Structured Quantum Search: Quantum Walks and Quantum Backtracking

Simon Martiel

November 5, 2021

Everything is based on:

- Ronald de Wolf's lecture note on quantum algorithms
- Ashley's Montanaro papers on quantum backtracking

Grover (unstructured search)
Quick reminder

Input: a predicate $f : \{0,1\}^n \to \{0,1\}$ that marks elements in $M \subseteq X$

Goal: find x s.t. $f(x) = 1$

Grover (unstructured search)

Quick reminder

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Goal: find x s.t. $f(x) = 1$

Classical algorithm:

1. pick x uniformly at random
2. if $f(x)$, output x, else go to (1)

$\implies O(\frac{1}{\varepsilon})$ calls to f where $\varepsilon = \frac{|M|}{|X|}$

Grover (unstructured search)
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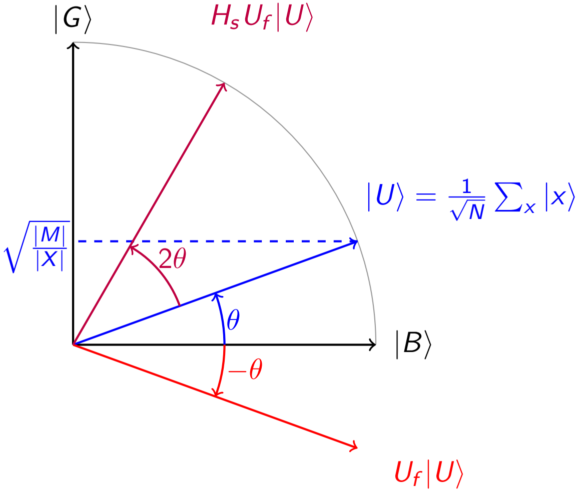
Quantum algorithm:

1. start from $|U\rangle = \frac{1}{\sqrt{|X|}} \sum_{x \in X} |x\rangle$
2. $O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ times:
   - 2.1 reflect through $|B\rangle = \frac{1}{\sqrt{|X| - |M|}} \sum_{x \in X, P(x)=0} |x\rangle$ (call to U_f)
   - 2.2 reflect through $|U\rangle$ ($-H_s$ from Emmanuel's talk)
3. measure the system (go to (1) if the result is invalid)

$\implies O(\sqrt{1/\varepsilon})$ calls to U_f

Grover (unstructured search)

Quick reminder



Moving to structured search

Grover assumes NO structure on f:

- X is simple to sample uniformly
- X is huge (we never "prune" the search space)

Real life problems have structure :

- the search space can usually be "pruned"
- but this "pruned" space is hard to sample uniformly

Example: SAT

- Grover will "explore" $\{0,1\}^n$
- a smart classical backtracking (e.g. DPLL) will explore a tiny portion of $\{0,1\}^n$
- thus Grover fails to bring a quadratic speed up

How can we "restrict" a quantum search to this "pruned" space ?

How to give structure to X ?

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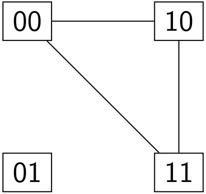
$\boxed{01}$ $\boxed{11}$

How to give structure to X ?



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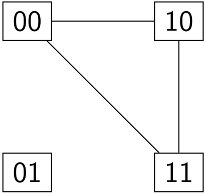

Natural structure: a graph $G(X, E)$

Random walk as a search algorithm

Question: how search for maked elements in a graph?

Random search in a graph:

1. Pick any vertex x of G (randomly, or not)
2. While not $f(x)$:
   2.1 pick a neighbor y in $\mathcal{N}_x$ at random
   2.2 set $x \leftarrow y$

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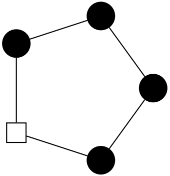

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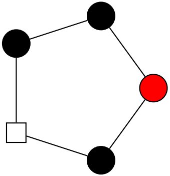

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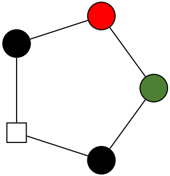

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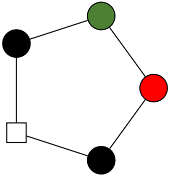

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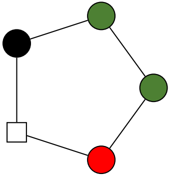

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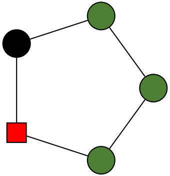

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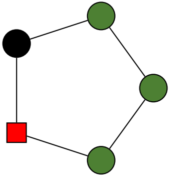


How can we analyze this algorithm ?

Random walk as a search algorithm

Theorem

If G is connected, not bipartite, and of constant degree:

- *this process converges to the uniform distribution over X*
- *the speed of convergence depends on the spectral gap δ of G*

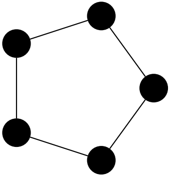
$$\delta = \lambda_0 - \max |\lambda_i|$$

where λ_i are eigenvalues of the (normalized) adjacency matrix of G, $\lambda_0 = 1$ being the largest one.

