Modeling and Solving Constraint Problems

Emmanuel Hebrard

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- The minimum about solving methods to allow for clever modeling
  - It turns out, it is already a lot!

Outline

1. **Language**
2. **Variables**
3. **Constraints**
4. **Modeling**
   - Ex: Golomb Ruler

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2. Variables
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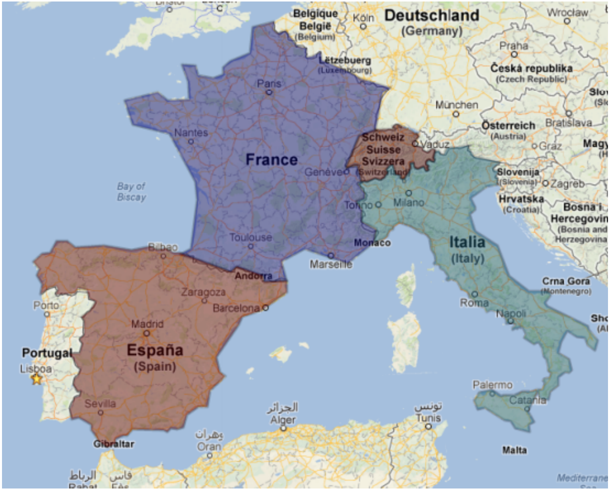
- Constraints: any relation between variables (e.g. $x = (\sqrt{y} \bmod 15)$)
- Objective: distinguished variable to minimize/maximize

Map Coloring



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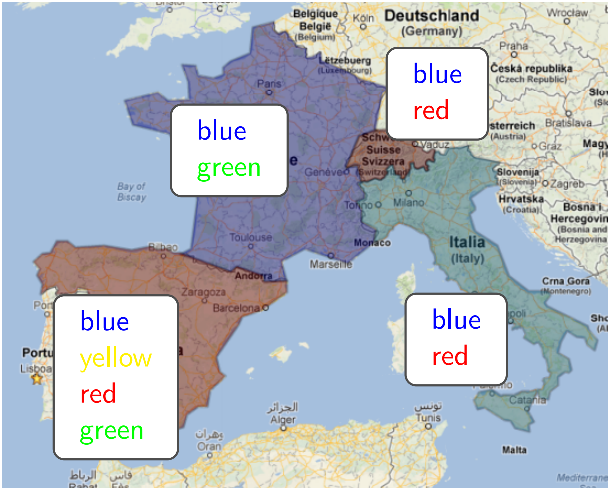

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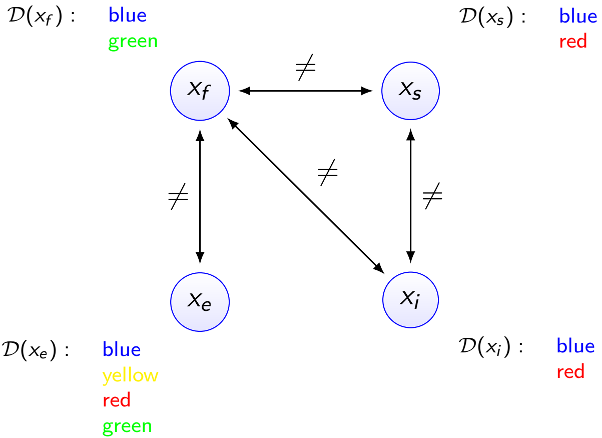

Map Coloring (Numberjack)

```
from Numberjack import *

france = Variable(['blue','green'], 'france')
switzerland = Variable(['blue','red'], 'switzerland')
spain = Variable(['blue','yellow','red','green'], 'spain')
italy = Variable(['blue','red'], 'italy')

model = Model(
    france != switzerland,
    france != italy,
    france != spain,
    italy != switzerland
    )

solver = model.load('Mistral2')

if solver.solve():
    for var in [france, switzerland, spain, italy]:
        print var.name(), 'in', var.get_value()
```

