, \text{div}\underline{s}'_h)_H = 0, \end{aligned} \right. \tag{5.24}$$

where $V_h = S_h$ and Q_h are the P_1 continuous finite elements.

5.3.3 Second error computation

Define the norm:

$$\begin{aligned} |||(v_h, q_h, s'_h)|||_{h2} = (&||v_h||_V^2 + ||q_h||_Q^2 + ||s'_h||_{V'}^2 + ||G^t s'_h||_{Q'}^2 \\ &+ \varepsilon ||s'_h||_H^2 + h^{2r} ||G^t s'_h - \Pi_{Q_h} G^t s'_h||_H^2)^{\frac{1}{2}}. \end{aligned} \tag{5.25}$$

Lemma 5.8 *Suppose that the solution (u, p, s) of (5.2) satisfies $G^t s \in H^n$ and that the interpolation inequalities of Lemma 5.4 and Lemma 5.7 hold. In the case $\beta_c < \beta$ choose $\delta \ge \frac{c_2}{8\beta_c}$, and in the case $\beta_c = \beta$ either suppose $||G^t u||_H \le ||u||_V$ and choose $\delta \ge \frac{c_2}{8\beta_c}$, or suppose the following inverse inequality:*

$$\exists c_i > 0, \quad \forall v_h \in V_h, \quad h^r ||G^t v_h||_H \le c_i ||v_h||_H$$

and choose $\delta \geq \min(\frac{1}{2\beta c_2}, \frac{1}{8c_i^2 \min(\frac{\beta}{16},\frac{1}{4})})$. *Then there exists a constant* $c_0 > 0$ *such that for any* $(u_h, p_h, s_h) \in V_h{\times}Q_h{\times}V_h$ *there exists* $(v_h, q_h, s'_h) \in V_h{\times}Q_h{\times}V_h$ *verifying:*

$$\psi((u_h, p_h, s_h), (v_h, q_h, s'_h)) \geq c_0|||(u_h, p_h, s_h)|||_{h2}|||(v_h, q_h, s'_h)||_{h2}, \tag{5.26}$$

where $c_0 = O(1)$ *as* $\lambda \to \infty$.

Proof. Choose any $(u_h, p_h, s_h) \in V_h \times Q_h \times V_h$.

1- Case $c{>}1$, i.e. $\beta_c{<}\beta$. We have, as soon as $\lambda \geq \beta_c$:

$$\begin{aligned} a(u_h, u_h){-}\beta_c||u_h||_H^2 &\geq \frac{\alpha}{c}||u_h||_V^2{-}\frac{\beta}{c}||u_h||_H^2{+}\frac{\alpha(c{-}1)}{c}||u_h||_V^2 \\ &\geq \frac{\alpha(c{-}1)}{c}||u_h||_V^2 \end{aligned}$$

so that, since $\lambda \geq \beta_c$ and then $\beta_c\frac{\lambda}{\lambda+\beta_c} \geq \frac{\beta_c}{2}$, for any given (u_h, p_h, s_h):

$$\begin{aligned} \psi_h((u_h, p_h, s_h), (u_h, p_h, s_h)) \geq \frac{\alpha(c{-}1)}{c}||u_h||_V^2 + \frac{\beta_c}{2}||p_h||_Q^2 + \varepsilon||s_h||_H^2 \\ + \delta h^{2r}||G^t s_h{-}\Pi_{Q_h}G^t s_h||_H^2. \end{aligned}$$

Then with (5.15) we have the existence of $v_s \in V$ such that:

$$\begin{aligned} &\psi_h((u_h, p_h, s_h), (v_s, 0, 0)) \\ &\geq -(||a||{+}\beta_c c_V^2)||u_h||_V||v_s||_V + c_1||s_h||_{V'}||v_s||_V, \end{aligned}$$

where we can choose v_s such that $||v_s||_V = ||s_h||_{V'}$. Then, with $(||a||{+}\beta_c c_V^2)||u_h||_V||v_s||_V \leq \frac{1}{2c_1}(||a||{+}\beta_c c_V^2)^2||u_h||_V^2 + \frac{c_1}{2}||v_s||_V^2$, we get:

$$\psi_h((u_h, p_h, s_h), (v_s, 0, 0)) \geq -\frac{(||a||{+}\beta_c c_V^2)^2}{2c_1}||u_h||_V^2 + \frac{c_1}{2}||s_h||_{V'}^2. \tag{5.27}$$

