A short and comprehensive interpretation of the Grover's algorithm in terms of basis changes

Gérard Fleury and Philippe Lacomme

Université Clermont Auvergne,

LIMOS UMR CNRS 6158, 63178 Aubière Cedex, France

gerard.fleury@isima.fr, placomme@isima.fr

1   Introduction

This article provides a description of the amplitude amplification quantum algorithm mechanism [1] in a very short computational way based on tensor products and provides a geometric presentation of the successive system states. To make the drawings simpler and therefore more effective, the paper alternates between general considerations to gives an idea of the scientific justification and a practical point of view including at the end numerical experiments. This presentation is at the crossroad of OR community and quantum one: the contribution is a former readable description dedicated to the OR community. Let us consider an unsorted finite set B spanning a Hilbert space $E = Span(B)$ and a function

$$f: \quad B \quad \rightarrow \quad \{0; 1\}$$
$$x \quad \rightarrow \quad f(x)$$

referred to as an Oracle that characterizes the marked subsets of B with $E_1 = span(x \in B/f(x) = i)$ and for example we have $E = E_0 \oplus E_1$. The problem consist in finding at least one $x \in E_1$ avoiding the costly enumeration of all element of B one by one if no extra information is available on B. Because search procedures are the corner stone in computer science of advanced data structures, the Grover's algorithm that provides a quadratic speed-up received a considerable of attention. An amplification procedure consists in considering first an initial $|\psi\rangle \in E$ to return a state $|\psi\rangle \in E_1$ with a probability close to 1. Note that multiple marked elements by the Oracle do not change the Grover's principles but the number of amplifications has to be tuned (Grover, 1996) (Zalka, 1999). The Grover amplification efficiency lies on an expected number of calls to the Oracle. The Hadamard gate is an operation that maps the basis state $B(|0\rangle; |1\rangle)$ into $B(|p\rangle; |m\rangle)$ by a $\pi/2$ rotation on $Y - axe$ and a $2.\pi$ rotation on $X - axe$ and creates an equal superposition of the two basis states with

$H.|0\rangle = \frac{1}{\sqrt{2}}.(|0\rangle + |1\rangle) = |p\rangle$, $H.|1\rangle = \frac{1}{\sqrt{2}}.(|0\rangle - |1\rangle) = |m\rangle$, $H.|p\rangle = |0\rangle$ and $H.|m\rangle = |1\rangle$. Note

that for convenience, we use the notation $|p\rangle$ for $H.|0\rangle$ and $|m\rangle$ for $H.|1\rangle$ but the common notation is $|+\rangle$ and $|-\rangle$. The X-gate is a symmetry around the $\pi/4$ axis leading to the following basic transformations : $X.|0\rangle = |1\rangle$, $X.|1\rangle = |0\rangle$, $H.|p\rangle = |p\rangle$ and $H.|m\rangle = -|m\rangle$. The minus sign is used to define the opposite direction of one state $|\psi\rangle$ as stressed on figure 1 where $-|1\rangle$ is the opposite direction to $|1\rangle$ meaning that the phase of $-|1\rangle$ is π.

2 Grover description for $r = |10100\rangle$

Figure 1 gives the Grover circuit for $n = 5$ and $r = |10100\rangle$. We provide a geometric description where we alternate in computation with n and the specific value 5 depending on formulae and comments on the figures.
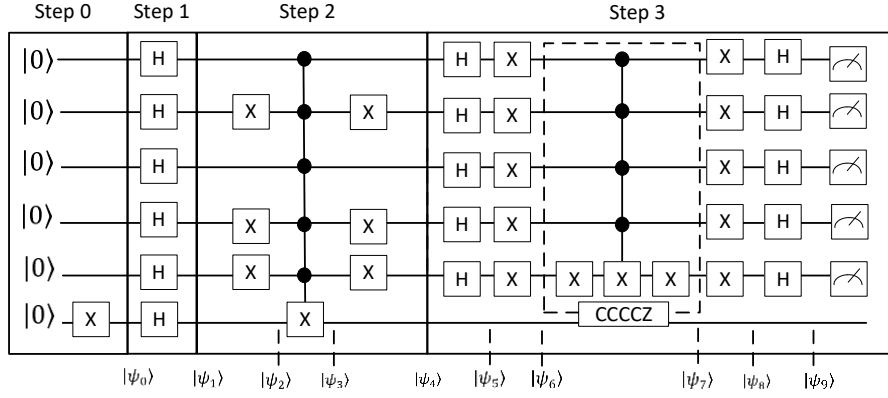

Figure 1. Circuit de Grover pour $n = 5$ et $r = |10100\rangle$

Step 0 and 1. Initialization

Step 0. $|\psi_0\rangle = |0\rangle^{\otimes 5} \otimes |1\rangle = |0\rangle^{\otimes 5} \otimes |1\rangle$

In the computational base with a amplitude of $\frac{1}{\sqrt{2^5}}$ .In the basis $B(|p\rangle^{\otimes 5}; |m\rangle)$, we have $|p\rangle^{\otimes 5}$ that is an eigenvector of $X^{\otimes n}$ assigned to the eigenvalue 1 and $|m\rangle$ is an eigenvector of X assigned to -1 which is the second eigenvalue of X.