Convergence "speed" in $O(\frac{1}{\delta})$

Random walk as a search algorithm

What is this spectral gap thing ?



$$P = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Eigenvalues of P: 1 0.309 0.309 - 0.809 - 0.809

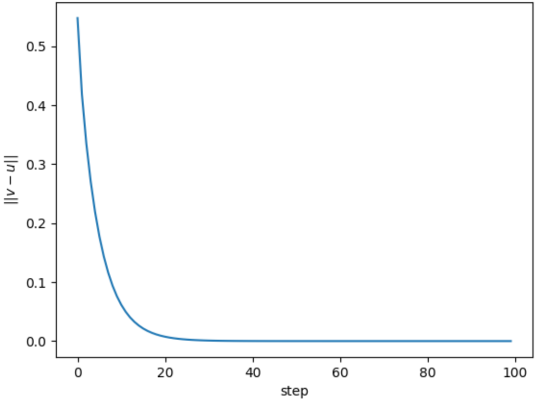
Hence $\delta = \lambda_0 - \max|\lambda_i| = 1 - 0.809 \approx 0.19$

Random walk as a search algorithm

In other words:

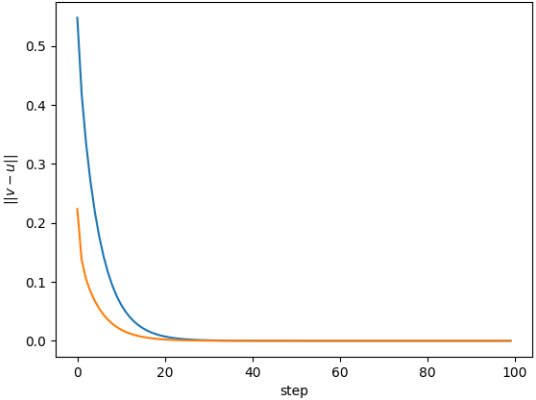
- we can associate to G some quantity δ
- if we "walk" randomly around G for $O(\frac{1}{\delta})$ steps we end up picking a vertex uniformly at random

Random walk as a search algorithm



Initial distribution: $[1, 0, 0, 0, 0]$

Random walk as a search algorithm



Initial distribution: $[1, 0, 0, 0, 0]$ $[0.5, 0.5, 0, 0, 0]$

Random walk as a search algorithm

Random search in a graph:

1. Pick any vertex x of G (randomly, or not)
2. repeat $O(\frac{1}{\delta})$ times:
   2.1 pick a neighbor y in $\mathcal{N}_x$ at random
   2.2 set $x \leftarrow y$
3. if $f(x)$ return x else go to (2)

Random walk as a search algorithm

Random search in a graph:

1. Pick any vertex x of G (randomly, or not)
2. repeat $O(\frac{1}{\delta})$ times:
   2.1 pick a neighbor y in $\mathcal{N}_x$ at random
   2.2 set $x \leftarrow y$
3. if $f(x)$ return x else go to (2) $\leftarrow$ x is picked almost uniformly

Expected cost of this procedure:

$$S + \frac{1}{\varepsilon}(C + \frac{1}{\delta}N)$$

S: cost of picking the first vertex
C: cost of a call to the predicate f
N: cost of picking a neighbor of a vertex

Random walk as a search algorithm

Some simple remarks:

- if G is the complete graph, this is the classical algorithm of the first slide
$$\delta_{K_N} = 1 - \frac{1}{N}$$
$\implies$ very large gap
- this is a cheap algorithm ($O(log(|X|))$ space)
- this is a tool to uniformly sample structured spaces : e.g. sampling valid partial assignments in a CSP

There is a "quantization" of this approach: Quantum Walks !

Quantum walks

We need two registers storing vertices:

$$|x\rangle|y\rangle, x, y \in X$$

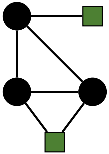
Morally:

- $|x\rangle$ stores the previous vertex
- $|y\rangle$ stores the current vertex

Notation:

$$|p_x\rangle = \frac{1}{\sqrt{d(x)}} \sum_{y \in \mathcal{N}_x} |y\rangle$$

$$|\phi_x\rangle = |x\rangle|p_x\rangle$$



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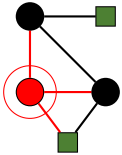


Quantum walks

Good state:

$$|G\rangle = \frac{1}{\sqrt{|M|}} \sum_{x \in M} |x\rangle |p_x\rangle$$

Bad state:

$$|B\rangle = \frac{1}{\sqrt{|X| - |M|}} \sum_{x \notin M} |x\rangle |p_x\rangle$$