Then with (5.21) we have the existence of $q_s \in Q$ such that, since $\frac{\lambda}{\lambda+\beta_c} \leq 1$:

$$\begin{aligned} &\psi_h((u_h, p_h, s_h), (0, q_s, 0)) \\ &\geq -\beta_c||p_h||_Q||q_s||_Q{+}c_2||G^t s_h||_{Q'}||q_s||_Q \\ &\quad -h^r||G^t s_h{-}\Pi_{Q_h}G^t s_h||_H||q_s||_Q, \end{aligned}$$

where we can choose q_s such that $||q_s||_Q = ||G^t s_h||_{Q'}$. Then, with $\beta_c||p_h||_Q||q_s||_Q \leq \frac{{\beta_c}^2}{c_2}||p_h||_Q^2 + \frac{c_2}{4}||q_s||_Q^2$, and

$h^r||G^ts_h - \Pi_{Q_h}G^ts_h||_H||q_s||_Q \leq \frac{h^{2r}}{c_2}||G^ts_h - \Pi_{Q_h}G^ts_h||_H^2 + \frac{c_2}{4}||q_s||_Q^2$ we get:

$$\begin{aligned}&\psi_h((u_h,p_h,s_h),(0,q_s,0))\\&\geq -\frac{{\beta_c}^2}{c_2}||p_h||_Q^2 + \frac{c_2}{2}||G^ts||_{Q'}^2 - \frac{h^{2r}}{c_2}||G^ts_h - \Pi_{Q_h}G^ts_h||_H^2.\end{aligned}$$

Then we get, for any $\kappa_1, \kappa_2 > 0$:

$$\begin{aligned}&\psi_h((u_h,p_h,s_h),(u_h,p_h,s_h)+(\kappa_1 v_s,\kappa_2 q_s,0))\\&\geq \Big(\frac{\alpha(c-1)}{c} - \frac{\kappa_1(||a||+\beta_c c_V^2)^2}{2c_1}\Big)||u_h||_V^2 + \Big(\frac{\beta_c}{2} - \frac{\kappa_2{\beta_c}^2}{c_2}\Big)||p_h||_Q^2\\&\quad + \varepsilon||s_h||_H^2 + \frac{\kappa_1 c_1}{2}||s_h||_{V'}^2 + \frac{\kappa_2 c_2}{2}||G^ts_h||_{Q'}^2\\&\quad + \Big(\delta - \frac{\kappa_2}{c_2}\Big)h^{2r}||G^ts_h - \Pi_{Q_h}G^ts_h||_H^2.\end{aligned} \tag{5.28}$$

Choose $\kappa_1 = \frac{1}{2}\frac{\alpha(c-1)2c_1}{c(||a||+\beta_c c_V^2)^2}$, $\kappa_2 = \frac{1}{2}\frac{\beta_c c_2}{2{\beta_c}^2}$, and δ such that $\delta = \frac{\kappa_2}{2c} = \frac{c_2}{8\beta_c}$ to get with $(v_h, q_h, s'_h) = (u_h, p_h, s_h) + (\kappa_1 v_s, \kappa_2 q_s, 0)$:

$$\begin{aligned}&\psi_h((u_h,p_h,s_h),(v_h,q_h,s'_h))\\&\geq \frac{\alpha(c-1)}{2c}||u_h||_V^2 + \frac{\beta_c}{4}||p_h||_Q^2 + \varepsilon||s_h||_H^2 + \frac{\kappa_1 c_1}{2}||s_h||_{V'}^2\\&\quad + \frac{\kappa_2 c_2}{2}||G^ts_h||_{Q'}^2 + \frac{\delta}{2}h^{2r}||G^ts_h - \Pi_{Q_h}G^ts_h||_H^2.\end{aligned}$$

Then with $|||(v_h, q_h, s'_h)||| = O(|||(u_h, p_h, s_h)|||)$ (easy to check) we get (5.26).