Map Coloring (Choco)

```
static final String[] colorname = {"red", "blue", "green", "yellow"};
static final Map<String, Integer> colorindex = new HashMap<String, Integer>();

public static void main(String[] args) {
	for(int i=0; i<colorname.length; ++i) colorindex.put(colorname[i], i);

	Model model = new Model("Map coloring example");

	IntVar france = model.intVar("france", new int[]{colorindex.get("blue"), colorindex.get("green")});
	IntVar switzerland = model.intVar("switzerland", new int[]{colorindex.get("blue"), colorindex.get("red"
	IntVar spain = model.intVar("spain", new int[]{colorindex.get("blue"), colorindex.get("yellow"), colori
	IntVar italy = model.intVar("italy", new int[]{colorindex.get("blue"), colorindex.get("red")});

	model.arithm(france, "!=", switzerland).post();
	model.arithm(france, "!=", italy).post();
	model.arithm(france, "!=", spain).post();
	model.arithm(italy, "!=", switzerland).post();

	if(model.getSolver().solve()){
		for(IntVar x : new IntVar[]{france, switzerland, spain, italy})
			System.out.printf("%s in %s\n", x.getName(), color_name[x.getValue()]);
	}
}
```

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  - Any Expression tree of the above

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- Consider the graph coloring example

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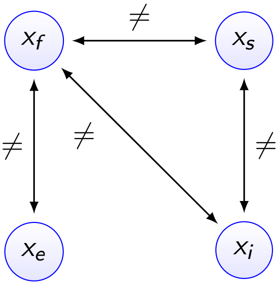

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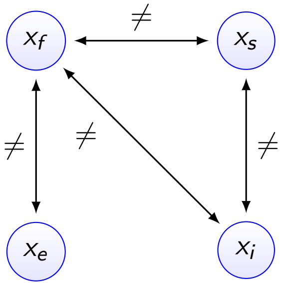

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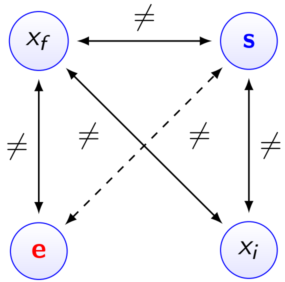

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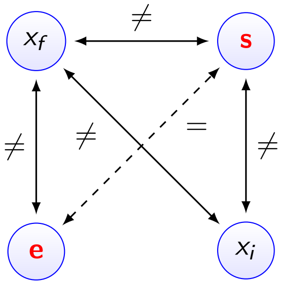

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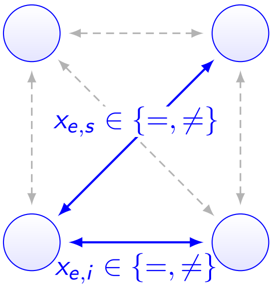

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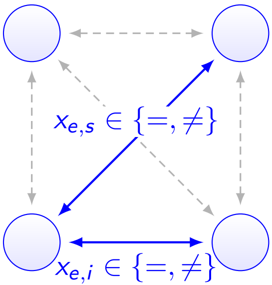

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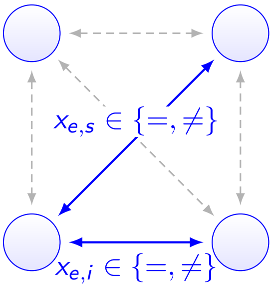

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  - either e and s take a different color, so adding the edge would not hurt
  - or e and s take the same color, so merging them (adding an equality constraint) would not hurt
- Instead of assigning colors to nodes, we can assign $\{=, \neq\}$ to non-edges
- No color symmetry anymore!
- But stating the constraints is difficult

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What is a good constraint set?

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Combining constraints (logically)

- Most logic operators
  - can be used as a relation $(x \neq y)$...
  - or as a predicate $((x \neq y) \implies y \leq 12)$
- Two different constraints: $x \neq y$ and $(x \neq y) \iff z$ (reification)

$$(x \neq y) \implies y \leq 12 \quad \text{encoded as} \quad \begin{array}{l} (x \neq y) \iff z \\ z \implies (y \leq 12) \end{array}$$

- Which you can write $(x \neq y) \implies y \leq 12$ (and let the system insert extra variables)

Combining constraints (functionally)

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- There are also function operators that must be combined similarly
  - For instance $(|x - y| * z) \leq (z + 12)$

$$(|x - y| * z) \leq (z + 12) \quad \text{encoded as} \quad \begin{aligned} &(x - y) = a_1 \\ &|a_1| = a_2 \\ &a_2 * z = a_3 \\ &z + 12 = a_4 \\ &a_3 \leq a_4 \end{aligned}$$

Constraints - Root of the expression tree

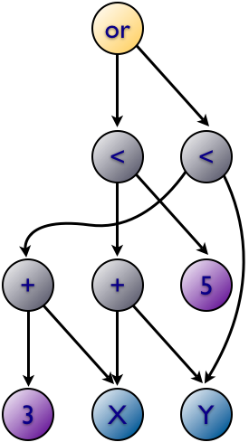
```
C1 = (X+Y < 5) | (X+3 < Y)
C2 = AllDiff([x,y,z])
C3 = Sum([a,b,c,d]) >= e
```

