

Mechanics: Surfaces, geodesics, connections, Riemannian curvature tensor

Gilles LEBORGNE

July 1, 2020

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In surfaces in \mathbb{R}^n , or more generally in varieties:

- 1- The branch of mathematics which adds to a geometry the connections (derivations), parallel transport, geodesics and curvature is called “affine geometry” (quantification of “variations”).
- 2- The branch of mathematics which adds a metric to a geometry is called “Riemannian geometry”.
- 3- The union of these two branches of mathematics gives “differential geometry”.

Part I

Surfaces in \mathbb{R}^n

1 Surfaces and bases

1.1 Surface

Let $n = 1, 2$ or 3 , let \mathbb{R}^n be the affine geometric space, and let $\vec{\mathbb{R}}^n$ be the associated vector space.

Let $m \in \mathbb{N}^*$, $m \leq n$, and $\vec{\mathbb{R}}^m = \mathbb{R} \times \dots \times \mathbb{R}$, (the Cartesian space of “parameters”). Let $(\vec{A}_i)_{i=1, \dots, m} = \text{written}(\vec{A}_i)$ be the canonical basis in $\vec{\mathbb{R}}^m$, that is, $\vec{A}_1 = (1, 0, \dots, 0)$, \dots , $\vec{A}_m = (0, \dots, 0, 1)$. Let U be an open set in $\vec{\mathbb{R}}^m$ (set of parameters).

Definition 1.1 A (parametric) regular surface in $\vec{\mathbb{R}}^n$ is a C^∞ map

$$\Phi : \begin{cases} U \subset \vec{\mathbb{R}}^m \rightarrow S \subset \mathbb{R}^n, \\ \vec{q} \rightarrow p = \Phi(\vec{q}), \end{cases} \quad (1.1)$$

which is surjective on S and of rank m for all $\vec{q} \in U$ (the surface S is m -dimensional). And Φ is called a coordinate system on S . The geometric surface S is the range (or image) of Φ , that is,

$$S = \Phi(U) = \text{Im}\Phi = \bigcup_{\vec{q} \in U} \{p = \Phi(\vec{q})\}. \quad (1.2)$$

Example 1.2 (Polar coordinates in \mathbb{R}^2 .) $m=n=2$. E.g. $U = \mathbb{R}_+^* \times]-\pi, \pi[\subset \vec{\mathbb{R}}^2$. Let (\vec{A}_1, \vec{A}_2) be the canonical basis in \mathbb{R}^2 . Let $\vec{q} \in U$ and $\vec{q} = r\vec{A}_1 + \theta\vec{A}_2 = \text{written}(r, \theta)$ (parameter). Let O be an origin (the center of the disk in the following) and (\vec{E}_1, \vec{E}_2) be a Euclidean basis in the geometric space \mathbb{R}^2 . And consider:

$$\Phi : \begin{cases} U \rightarrow \mathbb{R}^2 \\ \vec{q} = (r, \theta) \rightarrow p = \Phi(\vec{q}) = O + r \cos \theta \vec{E}_1 + r \sin \theta \vec{E}_2 = O + \vec{x}, \end{cases} \quad (1.3)$$

so $\vec{x} = \vec{Op}$ and, column matrix representing \vec{x} relative to (\vec{E}_1, \vec{E}_2) ,

$$[\vec{x}]_{\vec{E}} = [\vec{Op}]_{\vec{E}} = [O\Phi(\vec{q})]_{\vec{E}} = \begin{pmatrix} x = r \cos \theta \\ y = r \sin \theta \end{pmatrix}. \quad (1.4)$$

The polar coordinates are the parameters $r = \|\vec{x}\|$ (the “length” of $\|\vec{x}\|$) and $\theta =$ (the angle between \vec{x} and \vec{E}_1). Here $S = \text{Im}\Phi$ is \mathbb{R}^2 without the left axis $\{(x, y) : x \leq 0, y = 0\}$. \blacksquare

Example 1.3 (Circle.) $n=2$ and $m=1$. Let $R > 0$ and $U =]-\pi, \pi[$. With (1.3), let

$$\Phi_R(\theta) = \Phi(R, \theta). \quad (1.5)$$

So:

$$\vec{x} = R \cos \theta \vec{E}_1 + R \sin \theta \vec{E}_2, \quad [\vec{x}]_{\vec{E}} = \begin{pmatrix} R \cos \theta \\ R \sin \theta \end{pmatrix}. \quad (1.6)$$

θ is the parameter. The geometric surface $S = \text{Im}\Phi_R$ is the circle $C(\vec{0}, R)$ with center $\vec{0}$ and radius R without the point $(-1, 0)$. Here, in \mathbb{R}^2 , the surface Φ_R is a curve. \blacksquare

Example 1.4 See § 3 for other usual examples. \blacksquare

Definition 1.5 A material coordinate system is a coordinate system which depends on time:

$$\Phi : \begin{cases} \mathbb{R} \times U \rightarrow \mathbb{R}^n, \\ (t, \vec{q}) \rightarrow p = \Phi(t, \vec{q}). \end{cases} \quad (1.7)$$

1.2 Coordinate lines, and the basis of a coordinate system

Definition 1.6 Let $\vec{q}_0 \in U$, $\vec{q}_0 = \sum_{i=1}^m q_0^i \vec{A}_i$, and $p_0 = \Phi(\vec{q}_0)$. The j -th coordinate line at p_0 , $j \in [1, m]_{\mathbb{N}}$, is the curve

$$c_{p_0}^{(j)} : \begin{cases}]-\varepsilon, \varepsilon[\longrightarrow S \subset \mathbb{R}^n \\ u \longrightarrow p = c_{p_0}^{(j)}(u) = \Phi(\vec{q}_0 + u\vec{A}_j) = \Phi(q_0^1, \dots, q_0^{j-1}, q_0^j + u, q_0^{j+1}, \dots, q_0^n). \end{cases} \quad (1.8)$$

In particular $c_{p_0}^{(j)}(0) = p_0$ (and ε has been chosen small enough for $\vec{q}_0 + u\vec{A}_j$ to be in U). And the j -th basis vector at $p_0 = \Phi(\vec{q}_0)$ of the coordinate system Φ is the vector

$$\vec{e}_j(p_0) := (c_{p_0}^{(j)})'(0) = \left(= \lim_{h \rightarrow 0} \frac{c_{p_0}^{(j)}(h) - c_{p_0}^{(j)}(0)}{h} \right), \quad (1.9)$$

tangent vector of the j -th coordinate curve at $p_0 = \Phi(\vec{q}_0)$.

So

$$\vec{e}_j(p_0) = d\Phi(\vec{q}) \cdot \vec{A}_j \quad \left(= \frac{\partial \Phi}{\partial q^j}(\vec{q}_0) = \lim_{h \rightarrow 0} \frac{\Phi(\vec{q}_0 + h\vec{A}_j) - \Phi(\vec{q}_0)}{h} \right). \quad (1.10)$$

And, Φ being regular, for all $p \in S$, the $\vec{e}_i(p)$ are independent ($i = 1, \dots, m$).

Example 1.7 Continuation of the example 1.2: $\vec{O}p = \overrightarrow{O\Phi(r, \theta)} = r \cos \theta \vec{E}_1 + r \sin \theta \vec{E}_2$, $c_p^{(1)} : r \rightarrow c_p^{(1)}(r) = \Phi_\theta(r)$ (radius), $c_p^{(2)} : \theta \rightarrow c_p^{(2)}(\theta) = \Phi_r(\theta)$ (circle), and (1.10) give

$$\begin{cases} \vec{e}_1(p) = d\Phi(p) \cdot \vec{A}_1 = \frac{\partial \Phi}{\partial r}(r, \theta) = \cos \theta \vec{E}_1 + \sin \theta \vec{E}_2, & [\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \\ \vec{e}_2(p) = d\Phi(p) \cdot \vec{A}_2 = \frac{\partial \Phi}{\partial \theta}(r, \theta) = -r \sin \theta \vec{E}_1 + r \cos \theta \vec{E}_2, & [\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix}. \end{cases} \quad (1.11)$$

▀

Example 1.8 Continuation of the example 1.3. $m = 1$ and the canonical vector basis in \mathbb{R} is written $A_1 = 1$. The coordinate line is $\theta \rightarrow p = \Phi_R(\theta)$ (circle), and the tangent vector to the coordinate line at p is $\vec{f}_1(p) = d\Phi_R(\theta) \cdot A_1 = \frac{d\Phi}{d\theta}(\theta) = \Phi'_R(\theta) = -R \sin \theta \vec{E}_1 + R \cos \theta \vec{E}_2$, cf. (1.10), with $\|\vec{f}_1(p)\| = R$. Here, relatively to (1.11), $\vec{f}_1(p) = \vec{e}_2(p)$ at $p = \Phi(R, \theta)$. ▀

Exercise 1.9 Prove the mean value theorem in S surface in \mathbb{R}^n : If $T > 0$, if $c : t \in [-T, T] \rightarrow p_t = c(t) \in S$ is a regular curve in S and $\vec{v}(p_t) = c'(t)$ is the tangent vector at $p_t = c(t)$, then there exists $u \in]t, t+h[$ s.t. $p_u = c(u)$ satisfies

$$f(p_{t+h}) - f(p) = h df(p_u) \cdot \vec{v}(p_u). \quad (1.12)$$

Answer. Let $g(t) = f(c(t))$, thus $g'(t) = df(c(t)) \cdot c'(t)$, thus $\exists u \in]t, t+h[$ s.t. $g(t+h) - g(t) = h g'(u)$ (mean value theorem in \mathbb{R}), i.e. (1.12). ▀

1.3 The tangent space at p

Definition 1.10 The tangent space at p of $S = \Phi(U)$ is

$$T_p S = \text{Vect}\{\vec{e}_1(p), \dots, \vec{e}_m(p)\} \subset \mathbb{R}^n. \quad (1.13)$$

Φ being regular, for all $p \in S$ the $\vec{e}_j(p)$ are linearly independent, and $(\vec{e}_i(p))_{i=1, \dots, m} = \text{written}(\vec{e}_i(p))$ is a basis for $T_p S$, so $\dim T_p S = m$.

(If $m = 2$ and $n = 3$ then $T_p S$ is the tangent plane at p , and if $m = 1$ and $n = 2$ or 3 then $T_p S$ is the tangent line at p .)

1.4 The fiber at p

Definition 1.11 Using (1.13), the fiber at p is the couple

$$(p, T_p S) \in S \times \mathbb{R}^n. \quad (1.14)$$

(A vector in \mathbb{R}^n is “drawn anywhere”. A element (p, \vec{w}_p) of the fiber $(p, T_p S)$ is “a vector drawn at p ”.)

1.5 The tangent bundle TS

Definition 1.12 The tangent bundle TS is the set of fibers:

$$TS = \bigcup_{p \in S} (p, T_p S) \subset S \times \mathbb{R}^n. \quad (1.15)$$

(Subset of the cross product “affine space \mathbb{R}^n ” \times “vector space \mathbb{R}^n ”.)

Example 1.13 In \mathbb{R}^2 , let S be the circle $C(\vec{0}, R)$, cf. (1.6). Then the fiber at $p \in S$ can be drawn as the tangent line at S at p . And the tangent bundle can be represented by the union of these fibers.

However, the cross-product $\mathbb{R}^n \times \mathbb{R}^n$ makes us represent a fiber as a “vertical line” (a line on the cylinder) at p , and the tangent bundle is then represented as the “vertical cylinder” through $C(\vec{0}, R)$. ■

A vector field is a map

$$\tilde{v}: \begin{cases} S \rightarrow TS = \bigcup_{p \in S} (p, T_p S) \\ p \mapsto \tilde{v}(p) = (p, \vec{v}(p)), \end{cases} \quad (1.16)$$

supposed to be C^∞ . Then the range (image) $\text{Im}(\tilde{v}) = \bigcup_{p \in S} (p, \vec{v}(p)) = \text{graph}(\vec{v})$ is the graph of \vec{v} . (So, in mechanics \vec{v} is a Eulerian function.)

The set of vector fields is named $\Gamma(S)$.

If there is no ambiguity, we simply write

$$\tilde{v}(p) = (p, \vec{v}(p)) \stackrel{\text{written}}{=} \vec{v}(p). \quad (1.17)$$

Example 1.14 in \mathbb{R}^2 . Polar coordinates: $\vec{O}p = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$, basis at p given by $\vec{e}_1(p) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ and $\vec{e}_2(p) = \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix}$: And \vec{e}_1 and \vec{e}_2 are vector fields in S . ■

1.6 Notation (dq^i) for the dual basis (A^i)

Let $(A^i) = (dq^i)$ be the dual basis of the canonical (\vec{A}_i) . Recall: $A^i = dq^i \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}) = \mathbb{R}^{m*}$ is the linear form defined by $A^i(\vec{A}_j) = \delta_j^i = dq^i \cdot \vec{A}_j$. The notation dq^i refers to the chosen name $\vec{q} = (q^1, \dots, q^m)$ for a variable in $U \subset \mathbb{R}^m$. (So, in case of multiple notations, use A^i , not dq^i .)

Let (\vec{E}_i) be a Cartesian basis in \mathbb{R}^n and let $(E^i) = (dx^i)$ be its dual basis.

With $\vec{q} \in U \subset \mathbb{R}^m$ and $p = \Phi(\vec{q}) = \sum_{i=1}^n \Phi^i(\vec{q}) \vec{E}_i \in \mathbb{R}^n$, let $d\Phi^i(\vec{q}) \cdot \vec{A}_j \stackrel{\text{written}}{=} \frac{\partial \Phi^i}{\partial q^j}$: The notations refers to the chosen name $\vec{q} = (q^1, \dots, q^m)$ for a variable in U (so, in case of multiple notations, use $d\Phi^i(\vec{q}) \cdot \vec{A}_j$, not $\frac{\partial \Phi^i}{\partial q^j}$.) For $j \in [1, m]_{\mathbb{N}}$ we get

$$d\Phi(\vec{q}) \cdot \vec{A}_j = \sum_{i=1}^n (d\Phi^i(\vec{q}) \cdot \vec{A}_j) \vec{E}_i = \sum_{i=1}^n \frac{\partial \Phi^i}{\partial q^j}(\vec{q}) \vec{E}_i, \quad \text{and} \quad [d\Phi(\vec{q})]_{|\vec{A}, \vec{E}} = \left[\frac{\partial \Phi^i}{\partial q^j}(\vec{q}) \right] \quad (1.18)$$

is the Jacobian matrix of Φ relative to (\vec{A}_i) and (\vec{E}_i) . Tensorial notation:

$$d\Phi(\vec{q}) = \sum_{i=1}^n \sum_{j=1}^m \frac{\partial \Phi^i}{\partial q^j}(\vec{q}) \vec{E}_i \otimes A^j = \sum_{i=1}^n \sum_{j=1}^m \frac{\partial \Phi^i}{\partial q^j}(\vec{q}) \vec{E}_i \otimes dq^j, \quad (1.19)$$

the choice of the bases being explicit. And with contractions we get, with $\vec{u} \in \mathbb{R}^m$ and $\vec{u} = \sum_{j=1}^m u^j \vec{A}_j$,

$$d\Phi(\vec{q}) \cdot \vec{u} = \left(\sum_{i,j} \frac{\partial \Phi^i}{\partial q^j}(\vec{q}) \vec{E}_i \otimes dq^j \right) \cdot \vec{u} = \sum_{i,j} \frac{\partial \Phi^i}{\partial q^j}(\vec{q}) \vec{E}_i (dq^j \cdot \vec{u}) = \sum_{i,j} \frac{\partial \Phi^i}{\partial q^j}(\vec{q}) u^j \vec{E}_i,$$

And $[d\Phi(\vec{q}) \cdot \vec{u}]_{|\vec{E}} = [d\Phi(\vec{q})]_{|\vec{A}, \vec{E}} [\vec{u}]_{|\vec{A}}$.

1.7 Notation ($dq^i(p)$) for the dual basis ($e^i(p)$)

One of the difficulty is notations... The basis ($\vec{e}_i(p)$) in T_pS , of the coordinate system has been defined at (1.10). Its dual basis ($e^i(p)_{i=1,\dots,m} \stackrel{\text{written}}{=} e^i(p)$) is made of the linear forms $e^i(p) \in \mathcal{L}(T_pS; \mathbb{R})$ defined by

$$\forall i, j = 1, \dots, m, e^i(p) \cdot \vec{e}_j(p) = \delta_j^i, \quad \text{and} \quad e^i(p) \stackrel{\text{written}}{=} dq^i(p). \quad (1.20)$$

Why? With (1.3) and $S = \Phi(U)$, consider

$$\Phi^{-1} : \begin{cases} S \rightarrow U \\ p \rightarrow \vec{q} = \Phi^{-1}(p) \stackrel{\text{written}}{=} \vec{q}(p) = \sum_{i=1}^m q^i(p) \vec{A}_i. \end{cases} \quad (1.21)$$

So $d\Phi^{-1}(p) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ and $d\Phi^{-1}(p) \cdot \vec{w}_p = \sum_{i=1}^m (dq^i(p) \cdot \vec{w}_p) \vec{A}_i$ for all vector \vec{w}_p at p . In particular

$$\vec{A}_j = d\Phi^{-1}(p) \cdot \vec{e}_j(p) = \sum_{i=1}^m (dq^i(p) \cdot \vec{e}_j(p)) \vec{A}_i \implies dq^i(p) \cdot \vec{e}_j(p) = \delta_j^i, \quad \forall i, j, \quad (1.22)$$

hence $dq^i(p) = e^i(p)$, hence the notation (1.20)₂.

So: In \mathbb{R}^m , the dual basis of (\vec{A}_i) is (dq^i) since here the variable name is \vec{q} . And in T_pS the dual basis of ($\vec{e}_i(p)$) is ($dq^i(p)$) with dq^i the exact differential form = the differential of $q^i : S \rightarrow \mathbb{R}$. So beware of the context.

1.8 Bidual basis (∂_i) = ($\frac{\partial}{\partial q^i}$)

Let $p \in S$. Let ($\partial_j(p)$) be the dual basis of ($dq^i(p)$) (the bidual basis of ($\vec{e}_j(p)$)): The $\partial_j(p) \in (T_pS^*)^* \stackrel{\text{written}}{=} T_pS^{**}$ (dual of T_pS^* and bidual of T_pS) satisfy, for all $i, j = 1, \dots, m$,

$$\partial_j(p) \cdot dq^i(p) = \delta_j^i \quad (\text{thus} = dq^i(p) \cdot \vec{e}_j(p)). \quad (1.23)$$

The canonical isomorphism

$$J : \begin{cases} E \rightarrow E^{**} \\ \vec{v} \rightarrow J(\vec{v}) \quad \text{s.t.} \quad J(\vec{v}) \cdot \ell = \ell \cdot \vec{v}, \quad \forall \vec{v} \in E, \end{cases} \quad (1.24)$$

is natural (independent of an observer, see Spivak [17]). Then

$$J(\vec{v}) \stackrel{\text{written}}{=} \vec{v}. \quad (1.25)$$

In particular with $E = T_pS$, we get

$$\partial_i(p) = \mathcal{J}(\vec{e}_i(p)) \stackrel{\text{written}}{=} \vec{e}_i(p), \quad \text{and} \quad \partial_i \stackrel{\text{written}}{=} \vec{e}_i. \quad (1.26)$$

1.9 The notation $\frac{\partial g}{\partial q^j}(p)$

Let $f_S : p \in S \rightarrow f_S(p) \in \mathbb{R}$ be a C^1 function (defined in S), and consider the function $f_U : \vec{q} \in U \rightarrow f_U(\vec{q}) \in \mathbb{R}$ defined (in U) by

$$f_U := f_S \circ \Phi, \quad \text{i.e.} \quad f_U(\vec{q}) = f_S(p) \quad \text{when} \quad p = \Phi(\vec{q}). \quad (1.27)$$

So $f_U(\vec{q}) = f_S(\Phi(\vec{q}))$, thus $df_U(\vec{q}) \cdot \vec{A}_j = df_S(\Phi(\vec{q})) \cdot d\Phi(\vec{q}) \cdot \vec{A}_j$, that is,

$$df_S(p) \cdot \vec{e}_j(p) = df_U(\vec{q}) \cdot \vec{A}_j \quad \text{when} \quad p = \Phi(\vec{q}). \quad (1.28)$$

Definition (notation):

$$\frac{\partial f_S}{\partial q^j}(p) := \frac{\partial f_U}{\partial q^j}(\vec{q}), \quad \text{i.e.} \quad \frac{\partial f_S}{\partial q^j}(p) := df_S(p) \cdot \vec{e}_j(p) \quad (= \frac{\partial (f_S \circ \Phi)}{\partial q^j}(p) = df_U(\vec{q}) \cdot \vec{A}_j). \quad (1.29)$$

NB: f_S is a function of p , not a function of \vec{q} , thus the notation $\frac{\partial f_S}{\partial q^j}$ would be meaningless without the definition (notation) (1.29).

(1.29) gives

$$df_S(p) = \sum_{i=1}^m \frac{\partial f_S}{\partial q^i}(p) dq^i(p) \quad (= \sum_{i=1}^m \frac{\partial f_S}{\partial q^i}(p) e^i(p)). \quad (1.30)$$

Indeed, $(e^i(p)) = (dq^i(p))$ being the dual basis of the basis $(\vec{e}_i(p))$ in $T_p S$, and $df_S(p)$ being a linear form $\in \mathcal{L}(T_p S; \mathbb{R})$, there exists $\alpha_1(p), \dots, \alpha_m(p) \in \mathbb{R}$ (components) such that $df_S(p) = \sum_{i=1}^m \alpha_i(p) e^i(p)$; And $df_S(p) \cdot \vec{e}_j(p) = (\sum_{i=1}^m \alpha_i(p) e^i(p)) \cdot \vec{e}_j(p) = \sum_{i=1}^m \alpha_i(p) (e^i(p) \cdot \vec{e}_j(p)) = \sum_{i=1}^m \alpha_i(p) \delta_j^i = \alpha_j(p)$.

Interpretation of $\partial_i(p)$, cf. (1.23): At p , $\partial_i(p)$ is the directional derivative along $\vec{e}_i(p)$: For $i = 1, \dots, m$,

$$\partial_i(p) = \frac{\partial}{\partial q^i}(p) : \begin{cases} C^1(S; \mathbb{R}) \rightarrow \mathbb{R}, \\ f_S \rightarrow \partial_i(p)(f_S) = \left(\frac{\partial}{\partial q^i}(p)\right)(f_S) := df_S(p) \cdot \vec{e}_i(p) = \frac{\partial f_S}{\partial q^i}(p). \end{cases} \quad (1.31)$$

In other words, $\frac{\partial f_S}{\partial q^i}(p)$ is the derivative of f_S along the i -th coordinate line, cf. (1.8):

$$\lim_{h \rightarrow 0} \frac{(f_S \circ c_p^{(i)})(h) - (f_S \circ c_p^{(i)})(0)}{h} = (f_S \circ c_p^{(i)})'(0) = df_S(p) \cdot \vec{e}_i(p) \stackrel{\text{written}}{=} \frac{\partial f_S}{\partial q^i}(p). \quad (1.32)$$

Example 1.15 With (1.4): $x = r \cos \theta$, $y = r \sin \theta$, $\vec{O}\vec{p} = (x, y)$. Let $f_S(p) = \text{written } f_S(x, y)$. E.g. with $f_S(p) = xy^2 = f_U(r, \theta) = r^3 \cos \theta \sin^2 \theta$, we get $\frac{\partial f_S}{\partial r}(x, y) := \frac{\partial f_U}{\partial r}(r, \theta) = 3r^2 \cos \theta \sin^2 \theta = 3 \frac{xy^2}{\sqrt{x^2+y^2}} = \text{written } \frac{\partial f_S}{\partial r}(x, y)$ and $\frac{\partial f_S}{\partial \theta}(x, y) := \frac{\partial f_U}{\partial \theta}(r, \theta) = r^3(-\sin^3 \theta + 2 \cos^2 \theta \sin \theta) = -y^3 + 2x^2y = \text{written } \frac{\partial f_S}{\partial \theta}(x, y)$.

Check: $f_S(p) = xy^2$ gives $df_S(x, y) = y^2 dx + 2xy dy$; Thus, with $\vec{e}_1(p) = \cos \theta \vec{E}_1 + \sin \theta \vec{E}_2$, we get $\frac{\partial f_S}{\partial r}(p) := df_S(p) \cdot \vec{e}_1(\vec{x}) = y^2 \cos \theta + 2xy \sin \theta = r^2 \sin^2 \theta \cos \theta + 2r^2 \cos \theta \sin \theta \sin \theta$; and, with $\vec{e}_2(p) = -r \sin \theta \vec{E}_1 + r \cos \theta \vec{E}_2$, we get $\frac{\partial f_S}{\partial \theta}(p) := df_S(p) \cdot \vec{e}_2(\vec{x}) = y^2(-r \sin \theta) + 2xy(r \cos \theta) = -r^3 \sin^3 \theta + 2r^3 \cos \theta \sin \theta \cos \theta$. \blacksquare

Remark 1.16 Classic problem when $f_S \in \mathcal{F}(S; \mathbb{R})$ and $f_S = 0$ outside S : The classic definition $\frac{\partial f_S}{\partial x^j}(p) = \lim_{h \rightarrow 0} \frac{f_S(p+h\vec{E}_j) - f_S(p)}{h} = df_S(p) \cdot \vec{E}_j$ is not defined in general, since $p + h\vec{E}_j \notin S$ give $f_S(p + h\vec{E}_j) = 0$ and then $\lim_{h \rightarrow 0} \frac{f_S(p+h\vec{E}_j) - f_S(p)}{h} = \pm\infty$ in general.

On the other hand, $\frac{\partial f_S}{\partial q^j}(p) = df_S(p) \cdot \vec{e}_j(p) := \lim_{h \rightarrow 0} \frac{(f_S \circ c_p^{(j)})(h) - (f_S \circ c_p^{(j)})(0)}{h} = \lim_{h \rightarrow 0} \frac{(f_U(\vec{q}+h\vec{A}_j) - f_U(\vec{q}))}{h}$, cf. (1.32) (the curve $c^{(j)}$ is in S). \blacksquare

Exercise 1.17 Matrix calculations: Let $\Phi(\vec{q}) = \sum_{i=1}^n \Phi^i(\vec{q}) \vec{E}_i$ and $[d\Phi(\vec{q})]_{|\vec{A}, \vec{E}} = \left[\frac{\partial \Phi^i}{\partial q^j}(\vec{q})\right]_{\substack{i=1, \dots, n \\ j=1, \dots, m}}$. Let $f_U = f_S \circ \Phi$. Prove:

$$[df_U(\vec{q})]_{|\vec{A}} = [df_S(p)]_{|\vec{E}} \cdot [d\Phi(\vec{q})]_{|\vec{A}, \vec{E}} = [df_S(p)]_{|\vec{e}}. \quad (1.33)$$

Answer. We have $\frac{\partial f_U}{\partial q^j}(\vec{q}) = df_U(\vec{q}) \cdot \vec{A}_j$ for $j = 1, \dots, m$ (usual Cartesian notation). And $\frac{\partial f_S}{\partial x^j}(p) = df_S(p) \cdot \vec{E}_j$ for $j = 1, \dots, n$ (usual Cartesian notation). And $\frac{\partial f_S}{\partial q^j}(p) = df_S(p) \cdot \vec{e}_j(p)$ for $j = 1, \dots, m$, cf. (1.29). So:

$$df_U(\vec{q}) = \sum_{j=1}^m \frac{\partial f_U}{\partial q^j}(\vec{q}) dq^j, \quad \text{and} \quad df_S(p) = \sum_{i=1}^n \frac{\partial f_S}{\partial x^i}(p) dx^i = \sum_{j=1}^m \frac{\partial f_S}{\partial q^j}(p) dq^j(p). \quad (1.34)$$

Consider the matrices $[df_U(\vec{q})]_{|\vec{A}} = \left[\frac{\partial f_U}{\partial q^j}(\vec{q})\right]_{j=1, \dots, m}$, $[df_S(p)]_{|\vec{E}} = \left[\frac{\partial f_S}{\partial x^i}(p)\right]_{i=1, \dots, n}$, and $[df_S(p)]_{|\vec{e}} = \left[\frac{\partial f_S}{\partial q^j}(p)\right]_{j=1, \dots, m} = [df_U(\vec{q})]_{|\vec{A}}$, cf. (1.29) (line matrices for linear forms). $f_U = f_S \circ \Phi$ gives $df_U(\vec{q}) \cdot \vec{A}_j = df_S(p) \cdot d\Phi(\vec{q}) \cdot \vec{A}_j$, that is, $\frac{\partial f_U}{\partial q^j}(\vec{q}) = df_S(p) \cdot (\sum_{i=1}^n \frac{\partial \Phi^i}{\partial q^j}(\vec{q}) \vec{E}_i) = \sum_{i=1}^n \frac{\partial \Phi^i}{\partial q^j}(\vec{q}) df_S(p) \cdot \vec{E}_i = \sum_{i=1}^n \frac{\partial \Phi^i}{\partial q^j}(\vec{q}) \frac{\partial f_S}{\partial x^i}(p)$, for all $j = 1, \dots, m$, that is, $[df_U(\vec{q})]_{|\vec{A}} = [df_S(p)]_{|\vec{E}} \cdot [d\Phi(\vec{q})]_{|\vec{A}, \vec{E}} = [df_S(p)]_{|\vec{e}}$. \blacksquare

1.10 Notation $\frac{\partial \vec{w}}{\partial q^j}(p)$

Let $\vec{w}_S : p \in S \rightarrow f_S(p) \in \mathbb{R}^n$ be a C^1 function (defined in S), and consider the function $\vec{w}_U : \vec{q} \in U \rightarrow f_U(\vec{q}) \in \mathbb{R}^n$ defined (in U) by

$$\vec{w}_U := \vec{w}_S \circ \Phi, \quad \text{i.e.} \quad \vec{w}_U(\vec{q}) = \vec{w}_S(p) \quad \text{when} \quad p = \Phi(\vec{q}). \quad (1.35)$$

So $\vec{w}_U(\vec{q}) = \vec{w}_S(\Phi(\vec{q}))$, thus $d\vec{w}_U(\vec{q}) \cdot \vec{A}_j = d\vec{w}_S(\Phi(\vec{q})) \cdot d\Phi(\vec{q}) \cdot \vec{A}_j$, that is,

$$d\vec{w}_S(p) \cdot \vec{e}_j(p) = d\vec{w}_U(\vec{q}) \cdot \vec{A}_j \quad \text{when} \quad p = \Phi(\vec{q}). \quad (1.36)$$

Definition (notation):

$$\frac{\partial \vec{w}_S}{\partial q^j}(p) := \frac{\partial \vec{w}_U}{\partial q^j}(\vec{q}), \quad \text{i.e.} \quad \frac{\partial \vec{w}_S}{\partial q^j}(p) := d\vec{w}_S(p) \cdot \vec{e}_j(p) \quad (= \frac{\partial(\vec{w}_S \circ \Phi)}{\partial q^j}(p) = d\vec{w}_U(\vec{q}) \cdot \vec{A}_j). \quad (1.37)$$

NB: \vec{w}_S is a function of p , not a function of \vec{q} , thus the notation $\frac{\partial \vec{w}_S}{\partial q^j}$ would be meaningless without the definition (notation) (1.29).

Thus, with $p = \Phi(\vec{q})$ and $c_p^{(j)}(h) = \Phi(\vec{q} + h\vec{A}_j)$,

$$\begin{aligned} \frac{\partial \vec{w}_S}{\partial q^j}(p) &:= \frac{\partial \vec{w}_U}{\partial q^j}(\vec{q}) = \lim_{h \rightarrow 0} \frac{(\vec{w}_U(\vec{q} + h\vec{A}_j) - \vec{w}_U(\vec{q}))}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\vec{w}_S \circ c_p^{(j)})(h) - (\vec{w}_S \circ c_p^{(j)})(0)}{h} = d\vec{w}_S(p) \cdot \vec{e}_j(p) = (\vec{w}_S \circ c_p^{(j)})'(0). \end{aligned} \quad (1.38)$$

Let $\vec{w}_U(p) = \sum_{i=1}^n w_U^i(\vec{q}) \vec{A}_i$ and $\vec{w}_S(p) = \sum_{i=1}^n w_S^i(p) \vec{E}_i$. Then

$$d\vec{w}_S(p) = \sum_{i=1}^n \sum_{j=1}^m \frac{\partial w_S^i}{\partial q^j}(p) \vec{E}_i \otimes e^j(p), \quad [d\vec{w}_S(p)]_{|\vec{E}, \vec{e}(p)} = [\frac{\partial w_S^i}{\partial q^j}(p)] = [\frac{\partial w_U^i}{\partial q^j}(\vec{q})]. \quad (1.39)$$

Indeed, $d\vec{w}_S(p) = \sum_{i=1}^n \vec{E}_i \otimes dw_S^i(p)$ gives $d\vec{w}_S(p) \cdot \vec{e}_j(p) = (\sum_{i=1}^n \vec{E}_i \otimes dw_S^i(p)) \cdot \vec{e}_j(p) = \sum_{i=1}^n \vec{E}_i (dw_S^i(p) \cdot \vec{e}_j(p)) = \sum_{i=1}^n \vec{E}_i \frac{\partial w_S^i}{\partial q^j}(p)$, which is also the result of $(\sum_{i=1}^n \sum_{j=1}^m \frac{\partial w_S^i}{\partial q^j}(p) \vec{E}_i \otimes e^j(p)) \cdot \vec{e}_j(p)$.

Exercise 1.18 Polar coordinate (1.4): $[\vec{O}\vec{p}]_{|\vec{E}} = [\vec{x}]_{|\vec{E}} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$ and, cf. (1.11),

$$\vec{e}_1(p) = \cos \theta \vec{E}_1 + \sin \theta \vec{E}_2 \stackrel{\text{written}}{=} \vec{a}_1(r, \theta), \quad \text{and} \quad \vec{e}_2(p) = -r \sin \theta \vec{E}_1 + r \cos \theta \vec{E}_2 \stackrel{\text{written}}{=} \vec{a}_2(r, \theta). \quad (1.40)$$

With $\vec{q} = (r, \theta)$ and $p = \Phi(\vec{q}) = \Phi(r, \theta)$, prove:

$$\left[\frac{\partial \vec{e}_1}{\partial r}(p) \right]_{|\vec{E}} = \vec{0}, \quad \left[\frac{\partial \vec{e}_1}{\partial \theta}(p) \right]_{|\vec{E}} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}, \quad \left[\frac{\partial \vec{e}_2}{\partial r}(p) \right]_{|\vec{E}} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}, \quad \left[\frac{\partial \vec{e}_2}{\partial \theta}(p) \right]_{|\vec{E}} = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix}. \quad (1.41)$$

Deduce:

$$[d\vec{e}_1]_{|\vec{E}, \vec{e}(p)} = \begin{pmatrix} 0 & -\sin \theta \\ 0 & \cos \theta \end{pmatrix} = [d\vec{a}_1]_{|\vec{A}, \vec{E}}, \quad [d\vec{e}_2]_{|\vec{E}, \vec{e}(p)} = \begin{pmatrix} -\sin \theta & -r \cos \theta \\ \cos \theta & -r \sin \theta \end{pmatrix} = [d\vec{a}_2]_{|\vec{A}, \vec{E}}. \quad (1.42)$$

Answer. Let $\vec{a}_i(\vec{q}) := (\vec{e}_i \circ \Phi)(\vec{q}) = \vec{e}_i(p)$. Thus $d\vec{a}_i(\vec{q}) \cdot \vec{A}_j = d\vec{e}_i(p) \cdot \vec{e}_j(p)$.

$d\vec{a}_1(r, \theta) \cdot \vec{A}_1 = \frac{\partial \vec{a}_1}{\partial r}(r, \theta) = \vec{0}$ and $d\vec{a}_1(r, \theta) \cdot \vec{A}_2 = \frac{\partial \vec{a}_1}{\partial \theta}(r, \theta) = -\sin \theta \vec{E}_1 + \cos \theta \vec{E}_2$, thus $d\vec{a}_1(r, \theta) = -\sin \theta \vec{E}_1 \otimes \vec{A}_2 + \cos \theta \vec{E}_2 \otimes \vec{A}_2$.

$d\vec{a}_2(r, \theta) \cdot \vec{A}_1 = \frac{\partial \vec{a}_2}{\partial r}(r, \theta) = -\sin \theta \vec{E}_1 + \cos \theta \vec{E}_2$ and $d\vec{a}_2(r, \theta) \cdot \vec{A}_2 = \frac{\partial \vec{a}_2}{\partial \theta}(r, \theta) = -r \cos \theta \vec{E}_1 - r \sin \theta \vec{E}_2$, thus $d\vec{a}_2(r, \theta) = -\sin \theta \vec{E}_1 \otimes \vec{A}_1 + \cos \theta \vec{E}_2 \otimes \vec{A}_1 - r \cos \theta \vec{E}_1 \otimes \vec{A}_2 - r \sin \theta \vec{E}_2 \otimes \vec{A}_2$. \blacksquare

2 Christoffel symbols γ_{ij}^k

2.1 Definition

Let $\Phi : U \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a parametric surface, cf. (1.1), and let $S = \Phi(U)$, cf. (1.2).

Hypothesis (makes the presentation easier): Φ is the restriction to S of a C^2 -diffeomorphism

$$\Phi_+ : \left\{ \begin{array}{l} U_+ \subset \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow S_+ \in \mathbb{R}^n, \\ \vec{q}_+ = (\vec{q}, \vec{z}) \rightarrow p = \Phi(\vec{q}_+) \end{array} \right\} \quad \text{s.t.} \quad \Phi_+(\vec{q}, 0, \dots, 0) = \Phi(\vec{q}), \quad \forall \vec{q} \in U, \quad (2.1)$$

U_+ being an open set in \mathbb{R}^n and $U \subset U_+$; So $S = \Phi_+(\vec{q}, \vec{0})$ (and S_+ is a ‘‘thick surface’’).

Let $(\vec{A}_i)_{i=1, \dots, n}$ be the canonical basis in \mathbb{R}^n ; Thus the basis $(\vec{e}_i(p))_{i=1, \dots, n}$ of the system Φ_+ at $p = \Phi_+(\vec{q})$ is given by, cf. (1.10),

$$\forall j \in [1, n]_{\mathbb{N}}, \quad \vec{e}_j(p) = d\Phi_+(\vec{q}) \cdot \vec{A}_j = \frac{\partial \Phi_+}{\partial q^j}(\vec{q}), \quad \text{and} \quad \forall j \in [1, m]_{\mathbb{N}}, \quad \vec{e}_j(p) \in T_p S. \quad (2.2)$$

Definition 2.1 Let $i, j \in [1, n]_{\mathbb{N}}$. The Christoffel symbols $(\gamma_{ij}^k(p))_{k=1, \dots, n}$ at $p \in S_+$ are the components of the vector $d\vec{e}_j(p) \cdot \vec{e}_i(p) \in \mathbb{R}^n$ relative to the (local) basis $(\vec{e}_i(p))_{i \in [1, n]_{\mathbb{N}}}$, that is,

$$\forall i, j \in [1, n]_{\mathbb{N}}, \quad d\vec{e}_j(p) \cdot \vec{e}_i(p) = \sum_{k=1}^n \gamma_{ij}^k(p) \vec{e}_k(p), \quad \text{i.e.} \quad \gamma_{ij}^k = e^k \cdot (d\vec{e}_j \cdot \vec{e}_i), \quad (2.3)$$

or, for all $j, k \in [1, n]_{\mathbb{N}}$, $d\vec{e}_k(p) \cdot \vec{e}_j(p) = \sum_{i=1}^n \gamma_{jk}^i(p) \vec{e}_i(p)$, i.e. $\gamma_{jk}^i = e^i \cdot (d\vec{e}_k \cdot \vec{e}_j)$. So,

$$\forall k \in [1, n]_{\mathbb{N}}, \quad d\vec{e}_k = \sum_{i, j=1}^n \gamma_{jk}^i \vec{e}_i \otimes e^j, \quad \text{and} \quad [d\vec{e}_k]_{|\vec{e}} = [\gamma_{jk}^i]_{\substack{i=1, \dots, n \\ j=1, \dots, n}}. \quad (2.4)$$

Definition 2.2 Let $i, j \in [1, m]_{\mathbb{N}}$. The Christoffel symbols $(\gamma_{ij}^k(p))_{k=1, \dots, n}$ at $p \in S$ are the components of the vector $d\vec{e}_j(p) \cdot \vec{e}_i(p) \in \mathbb{R}^n$ relative to the basis $(\vec{e}_i(p))_{i \in [1, n]_{\mathbb{N}}}$, that is,

$$\forall i, j \in [1, m]_{\mathbb{N}}, \quad d\vec{e}_j(p) \cdot \vec{e}_i(p) = \sum_{k=1}^n \gamma_{ij}^k(p) \vec{e}_k(p), \quad \text{i.e.} \quad \gamma_{ij}^k = e^k \cdot (d\vec{e}_j \cdot \vec{e}_i), \quad k \in [1, n]_{\mathbb{N}}. \quad (2.5)$$

NB: Although \vec{e}_i and \vec{e}_j are in $T_p S$ in (2.5), the vector $d\vec{e}_j(p) \cdot \vec{e}_i(p)$ is not in $T_p S$ in general: See e.g. (1.42) which gives $d\vec{e}_2(p) \cdot \vec{e}_2(p) = -R\vec{e}_1(p)$ which is not tangent to the circle whereas $\vec{e}_2(p)$ is.