Uniform state:

$$|U\rangle = \frac{1}{\sqrt{|X|}} \sum_{x \in X} |x\rangle |p_x\rangle = \sin\theta |G\rangle + \cos\theta |B\rangle$$

with $\theta = \arcsin\left(\sqrt{\varepsilon}\right)$

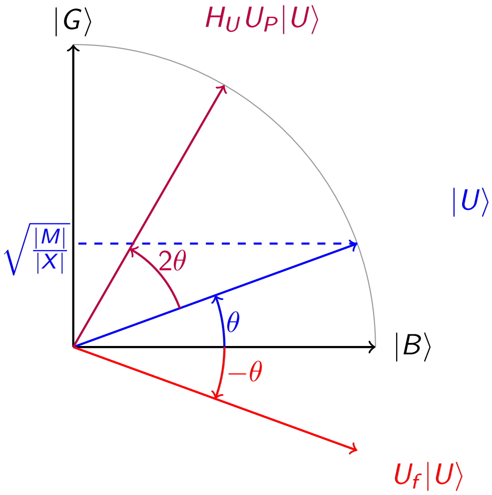
Quantum search with a Quantum Walk:

1. prepare the system in state $|U\rangle$
2. Repeat $O(\frac{1}{\sqrt{\varepsilon}})$ times
   2.1 reflect through $|B\rangle$
   2.2 reflect through $|U\rangle$
3. measure and check if the result is valid

Proof that it works: same as Grover

Quantum walk

Quantum Search



After p steps: $\sin\left((2p+1)\theta\right)|G\rangle + \cos\left((2p+1)\theta\right)|B\rangle$

If we manage to implement:

- a reflection through $|B\rangle$
- a reflection through $|U\rangle$

We have a similar argument as Grover

Reflection through $|B\rangle$:

$$|x\rangle \mapsto (-1)^{f(x)}|x\rangle$$

Reflection through $|U\rangle$: tricky

Theorem

Reflection through $|U\rangle$ can be implemented using $O(\frac{1}{\sqrt{\delta}})$ calls to:

$$N : |x\rangle|0\rangle \mapsto |x\rangle|p_x\rangle$$

Final complexity:

$$S + \frac{1}{\sqrt{\varepsilon}}(C + \frac{1}{\sqrt{\delta}}N)$$

Quantum Walk

Magniez Nayak Roland Santha (MRNS)

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Final complexity:

$$S + \frac{1}{\sqrt{\varepsilon}}(C + \frac{1}{\sqrt{\delta}}N)$$

Proof: using a Quantum Phase Estimation to distinguish $|U\rangle$ from other states

Aparté on Quantum Phase Estimation

Input: a unitary operator W and an eigenvector $|s\rangle$:

$$W|s\rangle = e^{2i\pi\theta_s}|s\rangle$$

Goal: produce an estimate of θ_s

Quantum Phase Estimation:

$$QPE|s\rangle|0\rangle \mapsto |s\rangle|\tilde{\theta}_s\rangle$$

where $\tilde{\theta}_s = \lfloor \theta_s \times 2^k \rfloor$

Aparté on Quantum Phase Estimation

$$W|s\rangle = e^{0.8125*2i\pi}|s\rangle$$

with $k = 0$?

$$0.8125 \times 2^0 = 0.8125 \approx 0$$

$$\tilde{\theta}_s = "0"$$

Aparté on Quantum Phase Estimation

$$W|s\rangle = e^{0.8125 * 2i\pi}|s\rangle$$

with $k = 1$?

$$0.8125 \times 2^1 = 1.625 \approx 1$$

$$\tilde{\theta}_s = "01"$$

Aparté on Quantum Phase Estimation

$$W|s\rangle = e^{0.8125*2i\pi}|s\rangle$$

with $k = 2$?

$$0.8125 \times 2^2 = 3.25 \approx 3$$

$$\tilde{\theta}_s = "011"$$

Aparté on Quantum Phase Estimation

$$W|s\rangle = e^{0.8125*2i\pi}|s\rangle$$

with $k = 3$?

$$0.8125 \times 2^3 = 6.5 \approx 6$$

$$\tilde{\theta}_s = "0110"$$

Aparté on Quantum Phase Estimation

$$W|s\rangle = e^{0.8125*2i\pi}|s\rangle$$

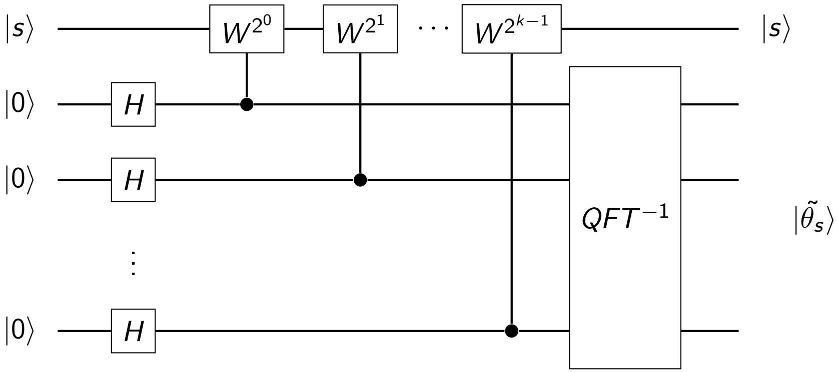
with $k = 4$?

$$0.8125 \times 2^4 = 13$$

$$\tilde{\theta}_s = "01101"$$

Aparté on Quantum Phase Estimation

How to?