Step 1. Application of $H^{\otimes 6}$.

Assigns a similar amplitude to all states with half of states with positive amplitude and half with negative one in the computational base $B(|0\rangle; |1\rangle)$: $|\psi_1\rangle = |p\rangle^{\otimes 5} \otimes |m\rangle$

Step 2. Oracle definition

The **first step** of the Oracle consist in application of $(Id \otimes X \otimes Id \otimes X \otimes X \otimes Id)$. Considering only the 5-first qubits, this quantum state means that the base $B(|r\rangle; |p\rangle^{\otimes 5})$ has been switched into $B(|1\rangle^{\otimes 5}; |p\rangle^{\otimes 5})$. The **second step** consists in application of $CCCCCX$ that does not affect $B(|1\rangle^{\otimes 5}; |p\rangle^{\otimes 5})$ and only switches $|m\rangle$ in $-|m\rangle$.

So : $CCCCCX. \left(|p\rangle^{\otimes 5} \otimes |m\rangle\right) = \left(|p\rangle^{\otimes 5} - \frac{2}{\sqrt{2^n}}|1\rangle^{\otimes 5}\right) \otimes |m\rangle$

The last step of the Oracle consist in application of $(Id \otimes X \otimes Id \otimes X \otimes X)$ that switches:
$$(Id \otimes X \otimes Id \otimes X \otimes X). \left(-|1\rangle^{\otimes 5}\right) = -|10100\rangle$$
And $(Id \otimes X \otimes Id \otimes X \otimes X). |p\rangle^{\otimes 5} = |p\rangle^{\otimes 5}$ meaning that the base $B(|1\rangle^{\otimes 5}; |p\rangle^{\otimes 5})$ turns back into $B(-|10100\rangle; |p\rangle^{\otimes 5})$
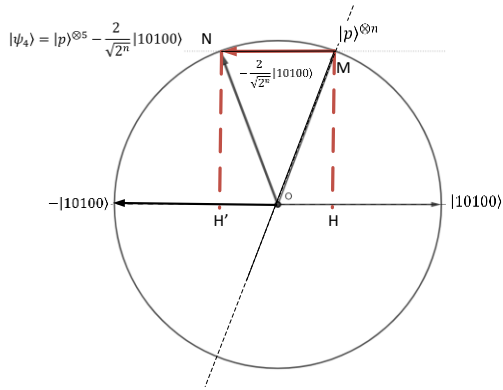
Step 3. Amplification

Step 3.1. Performing $H^{\otimes 5}$ permits to switch from $B(-|10100\rangle; |p\rangle^{\otimes 5})$ to basis $B(-|mpmpp\rangle ; |0\rangle^{\otimes 5})$.

Step 3.2. Performing $X^{\otimes 5}$ permits to switch from $B(-|mpmpp\rangle ; |0\rangle^{\otimes 5})$ to $B(-|mpmpp\rangle; |1\rangle^{\otimes 5})$.

Figure 3. $|\psi_4\rangle = |p\rangle^{\otimes 5} - \frac{2}{\sqrt{2^5}}.|10100\rangle$ at
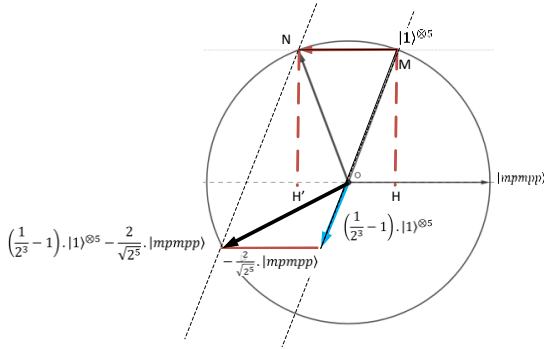
the end of Oracle

Step 3.3. Performing $CC\dots CZ$

The geometric construction of $|\psi_6\rangle = |1\rangle^{\otimes 5} - \frac{2}{\sqrt{2^5}}.|mpmpp\rangle$ makes it clear that the $CCC\dots CZ$ application should define a new state where $|\psi_7\rangle$ should be closest to $|1\rangle^{\otimes 5}$. Performing $CCCCZ$ gives:

$$|\psi_7\rangle = \left(\frac{1}{2^3} - 1\right).|1\rangle^{\otimes 5} - \frac{2}{\sqrt{2^5}}.|mpmpp\rangle$$

The quantum state is highlighted on figure 4.