In other words, in S ,

$$d\vec{e}_j = \sum_{k=1}^n \sum_{i=1}^m \gamma_{ij}^k \vec{e}_k \otimes e^i, \quad \text{i.e.} \quad d\vec{e}_k = \sum_{i=1}^n \sum_{j=1}^m \gamma_{jk}^i \vec{e}_i \otimes e^j \quad (2.6)$$

2.2 Symmetry: $d\vec{e}_j \cdot \vec{e}_i = d\vec{e}_i \cdot \vec{e}_j$

$\vec{e}_j(\Phi_+(\vec{q})) = d\Phi_+(\vec{q}) \cdot \vec{A}_j$, cf. (2.2), gives $d\vec{e}_j(p) \cdot d\Phi_+(\vec{q}) \cdot \vec{A}_i = (d^2\Phi_+(\vec{q}) \cdot \vec{A}_i) \cdot \vec{A}_j$ for all i, j , thus,

$$d\vec{e}_j(p) \cdot \vec{e}_i(p) = \frac{\partial^2 \Phi_+}{\partial q^i \partial q^j}(\vec{q}) = \frac{\partial^2 \Phi_+}{\partial q^j \partial q^i}(\vec{q}) = d\vec{e}_i(p) \cdot \vec{e}_j(p) \quad (= \frac{\partial \vec{e}_j}{\partial q^i}(p) = \frac{\partial \vec{e}_i}{\partial q^j}(p)) \quad (2.7)$$

when Φ_+ is supposed C^2 (Schwarz theorem). That is, $\sum_{k=1}^m \gamma_{ij}^k \vec{e}_k = \sum_{k=1}^m \gamma_{ji}^k \vec{e}_k$ for all i, j , so,

$$\forall i, j \in [1, n]_{\mathbb{N}}, \quad \gamma_{ij}^k = \gamma_{ji}^k, \quad (2.8)$$

and the Christoffel symbols are said to be (covariant) symmetric.

Corollary 2.3 Let $f \in C^2(S_+; \mathbb{R})$. Then, for all $i, j = 1, \dots, n$,

$$d(df \cdot \vec{e}_j) \cdot \vec{e}_i = d^2 f(\vec{e}_i, \vec{e}_j) + \sum_{k=1}^n \gamma_{ij}^k df \cdot \vec{e}_k. \quad (2.9)$$

(NB: a first order derivative is still alive.)

Proof.

$$\left(\frac{\partial f}{\partial q^j}(p) =\right) df(p) \cdot \vec{e}_j(p) = df(\Phi(\vec{q})) \cdot \frac{\partial \Phi}{\partial q^j}(\vec{q}) = \frac{\partial (f \circ \Phi)}{\partial q^j}(\vec{q}).$$

Thus,

$$\begin{aligned} \left(\frac{\partial^2 f}{\partial q^i \partial q^j}(p) =\right) d(df(p) \cdot \vec{e}_j(p)) \cdot \vec{e}_i(p) &= \left(d(df)(\Phi(\vec{q})) \cdot \frac{\partial \Phi}{\partial q^i}(\vec{q})\right) \cdot \frac{\partial \Phi}{\partial q^j}(\vec{q}) + df(\Phi(\vec{q})) \cdot \left(\frac{\partial^2 \Phi}{\partial q^i \partial q^j}(\vec{q})\right) \\ &= \left(d^2 f(p) \cdot \vec{e}_i(p)\right) \cdot \vec{e}_j(p) + df(p) \cdot (d\vec{e}_j(p) \cdot \vec{e}_i(p)), \end{aligned} \quad (2.10)$$

and $f \in C^2$, thus $(d^2 f(p) \cdot \vec{e}_i(p)) \cdot \vec{e}_j(p) = (d^2 f(p) \cdot \vec{e}_j(p)) \cdot \vec{e}_i(p) = d^2 f(\vec{e}_i, \vec{e}_j)$. \blacksquare

2.3 Geometric interpretation of $d\vec{e}_i \cdot \vec{e}_j = d\vec{e}_j \cdot \vec{e}_i$

It is the geometric interpretation of the Schwarz theorem that gives $\frac{\partial^2 \Phi_+}{\partial q^i \partial q^j}(\vec{q}) = \frac{\partial^2 \Phi_+}{\partial q^j \partial q^i}(\vec{q})$: Let $p_0 = \Phi(\vec{q}_0) \in S$, and consider the coordinate lines $\vec{c}_{p_0}^{(i)} : t \rightarrow \vec{c}_{p_0}^{(i)}(t) = \Phi(\vec{q}_0 + t\vec{A}_i)$; Then, with $\vec{q} = \vec{q}_0 + t\vec{A}_i$ and $p = \vec{c}_{p_0}^{(i)}(t) = \Phi(\vec{q})$, we get

$$\vec{c}_{p_0}^{(i)'}(t) = d\Phi(\vec{q}) \cdot \vec{A}_i = \vec{e}_i(p). \quad (2.11)$$

then look at figure 2.1: Let $i, j \in [1, m]_{\mathbb{N}}$, $i \neq j$, let $p_0 =$ written $P \in S$, let $h, k > 0$ ("small"), and let

$$P_i = \vec{c}_P^{(i)}(h), \quad P_j = \vec{c}_P^{(j)}(k). \quad (2.12)$$

Then let, see figure 2.1,

$$P_{ij} = \vec{c}_{P_i}^{(j)}(k), \quad P_{ji} = \vec{c}_{P_j}^{(i)}(h). \quad (2.13)$$

Thus to get to P_{ij} , start from P , follow the trajectory $\vec{c}_P^{(i)}$ to the point P_i , then follow the trajectory $\vec{c}_{P_i}^{(j)}$ to the point P_{ij} . And to get to $P_{ji}(p)$, start from P , follow the trajectory $\vec{c}_P^{(j)}$ to the point P_j , then follow the trajectory $\vec{c}_{P_j}^{(i)}$ to the point $P_{ji}(p)$. See example 2.4.

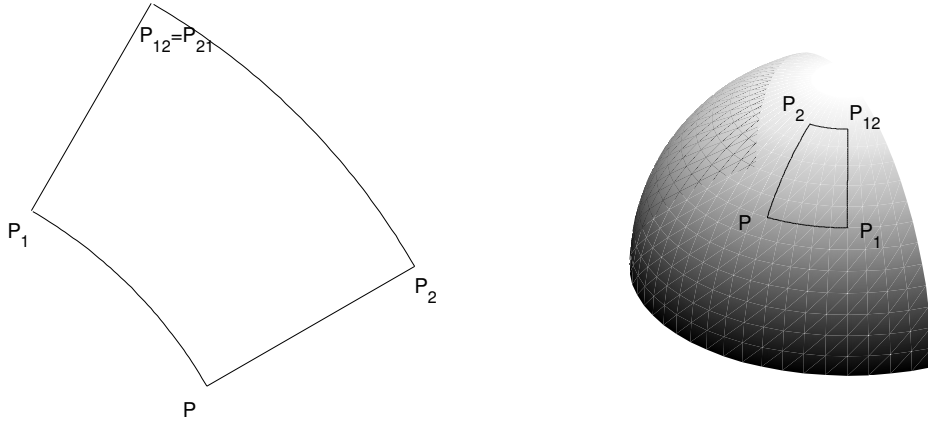


Figure 2.1: With coordinate lines, the curve is closed: $P_{12} = P_{21}$. On the left with polar coordinates, exemple 2.4, on the right with spherical coordinates along parallels and meridians.

Example 2.4 Polar coordinates: $P = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$, $P_1 = \begin{pmatrix} (r+h) \cos \theta \\ (r+h) \sin \theta \end{pmatrix}$, $P_{12} = \begin{pmatrix} (r+h) \cos(\theta+k) \\ (r+h) \sin(\theta+k) \end{pmatrix}$, $P_2 = \begin{pmatrix} r \cos(\theta+k) \\ r \sin(\theta+k) \end{pmatrix}$ and $P_{21} = \begin{pmatrix} (r+h) \cos(\theta+k) \\ (r+h) \sin(\theta+k) \end{pmatrix} = P_{12}$. \blacksquare

Proposition 2.5

$$P_{ij} = P_{ji}. \quad (2.14)$$

This is the interpretation of Schwarz theorem (for C^2 functions):

$$\frac{\partial^2 \Phi}{\partial q^j \partial q^i}(\vec{q}_0) = \frac{\partial^2 \Phi}{\partial q^i \partial q^j}(\vec{q}_0) = \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \frac{P_{ij} - P_j - P_i + P}{hk} = \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \frac{P_{ji} - P_j - P_i + P}{hk}. \quad (2.15)$$

Proof. $P_i = \vec{c}_p^{(i)}(h) = \Phi(\vec{q}_0 + h\vec{A}_i)$ and $P_{ij} = \Phi((\vec{q}_0 + h\vec{A}_i) + k\vec{A}_j)$.

And $P_j = \vec{c}_p^{(j)}(k) = \Phi(\vec{q}_0 + k\vec{A}_j)$ and $P_{ji} = \Phi((\vec{q}_0 + k\vec{A}_j) + h\vec{A}_i)$. Thus $P_{ij} = P_{ji}$.

And $\frac{\partial \Phi}{\partial q^i}(\vec{q}_0) = \lim_{h \rightarrow 0} \frac{\Phi(\vec{q}_0 + h\vec{A}_i) - \Phi(\vec{q}_0)}{h}$, thus,

$$\begin{aligned} \frac{\partial \frac{\partial \Phi}{\partial q^i}}{\partial q^j}(\vec{q}_0) &= \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \frac{\frac{\partial \Phi}{\partial q^i}(\vec{q}_0 + k\vec{A}_j) - \frac{\partial \Phi}{\partial q^i}(\vec{q}_0)}{k} \\ &= \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \frac{\Phi(\vec{q}_0 + k\vec{A}_j + h\vec{A}_i) - \Phi(\vec{q}_0 + k\vec{A}_j) - \Phi(\vec{q}_0 + h\vec{A}_i) + \Phi(\vec{q}_0)}{hk} \\ &= \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \frac{P_{ij} - P_j - P_i + P}{hk}. \end{aligned}$$

Similarly $\frac{\partial \frac{\partial \Phi}{\partial q^j}}{\partial q^i}(\vec{q}_0) = \lim_{\substack{h \rightarrow 0 \\ k \rightarrow 0}} \frac{P_{ji} - P_i - P_j + P}{hk}$, therefore $P_{ij} = P_{ji}$ gives $\frac{\partial \frac{\partial \Phi}{\partial q^i}}{\partial q^j}(\vec{q}_0) = \frac{\partial \frac{\partial \Phi}{\partial q^j}}{\partial q^i}(\vec{q}_0)$. \blacksquare

2.4 Identity $de^i \cdot \vec{e}_j + e^i \cdot d\vec{e}_j = 0$

We have seen that the dual basis ($e^i(p)$) is nothing but $(dq^i(p))$, cf. (1.20)-(1.22), where $\vec{q}(p) := \Phi^{-1}(p)$. Φ^{-1} being C^∞ , $de^i(p) = d^2(\Phi^{-1})(p)$ is well defined.

Proposition 2.6 For all $i, j = 1, \dots, m$, we have

$$de^i \cdot \vec{e}_j + e^i \cdot d\vec{e}_j = 0, \quad \text{i.e.} \quad (de^i \cdot \vec{v}) \cdot \vec{e}_j + e^i \cdot (d\vec{e}_j \cdot \vec{v}) = 0, \quad \forall \vec{v} \in TS. \quad (2.16)$$

Thus de^i is symmetric, that is, $(de^i \cdot \vec{u}) \cdot \vec{w} = (de^i \cdot \vec{w}) \cdot \vec{u}$ for all $\vec{u}, \vec{w} \in \Gamma(S)$, and

$$de^i = d(dq^i) \stackrel{\text{written}}{=} d^2q^i. \quad (2.17)$$

Proof. $e^i(p) \cdot \vec{e}_j(p) = \delta_j^i$ gives $(de^i(p) \cdot \vec{v}_p) \cdot \vec{e}_j(p) + e^i(p) \cdot (d\vec{e}_j(p) \cdot \vec{v}_p) = 0$ for all $\vec{v}_p \in T_pS$, i.e. (2.16). Thus $(de^i \cdot \vec{e}_k) \cdot \vec{e}_j = e^i \cdot (d\vec{e}_j \cdot \vec{e}_k) \stackrel{(2.7)}{=} e^i \cdot (d\vec{e}_k \cdot \vec{e}_j) \stackrel{(2.16)}{=} (de^i \cdot \vec{e}_j) \cdot \vec{e}_k$, thus de^i is symmetric. \blacksquare

2.5 Components of $de^i = d^2q^i$

$e^i(p) \cdot \vec{e}_k(p) = \delta_k^i$ gives $(de^i \cdot \vec{e}_j) \cdot \vec{e}_k + e^i \cdot (d\vec{e}_k \cdot \vec{e}_j) = 0$, thus $(de^i \cdot \vec{e}_j) \cdot \vec{e}_k = -e^i \cdot (\sum_{\ell=1}^n \gamma_{jk}^\ell \vec{e}_\ell) = -\gamma_{jk}^i$, hence

$$de^i \cdot \vec{e}_j = - \sum_{k=1}^n \gamma_{jk}^i e^k. \quad (2.18)$$

So, the components of $de^i \cdot \vec{e}_j$ are γ_{jk}^i .

$$d^2q^i = de^i = - \sum_{j,k=1}^n \gamma_{jk}^i e^j \otimes e^k. \quad (2.19)$$

N.B: for a computation restricted in the surface S , then (2.18) reads with i, j restricted to $i, j = 1, \dots, m$, and (2.19) reads $d^2q^i = de^i = - \sum_{j=1}^m \sum_{k=1}^n \gamma_{jk}^i e^j \otimes e^k$.

Example 2.7 Cartesian basis: \vec{e}_j and e^i are uniform, then γ_{jk}^i and $de^j = 0$. \blacksquare

Exercise 2.8 Polar coordinates: Give d^2r and $d^2\theta$.

Answer. (3.11) gives $d^2r = -\gamma_{11}^1 e^1 \otimes e^1 - \gamma_{21}^1 e^1 \otimes e^2 - \gamma_{12}^1 e^2 \otimes e^1 - \gamma_{22}^1 e^2 \otimes e^2 = re^2 \otimes e^2$.

And $d^2\theta = -\gamma_{11}^2 e^1 \otimes e^1 - \gamma_{21}^2 e^1 \otimes e^2 - \gamma_{12}^2 e^2 \otimes e^1 - \gamma_{22}^2 e^2 \otimes e^2 = -\frac{1}{r}(e^1 \otimes e^2 + e^2 \otimes e^1)$.

Thus

$$d^2r = r d\theta \otimes d\theta \quad \text{and} \quad d^2\theta = -\frac{1}{r}(dr \otimes d\theta + d\theta \otimes dr). \quad (2.20)$$

\blacksquare

2.6 Non holonomic basis

Consider m vector fields $\vec{b}_1, \dots, \vec{b}_m$ in S , that is, such that $\vec{b}_i(p) \in T_p S$ for all $i \in [1, m]_{\mathbb{N}}$ and $p \in S$.

Definition 2.9 Let $p_0 \in S$ and $\vec{q}_0 = \Phi^{-1}(p_0)$. If there exists an open set V_{p_0} in S , an open set $U_{\vec{q}_0}$ in U , and a diffeomorphism $\Psi : \left\{ \begin{array}{l} U_{\vec{q}_0} \rightarrow V_{p_0} \\ \vec{q} \rightarrow p = \Psi(\vec{q}) \end{array} \right\}$ such that

$$\forall i \in [1, m]_{\mathbb{N}}, \forall p \in V_{p_0}, \vec{b}_i(p) = \frac{\partial \Psi}{\partial q^i}(\vec{q}) \quad \text{where} \quad \Psi(\vec{q}) = p, \quad (2.21)$$

then the basis $(\vec{b}_i(p))$ is said to be holonomic in V_{p_0} , associated to the local coordinate system Ψ .

Otherwise, (\vec{b}_i) is not holonomic.

Example 2.10 In particular, the basis $(\vec{e}_i(p))$ of the coordinate system Φ is holonomic: Take $\Psi = \Phi$. E.g., the polar basis (3.3) is holonomic. \blacksquare

Exercise 2.11 (Fundamental). Consider the (widely used) normalized polar basis (\vec{b}_1, \vec{b}_2) given by

$$\vec{b}_1(p) = \frac{\vec{e}_1(p)}{\|\vec{e}_1(p)\|}, \quad \vec{b}_2(p) = \frac{\vec{e}_2(p)}{\|\vec{e}_2(p)\|}, \quad \text{i.e.} \quad [\vec{b}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad [\vec{b}_2(p)]_{|\vec{E}} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}, \quad (2.22)$$

Prove that (\vec{b}_1, \vec{b}_2) is nowhere holonomic: Give two proofs, the first one by proving

$$d\vec{b}_2 \cdot \vec{b}_1 - d\vec{b}_1 \cdot \vec{b}_2 = -\frac{1}{r} \vec{b}_2, \quad \text{so} \quad d\vec{b}_2 \cdot \vec{b}_1 - d\vec{b}_1 \cdot \vec{b}_2 \neq \vec{0}, \quad (2.23)$$

(and use the Schwarz equality), the second one supposing that (\vec{b}_1, \vec{b}_2) is holonomic.

Answer. 1) $\vec{b}_1(p) = \cos \theta \vec{E}_1 + \sin \theta \vec{E}_2$ and $\vec{b}_2(p) = -\sin \theta \vec{E}_1 + \cos \theta \vec{E}_2$ give, with (1.20), $d\vec{b}_1(p) = -\sin \theta \vec{E}_1 \otimes d\theta + \cos \theta \vec{E}_2 \otimes d\theta$ and $d\vec{b}_2(p) = -\cos \theta \vec{E}_1 \otimes d\theta - \sin \theta \vec{E}_2 \otimes d\theta$. Thus, $d\theta \cdot \vec{e}_1 = 0$ and $d\theta \cdot \vec{e}_2 = 1$ give

$$\left\{ \begin{array}{l} d\vec{b}_1 \cdot \vec{b}_2 = d\vec{b}_1 \cdot \frac{\vec{e}_2}{r} = -\frac{1}{r} \sin \theta \vec{E}_1 + \frac{1}{r} \cos \theta \vec{E}_2 = -\frac{1}{r} \vec{b}_2, \\ d\vec{b}_2 \cdot \vec{b}_1 = 0 + 0 = 0, \end{array} \right\}, \quad \text{and} \quad d\vec{b}_2 \cdot \vec{b}_1 - d\vec{b}_1 \cdot \vec{b}_2 = -\frac{1}{r} \vec{b}_2. \quad (2.24)$$

If (\vec{b}_1, \vec{b}_2) is a basis of a coordinate system Ψ , then, at $p = \Psi(\vec{q})$, $\vec{b}_i(p) = \frac{\partial \Psi}{\partial q^i}(\vec{q})$, and the Schwarz equality $\frac{\partial \frac{\partial \Psi}{\partial q^1}}{\partial q^2} = \frac{\partial \frac{\partial \Psi}{\partial q^2}}{\partial q^1}$ gives $d\vec{b}_2 \cdot \vec{b}_1 = d\vec{b}_1 \cdot \vec{b}_2$: But we have $d\vec{b}_2 \cdot \vec{b}_1 \neq d\vec{b}_1 \cdot \vec{b}_2$.

2) Suppose $\exists \Psi : \vec{q} = (q^1, q^2) \in Z \rightarrow p = \Psi(\vec{q})$ diffeomorphism, Z being a non empty open set, s.t. $\vec{b}_i(p) = \frac{\partial \Psi}{\partial q^i}(\vec{q})$. Let $\vec{O}\vec{p} = \Psi^1(\vec{q})\vec{E}_1 + \Psi^2(\vec{q})\vec{E}_2$. Since $\vec{b}_1(p) = \vec{e}_1(p)$ and $\vec{b}_2(p) = \frac{\vec{e}_2(p)}{\|\vec{e}_2(p)\|}$ with $p = \Phi(r, \theta)$ and Φ being a diffeomorphism, then eventually replacing Ψ with $\Psi \circ \Phi$, and consider $\Psi(\vec{q}) = \psi^1(\vec{q})\vec{E}_1 + \psi^2(\vec{q})\vec{E}_2$ to be a function of $\vec{q} = (r, \theta)$. So $\cos \theta \vec{E}_1 + \sin \theta \vec{E}_2 = \vec{b}_1 = \frac{\partial \Psi}{\partial r}(r, \theta) = \frac{\partial \psi^1}{\partial r}(r, \theta)\vec{E}_1 + \frac{\partial \psi^2}{\partial r}(r, \theta)\vec{E}_2$ and $-\sin \theta \vec{E}_1 + \cos \theta \vec{E}_2 = \vec{b}_2 = \frac{\partial \Psi}{\partial \theta}(r, \theta) = \frac{\partial \psi^1}{\partial \theta}(r, \theta)\vec{E}_1 + \frac{\partial \psi^2}{\partial \theta}(r, \theta)\vec{E}_2$. Thus $\frac{\partial \psi^1}{\partial r}(r, \theta) = \cos \theta$ and $\frac{\partial \psi^1}{\partial \theta}(r, \theta) = -\sin \theta$. But $\frac{\partial \psi^1}{\partial \theta}(r, \theta) = -\sin \theta$ gives $\psi^1(r, \theta) = \cos \theta + g(r)$ thus $\frac{\partial \psi^1}{\partial r}(r, \theta) = g'(r)$ with $\frac{\partial \psi^1}{\partial r}(r, \theta) = \cos \theta$, hence $\cos \theta = g'(r)$ for all $(r, \theta) \in Z$: Absurd in any (non empty) open set Z . Thus ψ^1 does not exist, thus Ψ does not exist. \blacksquare

Example 2.12 Consider $R > 0$ and the polar modified system $p = \Psi(r, s) = r \cos(\frac{s}{R})\vec{E}_1 + r \sin(\frac{s}{R})\vec{E}_2$, i.e., $[\Psi(r, s)]_{|\vec{E}} = \begin{pmatrix} r \cos(\frac{s}{R}) \\ r \sin(\frac{s}{R}) \end{pmatrix}$. Thus $[\vec{f}_1]_{|\vec{E}} = [\frac{\partial \Psi}{\partial r}(r, s)]_{|\vec{E}} = \begin{pmatrix} \cos(\frac{s}{R}) \\ \sin(\frac{s}{R}) \end{pmatrix}$, and $[\vec{f}_2]_{|\vec{E}} = [\frac{\partial \Psi}{\partial s}(r, s)]_{|\vec{E}} = \frac{r}{R} \begin{pmatrix} -\sin(\frac{s}{R}) \\ \cos(\frac{s}{R}) \end{pmatrix}$: And (\vec{f}_1, \vec{f}_2) is orthonormal only on the circle $C(\vec{0}, R)$. \blacksquare

3 Examples

3.1 Cartesian coordinate system

Here $U = \mathbb{R}^m$ and $\Phi : U = \mathbb{R}^m \rightarrow S \subset \mathbb{R}^n$ is affine, that is, $\exists L \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ s.t., for all $\vec{q}, \vec{q}_0 \in \mathbb{R}^m$,

$$\Phi(\vec{q}) = \Phi(\vec{q}_0) + L \cdot (\vec{q} - \vec{q}_0). \quad (3.1)$$

So $L = d\Phi(\vec{q}_0) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n)$ is independent of \vec{q}_0 , and $S = \Phi(U)$ is a affine sub-space in \mathbb{R}^n .

And with (\vec{A}_i) the canonical basis in \mathbb{R}^m , and with $p = \Phi(\vec{q})$, we get $d\Phi(\vec{q}).\vec{A}_i = \vec{e}_i(p) = L.\vec{A}_i = \text{written } \vec{e}_i \in \mathbb{R}^n$ independent of p , and (\vec{e}_i) is the basis of the coordinate system. Then the Christoffel, cf. (2.5) vanish: $\gamma_{ij}^k = 0$ for all i, j, k , since $d\vec{e}_i(p) = 0$.

And the coordinate lines through p are the straight line $t \rightarrow c_x^i(t) = p + t\vec{e}_i$ pour $i = 1, \dots, m$.

E.g., $m = n = 2$, O origin and (\vec{E}_1, \vec{E}_2) Cartesian basis in \mathbb{R}^2 (geometric), Φ given by $d\Phi(\vec{q}) = A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ (symmetric) definite positive ($a + d > 0$ and $ad - b^2 > 0$). Thus $p = \Phi(\vec{q})$, $\vec{x} = \vec{O}\vec{p} = x\vec{E}_1 + y\vec{E}_2$, that is, $[\vec{x}]_{|\vec{E}} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $[\vec{x}]_{|\vec{E}} = A \cdot \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, and $[\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} a \\ b \end{pmatrix}$, $[\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} b \\ d \end{pmatrix}$.

3.2 Polar coordinate system

$\mathbb{R}^n = \mathbb{R}^m = \mathbb{R}^2$. And (\vec{A}_i) is the canonical basis in \mathbb{R}^2 (parameter space), and (\vec{E}_i) is a Euclidean basis in \mathbb{R}^2 (geometric space).

3.2.1 The coordinate system

See (1.3): With $\vec{q} = (r, \theta)$,

$$\vec{O}\vec{p} = \vec{O}\Phi(\vec{q}) = r \cos \theta \vec{E}_1 + r \sin \theta \vec{E}_2 = \vec{x}, \quad [\vec{x}]_{|\vec{E}} = \begin{pmatrix} x = r \cos \theta \\ y = r \sin \theta \end{pmatrix}. \quad (3.2)$$

The coordinate lines through p are

$$\begin{cases} c_p^{(1)}(h) := \Phi(r+h, \theta), & \text{i.e. } \vec{O}c_p^{(1)}(h) = (r+h) \cos \theta \vec{E}_1 + (r+h) \sin \theta \vec{E}_2, \quad h > -r, \\ c_p^{(2)}(h) := \Phi(r, \theta+h), & \text{donc } \vec{O}c_p^{(2)}(h) = r \cos(\theta+h) \vec{E}_1 + r \sin(\theta+h) \vec{E}_2, \quad h \in \mathbb{R}. \end{cases}$$

That is,

$$[\vec{O}c_p^{(1)}(h)]_{|\vec{E}} = \begin{pmatrix} (r+h) \cos \theta \\ (r+h) \sin \theta \end{pmatrix}, \quad \text{et} \quad [\vec{O}c_p^{(2)}(h)]_{|\vec{E}} = \begin{pmatrix} r \cos(\theta+h) \\ r \sin(\theta+h) \end{pmatrix}.$$

The basis vectors at $p = \Phi(\vec{q})$ are

$$\begin{cases} \vec{e}_1(p) = d\Phi(\vec{q}).\vec{A}_1 = \frac{\partial \Phi}{\partial r}(\vec{q}) = \cos \theta \vec{E}_1 + \sin \theta \vec{E}_2, & [\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \\ \vec{e}_2(p) = d\Phi(\vec{q}).\vec{A}_2 = \frac{\partial \Phi}{\partial \theta}(\vec{q}) = -r \sin \theta \vec{E}_1 + r \cos \theta \vec{E}_2, & [\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix}. \end{cases} \quad (3.3)$$

With (A^1, A^2) the dual basis of the canonical basis in \mathbb{R}^m , we have

$$d\Phi(\vec{q}) = \cos \theta \vec{E}_1 \otimes A^1 - r \sin \theta \vec{E}_1 \otimes A^2 + \sin \theta \vec{E}_2 \otimes A^1 + r \cos \theta \vec{E}_2 \otimes A^2, \quad (3.4)$$

that is,

$$[d\Phi(\vec{q})]_{|\vec{A}, \vec{E}} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = ([\vec{e}_1(p)]_{|\vec{E}} \quad [\vec{e}_2(p)]_{|\vec{E}}). \quad (3.5)$$

Thus, cf. (1.41),

$$(d\vec{e}_1.\vec{e}_1)(p) = \frac{\partial \vec{e}_1}{\partial r}(p) = \vec{0}, \quad (3.6)$$

$$(d\vec{e}_1.\vec{e}_2)(p) = \frac{\partial \vec{e}_1}{\partial \theta}(p) = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} = \frac{1}{r} \vec{e}_2(p), \quad (3.7)$$

$$(d\vec{e}_2.\vec{e}_1)(p) = \frac{\partial \vec{e}_2}{\partial r}(p) = (d\vec{e}_1.\vec{e}_2)(p) = \frac{1}{r} \vec{e}_2(p), \quad (3.8)$$

$$(d\vec{e}_2.\vec{e}_2)(p) = \frac{\partial \vec{e}_2}{\partial \theta}(p) = \begin{pmatrix} -r \cos \theta \\ -r \sin \theta \end{pmatrix} = -r \vec{e}_1(p). \quad (3.9)$$

So,

$$d\vec{e}_1 = \frac{1}{r} \vec{e}_2 \otimes e^2 \quad \text{and} \quad d\vec{e}_2 = \frac{1}{r} \vec{e}_2 \otimes e^1 - r \vec{e}_1 \otimes e^2, \quad (3.10)$$

and

$$\gamma_{11}^1 = \gamma_{11}^2 = 0, \quad \gamma_{12}^1 = 0 = \gamma_{21}^1, \quad \gamma_{12}^2 = \frac{1}{r} = \gamma_{21}^2, \quad \gamma_{22}^1 = -r, \quad \gamma_{22}^2 = 0, \quad (3.11)$$

and

$$[d\vec{e}_1(p)]_{|\vec{e}(p), \vec{e}(p)} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix}, \quad [d\vec{e}_2(p)]_{|\vec{e}(p), \vec{e}(p)} = \begin{pmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{pmatrix}. \quad (3.12)$$

And (3.10) gives

$$\begin{aligned} d\vec{e}_1 &= -\sin \theta \vec{E}_1 \otimes e^2 + \cos \theta \vec{E}_2 \otimes e^2, \\ d\vec{e}_2(p) &= -\sin \theta \vec{E}_1 \otimes e^1(p) - r \cos \theta \vec{E}_1 \otimes e^2(p) + \cos \theta \vec{E}_2 \otimes e^1(p) - r \sin \theta \vec{E}_e^g \otimes e^2(p). \end{aligned} \quad (3.13)$$

Thus,

$$[d\vec{e}_1(p)]_{|\vec{e}(p), \vec{E}} = \begin{pmatrix} 0 & -\sin \theta \\ 0 & \cos \theta \end{pmatrix}, \quad [d\vec{e}_2(p)]_{|\vec{e}, \vec{E}} = \begin{pmatrix} -\sin \theta & -r \cos \theta \\ \cos \theta & -r \sin \theta \end{pmatrix}. \quad (3.14)$$

Exercise 3.1 Prove:

$$[d\vec{e}_1(p)]_{|\vec{E}, \vec{E}} = \frac{1}{r} \begin{pmatrix} \sin^2 \theta & -\cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos^2 \theta \end{pmatrix}, \quad [d\vec{e}_2(p)]_{|\vec{E}, \vec{E}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.15)$$

(And $d\vec{e}_2(p)$ is a rotation with angle $\frac{\pi}{2}$.)

Answer. The transition matrix from (\vec{E}_i) to $(\vec{e}_i(p))$ is $P = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$, cf. (3.3). Its inverse is $P^{-1} = \frac{1}{r} \begin{pmatrix} r \cos \theta & r \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. Thus, with (dx, dy) the dual basis of (\vec{E}_1, \vec{E}_2) ,

$$\begin{aligned} e^1(p) &= \cos \theta dx + \sin \theta dy, \\ e^2(p) &= -\frac{\sin \theta}{r} dx + \frac{\cos \theta}{r} dy. \end{aligned}$$

Then (3.13) gives

$$\begin{aligned} d\vec{e}_1(p) &= -\sin \theta \vec{E}_1 \otimes \left(-\frac{\sin \theta}{r} dx + \frac{\cos \theta}{r} dy\right) + \cos \theta \vec{E}_2 \otimes \left(-\frac{\sin \theta}{r} dx + \frac{\cos \theta}{r} dy\right), \\ d\vec{e}_2(p) &= -\sin \theta \vec{E}_1 \otimes (\cos \theta dx + \sin \theta dy) - r \cos \theta \vec{E}_1 \otimes \left(-\frac{\sin \theta}{r} dx + \frac{\cos \theta}{r} dy\right) \\ &\quad + \cos \theta \vec{E}_2 \otimes (\cos \theta dx + \sin \theta dy) - r \sin \theta \vec{E}_e^g \otimes \left(-\frac{\sin \theta}{r} dx + \frac{\cos \theta}{r} dy\right) \end{aligned}$$

■

3.2.2 Inverse diffeomorphism

Let the domain of Φ be restricted to, e.g., $U = \mathbb{R}_+^* \times]-\pi, \pi[$. Then Φ is a diffeomorphism, and $\Phi^{-1} : \Omega \rightarrow \mathbb{R}_+^* \times]-\pi, \pi[$ with $\Omega = \mathbb{R}^2 - (\mathbb{R}_- \times \{0\})$.

Notation: $\vec{O}\vec{p} = x\vec{E}_1 + y\vec{E}_2 = \vec{x}$. Then, if $x > 0$, i.e. if $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, then

$$[\Phi^{-1}(x, y)]_{\vec{A}} = \begin{pmatrix} r(x, y) = \sqrt{x^2 + y^2} \\ \theta(x, y) = \arctan \frac{y}{x} \end{pmatrix} \stackrel{\text{written}}{=} [\vec{q}(x, y)]_{\vec{A}}. \quad (3.16)$$

General case, with $\theta \neq \pi$,

$$[\Phi^{-1}(x, y)]_{\vec{A}} = \begin{pmatrix} r(x, y) = \sqrt{x^2 + y^2} \\ \theta(x, y) = 2 \arctan \frac{y}{x + \sqrt{x^2 + y^2}} \end{pmatrix} \stackrel{\text{written}}{=} [\vec{q}]_{\vec{A}}, \quad (3.17)$$

that is, $\Phi^{-1}(x, y) = \sqrt{x^2 + y^2} \vec{A}_1 + 2 \arctan \frac{y}{x + \sqrt{x^2 + y^2}} \vec{A}_2 = \text{written } \vec{q}(x, y)$.

Exercise 3.2 Prove:

$$d\Phi^{-1}(p) = \cos \theta \vec{A}_1 \otimes dx + \sin \theta \vec{A}_1 \otimes dy - \frac{1}{r} \sin \theta \vec{A}_2 \otimes dx + \frac{1}{r} \cos \theta \vec{A}_2 \otimes dy = d\vec{q}(p), \quad (3.18)$$

that is,

$$[d\Phi^{-1}(p)]|_{\vec{E}, \vec{A}} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\frac{1}{r} \sin \theta & \frac{1}{r} \cos \theta \end{pmatrix} \stackrel{\text{written}}{=} [d\vec{q}(p)]|_{\vec{E}, \vec{A}}. \quad (3.19)$$

Answer. With (3.16) to simplify. $\Phi^{-1}(p) = \vec{q}(p) = (r(p), \theta(p)) = r(p)\vec{A}_1 + \theta(p)\vec{A}_2$, gives $d\vec{q}(p) = \vec{A}_1 \otimes dr(p) + \vec{A}_2 \otimes d\theta(p)$. And $\frac{\partial r}{\partial x}(x, y) = \frac{x}{\sqrt{x^2+y^2}} = \cos \theta$, $\frac{\partial r}{\partial y}(x, y) = \frac{y}{\sqrt{x^2+y^2}} = \sin \theta$, $\frac{\partial \theta}{\partial x}(x, y) = -\frac{y}{x^2} \frac{1}{1+\frac{y^2}{x^2}} = -\frac{y}{r^2} = -\frac{\sin \theta}{r}$, $\frac{\partial \theta}{\partial y}(x, y) = \frac{1}{x} \frac{1}{1+\frac{y^2}{x^2}} = \frac{x}{r^2} = \frac{\cos \theta}{r}$, which gives (3.19). \blacksquare

Exercise 3.3 Ellipse coordinate system: let pour $a, b > 0$, $r \geq 0$, $\theta \in \mathbb{R}$, and

$$p = \Phi(r, \theta), \quad [\vec{x}]|_{\vec{E}} = \begin{pmatrix} ar \cos \theta \\ br \sin \theta \end{pmatrix}, \quad (3.20)$$

Find \vec{e}_1 and \vec{e}_2 , and give $d\vec{e}_1$ et $d\vec{e}_2$.

Answer. Let $\vec{q} = (r, \theta)$ and $p = \Phi(\vec{q}) = \Phi(r, \theta)$.

$$[\vec{e}_1(p)]|_{\vec{E}} = \begin{pmatrix} a \cos \theta \\ b \sin \theta \end{pmatrix}, \quad [\vec{e}_2(p)]|_{\vec{E}} = \begin{pmatrix} -ar \sin \theta \\ br \cos \theta \end{pmatrix}, \quad [d\Phi(\vec{q})]|_{\vec{A}, \vec{E}} = \begin{pmatrix} a \cos \theta & -ar \sin \theta \\ b \sin \theta & br \cos \theta \end{pmatrix}. \quad (3.21)$$

Hence,

$$\begin{aligned} [d\vec{e}_1(p) \cdot \vec{e}_1(p)] &= \left[\frac{\partial \vec{e}_1}{\partial r}(p) \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}, \\ [d\vec{e}_1(p) \cdot \vec{e}_2(p)] &= \left[\frac{\partial \vec{e}_1}{\partial \theta}(p) \right] = \begin{pmatrix} -a \sin \theta \\ b \cos \theta \end{pmatrix} = [d\vec{e}_2(p) \cdot \vec{e}_1(p)] = \frac{1}{r} [\vec{e}_2(p)], \\ [d\vec{e}_2(p) \cdot \vec{e}_2(p)] &= \left[\frac{\partial \vec{e}_2}{\partial \theta}(p) \right] = \begin{pmatrix} -ar \cos \theta \\ -br \sin \theta \end{pmatrix} = -r [\vec{e}_1(p)]. \end{aligned} \quad (3.22)$$

So,

$$d\vec{e}_1 = \frac{1}{r} \vec{e}_2 \otimes e^2, \quad d\vec{e}_2 = \frac{1}{r} \vec{e}_2 \otimes e^1 - r \vec{e}_1 \otimes e^2. \quad (3.23)$$

And the Christoffel symbols are those of the polar coordinate system. (NB: If $a \neq b$ then $\vec{e}_1(p) \not\perp \vec{e}_2(p)$, and the dual basis $(dr(p), d\theta(p))$ is not made of orthogonal projections. See next remark.) And the Jacobian matrices are

$$[d\vec{e}_1(p)]|_{\vec{e}, \vec{e}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix}, \quad [d\vec{e}_2(p)]|_{\vec{e}, \vec{e}} = \begin{pmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{pmatrix}. \quad (3.24)$$

\blacksquare

Remark 3.4 If $m = n = 2$, then (3.23) tells that the Christoffel symbols γ_{ij}^k , with $i, j, k \in \{1, 2\}$, of the elliptic coordinate system (3.20) are those of the polar coordinate system (3.2), cf. (3.10) and (3.23).