Predicates & functions - Internal nodes

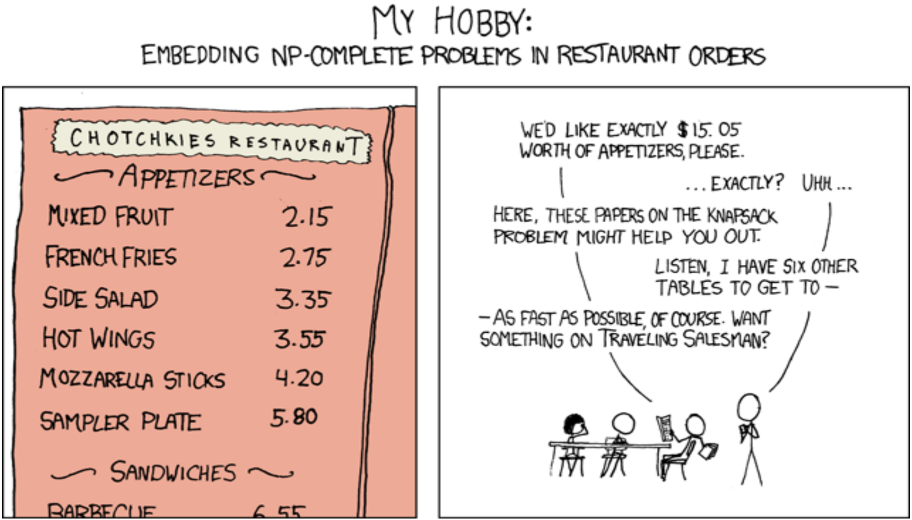
```
P = X+Y          # arythmetic value
Q = X+3 <= Y     # truth (logic) value
```

Variables - Leaves of the expression tree

```
X = Variable(0,10)
X = Variable([1,3,5,7])
```



XKCD Knapsack



XKCD Knapsack

```
from Numberjack import *

price = [215, 275, 335, 355, 420, 580]
appetizers = ["Mixed Fruit", "French Fries", "Side Salad",
              "Hot Wings", "Mozzarella Sticks", "Sample Plate"]
total = 1505
num_appetizers = len(appetizers)

quantities = [Variable(0, 1505/price[i], '#'+appetizers[i])
              for i in range(num_appetizers)]

model = Model(
    Sum([quantities[i] * price[i] for i in range(num_appetizers)]) == total
    )

solver = model.load('Mistral2')

solver.startNewSearch()
while solver.getNextSolution() == SAT:
    print "\nSOLUTION:\n", "\n".join("%s x %s ($%.2lf)" % (quantities[i], \
        appetizers[i], price[i] / 100.0) for i in xrange(num_appetizers))
```

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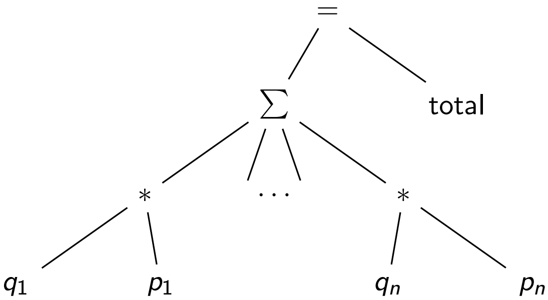

Solution

- Solution 1:

7	×	Mixed Fruit	($2.15)
0	×	French Fries	($2.75)
0	×	Side Salad	($3.35)
0	×	Hot Wings	($3.55)
0	×	Mozzarella Sticks	($4.20)
0	×	Sample Plate	($5.80)

- Solution 2:

1	×	Mixed Fruit	($2.15)
0	×	French Fries	($2.75)
0	×	Side Salad	($3.35)
2	×	Hot Wings	($3.55)
0	×	Mozzarella Sticks	($4.20)
1	×	Sample Plate	($5.80)

