Solving the Job Shop Scheduling Problem: QUBO model and Quantum Annealing

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Keywords : *quantum computing, optimisation, QUBO, job shop scheduling.*

1 Introduction

Quantum optimization which is the use of quantum computers and algorithms for solving complex optimization problems, is one of the topics in quantum computing with the highest potential. Two types of approaches are generally used to solve combinatorial optimization problems with quantum computers: exact methods such as the Grover's search algorithm [6], and meta-heuristics such as the Quantum Annealing [7] and the Quantum Approximate Optimization Algorithm (QAOA) [4]. Exact methods and variational methods like QAOA make use of universal gate-based quantum computers, such as the ones developed by the IBM company. Quantum annealing is designed for analogic quantum computers developed by D-Wave. Solving a combinatorial optimization problem with heuristics generally requires a transformation of the problem to a format suitable for the quantum computer. While some existing frameworks[1] can cope with Constrained Quadratic Models (CQM), Quadratic Unconstrained Binary Optimization (QUBO) is the best option to map an optimization problem to a quantum computer or simulator. We consider in this paper the job shop scheduling problem. We first review in the following section 2 the quantum-based methods recently proposed in the literature. Then in section 3 we propose a QUBO formulation of the scheduling problem. Finally, in section 4 numerical results obtained with D-Wave quantum annealing machines and conclusions are presented.

2 Problem definition and related works

The job shop scheduling problem can be stated as follows: A set of n jobs $J = \{J_1, J_2, \ldots, J_n\}$ has to be processed on a set of m machines $M = \{M_1, M_2, \ldots, M_m\}$. Each job J_i consists in a linear sequence of n_i operations $(O_{i1}, O_{i2}, \ldots, O_{in_i})$. Each machine can process only one operation at a time and each operation O_{ij} with a processing time of p_{ij} time units needs only one machine. Each job visits the machines according to its own predefined routing. This problem generalizes the flow shop scheduling problem, in which all the jobs are processed following the same routing $(M_1, M_2, \ldots, M_m)$. The objective is to determine the starting date of each operation O_{ij} so that the makespan noted C_{max} is minimized. The problem is NP-hard for $n > 2$ and $m > 2$ [5].

The scientific literature on quantum solutions for hard combinatorial optimization problems is becoming significant and diversified. The studies of shop scheduling problems are quite recent and scarce. The first paper in the domain is from Venturelli *et al.* [12]. The authors

[1] https ://cloud.dwavesys.com/leap

proposed a QUBO formulation and a quantum annealing solution for the job shop scheduling problem with the makespan objective. The method was implemented on a D-Wave quantum annealer. Their model has been re-used in several studies. For instance, in [8] the authors have proposed a hybrid quantum annealing heuristic to solve a particular instance of the job shop scheduling problem on the D-Wave 2000Q quantum annealing system. In [2], job shop instances with unitary operations have been tested on the D-Wave Advantage machine. Extensive experiments with the reverse annealing procedure and comparisons with simulated annealing are also described. A generalization of the job shop scheduling problem with pools of parallel machines available for processing operations was considered in [3]. The authors proposed a QUBO derived from the one of [12] and an iterative procedure to solve relatively large size instances on a specialized hardware[2]. Using the QUBO formulations proposed in [3], the authors in [11], tackle the flexible job shop scheduling problem with the D-Wave solvers. Another QUBO formulation is proposed in [9] for assigning dispatching rules to the machines and scheduling the operations in a flexible job shop system. The problem is solved using the leap hybrid solver. In [10], the authors propose a QUBO formulation for the job shop scheduling with worker assignment considerations. Possible ways to approximate the makespan are discussed and instances solved with the Fujitsu Digital Annealer are described. On the same environment, the autors in [13] solve efficiently large instances of the job shop scheduling problem with a hybrid approach that combines constraints programming and QUBO models for one-machine problems. Finally, the only study involving gate-based computers is due to [1]. The authors have proposed four variational quantum heuristics for solving a job shop scheduling problem with early and late delivery as well as production costs, adapted from a steel manufacturing process. They have compared the performance of the heuristics on two-machine instances using IBM quantum processors.

