

Control of a production-inventory system with returns under imperfect advance return information

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Abstract

We consider a production-inventory system with product returns that are announced in advance by the customers. Demands and announcements of returns occur according to independent Poisson processes. An announced return is either actually returned or cancelled after a random return leadtime. We consider both lost sale and backorder situations. Using a Markov decision formulation, the optimal production policy, with respect to the discounted cost over an infinite horizon, is characterized for situations with and without advance return information. We give insights in the potential value of this information. Also some attention is paid to combining advance return and advance demand information. Further applications of the model as well as topics for further research are indicated.

Keywords : Reverse logistics; Inventory control; Stochastic dynamic programming; Advance return information

1. Introduction

During the last 15 years a lot of attention has been paid to so called closed-loop supply chains, reverse logistics, product recovery, both in practice, as in academic literature, see e.g. Dekker et al. (2004) and Rubio et al. (2008). In this context, also attention has been paid to forecasting the reverse flows. Available publications use delivery/purchase information to forecast returns, see e.g. (Yuan and Cheung, 1998), sometimes taking into account information on actual returns, see e.g. de Brito and van der Laan (2009).

In this paper we neglect the use of the above information, but focus on return information supplied by the owner/user of a product after the initial delivery, purchase of this product. We study situations where customers have to announce the return of a product. Advance Return Information/Advance Supply Information (ARI/ASI) is among others required in practice for warranty returns, commercial returns, buy back contract returns, returns due to wrong delivery. An important reason for the above is to prevent unnecessary or incorrect returns. See e.g. Boykin (2001) for a general description of the Return Material Authorization process and the support offered for this process by SAP. Other examples of using ARI concern information related to the end of lease contracts, when the lessee has to indicate some time before whether or not (s)he will continue the contract or buy the leased product.

A number of authors paid already attention to the value of advance information in the context of product recovery, including the recent contribution by Khawam and Hausman (2009) with an up-to-date review of the literature in this field. Our paper differs from the above paper in a number of aspects including the origin of supply uncertainty, a finite production capacity, a continuous review of the inventory position, random leadtimes and lost sales.

We adopt a make-to-stock queue framework to model production capacity and uncertainty with respect to production, returns and demand. A make-to-stock queue refers to a make-to-stock system where the supply process is modeled by servers producing units one by one. Make-to-stock queues have been used to investigate issues such as stock allocation (de Véricourt et al, 2002), production scheduling (Zhao et al, 2008), dynamic pricing (Gayon et al., 2009b) and multi-echelon coordination (Veatch and Wein, 1994). A few make-to-stock papers include product

returns (see e.g. Heyman, 1977, Gayon and Dallery, 2007). However, none of them investigates the use of ARI. Our modeling of imperfect ARI is close to the modeling of imperfect Advance Demand Information (ADI) introduced by Gayon et al (2009a). In the latter paper, the customer announces his intention to buy a product but the actual ordering takes place after a stochastic demand leadtime, with a cancellation probability. In this paper, we assume that the customer announces his intention to return a product where the actual return occurs after a stochastic return leadtime, with a return cancellation probability. ADI and ARI have opposite impacts on production control. For ADI, production is planned when there are many pending orders. For ARI, production is not planned when there are many pending returns. Because of the increasing use of ADI, we also pay some attention to the combined use of ARI and ADI.

The rest of the paper is setup as follows. First, we describe the situation that we study as well as the objective function to be optimized. Next, we derive the optimal production policy for lost sales situations for an infinite horizon. Via numerical experiments we determine the sets of parameter values for which ARI may be useful. Next we show that the model developed for the lost sales situations can be amended to deal with backlog situations. We also derive the optimal production policy when both ARI and ADI are used. Then we explain how our model can be used for other applications than product returns. Finally we briefly summarize our main findings and indicate some interesting extensions of the model presented here.

2. Problem description

In this paper we focus on situations where individual products are produced and returned. Products that are returned are as good as new, and are stored in the stock of serviceable products together with the products that the company produces new.

We consider an M/M/1 make-to-stock queue for a single item (see Figure 1). The company can decide at any time to produce this item. The production time is exponentially distributed with mean $1/\mu$. After having been produced, products are stored in the serviceable products inventory. Demand for the serviceable products follows a Poisson process with rate λ . For the moment being, we assume lost sales: Demand that cannot be fulfilled immediately is lost. We will also consider backorder situations (see Section 4).

Besides the single server production mode, the company has an alternative procurement mode where the company receives products from another source that is not under her direct control. These products can not be distinguished from the products produced by the single server. We assume that the company has some advance information on the alternative procurement process.

The alternative source considered hereafter is customers that can return products, although, as we shall indicate in Section 5, the following also holds for other alternative sources. Before returning a product, the customer must announce that he will return the product. The announcements occur according to a Poisson process with rate δ , independently of the demand process. However, not every announced return becomes an actual return. Reasons for this in practice include forgetting to return, not at home at the moment of planned pickup, mind change. We assume that there is a probability p that an announced return is actually returned. There is a probability $q = 1 - p$ that an announced return is cancelled. All actual returns have to be accepted and a return can not be disposed. Therefore, to guarantee the stability of the on-hand stock of serviceable products, we assume that $p\delta < \lambda$.

We further assume that the time L that elapses between the announcement of a return and its actual receipt (or cancellation) is exponentially distributed with rate γ . This time does not depend on the number of announced returns. Note that a number of the earlier mentioned examples from practice concern situations with a predefined maximum return time. However, in practice, companies deviate from this time for all kinds of reasons, for instance to keep important customers. We make here the same approximation as many other authors, including Yuan and Cheung (1998).

Once received, a return is stored in the serviceable stock and can be sold. The state of the system can be described by $(X(t), Y(t))$ where $X(t)$ denotes the on-hand stock of new and returned products at time t , and $Y(t)$ denotes the number of returns that have been announced but still have not been received or cancelled at time t .

Insert Figure 1

We consider unit production cost, c_p , unit lost sale cost c_l , unit return cost c_r that only has to be paid for actual returns, and unit inventory holding cost per unit of time, c_h . We assume that $c_p < c_l$ in order to have an incentive to produce. The objective of the decision maker is to find a production control policy π minimizing the expected discounted cost over an infinite time horizon. The discount rate is denoted by α . The production control policy specifies, for each state of the system, when to produce. We define $v^\pi(x, y)$ as the expected total discounted cost associated with policy π , for initial state $(X(0), Y(0)) = (x, y)$.

We seek to find the optimal policy π^* minimizing $v^\pi(x, y)$, where we let $v^*(x, y) = v^{\pi^*}(x, y)$ denote the optimal value function. We restrict our analysis to stationary Markovian policies since there exists an optimal stationary Markovian policy (Puterman, 1994). In the following, we characterize the optimal policy for the case where ARI is used and for the case where ARI is ignored.