2- Case c=1, i.e. β_c=β. We have:

$$\begin{aligned}a(u_h,u_h) - \beta||u_h||_H^2 &= a(u_h,u_h) - \frac{\beta}{\gamma}||u_h||_H^2 - \frac{\beta(\gamma-1)}{\gamma}||u_h||_H^2\\&\geq \frac{\alpha(\gamma-1)}{\gamma}||u_h||_V^2 - \frac{\beta(\gamma-1)}{\gamma}||u_h||_H^2,\end{aligned}$$

for any γ>1. Then we recover coercivity in $||u_h||_H$ with:

$$\begin{aligned}&\psi_h((u_h,p_h,s_h),(0,0,-u_h))\\&\geq ||u_h||_H^2 - ||u_h||_H||Gp_h||_H - \varepsilon||s_h||_H||u_h||_H\\&\quad - \delta h^{2r}||G^ts_h - \Pi_{Q_h}G^ts_h||_H||G^tu_h||_H\end{aligned} \tag{5.29}$$

Then two cases:

21- If $||G^tu||_H \leq ||u||_V$ for any $u \in V$ and then any $u_h \in V_h$, we use:

$$\begin{aligned}\delta h^{2r}||G^ts_h - \Pi_{Q_h}G^ts_h||_H||u_h||_V &= \delta h^{\frac{3}{2}r}||G^ts_h - \Pi_{Q_h}G^ts_h||_H h^{\frac{1}{2}}||u_h||_V\\&\leq \delta^2 h^{3r}||G^ts_h - \Pi_{Q_h}G^ts_h||_H^2 + h||u_h||_V^2,\end{aligned}$$

and then:

$$\psi_h((u_h,p_h,s_h),(0,0,-u_h))$$
$$\geq \frac{1}{2}||u_h||_H^2 - ||p_h||_Q^2 - \varepsilon^2||s_h||_H^2 - \delta^2 h^{3r}||G^t s_h - \Pi_{Q_h} G^t s_h||_H^2 - h||u_h||_V^2.$$

Then with (5.17) and (5.28), for any $\kappa_1,\kappa_2,\kappa_3>0$:

$$\begin{aligned}
&\psi_h((u_h,p_h,s_h),(u_h,p_h,s_h)+(\kappa_1 v_s,\kappa_2 q_s,-\kappa_3 u_h))\\
&\geq (\frac{\alpha(\gamma-1)}{\gamma} - \frac{\kappa_1(||a||+\beta_\gamma c_V^2)^2}{2c_1} - \kappa_3 h)||u_h||_V^2\\
&\quad + (\frac{\beta_\gamma}{2} - \frac{\kappa_2\beta_\gamma^2}{c_2} - \kappa_3)||p_h||_Q^2 + (\frac{\kappa_3}{2} - \frac{\beta(\gamma-1)}{\gamma})||u_h||_H^2\\
&\quad + \varepsilon(1-\kappa_3\varepsilon)||s_h||_H^2 + \frac{\kappa_1 c_1}{2}||s_h||_{V'}^2 + \frac{\kappa_2 c_2}{2}||G^t s_h||_{Q'}^2\\
&\quad + (\delta - \frac{\kappa_2}{c_2} - \kappa_3 h)h^{2r}||G^t s_h - \Pi_{Q_h} G^t s_h||_H^2.
\end{aligned}$$

Take $\kappa_3 = \min(\frac{\beta}{8\gamma},\frac{1}{2\varepsilon})$, $\kappa_2 = \frac{c_2\gamma}{8\beta}$, $\kappa_1 = \frac{1}{4}\frac{\alpha(\gamma-1)2c_1}{\gamma(||a||+\beta_\gamma c_V^2)^2}$, δ such that $\delta = \frac{4\kappa_2}{c_2} = \frac{2\gamma}{\beta}$, γ such that $\frac{\kappa_3}{2} = \frac{\beta(\gamma-1)}{\gamma}$ i.e. $\gamma = \frac{\beta}{\beta - \frac{\kappa_3}{2}}$ (with $\frac{\kappa_3}{2} < \beta$ with the above value of κ_3), and h small enough so that:

$$\begin{aligned}
&\psi_h((u_h,p_h,s_h),(u_h,p_h,s_h)+(\kappa_1 v_s,\kappa_2 q_s,-\kappa_3 u_h))\\
&\geq \frac{\alpha(\gamma-1)}{2\gamma}||u_h||_V^2 + \frac{\beta}{4\gamma}||p_h||_Q^2 + \frac{\varepsilon}{2}||s_h||_H^2\\
&\quad + \frac{\kappa_1 c_1}{2}||s_h||_{V'}^2 + \frac{\kappa_2 c_2}{2}||G^t s_h||_{Q'}^2 + \frac{\delta}{2}h^{2r}||G^t s_h - \Pi_{Q_h} G^t s_h||_H^2.
\end{aligned}$$