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Element

$$\mathit{Element}(x_0, \ldots, x_{n-1}, y, z) \iff x_y = z$$

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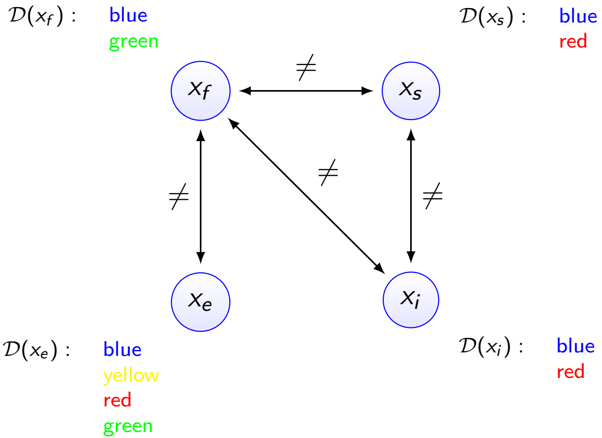
$\mathcal{D}(x_f)$: blue green
$\mathcal{D}(x_s)$: blue red
x_f
x_s
x_e
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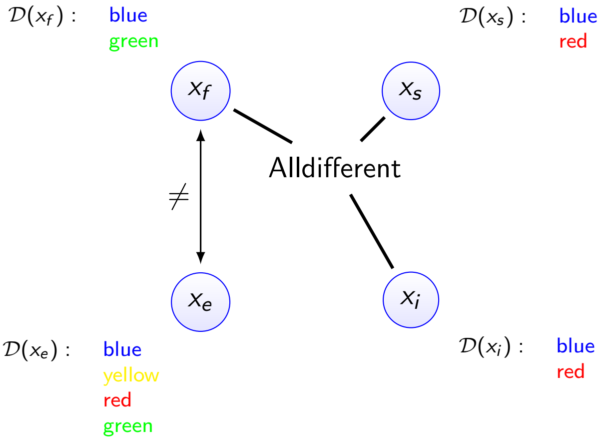

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constraint propagation

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- If $\mathcal{D}(x_f) = \{v\}$ then x_e cannot take the value v
- $x_f \in \{\mathbf{b}\}, x_e \in \{\mathbf{b}, \mathbf{r}, \mathbf{g}\} \implies x_f \in \{\mathbf{b}\}, x_e \in \{\mathbf{r}, \mathbf{g}\}$

Search Tree

$$x_f \in \{\mathbf{b}, \mathbf{g}\} \; - \; x_s \in \{\mathbf{b}, \mathbf{r}\}$$

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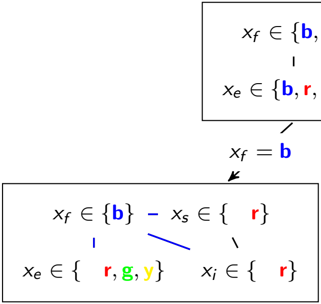
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$x_f \in \{b\}$ – $x_s \in \{ \; r\}$
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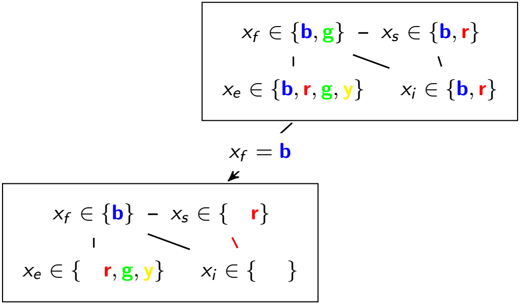
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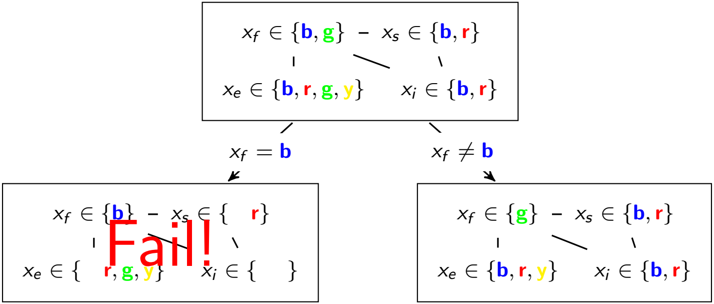
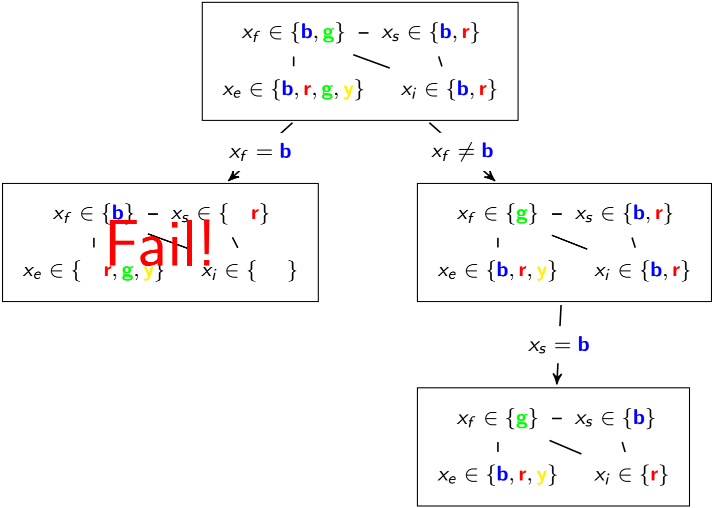
$x_i \in \{ \quad \}$