3. Lost sales situations

3.1. Optimal policy when ARI is used

When ARI is taken into account, decisions are based on both the on-hand stock of serviceable products, $X(t)$, and the number of announced returns, $Y(t)$. The situation can be modeled as a continuous-time Markov Decision Process (MDP). In order to uniformize this MDP (Lippman, 1975), we assume that the number of announced returns is bounded by M . This is not a crucial assumption since our results hold for any M . We choose a uniformization rate $C = \lambda + \mu + \delta + M\gamma$. The optimal value function can be shown (Puterman, 1994) to satisfy the optimality equations

$$v^*(x, y) = Tv^*(x, y), \forall (x, y)$$

where the operator T is a contraction mapping defined as

$$Tv(x, y) = \frac{1}{C + \alpha} [c_h x + \mu T_0 v(x, y) + \lambda T_1 v(x, y) + \delta T_2 v(x, y) + \gamma p T_3 v(x, y) + \gamma (1 - p) T_4 v(x, y)] \quad (1)$$

with

$$\begin{aligned}
T_0 v(x, y) &= \min \left[v(x, y), v(x+1, y) + c_p \right] \\
T_1 v(x, y) &= \begin{cases} v(x-1, y) & \text{if } x > 0 \\ v(x, y) + c_l & \text{if } x = 0 \end{cases} \\
T_2 v(x, y) &= \begin{cases} v(x, y+1) & \text{if } y < M \\ v(x, y) & \text{if } y = M \end{cases} \\
T_3 v(x, y) &= \begin{cases} y[v(x+1, y-1) + c_r] + (M-y)v(x, y) & \text{if } y > 0 \\ Mv(x, y) & \text{if } y = 0 \end{cases} \\
T_4 v(x, y) &= \begin{cases} yv(x, y-1) + (M-y)v(x, y) & \text{if } y > 0 \\ Mv(x, y) & \text{if } y = 0 \end{cases}
\end{aligned}$$

Operator T_0 is related to the production decision. Operator T_1 is associated with the demand. Operator T_2 corresponds to the announcements of the returns. Finally, operator T_3 (resp. T_4) is related to an announcement that will (resp. will not) actually lead to a return. Considering operator T_0 , we notice that the optimal production control policy is entirely determined by the sign of $(\Delta v^*(x, y) + c_p)$ where $\Delta v(x, y) = v(x+1, y) - v(x, y)$. In order to characterize the optimal policy, we prove that v^* belongs to the following set U of real-valued functions.

Definition 1 *If $v \in U$, then for all (x, y) :*

- C.1 $\Delta v(x, y) \leq \Delta v(x+1, y)$
- C.2 $\Delta v(x, y) \leq \Delta v(x, y+1), \forall y < M$
- C.3 $\Delta v(x, y+1) \leq \Delta v(x+1, y), \forall y < M$
- C.4 $\Delta v(x, y) \geq -c_l$

Lemma 1 *If $v \in U$, then $Tv \in U$. Moreover the optimal value function v^* belongs to U .*

The proof of Lemma 1, based on an induction argument, is given in Appendix A.1 at the end of this paper. As v^* satisfies C.1, we can define the threshold $S(y) = \min \left[x \mid \Delta v^*(x, y) + c_p > 0 \right]$ such that $\Delta v^*(x, y) + c_p \leq 0$ (i.e. it is optimal to produce) if and only if x is below this threshold. Conditions C.2 and C.3 imply a slope of $S(y)$ between 0 and 1 ($S(y) - 1 \leq S(y+1) \leq S(y)$), i.e. an additional announced return leads to at most a one unit decrease of the threshold.

Theorem 1 *There exists a switching curve $S(y)$ such that it is optimal to produce if and only if $x < S(y)$. Moreover, the switching curve has the following property:*

$$S(y) - 1 \leq S(y+1) \leq S(y)$$

Figure 2 illustrates the optimal policy when using ARI with respect to when to produce and when not to produce, for a given set of parameter values.

Insert Figure 2

3.2. Optimal policy when ARI is ignored

When ARI is ignored, the decision maker does not take into account the announced returns to make production decisions. Then the state of the system can be described by the single variable $X(t)$. The physical returns to the stock occur according to a Poisson process with rate $p\delta$. This is due to the property that the output process of an M/M/ ∞ queue is a Poisson process with rate equal to the arrival rate (Gross and Harris, 1998). Hence, when ARI is ignored, the system behaves like an M/M/1 make-to-stock queue with independent Poisson demand and returns (see Figure 3) where the parameter γ can be omitted.

Insert Figure 3

For this system, Zerhouni et al. (2009) show that the optimal policy is a base-stock policy with a single parameter S^* such that it is optimal to produce if and only if $x < S^*$. When a base-stock policy is applied, the dynamics of the system is rather simple. For a given base-stock level S , the on-hand stock $X(t)$ evolves as a continuous-time Markov chain (see Figure 4).

Insert Figure 4

It is straightforward to derive the steady-state probability $\pi_x(S)$ to be in state x when the base-stock level is S . Let $\rho_1 = \lambda / (\mu + p\delta)$ and $\rho_2 = p\delta / \lambda$. We have

$$\pi_0(S) = \rho_1^S \left(\frac{1 - \rho_1^{S+1}}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2} \right)^{-1}, \quad \pi_x(S) = \begin{cases} \rho_1^{-x} \pi_0(S) & \text{if } x \leq S \\ \rho_1^{-S} \rho_2^{x-S} \pi_0(S) & \text{if } x > S \end{cases}$$

Then the average cost $C(S)$ with respect to a base-stock level S can be expressed as

$$\begin{aligned} C(S) &= p\delta c_r + \lambda c_l \pi_0(S) + \mu c_p \sum_{x=0}^{S-1} \pi_x(S) + c_h \sum_{x=1}^{+\infty} x \pi_x(S) \\ &= p\delta c_r + \lambda c_l \pi_0(S) + \mu c_p \left(\frac{1 - \rho_1^{-S}}{1 - \rho_1^{-1}} \right) + c_h \pi_0(S) \rho_1^{-S} \left(\frac{\rho_1^{S+1} - \rho_1 - \rho_1 S + S}{(1 - \rho_1)^2} + \frac{\rho_2(1 + (1 - \rho_2)S)}{(1 - \rho_2)^2} \right) \end{aligned}$$

When $\rho_1 \leq 1$, Zerhouni et al. (2009) show that this average cost is convex. In this case, any convex optimization procedure can be used to find the optimal base-stock level. When $\rho_1 > 1$, Zerhouni et al. (2009) show that the average cost is bounded above by $S_u = (\lambda - \mu)c_l / c_h + [\ln(\lambda / \mu)]^{-1}$. In this case, an exhaustive search of the optimal base-stock level on the set $\{0, \dots, S_u\}$ can be executed.

3.3. Comparing the two policies

In a numerical study, we investigate the value of using ARI by comparing the results obtained for the two policies introduced in sections 3.1 and 3.2.

We focus on the average cost criterion. The average cost optimal policy can be shown to be the limit of the discounted cost optimal policy when the discount rate α goes to 0, by theorems 7.2.3 and 7.5.6 of Sennott (1999). Therefore the structures of the optimal policies are similar to the ones introduced in sections 3.1 and 3.2.

We denote by $g(ARI)$ and $g(no\ ARI)$ the optimal average costs when using ARI and not using ARI. In order to compare the two policies, we look at the percentage cost increase for not using ARI, defined as $\Delta g = (g(no\ ARI) - g(ARI)) / g(ARI)$

3.3.1. Computational procedure

To compute the optimal average costs, we use a value iteration algorithm (Puterman, 1994). The iteration is terminated when a six digit accuracy is achieved. To implement this algorithm, we need to truncate the state space. We repeat the value iteration algorithm with larger and larger state spaces until the cost is no longer sensitive to increasing the state space, with a six digit accuracy. With a standard PC, computing the optimal policy takes in general less than a minute. However, it also

may take several hours, e.g. for large return leadtimes or when the system approaches instability of inventory (i.e. when $p\delta$ is close to λ). The data to produce the figures can be found in the online appendix.

3.3.2. Experimental design

The model presented in Section 2 includes 9 parameters $(\lambda, \mu, \delta, \gamma, c_r, c_p, c_h, c_l, p)$. However, in this numerical study, we concentrate on varying 5 parameters $(\mu, \delta, \gamma, c_l, p)$. Without losing generality, we can set $\lambda = 1$ and $c_h = 1$ since this is equivalent to choosing the time and monetary unit to be used in the calculations.