For the restriction to the circle and to the ellipse, that is with $\Phi_R(\theta) = \Phi(R, \theta)$, this will be false for the association Riemannian connection: The Riemannian connection in \mathbb{R}^n on a surface (the usual connection) is the Euclidean orthogonal projection on the surface. Here $\vec{e}_1(p) \not\perp \vec{e}_2(p)$ when $a \neq b$, cf. (3.21). And, with E the ellipse curve $\Phi_R(\mathbb{R})$, cf. (3.20), and with $T_p E = \text{Vect}\{\vec{e}_2(p)\}$ the tangent line to the ellipse at p , we have $(\nabla_{\vec{e}_2} \vec{e}_2)(p) = \text{Proj}_{T_p E}(d\vec{e}_2(p) \cdot \vec{e}_2(p)) = \text{Proj}_{T_p E}(-R\vec{e}_1(p))$. So, $(\vec{e}_1(p), \vec{e}_2(p))_{\mathbb{R}^2} = r(b^2 - a^2) \cos \theta \sin \theta$, and $\|\vec{e}_2(p)\| = r(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{\frac{1}{2}}$, give

$$(\nabla_{\vec{e}_2} \vec{e}_2)(p) = \gamma_{22}^2(p) \vec{e}_2(p) \quad \text{with} \quad \gamma_{22}^2(p) = -R \frac{(b^2 - a^2) \cos \theta \sin \theta}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{\frac{1}{2}}} \vec{e}_2(p).$$

(And $\gamma_{22}^2 = 0$ when $a = b$.) \blacksquare

3.3 Cylindrical coordinate system

The cylindrical coordinate system is

$$\Phi : \begin{cases} \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^3 \\ \vec{q} = (r, \theta, z) \longmapsto p = \Phi(r, \theta, z), \quad [\vec{O}p]_{|\vec{E}} = \begin{pmatrix} x(r, \theta, z) = \Phi^1(r, \theta, z) = r \cos \theta \\ y(r, \theta, z) = \Phi^2(r, \theta, z) = r \sin \theta \\ z(r, \theta, z) = \Phi^3(r, \theta, z) = z \end{pmatrix}. \end{cases} \quad (3.25)$$

To get a diffeomorphism we can consider the restriction $\Phi : U \rightarrow \Omega$ where, e.g.,

$$U = \mathbb{R}_+^* \times]-\pi, \pi[\times \mathbb{R} \quad \text{and} \quad \Omega = \mathbb{R}^3 - (\mathbb{R}_- \times \{0\} \times \mathbb{R}). \quad (3.26)$$

Thus the coordinate curves are

$$c_p^{(1)}(h) = \begin{pmatrix} (r+h) \cos \theta \\ (r+h) \sin \theta \\ z \end{pmatrix}, \quad c_p^{(2)}(h) = \begin{pmatrix} r \cos(\theta+h) \\ r \sin(\theta+h) \\ z \end{pmatrix}, \quad c_p^{(3)}(h) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z+h \end{pmatrix}.$$

- 1- $c_p^{(1)}$ is the radius at altitude z at angle θ with \vec{E}_1 ,
- 2- $c_p^{(2)}$ is the cercle at altitude z centered at $(0, 0, z)$ with radius r ,
- 3- $c_p^{(3)}$ is the vertical line through p .

3.4 Spherical coordinate system

Let O be an origin and (\vec{E}_i) be a Euclidean basis in \mathbb{R}^3 . If $p \in \mathbb{R}^3$, let $\vec{x} = \vec{O}p = \sum_i x^i \vec{E}_i$. Let $\Phi : \vec{q} \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \rightarrow p \in \mathbb{R}^3$. There are two usual spherical coordinate systems.

3.4.1 The GPS system

(GPS = Global Positioning System.) O is the center of the Earth, \vec{E}_3 gives the axis of rotation of the Earth, and $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ is a Euclidean basis. Then, with r the distance to O , θ the longitude, and φ the latitude:

$$\text{GPS: } \vec{O}\Phi(r, \theta, \varphi) = [\vec{x}]_{|\vec{E}} = \begin{pmatrix} x = r \cos \theta \cos \varphi \\ y = r \sin \theta \cos \varphi \\ z = r \sin \varphi \end{pmatrix}, \quad (r, \theta, \varphi) \in \mathbb{R}_+ \times [-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]. \quad (3.27)$$

E.g., with r_0 the radius of the Earth and φ_0 a given latitude,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z = r_0 \sin \varphi_0 \end{pmatrix} = r_0 \cos \varphi_0 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad \theta \in [-\pi, \pi],$$

is the parallel at latitude φ_0 ; In particular $\varphi_0 = 0$ gives the equator.

And e.g., with r_0 the radius of the Earth and θ_0 a given longitude,

$$[\vec{x}]_{|\vec{E}} = \begin{pmatrix} x = r_0 \cos \theta_0 \cos \varphi \\ y = r_0 \sin \theta_0 \cos \varphi \\ z = r_0 \sin \varphi \end{pmatrix}, \quad \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}], \quad (3.28)$$

gives the meridian at longitude θ_0 .

Example 3.5 GPS Coordinates at ISIMA: $\theta \simeq 3^\circ 07'E$ and $\varphi \simeq 45^\circ 45'N$ in degrees and minute. With radian: $\theta \simeq 0,054$ and $\varphi \simeq -0,80$. And $r \simeq 6370$ km gives the distance to the center of the Earth, and the altitude is given relative to the (mean) level of the oceans ($\simeq 400$ meter at ISIMA). ■

The basis at $p = O + \vec{x}$ or the GPS system is given with the $\vec{e}_i(p) = \frac{\partial \vec{\Phi}}{\partial q^i}(\vec{q}) = c_p^{(i)'}(0)$, so,

$$[\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos \theta \cos \varphi \\ \sin \theta \cos \varphi \\ \sin \varphi \end{pmatrix}, \quad [\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} -r \sin \theta \cos \varphi \\ r \cos \theta \cos \varphi \\ 0 \end{pmatrix}, \quad [\vec{e}_3(p)]_{|\vec{E}} = \begin{pmatrix} -r \cos \theta \sin \varphi \\ -r \sin \theta \sin \varphi \\ r \cos \varphi \end{pmatrix}. \quad (3.29)$$

Thus, omitting the writing of (p) ,

$$[d\vec{e}_1 \cdot \vec{e}_1]_{|\vec{E}} = \left[\frac{\partial \vec{e}_1}{\partial r} \right]_{|\vec{E}} = 0, \quad (3.30)$$

$$[d\vec{e}_1 \cdot \vec{e}_2]_{|\vec{E}} = \left[\frac{\partial \vec{e}_1}{\partial \theta} \right]_{|\vec{E}} = \begin{pmatrix} -\sin \theta \cos \varphi \\ \cos \theta \cos \varphi \\ 0 \end{pmatrix} = \left[\frac{1}{r} \vec{e}_2 \right]_{|\vec{E}}, \quad (3.31)$$

$$[d\vec{e}_1 \cdot \vec{e}_3]_{|\vec{E}} = \left[\frac{\partial \vec{e}_1}{\partial \varphi} \right]_{|\vec{E}} = \begin{pmatrix} -\cos \theta \sin \varphi \\ -\sin \theta \sin \varphi \\ \cos \varphi \end{pmatrix} = \left[\frac{1}{r} \vec{e}_3 \right]_{|\vec{E}}, \quad (3.32)$$

$$[d\vec{e}_2 \cdot \vec{e}_2]_{|\vec{E}} = \left[\frac{\partial \vec{e}_2}{\partial \theta} \right]_{|\vec{E}} = \begin{pmatrix} -r \cos \theta \cos \varphi \\ -r \sin \theta \cos \varphi \\ 0 \end{pmatrix} - r \cos^2 \varphi [\vec{e}_1]_{|\vec{E}} + \cos \varphi \sin \varphi [\vec{e}_3]_{|\vec{E}}, \quad (3.33)$$

$$[d\vec{e}_2 \cdot \vec{e}_3]_{|\vec{E}} = \left[\frac{\partial \vec{e}_2}{\partial \varphi} \right]_{|\vec{E}} = \begin{pmatrix} r \sin \theta \sin \varphi \\ -r \cos \theta \sin \varphi \\ 0 \end{pmatrix} = [-\tan \varphi \vec{e}_2]_{|\vec{E}}, \quad (3.34)$$

$$[d\vec{e}_3 \cdot \vec{e}_3]_{|\vec{E}} = \left[\frac{\partial \vec{e}_3}{\partial \varphi} \right]_{|\vec{E}} = \begin{pmatrix} -r \cos \theta \cos \varphi \\ -r \sin \theta \cos \varphi \\ -r \sin \varphi \end{pmatrix} = [-r \vec{e}_1]_{|\vec{E}}. \quad (3.35)$$

Thus,

$$\gamma_{11}^1 = 0, \quad \gamma_{11}^2 = 0, \quad \gamma_{11}^3 = 0, \quad (3.36)$$

$$\gamma_{21}^1 = 0 = \gamma_{12}^1, \quad \gamma_{21}^2 = \frac{1}{r} = \gamma_{12}^2, \quad \gamma_{21}^3 = 0 = \gamma_{12}^3, \quad (3.37)$$

$$\gamma_{31}^1 = 0 = \gamma_{13}^1, \quad \gamma_{31}^2 = 0 = \gamma_{13}^2, \quad \gamma_{31}^3 = \frac{1}{r} = \gamma_{13}^3, \quad (3.38)$$

$$\gamma_{22}^1 = -r \cos^2 \varphi, \quad \gamma_{22}^2 = 0, \quad \gamma_{22}^3 = \cos \varphi \sin \varphi, \quad (3.39)$$

$$\gamma_{32}^1 = 0 = \gamma_{23}^1, \quad \gamma_{32}^2 = -\tan \varphi = \gamma_{23}^2, \quad \gamma_{32}^3 = 0 = \gamma_{23}^3, \quad (3.40)$$

$$\gamma_{33}^1 = -r, \quad \gamma_{33}^2 = 0, \quad \gamma_{33}^3 = 0. \quad (3.41)$$

Therefore, the non vanishing Christoffel with the index 1 are

$$\gamma_{12}^2 = \gamma_{21}^2 = \frac{1}{r} = \gamma_{13}^3 = \gamma_{31}^3, \quad \gamma_{22}^1 = -r \cos^2 \varphi, \quad \gamma_{33}^1 = -r, \quad (3.42)$$

and without the index 1 (so corresponding to spherical surface with r_0 fixed)

$$\gamma_{22}^3 = \cos \varphi \sin \varphi, \quad \gamma_{23}^2 = \gamma_{32}^2 = -\tan \varphi. \quad (3.43)$$

Exercise 3.6 Express the Euclidean dot product $g(\cdot, \cdot) = (\cdot, \cdot)_{\mathbb{R}^3}$ in the GPS basis.

Answer. We have to compute $g_{ij} = g(\vec{e}_i(p), \vec{e}_j(p)) = (\vec{e}_i(p), \vec{e}_j(p))_{\mathbb{R}^3}$. The spherical basis being orthogonal, we get $g_{ij} = 0$ if $i \neq j$, and $g_{11}(p) = \|\vec{e}_1(p)\|^2 = 1$, $g_{22}(p) = \|\vec{e}_2(p)\|^2 = r^2 \cos^2 \varphi$, and $g_{33}(p) = \|\vec{e}_3(p)\|^2 = r^2$.

$$\text{Thus } [g(p)]_{|\vec{E}} = [g_{ij}(p)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \cos^2 \varphi & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \quad \blacksquare$$

Exercise 3.7 Let $a, b, c > 0$. Ellipsoid $\vec{x} = \vec{Op} = \overrightarrow{O\Phi(r, \theta, \varphi)}$ with $[\vec{x}]_{|\vec{E}} = \begin{pmatrix} x = ar \cos \theta \cos \varphi \\ y = br \sin \theta \cos \varphi \\ z = cr \sin \varphi \end{pmatrix}$, $(r, \theta, \varphi) \in \mathbb{R}_+^* \times]-\pi, \pi[\times]-\frac{\pi}{2}, \frac{\pi}{2}[$. Prove that the Christoffel symbols are still given by (3.42)-(3.43). Express the Euclidean dot product $g(\cdot, \cdot) = (\cdot, \cdot)_{\mathbb{R}^3}$ in the corresponding basis, in particular if $a = b$.

Answer. $\vec{e}_1(p) = \begin{pmatrix} a \cos \theta \cos \varphi \\ b \sin \theta \cos \varphi \\ c \sin \varphi \end{pmatrix}$, $\vec{e}_2(p) = \begin{pmatrix} -ar \sin \theta \cos \varphi \\ br \cos \theta \cos \varphi \\ 0 \end{pmatrix}$, $\vec{e}_3(p) = \begin{pmatrix} -ar \cos \theta \sin \varphi \\ -br \sin \theta \sin \varphi \\ cr \cos \varphi \end{pmatrix}$. (The

basis $(\vec{e}_i(p))$ isn't orthogonal in general). Then $\frac{\partial \vec{e}_1}{\partial r}(p) = \vec{0}$, $\frac{\partial \vec{e}_1}{\partial \theta}(p) = \begin{pmatrix} -a \sin \theta \cos \varphi \\ b \cos \theta \cos \varphi \\ 0 \end{pmatrix} = \frac{\vec{e}_2(p)}{r}$,

$\frac{\partial \vec{e}_1}{\partial \varphi}(p) = \begin{pmatrix} -a \cos \theta \sin \varphi \\ -b \sin \theta \sin \varphi \\ c \cos \varphi \end{pmatrix} = \frac{\vec{e}_3(p)}{r}$, $\frac{\partial \vec{e}_2}{\partial r}(p) = \begin{pmatrix} -a \sin \theta \cos \varphi \\ b \cos \theta \cos \varphi \\ 0 \end{pmatrix} = \frac{\vec{e}_2(p)}{r}$, $\frac{\partial \vec{e}_2}{\partial \theta}(p) = \begin{pmatrix} -ar \cos \theta \cos \varphi \\ -br \sin \theta \cos \varphi \\ 0 \end{pmatrix} =$

$-r \cos^2 \varphi \vec{e}_1(p) + \cos \varphi \sin \varphi \vec{e}_3(p)$, $\frac{\partial \vec{e}_2}{\partial \varphi}(p) = \begin{pmatrix} ar \sin \theta \sin \varphi \\ -br \cos \theta \sin \varphi \\ 0 \end{pmatrix} = -\tan \varphi \vec{e}_2(p)$, $\frac{\partial \vec{e}_3}{\partial r}(p) = \begin{pmatrix} -a \cos \theta \sin \varphi \\ -b \sin \theta \sin \varphi \\ c \cos \varphi \end{pmatrix} =$

$\frac{\vec{e}_3(p)}{r}$, $\frac{\partial \vec{e}_3}{\partial \theta}(p) = \begin{pmatrix} ar \sin \theta \sin \varphi \\ -br \cos \theta \sin \varphi \\ 0 \end{pmatrix} = -\tan \varphi \vec{e}_2(p)$, $\frac{\partial \vec{e}_3}{\partial \varphi}(p) = \begin{pmatrix} -ar \cos \theta \cos \varphi \\ -br \sin \theta \cos \varphi \\ -cr \sin \varphi \end{pmatrix} = -r \vec{e}_1(p)$. Therefore the

Christoffel symbols.

Then $g_{11}(p) = \|\vec{e}_1(p)\|^2 = 1$, $g_{22}(p) = \|\vec{e}_2(p)\|^2 = r^2 \cos^2 \varphi$, et $g_{33}(p) = \|\vec{e}_3(p)\|^2 = r^2$.

$g_{12}(p) = (\vec{e}_1(p), \vec{e}_2(p))_{\mathbb{R}^3} = r \cos^2 \varphi \cos \theta \sin \theta (-a^2 + b^2)$

$g_{13}(p) = (\vec{e}_1(p), \vec{e}_3(p))_{\mathbb{R}^3} = r \cos \varphi \sin \varphi (c^2 - a^2 \cos^2 \theta - b^2 \sin^2 \theta)$

$g_{23}(p) = (\vec{e}_2(p), \vec{e}_3(p))_{\mathbb{R}^3} = r^2 (a^2 - b^2) \cos \theta \sin \theta \cos \varphi \sin \varphi$.

In particular, $a = b$ gives $[g(p)]_{|\vec{E}} = \begin{pmatrix} 1 & 0 & r \cos \varphi \sin \varphi (c^2 - a^2) \\ 0 & r^2 \cos^2 \varphi & 0 \\ r \cos \varphi \sin \varphi (c^2 - a^2) & 0 & r^2 \end{pmatrix}$. \blacksquare

3.4.2 GPS system on Earth

We have

$$\Phi_R : \begin{cases} U \subset \mathbb{R}^2 \longrightarrow S \subset \mathbb{R}^3 \\ \vec{q} = (\theta, \varphi) \longmapsto p = \Phi(\theta, \varphi) = \Phi_R(\theta, \varphi), \quad [\vec{x}] = \begin{pmatrix} \Phi^1(\theta, \varphi) = R \cos \theta \cos \varphi \\ \Phi^2(\theta, \varphi) = R \sin \theta \cos \varphi \\ \Phi^3(\theta, \varphi) = R \sin \varphi \end{pmatrix}. \end{cases} \quad (3.44)$$

And the non vanishing Christoffel symbols are given in (3.42)-(3.43):

$$\gamma_{22}^1 = -R \cos^2 \varphi, \quad \gamma_{33}^1 = -R, \quad \gamma_{22}^3 = \cos \varphi \sin \varphi, \quad \gamma_{23}^2 = \gamma_{32}^2 = -\tan \varphi. \quad (3.45)$$

NB:

$$d\vec{e}_3(p) \cdot \vec{e}_3(p) = -R \vec{e}_1(p) = \gamma_{33}^1 \vec{e}_1(p) \quad (3.46)$$

is not tangent to the surface, although $\vec{e}_3(p)$ is.

3.4.3 The “classical mechanics” spherical system

Here, instead of the latitude (φ above), we use the colatitude $\frac{\pi}{2} - \varphi$ (so $\psi = 0$ gives the North pole), and the colatitude is named φ , so, $\vec{x} = \vec{Op}$ and

$$\text{Mechanic: } [\vec{x}]_{|\vec{E}} = [\Phi(r, \theta, \varphi)]_{|\vec{E}} = \begin{pmatrix} x = r \cos \theta \sin \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \varphi \end{pmatrix}, \quad r \in \mathbb{R}_+, \quad \theta \in [-\pi, \pi], \quad \varphi \in [0, \pi]. \quad (3.47)$$

Thus

$$\begin{aligned} \vec{e}_1(p) &= \cos \theta \sin \varphi \vec{E}_1 + \sin \theta \sin \varphi \vec{E}_2 + \cos \varphi \vec{E}_3, \\ \vec{e}_2(p) &= -r \sin \theta \sin \varphi \vec{E}_1 + r \cos \theta \sin \varphi \vec{E}_2, \\ \vec{e}_3(p) &= r \cos \theta \cos \varphi \vec{E}_1 + r \sin \theta \cos \varphi \vec{E}_2 - r \sin \varphi \vec{E}_3, \end{aligned} \quad (3.48)$$

$$[\vec{e}_1(p)]_{|\vec{E}} = \begin{pmatrix} \cos \theta \sin \varphi \\ \sin \theta \sin \varphi \\ \cos \varphi \end{pmatrix}, \quad [\vec{e}_2(p)]_{|\vec{E}} = \begin{pmatrix} -r \sin \theta \sin \varphi \\ r \cos \theta \sin \varphi \\ 0 \end{pmatrix}, \quad [\vec{e}_3(p)]_{|\vec{E}} = \begin{pmatrix} r \cos \theta \cos \varphi \\ r \sin \theta \cos \varphi \\ -r \sin \varphi \end{pmatrix}. \quad (3.49)$$

Then

$$\begin{aligned} d\vec{e}_1 \cdot \vec{e}_1 &= \frac{\partial \vec{e}_1}{\partial r} = \vec{0}, & d\vec{e}_1 \cdot \vec{e}_2 &= \frac{\partial \vec{e}_1}{\partial \theta} = \frac{\vec{e}_2}{r}, & d\vec{e}_1 \cdot \vec{e}_3 &= \frac{\partial \vec{e}_1}{\partial \varphi} = \frac{\vec{e}_3}{r}, \\ d\vec{e}_2 \cdot \vec{e}_1 &= \frac{\vec{e}_2}{r}, & d\vec{e}_2 \cdot \vec{e}_2 &= -r \sin^2 \varphi \vec{e}_1 - \cos \varphi \sin \varphi \vec{e}_3, & d\vec{e}_2 \cdot \vec{e}_3 &= \cot \varphi \vec{e}_2, \\ d\vec{e}_3 \cdot \vec{e}_1 &= \frac{\vec{e}_3}{r}, & d\vec{e}_3 \cdot \vec{e}_2 &= \cot \varphi \vec{e}_2, & d\vec{e}_3 \cdot \vec{e}_3 &= -r \vec{e}_1, \end{aligned} \quad (3.50)$$

and the non-vanishing christoffel symbols are

$$\gamma_{12}^2 = \frac{1}{r} = \gamma_{21}^2, \quad \gamma_{13}^3 = \frac{1}{r} = \gamma_{31}^3, \quad \gamma_{22}^1 = -r \sin^2 \varphi, \quad \gamma_{22}^3 = -\cos \varphi \sin \varphi, \quad \gamma_{23}^2 = \cot \varphi = \gamma_{32}^2, \quad ; \gamma_{33}^1 = -r. \quad (3.51)$$

Part II

Derivation operator on $\mathcal{F}(S)$

4 Operator $\tilde{\nabla}_{\vec{v}}f = \tilde{\mathcal{L}}_{\vec{v}}f = df \cdot \vec{v}$

Let $\Phi : U \subset \mathbb{R}^m \rightarrow S$ be a coordinate system in S , cf. (1.1), and $(\vec{e}_i(p))_{i=1,\dots,m}$ be the coordinate basis at $p \in S$, cf. (1.10).

4.1 Derivation operator at a point

Definition 4.1 Let $p_0 \in S$. A \mathbb{R} -linear form $\tilde{\mathcal{L}}_{p_0} \in L(\mathcal{F}(S); \mathbb{R})$ is a derivation in S at p_0 iff for all $f, g \in \mathcal{F}(S)$,

$$\tilde{\mathcal{L}}_{p_0}(fg) = \tilde{\mathcal{L}}_{p_0}(f)g(p_0) + f(p_0)\tilde{\mathcal{L}}_{p_0}(g). \quad (4.1)$$

(= Derivation formula $(fg)' = f'g + fg'$).

Let $1_S : p \in \mathbb{R}^n \rightarrow 1_S(p) \begin{cases} = 1 & \text{if } p \in S \\ = 0 & \text{if } p \notin S \end{cases}$ (indicator function).

Proposition 4.2 If $c \in \mathbb{R}$, if $f = c1_S = \text{written } c$ and if $\tilde{\mathcal{L}}_{p_0}$ is a derivation at $p_0 \in S$, then

$$\tilde{\mathcal{L}}_{p_0}(c) = 0. \quad (4.2)$$

Proof. (4.1) gives $\tilde{\mathcal{L}}_{p_0}(1_S) = \tilde{\mathcal{L}}_{p_0}(1_S 1_S) = \tilde{\mathcal{L}}_{p_0}(1_S)1_S + 1_S \tilde{\mathcal{L}}_{p_0}(1_S) = 2\tilde{\mathcal{L}}_{p_0}(1_S)$, thus $\tilde{\mathcal{L}}_{p_0}(1_S) = 0$. And the linearity of $\tilde{\mathcal{L}}_{p_0}$ gives $\tilde{\mathcal{L}}_{p_0}(c1_S) = c\tilde{\mathcal{L}}_{p_0}(1_S) = 0$. \blacksquare

4.2 Characterization

Proposition 4.3 Let $p_0 \in S$. If $\vec{v}_{p_0} \in T_{p_0}S$, then the map $\tilde{\mathcal{L}}_{p_0} : \mathcal{F} \rightarrow \mathbb{R}$ defined by

$$\tilde{\mathcal{L}}_{p_0}f := df(p_0) \cdot \vec{v}_{p_0} \stackrel{\text{written}}{=} \tilde{\nabla}_{\vec{v}}f(p_0) \stackrel{\text{written}}{=} \vec{v}[f](p_0) \quad (4.3)$$

is a derivation at p_0 on $\mathcal{F}(S)$ (we say that the vector field \vec{v} acts on f as a derivation operator).

Conversely:

A derivation $\tilde{\mathcal{L}}_{p_0} \in L(\mathcal{F}(S); \mathbb{R})$ is a directional derivative,

that is, there exists $\vec{v}_{p_0} \in T_{p_0}S$ s.t. $\tilde{\mathcal{L}}_{p_0}$ is given by (4.3).

Thus, considering the coordinate basis $(\vec{e}_i(p_0))$ in $T_{p_0}S$, a derivation $\tilde{\mathcal{L}}_{p_0}$ is a linear combination of the elementary derivations $\partial_i(p_0) = \frac{\partial}{\partial q^i}(p_0)$ (bidual basis of $(\vec{e}_i(p_0))$): $\exists v_{p_0}^1, \dots, v_{p_0}^m \in \mathbb{R}$ s.t.

$$\tilde{\mathcal{L}}_{p_0}f = \sum_{i=1}^m v_{p_0}^i \frac{\partial f}{\partial q^i}(p_0) \quad (= \sum_{i=1}^m v_{p_0}^i df(p_0) \cdot \vec{e}_i(p_0) = df(p_0) \cdot \vec{v}_{p_0}). \quad (4.4)$$

Proof. \Rightarrow : The $\frac{\partial}{\partial q^i}(p_0)$ are derivations: $\frac{\partial(fg)}{\partial q^i}(p_0) = d(fg)(p_0) \cdot \vec{e}_i(p_0) = df(p_0) \cdot \vec{e}_i(p_0)g(p_0) + f(p_0)dg(p_0) \cdot \vec{e}_i(p_0) = \frac{\partial f}{\partial q^i}(p_0)g(p_0) + f(p_0)\frac{\partial g}{\partial q^i}(p_0)$. Thus $\tilde{\mathcal{L}}_{p_0} : f \rightarrow \tilde{\mathcal{L}}_{p_0}f = df(p_0) \cdot \vec{v}_{p_0} = \sum_{i=1}^m \frac{\partial f}{\partial q^i}(p_0)v_{p_0}^i$ is a derivation (trivial).

\Leftarrow : Converse: Let $f \in \mathcal{F}(S)$ and $f_U := f \circ \Phi \in \mathcal{F}(U)$, that is $f_U(\vec{q}) = f(p)$ when $p = \Phi(\vec{q})$. Let B be an open ball in S , let $p_0, p \in B$, let $\vec{q}_0 = \Phi^{-1}(p_0)$ and $\vec{q} = \Phi^{-1}(p)$. Consider $\alpha : t \rightarrow \alpha(t) = f_U(\vec{q}_0 + t(\vec{q} - \vec{q}_0))$, hence $\alpha'(t) = df_U(\vec{q}_0 + t(\vec{q} - \vec{q}_0)) \cdot (\vec{q} - \vec{q}_0)$. With $\vec{q} - \vec{q}_0 = \sum_{i=1}^m (q^i - q_0^i) \vec{A}_i$ and

$df_U \cdot \vec{A}_i = \frac{\partial f_U}{\partial q^i}$ we get

$$f_U(\vec{q}) - f_U(\vec{q}_0) = \alpha(1) - \alpha(0) = \int_0^1 \alpha'(t) dt = \sum_{i=1}^m (q^i - q_0^i) \int_0^1 \frac{\partial f_U}{\partial q^i}(\vec{q}_0 + t(\vec{q} - \vec{q}_0)) dt.$$

Thus, with $p = \Phi(\vec{q})$ (and $\vec{q} = \Phi^{-1}(p) = \vec{q}(p)$),

$$f(p) - f(p_0) = \sum_{i=1}^m (q^i(p) - q_0^i) g_i(p), \quad \text{where } g_i(p) = \int_0^1 \frac{\partial f_U}{\partial q^i}(\vec{q}_0 + t(\vec{q}(p) - \vec{q}_0)) dt$$

Thus, $\tilde{\mathcal{L}}_p$ being \mathbb{R} -linear, and with (4.2) and (4.1), we get

$$\tilde{\mathcal{L}}_p f - 0 = \sum_{i=1}^m g_i(p) (\tilde{\mathcal{L}}_p q^i - 0) + (q^i(p) - q_0^i) \tilde{\mathcal{L}}_p g_i,$$

hence, at $p = p_0$,

$$\tilde{\mathcal{L}}_{p_0} f = \sum_{i=1}^m g_i(p_0) \tilde{\mathcal{L}}_{p_0} q^i + 0, \quad \text{with } g_i(p_0) = \frac{\partial f_U}{\partial q^i}(\vec{q}_0).$$

Then $v_{p_0}^i = \tilde{\mathcal{L}}_{p_0} q^i = \tilde{\mathcal{L}}_{p_0}(\Phi^{-1})^i$ and $\vec{v}_{p_0} = \sum_{i=1}^m v_{p_0}^i \vec{e}_i(p_0)$ give $\tilde{\mathcal{L}}_{p_0} f = \sum_{i=1}^m v_{p_0}^i \frac{\partial(f \circ \Phi)}{\partial q^i}(\vec{q}_0)$, i.e., $\tilde{\mathcal{L}}_{p_0} f = \sum_{i=1}^m v_{p_0}^i df(p_0) \cdot d\Phi(p_0) \cdot \vec{e}_i = \sum_{i=1}^m df(p_0) \cdot \vec{v}_{p_0}$. \blacksquare

Example 4.4 Counter-example. A second order derivative does not define a derivation: (4.1) is not satisfied; Indeed, $(fg)'' \neq f''g + fg''$ (in general) since $(fg)'' = ((fg)')' = f''g + 2f'g' + fg''$. \blacksquare

4.3 Derivation operator in S

Definition 4.5 A linear map $\mathcal{L} \in L(\mathcal{F}(S); \mathcal{F}(S))$ is a derivation iff, for all $f, g \in \mathcal{F}(S)$,

$$\mathcal{L}(fg) = \mathcal{L}(f)g + f\mathcal{L}(g) \in \mathcal{F}(S), \quad (4.5)$$

that is, for all $p \in S$, $\mathcal{L}(fg)(p) = \mathcal{L}(f)(p)g(p) + f(p)\mathcal{L}(g)(p) \in \mathbb{R}$.

Remark 4.6 With $\mathcal{F}(\mathbb{R})$, the first order derivation $f \rightarrow f'$ is a derivation: $(fg)' = f'g + fg'$. But the second order derivation $f \rightarrow f''$ is not: $(fg)'' \neq f''g + fg''$ (in general) since $(fg)'' = f''g + 2f'g' + fg''$. \blacksquare

Proposition 4.7 If \mathcal{L} is a derivation on $\mathcal{F}(S)$, then, for all $c \in \mathbb{R}$, $\mathcal{L}(c1_S) = 0$.

Proof. (4.5) gives $\mathcal{L}(1_S) = \mathcal{L}(1_S 1_S) = \mathcal{L}(1_S)1 + 1\mathcal{L}(1_S) = 2\mathcal{L}(1_S)$, thus $\mathcal{L}(1_S) = 0$, thus $\mathcal{L}(c1_S) = c\mathcal{L}(1_S) = 0$ (linearity). \blacksquare

4.4 Autonomous Lie derivative $\tilde{\mathcal{L}}_{\vec{v}}^0 f = df \cdot \vec{v} = \tilde{\nabla}_{\vec{v}} f$

Definition 4.8 Let $\vec{v} \in \Gamma(S)$, $f \in \mathcal{F}(S)$. The autonomous Lie derivative of f along \vec{v} is the map $\tilde{\mathcal{L}}_{\vec{v}}^0 : \mathcal{F}(S) \rightarrow \mathcal{F}(S)$ defined by, for all $p \in S$,

$$\tilde{\mathcal{L}}_{\vec{v}}^0(f)(p) = df(p) \cdot \vec{v}(p), \quad \text{and} \quad \tilde{\mathcal{L}}_{\vec{v}}^0(f) \stackrel{\text{written}}{=} \tilde{\nabla}_{\vec{v}}(f) \quad (= df \cdot \vec{v} \stackrel{\text{written}}{=} \vec{v}(f)). \quad (4.6)$$

(So $\tilde{\mathcal{L}}_{\vec{v}}^0(f)(p)$ is the directional derivative of f at p along $\vec{v}(p)$.)

Proposition 4.9 $\tilde{\mathcal{L}}_{\vec{v}}^0$ is a derivation on $\mathcal{F}(S)$.

Conversely: If $\mathcal{L} : \mathcal{F}(S) \rightarrow \mathcal{F}(S)$ is a derivation, then \mathcal{L} is an autonomous Lie derivative, that is, there exists $\vec{v} \in \Gamma(S)$ s.t.

$$\mathcal{L}(f) = \tilde{\mathcal{L}}_{\vec{v}}^0 f \quad (= df \cdot \vec{v} = \tilde{\nabla}_{\vec{v}} f = \vec{v}(f)), \quad (4.7)$$

Proof. $\tilde{\mathcal{L}}_{\vec{v}}^0(fg)(p) = d(fg)(p) \cdot \vec{v}(p) = (df(p) \cdot \vec{v}(p))g(p) + f(p)(dg(p) \cdot \vec{v}(p)) = (\tilde{\mathcal{L}}_{\vec{v}}^0(f)g + f\tilde{\mathcal{L}}_{\vec{v}}^0(g))(p)$.

Converse. Let \mathcal{L} be a derivation on S , and let $p \in S$. Then $\mathcal{L}^p : \mathcal{F}(S) \rightarrow \mathbb{R}$ defined by $\mathcal{L}^p(f) = \mathcal{L}(f)(p)$ is a derivation at p (trivial). Thus there exists $\vec{v}(p)$ s.t. $\mathcal{L}^p = \sum_{i=1}^n v^i(p) \frac{\partial}{\partial q^i}(p)$, cf. proposition 4.3. This defines the map $\vec{v} : S \rightarrow TS$, and (4.3) gives $\mathcal{L}(f)(p) = \sum_{i=1}^n v^i(p) \frac{\partial f}{\partial q^i}(p) = \sum_{i=1}^n v^i(p) df(p) \cdot \vec{e}_i(p) = \tilde{\mathcal{L}}_{\vec{v}}^0(f)(p)$. And $\mathcal{L}(f)$ and $\vec{e}_i C^\infty$, gives $\vec{v} = \sum_i v^i \vec{e}_i C^\infty$: $\vec{v} \in \Gamma(S)$. \blacksquare

Part III

Usual Riemannian connection on vector fields

The usual connection $\nabla_{\vec{v}}$ in S , called the Riemannian connection, is given by $\nabla_{\vec{v}}\vec{w} := \text{Proj}_{TS}(d\vec{w} \cdot \vec{v}) =$ the orthogonal projection of $d\vec{w} \cdot \vec{v}$ on TS . This projection, called the covariant derivative of \vec{w} along \vec{v} , is the only component of $d\vec{w} \cdot \vec{v}$ we have access to, when the “outside” of S is inaccessible.

A general connection is defined s.t. it “looks like” the usual Riemannian connection.

5 Connection $\tilde{\nabla}$ on $\mathcal{F}(S)$

5.1 The classical definition is problematic in S

Let S be a sphere in \mathbb{R}^3 and $f = 1_S$ (surface density vanishing outside S). Let $p \in S$ and $\vec{v}_p \in T_p S$. If we try to use the usual definition

$$df(p) \cdot \vec{v} = \lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{f(p + h\vec{v}) - f(p)}{h} = \lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{1 - 0}{h} = \infty, \quad (5.1)$$

we get an absurd result: If we walk on the Earth surface, then $f = 1_S$ does not vary, so we expect $df = 0$, not ∞ . Thus the classical definition is inadequate.

5.2 Covariant derivative, or autonomous Lie derivative, on $\mathcal{F}(S)$

Let $c : \begin{cases} [a, b] \rightarrow S \subset \mathbb{R}^n \\ t \rightarrow c(t) \end{cases}$ be a regular curve in S . Let $p = c(t) \in \text{Im}c \subset S$ and name

$$\vec{v}(p) = \vec{c}'(t) \quad (= \lim_{h \rightarrow 0} \frac{c(t+h) - c(t)}{h} \in \vec{\mathbb{R}}^n), \quad (5.2)$$

the tangent vector to $\text{Im}c$ at $p = c(t)$.

Definition 5.1 Let $f : S \rightarrow \mathbb{R}$. At $p = c(t)$, the covariant derivative $\nabla_{\vec{v}}f(p)$, also called the autonomous Lie derivative $\mathcal{L}_{\vec{v}}^0f(p)$, is the scalar defined by (if it exists)

$$\begin{aligned} \tilde{\nabla}_{\vec{v}}f(p) = \mathcal{L}_{\vec{v}}^0f(p) &:= \frac{d(f \circ c)}{dt}(t) = \lim_{h \rightarrow 0} \frac{f(c(t+h)) - f(c(t))}{h} \\ &\stackrel{\text{written}}{=} \frac{Df}{dt}(p) = df(p) \cdot \vec{v}(p). \end{aligned} \quad (5.3)$$

(To compare with (5.1).)

And if $\vec{v} \in \Gamma(S)$, then we dispose of its integral curves in S , and the covariant derivative on $\mathcal{F}(S)$ along \vec{v} , also called the autonomous Lie derivative along \vec{v} , is the map, if it exists,

$$\tilde{\nabla}_{\vec{v}} = \mathcal{L}_{\vec{v}}^0 : \begin{cases} \mathcal{F}(S) \rightarrow \mathcal{F}(S), \\ f \mapsto \tilde{\nabla}_{\vec{v}}f = \mathcal{L}_{\vec{v}}^0f := \tilde{\nabla}_{\vec{v}}f = \frac{Df}{dt} = df \cdot \vec{v}. \end{cases} \quad (5.4)$$

Proposition 5.2 If $f, g \in \mathcal{F}(S)$ and $\vec{v}, \vec{w} \in \Gamma(S)$, then

$$\tilde{\nabla}_{f\vec{v}}g = f(\tilde{\nabla}_{\vec{v}}g) \stackrel{\text{written}}{=} f\tilde{\nabla}_{\vec{v}}g, \quad \text{and} \quad \tilde{\nabla}_{\vec{v}+\vec{w}}g = \tilde{\nabla}_{\vec{v}}g + \tilde{\nabla}_{\vec{w}}g, \quad (5.5)$$

and (derivation formula)

$$\tilde{\nabla}_{\vec{v}}(fg) = f(\tilde{\nabla}_{\vec{v}}g) + (\tilde{\nabla}_{\vec{v}}f)g \stackrel{\text{written}}{=} f\tilde{\nabla}_{\vec{v}}g + g\tilde{\nabla}_{\vec{v}}f. \quad (5.6)$$

Proof. Adapt the classical proposition in \mathbb{R}^n , cf. (5.1), to the definition given in (5.3). \blacksquare

5.3 Differential $\tilde{\nabla}f$ on $\mathcal{F}(S)$

Definition 5.3 If $f \in C^1(S; \mathbb{R})$, then, with (5.3), the differential of f in S is the map, if it exists,

$$\tilde{\nabla}f : \begin{cases} \Gamma(S) \rightarrow \mathcal{F}(S), \\ \vec{v} \mapsto \tilde{\nabla}f \cdot \vec{v} := \tilde{\nabla}_{\vec{v}}f \stackrel{\text{written}}{=} df \cdot \vec{v}, \end{cases} \quad (5.7)$$

And $\tilde{\nabla}f \stackrel{\text{written}}{=} df$ (although S is not an open set in \mathbb{R}^n if $m < n$).

5.4 Connection $\tilde{\nabla}$ in S

(5.7) enables to define the connection for scalar functions in S : It is the map

$$\tilde{\nabla} : \begin{cases} \Gamma(S) \times \mathcal{F}(S) \rightarrow \mathcal{F}(S), \\ (\vec{v}, f) \mapsto \tilde{\nabla}(\vec{v}, f) := \tilde{\nabla}_{\vec{v}}f = \tilde{\nabla}f \cdot \vec{v} \quad (= \mathcal{L}_{\vec{v}}^0 f = df \cdot \vec{v}), \end{cases} \quad (5.8)$$

NB: Usually $\tilde{\nabla} \stackrel{\text{written}}{=} \nabla$, the context removing ambiguities. However, we will stick to $\tilde{\nabla}$ to avoid confusions with a connection on vectors fields.

Proposition 5.4 $\tilde{\nabla}$ is $\mathcal{F}(S)$ -linear in the first variable, and is derivation in the second variable.

Proof. See (5.5) and (5.6). ▀

5.5 Quantification of $\tilde{\nabla}_{\vec{v}}f$ and $\tilde{\nabla}f$

(\vec{A}_i) being the canonical basis in \mathbb{R}^m , the coordinate basis ($\vec{e}_i(p)$) at $p = \Phi(\vec{q})$ is given by $\vec{e}_i(p) = d\Phi(\vec{q}) \cdot \vec{A}_i$, and ($e^i(p)$) ($dq^i(p)$) is its dual basis, cf. (1.20). Then the $\frac{\partial f}{\partial q^i}(p)$ are the components of $\tilde{\nabla}f(p)$ relative to the basis ($dq^i(p)$): If $\vec{v} = \sum_{i=1}^m v^i \vec{e}_i$ and $p = \Phi(\vec{q}) \in S$, then

$$(df(p) =) \quad \tilde{\nabla}f(p) = \sum_{i=1}^m \frac{\partial f}{\partial q^i}(p) dq^i(p), \quad \text{and} \quad \tilde{\nabla}_{\vec{v}}f = \sum_{i=1}^m \frac{\partial f}{\partial q^i} v^i. \quad (5.9)$$

Indeed, the i -th coordinate curve at p is $c^{(i)} : t \rightarrow c^{(i)}(t) = \Phi(\vec{q} + t\vec{A}_i)$, and $c^{(i)'}(t) = d\Phi(\vec{q} + t\vec{A}_i) \cdot \vec{A}_i$ gives $c^{(i)'}(0) = d\Phi(\vec{q}) \cdot \vec{A}_i = \vec{e}_i(p)$; Thus $\tilde{\nabla}f(p) \cdot \vec{e}_i(p) = \frac{d(f \circ c^{(i)})}{dt}(t) \Big|_{t=0} = \lim_{h \rightarrow 0} \frac{f(c^{(i)}(t+h)) - f(c^{(i)}(t))}{h} = df(p) \cdot \vec{e}_i(p) = \frac{\partial f}{\partial q^i}(p)$, and $(\sum_{j=1}^m \frac{\partial f}{\partial q^j}(p) dq^j(p)) \cdot \vec{e}_i(p) = \sum_{j=1}^m \frac{\partial f}{\partial q^j}(p) (dq^j(p) \cdot \vec{e}_i(p)) = \sum_{j=1}^m \frac{\partial f}{\partial q^j}(p) \delta_i^j = \frac{\partial f}{\partial q^i}(p)$ as well. And $\tilde{\nabla}f(p) \cdot \sum_{i=1}^m v^i(p) \vec{e}_i(p) = \sum_{i=1}^m v^i(p) \tilde{\nabla}f(p) \cdot \vec{e}_i(p)$, cf. (5.5), gives (5.9).