Search Tree


$x_f \in \{b, g\}$ – $x_s \in \{b, r\}$
$x_e \in \{b, r, g, y\}$
$x_i \in \{b, r\}$
$x_f = b$
$x_f \neq b$
$x_f \in \{b\}$ – $x_s \in \{\ \ r\}$
$x_e \in \{\ \ r, g, y\}$
$x_i \in \{\ \ \}$
Fail!
$x_f \in \{g\}$ – $x_s \in \{b, r\}$
$x_e \in \{b, r, y\}$
$x_i \in \{b, r\}$

Search Tree



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Example: global constraint

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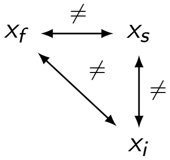

$$
\begin{aligned}
x_f &\in \{\mathbf{b}, \mathbf{g}\} \\
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$$

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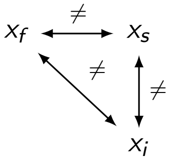

$$x_f \in \{\mathbf{b}, \mathbf{g}\}$$
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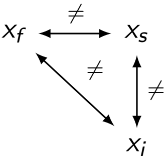
- Every inequality is consistent

Example: global constraint



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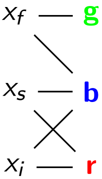
- Every inequality is consistent
- AllDifferent is not consistent!

Propagation of AllDifferent($\bar{x}$)

- A support is a perfect matching in the graph



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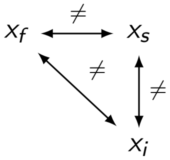

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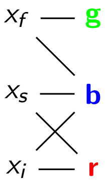
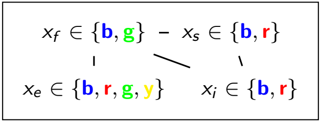
- Every inequality is consistent
- AllDifferent is not consistent!

Propagation of AllDifferent($\bar{x}$)

- A support is a perfect matching in the graph
- The edge (x_f, **b**) does not belong to any perfect matching
- AllDifferent(x_f, x_s, x_i) is consistent for $x_f \in \{\mathbf{g}\}$ $x_s \in \{\mathbf{b}, \mathbf{r}\}$ $x_i \in \{\mathbf{b}, \mathbf{r}\}$



Search Tree (AllDifferent)



Search Tree (AllDifferent)

$x_f \in \{\ \ \mathbf{g}\}$ – $x_s \in \{\mathbf{b}, \mathbf{r}\}$

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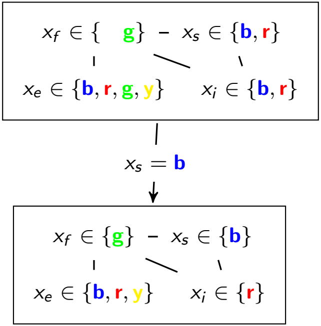
Search Tree (AllDifferent)



Propagation algorithm

- Every constraint has a propagation algorithm

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- How do we know what inference we can expect from a propagation algorithm?

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Every possible deduction w.r.t a single constraint on its variable's domain

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Every possible deduction w.r.t a single constraint on its variable's domain

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  - Does there exist a support for $x = v$ (a solution of the constraint involving $x = v$)
  - Otherwise, remove v from $\mathcal{D}(x)$

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- For every value v of every variable x
  - Does there exist a support for $x = v$ (a solution of the constraint involving $x = v$)
  - Otherwise, remove v from $\mathcal{D}(x)$
- The bigger (more global) the stronger! (and the slower...)