In the following, we show that we can set $c_r = 0$ and $c_p = 0$ when we investigate the added value of ARI. Consider Problem A with parameter values $(\lambda, \mu, \delta, \gamma, c_r, c_p, c_h, c_l, p)$ and Problem B with parameter values $(\bar{\lambda}, \bar{\mu}, \bar{\delta}, \bar{\gamma}, \bar{c}_r, \bar{c}_h, \bar{c}_l, \bar{p})$. Assume that the parameter values are identical for both problems except for the production cost, lost sale cost and return cost: $\bar{c}_l = c_l - c_p, \bar{c}_p = 0, \bar{c}_r = 0$

Property 1

- i. When ARI is used (resp. ARI is not used), the optimal production policy for problems A is also optimal for problem B.
- ii. We have the following relation between the optimal average cost of problem A, $g(ARI)$, and the optimal average cost of problem B, $\bar{g}(ARI)$,

$$g^A(ARI) = g^B(ARI) + \delta p(c_r - c_p) + \lambda c_p$$

$$g^A(\text{no ARI}) = g^B(\text{no ARI}) + \delta p(c_r - c_p) + \lambda c_p$$

- iii. The percentage cost increase for problem A, Δg , is smaller than the one for problem B, $\Delta \bar{g}$

The proof is detailed in Appendix A.2. For the remaining parameters $(\mu, \delta, \gamma, c_l, p)$, we have considered the following values: $c_l \in \{1, 10, 100\}$, $\delta \in \{0, 0.25, 0.5, 0.75, 0.95\}$, $\gamma \in \{0.1, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2\}$, $\mu \in \{0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, 10\}$, $p \in \{0, 0.25, 0.5, 0.75, 1\}$.

We did numerical experiments for the 4800 possible combinations of the above values for $\lambda = 1$, $c_h = 1$, $c_r = 0$ and $c_p = 0$.

3.3.3. Discussion results

When does ARI make sense, i.e. when does ARI result in "significantly" smaller average cost when compared with not using ARI, and when not?

The maximum Δg over all scenarios is 4.5 %, which is observed for the following combination of parameter values: $\mu = 1, \delta = 0.75, \gamma = 0.1, c_l = 100, p = 1$. Observe that, thanks to Property 1, this result remains valid for any positive return cost c_r and any combination of c_p and c_l such that $(c_l - c_p) \in \{1, 10, 100\}$.

In 91% of the examined scenarios, $\Delta g < 1\%$ and in 97% of the scenarios, $\Delta g < 2\%$. Whether ARI is useful depends on the exact combination of the parameter values. We have observed that Δg is non-monotonic with respect to all parameters. Figure 5 shows the influence of $1/\gamma$ (i.e. the average time between the announcement of a return and the actual receipt of the return or decline of the return by the customer) on Δg for some values of p . We observe that the relative benefit for using ARI tends to be insignificant when $1/\gamma$ is either small or large. When $1/\gamma$ is small, the explanation is simple: Returns are announced right before they arrive and ARI is useless. When $1/\gamma$ is large, returns are announced far in advance. Due to the exponential distribution choice for the return leadtime, when the expected value of the return leadtime increases, so does the variance and this makes ARI less useful. For intermediate values of $1/\gamma$, using ARI is most beneficial. We obtain similar results for the other combinations of parameter values examined in this paper.

Insert Figure 5

The previous observations hold for exponential return leadtimes but do they also pertain to other return leadtime distributions? To be able to compute the optimal policy, we need to consider leadtime distributions that are combinations of exponential distributions. Hereafter, we consider a return leadtime L distributed according to an Erlang- k distribution. Let $L = L_1 + \dots + L_k$ where L_1, \dots, L_k are

independently and exponentially distributed with rate $k\gamma$. Then the expected value of L is $1/\gamma$ and its standard deviation is $1/\sqrt{k}\gamma$. Figure 6 plots the effect of $1/\gamma$ for some values of k . We observe that the benefit of ARI is increasing with k . This is logical since the variance of the return leadtime is decreasing with k and the ‘quality’ of ARI is getting better. We also observe that the benefit of ARI remains maximal for intermediate values of $1/\gamma$. This may not be the case in general and especially for large values of k for which the standard deviation is smaller. Computing the optimal policy for large values of k is intractable due to the curse of dimensionality. When k goes to infinity, the return leadtime is converging to the constant leadtime $L=1/\gamma$. In this case, the decision maker has, at time t , exact information on the timing of all returns in the time window $[t, t+L]$. Increasing the horizon of visibility L will necessarily decrease the average cost. Hence, the benefit of ARI should be nondecreasing with the return leadtime, when deterministic.

Insert figure 6

Figure 7 shows the influence of p on Δg for different values of δ . When $p\delta$ goes to λ or to 0, we observe that Δg goes to 0. When $p\delta$ goes to λ , returns are sufficient to satisfy the demand and it is no more necessary to produce. With or without ARI, the optimal policy consists in not producing all the time and ARI is again useless. When $p\delta = 0$, there are no actual returns and ARI is useless. The curves in Figure 7 show irregularities which are due to the discrete nature of the base-stock levels. To make this relation clear, we plotted Δg and the optimal base-stock level S^* for the situation without ARI (Figure 8). We observe that the irregularities in the curve Δg coincide with changes in the optimal base-stock level.

Insert figures 7 and 8

Now we look at the influence of γ and p on the optimal policy. Figure 9 (respectively 10) plots the state-dependent base-stock level $S(y)$ as a function of y for different values of γ (respectively p). We observe in Figure 9 that $S(y)$ is nonincreasing with the return rate γ . The larger the return lead time, the more we should produce. Moreover, when the return lead time is very short ($\gamma=100$), the

optimal policy consists in producing when $x + y < 4$ where 4 is precisely the optimal base-stock level when not using ARI. In this case, announced returns can be considered to be already in stock. In Figure 10, we observe that $S(y)$ is nonincreasing with the return probability p . It seems logical that we produce less when there is a higher probability that returns arrive. When $p = 0$, there is no return at all and the base-stock level $S(y)$ is independent of y .

Insert figures 9 and 10

4. Backorder situations

4.1. Optimal policy when ARI is used

We also consider situations with backorders (with linear backorder cost c_b per unit of time) instead of lost sales. The model formulation is slightly different but leads to similar results. In this case, on-hand stock $X(t)$ is replaced by net stock $\tilde{X}(t)$ with $\tilde{X}^+(t) = \max[0, \tilde{X}(t)]$, the on-hand stock of serviceable products, and $\tilde{X}^-(t) = \max[0, -\tilde{X}(t)]$, the number of backorders. For the discounted cost problem, the optimal value function \tilde{v}^* satisfies the optimality equation:

$$\tilde{v}^*(x, y) = \tilde{T}\tilde{v}^*(x, y), \forall (x, y)$$

where operator \tilde{T} is defined as

$$\tilde{T}v(x, y) = \frac{1}{C + \alpha} \left[c_h x^+ + c_b x^- + \mu T_0 v(x, y) + \lambda \tilde{T}_1 v(x, y) + \delta T_2 v(x, y) + \gamma p T_3 v(x, y) + \gamma(1 - p) T_4 v(x, y) \right]$$

Operators T_0, T_2, T_3, T_4, C are defined as in Section 3 and operator \tilde{T}_1 is defined as

$$\tilde{T}_1 v(x, y) = v(x - 1, y)$$

We define a set of functions \tilde{U} satisfying all conditions of U except Condition C.4. Then the proof is similar to the one of Lemma 2 and the optimal value function can be shown to satisfy conditions C.1, C.2 and C.3. We conclude that Theorem 1 can be extended to the case with backorders.

Theorem 2 *There exists a switching curve $\tilde{S}(y)$ such that it is optimal to produce if and only if $x < \tilde{S}(y)$. Moreover, the switching curve has the following property:*

$$\tilde{S}(y) - 1 \leq \tilde{S}(y+1) \leq \tilde{S}(y)$$

Note that the switching curve $\tilde{S}(y)$ can take negative values in the backorder case. This occurs when the number of announced returns is large. In this case, it can be optimal not to produce, even if there are orders waiting to be satisfied.