6 Riemannian connection ∇ in S

6.1 The classical definition is problematic in S

As in paragraph 5.1, if $\vec{w} \in \Gamma(S)$ and is defined to be $\vec{0}$ outside of S , then $\lim_{h \rightarrow 0} \frac{\vec{w}(p+h\vec{v}) - \vec{w}(p)}{h} = \infty$, and the classical definition of the covariant derivative $d\vec{w} \cdot \vec{v} = \lim_{h \rightarrow 0} \frac{\vec{w}(p+h\vec{v}) - \vec{w}(p)}{h}$ is inappropriate. Thus, as for (5.3), we consider an integral curve $c : t \rightarrow c(t)$ of \vec{v} , so $\vec{v}(c(t)) = \vec{c}'(t)$, and we define

$$d\vec{w}(p) \cdot \vec{v}(p) := \frac{d(\vec{w} \circ c)}{dt}(t). \quad (6.1)$$

But a new problem arises: Even if $\vec{v}, \vec{w} \in TS$ (tangent to S), the covariant derivative $d\vec{w} \cdot \vec{v} \notin TS$ (not tangent to S) in general; E.g., (3.9) gives $d\vec{e}_2 \cdot \vec{e}_2 = -r\vec{e}_1$ (the centrifugal force), which is not tangent to the circle, whereas \vec{e}_2 is.

So one more step is required to get a derivation in S for vector fields in $\Gamma(S)$: To get rid of the orthogonal component of $d\vec{w} \cdot \vec{v}$ on S , and only keep the tangent part of $d\vec{w} \cdot \vec{v}$.

6.2 Projections $\text{Proj}_{T_p S}$ and Proj_{TS}

Let $(\cdot, \cdot)_{\mathbb{R}^n}$ be a Euclidean dot product in \mathbb{R}^n . Let $p \in S$. The orthogonal projection in S at p is the map

$$\text{Proj}_{T_p S} : \begin{cases} \mathbb{R}^n & \rightarrow T_p S \subset \mathbb{R}^n \\ \vec{u}_p & \rightarrow \text{Proj}_{T_p S}(\vec{u}_p), \end{cases} \quad (6.2)$$

where the projection $\text{Proj}_{T_p S}(\vec{u}_p)$ is the unique vector in $T_p S$ such that

$$(\text{Proj}_{T_p S}(\vec{u}_p), \vec{v}_p)_{\mathbb{R}^n} = (\vec{u}_p, \vec{v}_p)_{\mathbb{R}^n}, \quad \forall \vec{v}_p \in T_p S. \quad (6.3)$$

(That is, $(\text{Proj}_{T_p S}(\vec{u}_p) - \vec{u}_p, \vec{v}_p)_{\mathbb{R}^n} = 0$ for all $\vec{v}_p \in T_p S$, i.e. $\text{Proj}_{T_p S}(\vec{u}_p) - \vec{u}_p \perp T_p S$.)

This define the operator (the linear map)

$$\text{Proj}_{TS} : \begin{cases} \Gamma(\mathbb{R}^n) & \rightarrow \Gamma(S) \\ \vec{u} & \rightarrow \text{Proj}_{TS}(\vec{u}) \quad \text{s.t.} \quad \text{Proj}_{TS}(\vec{u})(p) := \text{Proj}_{T_p S}(\vec{u}(p)), \quad \forall p \in S. \end{cases} \quad (6.4)$$

6.3 Riemannian covariant derivative $\nabla_{\vec{v}} \vec{w} = \frac{D\vec{w}}{dt}$

Let $c : [a, b] \rightarrow S$ be regular curve in $S \subset \mathbb{R}^n$, let $p = c(t)$ and $\vec{v}(p) = \vec{c}'(t)$, cf. (5.2).

Definition 6.1 If $\vec{w} \in \Gamma(S)$, cf.(2.1), then its Riemannian covariant derivative along \vec{v} at p is

$$\nabla_{\vec{v}} \vec{w}(p) := \text{Proj}_{T_p S} \left(\frac{d(\vec{w} \circ c)}{dt}(t) \right) = \text{Proj}_{T_p S} (d\vec{w}(p) \cdot \vec{v}(p)) \stackrel{\text{written}}{=} \frac{D\vec{w}}{dt}(p), \quad (6.5)$$

And the Riemannian covariant derivative operator on $\Gamma(S)$ along \vec{v} is the (linear) map

$$\nabla_{\vec{v}} : \begin{cases} \Gamma(S) & \rightarrow \Gamma(S), \\ \vec{w} & \mapsto \nabla_{\vec{v}} \vec{w} = \text{Proj}_{TS} (d\vec{w} \cdot \vec{v}) = \frac{D\vec{w}}{dt}. \end{cases} \quad (6.6)$$

6.4 Riemannian differential $\nabla \vec{w}$ on $\Gamma(S)$

If $\vec{v} \in \Gamma(S)$ then we consider its integral curves in S ; Hence, with (6.6),

Definition 6.2 Let $\vec{w} \in \Gamma(S)$; its Riemannian differential at p is the map

$$\nabla \vec{w} : \begin{cases} \Gamma(S) & \rightarrow \Gamma(S), \\ \vec{v} & \mapsto \nabla \vec{w} \cdot \vec{v} := \nabla_{\vec{v}} \vec{w} \quad (= \text{Proj}_{T_p S} (d\vec{w}(p) \cdot \vec{v}(p))). \end{cases} \quad (6.7)$$

6.5 Riemannian connection ∇ in S

Definition 6.3 With (6.6), the Riemannian connection ∇ in S is the map

$$\nabla : \begin{cases} \Gamma(S) \times \Gamma(S) & \rightarrow \Gamma(S), \\ (\vec{v}, \vec{w}) & \mapsto \nabla(\vec{v}, \vec{w}) := \nabla_{\vec{v}} \vec{w} \quad (= \nabla \vec{w} \cdot \vec{v} = \text{Proj}_{T_p S} (d\vec{w}(p) \cdot \vec{v}(p))). \end{cases} \quad (6.8)$$

6.6 Properties

Proposition 6.4 If $f \in \mathcal{F}(S)$ and $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$ then

$$\nabla_{f\vec{v}} \vec{w} = f \nabla_{\vec{v}} \vec{w}, \quad \text{and} \quad \nabla_{\vec{u} + \vec{v}} \vec{w} = \nabla_{\vec{u}} \vec{w} + \nabla_{\vec{v}} \vec{w}, \quad (6.9)$$

and (derivation property)

$$\nabla_{\vec{v}} (f\vec{w}) = f \nabla_{\vec{v}} (\vec{w}) + \tilde{\nabla}_{\vec{v}}(f) \vec{w}. \quad (6.10)$$

Proof. Easy. ▀

Corollary 6.5 The derivation operator $\nabla_{\vec{v}} : \Gamma(S) \rightarrow \Gamma(S)$ is not tensorial.

Proof. It would require $\nabla_{\vec{v}}(f\vec{w}) = f\nabla\vec{w}$, which is wrong if f is not constant, cf. (6.10). \blacksquare

On the other hand

Proposition 6.6 If $\vec{w} \in \Gamma(S)$, the $\nabla\vec{w} : \begin{cases} \Gamma(S) \rightarrow \Gamma(S) \\ \vec{v} \rightarrow \nabla\vec{w}.\vec{v} \end{cases}$ is a tensor.

Proof. Corollary of $\nabla\vec{w}.(f\vec{v}) = f\nabla\vec{w}.\vec{v}$, cf. (6.9). \blacksquare

6.7 Notation $\frac{D^2\vec{w}}{dt^2}$

With $\frac{D\vec{w}}{dt} = \nabla_{\vec{v}}\vec{w} \in \Gamma(S)$, cf. (6.13), then

$$\frac{D\frac{D\vec{w}}{dt}}{dt} \stackrel{\text{written}}{=} \frac{D^2\vec{w}}{dt^2} \quad (= \nabla_{\vec{v}}(\nabla_{\vec{v}}\vec{w})) \quad (6.11)$$

gives the second order variations of \vec{w} along an integral curve of \vec{v} (will give the Riemann tensor).

6.8 Unstationary vector fields and $\frac{D\vec{w}}{Dt}$

Let $\vec{v}, \vec{w} : [t_0, T] \times S \rightarrow \vec{v}(t, p) \in TS$.

Definition 6.7 The material derivative in S is

$$\frac{D\vec{w}}{Dt}(t, p) := \frac{\partial\vec{w}}{\partial t}(t, p) + \nabla_{\vec{v}}\vec{w}(t, p). \quad (6.12)$$

So,

$$\frac{D\vec{w}}{Dt}(t, p) = \frac{\partial\vec{w}}{\partial t}(t, p) + \text{Proj}_{T_p S}(d\vec{w}(t, p).\vec{v}(t, p)). \quad (6.13)$$

In particular if $m = n$, then S is open in \mathbb{R}^n and $\frac{D\vec{w}}{Dt}(t, p) = \frac{\partial\vec{w}}{\partial t}(t, p) + d\vec{w}(t, p).\vec{v}(t, p)$ is the usual material derivative in \mathbb{R}^n .

6.9 Acceleration $\frac{D\vec{v}}{Dt}$

Let \vec{v} be an unstationary vector field and let c be an integral curve, that is, s.t. $\vec{c}'(t) = \vec{v}(t, c(t))$. Then the acceleration in $\Gamma(S)$ is

$$\vec{\gamma}_S(c(t)) = \text{Proj}_{T_p S}(\vec{c}''(t)) = \frac{\partial\vec{v}}{\partial t}(t, p) + \nabla_{\vec{v}}\vec{v}(t, p) = \frac{D\vec{v}}{Dt}(t, p) \quad (= \text{Proj}_{T_p S}(\frac{\partial\vec{v}}{\partial t} + d\vec{v}.\vec{v})(t, p)). \quad (6.14)$$

7 Christoffel symbols

7.1 Definition

Definition 7.1 The Christoffel symbols $(\gamma_{ij}^k(p))_{k=1, \dots, n}$ at $p \in S \subset \mathbb{R}^n$ relative to the connection ∇ , cf. (6.8), are the components of the vector $\nabla_{\vec{e}_i}\vec{e}_j(p) = \text{Proj}_{T_p S}(d\vec{e}_j(p).\vec{e}_i(p)) \in T_p S$ relative to $(\vec{e}_i(p))_{i \in [1, m]_{\mathbb{N}}}$, that is, for $i, j = 1, \dots, m$,

$$(\text{Proj}_{T_p S}(d\vec{e}_j.\vec{e}_i) =) \quad \nabla_{\vec{e}_i}\vec{e}_j = \sum_{k=1}^m \gamma_{ij}^k \vec{e}_k, \quad \text{and} \quad \gamma_{ij}^k = e^k.(d\vec{e}_j.\vec{e}_i), \quad (7.1)$$

i.e., for all $p \in S$, $(\text{Proj}_{T_p S}(d\vec{e}_j(p).\vec{e}_i(p)) =) \quad \nabla_{\vec{e}_i}\vec{e}_j(p) = \sum_{k=1}^m \gamma_{ij}^k(p) \vec{e}_k(p)$, and $\gamma_{ij}^k = e^k.(d\vec{e}_j.\vec{e}_i)$.

(Comparison with definition 2.1: The sum is limited to $k = 1, \dots, m$.)

With $d\vec{e}_j(p).\vec{e}_i(p) = \frac{\partial^2 \Phi}{\partial q^i \partial q^j}(\vec{q}) = d\vec{e}_i(p).\vec{e}_j(p)$ we get $\nabla_{\vec{e}_i}\vec{e}_j = \text{Proj}_{T_p S}(d\vec{e}_j.\vec{e}_i) = \text{Proj}_{T_p S}(d\vec{e}_i.\vec{e}_j) = \nabla_{\vec{e}_j}\vec{e}_i$, so

$$\forall i, j, k \in [1, m]_{\mathbb{N}}, \quad \gamma_{ij}^k = \gamma_{ji}^k. \quad (7.2)$$

Example 7.2 Polar coordinate system, cf. (3.10), $d\vec{e}_2.\vec{e}_2 = -r\vec{e}_1$. And restricted to the circle, $\nabla_{\vec{e}_2}\vec{e}_2 = \text{Proj}_{T_p S}(d\vec{e}_2.\vec{e}_2) = \text{Proj}_{T_p S}(-r\vec{e}_1) = \vec{0}$, thus $\gamma_{22}^2 = 0$.

For spherical coordinates restricted to the sphere, the γ_{ij}^k are given at (3.43). \blacksquare

7.2 Components $w_{|j}^i$ of $\nabla\vec{w}$

Let $\vec{w} \in \Gamma(S)$ and $\vec{w} = \sum_{i=1}^m w^i \vec{e}_i$, that is $\vec{w}(p) = \sum_{i=1}^m w^i(p) \vec{e}_i(p)$ for all $p \in S$; Then, for $j = 1, \dots, m$,

$$\begin{aligned} \nabla_{\vec{e}_j} \vec{w} &= \text{Proj}_{T_p S}(d\vec{w} \cdot \vec{e}_j) = \text{Proj}_{T_p S}\left(\sum_{i=1}^m (dw^i \cdot \vec{e}_j) \vec{e}_i + \sum_{i=1}^m w^i d\vec{e}_i \cdot \vec{e}_j\right) \\ &= \text{Proj}_{T_p S}\left(\sum_{i=1}^m \frac{\partial w^i}{\partial q^j} \vec{e}_i + \sum_{i=1}^m \sum_{k=1}^n w^i \gamma_{ji}^k \vec{e}_k\right). \end{aligned} \quad (7.3)$$

Thus,

$$\nabla_{\vec{e}_j} \vec{w} = \sum_{i=1}^m w_{|j}^i \vec{e}_i \quad \text{with} \quad w_{|j}^i = \frac{\partial w^i}{\partial q^j} + \sum_{k=1}^m \gamma_{jk}^i w^k, \quad \forall i = 1, \dots, m. \quad (7.4)$$

Hence, if $\vec{v} \in \Gamma(S)$ and $\vec{v} = \sum_{j=1}^m v^j \vec{e}_j$, then $\nabla_{\vec{v}} \vec{w} = \sum_{i=1}^m v^j \nabla_{\vec{e}_j} \vec{w}$ gives

$$\nabla_{\vec{v}} \vec{w} = \sum_{i,j=1}^m v^j w_{|j}^i \vec{e}_i = \sum_{i,j=1}^m \left(\frac{\partial w^i}{\partial q^j} + \sum_{k=1}^m \gamma_{jk}^i w^k\right) v^j \vec{e}_i = \nabla \vec{w} \cdot \vec{v}. \quad (7.5)$$

Thus the $w_{|j}^i = (\nabla \vec{w})_{|j}^i$ are the components of the tensor $\nabla \vec{w}$ relative to (\vec{e}_i) :

$$\nabla \vec{w} = \sum_{i,j=1}^m w_{|j}^i \vec{e}_i \otimes e^j, \quad \text{and} \quad [\nabla \vec{w}]_{|\vec{e}} = [w_{|j}^i]_{\substack{i=1,\dots,m \\ j=1,\dots,m}}. \quad (7.6)$$

(So $\nabla \vec{w} \cdot \vec{v} = \sum_{i,j=1}^m w_{|j}^i \vec{e}_i (e^j \cdot \vec{v}) = \sum_{i,j=1}^m w_{|j}^i v^j \vec{e}_i$ with the contraction rules.)

Exercise 7.3 Prove:

$$\nabla_{\vec{v}} \vec{w} - \nabla_{\vec{w}} \vec{v} = \sum_{i,j=1}^m (v^j \frac{\partial w^i}{\partial q^j} - w^j \frac{\partial v^i}{\partial q^j}) \vec{e}_i. \quad (7.7)$$

(The Christoffel symbols “disappear”, and $\mathcal{L}_{\vec{v}} \vec{w} := \nabla_{\vec{v}} \vec{w} - \nabla_{\vec{w}} \vec{v} \in \Gamma(S)$.) ▀

7.3 Divergence in S : $\text{div} \vec{w} = \sum w_{|i}^i = \text{Tr}(\nabla \vec{w})$

Definition 7.4 Let $\vec{w} \in \Gamma(S)$. Its divergence is the trace of its differential $\nabla \vec{w}$, that is, with (7.6),

$$\text{div} \vec{w} = \text{Tr}(\nabla \vec{w}) = \sum_{i=1}^m w_{|i}^i \quad (= \sum_{i=1}^m \frac{\partial w^i}{\partial q^i} + \sum_{i,k=1}^m \gamma_{ik}^i w^k). \quad (7.8)$$

(The divergence is independent of the coordinate basis, since $\nabla \vec{w}$ is a $\binom{1}{1}$ tensor, or apply (8.15).)

Exercise 7.5 Prove:

$$\text{div}(f\vec{w}) = f \text{div} \vec{w} + df \cdot \vec{w}. \quad (7.9)$$

Answer. $f\vec{w} = \sum_i f w^i \vec{e}_i$, thus $(f\vec{w})^i = f w^i$, thus $\sum_{i=1}^m (f w^i)_{|i} = \sum_i f_{|i} w^i + f \sum_i w_{|i}^i$. ▀

8 Change of coordinate system in S

8.1 Change of coordinate basis and transition matrix

Consider two coordinate systems describing (locally) S :

$$\Phi_a : \left\{ \begin{array}{l} U \subset \mathbb{R}^m \rightarrow S \\ \vec{q}_a \rightarrow p = \Phi_a(\vec{q}_a) \end{array} \right\} \quad \text{and} \quad \Phi_b : \left\{ \begin{array}{l} V \subset \mathbb{R}^m \rightarrow S \\ \vec{q}_b \rightarrow p = \Phi_b(\vec{q}_b) \end{array} \right\}. \quad (8.1)$$

(E.g., $\Phi_a = \text{Cartesian}$ and $\Phi_b = \text{polar}$ to describe $S \subset \mathbb{R}^2$.)

Consider $p \in S$, $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$, and the change of parameter diffeomorphism

$$\vec{\psi} := \Phi_a^{-1} \circ \Phi_b : \begin{cases} V \subset \mathbb{R}^m \rightarrow U \subset \mathbb{R}^m, \\ \vec{q}_b \rightarrow \vec{q}_a = \vec{\psi}(\vec{q}_b) := \Phi_a^{-1}(\Phi_b(\vec{q}_b)) \stackrel{\text{written}}{=} \vec{q}_a(\vec{q}_b). \end{cases} \quad (8.2)$$

Thus $\vec{q}_a(\vec{q}_b) = \vec{\psi}(\vec{q}_b) = \Phi_a^{-1}(p)$ when $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$.

With (\vec{A}_i) be the canonical basis in \mathbb{R}^m , the bases of the coordinate systems Φ_a and Φ_b are, at $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$,

$$\vec{e}_{i,a}(p) = d\Phi_a(\vec{q}_a) \cdot \vec{A}_i \quad \text{and} \quad \vec{e}_{i,b}(p) = d\Phi_b(\vec{q}_b) \cdot \vec{A}_i. \quad (8.3)$$

Consider the change of basis endomorphism $\mathcal{P}(p) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ defined by, for all $j \in [1, m]_{\mathbb{N}}$,

$$\vec{e}_{j,b}(p) = \mathcal{P}(p) \cdot \vec{e}_{j,a}(p) = \sum_{i=1}^m P_j^i(p) \vec{e}_{i,a}(p), \quad [\mathcal{P}(p)]|_{\vec{e}_a} = [P_j^i(p)]. \quad (8.4)$$

Proposition 8.1 With $\vec{\psi}(\vec{q}_b) = \sum_{i=1}^m \psi^i(\vec{q}_b) \vec{A}_i = \sum_{i=1}^m q_a^i(\vec{q}_b) \vec{A}_i$, then, at $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$, for all $i, j \in [1, m]_{\mathbb{N}}$,

$$P_j^i(p) = \frac{\partial q_a^i}{\partial q_b^j}(\vec{q}_b), \quad [\mathcal{P}(p)]|_{\vec{e}_a} = [d\vec{q}_a(\vec{q}_b)]|_{\vec{A}} \stackrel{\text{written}}{=} \left[\frac{\partial \vec{q}_a}{\partial \vec{q}_b}(\vec{q}_b) \right]. \quad (8.5)$$

Proof. $p = \Phi_b(\vec{q}_b) = \Phi_a(\vec{\psi}(\vec{q}_b))$ gives $d\Phi_b(\vec{q}_b) \cdot \vec{A}_j = d\Phi_a(\vec{\psi}(\vec{q}_b)) \cdot d\vec{\psi}(\vec{q}_b) \cdot \vec{A}_j$. And $\vec{\psi}(\vec{q}_b) = \sum_{i=1}^m \psi^i(\vec{q}_b) \vec{A}_i$ gives $d\vec{\psi}(\vec{q}_b) \cdot \vec{A}_j = \sum_{i=1}^m (d\psi^i(\vec{q}_b) \cdot \vec{A}_j)(\vec{q}_b) \vec{A}_i = \sum_{i=1}^m \frac{\partial \psi^i}{\partial q_b^j}(\vec{q}_b) \vec{A}_i$. Thus

$$d\Phi_b(\vec{q}_b) \cdot \vec{A}_j = \sum_{i=1}^m \frac{\partial \psi^i}{\partial q_b^j}(\vec{q}_b) d\Phi_a(\vec{q}_a(\vec{q}_b)) \cdot \vec{A}_i, \quad \text{i.e.} \quad \vec{e}_{i,b}(p) = \sum_{j=1}^m \frac{\partial \psi^i}{\partial q_b^j}(\vec{q}_b) \vec{e}_{j,a}(p). \quad (8.6)$$

So, with (8.4), we get $P_j^i(p) = \frac{\partial \psi^i}{\partial q_b^j}(\vec{q}_b)$ for all i, j , and $\psi^i(\vec{q}_b) = q_a^i(\vec{q}_b)$. \blacksquare

Example 8.2 In \mathbb{R}^2 : Φ_a Cartesian: $\left\{ \begin{array}{l} \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ \vec{q}_a = (x, y) \rightarrow p = \Phi_a(\vec{q}_a) = O + x\vec{E}_1 + y\vec{E}_2 \end{array} \right\}$, thus $\vec{e}_{i,a}(p) = \vec{E}_i$. And Φ_b polar: $\left\{ \begin{array}{l} \mathbb{R}_+ \times]-\pi, \pi] \rightarrow \mathbb{R}^2 \\ \vec{q}_b = (r, \theta) \rightarrow p = \Phi_b(\vec{q}_b) = O + r \cos \theta \vec{E}_1 + r \sin \theta \vec{E}_2 \end{array} \right\}$, thus $\vec{e}_{i,b}(p)$ given by (3.3). Thus $\vec{q}_a(\vec{q}_b) = (x = r \cos \theta, y = r \sin \theta)$, and $P = [\frac{\partial \vec{q}_a}{\partial \vec{q}_b}] = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = ([\vec{e}_{1,b}(p)]|_{\vec{E}} \quad [\vec{e}_{2,b}(p)]|_{\vec{E}}) = [P]|_{\vec{e}_a}$ goes from the Cartesian basis (\vec{E}_i) to the polar basis $(\vec{e}_i(p))$ as expected, cf. (3.3). \blacksquare

8.2 Change of basis formula for vectors

Let $\mathcal{Q}(p) = \mathcal{P}(p)^{-1}$. So, cf. (8.5),

$$\vec{e}_{j,a}(p) = \mathcal{Q}(p) \cdot \vec{e}_{j,b}(p) = \sum_{i=1}^m Q_j^i(p) \vec{e}_{i,b}(p), \quad [\mathcal{Q}(p)]|_{\vec{e}_b} = [Q_j^i(p)] = \mathcal{Q}(p) = P(p)^{-1}, \quad (8.7)$$

the last equality since $\vec{e}_{j,a} = \sum_{i=1}^m Q_j^i \sum_{k=1}^m P_i^k \vec{e}_{k,a} = \sum_{k=1}^m (\sum_{i=1}^m P_i^k Q_j^i) \vec{e}_{k,a}$ for all j , thus $(\sum_{i=1}^m P_i^k Q_j^i) = \delta_j^k$ for all j, k , i.e. $P \cdot \mathcal{Q} = I$. And, as for (8.5),

$$\mathcal{Q}(p) = [d\vec{q}_b(\vec{q}_a)]|_{\vec{A}} = [Q_j^i(p)] \stackrel{\text{written}}{=} \left[\frac{\partial \vec{q}_b}{\partial \vec{q}_a}(\vec{q}_a) \right]. \quad (8.8)$$

Proposition 8.3 Let $\vec{v} \in TS$, $\vec{v} = \sum_i v_a^i \vec{e}_{i,a} = \sum_i v_b^i \vec{e}_{i,b}$. Then

$$[\vec{v}]|_{\vec{e}_b} = P^{-1} \cdot [\vec{v}]|_{\vec{e}_a}, \quad \text{i.e.} \quad v_b^i = \sum_{j=1}^m Q_j^i v_a^j, \quad \forall i = 1, \dots, m. \quad (8.9)$$

Proof. (8.5) gives $\sum_{i=1}^m v_a^i \vec{e}_{i,a} = \vec{v} = \sum_{j=1}^m v_b^j \vec{e}_{j,b} = \sum_{j=1}^m v_b^j (\sum_{i=1}^m P_j^i \vec{e}_{i,a}) = \sum_{i=1}^m (\sum_{j=1}^m P_j^i v_b^j) \vec{e}_{i,a}$, thus $v_a^i = \sum_{j=1}^m P_j^i v_b^j$ for all i , i.e., $[\vec{v}]_a = P \cdot [\vec{v}]_b$, thus (8.9). \blacksquare

8.3 Change of dual basis formula

Proposition 8.4 Let $p \in S$. For all $j = 1, \dots, m$, we have

$$e_b^i = \sum_{j=1}^m Q_j^i e_a^j. \quad (8.10)$$

Proof. $(\sum_{\ell=1}^m Q_\ell^i e_a^\ell) \cdot \vec{e}_{j,b} = (\sum_{\ell=1}^m Q_\ell^i e_a^\ell) \cdot (\sum_{k=1}^m P_j^k \vec{e}_{k,a}) = \sum_{\ell,k=1}^m Q_\ell^i P_j^k \delta_k^\ell = \sum_{k=1}^m Q_k^i P_j^k = (P \cdot Q)_j^i = \delta_j^i$, for all $i, j = 1, \dots, m$. Thus $\sum_{\ell=1}^m Q_\ell^i e_a^\ell = e_b^i$ (by definition of the dual basis), i.e. (8.10). \blacksquare

8.4 Change of component formula for linear forms

Let $p \in S$ and $\ell_p \in (T_p S)^* = \mathcal{L}(T_p S; \mathbb{R})$ a linear form at p . Let $\ell = \sum_{i=1}^m \ell_{i,a} e_a^i = \sum_{i=1}^m \ell_{i,b} e_b^i$, so $[\ell_a] = (\ell_{1,a} \ \dots \ \ell_{m,a})$ and $[\ell_b] = (\ell_{1,b} \ \dots \ \ell_{m,b})$ (line matrices since ℓ_p is a linear form).

Proposition 8.5

$$[\ell_b] = [\ell_a] \cdot P, \quad \text{i.e.} \quad \ell_{j,b} = \sum_{i=1}^m \ell_{i,a} P_j^i, \quad j = 1, \dots, m. \quad (8.11)$$

Proof. (8.10) gives $\sum_{j=1}^m \ell_{j,a} e_a^j = \sum_{i=1}^m \ell_{i,b} e_b^i = \sum_{i=1}^m \ell_{i,b} (\sum_{j=1}^m Q_j^i e_a^j) = \sum_{j=1}^m (\sum_{i=1}^m \ell_{i,b} Q_j^i) e_a^j$, for all i, j , thus $\ell_{j,a} = \sum_{i=1}^m \ell_{i,b} Q_j^i$, so $[\ell_a] = [\ell_b] \cdot Q$. \blacksquare

8.5 Change of basis for the γ_{ij}^k

Consider the Riemannian connection $\nabla_{\vec{v}} \vec{w} = \text{Proj}_{TS}(d\vec{w} \cdot \vec{v})$ for all $\vec{v}, \vec{w} \in \Gamma(S)$ (derivation along \vec{v} in S). The Christoffel symbols relative to the coordinate systems Φ_a and Φ_b are, at $p = \Phi_a(\vec{q}_a) = \Phi_b(\vec{q}_b)$, cf. (2.3),

$$\nabla_{\vec{e}_{i,a}} \vec{e}_{j,a} = \sum_{k=1}^m \gamma_{ij,a}^k \vec{e}_{k,a} \quad \text{and} \quad \nabla_{\vec{e}_{i,b}} \vec{e}_{j,b} = \sum_{k=1}^m \gamma_{ij,b}^k \vec{e}_{k,b}. \quad (8.12)$$

Proposition 8.6

$$\gamma_{ij,b}^k = \sum_{\alpha,\beta,\lambda=1}^m P_i^\alpha P_j^\beta Q_\lambda^k \gamma_{\alpha\beta,a}^\lambda + \sum_{\alpha=1}^m \frac{\partial^2 q_a^\alpha}{\partial q_b^i \partial q_b^j} \frac{\partial q_b^k}{\partial q_a^\alpha}, \quad (8.13)$$

(Thus a connection is not a tensor because of the last term.)

Proof. (8.4) and (8.5) give $\vec{e}_{j,b}(p) = \sum_{\alpha=1}^m \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) \vec{e}_{\alpha,a}(p)$, thus, with $p = \Phi_b(\vec{q}_b)$,

$$d\vec{e}_{j,b}(p) \cdot d\Phi_b(\vec{q}_b) \cdot \vec{A}_i = \sum_{\alpha=1}^m (d(\frac{\partial q_a^\alpha}{\partial q_b^j})(\vec{q}_b) \cdot \vec{A}_i) \vec{e}_{\alpha,a}(p) + \sum_{\alpha=1}^m \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) d\vec{e}_{\alpha,a}(p) \cdot d\Phi_b(\vec{q}_b) \cdot \vec{A}_i,$$

i.e. $d\vec{e}_{j,b}(p) \cdot \vec{e}_{i,b}(p) = \sum_{\alpha=1}^m \frac{\partial^2 q_a^\alpha}{\partial q_b^j \partial q_b^i}(\vec{q}_b) \vec{e}_{\alpha,a}(p) + \sum_{\alpha=1}^m \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) d\vec{e}_{\alpha,a}(p) \cdot \vec{e}_{i,b}(p)$, thus $d\vec{e}_{j,b}(p) \cdot \vec{e}_{i,b}(p) = \sum_{\alpha=1}^m \frac{\partial^2 q_a^\alpha}{\partial q_b^j \partial q_b^i}(\vec{q}_b) \vec{e}_{\alpha,a}(p) + \sum_{\alpha,\beta=1}^m \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) \frac{\partial q_a^\beta}{\partial q_b^i}(\vec{q}_b) d\vec{e}_{\alpha,a}(p) \cdot \vec{e}_{\beta,a}(p)$. Hence $\vec{e}_{\alpha,a}(p) = \sum_{k=1}^m \frac{\partial q_b^k}{\partial q_a^\alpha}(\vec{q}_b) \vec{e}_{k,b}(p)$ gives

$$\sum_{k=1}^n \gamma_{ij,b}^k \vec{e}_{k,b} = \sum_{\alpha=1}^m \frac{\partial^2 q_a^\alpha}{\partial q_b^j \partial q_b^i}(\vec{q}_b) \vec{e}_{\alpha,a}(p) + \sum_{\alpha,\beta=1}^m \sum_{\lambda=1}^n \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) \frac{\partial q_a^\beta}{\partial q_b^i}(\vec{q}_b) \gamma_{\alpha\beta,a}^\lambda \vec{e}_{\lambda,a}(p),$$

thus (with projection on TS)

$$\sum_{k=1}^m \gamma_{ij,b}^k \vec{e}_{k,b} = \sum_{\alpha,k=1}^m \frac{\partial^2 q_a^\alpha}{\partial q_b^j \partial q_b^i}(\vec{q}_b) \frac{\partial q_b^k}{\partial q_a^\alpha}(\vec{q}_b) \vec{e}_{k,b}(p) + \sum_{\alpha,\beta,\lambda,k=1}^m \frac{\partial q_a^\alpha}{\partial q_b^j}(\vec{q}_b) \frac{\partial q_a^\beta}{\partial q_b^i}(\vec{q}_b) \gamma_{\alpha\beta,a}^\lambda \frac{\partial q_b^k}{\partial q_a^\lambda}(\vec{q}_b) \vec{e}_{k,b}(p).$$

\blacksquare

8.6 Change of basis formula for $\nabla\vec{w}$

Let $\vec{w} \in \Gamma(S)$, $\vec{w} = \sum_{i=1}^m w_a^i \vec{e}_{i,a} = \sum_{i,j=1}^m w_b^i \vec{e}_{i,b}$, and

$$\nabla\vec{w} = \sum_{i,j=1}^m w_{a|j}^i \vec{e}_{i,a} \otimes e_a^j = \sum_{i,j=1}^m w_{b|j}^i \vec{e}_{i,b} \otimes e_b^j. \quad (8.14)$$

Then (8.4) and (8.10) give $\sum_{i,j=1}^m w_{b|j}^i \vec{e}_{i,b} \otimes e_b^j = \sum_{k,\ell=1}^m w_{a|\ell}^k \vec{e}_{k,a} \otimes e_a^\ell = \sum_{k,\ell,i,j=1}^m w_{a|\ell}^k Q_i^k P_\ell^j \vec{e}_{i,b} \otimes e_b^j$, thus $w_{b|j}^i = \sum_{k,\ell=1}^m Q_i^k w_{a|\ell}^k P_\ell^j$, i.e.

$$[\nabla\vec{w}]_{|\vec{e}_b} = P^{-1} \cdot [\nabla\vec{w}]_{|\vec{e}_a} \cdot P, \quad (8.15)$$

as expected for endomorphisms.

Part IV

Lie autonomous derivative and Lie bracket

For the mechanical interpretation, see manuscript ‘‘Objectivity’’.

9 Lie autonomous derivative

9.1 Second order derivation, and issues

Consider Φ_+ , cf. (2.1). Let $f \in \mathcal{F}(S_+)$ and $\vec{v}, \vec{w} \in \Gamma(S_+)$. Then we have

$$d(df.\vec{w}).\vec{v} = d^2f(\vec{v}, \vec{w}) + df.(d\vec{w}.\vec{v}). \quad (9.1)$$

And the first order term $df.(d\vec{w}.\vec{v})$ remains on the right side (in general). Issue: If $f \in \mathcal{F}(S)$ is extended by $f = 0$ outside S , then $df.(d\vec{w}.\vec{v})$ is meaningless since $d\vec{w}.\vec{v} \notin \Gamma(S)$, even if $\vec{v}|_S, \vec{w}|_S \in \Gamma(S)$:

Example 9.1 Polar coordinates (3.2), $S_+ = \mathbb{R}^2$ and $S = C(\vec{0}, R)$: If $\vec{w} = \vec{v} = \vec{e}_2$, then $d\vec{e}_2(p).\vec{e}_2(p) = -r\vec{e}_1(p)$, cf. (3.9), and the right hand side of (9.1) gives

$$d^2f(\vec{e}_2, \vec{e}_2) + df.d\vec{e}_2.\vec{e}_2 = d^2f(\vec{e}_2, \vec{e}_2) - r df.\vec{e}_1. \quad (9.2)$$

And if $f \in \mathcal{F}(S)$ is extended by $f = 0$ outside S then $df.\vec{e}_1 = \infty$, cf. (5.1); Whereas the left-hand side of (9.1) is meaningful, since $\vec{e}_2 \in \Gamma(S)$ gives $df.\vec{e}_2 = \frac{\partial f}{\partial \theta} \in \mathcal{F}(S)$, cf. (5.3), and then $d(df.\vec{e}_2).\vec{e}_2 = \frac{\partial^2 f}{\partial \theta^2} \in \mathcal{F}(S)$ (meaning $\frac{\partial^2 f_U}{\partial \theta^2}(r, \theta)$ when $f_U = f \circ \Phi$). \blacksquare

So, in a surface S , the right hand side of (9.1) cannot be used to compute $d(df.\vec{e}_2).\vec{e}_2$. In fact, $d^2f(\vec{e}_2, \vec{e}_2)$ is not defined.

9.2 Lie autonomous derivative $\mathcal{L}_{\vec{v}}^0 \vec{w}$ on $\mathcal{F}(S)$

Definition 9.2 Let $\vec{v}, \vec{w} \in \Gamma(S)$. The autonomous Lie derivative $\mathcal{L}_{\vec{v}}^0 \vec{w}$ is the derivative operator

$$\mathcal{L}_{\vec{v}}^0 \vec{w} : \begin{cases} \mathcal{F}(S) & \rightarrow \mathcal{F}(S) \\ f & \rightarrow \mathcal{L}_{\vec{v}}^0 \vec{w}(f) = d(df.\vec{w}).\vec{v} - d(df.\vec{v}).\vec{w}. \end{cases} \quad (9.3)$$

That is, cf. (5.8),

$$\mathcal{L}_{\vec{v}}^0 \vec{w}(f) = \tilde{\nabla}_{\vec{v}}(\tilde{\nabla}_{\vec{w}} f) - \tilde{\nabla}_{\vec{w}}(\tilde{\nabla}_{\vec{v}} f) \stackrel{\text{written}}{=} [\tilde{\nabla}_{\vec{v}}, \tilde{\nabla}_{\vec{w}}](f). \quad (9.4)$$

Proposition 9.3 $\mathcal{L}_{\vec{v}}^0 \vec{w}$ is a derivation on $\mathcal{F}(S)$, i.e., if $f, g \in \mathcal{F}(S)$, then $\mathcal{L}_{\vec{v}}^0 \vec{w}(fg) = \mathcal{L}_{\vec{v}}^0 \vec{w}(f)g + f\mathcal{L}_{\vec{v}}^0 \vec{w}(g)$. Hence, there exists a vector $\vec{z} \in \Gamma(S)$ such that, for all $f \in \mathcal{F}(S)$,

$$\mathcal{L}_{\vec{v}}^0 \vec{w}(f) = \mathcal{L}_{\vec{z}}(f) \quad (= df.\vec{z} = \tilde{\nabla}_{\vec{z}} f). \quad (9.5)$$

Proof. $d(fg).\vec{w} = (df.\vec{w})g + f(dg.\vec{w})$.

Thus $d(d(fg).\vec{w}).\vec{v} = d(df.\vec{w}).\vec{v}g + (df.\vec{v})(dg.\vec{w}) + (df.\vec{w})(dg.\vec{v}) + fd(dg.\vec{w}).\vec{v}$.

And $d(d(fg).\vec{v}).\vec{w} = d(df.\vec{v}).\vec{w}g + (df.\vec{w})(dg.\vec{v}) + (df.\vec{v})(dg.\vec{w}) + fd(dg.\vec{v}).\vec{w}$.

Thus $\mathcal{L}_{\vec{v}}^0\vec{w}(fg) = \left(d(df.\vec{w}).\vec{v} - d(df.\vec{v}).\vec{w}\right)g + f\left(d(dg.\vec{w}).\vec{v} - d(dg.\vec{v}).\vec{w}\right) = g\mathcal{L}_{\vec{v}}^0\vec{w}(f) + f\mathcal{L}_{\vec{v}}^0\vec{w}(g)$.

And (4.7) gives \vec{z} . \blacksquare

Exercise 9.4 Consider the normalized polar basis $(\vec{b}_1(p), \vec{b}_2(p))$ given by $\vec{b}_1(p) = \vec{e}_1(p)$ and $\vec{b}_2(p) = \frac{\vec{e}_2(p)}{r}$, cf. (3.3). Compute $d\vec{b}_2.\vec{b}_1$ and $d\vec{b}_1.\vec{b}_2$ and $\mathcal{L}_{\vec{b}_1}^0\vec{b}_2$.