Aparté on Quantum Phase Estimation

Step by step

$$
\begin{aligned}
|s\rangle|0\rangle &\mapsto \frac{1}{\sqrt{2^k}} \sum_{x\in\{0,1\}^k} |s\rangle|x\rangle && H^{\otimes k} \\
&\mapsto \frac{1}{\sqrt{2^k}} \sum_{x\in\{0,1\}^k} W^{x_0}|s\rangle|x\rangle && C-W^{2^0}
\end{aligned}
$$

Aparté on Quantum Phase Estimation

Step by step

$$|s\rangle|0\rangle \mapsto \frac{1}{\sqrt{2^k}} \sum_{x\in\{0,1\}^k} |s\rangle|x\rangle \qquad H^{\otimes k}$$

$$\mapsto \frac{1}{\sqrt{2^k}} \sum_{x\in\{0,1\}^k} e^{2i\pi x_0\theta_s}|s\rangle|x\rangle \qquad C-W^{2^0}$$

Aparté on Quantum Phase Estimation

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&\mapsto \frac{1}{\sqrt{2^k}} \sum_{x\in\{0,1\}^k} e^{2i\pi x_0\theta_s} W^{2^{x_1}}|s\rangle|x\rangle && C-W^{2^1}
\end{aligned}
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&\mapsto \frac{1}{\sqrt{2^k}} \sum_{x\in\{0,1\}^k} e^{2i\pi (x_0+2x_1)\theta_s}|s\rangle|x\rangle && C-W^{2^1}
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Aparté on Quantum Phase Estimation

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|s\rangle|0\rangle &\mapsto \frac{1}{\sqrt{2^k}} \sum_{x\in\{0,1\}^k} |s\rangle|x\rangle && H^{\otimes k} \\
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&\vdots \\
&\mapsto \frac{1}{\sqrt{2^k}} \sum_{x\in\{0,1\}^k} e^{2i\pi x\theta_s}|s\rangle|x\rangle && C-W^{2^{k-1}}
\end{aligned}
$$

$$|s\rangle \left[\frac{1}{\sqrt{2^k}} \sum_{x \in \{0,1\}^k} e^{2i\pi x \theta_s} |x\rangle \right]$$

The thing on the right can be written as :

$$\frac{1}{\sqrt{2^k}} \sum_{x \in \{0,1\}^k} e^{2i\pi x \tilde{\theta}_s / 2^k} |x\rangle$$

$$|s\rangle \left[\frac{1}{\sqrt{2^k}} \sum_{x \in \{0,1\}^k} e^{2i\pi x \theta_s} |x\rangle \right]$$

The thing on the right can be written as :

$$\frac{1}{\sqrt{2^k}} \sum_{x \in \{0,1\}^k} e^{\frac{2i\pi}{2^k} x \tilde{\theta}_s} |x\rangle$$

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The thing on the right can be written as :

$$\frac{1}{\sqrt{2^k}} \sum_{x \in \{0,1\}^k} \omega_{2^k}^{x\tilde{\theta}_s} |x\rangle$$

where ω_{2^k} is the 2^kth root of the unity

$$|s\rangle \left[\frac{1}{\sqrt{2^k}} \sum_{x \in \{0,1\}^k} e^{2i\pi x \theta_s} |x\rangle \right]$$

The thing on the right can be written as :

$$\frac{1}{\sqrt{2^k}} \sum_{x \in \{0,1\}^k} y_x |x\rangle$$

where y_x is the xth coefficient of the Fourier transform of $\tilde{\theta}_s$

Quantum Fourier Transform

In one slide :/

Theorem

One can build a circuit with $O(n^2)$ gates that performs a Fourier Transform:

$$QFT|x\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_y \omega_{2^n}^{xy} |y\rangle$$

Tiny remark: classically this requires $O(n2^n)$ classical operations (FFT)

Consequently (for our application), one can inverse the QFT in $O(k^2)$:

$$|s\rangle \underbrace{\left[\frac{1}{\sqrt{2^k}} \sum_{x \in \{0,1\}^k} e^{ix\theta_s} |x\rangle\right]}_{\approx QFT|\tilde{\theta}_s\rangle} \mapsto |s\rangle|\tilde{\theta}_s\rangle \qquad QFT^{-1}$$