4.2. Optimal policy when ARI is ignored

When ARI is ignored, the optimal policy is again a base-stock policy and the optimal base-stock level can be computed explicitly (for details, see Gayon and Vercaene, 2011). If we let $W = S - X$, the average cost can be written as

$$\tilde{C}(S) = c_h E(X)^+ + c_b E(X)^- = hE(S - W)^+ + bE(S - W)^-$$

It is precisely the objective function of a newsboy problem with stochastic demand W . Let F_W denote the probability distribution function (p.d.f.) of W . The optimal base-stock level \tilde{S}^* is then $\tilde{S}^* = \min \{z : F_W(z) > c_h / (c_h + c_b)\}$

As the Markov chain $\tilde{X}(t)$ has a simple birth-death structure, the p.d.f. of W can be easily derived:

$$F_W(z) = \begin{cases} \frac{(1 - \rho_1)\rho_2^{-z}}{1 - \rho_1\rho_2} & \text{if } z \leq 0 \\ 1 - \frac{(1 - \rho_2)\rho_1^{z+1}}{1 - \rho_1\rho_2} & \text{if } z \geq 0 \end{cases}$$

We finally obtain an explicit formula for the optimal base-stock level. When $F_W(0) \geq c_h / (c_h + c_b)$, the optimal base-stock level is nonpositive and is given by

$$\tilde{S}^* = \left\lfloor \frac{\ln \left(\frac{1 - \rho_1}{1 - \rho_1\rho_2} \frac{c_b + c_h}{c_b} \right)}{\ln(\rho_2)} \right\rfloor$$

When $F_W(0) \leq c_h / (c_h + c_b)$, the optimal base-stock level is nonnegative and is given by

$$\tilde{S}^* = \left\lceil \frac{\ln \left(\frac{1 - \rho_1\rho_2}{1 - \rho_2} \frac{c_h}{c_h + c_b} \right)}{\ln(\rho_1)} \right\rceil$$

4.3. Comparing the two policies

Similarly to the lost sales case, we can compute the percentage cost increase for not using ARI in backorder situations. We keep the same system parameter values for $(\lambda, \mu, \delta, \gamma, c_r, c_p, c_h, p)$ and we vary the backorder cost c_b in $\{1, 10, 100\}$. Interestingly, the percentage cost increase Δg is higher in backorder situations and attain 23% for the following combination of parameter values: $\mu = 10, \delta = 0.5, \gamma = 1, c_b = 1, p = 1$. For this instance, the optimal policy without ARI works in a make-to-order fashion (produce if and only if the net stock is negative). Such a situation does not occur in lost sales situation. In 78% of the examined scenarios, $\Delta g < 1\%$ and in 96% of the scenarios, $\Delta g < 5\%$. The other insights discussed in Section 3.3.3 for lost sales situations pertain to backorder situations. Due to space limitation, we have not reported these results.

5. Combining ARI and ADI

So far, only returns where announced in advance. In this section, we extend our analysis to include Advance Demand Information (ADI). Following the framework of Gayon et al. (2009a), we assume now that customers place orders according to a Poisson process with rate λ . After a demand lead time exponentially distributed with rate ν , an order becomes due with probability a or is cancelled with probability $(1-a)$. The state of the system can now be described by a triplet $(X(t), Y(t), Z(t))$ where $Z(t)$ is the number of orders that have been announced but that are not due yet (or cancelled) at time t . We assume that the number of orders $Z(t)$ is bounded by N . In what follows, we focus on lost sales situations but the analysis can be extended to backorder situations.

We choose an uniformization rate $D = \lambda + \mu + \delta + M\gamma + N\nu$. The optimal value function \bar{v}^* satisfies the optimality equations $\bar{v}^* = \bar{T}\bar{v}^*$ where \bar{T} is defined as

$$\bar{T}\bar{v}(x, y, z) = \frac{1}{D + \alpha} \left[c_h x + \mu T_0 v(x, y, z) + \delta T_2 v(x, y, z) + \gamma p T_3 v(x, y, z) + \gamma(1-p) T_4 v(x, y, z) \right. \\ \left. + \lambda \bar{T}_1 v(x, y, z) + \nu a T_5 v(x, y, z) + \nu(1-a) T_6 v(x, y, z) \right]$$

Operators T_0, T_2, T_3, T_4 are defined as in Section 3 and operator \bar{T}_1, T_5, T_6 are defined as

$$\begin{aligned}\bar{T}_1 v(x, y, z) &= \begin{cases} v(x, y, z+1) & \text{if } z < N \\ v(x, y, z) + c_l & \text{if } z = N \end{cases} \\ T_5 v(x, y) &= \begin{cases} z[v(x-1, y, z-1)] + (N-z)v(x, y, z) & \text{if } z > 0, x > 0 \\ z[v(x, y, z-1) + c_l] + (N-z)v(x, y, z) & \text{if } z > 0, x = 0 \\ Nv(x, y, z) & \text{if } z = 0 \end{cases} \\ T_6 v(x, y) &= \begin{cases} zv(x, y, z-1) + (N-z)v(x, y, z) & \text{if } z > 0 \\ Nv(x, y, z) & \text{if } z = 0 \end{cases}\end{aligned}$$

Operator \bar{T}_1 is associated to the announcement of customer orders. Operator T_5 (resp. T_6) is related to actual demands (resp. cancellations). The optimal production control is entirely determined by the sign of $(\Delta \bar{v}^*(x, y, z) + c_p)$ where $\Delta v(x, y, z) = v(x+1, y, z) - v(x, y, z)$. In order to characterize the optimal policy, we prove that \bar{v}^* belongs to the following set V of real-valued functions.

Definition 2 *If $v \in V$, then for all (x, y, z) :*

- C.1 $\Delta v(x, y, z) \leq \Delta v(x+1, y, z)$
- C.2 $\Delta v(x, y, z) \leq \Delta v(x, y+1, z), \forall y < M$
- C.3 $\Delta v(x, y+1, z) \leq \Delta v(x+1, y, z), \forall y < M$
- C.4 $\Delta v(x, y, z) \geq -c_l$
- C.5 $\Delta v(x, y, z) \geq \Delta v(x, y, z+1), \forall z < N$
- C.6 $\Delta v(x, y, z) \leq \Delta v(x+1, y, z+1), \forall z < N$

We show in Appendix A.3 that the optimal value function \bar{v}^* belongs to V by combining the proof when only ARI is used (Lemma 1, Section 3) and the proof when only ADI is used (Lemma 1 in Gayon et al. (2009a)). It implies that the optimal policy as the following characteristics.

Theorem 3 *There exists a switching curve $\bar{S}(y, z)$ such that it is optimal to produce if and only if $x < \bar{S}(y, z)$. Moreover, the switching curve has the following properties:*

$$\begin{aligned}\bar{S}(y, z) - 1 &\leq \bar{S}(y+1, z) \leq \bar{S}(y, z) \\ \bar{S}(y, z) &\leq \bar{S}(y, z+1) \leq \bar{S}(y, z) + 1\end{aligned}$$

The first (respectively second) property of the switching curve in Theorem 3 corresponds to the property for the switching curve when only ARI (respectively ADI)

is used. We can easily adapt Theorem 3 to backorder situations. It suffices to remove condition C.4 in the definition of set V .

In Figure 11, we show illustrative results depicting the impact of ARI alone, ADI alone and joint ARI and ADI. The results indicate that the benefits of ARI and ADI are complementary. The benefit due to ADI is more significant for the instances we have tested.

Insert Figure 11

6. Other applications

In this paper we focused on the value of ARI for situations with returns. There are many more situations where using advance supply information (ASI) may be profitable, which are not related to return flows. The model presented in this paper can be applied to some other situations, after minor changes.

One other application concerns production planning in situations where, apart from the primary process for generating a certain product P1, the production of other products P2, P3,... via other processes may result in P1 as well, as an undesired co- or by-product (production with a variable quality yield). Such a situation can among others be found in the process industries, where complete batches from one process can not have the desired quality from the point of view of this process, but can have the correct quality for another application for which normally a separate second production process is started on another facility, where customers buy complete batches one by one. In this case an announcement corresponds with the announcement of the startup of a batch for product P2, P3,... where δ indicates the arrival process of the above announcements. In this case, L denotes the time between the above announcements and the moment that the results of the related quality measurements become available.