Answer. (2.24) gives

$$d\vec{b}_2.\vec{b}_1 - d\vec{b}_1.\vec{b}_2 = -\frac{1}{r}\vec{b}_2 = -\frac{1}{r^2}\vec{e}_2, \quad \text{and} \quad \mathcal{L}_{\vec{b}_1}^0\vec{b}_2(f) = -\frac{1}{r^2}df.\vec{e}_2 = -\frac{1}{r^2}\frac{\partial f}{\partial\theta}, \quad (9.6)$$

first order derivative $\frac{\partial}{\partial\theta} : f \rightarrow \frac{\partial f}{\partial\theta} = df.\vec{e}_2$. \blacksquare

Proposition 9.5 For all $\vec{v}, \vec{w} \in \Gamma(S)$ and $\varphi \in \mathcal{F}(S)$,

$$\begin{cases} \mathcal{L}_{\varphi\vec{v}}^0\vec{w} = \varphi\mathcal{L}_{\vec{v}}^0\vec{w} - \mathcal{L}_{\vec{w}}^0\varphi\mathcal{L}_{\vec{v}}^0 & (= \varphi\mathcal{L}_{\vec{v}}^0\vec{w} - (d\varphi.\vec{w})\mathcal{L}_{\vec{v}}^0), \\ \mathcal{L}_{\vec{v}}^0(\varphi\vec{w}) = \varphi\mathcal{L}_{\vec{v}}^0\vec{w} + \mathcal{L}_{\vec{v}}^0\varphi\mathcal{L}_{\vec{w}}^0 & (= \varphi\mathcal{L}_{\vec{v}}^0\vec{w} + (d\varphi.\vec{v})\mathcal{L}_{\vec{w}}^0), \end{cases} \quad (9.7)$$

that is, for all $f \in \mathcal{F}(S)$,

$$\begin{cases} \mathcal{L}_{\varphi\vec{v}}^0\vec{w}(f) = \varphi\mathcal{L}_{\vec{v}}^0\vec{w}(f) - \mathcal{L}_{\vec{w}}^0\varphi\mathcal{L}_{\vec{v}}^0f & (= \varphi df.\vec{z} - (d\varphi.\vec{w})(df.\vec{v})), \\ \mathcal{L}_{\vec{v}}^0(\varphi\vec{w})(f) = \varphi\mathcal{L}_{\vec{v}}^0\vec{w}(f) + \mathcal{L}_{\vec{v}}^0\varphi\mathcal{L}_{\vec{w}}^0f & (= \varphi df.\vec{z} + (d\varphi.\vec{v})(df.\vec{w})), \end{cases} \quad (9.8)$$

Proof.

$\mathcal{L}_{\varphi\vec{v}}^0\vec{w}(f) = \tilde{\nabla}_{\varphi\vec{v}}(\tilde{\nabla}_{\vec{w}}f) - \tilde{\nabla}_{\vec{w}}(\tilde{\nabla}_{\varphi\vec{v}}f) = d(df.\vec{w}).(\varphi\vec{v}) - d(df.(\varphi\vec{v})).\vec{w} = \varphi d(df.\vec{w}).\vec{v} - d(\varphi(df.\vec{w})).\vec{w} = \varphi d(df.\vec{w}).\vec{v} - (d\varphi.\vec{w})(df.\vec{v}) - \varphi d(df.\vec{v}).\vec{w} = \varphi\mathcal{L}_{\vec{v}}^0\vec{w}(f) - \mathcal{L}_{\vec{w}}^0\varphi\mathcal{L}_{\vec{v}}^0f$.

$\mathcal{L}_{\vec{v}}^0(\varphi\vec{w})(f) = \tilde{\nabla}_{\vec{v}}(\tilde{\nabla}_{\varphi\vec{w}}f) - \tilde{\nabla}_{\varphi\vec{w}}(\tilde{\nabla}_{\vec{v}}f) = d(df.(\varphi\vec{w})).\vec{v} - d(df.\vec{v}).(\varphi\vec{w}) = d(\varphi(df.\vec{w})).\vec{v} - \varphi d(df.\vec{v}).\vec{w} = (d\varphi.\vec{v})(df.\vec{w}) + \varphi d(df.\vec{w}).\vec{v} - \varphi d(df.\vec{v}).\vec{w} = (d\varphi.\vec{v})(df.\vec{w}) + \varphi\mathcal{L}_{\vec{v}}^0\vec{w}(f)$. \blacksquare

10 Lie bracket

10.1 Lie bracket $[\vec{v}, \vec{w}]$

$\mathcal{L}_{\vec{v}}^0\vec{w} : \mathcal{F}(S) \rightarrow \mathcal{F}(S)$ being a derivation, cf. prop. 9.3, the vector \vec{z} in (9.5) is denoted

$$\vec{z} = [\vec{v}, \vec{w}] \in \Gamma(S) \quad (10.1)$$

and $[\vec{v}, \vec{w}] \in \Gamma(S)$ is called the Lie bracket of $\vec{v}, \vec{w} \in \Gamma(S)$. So, for all $f \in \mathcal{F}(S)$,

$$\mathcal{L}_{\vec{v}}^0\vec{w}(f) = df.[\vec{v}, \vec{w}] \stackrel{\text{written}}{=} [\vec{v}, \vec{w}](f), \quad (10.2)$$

action of the vector $[\vec{v}, \vec{w}] \simeq \mathcal{L}_{\vec{v}}^0\vec{w}$ on f , where $[\vec{v}, \vec{w}]$ is here considered as a derivation on $\mathcal{F}(S)$ along $[\vec{v}, \vec{w}]$, cf. (4.7).

E.g., in S_+ , with ∇ the usual connection ($\nabla_{\vec{v}}\vec{w} = d\vec{w}.\vec{v}$), (9.1) gives

$$[\vec{v}, \vec{w}] = d\vec{w}.\vec{v} - d\vec{v}.\vec{w} = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}, \quad (10.3)$$

E.g., (9.6) gives $[\vec{b}_1, \vec{b}_2] = -\frac{1}{r}\vec{b}_2$, and $[\vec{b}_1, \vec{b}_2](f) = df.[\vec{b}_1, \vec{b}_2] = -\frac{1}{r^2}\frac{\partial f}{\partial\theta}$.

E.g., in S , with ∇ the usual connection ($\nabla_{\vec{v}}\vec{w} = \text{Proj}_{TS}(d\vec{w}.\vec{v})$), (9.1) gives

$$[\vec{v}, \vec{w}] = \text{Proj}_{TS}(d\vec{w}.\vec{v} - d\vec{v}.\vec{w}) = \nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}. \quad (10.4)$$

Corollary 10.1 (9.7) reads

$$\begin{cases} [\varphi\vec{v}, \vec{w}](f) = \varphi[\vec{v}, \vec{w}](f) - (d\varphi.\vec{w})(df.\vec{v}) & (= \varphi df.[\vec{v}, \vec{w}] - (d\varphi.\vec{w})(df.\vec{v})), \\ [\vec{v}, \varphi\vec{w}](f) = \varphi[\vec{v}, \vec{w}](f) + (d\varphi.\vec{v})(df.\vec{w}) & (= \varphi df.[\vec{v}, \vec{w}] + (d\varphi.\vec{v})(df.\vec{w})). \end{cases} \quad (10.5)$$

10.2 $[\vec{v}, \vec{w}]$ is tangential

Theorem 10.2 *If $\vec{v}, \vec{w} \in \Gamma(S_+)$ are such that $\vec{v}|_S, \vec{w}|_S \in \Gamma(S)$ (tangent to S), then*

$$[\vec{v}, \vec{w}]|_S \in \Gamma(S) \quad (\text{tangent to } S). \quad (10.6)$$

More precisely,

$$[\vec{v}, \vec{w}]|_S = \text{Proj}_{TS}(d\vec{w} \cdot \vec{v}) - \text{Proj}_{TS}(d\vec{v} \cdot \vec{w}) = \nabla_{\vec{v}} \vec{w} - \nabla_{\vec{w}} \vec{v}, \quad (10.7)$$

with ∇ the usual connection in S , cf. (6.6). And $[\vec{v}, \vec{w}]$ is antisymmetric.

Quantification: $(\vec{e}_i)_{i=1, \dots, m}$ being the coordinate basis of Φ , if $\vec{v} = \sum_{i=1}^m v^i \vec{e}_i$ and $\vec{w} = \sum_{j=1}^m w^j \vec{e}_j$ in S , then, in S ,

$$[\vec{v}, \vec{w}]|_S = (d\vec{w} \cdot \vec{v} - d\vec{v} \cdot \vec{w})|_S = \sum_{i,j=1}^m \left(\frac{\partial w^i}{\partial q^j} v^j - \frac{\partial v^i}{\partial q^j} w^j \right) \vec{e}_i \in \Gamma(S). \quad (10.8)$$

Proof. With the extended basis $(\vec{e}_1, \dots, \vec{e}_n)$ of S_+ , we have in S :

$$\begin{aligned} d\vec{w} \cdot \vec{v} &= \sum_{i=1}^m (dw^i \cdot \vec{v}) \vec{e}_i + \sum_{i=1}^m w^i (d\vec{e}_i \cdot \vec{v}) = \sum_{i,j=1}^m v^j (dw^i \cdot \vec{e}_j) \vec{e}_i + \sum_{i,j=1}^m w^i v^j d\vec{e}_i \cdot \vec{e}_j \\ &= \sum_{i,j=1}^m \frac{\partial w^i}{\partial q^j} v^j \vec{e}_i + \sum_{i,j=1}^m w^i v^j \left(\sum_{k=1}^n \gamma_{ij}^k \vec{e}_k \right). \end{aligned} \quad (10.9)$$

And $\gamma_{ij}^k = \gamma_{ji}^k$ since (\vec{e}_i) is a coordinate system, thus the γ_{ij}^k terms vanish in $d\vec{w} \cdot \vec{v} - d\vec{v} \cdot \vec{w}$, hence (10.8), thus (10.7), (10.6), and the antisymmetry (trivial). \blacksquare

10.3 Jacobi identity

Proposition 10.3 (Jacobi identity) *The Lie bracket satisfies: For all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$,*

$$[\vec{u}, [\vec{v}, \vec{w}]] + [\vec{v}, [\vec{w}, \vec{u}]] + [\vec{w}, [\vec{u}, \vec{v}]] = 0. \quad (10.10)$$

Proof. $[\vec{v}, \vec{w}] = \tilde{\nabla}_{\vec{v}} \vec{w} - \tilde{\nabla}_{\vec{w}} \vec{v}$ gives

$$\begin{aligned} [\vec{u}, [\vec{v}, \vec{w}]] &= \tilde{\nabla}_{\vec{u}} \circ (\tilde{\nabla}_{\vec{v}} \circ \tilde{\nabla}_{\vec{w}} - \tilde{\nabla}_{\vec{w}} \circ \tilde{\nabla}_{\vec{v}}) - (\tilde{\nabla}_{\vec{v}} \circ \tilde{\nabla}_{\vec{w}} - \tilde{\nabla}_{\vec{w}} \circ \tilde{\nabla}_{\vec{v}}) \circ \tilde{\nabla}_{\vec{u}} \\ &= \tilde{\nabla}_{\vec{u}} \circ \tilde{\nabla}_{\vec{v}} \circ \tilde{\nabla}_{\vec{w}} - \tilde{\nabla}_{\vec{u}} \circ \tilde{\nabla}_{\vec{w}} \circ \tilde{\nabla}_{\vec{v}} - \tilde{\nabla}_{\vec{v}} \circ \tilde{\nabla}_{\vec{w}} \circ \tilde{\nabla}_{\vec{u}} + \tilde{\nabla}_{\vec{w}} \circ \tilde{\nabla}_{\vec{v}} \circ \tilde{\nabla}_{\vec{u}} \end{aligned} \quad (10.11)$$

Idem for the two other terms (circular permutation). Thus (10.10). \blacksquare

Definition 10.4 A Lie algebra is a quadruplet $(V, +, \cdot, a)$ where $(V, +, \cdot)$ is a vector space and $a : V \times V \rightarrow V$ is an antisymmetric bilinear map which satisfies Jacobi identity, that is,

$$a(a(\vec{u}, \vec{v}), \vec{w}) + a(a(\vec{v}, \vec{w}), \vec{u}) + a(a(\vec{w}, \vec{u}), \vec{v}) = \vec{0}. \quad (10.12)$$

Example 10.5 $\{n \times n \text{ matrices}\}$ with $[A, B] = AB - BA$ is a Lie algebra. $(\mathbb{R}^n, +, \cdot, [\cdot, \cdot])$ is a Lie algebra. \blacksquare

10.4 Derivation formula $\mathcal{L}_{\vec{u}}^0[\vec{v}, \vec{w}] = [\mathcal{L}_{\vec{u}}^0 \vec{v}, \vec{w}] + [\vec{v}, \mathcal{L}_{\vec{u}}^0 \vec{w}]$

With $\mathcal{L}_{\vec{u}}^0 \vec{v} = [\vec{u}, \vec{v}]$ and with $[\vec{u}, \vec{v}] = -[\vec{v}, \vec{u}]$, the Jacobi identity (10.10) gives the derivation rule

$$\mathcal{L}_{\vec{u}}^0[\vec{v}, \vec{w}](f) = [\mathcal{L}_{\vec{u}}^0 \vec{v}, \vec{w}](f) + [\vec{v}, \mathcal{L}_{\vec{u}}^0 \vec{w}](f). \quad (10.13)$$

10.5 Geometric interpretation of $[\vec{v}, \vec{w}] \neq \vec{0}$

Let $\vec{v}, \vec{w} \in \Gamma(S)$, let $p = \Phi(\vec{q}) \in S$, and let α_p and β_p be associated the integral curves at p , that is,

$$\frac{d\alpha_p}{ds}(s) = \vec{v}(\alpha_p(s)), \quad \alpha_p(0) = p, \quad \text{and} \quad \frac{d\beta_p}{dt}(t) = \vec{w}(\beta_p(t)), \quad \beta_p(0) = p, \quad (10.14)$$

for s close to 0. And consider the associated family of curves $\alpha(s, p) = \alpha_p(s)$ and $\beta(t, p) := \beta_p(t)$, so

$$\frac{\partial \alpha}{\partial s}(s, p) := \vec{v}(\alpha(s, p)), \quad \text{and} \quad \frac{\partial \beta}{\partial t}(t, p) := \vec{w}(\beta(t, p)). \quad (10.15)$$

Let $p = P_0 = \alpha(0, P_0) = \beta(0, P_0) \in S$ and, see figure 10.1,

$$\begin{cases} P_1 = \alpha_s(P_0), \\ P_2 = \beta_t(P_0), \\ P_{12} = \beta_t(P_1) = (\beta_t \circ \alpha_s)(P_0) \stackrel{\text{written}}{=} \varphi(s, t), \\ P_{21} = \alpha_s(P_2) = (\alpha_s \circ \beta_t)(P_0) \stackrel{\text{written}}{=} \psi(t, s). \end{cases} \quad (10.16)$$

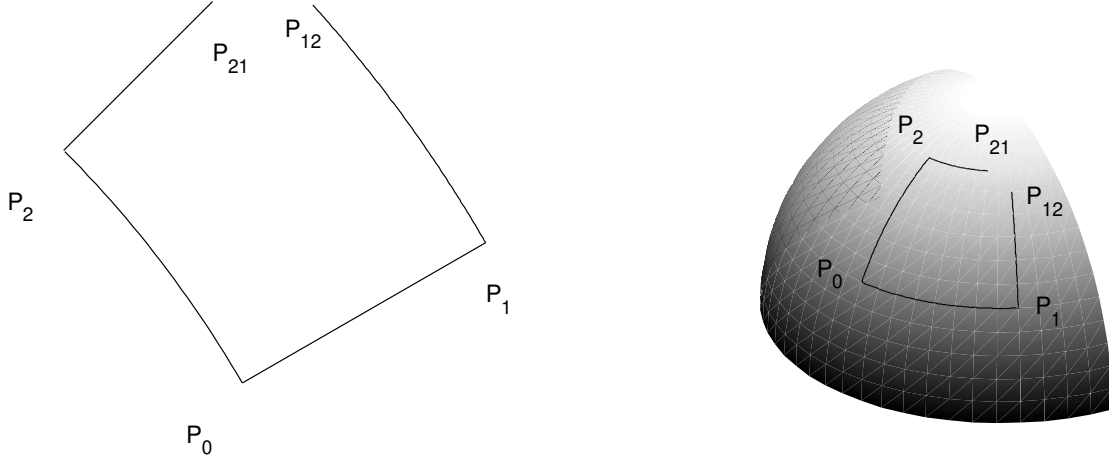


Figure 10.1: Along curves which are not curves of a coordinate system, the path joining P_{12} to P_{21} is not closed, and $\overrightarrow{P_{12}P_{21}} = st[\vec{v}, \vec{w}](p) + o(s^2+t^2)$, thus $[\vec{v}, \vec{w}]$ gives a measure of measure of the “aperture”.

Proposition 10.6 With $\varphi(s, t) = \beta_t(\alpha_s(P_0))$ and $\psi(t, s) = \alpha_s(\beta_t(P_0))$, cf. (10.16), the second order Taylor expansion are we have

$$\begin{cases} \varphi(s, t) = P_0 + (s\vec{v} + t\vec{w})(P_0) + (s^2 d\vec{v} \cdot \vec{v} + st d\vec{v} \cdot \vec{w} + t^2 d\vec{w} \cdot \vec{w})(P_0) + o(s^2+t^2), \\ \psi(t, s) = P_0 + (t\vec{w} + s\vec{v})(P_0) + (t^2 d\vec{w} \cdot \vec{w} + st d\vec{w} \cdot \vec{v} + t^2 d\vec{v} \cdot \vec{v})(P_0) + o(s^2+t^2), \end{cases} \quad (10.17)$$

thus,

$$(\varphi(s, t) - \psi(t, s)) = P_{12} - P_{21} = st[\vec{v}, \vec{w}](p) + o(s^2+t^2). \quad (10.18)$$

Thus $[\vec{v}, \vec{w}](P_0)$ is a measure of $\overrightarrow{P_{12}P_{21}}$ (the “aperture”, see figure 10.1).

Proof. $\varphi(s, t) = \beta(t, \alpha(s, P_0))$, $\frac{\partial \alpha}{\partial s}(s, P_0) = \vec{v}(\alpha(s, P_0))$ and $\frac{\partial \beta}{\partial t}(t, \alpha(s, P_0)) = \vec{w}(\beta(t, \alpha(s, P_0)))$ give

$$\begin{cases} \frac{\partial \varphi}{\partial s}(s, t) = d\beta(t, \alpha(s, P_0)) \cdot \vec{v}(\alpha(s, P_0)), \\ \frac{\partial \varphi}{\partial t}(s, t) = \vec{w}(\beta(t, \alpha(s, P_0))). \end{cases}$$

Thus,

$$\begin{cases} \frac{\partial^2 \varphi}{\partial s^2}(s, t) = \left(d^2 \beta(t, \alpha(s, P_0)) \cdot \vec{v}(\alpha(s, P_0)) \right) \cdot \vec{v}(\alpha(s, P_0)) + d\beta(t, \alpha(s, P_0)) \cdot \left(d\vec{v}(\alpha(s, P_0)) \cdot \vec{v}(\alpha(s, P_0)) \right), \\ \frac{\partial^2 \varphi}{\partial t \partial s}(s, t) = d \left(\frac{\partial \beta}{\partial t} \right) (t, \alpha(s, P_0)) \cdot \vec{v}(\alpha(s, P_0)) = d\vec{w}(\beta(t, \alpha(s, P_0))) \cdot \vec{v}(\alpha(s, P_0)), \\ \frac{\partial^2 \varphi}{\partial t^2}(s, t) = d\vec{w}(\beta(t, \alpha(s, P_0))) \cdot \vec{w}(\beta(t, \alpha(s, P_0))). \end{cases}$$

In particular,

$$\begin{cases} \varphi(0, 0) = P_0, & \frac{\partial \varphi}{\partial s}(0, 0) = \vec{v}(P_0), & \frac{\partial \varphi}{\partial t}(0, 0) = \vec{w}(P_0), \\ \frac{\partial^2 \varphi}{\partial s^2}(0, 0) = (d\vec{v} \cdot \vec{v})(P_0), & \frac{\partial^2 \varphi}{\partial t \partial s}(0, 0) = (d\vec{w} \cdot \vec{v})(P_0) = \frac{\partial^2 \varphi}{\partial s \partial t}(0, 0), & \frac{\partial^2 \varphi}{\partial t^2}(0, 0) = (d\vec{w} \cdot \vec{w})(P_0). \end{cases}$$

Idem for ψ . Thus the second order Taylor expansion are:

$$\begin{cases} \varphi(s, t) = P_0 + s \vec{v}(P_0) + t \vec{w}(P_0) + \frac{s^2}{2} (d\vec{v} \cdot \vec{v})(P_0) + st (d\vec{w} \cdot \vec{v})(P_0) + \frac{t^2}{2} (d\vec{w} \cdot \vec{w})(P_0) + o(s^2 + t^2), \\ \psi(t, s) = P_0 + t \vec{w}(P_0) + s \vec{v}(P_0) + \frac{t^2}{2} (d\vec{w} \cdot \vec{w})(P_0) + st (d\vec{v} \cdot \vec{w})(P_0) + \frac{s^2}{2} (d\vec{v} \cdot \vec{v})(P_0) + o(s^2 + t^2). \end{cases}$$

Thus $\varphi(s, t) - \psi(t, s) = st((d\vec{w} \cdot \vec{v})(P_0) - (d\vec{v} \cdot \vec{w})(P_0)) + o(s^2 + t^2) = st[\vec{v}, \vec{w}](P_0) + o(s^2 + t^2)$. \blacksquare

Part V

Connections

11 General connection ∇ on $\Gamma(S)$

11.1 Definition

If $\vec{v} \in \Gamma(S)$ then we have defined $\tilde{\nabla} : \left\{ \begin{array}{l} \Gamma(S) \times \mathcal{F}(S) \rightarrow \mathcal{F}(S) \\ (\vec{v}, f) \rightarrow \tilde{\nabla}(\vec{v}, f) = \tilde{\nabla}_{\vec{v}} f := df \cdot \vec{v} = \tilde{\mathcal{L}}_{\vec{v}}^0(f) \end{array} \right\}$ the covariant derivation of functions f along \vec{v} , cf. (5.3). Now:

Definition 11.1 An (affine) connection ∇ on $\Gamma(S)$ is a \mathbb{R} -bilinear map

$$\nabla : \begin{cases} \Gamma(S) \times \Gamma(S) \rightarrow \Gamma(S), \\ (\vec{v}, \vec{w}) \mapsto \nabla(\vec{v}, \vec{w}) \stackrel{\text{written}}{=} \nabla_{\vec{v}} \vec{w} \stackrel{\text{written}}{=} \nabla \vec{w} \cdot \vec{v}, \end{cases} \quad (11.1)$$

such that

1. For all $\vec{w} \in \Gamma(S)$, the map $\nabla(\cdot, \vec{w}) = \nabla \vec{w}(\cdot) : \Gamma(S) \rightarrow \Gamma(S)$ is $\mathcal{F}(S)$ -linear, i.e., for all $f \in \mathcal{F}(S)$ and all $\vec{u} \in \Gamma(S)$,

$$\nabla_{f\vec{u}} \vec{w} = f \nabla_{\vec{u}} \vec{w}, \quad (11.2)$$

i.e. $\nabla \vec{w} \cdot (f\vec{u}) = f \nabla \vec{w} \cdot \vec{u}$, or $\nabla(f\vec{u}, \vec{w}) = f \nabla(\vec{u}, \vec{w})$,

2. For all $\vec{v} \in \Gamma(S)$, the map $\nabla(\vec{v}, \cdot) = \nabla_{\vec{v}}(\cdot) : \Gamma(S) \rightarrow \Gamma(S)$ is such that, for all $f \in \mathcal{F}(S)$ and all $\vec{w} \in \Gamma(S)$ (derivation formula),

$$\nabla_{\vec{v}}(f\vec{w}) = (\tilde{\nabla}_{\vec{v}} f)\vec{w} + f \nabla_{\vec{v}} \vec{w}, \quad (11.3)$$

i.e. $\nabla(f\vec{w}) \cdot \vec{v} = (df \cdot \vec{v})\vec{w} + f \nabla \vec{w} \cdot \vec{v}$, or $\nabla(\vec{v}, f\vec{w}) = \tilde{\nabla}(\vec{v}, f)\vec{w} + f \nabla(\vec{v}, \vec{w})$, cf. (5.8).

And $\nabla_{\vec{v}} \vec{w}$ is called the covariant derivative along \vec{v} of the vector field \vec{w} .

Example 11.2 The Riemannian connection (6.6) satisfy (11.2) and (11.3), cf. (6.9)-(6.10): It is a connection. However the Riemannian connection (6.6) can only be defined if S is a surface in \mathbb{R}^n ; It cannot be “naturally” defined if S lives on its own (e.g. S is the Earth surface and a “vertical” is not accessible, or e.g. S = our curved space-time of general relativity). \blacksquare

Remark 11.3 The notation $\nabla_{\vec{v}}(\vec{w}) = \nabla_{\vec{v}} \vec{w} = \nabla \vec{w} \cdot \vec{v}$ seems to be universal, but the notation $\nabla(\vec{v}, \vec{w})$ depends on authors; Here we use Abraham and Marsden [1] notation, that is, $\nabla(\vec{v}, \vec{w}) = \nabla_{\vec{v}} \vec{w}$ (motivation: See § 11.8). (Misner–Thorne–Wheeler [15] use the notation $\nabla(\vec{w}, \vec{v}) = \nabla_{\vec{v}} \vec{w}$...). \blacksquare

11.2 Covariant derivative $\nabla_{\vec{v}}$ on $\Gamma(S)$ and $\frac{D\vec{w}}{dt}$

(11.1) and (11.3) enable to define the covariant derivative along \vec{v} :

$$\nabla_{\vec{v}} : \begin{cases} \Gamma(S) \rightarrow \Gamma(S), \\ \vec{w} \mapsto \nabla_{\vec{v}}(\vec{w}) := \nabla(\vec{v}, \vec{w}) \stackrel{\text{written}}{=} \nabla_{\vec{v}}\vec{w} \stackrel{\text{written}}{=} \frac{D\vec{w}}{dt} \quad (\stackrel{\text{written}}{=} \nabla\vec{w}.\vec{v}). \end{cases} \quad (11.4)$$

In \mathbb{R}^n , $\nabla_{\vec{v}}\vec{w} := d\vec{w}.\vec{v}$ is the usual covariant derivative.

In $S \subset \mathbb{R}^n$, $\nabla_{\vec{v}}\vec{w} := \text{Proj}_{TS}(d\vec{w}.\vec{v})$ is the usual (Riemannian) covariant derivative.

11.3 Differential $\nabla\vec{w}$ on $\Gamma(S)$

(11.1) and (11.2) enable to define the ∇ -differential of \vec{w} :

$$\nabla\vec{w} : \begin{cases} \Gamma(S) \rightarrow \Gamma(S) \\ \vec{v} \rightarrow \nabla\vec{w}.\vec{v} := \nabla_{\vec{v}}\vec{w} \quad (= \nabla(\vec{v}, \vec{w})). \end{cases} \quad (11.5)$$

Proposition 11.4 Let $\vec{w} \in \Gamma(S)$. Then $\nabla\vec{w} : \Gamma(S) \rightarrow \Gamma(S)$ is a $\binom{1}{1}$ tensor in S .

Proof. Thanks to the $\mathcal{F}(S)$ -linearity (11.2), i.e., $\nabla\vec{w}.(f\vec{u}) = f\nabla\vec{w}(\vec{u})$. \blacksquare

11.4 Naive connection ∇^0

Let $(\vec{a}_i)_{i=1, \dots, m}$ be a basis in TS (not necessary holonomous).

Definition 11.5 Let $\vec{v}, \vec{w} \in \Gamma(S)$ and $\vec{w} = \sum_{i=1}^m w^i \vec{a}_i \in \Gamma(S)$. The naive connection ∇^0 relative to the basis (\vec{a}_i) is defined by

$$\nabla_{\vec{v}}^0 \vec{w} = \sum_{i=1}^m \tilde{\nabla}_{\vec{v}} w^i \vec{a}_i = \sum_{i=1}^m (dw^i.\vec{v}) \vec{a}_i = \nabla^0(\vec{v}, \vec{w}) = \nabla^0 \vec{w}.\vec{v}. \quad (11.6)$$

In particular, for all $\vec{v} \in \Gamma(S)$ and all $i, j \in [1, m]_{\mathbb{N}}$,

$$\nabla_{\vec{a}_i}^0 \vec{a}_j = \vec{0} = \nabla^0(\vec{a}_i, \vec{a}_j) = \nabla^0 \vec{a}_j.\vec{a}_i. \quad (11.7)$$

Proposition 11.6 The naive connection is a connection.

Proof. 1. $\nabla_{f\vec{u}+\vec{v}}^0 \vec{w} = \sum_{i=1}^m (dw^i.(f\vec{u} + \vec{v})) \vec{a}_i = \sum_{i=1}^m f(dw^i.\vec{u}) \vec{a}_i + \sum_{i=1}^m (dw^i.\vec{v}) \vec{a}_i = f\nabla_{\vec{u}}^0 \vec{w} + \nabla_{\vec{v}}^0 \vec{w}$, i.e (11.2).

2. $\nabla_{\vec{v}}^0(f\vec{w}) = \sum_i (d(fw^i).\vec{v}) \vec{a}_i = \sum_i (df.\vec{v}) w^i \vec{a}_i + \sum_i f(dw^i.\vec{v}) \vec{a}_i = (df.\vec{v})\vec{w} + f\nabla_{\vec{v}}^0 \vec{w}$, i.e. (11.3). \blacksquare

Proposition 11.7 A naive connection is not a tensor.

Proof. $\nabla_{\vec{v}}^0(f\vec{w}) = (df.\vec{v})\vec{w} + f\nabla_{\vec{v}}^0 \vec{w} \neq f\nabla_{\vec{v}}^0(\vec{w})$ in general. \blacksquare

11.5 Torsion of a connection

The lie Bracket $[\vec{v}, \vec{w}] \in \Gamma(S)$ of $\vec{v}, \vec{w} \in \Gamma(S)$ has been defined on $\mathcal{F}(S)$ by, cf. (10.2),

$$\begin{aligned} \mathcal{L}_{\vec{v}}^0 \vec{w}(f) &= df.[\vec{v}, \vec{w}] = [\vec{v}, \vec{w}](f) \stackrel{\text{written}}{=} [\tilde{\nabla}_{\vec{v}}, \tilde{\nabla}_{\vec{w}}](f) \\ &= d(df.\vec{w}).\vec{v} - d(df.\vec{v}).\vec{w} = \tilde{\nabla}_{\vec{v}}(\tilde{\nabla}_{\vec{w}}f) - \tilde{\nabla}_{\vec{w}}(\tilde{\nabla}_{\vec{v}}f). \end{aligned} \quad (11.8)$$

Definition 11.8 Let ∇ be a connection, cf. (11.1)-(11.2)-(11.3), and let $\vec{v}, \vec{w} \in \Gamma(S)$. The torsion $T(\vec{v}, \vec{w}) \in \Gamma(S)$ due to \vec{v} and \vec{w} is the vector field considered as a derivation operator defined by, for all $f \in \mathcal{F}(S)$,

$$\begin{aligned} T(\vec{v}, \vec{w})(f) &= (\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v} - [\vec{v}, \vec{w}])(f) \\ &= df.(\nabla_{\vec{v}}\vec{w}) - df.(\nabla_{\vec{w}}\vec{v}) - d(df.\vec{w}).\vec{v} + d(df.\vec{v}).\vec{w} \end{aligned} \quad (11.9)$$

And the torsion T of ∇ is the map

$$T : \begin{cases} \Gamma(S) \times \Gamma(S) \rightarrow \Gamma(S) \\ (\vec{v}, \vec{w}) \rightarrow T(\vec{v}, \vec{w}) = (\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}) - [\vec{v}, \vec{w}]. \end{cases} \quad (11.10)$$

Example 11.9 With $\nabla = \nabla^0$ the naive connection relative to the the normalized polar basis (\vec{b}_1, \vec{b}_2) , we have $[\vec{b}_1, \vec{b}_2] = -\frac{1}{r}\vec{b}_2 \neq \vec{0}$, cf. (2.24), and we have $\nabla_{\vec{b}_1}^0 \vec{b}_2 - \nabla_{\vec{b}_2}^0 \vec{b}_1 = \vec{0}$, cf. (11.7). Thus $T(\vec{b}_1, \vec{b}_2) = \frac{1}{r}\vec{b}_2 \neq \vec{0}$: The torsion of the naive connection does not vanish. \blacksquare

11.6 Torsion-free connection

Definition 11.10 A connection ∇ is torsion free iff $T = 0$, that is, iff ∇ and $\tilde{\nabla}$ satisfy, for all $\vec{v}, \vec{w} \in \Gamma(S)$,

$$\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v} = [\vec{v}, \vec{w}] \quad (= \tilde{\nabla}_{\vec{v}} \circ \tilde{\nabla}_{\vec{w}} - \tilde{\nabla}_{\vec{w}} \circ \tilde{\nabla}_{\vec{v}}), \quad (11.11)$$

that is, for all $\vec{v}, \vec{w} \in \Gamma(S)$ and all $f \in \mathcal{F}(S)$,

$$(\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v})(f) = [\vec{v}, \vec{w}](f). \quad (11.12)$$

(Also called a symmetric connection with reference to the Christoffel symbols: With the basis of a coordinate system, $\gamma_{ij}^k = \gamma_{ji}^k$ for all i, j, k , see (11.20).)

Example 11.11 The Riemannian connection in a surface in \mathbb{R}^n is torsion-free: Indeed, if $\vec{v}, \vec{w} \in \Gamma(S)$ then $[\vec{v}, \vec{w}] = d\vec{w} \cdot \vec{v} - d\vec{v} \cdot \vec{w} \in \Gamma(S)$, cf. (10.4)-(10.6), and, with (6.6) and (10.7), $\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v} = \text{Proj}_{TS}(d\vec{w} \cdot \vec{v}) - \text{Proj}_{TS}(d\vec{v} \cdot \vec{w}) = \text{Proj}_{TS}(d\vec{w} \cdot \vec{v} - d\vec{v} \cdot \vec{w}) = [\vec{v}, \vec{w}]$, thus, $T = 0$. \blacksquare

Exercise 11.12 We will only use torsion free connection in the sequel. However, prove that, in all cases, the torsion T is an antisymmetric $\binom{1}{2}$ tensor, i.e. $T(\vec{v}, \vec{w}) = -T(\vec{w}, \vec{v})$ for all $\vec{v}, \vec{w} \in \Gamma(S)$.

Answer. Antisymmetry is trivial since $[\cdot, \cdot]$ is. And T is a tensor iff the associated map $\tilde{T} : \Omega^1(S) \times \Gamma(S) \times \Gamma(S) \rightarrow \mathbb{R}$ defined by $\tilde{T}(\alpha, \vec{v}, \vec{w}) = \alpha.T(\vec{v}, \vec{w})$ is $\mathcal{F}(S)$ -multilinear. For the first component α it is trivial. And

$$\begin{aligned} T(\vec{v}, f\vec{w}) &= (\nabla_{\vec{v}}(f\vec{w}) - \nabla_{f\vec{w}}\vec{v}) - ([\vec{v}, f\vec{w}]) \\ &= \left(f \nabla_{\vec{v}}\vec{w} + (df \cdot \vec{v})\tilde{\nabla}\vec{w} - f \nabla_{\vec{w}}\vec{v} \right) - \left(f [\vec{v}, \vec{w}] + (df \cdot \vec{v})\tilde{\nabla}\vec{w} \right) \\ &= f(\nabla_{\vec{v}}\vec{w} - \nabla_{\vec{w}}\vec{v}) - f[\vec{v}, \vec{w}] = fT(\vec{v}, \vec{w}). \end{aligned}$$

And T is antisymmetric, thus $T(f\vec{v}, \vec{w}) = -T(\vec{w}, f\vec{v}) = -fT(\vec{w}, \vec{v}) = fT(\vec{v}, \vec{w})$. Thus T is a tensor. \blacksquare

11.7 Tensor $\gamma \dots$

Proposition 11.13 If ∇ and N are two connections then

$$\gamma = \nabla - N \text{ is a tensor.} \quad (11.13)$$

Proof. Consider the associated map $\tilde{\gamma} : \Omega^1(S) \times \Gamma(S) \times \Gamma(S) \rightarrow \mathbb{R}$ given by $\tilde{\gamma}(\alpha, \vec{v}, \vec{w}) := \alpha.\gamma(\vec{v}, \vec{w})$. The affirmation “ $\gamma = \nabla - N$ is a tensor” means that “ $\tilde{\gamma}$ is a $\binom{1}{2}$ tensor”, that is, $\tilde{\gamma}$ is $\mathcal{F}(S)$ -multilinear. For the first component α it is trivial. And $\gamma(\vec{v}, f\vec{w}) = \nabla_{\vec{v}}(f\vec{w}) - N_{\vec{v}}(f\vec{w}) = f\nabla_{\vec{v}}\vec{w} + (df \cdot \vec{v})\vec{w} - (fN_{\vec{v}}\vec{w} + (df \cdot \vec{v})\vec{w}) = f(\nabla_{\vec{v}}\vec{w} - N_{\vec{v}}\vec{w}) = f\gamma(\vec{v}, \vec{w})$: $\tilde{\gamma}$ is $\mathcal{F}(S)$ -multilinear in \vec{w} . And $\gamma(f\vec{v}, \vec{w}) = -\gamma(\vec{w}, f\vec{v}) = -f\gamma(\vec{w}, \vec{v}) = f\gamma(\vec{v}, \vec{w})$: $\tilde{\gamma}$ is $\mathcal{F}(S)$ -multilinear in \vec{v} . \blacksquare

Corollary 11.14 If ∇ is a connection and $\nabla \neq 0$, then ∇ is not a tensor.

Proof. If ∇ was a tensor, then $\nabla^0 = \nabla - \gamma$ would be a tensor since $\gamma = \nabla - \nabla^0$ is a tensor. But ∇^0 is not a tensor. \blacksquare

Exercise 11.15 Let ∇ be a connection, $\nabla \neq 0$. Prove that $A = 2\nabla$ is not a connection.

Answer. If A was, then $A - \nabla = \nabla$ is a tensor, cf. (11.13): But ∇ is not. cf. corollary 11.14. \blacksquare

11.8 ... and Christoffel symbols γ_{jk}^i

Let ∇ be a connection, let (\vec{a}_i) be a basis, let ∇^0 be the naive connection relative to (\vec{a}_i) , and consider the tensor $\gamma = \nabla - \nabla^0$, cf. prop. 11.13, and the associated tensor $\tilde{\gamma} \in T_2^1(S)$ given by $\tilde{\gamma}(\alpha, \vec{v}, \vec{w}) := \alpha \cdot \gamma(\vec{v}, \vec{w})$ (cf. proof of prop. 11.13). Let C_{jk}^i be the components of $\tilde{\gamma}$ relative to the basis (\vec{a}_i) , that is,

$$\tilde{\gamma} = \sum_{i,j,k=1}^m C_{jk}^i \vec{a}_i \otimes a^j \otimes a^k, \quad \text{and} \quad \tilde{\gamma}(a^i, \vec{a}_j, \vec{a}_k) = C_{jk}^i \quad (= a^i \cdot \gamma(\vec{a}_j, \vec{a}_k)), \quad (11.14)$$

i.e.,

$$\forall j, k, \gamma_{\vec{a}_j} \vec{a}_k = \gamma(\vec{a}_j, \vec{a}_k) = \sum_{i=1}^m C_{jk}^i \vec{a}_i, \quad \text{or} \quad \forall i, j, \gamma_{\vec{a}_i} \vec{a}_j = \sum_{k=1}^m C_{ij}^k \vec{a}_k. \quad (11.15)$$

With (11.7) and $\gamma = \nabla - \nabla^0$, we have, for all $i, j \in [1, m]_{\mathbb{N}}$,

$$\gamma_{\vec{a}_i} \vec{a}_j = \nabla_{\vec{a}_i} \vec{a}_j - 0 = \sum_{k=1}^m C_{ij}^k \vec{a}_k \quad (= \gamma(\vec{a}_i, \vec{a}_j)). \quad (11.16)$$

Definition 11.16 If $(\vec{a}_i) = (\vec{e}_i)$ is the basis of a coordinate system, then $C_{ij}^k = \text{written } \gamma_{ij}^k$ are called the Christoffel symbols of the connection ∇ relative to (\vec{e}_i) .