Outline

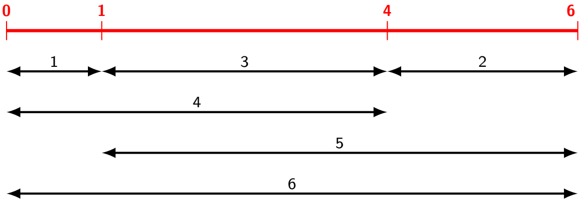
The art of modeling

Techniques to strenghthen propagation

- Common sub-expressions
- Global constraints
- Implied constraints
- Symmetry breaking
- Dominance

Problem definition

- Place *m* marks on a ruler
- Distance between each pair of marks is different
- Goal is to minimise the size of the ruler
- Proposed by Sidon [1932] then independently by Golomb and Babcock



A First Model (Numberjack)

```
import sys
from Numberjack import *

m = int(sys.argv[1]) if len(sys.argv)>1 else 6
n = 2 ** (m - 1)

marks = VarArray(m, n, 'm')
distance = [Abs(marks[i] - marks[j]) for i in range(1, m) for j in range(i)]

model = Model(
    Minimise(Max(marks)),  # objective function

    [m1 != m2 for m1,m2 in pair_of(marks)],
    [d1 != d2 for d1,d2 in pair_of(distance)]
)

solver = model.load('Mistral2', marks)
if solver.solve():
    print marks, [d.get_value() for d in distance]
```

A First Model (Choco)

```
Model model = new Model();

IntVar[] marks = model.intVarArray("m", m, 0, n);
IntVar[] distance = model.intVarArray("d",m * (m - 1) / 2, 1, n);

int k = 0;
for(int i=0; i<m; ++i) {
        for(int j=i+1; j<m; ++j) {
                model.distance(marks[i], marks[j], "=", distance[k++]).post();
                model.arithm(marks[i], "!=", marks[j]).post(); }}

for(int i=0; i<distance.length; ++i)
        for(int j=i+1; j<distance.length; ++j)
                model.arithm(distance[i], "!=", distance[j]).post();

IntVar objective = model.intVar("obj", 0, n);
model.max(objective, marks).post();

model.setObjective(Model.MINIMIZE, objective);
```

Branch & Bound

- An objective variable

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model.setObjective(Model.MINIMIZE, objective);
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- The lower bound is maintained via constraint propagation

```
model.max(objective, marks).post();
```

- Different models may entail different lower bounds for the same objective function

Global Constraints (Numberjack)

```python
import sys
from Numberjack import *

m = int(sys.argv[1]) if len(sys.argv)>1 else 6
n = 2 ** (m - 1)

marks = VarArray(m, n, 'm')
distance = [Abs(marks[i] - marks[j]) for i in range(m-1) for j in range(i+1,m)]

model = Model(
    Minimise(Max(marks)),  # objective function

    AllDiff(marks),
    AllDiff(distance)
)

solver = model.load('Mistral2', marks)
if solver.solve():
    print marks, [d.get_value() for d in distance]
```

Global Constraints (Choco)

```
Model model = new Model();

IntVar[] marks = model.intVarArray("m", m, 0, n);
IntVar[] distance = model.intVarArray("d",m * (m - 1) / 2, 1, n);

int k = 0;
for(int i=0; i<m; ++i)
        for(int j=i+1; j<m; ++j)
                model.distance(marks[i], marks[j], "=", distance[k++]).post();

model.allDifferent(marks).post();
model.allDifferent(distance).post();

IntVar objective = model.intVar("obj", 0, n);
model.max(objective, marks).post();

model.setObjective(Model.MINIMIZE, objective);
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Symmetry breaking

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- Solution symmetries $\Rightarrow$ symmetric (suboptimal) branches in the search tree

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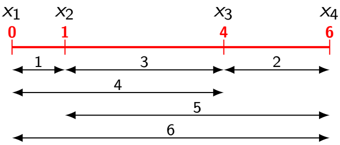

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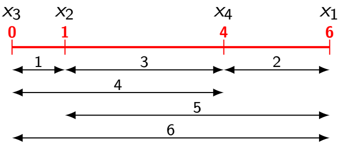

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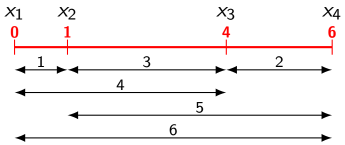

- Variable symmetries: `marks`, `distance`
- We can swap the marks or the distances of a solution (but not both)
- Force an arbitrary ordering
  - ★ `marks[1]` $<$ `marks[2]` $< \ldots <$ `marks[`m`]`

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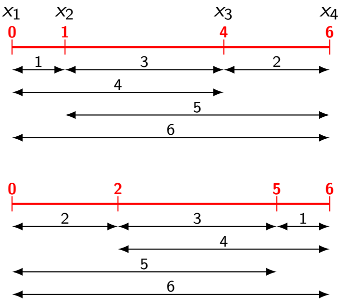

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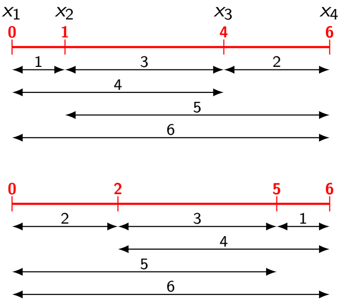

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- Force an arbitrary ordering
  - ★ `marks[1]` $<$ `marks[2]` $< \ldots <$ `marks[`m`]`
- Distances are still symmetric by reflection
  - ★ `distance[0,1]` $<$ `distance[`$m-2, m-1$`]`

Symmetry breaking (Numberjack)