The presented model can also be useful for companies that don't produce but buy from an external supplier with limited capacity. Some of these companies simultaneously try to buy individual (un)used products via e.g. the internet, spot markets, auctions, where it is uncertain whether or not a company will receive the desired products because bids by other companies may be higher. Examples are

airlines, transportation companies with big truck fleets, which follow the above strategy for expensive parts like engines, requiring a negligible effort to make them as good as new. In this case, δ indicates the arrival process of interesting announcements for which the company always bids (e.g. announcements by other companies in the same sector that decide to replace part of their fleet earlier than expected and due to this are confronted with obsolete stocks that they want to sell) whereas L denotes the time between the announcement/bid and the result of the bid.

7. Summary and conclusions

In this paper, we have examined the potential value of using imperfect advance supply information from a number of autonomous external sources for a company supplying one item, where the company also has one own production facility under complete control. We focused on the information that becomes available after products have been sold or lease contracts have started.

For both lost sales and backorder situations, we have characterized the optimal policy with and without ARI. In case of lost sales, it was shown that for the many sets of parameter values studied, using ARI as indicated in this paper can result in a cost reduction of 5% at most, but in 91% of the examined scenarios, the cost reduction was less than 1% and in 97% of the scenarios less than 2%. Although this may not seem much, as always we should compare this reduction with the net profit of a company, due to which the gain may be considerable. In general, our research shows that ARI, as used in this paper, seems to be most advantageous in situations where return times are not very short or very long, where the probability that an announced return becomes an actual return is also not very high or very low. When backorders are allowed, it was shown that the cost reduction can be higher, up to 23%. We have extended our analysis when both ARI and ADI are available. We also have indicated that our model can not only be useful in situations with returns from customers, but also in many other situations, like in situations with co- and by-products as well as for fleet owners.

Our model can be extended in several ways. One extension (for both lost sales and backorder situations) is to include an admission control for returns. When a customer announces the intended return of a product, the decision maker decides whether or not to accept the potential return. One reason for rejecting a return is e.g. to avoid

excess inventory. Then, the optimal policy is expected to consist of two switching curves, $R(y)$ and $S(y)$ such that it is optimal to accept a return (resp. to produce) if and only if the on-hand stock of serviceable products is below $R(y)$ (resp. $S(y)$). It is also possible to consider several types of returns and to investigate how to coordinate production and admission of the different return flows. Another extension would be to study the effect of the price offered for returns.

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Appendix

A.1. Proof of Lemma 1

As operator T is a contraction mapping (Puterman,1994), the fixed point theorem ensures that the sequence of value functions $v_{n+1} = Tv_n$ converges to v^* , for any v_0 , and in particular if v_0 is the null value function which belongs to U . In the following, we show that operator T preserves conditions of U , i.e. if $v \in U$, then $Tv \in U$. By induction, we can then conclude that $v^* \in U$.

Define $S_y = \min [x | \Delta v(x, y) + c_p \geq 0]$. Operator T_0 can be rewritten as follows.

$$T_0 v(x, y) = \begin{cases} v(x+1, y) & \text{if } x < S_y \\ v(x, y) & \text{if } x \geq S_y \end{cases}$$

We can then compute $\Delta T_0 v$.

$$\Delta T_0 v(x, y) = \begin{cases} \Delta v(x+1, y) & \text{if } x < S_y - 1 \\ -c_p & \text{if } x = S_y - 1 \\ \Delta v(x, y) & \text{if } x \geq S_y \end{cases}$$

As v satisfies condition C.4 ($\Delta v(x, y) \geq -c_l$) and $c_p < c_l$, we immediately obtain that $T_0 v$ satisfies condition C.4 ($\Delta T_0 v(x, y) \geq -c_l$).

We can now prove that $T_0 v$ satisfies conditions C.1, C.2 and C.3 of U . All the inequalities below are obtained by using the assumption that $v \in U$.

$$\Delta^2 T_0 v(x, y) = \begin{cases} \Delta^2 v(x+1, y) \geq 0 & \text{if } x < S_y - 2 \\ -\Delta v(x+1, y) - c_p \geq 0 & \text{if } x = S_y - 2 \\ \Delta v(x+1, y) + c_p \geq 0 & \text{if } x = S_y - 1 \\ \Delta^2 v(x, y) & \text{if } x \geq S_y \end{cases}$$

$$\Delta T_0 v(x, y+1) - \Delta T_0 v(x, y) = \begin{cases} \Delta v(x+1, y+1) - \Delta v(x+1, y) \geq 0 & \text{if } x < S_y - 1 \\ \Delta v(x, y+1) + c_p \geq 0 & \text{if } x = S_y - 1, x = S_{y+1} \\ 0 \geq 0 & \text{if } x = S_y - 1, x = S_{y+1} - 1 \\ \Delta v(x, y+1) - \Delta v(x, y) & \text{if } x \geq S_y \end{cases}$$

$$\Delta T_0 v(x+1, y) - \Delta T_0 v(x, y+1) = \begin{cases} \Delta v(x+2, y) - \Delta v(x+1, y+1) \geq 0 & \text{if } x < S_y - 2 \\ -c_p - \Delta v(x+1, y+1) \geq 0 & \text{if } x = S_y - 2, x = S_{y+1} - 2 \\ 0 & \text{if } x = S_y - 2, x = S_{y+1} - 1 \\ \Delta v(x+1, y) - \Delta v(x, y+1) \geq 0 & \text{if } x = S_y - 1, x = S_{y+1} \\ \Delta v(x+1, y) + c_p \geq 0 & \text{if } x = S_y - 1, x = S_{y+1} - 1 \\ \Delta v(x+1, y) - \Delta v(x, y+1) \geq 0 & \text{if } x \geq S_y \end{cases}$$

We conclude that $T_0 v$ belongs to U .

Similarly, we prove that $T_1 v$ belongs to U .

$$\Delta T_1 v(x, y) = \begin{cases} \Delta v(x-1, y) \geq -c_l & \text{if } x > 0 \\ -c_l \geq -c_l & \text{if } x = 0 \end{cases}$$

$$\Delta^2 T_1 v(x, y) = \begin{cases} \Delta^2 v(x-1, y) \geq 0 & \text{if } x > 0 \\ 0 \geq 0 & \text{if } x = 0 \end{cases}$$

$$\Delta T_1 v(x, y+1) - \Delta T_1 v(x, y) = \begin{cases} \Delta v(x-1, y+1) - \Delta v(x+1, y) \geq 0 & \text{if } x > 0 \\ 0 \geq 0 & \text{if } x = 0 \end{cases}$$

$$\Delta T_1 v(x+1, y) - \Delta T_1 v(x, y+1) = \begin{cases} \Delta v(x, y) - \Delta v(x-1, y+1) \geq 0 & \text{if } x > 0 \\ 0 \geq 0 & \text{if } x = 0 \end{cases}$$

We also obtain that $T_2 v \in U$.

$$\Delta T_2 v(x, y) = \begin{cases} \Delta v(x, y+1) \geq -c_l & \text{if } y < M \\ \Delta v(x, y) \geq -c_l & \text{if } y = M \end{cases}$$

$$\Delta^2 T_2 v(x, y) = \begin{cases} \Delta^2 v(x, y+1) \geq 0 & \text{if } y < M \\ \Delta^2 v(x, y) \geq 0 & \text{if } y = M \end{cases}$$

$$\Delta T_2 v(x, y+1) - \Delta T_2 v(x, y) = \begin{cases} \Delta v(x, y+2) - \Delta v(x, y+1) \geq 0 & \text{if } y < M-1 \\ 0 & \text{if } y = M-1 \end{cases}$$

$$\Delta T_2 v(x+1, y) - \Delta T_2 v(x, y+1) = \begin{cases} \Delta v(x+1, y+1) - \Delta v(x, y+2) \geq 0 & \text{if } y < M-1 \\ \Delta v(x+1, y+1) - \Delta v(x, y+1) \geq 0 & \text{if } y = M-1 \end{cases}$$

and that $T_3 v \in U$. Moreover, we prove that $\Delta T_3 v(x, y) \geq -M c_l$.