3 QUBO formulation

The boolean variable x_{ij}^t takes the value 1 if the operation j of the job i starts in period t, with $i = 1..n$, $j = 1..n_i$, $t = 1..T$, and takes the value 0 otherwise. We note $M_{ij}, i = 1..n, j = 1..n_i$, the required machine by the operation j of the job i. The minimization of the Objective function (1) forces the last operations of all jobs to start globally as soon as possible.

$$\sum_i \sum_t t.x_{in_i}^t \tag{1}$$

We force each operation to start exactly once with the following set of constraints (2) to be relaxed in the Objective function.

$$(\sum_t x_{ij}^t - 1)^2, \qquad i = 1..n, j = 1..n_i \tag{2}$$

Constraints (3) forbid to have more than one operation at a time on a given machine.

$$\begin{gathered} x_{ij}^t x_{i'j'}^{t'} = 0, \\ \forall (i,j,t) \cup (i',j',t') : i,i' = 1..n, j,j' = 1..n_i, (i,j) \neq (i',j'), \\ M_{ij} = M_{i'j'}, (t,t') \in T^2, 0 \leq t' - t < p_{ij} \end{gathered} \tag{3}$$

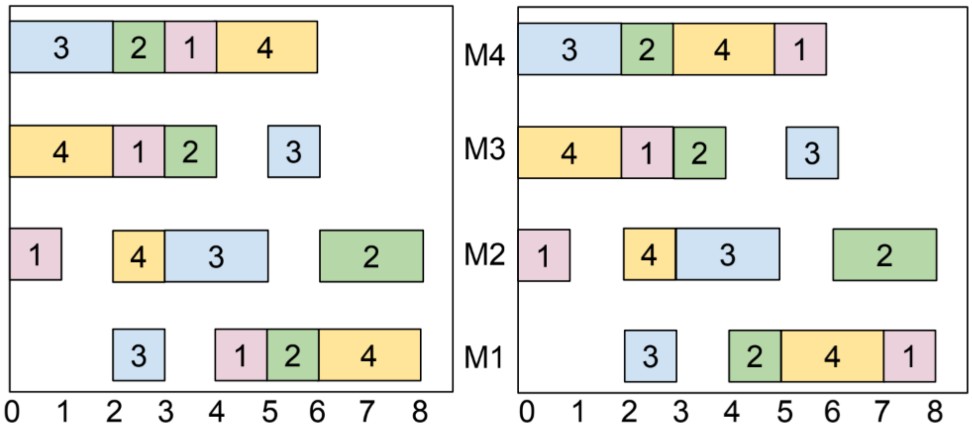
Constraints (4) forbid consecutive operations to start before the previous one is finished.

$$x_{ij}^t x_{ij+1}^{t'} = 0, \quad i = 1..n, j = 1..(n_i - 1), (t,t') \in T^2, t + p_{ij} > t' \tag{4}$$

The Quadratic Unconstrained Binary Optimisation (QUBO) model is given with the Objective function (5) to minimize with 4 multipliers, λ_1, λ_2, λ_3 and λ_4, balancing the relaxation of the 4 sets of constraints.

[2]https://www.fujitsu.com/global/services/business-services/digital-annealer

FIG. 1: Representation of the literature optimal solution (left, [2]) and the optimal solution we obtained with the D-Wave Hybrid Solution (right). The instance has 4 machines (M1, M2, M3, M4), 4 jobs, each represented by a different color, and 4 operations per job.



$$
\begin{aligned}
&\sum_i \sum_t t.x^t_{in_i} \\
&+\lambda_1 \sum_i \sum_j^{n_i} (\sum_t x^t_{ij} - 1)^2 \\
&+\lambda_2 \sum_{(i,j,t)\cup(i',j',t')\in T1} x^t_{ij} x^{t'}_{i'j'} \\
&+\lambda_3 \sum_{(i,j,t,t')\in T2} x^t_{ij} x^{t'}_{ij+1} \\
&\text{with:} \\
&T1 = (i,j,t) \cup (i',j',t') : i,i' = 1..n, j,j' = 1..n_i, (i,j) \neq (i',j'), \\
&M_{ij} = M_{i'j'}, (t,t') \in T^2, 0 \leq t' - t < p_{ij} \\
&T2 = (i,j,t,t') : i = 1..n, j = 1..(n_i - 1), (t,t') \in T^2, t + p_{ij} > t'
\end{aligned}
\tag{5}
$$

4 Results and discussion

As an early result, we present in Figure 1 the solution we obtained with the D-Wave Hybrid Solution on an instance from the literature ([2]). In this instance, 4 jobs with 4 operations have to be scheduled on 4 different machines. The next step will be the direct implementation of the QUBOs for the job shop instances and their solution with the full quantum machines.

In this paper, we first reviewed the recent solutions proposed to solve the job shop scheduling problem on quantum computers, with a focus on QUBO formulations that help integrating constraints that are relevant in practice. We have also proposed a QUBO model that has been solved using D-Wave quantum annealing machines. While the existing hardware are still limited in their ability to handle the number of variables generated, it remains important to progress on the modeling of practical problems. Future works include investigating mechanisms for keeping the number of variables low while integrating efficiently the constraints.

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