Summing up

$$QPE|s\rangle|0\rangle \mapsto |s\rangle|\tilde{\theta}_s\rangle$$

Aparté on Quantum Phase Estimation

Step by step

Summing up

$$QPE|s\rangle|0\rangle \mapsto |s\rangle|\tilde{\theta}_s\rangle$$

Complexity:

$$O(2^k) \text{ calls to } W \text{ for a resolution of } 2^{-k}$$

Summing up

$$QPE|s\rangle|0\rangle \mapsto |s\rangle|\tilde{\theta}_s\rangle$$

Complexity:

$$O(\tfrac{1}{\nu}) \text{ calls to } W \text{ for a resolution of } \nu$$

Back to quantum walks

We want to perform a reflection through:

$$|U\rangle = \sum_x |x\rangle|p_x\rangle$$

One can build a $W(G)$ such that:

$$W(G)|U\rangle = |U\rangle$$

using 4 calls to:

$$N : |x\rangle|0\rangle \mapsto |x\rangle|p_x\rangle$$

Back to quantum walks

What is this $W(G)$?

- a reflection through $\mathcal{A} = span\{|x\rangle|p_x\rangle\}$
- a reflection through $\mathcal{B} = span\{|p_y\rangle|y\rangle\}$

Implementing reflections

What is a reflection $R(\mathcal{C})$ through some subspace $\mathcal{C}$?
if $w \in \mathcal{C}$:

$$R(\mathcal{C}) \cdot w = w$$

if w is orthogonal to $\mathcal{C}$:

$$R(\mathcal{C}) \cdot w = -w$$

Implementing reflections

How to implement $R(\mathcal{A}) = R(span\{|x\rangle|p_x\rangle\})$?
Using our "graph" oracle:

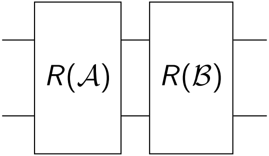
$$N : |x\rangle|0\rangle \mapsto |x\rangle|p_x\rangle$$

1. apply $N^{-1} : |x\rangle|p_x\rangle \mapsto |x\rangle|0\rangle$
   vectors in $\mathcal{A}$ now have a $|0\rangle$ in the second register
2. If the second register is NOT $|0\rangle$, inverse the phase
3. apply N

$R(\mathcal{B})$: same thing, just invert the two registers

Back to quantum walks

Now, we know how to implement $W(G)$:



Back to quantum walks

Some people can prove that:

$$W(G)|U\rangle = |U\rangle = e^{i0}|U\rangle$$

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and that the other eigenvalues of $W(G)$ are:

$$e^{\pm 2i \arccos(|\lambda_j|)} = e^{\pm i\theta_j}$$

where λ_j are the eigenvalues of G.

Back to quantum walks

Some people can prove that:

$$W(G)|U\rangle = |U\rangle = e^{i0}|U\rangle$$

and that the other eigenvalues of $W(G)$ are:

$$e^{\pm 2i \arccos(|\lambda_j|)} = e^{\pm i\theta_j}$$

where λ_j are the eigenvalues of G.
Thus we can bound:

$$\cos(\theta_j) = |\lambda_j| \leq 1 - \delta$$

and using:

$$\cos(\theta) \geq 1 - \frac{\theta^2}{2}$$

we get:

$$\theta_j \geq \sqrt{2\delta}$$

Implementing $R(|U\rangle)$

We have a $W(G)$ and we need to distinguish:

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- from other eigenvectors with eigenvalues $e^{\pm i\theta_j}$ with

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We can use a QPE:

$$\begin{aligned}
|w\rangle|0\rangle &\mapsto |w\rangle|\tilde{\theta_w}\rangle && QPE \\
&\mapsto (-1)^{\tilde{\theta_w}\neq 0}|w\rangle|\tilde{\theta_w}\rangle && \text{simple oracle} \\
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We will need a precision of about $\sqrt{\delta}/2$!
Hence, this will require $O(\frac{1}{\sqrt{\delta}})$ calls to $W(G)$

Summing up the algorithm

Step 1: prepare the unform state

$$|U\rangle = \frac{1}{\sqrt{|X|}} \sum_x |x\rangle |p_x\rangle$$

Step 2: perform $O(\frac{1}{\sqrt{\varepsilon}})$ "amplification" steps:

- use U_f to reflect through $|B\rangle$
- use $R(|U\rangle)$ to reflect through $|U\rangle$:
  - use $O(\frac{1}{\sqrt{\delta}})$ calls to N to perform a QPE of $W(G)$
  - flip the phase if the estimate of θ_j is not 0
  - undo the QPE

Step 3: measure the first register

Example

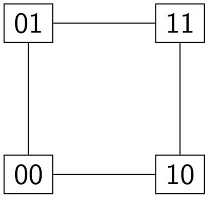
Here I should alt-tab and show some implementation

github.com $\rightarrow$ smartiel + quantum_walks_example

We will use: myqlm.github.io

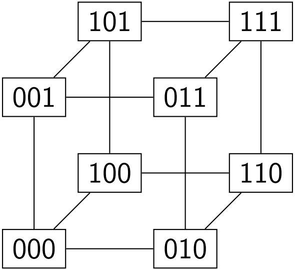
An hypercube of dimension n

Dimension 2:



An hypercube of dimension n

Dimension 3:



An hypercube of dimension n

Dimension n:
neighbors of $x = (x_1, ..., x_i, ..., x_n)$ are $(x_1, ..., \neg x_i, ..., x_n)$
For our QWalk we need to implement:

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An hypercube of dimension n

$$N : |x\rangle|0\rangle \mapsto |x\rangle|p_x\rangle$$

An hypercube of dimension n

$$N : |x\rangle|0\rangle \mapsto \frac{1}{n}|x\rangle \sum_{y \in \mathcal{N}_x} |y\rangle$$

Step 1:

$$|x\rangle|0\rangle \mapsto \frac{1}{\sqrt{n}}|x\rangle \sum_i |e_i\rangle$$

Step 2:

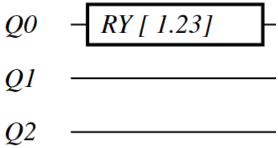
$$\frac{1}{\sqrt{n}}|x\rangle \sum_i |e_i\rangle \mapsto \frac{1}{\sqrt{n}}|x\rangle \sum_i |x \oplus e_i\rangle$$

An hypercube of dimension n

$1|000\rangle$

An hypercube of dimension n

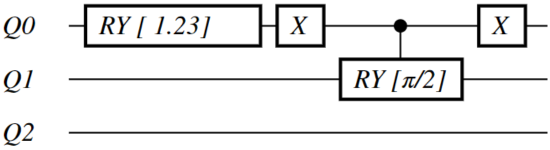
$$\begin{tikzcd} 1|000\rangle \arrow[r] \arrow[dr] & \frac{1}{\sqrt{3}}|100\rangle \\ & \sqrt{\frac{2}{3}}|000\rangle \end{tikzcd}$$



$$2\arcsin\left(\frac{1}{\sqrt{3}}\right) \approx 1.23$$

An hypercube of dimension n

$$
\begin{tikzcd}
1|000\rangle \arrow[r] \arrow[dr] & \frac{1}{\sqrt{3}}|100\rangle \arrow[r] & \frac{1}{\sqrt{3}}|100\rangle \\
& \sqrt{\frac{2}{3}}|000\rangle \arrow[r] \arrow[dr] & \frac{1}{\sqrt{3}}|010\rangle \\
& & \frac{1}{\sqrt{3}}|000\rangle
\end{tikzcd}
$$



$$2\arcsin\left(\frac{1}{\sqrt{2}}\right) = \pi/2$$

An hypercube of dimension n

$$\begin{array}{ccccccc}
1|000\rangle & \longrightarrow & \frac{1}{\sqrt{3}}|100\rangle & \longrightarrow & \frac{1}{\sqrt{3}}|100\rangle & \longrightarrow & \frac{1}{\sqrt{3}}|100\rangle \\
& \searrow & & & & & \\
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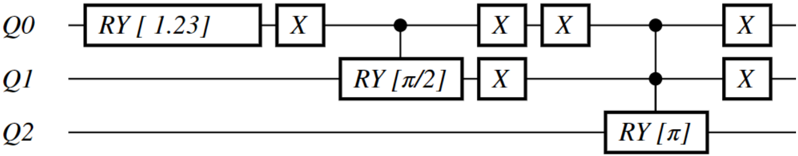

$$2\arcsin(1) = \pi$$

Le retour de l'arnaque

We presented a particular framework :

$$S + \frac{1}{\sqrt{\varepsilon}}(C + \frac{1}{\sqrt{\delta}}N)$$

We assumed that we can prepare $|U\rangle = \frac{1}{\sqrt{|X|}} \sum_x |x\rangle|p_x\rangle$

There are many quantum walk frameworks:

Framework	Complexity
Hitting time framework [Sze04, KMOR16, AGJK19]	$\mathsf{S} + \sqrt{\mathrm{HT}(P, M)}(\mathsf{U} + \mathsf{C})$
MNRS framework [MNRS11]	$\mathsf{S} + \frac{1}{\sqrt{\varepsilon}}(\frac{1}{\sqrt{\delta}}\mathsf{U} + \mathsf{C})$
Electric network framework [BCJ+13, Bel13]	$\mathsf{S}(\sigma) + \sqrt{C_{\sigma,M}}(\mathsf{U}(\sigma) + \mathsf{C})$
Controlled quantum amplification [DH17]	$\mathsf{S} + \sqrt{\mathrm{HT}(P, \{m\})}\mathsf{U} + \frac{1}{\sqrt{\varepsilon}}\mathsf{C}$

Table 1: Comparison of different quantum walk frameworks.

They were unified recently !