Hence (11.16) reads,

$$\nabla_{\vec{e}_i} \vec{e}_j = \sum_{k=1}^m \gamma_{ij}^k \vec{e}_k, \quad (11.17)$$

Then, with $\vec{w} = \sum_{j=1}^m w^j \vec{e}_j$, (11.3) gives $\nabla_{\vec{e}_i} \vec{w} = \sum_{j=1}^m (\tilde{\nabla}_{\vec{e}_i} w^j) \vec{e}_j + \sum_{i=1}^m w^j (\nabla_{\vec{e}_i} \vec{e}_j) = \sum_{j=1}^m \frac{\partial w^j}{\partial q^i} \vec{e}_j + \sum_{j,k=1}^m w^j \gamma_{ij}^k \vec{e}_k$, so

$$\nabla_{\vec{e}_i} \vec{w} = \sum_{i,j=1}^m w_{|i}^j \vec{e}_j \quad \text{with} \quad w_{|i}^j = \frac{\partial w^j}{\partial q^i} + \sum_{k=1}^m \gamma_{ik}^j w^k. \quad (11.18)$$

Then, with $\vec{v} = \sum_{i=1}^m v^i \vec{e}_i$, (11.2) gives $\nabla_{\vec{v}} \vec{w} = \nabla_{\sum_{i=1}^m v^i \vec{e}_i} \vec{w} = \sum_{i=1}^m v^i \nabla_{\vec{e}_i} \vec{w}$, so

$$\nabla_{\vec{v}} \vec{w} = \sum_{i,j=1}^m w_{|i}^j v^i \vec{e}_j = \sum_{i,j=1}^m \left(\frac{\partial w^j}{\partial q^i} v^i + \sum_{k=1}^m \gamma_{ik}^j w^k v^i \right) \vec{e}_j. \quad (11.19)$$

Proposition 11.17 If ∇ is torsion-free, and if (\vec{e}_i) is the basis of a coordinate system, then

$$\forall i, j, k = 1, \dots, m, \quad \gamma_{ij}^k = \gamma_{ji}^k. \quad (11.20)$$

Proof. $\nabla_{\vec{v}} \vec{w} - \nabla_{\vec{w}} \vec{v} = [\vec{v}, \vec{w}]$, cf. (11.12), gives $(\nabla_{\vec{e}_i} \vec{e}_i - \nabla_{\vec{e}_i} \vec{e}_i)(f) = (\tilde{\nabla}_{\vec{e}_i} \circ \tilde{\nabla}_{\vec{e}_i} - \tilde{\nabla}_{\vec{e}_i} \circ \tilde{\nabla}_{\vec{e}_i})(f) = \frac{\partial}{\partial q^j} \frac{\partial f}{\partial q^i} - \frac{\partial}{\partial q^i} \frac{\partial f}{\partial q^j} = 0$ (Schwarz for $f_U = f \circ \Phi^{-1}$, cf. (1.29)). Hence $\sum_{k=1}^m \gamma_{ij}^k \vec{e}_k - \sum_{k=1}^m \gamma_{ji}^k \vec{e}_k = 0$. \blacksquare

12 Connection $\tilde{\nabla}$ on $\Omega^1(S)$

12.1 Directional derivative $\tilde{\nabla}_{\vec{v}} \alpha = \frac{D\alpha}{dt}$

In \mathbb{R}^n , if $\alpha \in \Omega^1(\mathbb{R}^n)$ and $\vec{w} \in \Gamma(\mathbb{R}^n)$ then $\alpha \cdot \vec{w} \in \mathcal{F}(\mathbb{R}^n)$, and

$$d(\alpha \cdot \vec{w}) \cdot \vec{v} = (d\alpha \cdot \vec{v}) \cdot \vec{w} + \alpha \cdot (d\vec{w} \cdot \vec{v}) \in \mathcal{F}(S). \quad (12.1)$$

In S , with $\tilde{\nabla}_{\vec{v}} f = df \cdot \vec{v}$ and a connection ∇ on $\Gamma(S)$, we define $\tilde{\nabla}_{\vec{v}}$ on $\Omega^1(S)$ such that

$$\tilde{\nabla}_{\vec{v}}(\alpha \cdot \vec{w}) = \tilde{\nabla}_{\vec{v}} \alpha \cdot \vec{w} + \alpha \cdot \nabla_{\vec{v}} \vec{w} \quad (12.2)$$

(derivation of product):

Definition 12.1 The covariant derivation on $\Omega^1(S)$ along \vec{v} is

$$\tilde{\nabla}_{\vec{v}} : \begin{cases} \Omega^1(S) \rightarrow \Omega^1(S), \\ \alpha \mapsto \tilde{\nabla}_{\vec{v}}\alpha, \quad \text{s.t.} \quad \tilde{\nabla}_{\vec{v}}\alpha.\vec{w} := \tilde{\nabla}_{\vec{v}}(\alpha.\vec{w}) - \alpha.\nabla_{\vec{v}}\vec{w}, \quad \forall \vec{w} \in \Gamma(S). \end{cases} \quad (12.3)$$

E.g., for the Riemannian connection ∇ ,

$$\tilde{\nabla}_{\vec{v}}\alpha.\vec{w} = d(\alpha.\vec{w}).\vec{v} - \alpha.\text{Proj}_{TS}(d\vec{w}.\vec{v}), \quad (12.4)$$

Notation:

$$\tilde{\nabla}_{\vec{v}}\alpha \stackrel{\text{written}}{=} \frac{D\alpha}{dt}, \quad \text{so} \quad \frac{D\alpha}{dt}.\vec{w} = \tilde{\nabla}_{\vec{v}}\alpha.\vec{w}, \quad (12.5)$$

and

$$\frac{D\alpha}{dt}.\vec{w} = \frac{D(\alpha.\vec{w})}{dt} - \alpha.\frac{D\vec{w}}{dt}. \quad (12.6)$$

Proposition 12.2 If $f \in \mathcal{F}(S)$ and $\alpha \in \Omega^1(S)$, then

$$\tilde{\nabla}_{f\vec{v}}\alpha = f\tilde{\nabla}_{\vec{v}}\alpha, \quad \text{and} \quad \tilde{\nabla}_{\vec{v}}(f\alpha) = (\tilde{\nabla}_{\vec{v}}f)\alpha + f(\tilde{\nabla}_{\vec{v}}\alpha), \quad (12.7)$$

that is, for all $\vec{w} \in \Gamma(S)$, $\tilde{\nabla}_{f\vec{v}}\alpha.\vec{w} = f(\tilde{\nabla}_{\vec{v}}\alpha.\vec{w})$ and $\tilde{\nabla}_{\vec{v}}(f\alpha).\vec{w} = (\tilde{\nabla}_{\vec{v}}f)(\alpha.\vec{w}) + f(\tilde{\nabla}_{\vec{v}}\alpha.\vec{w})$.

Proof. (12.3) gives $\tilde{\nabla}_{f\vec{v}}\alpha.\vec{w} = \tilde{\nabla}_{f\vec{v}}(\alpha.\vec{w}) - \alpha.\nabla_{f\vec{v}}\vec{w} = f\tilde{\nabla}_{\vec{v}}(\alpha.\vec{w}) - \alpha.(f\nabla_{\vec{v}}\vec{w}) = f\tilde{\nabla}_{\vec{v}}(\alpha.\vec{w}) - f\alpha.\nabla_{\vec{v}}\vec{w} = f\tilde{\nabla}_{\vec{v}}\alpha.\vec{w}$, and $\tilde{\nabla}_{\vec{v}}(f\alpha).\vec{w} = \tilde{\nabla}_{\vec{v}}(f\alpha.\vec{w}) - f\alpha.\nabla_{\vec{v}}\vec{w} = (\tilde{\nabla}_{\vec{v}}f)(\alpha.\vec{w}) + f\tilde{\nabla}_{\vec{v}}(\alpha.\vec{w}) - f\alpha.\nabla_{\vec{v}}\vec{w} = (\tilde{\nabla}_{\vec{v}}f)(\alpha.\vec{w}) + f(\tilde{\nabla}_{\vec{v}}\alpha.\vec{w} + \alpha.\nabla_{\vec{v}}\vec{w}) - f\alpha.\nabla_{\vec{v}}\vec{w} = (\tilde{\nabla}_{\vec{v}}f)(\alpha.\vec{w}) + f\tilde{\nabla}_{\vec{v}}\alpha.\vec{w}$. \blacksquare

12.2 Connection on $\Omega^1(S)$

Definition 12.3 The associated connection on $\Gamma(S) \times \Omega^1(S)$ is

$$\tilde{\tilde{\nabla}} : \begin{cases} \Gamma(S) \times \Omega^1(S) \rightarrow \Omega^1(S), \\ (\vec{v}, \alpha) \mapsto \tilde{\tilde{\nabla}}(\vec{v}, \alpha) := \tilde{\nabla}_{\vec{v}}\alpha. \end{cases} \quad (12.8)$$

12.3 Differential $\tilde{\tilde{\nabla}}\alpha$ on $\Omega^1(S)$

Definition 12.4 The associated differential is

$$\tilde{\tilde{\nabla}}\alpha : \begin{cases} \Gamma(S) \rightarrow \Omega^1(S), \\ \vec{v} \mapsto \tilde{\tilde{\nabla}}\alpha.\vec{v} := \tilde{\nabla}_{\vec{v}}\alpha \stackrel{\text{written}}{=} \frac{D\alpha}{dt}. \end{cases} \quad (12.9)$$

Proposition 12.5 If $\alpha \in \Omega^1(S)$, then $\tilde{\tilde{\nabla}}\alpha$ is a tensor in $T_2^0(S)$.

Proof. $\tilde{\tilde{\nabla}}\alpha.(f\vec{u} + \vec{v}) = f\tilde{\tilde{\nabla}}\alpha.\vec{u} + \tilde{\tilde{\nabla}}\alpha.\vec{v}$, for all $f \in \mathcal{F}(S)$ and $\vec{u}, \vec{v} \in \Gamma(S)$, cf. (11.2). \blacksquare

Example 12.6 In S_+ and ∇ the Riemannian connection, we get (the usual result)

$$\begin{aligned} (\tilde{\tilde{\nabla}}\alpha).\vec{w} &= d(\alpha.\vec{w}).\vec{v} - \alpha.(d\vec{w}.\vec{v}) = (d\alpha.\vec{v}).\vec{w} + \alpha.(d\vec{w}.\vec{v}) - \alpha.(d\vec{w}.\vec{v}) \\ &= (d\alpha.\vec{v}).\vec{w} \quad (= d\alpha(\vec{v}, \vec{w})), \end{aligned} \quad (12.10)$$

In particular with $f \in \mathcal{F}(S)$ and $\alpha = df$, we get (the usual result, cf. (9.1))

$$(\tilde{\tilde{\nabla}}df).\vec{w} = d(df.\vec{v}).\vec{w} = d^2f(\vec{w}, \vec{v}) + df.(d\vec{v}.\vec{w}). \quad (12.11)$$

\blacksquare

13 Christoffel symbols for differential forms: $\nabla_{\vec{e}_j} e^i = -\sum_k \gamma_{jk}^i e^k$

Let ∇ be a connection in S . Let γ_{ij}^k be the Christoffel symbols, that is, $\nabla_{\vec{e}_j} \vec{e}_k = \sum_{i=1}^m \gamma_{jk}^i \vec{e}_i$. Then

Proposition 13.1

$$\tilde{\nabla}_{\vec{e}_j} e^i = -\sum_{k=1}^m \gamma_{jk}^i e^k = \tilde{\nabla} e^i \cdot \vec{e}_j, \quad \text{i.e.} \quad \gamma_{jk}^i = -(\tilde{\nabla}_{\vec{e}_j} e^i) \cdot \vec{e}_k, \quad (13.1)$$

i.e.,

$$\tilde{\nabla} e^i = \sum_{j=1}^m (\tilde{\nabla}_{\vec{e}_j} e^i) \otimes e^j = -\sum_{j,k=1}^m \gamma_{jk}^i e^k \otimes e^j. \quad (13.2)$$

Proof. $e^i \cdot \vec{e}_k = \delta_k^i$ gives $d(e^i \cdot \vec{e}_k) \cdot \vec{e}_j = 0 = (\tilde{\nabla}_{\vec{e}_j} e^i) \cdot \vec{e}_k + e^i \cdot (\nabla_{\vec{e}_j} \vec{e}_k)$, thus $(\tilde{\nabla}_{\vec{e}_j} e^i) \cdot \vec{e}_k = -\gamma_{jk}^i$. \blacksquare

Example 13.2 Polar system and Riemannian connection: (3.6) gives

$$\left\{ \begin{array}{l} d^2 r \cdot \vec{e}_1 = de^1 \cdot \vec{e}_1 = \nabla_{\vec{e}_1} e^1 = -\gamma_{11}^1 e^1 - \gamma_{12}^1 e^2 = 0, \\ d^2 r \cdot \vec{e}_2 = de^1 \cdot \vec{e}_2 = \nabla_{\vec{e}_2} e^1 = -\gamma_{21}^1 e^1 - \gamma_{22}^1 e^2 = r e^2 = r d\theta, \\ d^2 \theta \cdot \vec{e}_1 = de^2 \cdot \vec{e}_1 = \nabla_{\vec{e}_1} e^2 = -\gamma_{11}^2 e^1 - \gamma_{12}^2 e^2 = -\frac{1}{r} e^2 = -\frac{1}{r} d\theta, \\ d^2 \theta \cdot \vec{e}_2 = de^2 \cdot \vec{e}_2 = \nabla_{\vec{e}_2} e^2 = -\gamma_{21}^2 e^1 - \gamma_{22}^2 e^2 = -\frac{1}{r} e^1 = -\frac{1}{r} dr, \end{array} \right. \quad (13.3)$$

so $d^2 r = de^1 = \tilde{\nabla} e^1 = r e^2 \otimes e^2 = r d\theta \otimes d\theta$,

and $d^2 \theta = de^2 = \tilde{\nabla} e^2 = -\frac{1}{r} e^2 \otimes e^1 - \frac{1}{r} e^1 \otimes e^2 = -\frac{1}{r} d\theta \otimes dr - \frac{1}{r} dr \otimes d\theta$. \blacksquare

Example 13.3 Spherical GPS system and Riemannian connection: (3.30) and following give

$$\left\{ \begin{array}{l} d^2 r \cdot \vec{e}_1 = de^1 \cdot \vec{e}_1 = \nabla_{\vec{e}_1} \vec{e}_1 = -\gamma_{11}^1 e^1 - \gamma_{12}^1 e^2 - \gamma_{13}^1 e^3 = 0, \\ d^2 r \cdot \vec{e}_2 = de^1 \cdot \vec{e}_2 = -\gamma_{21}^1 e^1 - \gamma_{22}^1 e^2 - \gamma_{23}^1 e^3 = +r \cos^2 \varphi e^2 = r \cos^2 \varphi d\theta, \\ d^2 r \cdot \vec{e}_3 = de^1 \cdot \vec{e}_3 = -\gamma_{31}^1 e^1 - \gamma_{32}^1 e^2 - \gamma_{33}^1 e^3 = +r e^3 = r d\varphi, \\ d^2 \theta \cdot \vec{e}_1 = de^2 \cdot \vec{e}_1 = -\gamma_{11}^2 e^1 - \gamma_{12}^2 e^2 - \gamma_{13}^2 e^3 = -\frac{1}{r} e^2 = -\frac{1}{r} d\theta, \\ d^2 \theta \cdot \vec{e}_2 = de^2 \cdot \vec{e}_2 = -\gamma_{21}^2 e^1 - \gamma_{22}^2 e^2 - \gamma_{23}^2 e^3 = -\frac{1}{r} e^1 - \tan \varphi e^3 = -\frac{1}{r} dr - \tan \varphi d\varphi, \\ d^2 \theta \cdot \vec{e}_3 = de^2 \cdot \vec{e}_3 = -\gamma_{31}^2 e^1 - \gamma_{32}^2 e^2 - \gamma_{33}^2 e^3 = +\tan \varphi e^2 = \tan \varphi d\theta, \\ d^2 \varphi \cdot \vec{e}_1 = de^3 \cdot \vec{e}_1 = -\gamma_{11}^3 e^1 - \gamma_{12}^3 e^2 - \gamma_{13}^3 e^3 = -\frac{1}{r} e^3 = -\frac{1}{r} d\varphi, \\ d^2 \varphi \cdot \vec{e}_2 = de^3 \cdot \vec{e}_2 = -\gamma_{21}^3 e^1 - \gamma_{22}^3 e^2 - \gamma_{23}^3 e^3 = -\cos \varphi \sin \varphi e^2 = -\cos \varphi \sin \varphi d\theta, \\ d^2 \varphi \cdot \vec{e}_3 = de^3 \cdot \vec{e}_3 = -\gamma_{31}^3 e^1 - \gamma_{32}^3 e^2 - \gamma_{33}^3 e^3 = -\frac{1}{r} e^1 = -\frac{1}{r} dr, \end{array} \right. \quad (13.4)$$

\blacksquare

Let $\alpha \in \Omega^1(S)$ and $\alpha = \sum_i \alpha_i e^i \in \Omega^1(S)$. And let

$$\tilde{\nabla} \alpha = \sum_{i,j=1}^m \alpha_{i|j} e^i \otimes e^j, \quad (13.5)$$

that is, $[\nabla \alpha] = [\alpha_{i|j}]_{\substack{i=1,\dots,m \\ j=1,\dots,m}}$ is the Jacobian matrix of α relative to (\vec{e}_i) . Let $\vec{v} \in \Gamma(S)$ and $\vec{v} = \sum_j v^j \vec{e}_j \in \Gamma(S)$. Then,

$$\nabla_{\vec{v}} \alpha = \nabla \alpha \cdot \vec{v} = \sum_{i,j=1}^m \alpha_{i|j} v^j e^i, \quad (13.6)$$

In particular,

$$\tilde{\nabla} \alpha \cdot \vec{e}_j = \tilde{\nabla}_{\vec{e}_j} \alpha = \sum_{i=1}^m \alpha_{i|j} e^i, \quad \text{and} \quad \alpha_{i|j} = (\tilde{\nabla}_{\vec{e}_j} \alpha) \cdot \vec{e}_i. \quad (13.7)$$

Corollary 13.4

$$\alpha_{i|j} = \frac{\partial \alpha_i}{\partial q^j} - \sum_{k=1}^m \alpha_k \gamma_{ji}^k. \quad (13.8)$$

(In particular we find the previous result $\tilde{\nabla}_{\vec{e}_j} e^k = -\sum_i \gamma_{ji}^k e^i$, cf. (13.1).)

Proof. $\alpha_i = \alpha \cdot \vec{e}_i$ gives $d\alpha_i \cdot \vec{e}_j = \tilde{\nabla}_{\vec{e}_j} \alpha_i = \tilde{\nabla}_{\vec{e}_j} (\alpha \cdot \vec{e}_i) = (\tilde{\nabla}_{\vec{e}_j} \alpha) \cdot \vec{e}_i + \alpha \cdot (\nabla_{\vec{e}_j} \vec{e}_i)$, thus

$$\alpha_{i|j} = (\tilde{\nabla}_{\vec{e}_j} \alpha) \cdot \vec{e}_i = d\alpha_i \cdot \vec{e}_j - \alpha \cdot \nabla_{\vec{e}_j} \vec{e}_i = \frac{\partial \alpha_i}{\partial q^j} - \left(\sum_{\ell=1}^m \alpha_\ell e^\ell \right) \cdot \left(\sum_{k=1}^m \gamma_{ji}^k \vec{e}_k \right) = \frac{\partial \alpha_i}{\partial q^j} - \sum_{k=1}^m \alpha_k \gamma_{ji}^k. \quad (13.9)$$

▀

Example 13.5 If $\alpha = df = \sum_i \frac{\partial f}{\partial q^i} dq^i$, then $\alpha_i = \frac{\partial f}{\partial q^i}$, and we get

$$\nabla_{\vec{e}_j} (df) = \sum_{i=1}^m \left(\frac{\partial f}{\partial q^i} \right)_{|j} dq^i \quad \text{et} \quad \nabla(df) = \sum_{i=1}^m \frac{\partial f}{\partial q^i} |_{|j} dq^i \otimes dq^j, \quad (13.10)$$

with

$$\left(\frac{\partial f}{\partial q^i} \right)_{|j} = \frac{\partial^2 f}{\partial q^j \partial q^i} - \sum_{k=1}^m \frac{\partial f}{\partial q^k} \gamma_{ji}^k \quad (= (\nabla_{\vec{e}_j} df)_i = (\nabla_{\vec{e}_j} df) \cdot \vec{e}_i). \quad (13.11)$$

And $\vec{v} = \sum_j v^j \vec{e}_j$ gives

$$\nabla_{\vec{v}} (df) = \sum_{i,j=1}^m \left(\frac{\partial f}{\partial q^i} \right)_{|j} v^j dq^i. \quad (13.12)$$

▀

14 Lie autonomous derivative of a differential form

14.1 Definition

Let $\vec{v}, \vec{w} \in \Gamma(S)$ and $\alpha \in \Omega^1(S)$. Thus $f = \alpha \cdot \vec{w} \in \mathcal{F}(S)$, and (12.2) gives

$$\begin{aligned} \mathcal{L}_{\vec{v}}^0 (\alpha \cdot \vec{w}) &= \tilde{\nabla}_{\vec{v}} (\alpha \cdot \vec{w}) = (\tilde{\nabla}_{\vec{v}} \alpha) \cdot \vec{w} + \alpha \cdot (\nabla_{\vec{v}} \vec{w}) = (\tilde{\nabla}_{\vec{v}} \alpha) \cdot \vec{w} + \alpha \cdot (\nabla_{\vec{w}} \vec{v} + \mathcal{L}_{\vec{v}}^0 \vec{w}) \\ &= (\mathcal{L}_{\vec{v}}^0 \alpha) \cdot \vec{w} + \alpha \cdot (\mathcal{L}_{\vec{v}}^0 \vec{w}) \end{aligned} \quad (14.1)$$

as soon as:

Definition 14.1 The autonomous Lie derivative of a differential form $\alpha \in \Omega^1(S)$ along $\vec{v} \in \Gamma(S)$ is the differential form $\mathcal{L}_{\vec{v}}^0 \alpha \in \Omega^1(S)$ defined by, for all $\vec{w} \in \Gamma(S)$,

$$\begin{aligned} (\tilde{\mathcal{L}}_{\vec{v}}^0 \alpha) \cdot \vec{w} &:= (\tilde{\nabla}_{\vec{v}} \alpha) \cdot \vec{w} + \alpha \cdot (\nabla_{\vec{w}} \vec{v}) \\ &= (\tilde{\nabla} \alpha \cdot \vec{v}) \cdot \vec{w} + \alpha \cdot (\nabla \vec{v} \cdot \vec{w}) \stackrel{\text{written}}{=} (\mathcal{L}_{\vec{v}}^0 \alpha) \cdot \vec{w}, \end{aligned} \quad (14.2)$$

the last notation (without tilde) if there is no ambiguity.

Doing so, we have defined

$$\mathcal{L}_{\vec{v}}^0 : \begin{cases} \Omega^1(S) & \rightarrow \Omega^1(S) \\ \alpha & \rightarrow \mathcal{L}_{\vec{v}}^0 \alpha = \tilde{\nabla} \alpha \cdot \vec{v} + \alpha \cdot \nabla \vec{v} = \frac{D\alpha}{dt} + \alpha \cdot \nabla \vec{v}. \end{cases} \quad (14.3)$$

14.2 Components of $\mathcal{L}_{\vec{v}}^0 \alpha := \nabla_{\vec{v}} \alpha + \alpha \nabla \vec{v}$

Let $\vec{v} \in \Gamma(S)$ and $\vec{v} = \sum_{i=1}^m v^i \vec{e}_i$. Let $\nabla \vec{v} = \sum_{i,j=1}^m v_{|j}^i \vec{e}_i \otimes e^j$, where $v_{|j}^i = \frac{\partial v^i}{\partial q^j} + \sum_{k=1}^m \gamma_{jk}^i v^k$, cf. (7.4), and $[\nabla \vec{v}(p)]_{|\vec{e}} = [v_{|j}^i(p)]$ is the Jacobian matrix of \vec{v} at p .

Let $\alpha \in \Omega^1(S)$ and $\alpha = \sum_{i=1}^m \alpha_i e^i$. Let $\tilde{\nabla} \alpha = \sum_{i,j=1}^m \alpha_{i|j} e^i \otimes e^j$, so $\alpha_{i|j} = \frac{\partial \alpha_i}{\partial q^j} - \sum_{k=1}^m \alpha_k \gamma_{ji}^k$, cf. (13.8), and $[\tilde{\nabla} \alpha(p)]_{|\vec{e}} = [\alpha_{i|j}(p)]$ is the Jacobian matrix of α at p .

Corollary 14.2

$$\mathcal{L}_{\vec{v}}^0 \alpha = \sum_{i,j=1}^m (\alpha_i \frac{\partial v^i}{\partial q^j} + v^i \frac{\partial \alpha_j}{\partial q^i}) e^j. \quad (14.4)$$

Proof. $\tilde{\nabla} \alpha \cdot \vec{v} = \sum_{i,j=1}^m \alpha_{i|j} v^j e^i = \sum_{i,j=1}^m \alpha_{j|i} v^i e^j$ and $\alpha \cdot \nabla \vec{v} = \sum_{i,j=1}^m \alpha_i v_{|j}^i e^j$, thus $\mathcal{L}_{\vec{v}}^0 \alpha = \sum_{i,j=1}^m (\alpha_{j|i} v^i + \alpha_i v_{|j}^i) e^j$; So $\mathcal{L}_{\vec{v}}^0 \alpha = \sum_j (\mathcal{L}_{\vec{v}}^0 \alpha)_j e^j$ gives

$$(\mathcal{L}_{\vec{v}}^0 \alpha)_j = \sum_{i=1}^m \alpha_{j|i} v^i + \alpha_i v_{|j}^i = \sum_{i=1}^m \frac{\partial \alpha_j}{\partial q^i} v^i - \sum_{i,k=1}^m \alpha_k \gamma_{ij}^k v^j + \sum_{i=1}^m \alpha_i \frac{\partial v^i}{\partial q^j} + \sum_{i,k=1}^m \alpha_i \gamma_{jk}^i v^k. \quad (14.5)$$

And $\gamma_{ij}^k = \gamma_{ji}^k$ (coordinate system), hence (14.4). \blacksquare

In particular

$$\mathcal{L}_{\vec{v}}^0 e^i = \sum_{j=1}^m \frac{\partial v^i}{\partial q^j} e^j, \quad \text{i.e. } (\mathcal{L}_{\vec{v}}^0 e^i)_j = \frac{\partial v^i}{\partial q^j}, \quad \text{i.e. } [\mathcal{L}_{\vec{v}}^0 e^i] = \begin{pmatrix} \frac{\partial v^i}{\partial q^1} & \dots & \frac{\partial v^i}{\partial q^m} \end{pmatrix}, \quad (14.6)$$

and

$$\mathcal{L}_{\vec{e}_i}^0 \alpha = \sum_{j=1}^m \frac{\partial \alpha_i}{\partial q^j} e^j, \quad \text{i.e. } (\mathcal{L}_{\vec{e}_i}^0 \alpha)_j = \frac{\partial \alpha_i}{\partial q^j}, \quad \text{i.e. } [\mathcal{L}_{\vec{e}_i}^0 \alpha] = \begin{pmatrix} \frac{\partial \alpha_i}{\partial q^1} & \dots & \frac{\partial \alpha_i}{\partial q^m} \end{pmatrix}, \quad (14.7)$$

and, for all i, j ,

$$\mathcal{L}_{\vec{e}_i}^0 e^j = 0. \quad (14.8)$$

14.3 “Universal” derivation property

Hence,

$$\mathcal{L}_{\vec{v}}^0 (b.c) = (\mathcal{L}_{\vec{v}}^0 b).c + b.(\mathcal{L}_{\vec{v}}^0 c), \quad (14.9)$$

whenever $b.c$ is meaningful, that is, whenever

- 1- $b, c \in \mathcal{F}(S)$ (and $b.c = bc \in \mathcal{F}(S)$),
- 2- $b \in \mathcal{F}(S)$ and $c \in \Gamma(S)$ (and $b.c = bc \in \Gamma(S)$),
- 3- $b \in \mathcal{F}(S)$ and $c \in \Omega^1(S)$ (and $b.c = bc \in \Omega^1(S)$),
- 4- $b \in \Omega^1(S)$ and $c \in \Gamma(S)$ (and $b.c \in \mathcal{F}(S)$).

15 Connection ∇ on $T_s^r(S)$

15.1 Covariant derivative $\nabla_{\vec{v}} T$

Let ∇ be a connection in S . If $\vec{v} \in \Gamma(S)$, if $f \in \mathcal{F}(S)$, $\vec{w} \in \Gamma(S)$ and $\alpha \in \Omega^1(S)$, then the following covariant derivative along \vec{v} have been defined: $\tilde{\nabla}_{\vec{v}} f = \text{written } \nabla_{\vec{v}} f$, $\nabla_{\vec{v}} \vec{w}$, $\tilde{\nabla}_{\vec{v}} \alpha = \text{written } \nabla_{\vec{v}} \alpha$.

Definition 15.1 Let $\vec{v} \in \Gamma(S)$. Let $T_1 \in T_{s_1}^{r_1}(S)$ and $T_2 \in T_{s_2}^{r_2}(S)$. The covariant derivative of $T = T_1 \otimes T_2$ along \vec{v} is defined by

$$\nabla_{\vec{v}} T = \nabla_{\vec{v}} (T_1 \otimes T_2) := (\nabla_{\vec{v}} T_1) \otimes T_2 + T_1 \otimes (\nabla_{\vec{v}} T_2) \stackrel{\text{written}}{=} \frac{DT}{dt}. \quad (15.1)$$

This defines $\nabla_{\vec{v}} : T_s^r(S) \rightarrow T_s^r(S)$ the covariant derivative along \vec{v} for tensors.

Example 15.2 Coordinate system Φ , and with $(\vec{e}_i(p))$ the coordinate basis at $p \in S$; If $T(p) = e^i(p) \otimes e^j(p)$, then

$$\begin{aligned}\nabla_{\vec{e}_k}(e^i \otimes e^j) &= (\nabla_{\vec{e}_k} e^i) \otimes e^j + e^i \otimes (\nabla_{\vec{e}_k} e^j) \\ &= -\sum_{\ell=1}^m \gamma_{k\ell}^i e^\ell \otimes e^j - \sum_{\ell=1}^m \gamma_{k\ell}^j e^i \otimes e^\ell.\end{aligned}\quad (15.2)$$

So, if $\vec{v} = \sum_k v^k \vec{e}_k$ then

$$\nabla_{\vec{v}}(e^i \otimes e^j) = -\sum_{k,\ell=1}^m \gamma_{k\ell}^i v^k e^\ell \otimes e^j - \sum_{\ell=1}^m \gamma_{k\ell}^j v^k e^i \otimes e^\ell. \quad (15.3)$$

▀

15.2 Differential ∇T

Definition 15.3 If $T \in T_s^r(S)$ then its differential is the tensor $\nabla T \in T_{s+1}^r(S)$ defined by, for all $\vec{v} \in \Gamma(S)$,

$$\nabla T \cdot \vec{v} = \nabla_{\vec{v}} T, \quad (15.4)$$

That is, for all $\alpha_i \in \Omega^1(S)$ and all $\vec{w}_i \in \Gamma(S)$,

$$(\nabla T \cdot \vec{v})(\alpha_1, \dots, \alpha_r, \vec{w}_1, \dots, \vec{w}_s) = \nabla_{\vec{v}} T(\alpha_1, \dots, \alpha_r, \vec{w}_1, \dots, \vec{w}_s). \quad (15.5)$$

Remark: we have implicitly used the expression “ ∇T is a tensor $T_{s+1}^r(S)$ ” to mean that the associated multilinear form $Z : (\Omega^1(S))^r \times (\Gamma(S))^{s+1} \rightarrow \mathbb{R}$ defined by

$$Z(\alpha_1, \dots, \alpha_r, \vec{w}_1, \dots, \vec{w}_s, \vec{v}) := \nabla_{\vec{v}} T(\alpha_1, \dots, \alpha_r, \vec{w}_1, \dots, \vec{w}_s) = \nabla T(\alpha_1, \dots, \alpha_r, \vec{w}_1, \dots, \vec{w}_s) \cdot \vec{v}. \quad (15.6)$$

See next §: the components of $\nabla_{\vec{v}} T$ are indeed the components of Z .

Example 15.4 Continuing example 15.2: (15.3) gives

$$\nabla(e^i \otimes e^j) = -\sum_{i,j,k=1}^m \gamma_{k\ell}^i e^\ell \otimes e^j \otimes e^k - \sum_{i,j,k=1}^m \gamma_{k\ell}^j e^i \otimes e^\ell \otimes e^k \in T_3^0(S). \quad (15.7)$$

▀

15.3 Components $T_{j_1 \dots j_s | k}^{i_1 \dots i_r}$ of ∇T

Let $T \in T_s^r(S)$ and

$$T = \sum_{i_1, \dots, i_r, j_1, \dots, j_s=1}^m T_{j_1 \dots j_s}^{i_1 \dots i_r} \vec{e}_{i_1} \otimes \dots \otimes \vec{e}_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s}. \quad (15.8)$$

Let $T_{j_1 \dots j_s | k}^{i_1 \dots i_r}$ be the components of ∇T , that is,

$$\nabla T = \sum_{i_1, \dots, i_r, j_1, \dots, j_s, k=1}^m T_{j_1 \dots j_s | k}^{i_1 \dots i_r} \vec{e}_{i_1} \otimes \dots \otimes \vec{e}_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s} \otimes e^k. \quad (15.9)$$

So

$$\nabla_{\vec{e}_k} T = \nabla T \cdot \vec{e}_k = \sum_{i_1, \dots, i_r, j_1, \dots, j_s=1}^m T_{j_1 \dots j_s | k}^{i_1 \dots i_r} \vec{e}_{i_1} \otimes \dots \otimes \vec{e}_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s}, \quad (15.10)$$

and

$$\nabla_{\vec{v}} T = \nabla T \cdot \vec{v} = \sum_{i_1, \dots, i_r, j_1, \dots, j_s, k=1}^m T_{j_1 \dots j_s | k}^{i_1 \dots i_r} v^k \vec{e}_{i_1} \otimes \dots \otimes \vec{e}_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s}. \quad (15.11)$$

And (15.8) gives

$$\begin{aligned}\nabla T \cdot \vec{e}_k &= \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_s=1}}^m (dT_{j_1, \dots, j_s}^{i_1, \dots, i_r} \cdot \vec{e}_k) \vec{e}_{i_1} \otimes \dots \otimes e^{j_s} \\ &\quad + T_{j_1, \dots, j_s}^{i_1, \dots, i_r} (\nabla_{\vec{e}_k} \vec{e}_{i_1}) \otimes \vec{e}_{i_2} \otimes \dots \otimes e^{j_s} + \dots + T_{j_1, \dots, j_s}^{i_1, \dots, i_r} \vec{e}_{i_1} \otimes \dots \otimes e^{j_{s-1}} \otimes (\nabla_{\vec{e}_k} e^{j_s}).\end{aligned}$$

Thus, with $\nabla_{\vec{e}_k} \vec{e}_i = \sum_{\ell} \gamma_{ki}^{\ell} \vec{e}_{\ell}$ and $\nabla_{\vec{e}_k} \vec{e}^j = -\sum_{\ell} \gamma_{j\ell}^k e^{\ell}$ we get

$$\begin{aligned} T_{j_1 \dots j_s}^{i_1 \dots i_r} |k = & \frac{\partial T_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial q^k} + \sum_{\ell=1}^m T_{j_1 \dots j_s}^{\ell i_2 \dots i_r} \gamma_{k\ell}^{i_1} + \sum_{\ell=1}^m T_{j_1 \dots j_s}^{i_1 \ell i_3 \dots i_r} \gamma_{k\ell}^{i_2} + \dots \\ & - \sum_{\ell=1}^m T_{\ell j_2 \dots j_{s-1} \ell}^{i_1 \dots i_r} \gamma_{j_1 k}^{\ell} - \sum_{\ell=1}^m T_{j_1 \ell j_3 \dots j_{s-1} \ell}^{i_1 \dots i_r} \gamma_{j_2 k}^{\ell} - \dots \end{aligned} \quad (15.12)$$

16 Riemannian metric

Definition 16.1 A Riemannian metric is a symmetric positive-definite tensor $g \in T_2^0(S)$, that is, $g \in T_2^0(S)$ is C^∞ and $g_p = g(p)$ is a dot product in $T_p S$ for all $p \in S$.

E.g., in \mathbb{R}^n , the usual Riemannian metric relative to a Euclidean basis (\vec{E}_i) is the Euclidean dot product: $(\cdot, \cdot)_{\mathbb{R}^n} = g = \sum_{i=1}^n dx^i \otimes dx^i$, independent of p , where $(dx^i) = (E^i)$ is the dual basis of (\vec{E}_i) .

And if S is a surface in \mathbb{R}^n , the usual Riemannian metric is the restriction to S of a Euclidean dot product in \mathbb{R}^n (depends on the choice of a unit of measurement), cf. (6.6).

Quantification: Let $\Phi : U \subset \mathbb{R}^m \rightarrow S$ be a coordinate system for S , let $(\vec{e}_i(p))_{i=1, \dots, m}$ be the associated basis and $(e^i(p))_{i=1, \dots, m}$ be the dual basis at $p \in S$. Let $g \in T_2^0(S)$ be a metric and let g_{ij} be its components relative to (\vec{e}_i) , that is,

$$g = \sum_{i,j=1}^m g_{ij} e^i \otimes e^j, \quad g_{ij} = g(\vec{e}_i, \vec{e}_j), \quad [g]_{|\vec{e}} = [g_{ij}]. \quad (16.1)$$

Hence, if $\vec{v}, \vec{w} \in \Gamma(S)$, $\vec{v} = \sum_{i=1}^m v^i \vec{e}_i$, $\vec{w} = \sum_{i=1}^m w^i \vec{e}_i$, then

$$g(\vec{v}, \vec{w}) = \sum_{i,j=1}^m g_{ij} v^i w^j = [\vec{v}]_{|\vec{e}}^T [g]_{|\vec{e}} [\vec{w}]_{|\vec{e}}. \quad (16.2)$$

Example 16.2 \mathbb{R}^2 , polar system, usual Riemannian metric:

$$(\cdot, \cdot)_{\mathbb{R}^n} = g_p = dr \otimes dr + r^2 d\theta \otimes d\theta, \quad [g_p]_{pol} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad (16.3)$$

since $(\vec{e}_1(p), \vec{e}_2(p))_{\mathbb{R}^n} = 0 = g_{12}$, $\|\vec{e}_1(p)\|_{\mathbb{R}^n}^2 = 1 = g_{11}$ and $\|\vec{e}_2(p)\|_{\mathbb{R}^n}^2 = r^2 = g_{22}$.

Other calculation: $[g_p]_{pol} = P^T \cdot [(\cdot, \cdot)_{\mathbb{R}^n}]_{\mathbb{R}^n} \cdot P$ (change of basis formula for bilinear forms), thus $[g_p]_{pol} = P^T \cdot I \cdot P = P^T \cdot P$, with $P = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$ the transition matrix from (\vec{E}_i) to $(\vec{e}_i(p))$. \blacksquare

Example 16.3 \mathbb{R}^2 , polar system, $S = C(\vec{0}, R)$, usual Riemannian metric:

$$g_p = R^2 d\theta \otimes d\theta, \quad [g_p] = (R^2),$$

since $(\vec{e}_2(p), \vec{e}_2(p))_{\mathbb{R}^n} = R^2$. \blacksquare

Example 16.4 \mathbb{R}^3 , GPS system $\Phi(r, \theta, \varphi) = \begin{pmatrix} x = r \cos \theta \cos \varphi \\ y = r \sin \theta \cos \varphi \\ z = r \sin \varphi \end{pmatrix}$, usual Riemannian metric:

$$g_p = dr \otimes dr + r^2 \cos^2 \varphi d\theta \otimes d\theta + r^2 d\varphi \otimes d\varphi, \quad [g_{ij}(p)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 \cos^2 \varphi & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \quad (16.4)$$

\blacksquare

Example 16.5 \mathbb{R}^3 , GPS spherical system, $S = S(\vec{0}, R)$, usual Riemannian metric:

$$g_p = R^2 \cos^2 \varphi d\theta \otimes d\theta + R^2 d\varphi \otimes d\varphi, \quad [g_{ij}(p)] = \begin{pmatrix} R^2 \cos^2 \varphi & 0 \\ 0 & R^2 \end{pmatrix}. \quad (16.5)$$

\blacksquare

Exercise 16.6 Check $\nabla g = 0$ for polar coordinate, directly from $g_p = dr \otimes dr + r^2 d\theta \otimes d\theta$, cf. (16.3).