$$\Delta T_3 v(x, y) = \begin{cases} y \Delta v(x+1, y-1) + (M-y) \Delta v(x, y) \geq -M c_l & \text{if } y > 0 \\ M \Delta v(x, y) \geq -M c_l & \text{if } y = 0 \end{cases}$$

$$\Delta^2 T_3 v(x, y) = \begin{cases} y (\Delta^2 v(x+1, y-1) + c_r) + (M-y) \Delta^2 v(x, y) \geq 0 & \text{if } y > 0 \\ M \Delta^2 v(x, y) \geq 0 & \text{if } y = 0 \end{cases}$$

$$\Delta T_3 v(x, y+1) - \Delta T_3 v(x, y) = \begin{cases} y (\Delta v(x+1, y) - \Delta v(x+1, y-1)) + (M-y) (\Delta v(x, y+1) - \Delta v(x, y)) + \Delta v(x+1, y) - \Delta v(x, y+1) \geq 0 & \text{if } y > 0 \\ (y+1) (\Delta v(x+1, y) - \Delta v(x, y+1)) + M (\Delta v(x, y+1) - \Delta v(x, y)) \geq 0 & \text{if } y = 0 \end{cases}$$

$$\Delta T_3 v(x+1, y) - \Delta T_3 v(x, y+1) = \begin{cases} y (\Delta v(x+2, y-1) - \Delta v(x+1, y)) + (M-y-1) (\Delta v(x+1, y) - \Delta v(x, y+1)) \geq 0 & \text{if } y > 0 \\ (M-y-1) (\Delta v(x+1, y) - \Delta v(x, y+1)) \geq 0 & \text{if } y = 0 \end{cases}$$

We also show that $T_4 v \in U$. Moreover, we prove that $\Delta T_4 v(x, y) \geq -M c_l$.

$$\Delta T_4 v(x, y) = \begin{cases} y \Delta v(x, y-1) + (M-y) \Delta v(x, y) \geq -M c_l & \text{if } y > 0 \\ M \Delta v(x, y) \geq -M c_l & \text{if } y = 0 \end{cases}$$

$$\Delta^2 T_4 v(x, y) = \begin{cases} y \Delta^2 v(x, y-1) + (M-y) \Delta^2 v(x, y) \geq 0 & \text{if } y > 0 \\ M \Delta^2 v(x, y) \geq 0_l & \text{if } y = 0 \end{cases}$$

$$\Delta T_4 v(x, y+1) - \Delta T_4 v(x, y) = \begin{cases} y (\Delta v(x, y) - \Delta v(x, y-1)) + (M-y-1) (\Delta v(x, y+1) - \Delta v(x, y)) \geq 0 & \text{if } y > 0 \\ (M-y-1) (\Delta v(x, y+1) - \Delta v(x, y)) \geq 0 & \text{if } y = 0 \end{cases}$$

$$\Delta T_4 v(x+1, y) - \Delta T_4 v(x, y+1) = \begin{cases} y (\Delta v(x+1, y-1) - \Delta v(x, y)) + (M-y) (\Delta v(x+1, y) - \Delta v(x, y+1)) + \Delta v(x, y+1) - \Delta v(x, y) \geq 0 & \text{if } y > 0 \\ M (\Delta v(x+1, y) - \Delta v(x, y+1)) + (y+1) (\Delta v(x, y+1) - \Delta v(x, y)) \geq 0 & \text{if } y = 0 \end{cases}$$

As T is a positive linear combination of operators T_i , $i = 0,1,\dots,4$, it comes that Tv satisfies conditions C.1, C.2, and C.3. To show that Tv satisfies C.4, it is slightly more tricky.

From optimality equations, we have:

$$\Delta T v(x, y) = \frac{1}{C + \alpha} [c_h + \mu \Delta T_0 v(x, y) + \lambda \Delta T_1 v(x, y) + \delta \Delta T_2 v(x, y) + \gamma p \Delta T_3 v(x, y) + \gamma p \Delta T_4 v(x, y)]$$

Using the fact that $\Delta T_i v \geq -c_l$ for $i = 0,1,2$ and $\Delta T_i v \geq -M c_l$ for $i = 3,4$, we obtain

$$\Delta T v(x, y) \geq \frac{-\mu c_l - \lambda c_l - \delta c_l - \gamma M p c_l - \gamma M q c_l}{C + \mu + \lambda + \delta + M \gamma} \geq -c_l$$

and Tv satisfies C.4. Finally, we conclude that Tv belongs to U .

A.2. Proof of Property 1

i. For problem A and production policy π , we adopt the following notations: rate of lost sales (λ_{uns}^π), rate of demands that are satisfied by produced items (λ_{prod}^π), rate of demands that are satisfied by returned items (λ_{ret}^π), average inventory level (I^π), average cost (g^π). Note that $\lambda_{ret}^\pi = p\delta$ is policy independent while the other quantities are policy dependent. For problem B and production policy π , we adopt similar notations with a bar. A demand is either lost or satisfied. When it is satisfied, it can be by a produced item or by a return. Therefore we have the following balance equation

$$\lambda = \lambda_{uns}^\pi + \lambda_{prod}^\pi + \lambda_{ret}^\pi$$

The average cost for instance A can then be related to the average cost for problem B:

$$\begin{aligned} g^\pi &= \lambda_{uns}^\pi c_l + \lambda_{prod}^\pi c_p + \delta p c_r + c_h I^\pi = \lambda_{uns}^\pi c_l + (\lambda - \lambda_{uns}^\pi - \delta p) c_p + \delta p c_r + c_h I^\pi \\ &= \lambda_{uns}^\pi (c_l - c_p) + c_h I^\pi + \underbrace{\lambda c_p + \delta p (c_r - c_p)}_{=K>0} = \bar{g}^\pi + K \end{aligned}$$

Where K is a positive constant since, by assumption, $\lambda > p\delta$. Therefore, a policy minimizing g^π also minimizes \bar{g}^π .

ii. Follows directly from the proof of i. As $g^\pi = \bar{g}^\pi + K$ and the optimal policies are identical for A and B, we have immediately the desired relations between the average costs.

iii.

$$\Delta g = \frac{g(\text{no ARI}) - g(\text{ARI})}{g(\text{ARI})} = \frac{\overline{g(\text{no ARI})} + K - \overline{g(\text{ARI})} - K}{\overline{g(\text{ARI})} + K}$$

$$\leq \frac{\overline{g(\text{no ARI})} - \overline{g(\text{ARI})}}{\overline{g(\text{ARI})}} = \Delta \overline{g}$$

A.3. Proof of Theorem 3

The structure of the proof is similar to the one of Theorem 1. We have to prove that \overline{T} propagates condition C.1 to C.6. We can prove it by combine the proof when only ARI is used (Lemma 1, Section 3) and the proof when only ADI is used (Lemma 1 in Gayon et al. (2009a)).

In the proof of Lemma 1, we show that operators T_0, T_2, T_3, T_4 propagate conditions C.1, C.2, C.3 and C.4. Gayon et al. (2009a) show that operators $T_0, \overline{T}_1, T_5, T_6$ propagate conditions C.1, C.4, C.5 and C.6. Remains to prove that operators T_2, T_3, T_4 propagate conditions C.5 and C.6, and that \overline{T}_1, T_5, T_6 propagate conditions C.2 and C.3. The proof of these propagation results is trivial because the state transitions involved in the operators are not involved in the conditions.