Figure 4. $\left(\frac{1}{2^3} - 1\right).|1\rangle^{\otimes 5} -$

$\frac{2}{\sqrt{2^5}}.|mpmpp\rangle$

Step 3.4. Performing $X^{\otimes 5}$

The $X^{\otimes 5}$ gate application at this point of the algorithm gives a $|\psi_8\rangle$ spanned by $|0\rangle^{\otimes 5}$ and $|pmpmm\rangle$: $|\psi_8\rangle = \left(\frac{1}{2^3} - 1\right).|0\rangle^{\otimes 5} - \frac{2}{\sqrt{2^5}}.|mpmpp\rangle$
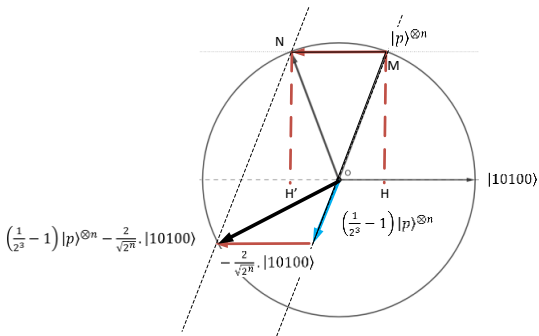
Figure 5. $|\psi_9\rangle = \left(\frac{1}{2^3} - 1\right)|p\rangle^{\otimes 5} -$

$\frac{2}{\sqrt{2^5}}.|10100\rangle$

Step 3.5. Performing $H^{\otimes 5}$

The $H^{\otimes 5}$ gate application at this point of the algorithm make a turn back to the initial bases $B(|10100\rangle; |p\rangle^{\otimes 5})$: $|\psi_9\rangle = \left(\frac{1}{2^3} - 1\right)|p\rangle^{\otimes 5} - \frac{2}{\sqrt{2^n}}.|10100\rangle$ The state $|\psi\rangle$ is defined by

$\begin{pmatrix} \frac{1}{2^4} - 1 \\ -\frac{2}{\sqrt{2^5}} \end{pmatrix}$ in $B(|p\rangle^{\otimes 5}; |10100\rangle)$ as stressed on the figure 8, in the plane $P_{|p\rangle^{\otimes n}; |10100\rangle}$.

3   Conclusion

In this paper we investigate a description of the Grover's algorithm using geometric considerations only and a tensorial computations. We introduce the Grover algorithm in a new way considering geometric considerations only to illustrate how the successive quantum states are computed and how the planes are spanned by the different basis vectors.

References

[1] Grover L.K. "Synthesis of Quantum Superpositions by Quantum Computation". Physical Review Letters. Vol. 85(6) pp. 1334-1337. 2000.

Appendix: successive states for $|001\rangle$

	Quantum State
Step 0.	$|\psi_0\rangle = |000\rangle$
Step 1.Application of $H^{\otimes 3}$	$|\psi_1\rangle = |ppp\rangle$
Oracle	
Step 2.1. Application of $X^{\otimes 2} \otimes Id$	$|\psi_2\rangle = |ppp\rangle$
Step 2.2. Application of $CZ(q_1, q_2; q_3)$	$|\psi_3\rangle = |ppp\rangle - \frac{1}{\sqrt{2}}.|111\rangle$
Step 2.3. Application of $X^{\otimes 2} \otimes Id$	$|\psi_4\rangle = |ppp\rangle - \frac{1}{\sqrt{2}}.|001\rangle$
Amplification	
Step 3.1. Application of $H^{\otimes 3}$	$|\psi_5\rangle = |000\rangle - \frac{1}{\sqrt{2}}.|ppm\rangle$
Step 3.2. Application of $X^{\otimes 3}$	$|\psi_6\rangle = |111\rangle + \frac{1}{\sqrt{2}}.|ppm\rangle$
Step 3.3. Application of $CZ(q_1, q_2; q_3)$	$|\psi_7\rangle = \frac{1}{\sqrt{2}}.|ppm\rangle - \frac{1}{2}.|111\rangle$
Step 3.4. Application of $X^{\otimes 3}$	$|\psi_8\rangle = -\frac{1}{\sqrt{2}}.|ppm\rangle - \frac{1}{2}.|000\rangle$
Step 3.5. Application of $H^{\otimes 3}$	$|\psi_9\rangle = -\frac{1}{\sqrt{2}}.|001\rangle - \frac{1}{2}.|ppp\rangle$