Answer.

$$dg_p \cdot \vec{v} = (d^2 r \cdot \vec{v}) \otimes dr + dr \otimes (d^2 r \cdot \vec{v}) + (d(r^2) \cdot \vec{v}) d\theta \otimes d\theta + r^2 (d^2 \theta \cdot \vec{v}) \otimes d\theta + r^2 d\theta \otimes (d^2 \theta \cdot \vec{v}).$$

And $d(r^2) = 2r dr$ with $d^2 r = r d\theta \otimes d\theta$ and $d^2 \theta = -\frac{1}{r}(dr \otimes d\theta + d\theta \otimes dr)$, cf. (2.20), thus,

$$\begin{aligned} dg_p \cdot \vec{e}_1 &= 0 + 0 + 2r d\theta \otimes d\theta + r^2 \left(-\frac{1}{r}\right) d\theta \otimes d\theta + r^2 \left(-\frac{1}{r}\right) d\theta \otimes d\theta = 0, \\ dg_p \cdot \vec{e}_2 &= r d\theta \otimes dr + r dr \otimes d\theta + 0 - r^2 \frac{1}{r} dr \otimes d\theta - r^2 \frac{1}{r} d\theta \otimes dr = 0. \end{aligned}$$

Thus $dg_p \cdot \vec{v} = 0$ for all \vec{v} . ▀

Exercise 16.7 \mathbb{R}^3 , GPS system, usual Riemannian metric: calculate dg with (16.4).

Answer. $g \in T_2^0(\mathbb{R}^3)$, thus $dg \in T_3^0(\mathbb{R}^3)$, $dg = \sum_{ijk} h_{ijk} e^i \otimes e^j \otimes e^k$, and $dg \cdot \vec{v} \in T_2^0(\mathbb{R}^3)$ is given by

$$\begin{aligned} dg \cdot \vec{v} &= (d^2 r \cdot \vec{v}) \otimes dr + dr \otimes (d^2 r \cdot \vec{v}) \\ &+ [2r \cos^2 \varphi (dr \cdot \vec{v}) - r^2 \sin \varphi \cos \varphi (d\varphi \cdot \vec{v})] d\theta \otimes d\theta + r^2 \cos^2 \varphi ((d^2 \theta \cdot \vec{v}) \otimes d\theta + d\theta \otimes (d^2 \theta \cdot \vec{v})) \\ &+ 2r (dr \cdot \vec{v}) d\varphi \otimes d\varphi + r^2 ((d^2 \varphi \cdot \vec{v}) \otimes d\varphi + d\varphi \otimes (d^2 \varphi \cdot \vec{v})). \end{aligned}$$

Thus with the Christoffel symbols, cf. (3.42)-(3.43),

$$\begin{aligned} dg \cdot \vec{e}_1 &= 0 + 0 + [2r \cos^2 \varphi + 0] d\theta \otimes d\theta + r^2 \cos^2 \varphi \left(-\frac{1}{r} d\theta \otimes d\theta - \frac{1}{r} d\theta \otimes d\theta\right) \\ &+ 2r d\varphi \otimes d\varphi + r^2 \left(-\frac{1}{r} d\varphi \otimes d\varphi - \frac{1}{r} d\varphi \otimes d\varphi\right) = 0 \end{aligned}$$

Similarly $dg \cdot \vec{e}_2 = dg \cdot \vec{e}_3 = 0$. Thus $dg = 0$. (Trivial with $g = \sum_{i=1}^3 dx^i \otimes dx^i$.) ▀

17 Metric related to a connection

17.1 ∇g and $g_{ij|k}$

Let $g \in T_2^0(S)$ and $g = \sum_{i,j=1}^m g_{ij} e^i \otimes e^j$. Then (15.12) (or direct calculation) gives

$$\nabla_{\vec{e}_k} g = \sum_{i,j=1}^m g_{ij|k} e^i \otimes e^j \quad \text{with} \quad g_{ij|k} = \frac{\partial g_{ij}}{\partial q^k} - \sum_{\ell} g_{\ell j} \gamma_{ki}^{\ell} - \sum_{\ell} g_{i\ell} \gamma_{kj}^{\ell}, \quad (17.1)$$

and

$$\nabla_{\vec{v}} g = \sum_{i,j,k=1}^m g_{ij|k} v^k e^i \otimes e^j = \frac{Dg}{dt}. \quad (17.2)$$

(The last equality refers to the derivation along an integral curve of \vec{v} , as in (5.3).)

Proposition 17.1 Let $g \in T_2^0(S)$. For all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$:

$$\tilde{\nabla}_{\vec{v}}(g(\vec{u}, \vec{w})) = (\nabla_{\vec{v}} g)(\vec{u}, \vec{w}) + g(\nabla_{\vec{v}} \vec{u}, \vec{w}) + g(\vec{u}, \nabla_{\vec{v}} \vec{w}), \quad (17.3)$$

that is,

$$\frac{D(g(\vec{u}, \vec{w}))}{dt} = \left(\frac{Dg}{dt}\right)(\vec{u}, \vec{w}) + g\left(\frac{D\vec{u}}{dt}, \vec{w}\right) + g\left(\vec{u}, \frac{D\vec{w}}{dt}\right). \quad (17.4)$$

(Derivation formula $(fgh)' = f'gh + fg'h + fgh'$.)

Proof. $g(\vec{u}, \vec{w}) = \sum_{ij} g_{ij} u^i w^j \in \mathcal{F}(S)$ gives

$$\begin{aligned} \tilde{\nabla}_{\vec{e}_k} (g(\vec{u}, \vec{w})) &= \sum_{ij} (\tilde{\nabla}_{\vec{e}_k} g_{ij}) u^i w^j + \sum_{ij} g_{ij} (\tilde{\nabla}_{\vec{e}_k} u^i) w^j + \sum_{ij} g_{ij} u^i (\tilde{\nabla}_{\vec{e}_k} w^j) \\ &= \sum_{ij} \frac{\partial g_{ij}}{\partial q^k} u^i w^j + \sum_{ij} g_{ij} \frac{\partial u^i}{\partial q^k} w^j + \sum_{ij} g_{ij} u^i \frac{\partial w^j}{\partial q^k}. \end{aligned} \quad (17.5)$$

Thus (17.1) and (11.18) give

$$\begin{aligned}
\tilde{\nabla}_{\bar{e}_k}(g(\vec{u}, \vec{w})) &= \sum_{ij} \left(g_{ij|k} + \sum_{\ell} g_{\ell j} \gamma_{ki}^{\ell} + g_{i\ell} \gamma_{kj}^{\ell} \right) u^i w^j \\
&\quad + \sum_{ij} g_{ij} \left(u^i_{|k} - \sum_{\ell} \gamma_{k\ell}^i u^{\ell} \right) w^j + \sum_{ij} g_{ij} u^i \left(w^j_{|k} - \sum_{\ell} \gamma_{k\ell}^j w^{\ell} \right) \\
&= (\nabla_{\bar{e}_k} g)(\vec{u}, \vec{w}) + g(\nabla_{\bar{e}_k} \vec{u}, \vec{w}) + g(\vec{u}, \nabla_{\bar{e}_k} \vec{w}) \\
&\quad + \sum_{ij\ell} (g_{\ell j} \gamma_{ki}^{\ell} + g_{i\ell} \gamma_{kj}^{\ell}) u^i w^j - \sum_{ij\ell} g_{ij} \gamma_{k\ell}^i u^{\ell} w^j - \sum_{ij\ell} g_{ij} \gamma_{k\ell}^j u^i w^{\ell} \\
&= (\nabla_{\bar{e}_k} g)(\vec{u}, \vec{w}) + g(\nabla_{\bar{e}_k} \vec{u}, \vec{w}) + g(\vec{u}, \nabla_{\bar{e}_k} \vec{w}) \\
&\quad + \sum_{ij\ell} (g_{\ell j} \gamma_{ki}^{\ell} + g_{i\ell} \gamma_{kj}^{\ell} - g_{\ell j} \gamma_{ki}^{\ell} - g_{i\ell} \gamma_{kj}^{\ell}) u^i w^j,
\end{aligned}$$

and the last sum vanishes. With $\tilde{\nabla}_{\bar{v}} = \sum_k v^k \tilde{\nabla}_{\bar{e}_k}$ we get (17.3). \blacksquare

Exercise 17.2 Let $g(\cdot, \cdot)$ be a metric in \mathbb{R}^n . Let $\alpha \in T_1^0(\mathbb{R}^n)$. Let $\bar{\alpha}_g \in \Gamma(\mathbb{R}^n)$ be the $(\cdot, \cdot)_g$ -Riesz representation vector of α , that is, for all $\vec{w} \in \mathbb{R}^n$:

$$\alpha \cdot \vec{w} = g(\bar{\alpha}_g, \vec{w}). \quad (17.6)$$

Let $\bar{v} \in T_0^1(\mathbb{R}^n)$. Prove (with (12.3)):

$$\tilde{\nabla}_{\bar{v}} \alpha \cdot \vec{w} = g(\nabla_{\bar{v}} \bar{\alpha}_g, \vec{w}) + \nabla_{\bar{v}} g(\bar{\alpha}_g, \vec{w}). \quad (17.7)$$

In particular in \mathbb{R}^n with $(\cdot, \cdot)_g$ a dot product (a uniform metric), $\tilde{\nabla}_{\bar{v}} \alpha \cdot \vec{w} = g(\nabla_{\bar{v}} \bar{\alpha}_g, \vec{w})$.

Answer. $\tilde{\nabla}_{\bar{v}} \alpha \cdot \vec{w} = \tilde{\nabla}_{\bar{v}}(\alpha \cdot \vec{w}) - \alpha \cdot \nabla_{\bar{v}} \vec{w} = \tilde{\nabla}_{\bar{v}}(g(\bar{\alpha}_g, \vec{w})) - g(\bar{\alpha}_g, \nabla_{\bar{v}} \vec{w}) = \tilde{\nabla}_{\bar{v}} g(\bar{\alpha}_g, \vec{w}) + g(\tilde{\nabla}_{\bar{v}} \bar{\alpha}_g, \vec{w}) + g(\bar{\alpha}_g, \tilde{\nabla}_{\bar{v}} \vec{w}) - g(\bar{\alpha}_g, \nabla_{\bar{v}} \vec{w}) = \tilde{\nabla}_{\bar{v}} g(\bar{\alpha}_g, \vec{w}) + g(\tilde{\nabla}_{\bar{v}} \bar{\alpha}_g, \vec{w})$, thus (17.7). And if $g(\cdot, \cdot)$ is a uniform metric in \mathbb{R}^n then $\nabla_{\bar{v}} g = 0$ for all \bar{v} (indeed choose a Cartesian basis so that the g_{ij} are constants and use (17.1)). \blacksquare

Proposition 17.3 Let S be a surface in \mathbb{R}^n , $g(\cdot, \cdot) = (\cdot, \cdot)_g$ be the usual Riemannian metric in S and ∇ be the associated Riemannian connection ($\nabla_{\bar{v}} \vec{w} = \text{Proj}_{TS}(d\vec{w} \cdot \bar{v})$). Then

$$\nabla g = 0. \quad (17.8)$$

Proof. $\nabla g = 0$ (take a Cartesian basis), thus $0 = \sum_{i=1}^n g_{ij|k} e^i \otimes e^j \otimes e^k$ in \mathbb{R}^n , thus $(\nabla g)|_{\Gamma(S)} = \sum_{i=1}^m g_{ij|k} e^i \otimes e^j \otimes e^k = 0$. \blacksquare

17.2 Killing vectors and metrics (relative to a connection)

Definition 17.4 Let ∇ be a connection in S and $g \in T_2^0(S)$ be a metric in S . A Killing vector field $\bar{v} \in \Gamma(S)$, relative to $g(\cdot, \cdot)$ and ∇ , is a vector field such that

$$\nabla_{\bar{v}} g = 0 = \frac{Dg}{dt}. \quad (17.9)$$

(The last equality refers to the derivation along an integral curve of \bar{v} .)

Definition 17.5 Let ∇ be a connection in S . A metric of Killing $g \in T_2^0(S)$ relative to ∇ is a metric such that

$$\nabla g = 0, \quad (17.10)$$

that is, such that $\nabla_{\bar{v}} g = 0$ for all $\bar{v} \in \Gamma(S)$.

(The German mathematician Wilhlem Killing, early 20th century, was a student of Weierstrass.)

Example 17.6 A Riemannian metric on a surface $S \subset \mathbb{R}^n$ is a metric of Killing relative to the usual Riemannian connection, cf. (17.8). \blacksquare

In other words, a metric of Killing is uniform in S (equation $\nabla g = 0$). And with (17.3) we have, for all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$,

$$\tilde{\nabla}_{\bar{v}}(\vec{u}, \vec{w})_g = (\nabla_{\bar{v}} \vec{u}, \vec{w})_g + (\vec{u}, \nabla_{\bar{v}} \vec{w})_g, \quad \text{i.e.} \quad \frac{D}{dt}((\vec{u}, \vec{w})_g) = \left(\frac{D\vec{u}}{dt}, \vec{w} \right)_g + \left(\vec{u}, \frac{D\vec{w}}{dt} \right)_g. \quad (17.11)$$

Remark 17.7 If $(\cdot, \cdot)_g$ is a metric of Killing, then $\nabla_{\bar{v}} g = 0$ for all \bar{v} , that is, the first order derivatives vanish. But the second order derivatives don't vanish: They will give the curvature. \blacksquare

17.3 Levi-Civita theorem 1

Theorem 17.8 (Levi-Civita.) *If $g(\cdot, \cdot)$ is a metric of Killing relative to a connection ∇ , then, for all i, j, k ,*

$$((\nabla_{\vec{e}_k} g)_{ij} =) \quad g_{ij|k} = 0, \quad (17.12)$$

i.e.,

$$\frac{\partial g_{ij}}{\partial q^k} = \sum_{\ell=1}^m (g_{i\ell} \gamma_{jk}^\ell + g_{j\ell} \gamma_{ik}^\ell). \quad (17.13)$$

And:

$$2 \sum_{\ell=1}^m g_{i\ell} \gamma_{jk}^\ell = \frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i}, \quad (17.14)$$

that is, with $[g^{ij}] := [g_{ij}]^{-1}$,

$$\gamma_{jk}^i = \frac{1}{2} \sum_{\ell} g^{i\ell} \left(\frac{\partial g_{\ell j}}{\partial q^k} + \frac{\partial g_{\ell k}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell} \right). \quad (17.15)$$

Proof. $\nabla g = 0$ gives (17.12), thus (17.13), cf. (17.1). Thus (17.14) (circular permutation and sum). And, for all j, k , (17.14) reads $2[g] \cdot [\vec{a}] = [\vec{b}]$, thus $[\vec{a}] = \frac{1}{2}[g]^{-1} \cdot [\vec{b}]$, thus (17.15). \blacksquare

Corollary 17.9 *Let S be a surface of dimension $n-1$ in \mathbb{R}^n and $(\vec{e}_1, \dots, \vec{e}_{n-1})$ be the coordinate basis. Let $\vec{e}_n \in \Gamma(\mathbb{R}^n)$ be a vector field in \mathbb{R}^n such that $\|\vec{e}_n(p)\|_g = 1$ and $g(\vec{e}_n(p), \vec{e}_j(p)) = \delta_{nj}$ for all $j = 1, \dots, n-1$ and $p \in S$. Then let $g_+ = \sum_{i,j=1}^n g_{ij} e^i \otimes e^j$, hence $g_{nj} = \delta_{nj}$ for all $j \in [1, n]_{\mathbb{N}}$, and*

$$[g_+]_{|\vec{e}} = \begin{pmatrix} g_{11} & \cdots & g_{1,n-1} & 0 \\ \vdots & & \vdots & \vdots \\ g_{n-1,1} & \cdots & g_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \quad (17.16)$$

In particular,

$$\frac{\partial g_{nj}}{\partial q^k} = 0, \quad \text{and} \quad 2\gamma_{ij}^n = -\frac{\partial g_{ij}}{\partial q^n}. \quad (17.17)$$

Example 17.10 Sphere $S(\vec{0}, R) \subset \mathbb{R}^3$ and $\Phi : (r, \theta, \varphi) \rightarrow \Phi(r, \theta, \varphi)$ the GPS coordinate system, cf. (3.27). Let $\Psi(\theta, \varphi, r) := \Phi(r, \theta, \varphi)$, then $\vec{f}_1(p) = \Psi_{,1}(p) = \frac{\partial \Psi}{\partial \theta}(p) = \Phi_{,2}(p) = \vec{e}_2(p)$ (along a parallel), $\vec{f}_2(p) = \Psi_{,2}(p) = \frac{\partial \Psi}{\partial \varphi}(p) = \Phi_{,3}(p) = \vec{e}_3(p)$ (along a meridian, and $\vec{f}_3(p) = \Psi_{,3}(p) = \frac{\partial \Psi}{\partial r}(p) = \Phi_{,1}(p) = \vec{e}_1(p)$ (radial). Then

$$[g]_{|\vec{f}} = \begin{pmatrix} r^2 \cos^2 \varphi & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \stackrel{\text{written}}{=} [g_{ij}], \quad (17.18)$$

cf. exercise 16.4 (here $1 \leftrightarrow \theta$, $2 \leftrightarrow \varphi$, $3 \leftrightarrow r$). \blacksquare

18 Application 2: Endomorphisms

Corollary 18.1 *Let $T \in T_1^1(S)$ and*

$$T = \sum_{i,j=1}^m T_j^i \vec{e}_i \otimes e^j, \quad \nabla T = \sum_{i,j,k=1}^m T_{j|k}^i \vec{e}_i \otimes e^j \otimes e^k. \quad (18.1)$$

Then

$$T_{j|k}^i = \frac{\partial T_j^i}{\partial q^k} - \sum_{\beta} T_{\beta}^i \gamma_{kj}^{\beta} + \sum_{\alpha} \gamma_{k\alpha}^i T_j^{\alpha} \quad (= \nabla T(e^i, \vec{e}_j, \vec{e}_k)). \quad (18.2)$$

Proof. Apply (15.2) or (15.11). \blacksquare

Example 18.2 In \mathbb{R}^n , with a Euclidean basis and the usual Euclidean metric we get $T_j^i|_k = \frac{\partial T_j^i}{\partial x^k}$.
 And for all $(\alpha, \vec{w}) \in \Omega^1(S) \times \Gamma^1(S)$,

$$\tilde{\nabla}_{\vec{v}}(T(\alpha, \vec{w})) = (\nabla_{\vec{v}}T)(\alpha, \vec{w}) + T(\nabla_{\vec{v}}\alpha, \vec{w}) + T(\alpha, \nabla_{\vec{v}}\vec{w}). \quad (18.3)$$

And

$$\frac{D}{dt}(T(\alpha, \vec{w})) = \frac{DT}{dt}(\alpha, \vec{w}) + T\left(\frac{D\alpha}{dt}, \vec{w}\right) + T\left(\vec{w}, \frac{D\vec{w}}{dt}\right). \quad (18.4)$$

■

Part VI

Geodesics and parallel transport

19 Parallel transport in \mathbb{R}^n

Let $c : s \in [a, b] \rightarrow p = c(s) \in \mathbb{R}^n$ be a regular curve in \mathbb{R}^n and $\vec{v}(p) = \vec{c}'(s)$ when $p = c(s) \in \text{Im}c$.

19.1 Parallel transport of a scalar function

Definition 19.1 $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is parallel transported along c iff f is uniform on $\text{Im}c$, that is,

$$f \circ c \text{ is constant, i.e. } f(p) = f(\tilde{p}), \quad \forall p, \tilde{p} \in \text{Im}(c). \quad (19.1)$$

(So c is a level curve of f).

So $f \in \mathcal{F}(\mathbb{R}^n)$ is parallel transported along c iff $f(c(s)) = 0$ for all s , that is, iff

$$\forall p \in \text{Im}c, \quad df(p) \cdot \vec{v}(p) = 0, \quad \text{i.e.} \quad \tilde{\nabla}_{\vec{v}} f = 0 = \frac{Df}{ds}. \quad (19.2)$$

(See notation (5.3).) In the basis (\vec{e}_i) of a coordinate system, with $\vec{v} = \sum_{i=1}^n v^i \vec{e}_i$, we get, with $\frac{\partial f}{\partial q^i} := df \cdot \vec{e}_i$, cf. (1.29),

$$\sum_{i=1}^n \frac{\partial f}{\partial q^i} v^i = 0. \quad (19.3)$$

Example 19.2 If f is uniform in \mathbb{R}^n then f is parallel transported along any curve. ▀

Example 19.3 Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(p) = \sqrt{x^2 + y^2} = \|\vec{x}\|_{\mathbb{R}^2}$ when $\vec{x} = \vec{O}\vec{p}$. Let $c(s) = \begin{pmatrix} x(s) = R \cos s \\ y(s) = R \sin s \end{pmatrix}$ for $s \in [0, 2\pi]$. So $\vec{c}'(s) = \begin{pmatrix} -y(s) \\ x(s) \end{pmatrix}$, and $\vec{v}(p) = \begin{pmatrix} -y \\ x \end{pmatrix}$ along c . Thus $f \circ c$ is uniform along c (since $f(c(s)) = R$), so $\tilde{\nabla}_{\vec{v}} f(p) = \frac{Df}{ds}(p) = 0$ when $p \in \text{Im}c$, and f is parallel transported along c .

Or $p = c(s)$ gives $df(p) \cdot \vec{c}'(s) = \left(\frac{\partial f}{\partial x}(p) \quad \frac{\partial f}{\partial y}(p) \right) \cdot \begin{pmatrix} c'_1(s) \\ c'_2(s) \end{pmatrix} = \left(\frac{x}{R} \quad \frac{y}{R} \right) \cdot \begin{pmatrix} -y \\ x \end{pmatrix} = 0.$ ▀

19.2 Parallel transport of a vector field

Definition 19.4 A vector field $\vec{w} \in \Gamma(\mathbb{R}^n)$ is parallel transported along c iff \vec{w} is uniform on $\text{Im}c$, that is,

$$\vec{w} \circ c \text{ is constant, i.e. } \vec{w}(p) = \vec{w}(\tilde{p}), \quad \forall p, \tilde{p} \in \text{Im}(c). \quad (19.4)$$

So \vec{w} keeps its direction and its norm along c .

Hence $\vec{w} \in \mathcal{F}(\mathbb{R}^n)$ is parallel transported along c iff $\vec{w}(c(s)) = 0$ for all s , that is, iff

$$\forall p \in \text{Im}c, \quad d\vec{w}(p) \cdot \vec{v}(p) = 0, \quad \text{i.e.} \quad \nabla_{\vec{v}} \vec{w} = 0 = \frac{D\vec{w}}{ds}. \quad (19.5)$$

In the basis (\vec{e}_i) of a coordinate system, with $\vec{v} = \sum_{i=1}^n v^i \vec{e}_i$ and $\vec{w} = \sum_{i=1}^n w^i \vec{e}_i$, we get

$$\sum_{k=1}^n w^i_{|k} v^k = 0, \quad \text{i.e.} \quad \sum_{k=1}^n \left(\frac{\partial w^i}{\partial q^k} + \sum_{j=1}^n \gamma^i_{jk} w^j \right) v^k = 0. \quad (19.6)$$

Example 19.5 If \vec{w} is uniform in \mathbb{R}^n then \vec{w} is parallel transported along any curve. ▀

Example 19.6 \mathbb{R}^2 ; $p = c(\theta) = R \cos \theta \vec{E}_1 + R \sin \theta \vec{E}_2 = \Phi(R, \theta)$ with $\theta \in [0, 2\pi]$ (circle). Let $\vec{w}(p) = \vec{w}(c(\theta)) = \alpha(r, \theta) \vec{E}_1 + \beta(r, \theta) \vec{E}_2$. Then $(\vec{w} \circ c)'(\theta) = \frac{\partial \alpha}{\partial \theta}(r, \theta) \vec{E}_1 + \frac{\partial \beta}{\partial \theta}(r, \theta) \vec{E}_2$, and $\frac{D\vec{w}}{ds} = 0$ iff $\frac{\partial \alpha}{\partial \theta}(r, \theta) = \frac{\partial \beta}{\partial \theta}(r, \theta) = 0$, i.e. iff $\alpha(r, \theta) = \alpha(r)$ et $\beta(r, \theta) = \beta(r)$, i.e. iff \vec{w} is independent of θ , which means that $\vec{W}_R(\theta) := \vec{w}(\Phi(\vec{q}))$ is independent of θ , i.e., \vec{W} is uniform in $\text{Im}(c)$. (And $\vec{0} = \frac{\partial \vec{W}}{\partial \theta}(\vec{q}) = d\vec{w}(\Phi(R, \theta)) \cdot \frac{\partial \Phi}{\partial \theta}(R, \theta) = d\vec{w}(p) \cdot \vec{v}(p) = 0$, cf. (19.5).) ▀

Exercise 19.7 Polar system and $\vec{w}(p) = \alpha(p)\vec{e}_1(p) + \beta(p)\vec{e}_2(p)$: Prove that

$$d\vec{w}.\vec{e}_2 = \left(\frac{\partial\alpha}{\partial\theta} - r\beta\right)\vec{e}_1 + \left(\frac{\alpha}{r} + \frac{\partial\beta}{\partial\theta}\right)\vec{e}_2, \quad (19.7)$$

And that $\frac{D\vec{w}}{dt} = 0$ along the circle $C(\vec{0}, R)$ iff \vec{w} is uniform along $C(\vec{0}, R)$.

Answer. $d\vec{w}.\vec{e}_2 = (d\alpha.\vec{e}_2)\vec{e}_1 + \alpha(d\vec{e}_1.\vec{e}_2) + (d\beta.\vec{e}_2)\vec{e}_2 + \beta(d\vec{e}_2.\vec{e}_2)$ with $d\alpha.\vec{e}_2 = \frac{\partial\alpha}{\partial\theta}$, $d\beta.\vec{e}_2 = \frac{\partial\beta}{\partial\theta}$, $d\vec{e}_1.\vec{e}_2 = \frac{1}{r}\vec{e}_2$ and $d\vec{e}_2.\vec{e}_2 = -r\vec{e}_1$. Thus (19.7).

And $\frac{D\vec{w}}{dt} = 0$ along the circle $C(\vec{0}, R)$ iff $d\vec{w}.\vec{e}_2 = 0$. So $\frac{\partial\alpha}{\partial\theta} - R\beta = 0$ and $\frac{\alpha}{R} + \frac{\partial\beta}{\partial\theta} = 0$ at any $p \in C(\vec{0}, R)$. Thus $\frac{\partial^2\alpha}{\partial\theta^2} + \alpha = 0$ when $r = R$. Thus, with $\tilde{\alpha}_R(\theta) = \alpha(p)$ when $p = \Phi(R, \theta)$, we get $\tilde{\alpha}_R(\theta) = a_R \cos \theta + b_R \sin \theta$, with $a_R, b_R \in \mathbb{R}$. Thus, with $\tilde{\beta}_R(\theta) = \beta(p)$ when $p = \Phi(R, \theta)$, we get $\tilde{\beta}_R(\theta) = \frac{1}{R}(-a_R \sin \theta + b_R \cos \theta)$. Thus, with the Euclidean basis, at $p = \Phi(R, \theta)$ we get

$$\vec{w}(p) = (a_R \cos \theta + b_R \sin \theta) \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}_{|(\vec{E})} + (-a_R \sin \theta + b_R \cos \theta) \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}_{|(\vec{E})} = \begin{pmatrix} a_R \\ b_R \end{pmatrix}_{|(\vec{E})},$$

so \vec{w} is uniform along $C(\vec{0}, R)$. ▀

19.3 Geodesic in \mathbb{R}^n : A straight line

Definition 19.8 In \mathbb{R}^n , a geodesic is a curve $c : s \in [a, b] \rightarrow c(s) \in \mathbb{R}^n$ such that:

- 1- c is regular and s is an intrinsic paramater, that is, such that $\|\vec{c}'(s)\| = 1$ for all $s \in [a, b]$, and
- 2- $\vec{v}(p) = \vec{c}'(s)$ at $p = c(s)$ is parallel transported along c , that is,

$$(\vec{c}'(s) =) \quad (\vec{v} \circ c)(s) = (\vec{v} \circ c)(a), \quad \forall s \in [a, b], \quad (19.8)$$

i.e., for all $p \in \text{Im}c$,

$$d\vec{v}(p).\vec{v}(p) = 0 = \frac{D\vec{v}}{ds} \quad (\text{when } \|\vec{v}\| = 1). \quad (19.9)$$

And $\text{Im}c$ is also called a geodesic.

It is trivial that the straight lines are geodesic, since then $c(s) = c(s_0) + s\vec{v}_0$. Converse:

Proposition 19.9 If c is a geodesic, then

$$\vec{c}'' = 0, \quad (19.10)$$

and $\text{Im}c$ is a straight line.

Proof. $\vec{c}'(s) = \vec{c}'(a)$ gives $\vec{c}'' = 0$. Thus $c(s) = c(a) + s\vec{c}'(a)$, which is the equation of a straight line. ▀

Exercise 19.10 Let $\vec{v}_0 \in \mathbb{R}^n - \{\vec{0}\}$. Let $\alpha(t) = (t-a)^2\vec{v}_0 + \alpha_a$. Prove that $\text{Im}\alpha$ is a geodesic, and give the associated geodesic $c : s \in [a, b] \rightarrow c(s) \in \mathbb{R}^n$.

Answer. A trivial answer is $c(s) = s\frac{\vec{v}_0}{\|\vec{v}_0\|} + \alpha_a$.

Generic calculation: We look for an intrinsic parametrization of α . Let $s : t \rightarrow s(t)$ be a diffeomorphism. Let $c(s(t)) = \alpha(t)$. Thus $\vec{c}'(s(t))s'(t) = \alpha'(t)$, and we want $\|\vec{c}'\| = 1$, thus $s'(t) = \|\alpha'(t)\| = 2|t-a|\|\vec{v}_0\|$. Thus, $s(t) = (t-a)^2\|\vec{v}_0\|$ (up to a constant), hence $c(s) = s\frac{\vec{v}_0}{\|\vec{v}_0\|} + \alpha_a$, and $\vec{c}''(s) = 0$. ▀

20 Geodesic in a surface

20.1 Geodesic = a “straight line” in a surface

Definition 20.1 Let S be a surface in \mathbb{R}^n . A geodesic in S is a regular curve $c : s \in [a, b] \rightarrow c(s) \in S$ such that:

- 1- s is an intrinsic paramater, i.e. such that $\|\vec{c}'(s)\| = 1$ for all $s \in [a, b]$ (constant speed), and
- 2- The acceleration in S vanishes, that is,

$$\text{Proj}_{T_p S}(\vec{c}''(s)) = 0, \quad \forall s \in [a, b], \quad (20.1)$$

i.e., with $p = c(s)$ and $\vec{v}(p) = \vec{c}'(s)$, along $\text{Im}c$,

$$\nabla_{\vec{v}}\vec{v} = 0 \quad (= \text{Proj}_{T_p S}(d\vec{v}.\vec{v}) = \frac{D\vec{v}}{ds}), \quad (20.2)$$

i.e., $(\nabla_{\vec{v}}\vec{v})(p) = \vec{0}$ ($= \text{Proj}_{T_p S}(d\vec{v}(p).\vec{v}(p)) = \frac{D\vec{v}}{ds}(p)$) for all $p \in \text{Im}c$.

A geometric curve is a geodesic iff, parametrized with an intrinsic parameter, $\text{Proj}_{T_p S}(\vec{c}''(s)) = 0$.

Interpretation: On a geodesic, there is no “lateral acceleration” and no “longitudinal acceleration” (since $\|\vec{v}\| = \text{constant}$): we go “straight”. So, the acceleration can only be orthogonal to S , i.e. centrifugal relative to S . In other words, a geodesic is obtained by “applying a straight line” on the surface.

Example 20.2 Thus on Earth, driving at constant speed, we are on a geodesic iff we don’t feel any lateral forces or longitudinal forces (eventually we may feel vertical forces).

E.g., on the bridge over the Pontchartrain lake in Louisiana, we are on an Earth geodesic: In a car at constant speed, the bridge seems to be a straight line, but it is not, since the Earth is not flat. (And the length of the bridge makes it possible to see the roundness of the Earth.) \blacksquare

Remark 20.3 In the definition we may replace $\|\vec{c}'\| = 1$ by $\|\vec{c}'\| = v_0$ (constant speed) for any $v_0 > 0$. \blacksquare

20.2 Change of parameter

Let $\alpha : \left\{ \begin{array}{l} [\tilde{a}, \tilde{b}] \rightarrow S \\ t \rightarrow p = \alpha(t) \end{array} \right\}$ be a regular curve in S . To know if $\text{Im}\alpha$ is a geodesic, consider the increasing change of parameter $s : \left\{ \begin{array}{l} [\tilde{a}, \tilde{b}] \rightarrow [a, b] \\ t \rightarrow s = s(t) \end{array} \right\}$ (diffeomorphism with $s'(t) > 0$ for all t) such that the curve

$$c : \left\{ \begin{array}{l} [a, b] \rightarrow S \\ s \rightarrow p = c(s) = \alpha(t) \quad \text{when } s = s(t) \end{array} \right\} \quad \text{satisfies} \quad \|\vec{c}'(s)\|_{\mathbb{R}^n} = 1. \quad (20.3)$$

Thus

$$\vec{c}'(s(t)) s'(t) = \vec{\alpha}'(t), \quad \text{and} \quad s'(t) = \|\vec{\alpha}'(t)\|_{\mathbb{R}^n}, \quad (20.4)$$

and then

$$\vec{\alpha}''(t) = \vec{c}''(s(t))(s'(t))^2 + \vec{c}'(s(t))s''(t). \quad (20.5)$$

Since $(\vec{c}'(s), \vec{c}'(s))_{\mathbb{R}^n} = 1$, we have $2(\vec{c}''(s), \vec{c}'(s))_{\mathbb{R}^n} = 0$, so $\vec{c}''(s) \perp \vec{c}'(s)$. Thus

$$s''(t) = (\vec{\alpha}''(t), \vec{c}'(s(t)))_{\mathbb{R}^n} = \frac{(\vec{\alpha}''(t), \vec{\alpha}'(t))_{\mathbb{R}^n}}{\|\vec{\alpha}'(t)\|}. \quad (20.6)$$

Hence, with (20.1), $\text{Im}\alpha$ is a geodesic iff

$$\text{Proj}_{T_p S}(\alpha''(t)) = \frac{(\vec{\alpha}''(t), \vec{\alpha}'(t))_{\mathbb{R}^n}}{\|\vec{\alpha}'(t)\|^2} \vec{\alpha}'(t) \quad (= \vec{c}'(s(t))s''(t)). \quad (20.7)$$

(In particular, the acceleration in S is “purely longitudinal”.)

In other words, with $p = \alpha(t)$ and $\vec{v}_\alpha(p) = \vec{\alpha}'(t)$, we have $\vec{\alpha}''(t) = d\vec{v}_\alpha(p) \cdot \vec{v}_\alpha(p)$, thus with the usual Riemannian connection we have $\text{Proj}_{T_p S}(\alpha''(t)) = \nabla_{\vec{v}_\alpha} \vec{v}_\alpha = \frac{D\vec{v}_\alpha}{dt}$, and $\text{Im}\alpha$ is a geodesic iff

$$\left(\frac{D\vec{v}_\alpha}{dt}(p) = \right) \quad (\nabla_{\vec{v}_\alpha} \vec{v}_\alpha)(p) = \frac{(\vec{\alpha}''(t), \vec{\alpha}'(t))_{\mathbb{R}^n}}{\|\vec{\alpha}'(t)\|^2} \vec{\alpha}'(t). \quad (20.8)$$

Exercise 20.4 Definition: A great circle on a sphere $S \subset \mathbb{R}^3$ is the intersection of S with a plane containing the center of S . And a parallel is the intersection of S with a plane parallel to the equator.

Prove that a parallel is a geodesic iff it is the equator.

Answer. Choose the origin O of \mathbb{R}^n to be the center of S (we are interested in the derivatives, and the center won’t be used, but to simplify the writings). So $S = S(\vec{0}, R)$. Choose a Euclidean basis and choose the GPS parametrization of S , so that parallel at latitude φ_0 is given by $p = c(\theta) =$

$$\begin{pmatrix} R \cos \theta \cos \varphi_0 \\ R \sin \theta \cos \varphi_0 \\ R \sin \varphi_0 \end{pmatrix}. \quad \text{Thus } \vec{v}(p) = \vec{c}'(\theta) = \begin{pmatrix} -R \sin \theta \cos \varphi_0 \\ R \cos \theta \cos \varphi_0 \\ 0 \end{pmatrix} \quad \text{and } \|\vec{v}(p)\| = R \cos \varphi_0 \text{ is constant, and}$$

$$\vec{c}''(\theta) = \begin{pmatrix} -R \cos \theta \cos \varphi_0 \\ -R \sin \theta \cos \varphi_0 \\ 0 \end{pmatrix} = d\vec{e}_2(p) \cdot \vec{e}_2(p) = -r \cos^2 \varphi_0 \vec{e}_1(p) + \cos \varphi_0 \sin \varphi_0 \vec{e}_3(p), \quad \text{cf. (3.33).} \quad \text{Thus}$$

$\nabla_{\vec{v}} \vec{v}(p) = \text{Proj}_{T_p S}(\vec{c}''(\theta)) = \cos \varphi_0 \sin \varphi_0 \vec{e}_3(p)$. Therefore c is a geodesic iff $\cos \varphi_0 \sin \varphi_0 = 0$, i.e., iff $\varphi_0 = 0$ or $\frac{\pi}{2}$; But at $\frac{\pi}{2}$ the curve is reduced to a point (the North Pole): Not a regular curve. Thus only the equator is a geodesic. \blacksquare

Exercise 20.5 1- Give the equation of a great circle.

2- Give the equation of a great circle which makes an angle $\alpha \in]0, \pi[$ with a meridian.

Answer. Choose the origin at the center of S , so $S = S(\vec{0}, R)$.

1- A great circle is the intersection of S with a plane containing S , so is the set of $p = (x, y, z)$ s.t.

$$\begin{cases} x^2 + y^2 + z^2 = R^2, \\ ax + by + cz = 0. \end{cases} \quad (20.9)$$

(The unknowns are a, b, c .) Choose the GPS coordinates $p = c(\theta, \varphi) = \begin{pmatrix} x = R \cos \theta \cos \varphi \\ y = R \sin \theta \cos \varphi \\ z = R \sin \varphi \end{pmatrix}$, thus $x^2 + y^2 + z^2 = R^2$ is satisfied, and the great circles satisfy

$$a \cos \theta \cos \varphi + b \sin \theta \cos \varphi + c \sin \varphi = 0, \quad \theta \in [0, 2\pi], \quad \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \quad (20.10)$$

In particular, if $c = 0$ and $\theta = \theta_0$ then $a \cos \theta_0 \cos \varphi + b \sin \theta_0 \cos \varphi = 0$, and we can choose, e.g., $a = -\sin \theta_0$ and $b = \cos \theta_0$, so a meridian is a great circle.

Suppose $c \neq 0$; And eventually dividing (20.9)₂ by c , suppose $c = -1$. Thus

$$a \cos \theta + b \sin \theta = \tan \varphi, \quad \text{i.e.} \quad \varphi = \tan^{-1}(a \cos \theta + b \sin \theta) = \varphi(\theta), \quad (20.11)$$

and a great circle other than a meridian is a curve $p(\theta) = c(\theta, \varphi(\theta))$.

2- Let $p_0 = c(\theta_0, \varphi_0)$ be a point in S . Consider the meridian which passes through p_0 , that is, the curve $\varphi \rightarrow c(\theta_0, \varphi)$ which is normal to $\vec{e}_2(p_0) \parallel \vec{n}_m = \begin{pmatrix} \cos \theta_0 \\ \sin \theta_0 \\ 0 \end{pmatrix}$. A plane through p_0 other than the meridian, equation $ax + by - z = 0$, is normal to $\vec{n}_p = \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$. Thus

$$\cos \alpha = \left(\frac{\vec{n}_m}{\|\vec{n}_m\|}, \frac{\vec{n}_p}{\|\vec{n}_p\|} \right)_{\mathbb{R}^3} = \frac{-a \sin \theta_0 + b \cos \theta_0}{\sqrt{a^2 + b^2 + 1}}. \quad (20.12)$$

So, (20.11) and (20.12) give: a and b satisfy

$$\begin{cases} a \cos \theta_0 + b \sin \theta_0 = \tan \varphi_0, \\ -a \sin \theta_0 + b \cos \theta_0 = \sqrt{a^2 + b^2 + 1} \cos \alpha. \end{cases} \quad (20.13)$$

Thus,

$$\begin{cases} a^2 \cos^2 \theta_0 + b^2 \sin^2 \theta_0 + 2ab \cos \theta_0 \sin \theta_0 = \tan^2 \varphi_0, \\ a^2 \sin^2 \theta_0 + b^2 \cos^2 \theta_0 - 2ab \cos \theta_0 \sin \theta_0 = (a^2 + b^2) \cos^2 \alpha + \cos^2 \alpha. \end{cases}$$

so (summation),

$$(a^2 + b^2)(1 - \cos^2 \alpha) = \tan^2 \varphi_0 + \cos^2 \alpha$$

Let $\begin{cases} a = \rho \cos \gamma, \\ b = \rho \sin \gamma, \end{cases}$: Thus,

$$\rho^2 = \frac{\tan^2 \varphi_0 + \cos^2 \alpha}{1 - \cos^2 \alpha}, \quad (20.14)$$

and (20.13) gives

$$\begin{cases} \rho(\cos \gamma \cos \theta_0 + \sin \gamma \sin \theta_0) = \tan \varphi_0 & = \rho \cos(\gamma - \theta_0), \\ \rho(-\cos \gamma \sin \theta_0 + \sin \gamma \cos \theta_0) = \sqrt{\rho^2 + 1} \cos \alpha & = \rho \sin(\gamma - \theta_0). \end{cases} \quad (20.15)$$

If p is not on the equator ($\alpha \neq \frac{\pi}{2}$) then $\rho^2 + 1 = \frac{\tan^2 \varphi_0 + 1}{1 - \cos^2 \alpha}$ and

$$\tan(\gamma - \theta_0) = -\frac{\cos \alpha}{\tan \varphi_0} \sqrt{\frac{\tan^2 \varphi_0 + 1}{1 - \cos^2 \alpha}} \quad (20.16)$$

■

Exercise 20.6 With (20.11) prove that the great circles are geodesics.