22- And if the inverse inequality is used, we have:

$$\begin{aligned}
&\delta h^{2r}||G^t s_h - \Pi_{Q_h} G^t s_h||_H ||G^t u_h||_H\\
&\leq \frac{1}{2\kappa_4}\delta^2 h^{2r}||G^t s_h - \Pi_{Q_h} G^t s_h||_H^2 + \frac{\kappa_4}{2}c_i^2||u_h||_H^2,
\end{aligned}$$

for any $\kappa_4>0$. Then choose κ_4 such that $\frac{\kappa_4}{2}c_i^2 \leq \frac{1}{8}$, i.e. $\kappa_4 = \frac{1}{4c_i^2}$, so that with (5.18):

$$\begin{aligned}
&\psi_h((u_h,p_h,s_h),(0,0,-u_h))\\
&\geq ||u_h||_H^2 - ||u_h||_H||Gp_h||_H - \varepsilon||s_h||_H||u_h||_H\\
&- \delta h^r||G^t s_h - \Pi_{Q_h} G^t s_h||_H||u_h||_H\\
&\geq \frac{1}{2}||u_h||_H^2 - 2||p_h||_Q^2 - 2\varepsilon^2||s_h||_H^2 - 2c_i^2\delta^2 h^{2r}||G^t s_h - \Pi_{Q_h} G^t s_h||_H^2.
\end{aligned}$$

Then with (5.17) and (5.28), for any $\kappa_1, \kappa_2, \kappa_3 > 0$:

$$\begin{aligned}
&\psi_h((u_h,p_h,s_h),(u_h,p_h,s_h)+(\kappa_1 v_s, \kappa_2 q_s, -\kappa_3 u))\\
&\geq (\frac{\alpha(\gamma-1)}{\gamma} - \frac{\kappa_1(||a||+\beta_\gamma c_V^2)^2}{2c_1})||u_h||_V^2 + (\frac{\beta_\gamma}{2} - \frac{\kappa_2\beta_\gamma^2}{c_2} - 2\kappa_3)||p_h||_Q^2\\
&\quad + (\frac{\kappa_3}{2} - \frac{\beta(\gamma-1)}{\gamma})||u_h||_H^2 + \varepsilon(1-2\kappa_3\varepsilon)||s_h||_H^2 + \frac{\kappa_1 c_1}{2}||s_h||_{V'}^2\\
&\quad + \frac{\kappa_2 c_2}{2}||G^t s_h||_{Q'}^2 + (\delta - \frac{\kappa_2}{c_2} - 2\kappa_3 c_i^2\delta^2)h^{2r}||G^t s_h - \Pi_{Q_h} G^t s_h||_H^2.
\end{aligned}$$

Then we choose $\kappa_1 = \frac{1}{2}\frac{2\alpha(\gamma-1)}{\gamma(||a||+\beta c_V^2)^2}$, $\kappa_2 = \frac{1}{8\beta}$ and $\kappa_3 = \min(\frac{\beta}{16}, \frac{1}{4})$ (with the hypothesis $\varepsilon < 1$), so that we also now choose γ such that $\frac{\beta(\gamma-1)}{\gamma} = \frac{\kappa_3}{2}$, i.e. $\gamma = \frac{\beta}{\beta - \frac{\kappa_3}{2}} > 1$ (with $\frac{\kappa_3}{2} < \beta$ by the above choice of κ_3) to get, choosing δ such that $\delta = \min(4\frac{\kappa_2}{c_2}, \frac{1}{8\kappa_3 c_i^2})$, with $(v_h, q_h, s'_h) = (u_h + \kappa_1 v_s, p_h + \kappa_2 q_s, s_h - \kappa_3 u_h)$:

$$\begin{aligned}
&\psi_h((u_h,p_h,s_h),(v_h,q_h,s'_h))\\
&\geq \frac{\alpha(\gamma-1)}{2\gamma}||u_h||_V^2 + \frac{\beta}{4\gamma}||p_h||_Q^2 + \frac{\varepsilon}{2}||s_h||_H^2 + \frac{\kappa_1 c_1}{2}||s_h||_{V'}^2\\
&\quad + \frac{\kappa_2 c_2}{2}||G^t s_h||_{Q'}^2 + \frac{\delta}{2}h^{2r}||G^t s_h - \Pi_{Q_h} G^t s_h||_H^2.
\end{aligned}$$

And here we have $\delta = \min(\frac{1}{2\beta c_2}, \frac{1}{8c_i^2 \min(\frac{\beta}{16}, \frac{1}{4})})$ (as soon as $\varepsilon < 1$).