(Not so) Quantum Backtracking

Consider a CSP:

- with n variables $x_1, ..., x_n$
- each variable takes value in $[d]$
- a predicate $P : [d]^n \to \{0, 1\}$

Goal: find $x \in [d]^n$ s.t. $P(x) = 1$

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Goal: find $x \in [d]^n$ s.t. $P(x) = 1$

Ingredients for a backtracking:

- a variant of P that can check partial assignments:

$$P' : ([d] \cup \{\star\})^n \to \{0, 1, \mathrm{indeterminate}\}$$

- a heuristic h that tells me what to do next:

$$h : ([d] \cup \{\star\})^n \to [n]$$

(Not so) Quantum Backtracking

Backtracking algorithm:

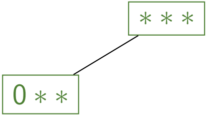
$\texttt{bt}(x)$:

1. If $P(x)$ is true, output x and return.
2. If $P(x)$ is false, or x is a complete assignment, return.
3. Set $j = h(x)$.
4. For each $w \in [d]$:
   (a) Set y to x with the j'th entry replaced with w.
   (b) Call $\texttt{bt}(y)$.

$\Longrightarrow$ call $bt(\star^n)$

(Not so) Quantum Backtracking

Example: satisfy $a \oplus b \wedge b \oplus c$



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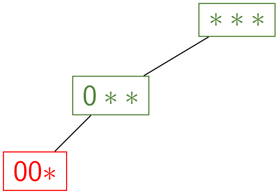

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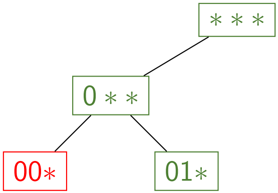

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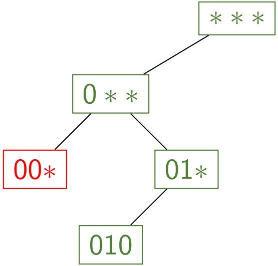

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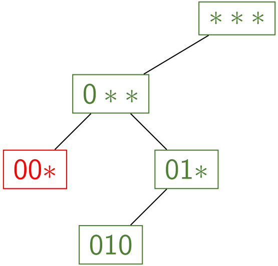
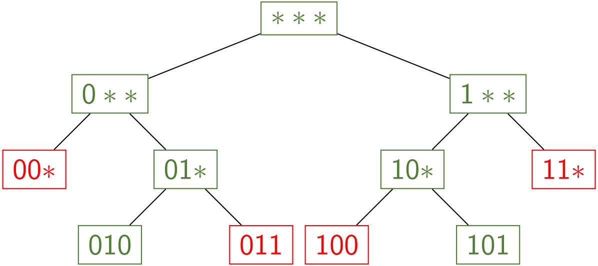


We explored 5 valuations !

(Not so) Quantum Backtracking

We could run it further to enumerate all the solutions:



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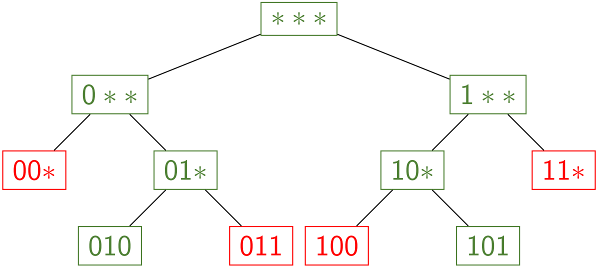

Total number of partial assignments:

$$3^3 = 27$$

Here we explored 11 partial assignments

Main idea:

- P/P' can be turned into quantum oracles
- h can be turned into a quantum oracle
- we have all we need to run a Quantum Walk on the backtracking tree !

Montanaro's Quantum Backtracking

We store assignments in the quantum memory as arrays:

$$|v_1\rangle \dots |v_n\rangle$$

$v_i \in [d] \cup \{\star\}$

Use U_P and U_h to implement:

- a reflection $R(\mathcal{A}) = R(span\{|x\rangle|p_x\rangle\})$
- a reflection $R(\mathcal{B}) = R(span\{|p_y\rangle|y\rangle\})$

This should seem familiar :)

Let's call it W !

Montanaro's Quantum Backtracking

First remark: we can't produce $|U\rangle$
(this would entail knowing the "pruned" tree)

Lemma

W admits as eigenvector of eigenvalue 1 superpositions of shape:

$$\sqrt{n}|\,\mathrm{root}\rangle + \sum_{x \neq r, x \to x_0} (-1)^{l(x)} |x\rangle$$

where x_0 is a valid leaf of the backtracking tree.

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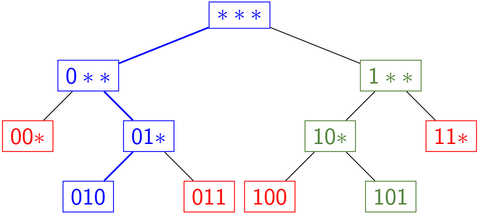

$$W\left(\sqrt{3}| * **\rangle - |0 * *\rangle + |01*\rangle - |010\rangle\right) = \sqrt{3}|***\rangle - |0**\rangle + |01*\rangle - |010\rangle$$

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For any $|\phi\rangle$ that is an eigenvector of eigenvalue 1 of W, we have:

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- in particular, we can run a Quantum Phase Estimation on W to see if we can detect an eigenvalue $1 = e^{i0}$

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- in particular, we can run a Quantum Phase Estimation on W to see if we can detect an eigenvalue $1 = e^{i0}$

Hence we can build a routine that:

- prepare an initial state $|\text{root}\rangle|p_{root}\rangle$
- perform a QPE on this state
- measure the phase estimation

If we measure $\theta = 0$, it means that there is a valid leaf !