```
import sys
from Numberjack import *

m = int(sys.argv[1]) if len(sys.argv)>1 else 6
n = 2 ** (m - 1)

marks = VarArray(m, n, 'm')
distance = [marks[j] - marks[i] for i in range(m-1) for j in range(i+1,m)]

model = Model(
    Minimise(marks[-1]),  # objective function

    [marks[i-1] < marks[i] for i in range(1, m)],
    marks[0] == 0,
    distance[0] < distance[-1],
    AllDiff(distance)
)

solver = model.load('Mistral2', marks)
solver.setHeuristic('MinDomainMinVal');
if solver.solve():
    print marks, [d.get_value() for d in distance]
```

Symmetry breaking (Choco)

```
Model model = new Model();

IntVar[] marks = model.intVarArray("m", m, 0, n);
IntVar[] distance = model.intVarArray("d",m * (m - 1) / 2, 1, n);

int k = 0;
for(int i=0; i<m-1; ++i) {
        model.arithm(marks[i], "<", marks[i+1]).post();
        for(int j=i+1; j<m; ++j)
                model.scalar(new IntVar[]{marks[i], marks[j]}, new int[]{-1,1}, "=", distance[k++]).post();
        model.arithm(marks[0], "=", 0).post();
        model.arithm(distance[0], "<", distance[distance.length-1]).post();
}

model.allDifferent(distance).post();

model.setObjective(Model.MINIMIZE, marks[m-1]);
```

Implied constraint

Implied by the model, does not change the set of solutions

Implied Constraints

Implied constraint

Implied by the model, does not change the set of solutions, ex:

- $x \neq y, y \neq z, x \neq z \implies \text{AllDifferent}(x, y, z)$
- $x \neq y, x \leq y \implies x < y$

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- $x < y$ is inconsistent

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- $x \neq y, y \neq z, x \neq z \implies \text{AllDifferent}(x, y, z)$
- $x \neq y, x \leq y \implies x < y$

Let $x \in \{1, \dots, 10\}, y \in \{1, \dots, 10\}$

- $x \neq y$ is consistent ($x = 10$ has $\langle 10, 9 \rangle$ as support)
- $x \leq y$ is consistent ($x = 10$ has $\langle 10, 10 \rangle$ as support)
- $x < y$ is inconsistent
  - consistent with $x \in \{1, \dots, 9\}, y \in \{2, \dots, 10\}$

Implied Constraints: Golomb Ruler

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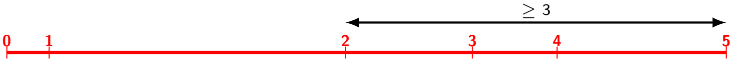

- `distance[i,j]` $\geq$ sum of $j - i$ distances

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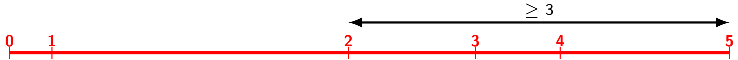

- `distance[i,j]` $\geq$ sum of $j - i$ distances
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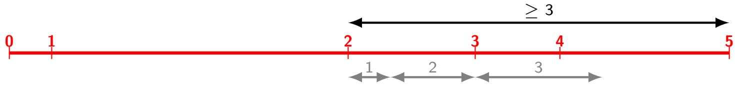

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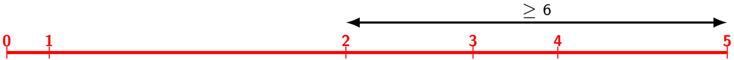

- `distance[i,j]` $\geq$ sum of $j - i$ distances
- The distances are all different `distance[i,j]` $\geq (j - i) * (j - i + 1)/2$

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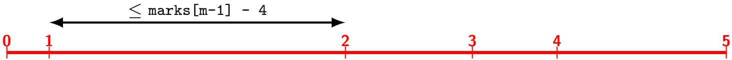

- `distance[i,j]` $\geq$ sum of $j - i$ distances
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- Same reasoning from the end (`marks[`$m - 1$`]`)
  - `distance[i,j]` $\leq$ `marks[m]` $-$ sum of $m - 1 - j + i$ distances

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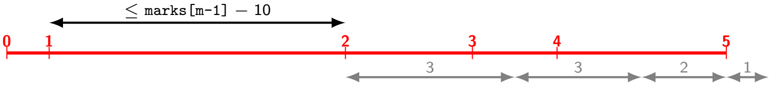

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  - distance[i,j] $\leq$ marks[m]$-$ sum of $m - 1 - j + i$ distances
  - distance[i,j] $\leq$ marks[m] $- (m - 1 - j + i) * (m - j + i)/2$

Implied Constraints: Golomb Ruler

- Implied constraints
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- How do we know that these constraints are useful (improving constraint propagation)

Implied Constraints: Golomb Ruler

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  - `distance[i,j]` $\geq (j-i)*(j-i+1)/2$
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- We need to combine the reasoning of two constraints (`AllDifferent(distance)` and `distance[i,j]` $= \sum_{k=i}^{j-1}$ `distance[k,k+1]`)

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- Domain reduction is not sufficient to "communicate" between the two constraints

  - The implied constraints reduce the domains at the root node

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- Implied constraints
  - distance[i,j] $\geq (j - i) * (j - i + 1)/2$
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- Domain reduction is not sufficient to "communicate" between the two constraints
  - The implied constraints reduce the domains at the root node

- In doubt, just try!

Implied Constraints (Numberjack)