FIGURES

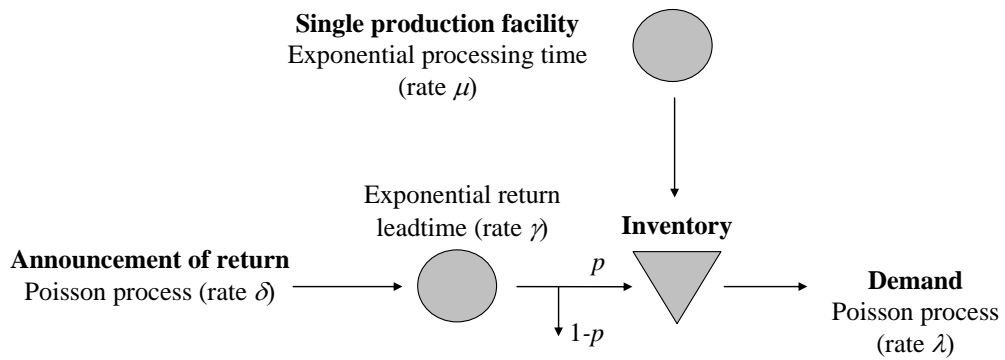


Figure 1: Inventory system with return flow and ARI

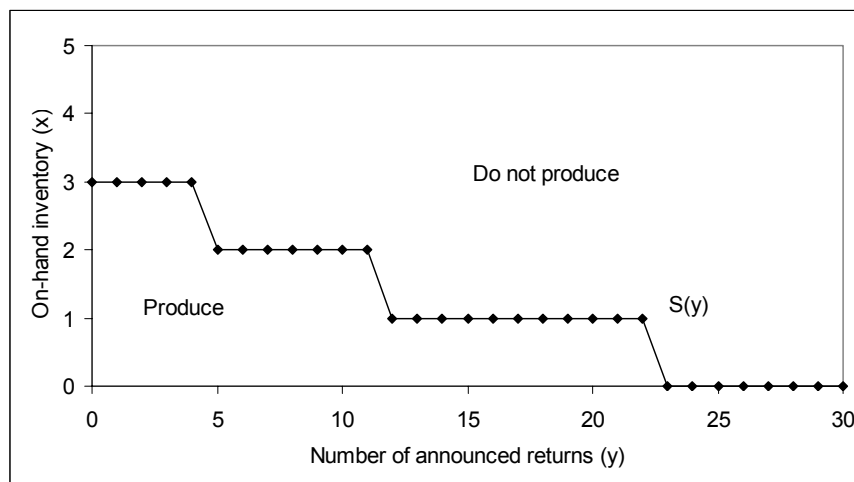


Figure 2: The structure of the optimal policy when $(\lambda = 1, \mu = 1, \delta = 0.25, \gamma = 0.75, c_r = 0, c_p = 0, c_h = 1, c_l = 10, p = 0.25)$

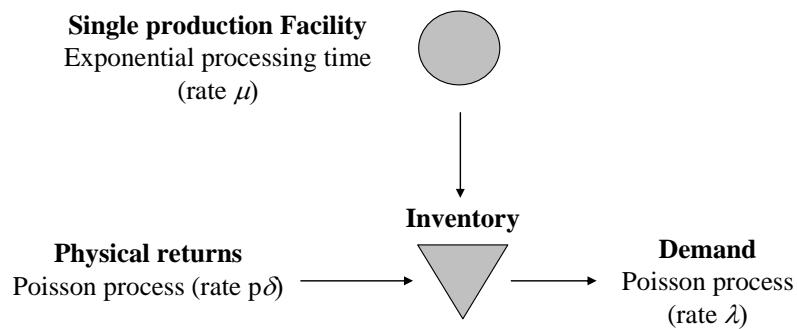


Figure 3: Inventory system with return flow and without ARI

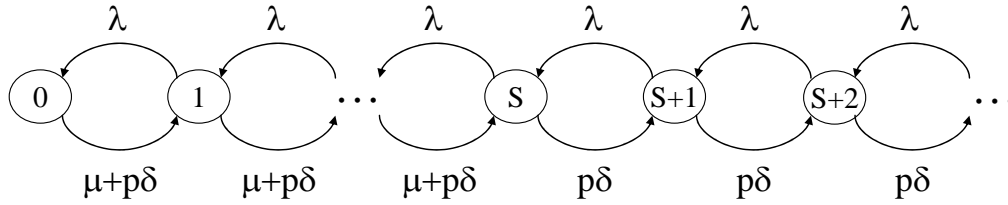


Figure 4: Markov chain for the system without ARI

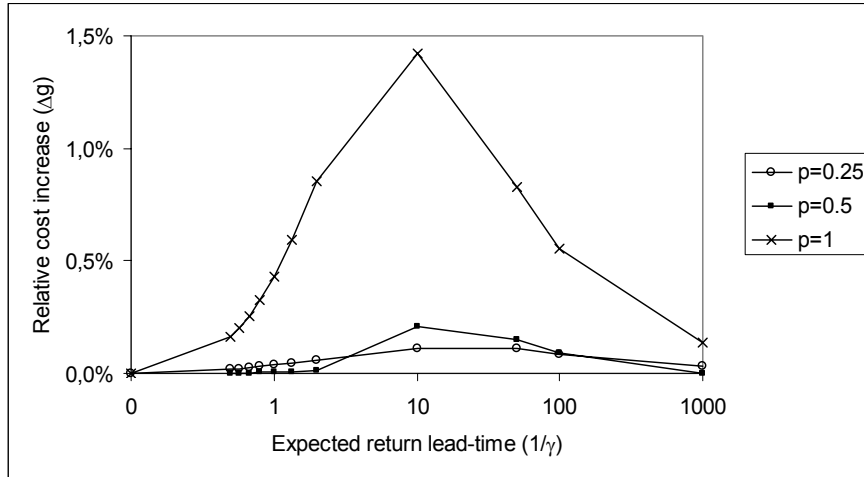


Figure 5: The effect of the expected return leadtime on the value of using ARI ($\lambda = 1, \mu = 1, \delta = 0.25, c_r = 0, c_p = 0, c_h = 1, c_l = 100$)

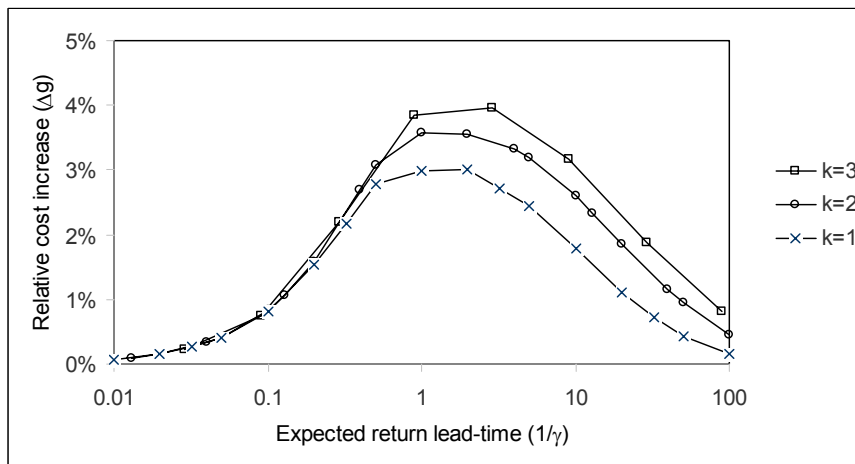


Figure 6: The effect of the expected return leadtime on the value of using ARI for Erlang- k distributions ($\lambda = 1, \mu = 2, \delta = 0.5, c_r = 0, c_p = 0, c_h = 1, c_l = 10, p = 1$)