Then with $|||(v_h, q_h, s'_h)||| = O(|||(u_h, p_h, s_h)|||)$ (easy to check) we get (5.26). ∎

Proposition 5.9 *Suppose that the hypotheses of Lemma 5.8 are satisfied. Then:*

$$\begin{aligned}
&||(u-u_h, p-p_h, s-s_h)||_{h2}^2\\
&\leq c \inf_{(v_i,q_i,s'_i)\in V_h\times Q_h\times V_h} ||(u-u_i, p-p_i, s-s'_i||_{h2}^2 + h^{2r}||G^t s - \Pi_{Q_h} G^t s||_H^2
\end{aligned}$$

where $c = O(1)$ as $\lambda \to \infty$.

Proof. We have: $\psi_h((u-u_h, p-p_h, s-s_h),(v_h, q_h, s'_h)) = h^{2r}(G^t s - \Pi_{Q_h} G^t s, G^t s_h - \Pi_{Q_h} G^t s_h)$, and then the result with the usual technique. ∎

Remark 5.10 The value c_i of the inverse inequality can be computed by an eigenvalue type computation (through a Raileigh quotient) as well as the value of β:

$$c_i^2 = h^2 \max_{v_h\in V_h} \frac{||v_h||_V^2}{||v_h||_H^2}, \quad \beta = \min_{v_h\in V_h} \frac{a(v_h, v_h)}{||v_h||_H^2}.$$

However, Proposition 5.19 shows that the computation of c_i is useless if we use β_c with $c > 1$. ∎

6 Numerical results

The results are computed with Matlab PDE Toolbox, thus with the only use of P_1-continuous finite elements (the only available elements up to now in the Toolbox).

The meshes used for the computations are shown in Figure 1, but since the results are quite independents of these meshes, the results shown are the one computed with the first mesh (obtained with the Matlab PDE Toolbox).

The problem under concern is (2.4). In the abstract setting, we then have $V = H_0^1(\Omega)^2$, $Q = H_0^1(\Omega)$, $H = L^2(\Omega)$, $G = \text{grad}$. And the problems that will be solved are:

1- The initial discretized problem (3.11),

2- The corrected discretized problem (3.17),

3- The discretized problem (3.24) (with the variable s_h),

4- The discretized stabilized problem (5.24) (with the variable s_h).

The numerics are performed on the square $\Omega =]0,1[^2$ to get the analytical solution:

$$p = x^2(1-x)^2y^2(1-y)^2 \quad \text{and} \quad \underline{u} = \text{grad}p.$$

We check that indeed p and $\underline{u}$ are in $H_0^1(\Omega)$. And the source terms are then given by:

$$\underline{g} = -\Delta\underline{u}, \quad f = 0.$$

And the choice for λ is:

$$\lambda = 100000$$

Finally β is computed as realizing the minimum of the Rayleigh quotient '$\frac{||\text{grad}v_h||^2_{L^2}}{||v_h||^2_{L^2}}$'. Its numerical value is close to 20. And we shall use $\beta_c = 10 \simeq \frac{\beta}{2}$.

The locked result is obtained using (3.11) (the initial discrete equations), and the unlocked result is obtained using either the first correction (3.17) or (3.24) (no significant differences), or the stabilization (5.24).

We show a generic result, see Figure 2, and then we plot the computed errors relative to the different computations, all results showing the $O(h)$ expected convergence:

1. The first correction (3.17) and its counterpart (3.24) yield very similar errors which will be represented just once.

2. The stabilization (5.24) yields some differences with the two previous corrected equations in the numerical values of the error and will be represented on its own, the computation being done with $\beta_c = 10 \simeq \frac{\beta}{2}$ to avoid the computation of the inverse inequality constant.