```
import sys
from Numberjack import *

m = int(sys.argv[1]) if len(sys.argv)>1 else 6
n = 2 ** (m - 1)

marks = VarArray(m, n, 'm')
dmap  = dict([((i,j), marks[j] - marks[i]) for i in range(m-1) for j in range(i+1,m)])
distance = [dmap[(i,j)] for i in range(m-1) for j in range(i+1,m)]

lbs = [(j - i) * (j - i + 1) / 2 for i in range(m-1) for j in range(i+1,m)]
ubs = [marks[-1] - (m - 1 - j + i) * (m - j + i) / 2 for i in range(m-1) for j in range(i+1,m)]

model = Model(
    Minimise(marks[-1]),  # objective function

    [marks[i-1] < marks[i] for i in range(1, m)],
    marks[0] == 0,
    distance[0] < distance[-1],
    AllDiff(distance),

    [d >= l for d,l in zip(distance, lbs)],
    [d <= u for d,u in zip(distance, ubs)],
    [dmap[(i,j)] == dmap[(i,j-1)] + dmap[(j-1,j)] for i in range(m-2) for j in range(i+2,m)]
)

solver = model.load('Mistral2',marks)
if solver.solve():
    print marks, [d.get_value() for d in distance]
```

Implied Constraints (Choco)

```
Model model = new Model();

IntVar[] marks = model.intVarArray("m", m, 0, n);
IntVar[] distance = model.intVarArray("d",m * (m - 1) / 2, 1, n);
% IntVar[][] dmap = new IntVar[m][m];

int k = 0;
for(int i=0; i<m-1; ++i) {
        model.arithm(marks[i], "<", marks[i+1]).post();
        for(int j=i+1; j<m; ++j) {
                dmap[i][j] = distance[k];
                model.arithm(distance[k], "<=", marks[m - 1], "-", ((m - 1 - j + i) * (m - j + i)) / 2).post();
                model.arithm(distance[k], ">=", (j - i) * (j - i + 1) / 2).post();
                model.scalar(new IntVar[]{marks[i], marks[j]}, new int[]{-1,1}, "=", distance[k++]).post();
        }
        model.arithm(marks[0], "=", 0).post();
        model.arithm(distance[0], "<", distance[distance.length-1]).post();
}

% for(int i=0; i<m-2; ++i)
%         for(int j=i+2; j<m; ++j)
%                 model.arithm(dmap[i][j], "=", dmap[i][j-1], "+", dmap[j-1][j]).post();

model.allDifferent(distance).post();

model.setObjective(Model.MINIMIZE, marks[m-1]);
```

Conclusions

Good modeling practices

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Conclusions

Good modeling practices

- What are the variables, what are the values?
  - Constraints will follow
  - Defines the shape of the search tree
- Key principle: **strengthen constraint propagation**
  - Global constraints
  - Implied constraints
  - Symmetry breaking

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