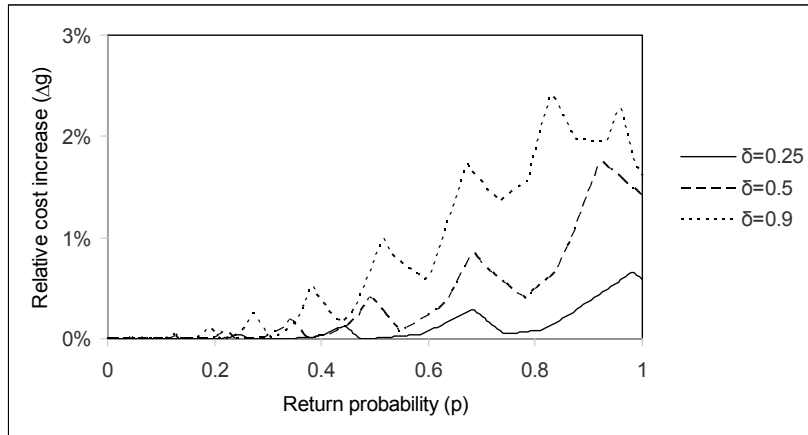


Figure 7: The effect of the expected return leadtime on the value of using ARI
 $(\lambda = 1, \mu = 1, \gamma = 0.8, c_r = 0, c_p = 0, c_h = 1, c_l = 100)$

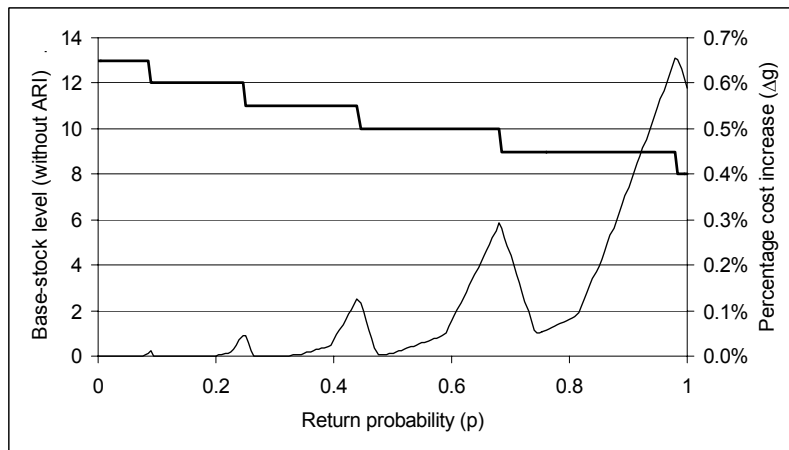


Figure 8: The effect of the expected return leadtime on the value of using ARI
 $(\lambda = 1, \mu = 1, \delta = 0.25, \gamma = 0.8, c_r = 0, c_p = 0, c_h = 1, c_l = 100)$

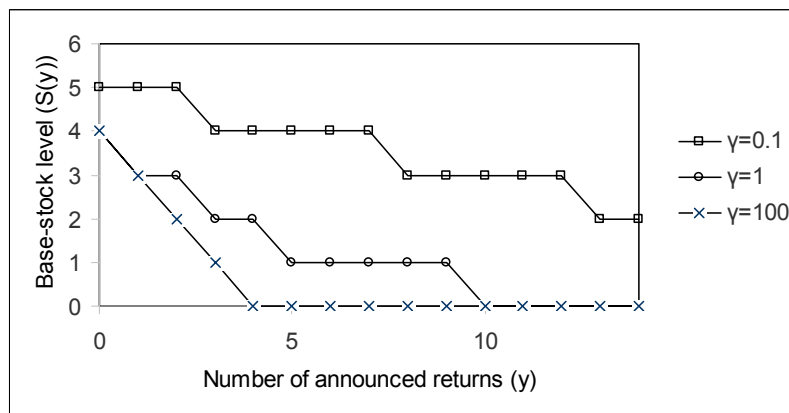


Figure 9: The effect of the return rate γ on the optimal policy
 $(\lambda = 1, \mu = 2, \delta = 0.5, c_r = 0, c_p = 0, c_h = 1, c_l = 100, p = 1)$

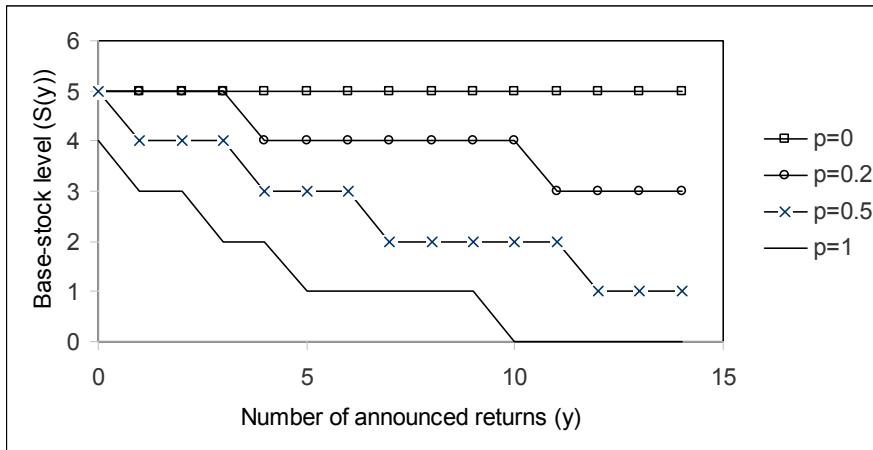
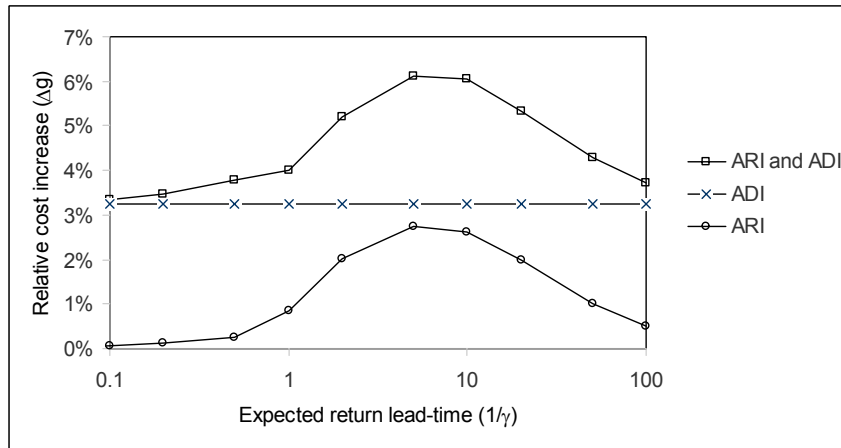
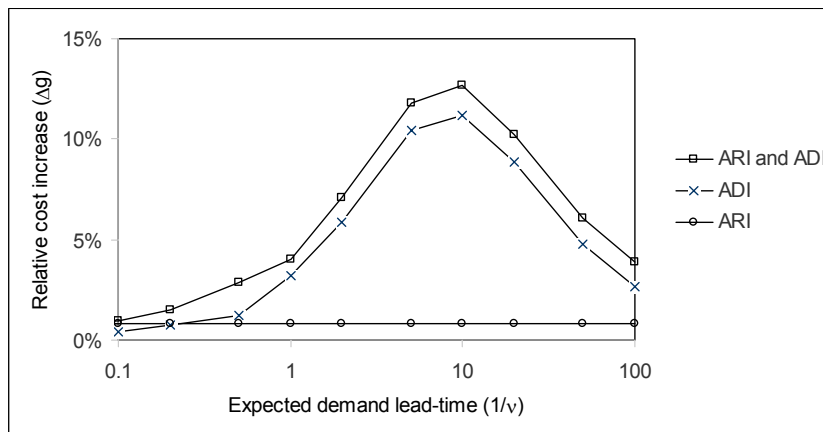


Figure 10: The effect of the return probability p on the optimal policy
 $(\lambda = 1, \mu = 2, \delta = 0.5, \gamma = 1, c_r = 0, c_p = 0, c_h = 1, c_l = 100)$



(a) ($\nu = 1$)



(b) ($\gamma = 1$)

Figure 11: The effect of ARI versus ADI
 $(\lambda = 1, \mu = 1, \delta = 0.5, c_r = 0, c_p = 0, c_h = 1, c_l = 100)$