Remark 6.1 The initial discrete locked equations (3.11) yields too large errors to be shown on the same plot as the errors of the unlocked equations. Thus they are not shown but their behaviour is classical. ∎

The conclusion is: For Mindlin–Reissner thick plate type equations, the use of P_1-continuous finite elements in each variable seems to be adequate once the modifications proposed are performed.

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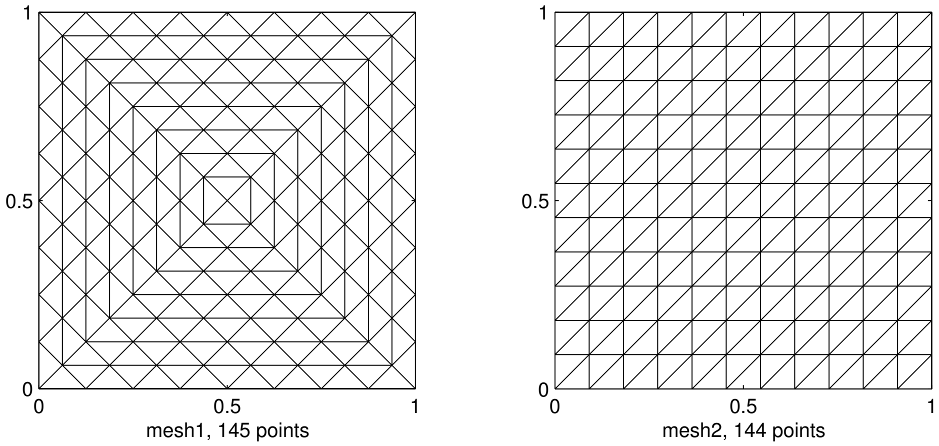

Figure 1: Meshes used: The results shown in the following figures have been obtained using the left mesh and its refinements which are given by Matlab PDE Toolbox. The results obtained with the right mesh and its refiniments are similars and not represented. Each triangle is half a square, each square having the same size, and the mesh size h has been taken to be the size of a side of a square. This size h is used in Figure 3 to compute the errors.