Montanaro's Quantum Backtracking

The cost is determined by the precision of the QPE.

Theorem

We need a precision of $O(\frac{1}{\sqrt{Tn}})$ where T is the size of the backtracking tree (or any upper bound on it)

Montanaro's Quantum Backtracking

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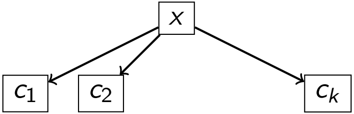
Theorem

We need $O(\sqrt{Tn})$ calls to W where T is the size of the backtracking tree (or any upper bound on it)

So we can detect the presence of a valid leaf in time $O(\sqrt{T})$.

Montanaro's Quantum Backtracking

An hybrid algorithm



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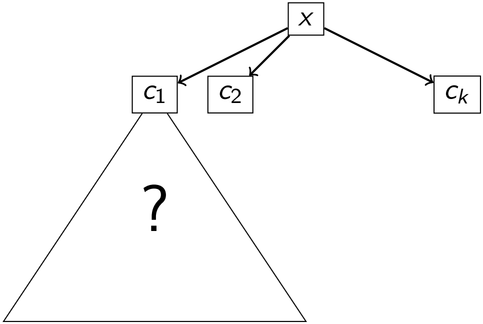

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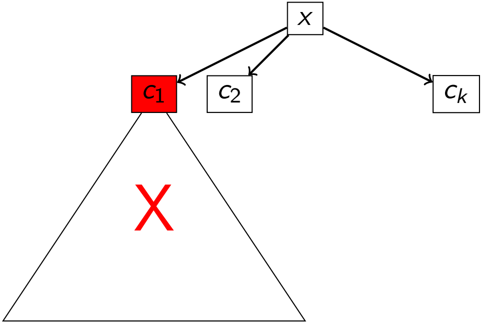

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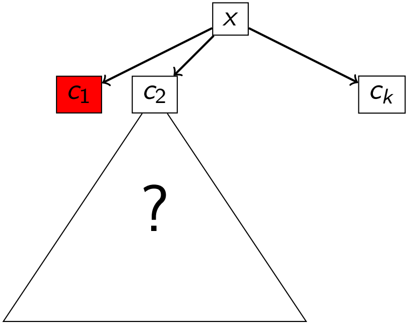

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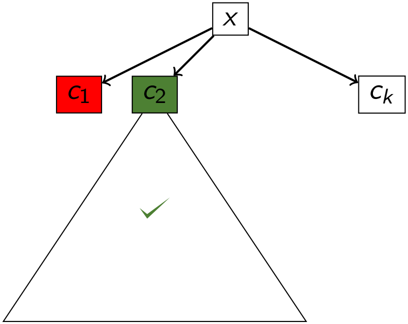

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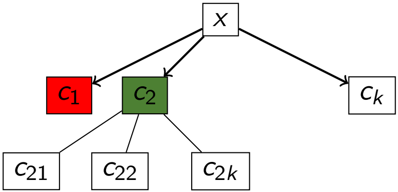

Montanaro's Quantum Backtracking

Complexity of finding a marked element (failure probability of δ):

$$O(\sqrt{T}n^{3/2}\log n \log 1/\delta)$$

Complexity of finding all marked elements:

$$O(k\sqrt{T}n^{3/2}\log n \log k/\delta)$$

Neat trick if there is a single marked element:

$$O(\sqrt{Tn}\log^3 n)$$

Applications

Campbell, Khurana, Montanaro

- estimate the size of the quantum circuit to implement W
- estimate the size of the backtracking tree for some problems (graph coloring and SAT solving)
- estimate the overhead due to quantum error correction
- deduce the effective advantage

Conclusion:

- it's (way) more competitive than Grover
- it needs lots of qubits (10^{13} for graphs that are not so big)
- for 1 day of computation: speed up of roughly 10^3 to 10^5 w.r.t to classical backtracking

A last word

Grover:

$$\frac{1}{\sqrt{\varepsilon}} C$$

Random walk:

$$S + \frac{1}{\varepsilon}(C + \frac{1}{\delta} N)$$

Quantum Walk (MRNS):

$$S + \frac{1}{\sqrt{\varepsilon}}(C + \frac{1}{\sqrt{\delta}} N)$$

Montanaro backtracking (more or less):

$$O(\sqrt{T})$$

Realistically:

- can provide speed up
- but the number of qubits is detrimental (10^{13})

Still gives hope for structured search !