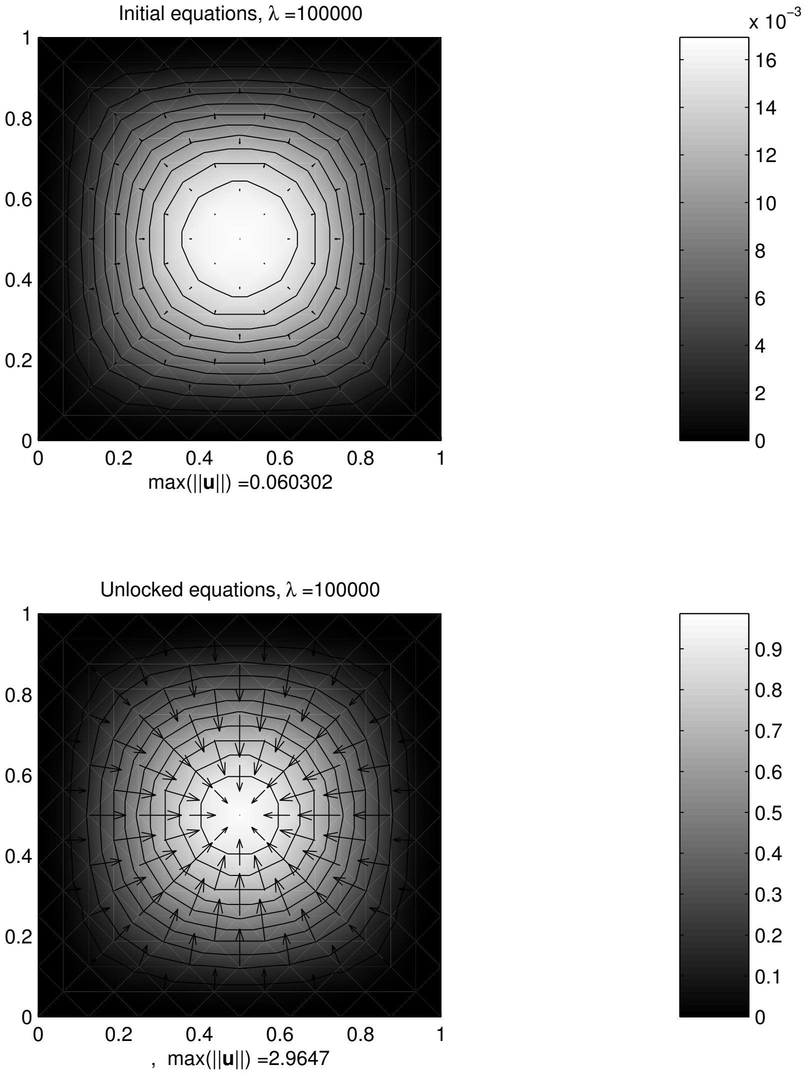

Figure 2: The values of p_h are given by the gray scale and the color bar on the right of the Figures whereas the values of $\underline{u}_h = \mathbf{u}$ are given by the arrows, the maximum length $\max(||\mathbf{u}||)$ being given under each Figure. The first Figure corresponds to equations (3.11) and shows values of p_h (and $\underline{u}_h$) too small which is typical of locked equations. And the second Figure corresponds to equations (3.17) and shows unlocked p_h and $\underline{u}_h$ fields, the visual results being similar for equations (3.24) or the stabilized equations (5.24). The finite elements used are the P_1-continuous finite elements in each variable.

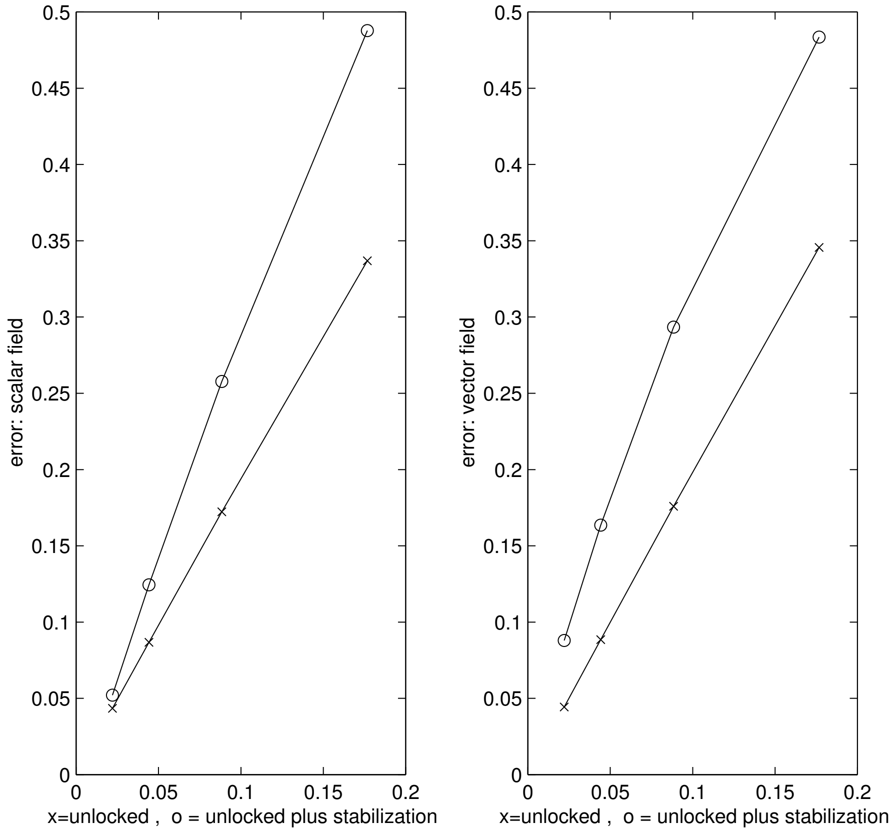

Figure 3: Error computations, corresponding to the case $\lambda = 100000$ are represented. The horizontal axis is the h axis. The left Figure correponds to the relative error $\frac{||\mathrm{grad} p - \mathrm{grad} p_h||_{L^2}}{||\mathrm{grad} p||_{L^2}}$ corresponding to the scalar field p, whereas the right Figure corresponds to the relative error $\frac{||\mathrm{grad} \underline{u} - \mathrm{grad} \underline{u}_h||_{L^2}}{||\mathrm{grad} \underline{u}||_{L^2}}$ corresponding to the vector field $\underline{u}$. And on both of these figures, the 'x' lines correspond to the unlocked equations (3.17), whereas the 'o' lines correspond to the unlocked and stabilized equations (5.24). The errors show the $O(h)$ behaviour.