Answer. A meridian $c(\theta_0, \varphi)$ is a geodesic (easy). Otherwise, $c = -1$ gives

$$\varphi = \varphi(\theta) = \tan^{-1}(a \cos \theta + b \sin \theta), \quad c(\theta) = R \begin{pmatrix} \cos \theta \cos(\varphi(\theta)) \\ \sin \theta \cos(\varphi(\theta)) \\ \sin(\varphi(\theta)) \end{pmatrix}.$$

$$\text{Thus } \vec{c}'(\theta) = R \begin{pmatrix} -\sin \theta \cos(\varphi(\theta)) \\ \cos \theta \cos(\varphi(\theta)) \\ 0 \end{pmatrix} + R\varphi'(\theta) \begin{pmatrix} -\cos \theta \sin(\varphi(\theta)) \\ -\sin \theta \sin(\varphi(\theta)) \\ \cos(\varphi(\theta)) \end{pmatrix}, \text{ i.e., with } p = c(\theta),$$

$$\vec{c}'(\theta) = \vec{e}_2(p) + \varphi'(\theta)\vec{e}_3(p).$$

Thus, with lightened notations,

$$\begin{aligned} \vec{c}'' &= d\vec{e}_2.\vec{c}' + \varphi''\vec{e}_3 + \varphi'd\vec{e}_3.\vec{c}' \\ &= d\vec{e}_2.\vec{e}_2 + \varphi'd\vec{e}_2.\vec{e}_3 + \varphi''\vec{e}_3 + \varphi'd\vec{e}_3.\vec{e}_2 + \varphi'^2 d\vec{e}_3.\vec{e}_3, \\ &= -R\cos^2\varphi\vec{e}_1 + \sin\varphi\cos\varphi\vec{e}_3 - 2\varphi'\tan\varphi\vec{e}_2 + \varphi''\vec{e}_3 - \varphi'^2 R\vec{e}_1. \end{aligned}$$

Thus,

$$\text{Proj}_{TS}(\vec{c}''(\theta)) = -2\varphi'\tan\varphi\vec{e}_2 + (\varphi'' + \sin\varphi\cos\varphi)\vec{e}_3.$$

We will use (20.7). \vec{e}_1, \vec{e}_2 and \vec{e}_3 are orthogonal, $\|\vec{e}_2\|^2 = R^2\cos^2\varphi$, $\|\vec{e}_3\|^2 = R^2$, thus, $\|\vec{c}'\|^2 = R^2(\cos^2\varphi + (\varphi')^2)$ and

$$\frac{(\vec{c}'', \vec{c}')_{\mathbb{R}^n}}{\|\vec{c}'\|^2} = \frac{-2\varphi'\tan\varphi\cos^2\varphi + \varphi'(\varphi'' + \sin\varphi\cos\varphi)}{\cos^2\varphi + (\varphi')^2} = \frac{-\varphi'\sin\varphi\cos\varphi + \varphi'\varphi''}{\cos^2\varphi + (\varphi')^2}.$$

Thus, with (20.7) we have to check that

$$\begin{cases} -2\varphi'\tan\varphi(\cos^2\varphi + (\varphi')^2) = -\varphi'\sin\varphi\cos\varphi + \varphi'\varphi'', \\ (\varphi'' + \sin\varphi\cos\varphi)(\cos^2\varphi + (\varphi')^2) = \varphi'(-\varphi'\sin\varphi\cos\varphi + \varphi'\varphi'') \end{cases}$$

that is,

$$\begin{cases} -\sin\varphi\cos\varphi - 2(\varphi')^2\tan\varphi = \varphi'', \\ \varphi''\cos^2\varphi + \sin\varphi\cos^3\varphi + 2(\varphi')^2\sin\varphi\cos\varphi = 0 \end{cases}$$

We have $\varphi(\theta) = \tan^{-1}(a \cos \theta + b \sin \theta)$ and $\tan^{-1}'(x) = \frac{1}{1+x^2}$.

Thus $\varphi' = \frac{-a \sin \theta + b \cos \theta}{1 + \tan^2(\varphi)} = \cos^2\varphi(-a \sin \theta + b \cos \theta)$.

Thus $\varphi'' = -2\sin\varphi\cos\varphi\varphi'(-a \sin \theta + b \cos \theta) - \cos^2\varphi(a \cos \theta + b \sin \theta) = -2\varphi'^2\tan\varphi - \cos\varphi\sin\varphi$ (the first equation).

Thus $\varphi'' + \sin\varphi\cos\varphi = -2(\varphi')^2\tan\varphi$ (the second equation). ▀

20.3 Curve in S and coordinate systems

Consider a regular curve

$$c : t \in]a, b[\longrightarrow p = c(t) = \Phi(\vec{q}(t)) \in S, \quad (20.17)$$

where $\vec{q} = \Phi^{-1} \circ c : t \in]a, b[\longrightarrow \vec{q}(t) \in U$ is a curve in $U \subset \mathbb{R}^m$ (space of parameters). Let $\vec{q}(t) = \sum_{i=1}^m q^i(t)\vec{A}_i$. Then,

$$\frac{d\vec{q}}{dt}(t) = \sum_{i=1}^m \frac{dq^i}{dt} \vec{A}_i, \quad \text{written } \vec{q}'(t) = \sum_{i=1}^m (q^i)'(t)\vec{A}_i.$$

And $\vec{c}'(t) = d\Phi(p).\vec{q}'(t) = \sum_{i=1}^m (q^i)'(t)d\Phi(p).\vec{A}_i = \sum_{i=1}^m (q^i)'(t)\vec{e}_i(p)$ gives the velocity along \vec{c} at $p = \vec{c}(t)$:

$$\vec{c}'(t) = \vec{v}(p) = \sum_{i=1}^m (q^i)'(t)\vec{e}_i(p), \quad \text{written } \vec{v} = \sum_{i=1}^m (q^i)' \vec{e}_i. \quad (20.18)$$

Thus

$$\vec{c}''(t) = \sum_{i=1}^m (q^i)''(t)\vec{e}_i(c(t)) + \sum_{j=1}^m (q^j)'(t)d\vec{e}_j(c(t)).\vec{c}'(t) = \sum_{i=1}^m (q^i)''(t)\vec{e}_i(p) + \sum_{j,k=1}^m (q^j)'(t)(q^k)'(t)d\vec{e}_j(p).\vec{e}_k(p),$$

and $d\vec{e}_j(p).\vec{e}_k(p) = \sum_{i=1}^n \gamma_{jk}^i \vec{e}_i$ gives $\nabla_{\vec{v}} \vec{e}_j(p) = \sum_{i=1}^m \gamma_{jk}^i (q^i)' \vec{e}_i$, and the acceleration at p in S (in $T_p S$) is

$$\vec{c}''(t) = \nabla_{\vec{v}} \vec{v}(p) = \frac{D\vec{v}}{dt}(p) = \sum_{i=1}^m \left((q^i)''(t) + \sum_{j,k=1}^m \gamma_{jk}^i (q^j)'(t)(q^k)'(t) \right) \vec{e}_i(p). \quad (20.19)$$

20.4 Geodesic in a coordinate system

Corollary 20.7 *The following propositions are equivalent:*

- (i) Imc is a geodesic,
- (ii) c being travelled at constant speed and, for all $i = 1, \dots, m$,

$$(q^i)'' + \sum_{j,k=1}^m \gamma_{jk}^i (q^j)' (q^k)' = 0, \quad (20.20)$$

meaning $(q^i)''(t) + \sum_{j,k=1}^m \gamma_{jk}^i(p) (q^j)'(t) (q^k)'(t) = 0$ for all t and all $i = 1, \dots, m$, with $p = c(t)$.

Proof. (i) \Rightarrow (ii). If $\text{Proj}_{TS}(\bar{c}''(t)) = 0$, then (20.19) gives (20.20).

(ii) \Rightarrow (i). If (20.20), then (20.19) implies $\text{Proj}_{TS}(\bar{c}''(t)) = 0$, so, the speed being constant, c is a geodesic. \blacksquare

Exercise 20.8 Converse of exercise 20.6. Prove with (20.20) that the geodesics on $S = S(0, R) \subset \mathbb{R}^3$ are great circles.

Answer. 1- ODE satisfied by a great circle: Consider a regular curve $t \in]-\varepsilon, \varepsilon[\rightarrow c(\theta(t), \varphi(t)) \in S$, let $c(0) = p_0$, and choose a basis such that in (20.10) we can take $c = -1$. Thus

$$a \cos \theta(t) + b \sin \theta(t) = \tan \varphi(t).$$

Thus

$$\theta'(-a \sin \theta + b \cos \theta) = \varphi'(1 + \tan^2 \varphi),$$

i.e., $\theta''(-a \sin \theta + b \cos \theta) + \theta'^2(-a \cos \theta - b \sin \theta) = \varphi''(1 + \tan^2 \varphi) + 2\varphi'^2 \tan \varphi(1 + \tan^2 \varphi)$,

i.e., $\theta''(-a \sin \theta + b \cos \theta) - \theta' \tan \varphi = \varphi''(1 + \tan^2 \varphi) + 2\varphi' \theta' \tan \varphi(-a \sin \theta + b \cos \theta)$,

i.e. θ and φ satisfy the ODE

$$(\theta'' - 2\varphi' \theta' \tan \varphi)(-a \sin \theta + b \cos \theta) = \frac{1}{\cos^2 \varphi}(\varphi'' + \theta' \cos \varphi \sin \varphi). \quad (20.21)$$

2- And consider a geodesic $t \in]-\varepsilon, \varepsilon[\rightarrow c(\theta(t), \varphi(t)) \in S$ with $\|\bar{c}'(t)\| = \text{cste}$. Then (20.20) gives

$$\begin{cases} \theta'' - 2\theta' \varphi' \tan \varphi = 0, \\ \varphi'' + \theta'^2 \sin \varphi \cos \varphi = 0. \end{cases}$$

And these equations trivially satisfy (20.21). \blacksquare

20.5 Geodesic: The shortest curve

Consider \mathbb{R}^n with a Euclidean dot product that defines the usual metric $(\cdot, \cdot)_g$, and $\|\vec{v}\| = \sqrt{g(\vec{v}, \vec{v})}$. Let $c : [a, b] \rightarrow S$ be a regular curve in S . Its length is

$$L(c) = \int_a^b \|\bar{c}'(t)\| dt \stackrel{\text{written}}{=} \tilde{L}(\text{Imc}), \quad (20.22)$$

the length being independent of the parametrization.

Proposition 20.9 *Let A and B be two close points in S , and let \mathcal{C} be the set of regular curves in S from A to B . The curve realizing the $\min_{\alpha \in \mathcal{C}}(\tilde{L}(\text{Im}\alpha))$ is a geodesic.*

Proof. Consider a family of curves $\alpha_u : t \in [a, b] \rightarrow \alpha_u(t) \in S$ for $u \in [-1, 1]$ in \mathcal{C} s.t. $\alpha_u(a) = A$ and $\alpha_u(b) = B$ for all $u \in [0, 1]$. Let $\alpha(u, t) := \alpha_u(t)$ (defined on $[-1, 1] \times [a, b]$). The length of α_u is

$$\ell(u) = L(\alpha_u) = \int_a^b \|\bar{\alpha}_u'(t)\| dt = \int_a^b \left\| \frac{\partial \alpha}{\partial t}(u, t) \right\| dt. \quad (20.23)$$

The curves being regular and $[a, b]$ being compact, we get $\ell'(u) = \int_a^b \frac{\partial}{\partial u} (\|\frac{\partial \alpha}{\partial t}(u, t)\|) dt$. And

$$\frac{\partial}{\partial u} (\|\frac{\partial \alpha}{\partial t}(u, t)\|) = \frac{\partial}{\partial u} ((\frac{\partial \alpha}{\partial t}(u, t), \frac{\partial \alpha}{\partial t}(u, t))_{\mathbb{R}^n})^{\frac{1}{2}} = \frac{(\frac{\partial^2 \alpha}{\partial u \partial t}(u, t), \frac{\partial \alpha}{\partial t}(u, t))_{\mathbb{R}^n}}{\|\frac{\partial \alpha}{\partial t}(u, t)\|}.$$

Thus,

$$\begin{aligned} \ell'(u) &= \int_a^b \left(\frac{\partial^2 \alpha}{\partial u \partial t}(u, t), \vec{w}_u(t) \right)_{\mathbb{R}^n} dt, \quad \text{where } \vec{w}_u(t) = \frac{\alpha_u'(t)}{\|\alpha_u'(t)\|}, \\ &= - \int_a^b \left(\frac{\partial \alpha}{\partial u}(u, t), \vec{w}_u'(t) \right)_{\mathbb{R}^n} dt + \left(\frac{\partial \alpha}{\partial u}(u, b), \vec{w}_u(b) \right)_{\mathbb{R}^n} - \left(\frac{\partial \alpha}{\partial u}(u, a), \vec{w}_u(a) \right)_{\mathbb{R}^n}. \end{aligned} \quad (20.24)$$

And $\alpha(u, a) = B$ constant for all u , thus $\frac{\partial \alpha}{\partial u}(u, a) = 0$, idem $\frac{\partial \alpha}{\partial u}(u, b) = 0$. Thus a curve α_{u_0} realizing the minimum satisfies

$$\ell'(u_0) = 0 = \int_a^b \left(\frac{\partial \alpha}{\partial u}(u_0, t), \vec{w}_{u_0}'(t) \right)_{\mathbb{R}^n} dt = 0.$$

And this is true for all $\int_a^{\tilde{b}}$ with $[\tilde{a}, \tilde{b}] \subset [a, b]$. Thus $\frac{\partial \alpha}{\partial u}(u_0, t) \perp \vec{w}_{u_0}'(t)$ for all t . Considering all the family of curves we get $\vec{w}_{u_0}'(t) \perp T_p S$ with $p = c_{u_0}(t)$, that is $\text{Proj}_{T_p S} \vec{w}_{u_0}'(t) = 0$ for all t .

With $\vec{w}_u(t) = z(t)\alpha_u'(t)$ where $z(t) = \|\alpha_u'(t)\|^{-1} = (\alpha_u'(t), \alpha_u'(t))_g^{-\frac{1}{2}}$, thus $z'(t) = (-\frac{1}{2})2(\alpha_u''(t), \alpha_u'(t))_g (\alpha_u'(t), \alpha_u'(t))_g^{-\frac{3}{2}} = -\|\alpha_u'(t)\|^{-3}(\alpha_u''(t), \alpha_u'(t))_g$. Thus $\vec{w}_u' = z'\alpha_u' + z\alpha_u''$ gives

$$\vec{w}_u' = -\|\alpha_u'\|^{-3}(\alpha_u'', \alpha_u')_g \alpha_u' + \|\alpha_u'\|^{-1} \alpha_u''$$

Thus $\text{Proj}_{T_p S} \vec{w}_{u_0}' = 0 = \text{Proj}_{T_p S} (\|\alpha_{u_0}'\|^3 \vec{w}_{u_0}')$ gives $\text{Proj}_{T_p S} (\|\alpha_{u_0}'\|^2 \alpha_{u_0}'') = (\alpha_{u_0}'', \alpha_{u_0}')_g \alpha_{u_0}'$, that is (20.7): α_{u_0} is a geodesic. \blacksquare

20.6 Geodesic: The minimum energy curve

Consider the (kinematic) energy along c :

$$E(c) = \frac{1}{2} \int_a^b \|\dot{c}'(t)\|^2 dt. \quad (20.25)$$

NB: The energy E depends on the parametrization of $\text{Im}c$. Indeed, if $t : u \in [c, d] \rightarrow t(u) \in [a, b]$ is a diffeomorphism, if $\alpha(u) = c(t(u))$, then $\dot{\alpha}'(u) = \dot{c}'(t(u)) t'(u)$, and

$$\int_{t \in [a, b]} \|\dot{c}'(t)\|^2 dt = \int_{u \in [c, d]} \|\dot{\alpha}'(u)\|^2 \frac{1}{|t'(u)|} |t'(u)| du \neq \int_{u \in [c, d]} \|\dot{\alpha}'(u)\|^2 du, \quad (20.26)$$

unless $|t'(u)| = 1$.

Proposition 20.10 *Let A and B be two close points in S . Let \mathcal{C} be the set of regular curves c connecting A and B such that $\|\dot{c}'(t)\| = 1$ (speed). If a curve realizes $\min_{c \in \mathcal{C}} E(c)$ then this curve is a geodesic.*

Proof. Consider a family of curves $(c_u : [a, b] \rightarrow S)_{u \in [-1, 1]}$ in \mathcal{C} (so $c_u(a) = A$ and $c_u(b) = B$ for all $u \in [0, 1]$ and $\|\dot{c}_u'(t)\| = 1$ for all t). Let $c(u, t) := c_u(t)$ (defined on $[-1, 1] \times [a, b]$). Thus

$$E(c_u) = \frac{1}{2} \int_a^b \left\| \frac{\partial c}{\partial t}(u, t) \right\|^2 dt. \quad (20.27)$$

And $\frac{\partial}{\partial u} (\|\frac{\partial c}{\partial t}(u, t)\|^2) = \frac{\partial}{\partial u} \left(\frac{\partial c}{\partial t}(u, t), \frac{\partial c}{\partial t}(u, t) \right)_{\mathbb{R}^n} = 2 \left(\frac{\partial^2 c}{\partial u \partial t}(u, t), \frac{\partial c}{\partial t}(u, t) \right)_{\mathbb{R}^n}$ gives

$$\begin{aligned} \frac{d}{du} (E(c_u)) &= \int_a^b \left(\frac{\partial^2 c}{\partial u \partial t}(u, t), \frac{\partial c}{\partial t}(u, t) \right)_{\mathbb{R}^n} dt \\ &= - \int_a^b \left(\frac{\partial c}{\partial u}(u, t), \frac{\partial^2 c}{\partial t^2}(u, t) \right)_{\mathbb{R}^n} dt + \left[\left(\frac{\partial c}{\partial u}(u, t), \frac{\partial c}{\partial t}(u, t) \right)_{\mathbb{R}^n} \right]_{t=a}^b. \end{aligned}$$

And $c(u, a) = B$ constant for all u , thus $\frac{\partial c}{\partial u}(u, a) = 0$, idem $\frac{\partial c}{\partial u}(u, b) = 0$. And this is true for all \int_c^d with $[c, d] \subset [a, b]$. Thus $\frac{\partial c}{\partial u}(u_0, t) \perp \frac{\partial^2 c}{\partial t^2}(u_0, t)$, for all $t \in [a, b]$. Considering all the family of curves we get $\text{Proj}_{T_p S} \frac{\partial^2 c}{\partial t^2}(u_0, t) = 0$ for all t : c_{u_0} is a geodesic. \blacksquare

Remark 20.11

$$L^2(c) \leq \frac{1}{2(b-a)} E(c). \quad (20.28)$$

Indeed, Cauchy–Schwarz theorem gives

$$L(c) = \int_a^b \|\vec{c}'(t)\| dt = (1_{[a,b]}, \|\vec{c}'\|)_{L^2([a,b])} \leq \|1\|_{L^2([a,b])} \|\vec{c}'\|_{L^2([a,b])},$$

with $\|1\|_{L^2([a,b])}^2 = \int_a^b 1^2 dt = b-a$ and $\|\vec{c}'\|_{L^2([a,b])}^2 = \int_a^b \|\vec{c}'(t)\|^2 dt = 2E$. \blacksquare

20.7 Geodesic and normal curvatures in \mathbb{R}^3

Consider \mathbb{R}^3 with a Euclidean dot product, S a dimension 2 regular surface, $\Phi : U \subset \mathbb{R}^2 \rightarrow S \subset \mathbb{R}^3$ a coordinate system, (\vec{A}_1, \vec{A}_2) the canonical basis in \mathbb{R}^2 , $(\vec{e}_i(p)) = (d\Phi(\vec{q}) \cdot \vec{A}_i)$ the system basis at $p = \Phi(\vec{q})$. Let $\vec{n}(p)$ be the unit normal vector at $T_p S$ such that $(\vec{e}_1(p), \vec{e}_2(p), \vec{n}(p))$ is a direct basis, that is,

$$\vec{n}(p) \perp T_p S, \quad \|\vec{n}_p\| = 1, \quad \det(\vec{e}_1(p), \vec{e}_2(p), \vec{n}(p)) > 0. \quad (20.29)$$

Let $c : s \in [0, L] \rightarrow p = c(s) \in S$ with s an intrinsic parameter: At any $p = c(s)$,

$$\vec{v}(p) = \vec{c}'(s) \stackrel{\text{written}}{=} \vec{t}(p), \quad \text{and} \quad \|\vec{t}(p)\| = \|\vec{v}(p)\| = 1, \quad (20.30)$$

$\vec{t}(p)$ being the tangent unit vector at p .

Definition 20.12 The normal geodesic vector at $p = c(s)$ is the vector $\vec{n}_g(p) \in T_p S$ defined by

$$\vec{n}_g(p) = \vec{n}(p) \wedge \vec{t}(p) \in T_p S. \quad (20.31)$$

Thus $(\vec{n}(p), \vec{t}(p), \vec{n}_g(p))$ is a direct orthogonal basis ins \mathbb{R}^3 (Serret–Frénet basis $(\vec{t}(p), \vec{n}_g(p), \vec{n}(p))$). $\|\vec{c}'(s)\|_{\mathbb{R}^n}^2 = 1$ gives $(\vec{c}'(s), \vec{c}''(s))_{\mathbb{R}^n} = 0$, so the acceleration $\vec{c}''(s)$ satisfies

$$\vec{c}''(s) \perp \vec{t}(p), \quad \text{and} \quad \vec{c}''(s) = \kappa_n(p) \vec{n}(p) + \kappa_g(p) \vec{n}_g(p) \in \text{Vect}\{\vec{n}_p\} \oplus^\perp \text{Vect}\{\vec{n}_g(p)\} \quad (20.32)$$

where

$$\left\{ \begin{array}{l} \kappa_n(p) = (\vec{c}''(s), \vec{n}(p))_{\mathbb{R}^3}, \\ \kappa_g(p) = (\vec{c}''(s), \vec{n}_g(p))_{\mathbb{R}^3}, \end{array} \right\} \quad \text{and} \quad \kappa_g(p) \vec{n}_g(p) = \text{Proj}_{T_p S}(\vec{c}''(s)). \quad (20.33)$$

Definition 20.13

The normal curvature at p is $\kappa_n(p)$ (normal acceleration when $\|\vec{v}(p)\| = 1$),

The geodesic curvature at p is $\kappa_g(p)$ (tangential acceleration in S when $\|\vec{v}(p)\| = 1$).

(If c is a geodesic, then $\kappa_g = 0$.)

(if c is not a geodesic then $|\kappa_g(p)| = \|\vec{c}''(s) \wedge \vec{n}(p)\|$ gives a measure of the “rotation about $\vec{n}(p)$ ”).

Example 20.14 Sphere $S = S(\vec{0}, R)$, and GPS coordinates. A parallel in intrinsic curvilinear coordinate is a curve

$$c(s) = \begin{pmatrix} R \cos\left(\frac{s}{R \cos \varphi}\right) \cos \varphi \\ R \sin\left(\frac{s}{R \cos \varphi}\right) \cos \varphi \\ R \sin \varphi \end{pmatrix}.$$

(Intrinsic since $\|\vec{c}'(s)\| = \left\| \begin{pmatrix} -\sin\left(\frac{s}{R \cos \varphi}\right) \\ \cos\left(\frac{s}{R \cos \varphi}\right) \\ 0 \end{pmatrix} \right\| = 1$.) The acceleration is $\vec{c}''(s) = -\frac{1}{R \cos \varphi} \begin{pmatrix} \cos \frac{s}{R \cos \varphi} \\ \sin \frac{s}{R \cos \varphi} \\ 0 \end{pmatrix}$. The unit normal vector s.t. $(\vec{e}_1(p), \vec{e}_2(p), \vec{n}_p)$ is direct (here $\vec{e}_1(p)$ is along

a parallel and $\vec{e}_2(p)$ is along a meridian) is $\vec{n}(p) = \begin{pmatrix} \cos \frac{s}{R \cos \varphi} \cos \varphi \\ \sin \frac{s}{R \cos \varphi} \cos \varphi \\ \sin \varphi \end{pmatrix}$. Thus

$$(\vec{c}''(s), \vec{n}(p))_{\mathbb{R}^3} = \frac{-1}{R} = \kappa_n(p). \quad (20.34)$$

And

$$\vec{c}''(s) + \frac{1}{R} \vec{n}(p) = \frac{1}{R} \tan \varphi \begin{pmatrix} \cos \frac{s}{R \cos \varphi} \sin \varphi \\ \sin \frac{s}{R \cos \varphi} \sin \varphi \\ \cos \varphi \end{pmatrix} = \frac{1}{R} \tan \varphi \vec{n}_g. \quad (20.35)$$

Thus

$$\kappa_g = \frac{1}{R} \tan \varphi. \quad (20.36)$$

In particular on the equator, $\varphi = 0$ and $\kappa_g = 0$ (the equator is a geodesic). And κ_g increases with φ . \blacksquare

21 Parallel transport in $S \subset \mathbb{R}^n$

21.1 Definition

In the 2-sphere $S = S(\vec{0}, R)$ in \mathbb{R}^3 , a vector field in S cannot be uniform: The sphere is not flat, thus the direction changes when you go from A to B from the point of view of \mathbb{R}^3 .

Definition 21.1 Let S be a m differential manifold. Let ∇ be a connection in S . A vector field $\vec{w} \in \Gamma(S)$ is parallel transported in S (relative to ∇) along a curve $c : t \in [a, b] \rightarrow c(t) \in S$ iff, for all $p = c(t) \in c([a, b])$,

$$(\nabla_{\vec{v}} \vec{w})(p) = 0, \quad \text{i.e.} \quad \frac{D\vec{w}}{dt}(p) = 0. \quad (21.1)$$

E.g., if S be a surface in \mathbb{R}^n and ∇ is the Riemannian usual connection, given at $p = c(t)$ by

$$(\nabla_{\vec{v}} \vec{w})(p) = \text{Proj}_{T_p S}(d\vec{w} \cdot \vec{v})(p) = \text{Proj}_{T_p S}\left(\frac{d(\vec{w} \circ c)}{dt}(t)\right) = 0 \quad (21.2)$$

then a vector field is parallel transported in S iff $\text{Proj}_{T_p S}(d\vec{w} \cdot \vec{v}) = 0$.

Example 21.2 Sphere, see figure 21.1, A = the north pole, B = the south pole. Consider two meridians $c_i : [0, \pi] \rightarrow S$ connecting A and B , $i = 1, 2$, the meridians making an angle of 90 degrees in the plane tangent at the north pole. Let $\vec{v}_i(c(t)) = c_i'(t)$ be the velocities. The parallel transport of $\vec{v}_1(A)$ along \vec{c}_1 (geodesic) gives $\vec{v}_1(B)$ (tangent vector to \vec{c}_1 at B). Whereas the parallel transport of $\vec{v}_1(A)$ (orthogonal to $\vec{v}_2(A)$) along \vec{c}_2 (geodesic) gives $\vec{w}_2(B)$ (orthogonal to $\vec{v}_2(B)$): And $\vec{v}_2(A) = -\vec{w}_2(B)$; Thus the result of a parallel transportation depends on the curve chosen to go from A to B . \blacksquare

Remark 21.3 Beware of vocabulary: a parallel on Earth is “parallel to the equator”, but along such a parallel a vector is not “parallel transported” since a parallel is not a geodesic. See proposition 22.14 and example 22.16. \blacksquare

Proposition 21.4 Let $c : [a, b] \rightarrow S$ be a curve in S surface in \mathbb{R}^n , and consider the usual Riemannian connection. Then the parallel transport along c in S does not depend on the parametrization of c .

Proof. Let $\alpha : s \in [c, d] \rightarrow p = \alpha(s)$ be another parametrization of $\text{Im}c$, and let $t : s \in [c, d] \rightarrow t(s) \in [a, b]$ be the change of parameter (diffeomorphism), so $\alpha = c \circ t$, that is $\alpha(s) = c(t(s)) = p$. Thus $\vec{w}(p) = (\vec{w} \circ \alpha)(s) = (\vec{w} \circ c)(t(s))$ gives

$$(\vec{w} \circ \alpha)'(s) = (\vec{w} \circ c)'(t(s)) t'(s).$$

Thus $\text{Proj}_{T_p S}(\vec{w} \circ \alpha)'(s) = \text{Proj}_{T_p S}(\vec{w} \circ c)'(t(s)) t'(s)$, that is, $\frac{D\vec{w}}{ds}(p) = \frac{D\vec{w}}{dt}(p) t'(s)$; Thus $\frac{D\vec{w}}{ds}(p) = 0$ iff $\frac{D\vec{w}}{dt}(p) = 0$. \blacksquare

With the basis of a coordinate system, (7.5) gives $\frac{D\vec{w}}{dt} = \nabla_{\vec{v}} \vec{w} = \sum_{i,j=1}^m v^j w_{|j}^i \vec{e}_i = \sum_{i,j=1}^m \left(\frac{\partial w^i}{\partial q^j} + \sum_{k=1}^m w^k \gamma_{jk}^i \right) v^j \vec{e}_i$. Thus \vec{w} is parallel transported along a curve c in S iff, for all i ,

$$\sum_{j=1}^m w_{|j}^i v^j = 0, \quad \text{i.e.} \quad \sum_{j=1}^m \left(\frac{\partial w^i}{\partial q^j} + \sum_{k=1}^m \gamma_{jk}^i w^k \right) v^j = 0. \quad (21.3)$$

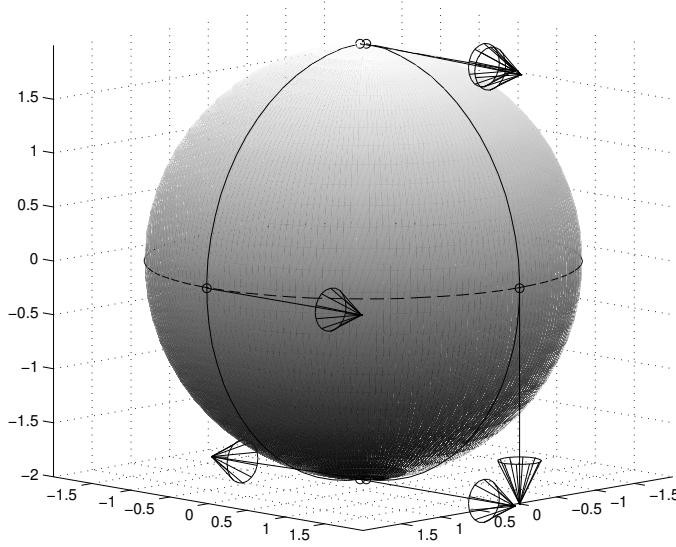


Figure 21.1: Parallel transport along the meridians 0 and 90 degrees: The result of a parallel transport along the meridians of a (tangent) vector at the north pole gives to different vectors at the south pole.

Remark 21.5 We cannot confuse

- The parallel transport equation $\nabla_{\vec{v}}\vec{w} = \frac{D\vec{w}}{dt} = 0$ which is independent of a parametrization of the curve, cf. proposition 21.4, with
- The geodesic equation $\nabla_{\vec{v}}\vec{v} = 0$ which requires $\|\vec{v}\| = 1$ (intrinsic parametrization): Cf. (20.8) if the parameter is not intrinsic. \blacksquare

Remark 21.6 The Levi-Civita theorem 22.22 will states that: If the connection parallel transports the metric, then proposition 21.4 will be valid (case of a usual Riemannian metric and connection in \mathbb{R}^n). \blacksquare

22 The parallel transport operator

22.1 The shifter $J_t^{t_0}$ in \mathbb{R}^n

Let $t_0, T \in \mathbb{R}$ s.t. $t_0 < T$, let $c : t \in [t_0, T] \rightarrow c(t) \in \mathbb{R}^n$ be a regular curve in \mathbb{R}^n (with t interpreted as a time).

Definition 22.1 In \mathbb{R}^n , let $t_0, t \in]a, b[$, $p_{t_0} = c(t_0)$ and $p_t = c(t)$. The shifter between t_0 and t is the translation

$$J_{c,t}^{t_0} : \begin{cases} \{p_{t_0}\} \times T_{p_{t_0}}\mathbb{R}^n & \rightarrow \{p_t\} \times T_{p_t}\mathbb{R}^n \\ (p_{t_0}, \vec{w}_{p_{t_0}}) & \rightarrow J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) := (p_t, \vec{w}_{p_t}), \quad \text{where } \vec{w}_{p_t} = \vec{w}_{p_{t_0}}, \end{cases} \quad (22.1)$$

that is, $J_{c,t}^{t_0}$ translates the vector $\vec{w}_{p_{t_0}}$ at p_{t_0} at t_0 toward the vector $\vec{w}_{p_t} = \vec{w}_{p_{t_0}}$ at p_t at t .

Thus, if $A = p_{t_0} = c(t_0) = \tilde{c}(t_0)$ and $B = p_t = c(t) = \tilde{c}(t)$ are two points connected by two curves c and \tilde{c} , then $J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) = J_{\tilde{c},t}^{t_0}(\vec{w}_{p_{t_0}})$: The result is independent of the curves that connect A and B :

$$J_{c,t}^{t_0} \stackrel{\text{written}}{=} J_{\tilde{c},t}^{t_0}. \quad (22.2)$$

Hence, with (22.1), we have defined $J^{t_0} : \begin{cases}]a, b[& \rightarrow \mathcal{F}(TS_{t_0}, T\mathbb{R}^n) \\ t & \rightarrow J^{t_0}(t) := J_t^{t_0}. \end{cases}$

If there is no ambiguity, then (shorten notation)

$$J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) \stackrel{\text{written}}{=} \vec{w}_{p_t}, \quad \text{and} \quad J_t^{t_0}(\vec{w}_{p_{t_0}}) = J^{t_0}(t, \vec{w}_{p_{t_0}}) \quad (22.3)$$

Example 22.2 In \mathbb{R}^2 , with $c : t \rightarrow \begin{pmatrix} R \cos t \\ R \sin t \end{pmatrix}$, if $t_0 = 0$ and $\vec{w}_{p_{t_0}} = \vec{E}_1$ at $p_{t_0} = O + \vec{E}_1$, then at $t = \frac{\pi}{2}$, $p_t = O + \vec{E}_2$, and we get $J_c^{t_0}(\vec{w}_{p_{t_0}}) = \vec{E}_1$ (parallel transport in \mathbb{R}^2 : And relative to the circle, the vector \vec{w}_{p_t} “turns to the right as t increases”, relative to the tangent vector $\vec{v}(p_t) = \vec{c}'(t)$. \blacksquare

22.2 Parallel transport $J_{c,t}^{t_0}$ in S

Setting (obvious but has to be given): Let S be a surface in \mathbb{R}^n , let $t_0, T \in \mathbb{R}$ s.t. $t_0 < T$, let S_{t_0} be an open set in S , let $\Phi^{t_0} : [t_0, T] \times S_{t_0} \rightarrow S$ be a regular map (a motion). Let $t \in [t_0, T]$, let $\Phi^{t_0}(t, p_{t_0}) = \text{written } \Phi_t^{t_0}(p_t)$, let $S_t = \Phi_t^{t_0}(S_{t_0})$, and $\Phi_t^{t_0} : S_{t_0} \rightarrow S_t$ is supposed to be a diffeomorphism. Let $c : [t_0, T] \rightarrow S$ be a regular curve in S such that $c(t) \in S_t$ for all t . And let ∇ be the usual Riemannian connection ∇ in S .

Proposition 22.3 Let $p_{t_0} = c(t_0)$, $\vec{w}_{p_{t_0}} \in T_{p_{t_0}}S$, and $p_t = c(t)$. There exists a unique vector field \vec{w}_c defined at all points p_t ($\vec{w}_c \in \Gamma(\text{Im}c)$) such that

$$(\nabla_{\vec{v}} \vec{w}_c(p_t)) = \frac{D\vec{w}_c}{dt}(p_t) = \vec{0} \quad \text{and} \quad \vec{w}_c(p_{t_0}) = \vec{w}_{p_{t_0}}. \quad (22.4)$$

(E.g., see figure 22.1.)

Proof. (22.4) is an ODE with initial condition: Apply Cauchy–Lipschitz theorem. \blacksquare

Definition 22.4 \vec{w}_c being the solution of (22.4), the vector $\vec{w}_c(p_t)$ at $p_t = c(t)$ at t , is called the parallel transported vector from $\vec{w}_{p_{t_0}}$ at $p_{t_0} = c(t_0)$ at t_0 in S along c . And $\vec{w}_c(p_t) = \text{written } J_{c,t}^{t_0}(\vec{w}_{p_{t_0}})$. This defines the shifter (the “parallel transport operator”) $J_{c,t}^{t_0}$ along c from $T_{p_{t_0}}S_{t_0}$ to $T_{p_t}S_t$:

$$J_{c,t}^{t_0} : \begin{cases} \{p_{t_0}\} \times T_{p_{t_0}}S_{t_0} & \rightarrow \{p_t\} \times T_{p_t}S_t \\ (p_{t_0}, \vec{w}_{p_{t_0}}) & \rightarrow J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) := (p_t, \vec{w}_{p_t}), \quad \text{where } \vec{w}_{p_t} = \vec{w}_c, \end{cases} \quad (22.5)$$

and the shifter (the “parallel transport operator”) $J_c^{t_0}$ along c .

$$J_c^{t_0} : \begin{cases}]a, b[\times \{p_{t_0}\} \times T_{p_{t_0}}S & \rightarrow TS \\ (t, p_{t_0}, \vec{w}_{p_{t_0}}) & \rightarrow J_c^{t_0}(t, \vec{w}_{p_{t_0}}) := (p_t, \vec{w}_c(p_t)) \stackrel{\text{written}}{=} J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}), \end{cases} \quad (22.6)$$

Shorten notation:

$$J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) = J_c^{t_0}(t, \vec{w}_{p_{t_0}}) = \vec{w}_{p_t} = \vec{w}_c. \quad (22.7)$$

Example 22.5 See § 22.6.4. \blacksquare

22.3 The shifter is \mathbb{R} -linear

Proposition 22.6 $J_{c,t}^{t_0}$ is \mathbb{R} -linear. Thus $dJ_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) = J_{c,t}^{t_0}$ for all $\vec{w}_{t_0} \in T_{p_{t_0}}S$.

Proof. Let $\vec{w}_{p_{t_0}}, \vec{z}_{p_{t_0}} \in T_{p_{t_0}}S$ and $h \in \mathbb{R}$. The ODE (22.4) being linear (the right hand side is linear since it vanishes), we get $J_{c,t}^{t_0}(\vec{w}_{p_{t_0}} + h\vec{z}_{p_{t_0}}) = J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) + hJ_{c,t}^{t_0}(\vec{z}_{p_{t_0}})$. \blacksquare

$J_{c,t}^{t_0}$ being linear, $J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) = \text{written } J_{c,t}^{t_0} \cdot \vec{w}_{p_{t_0}}$.

Quantification with a coordinate basis:

$$J_{c,t}^{t_0} = \sum_{i,j=1}^m (J_{c,t}^{t_0})_j^i \vec{e}_i(p_t) \otimes e^j(p_{t_0}), \quad (22.8)$$

so, with $\vec{w}_{p_{t_0}} = \sum_{j=1}^m w_{p_{t_0}}^j \vec{e}_j(p_{t_0})$, $\vec{w}_c(p_t) = J_{c,t}^{t_0} \cdot \vec{w}_{p_{t_0}} = \sum_{i,j=1}^m (J_{c,t}^{t_0})_j^i w_{p_{t_0}}^j (p_{t_0}) \vec{e}_i(p_t)$, i.e.,

$$[\vec{w}_c(p_t)]|_{\vec{e}_i(p_t)} = [J_{c,t}^{t_0}]|_{\vec{e}_i(p_{t_0}), \vec{e}_i(p_t)} \cdot [\vec{w}_{p_{t_0}}]|_{\vec{e}_i(p_{t_0})}. \quad (22.9)$$

22.4 Composition of shifters S

Let $J_c(t, t_0) := J_{c,t}^{t_0}$.

Proposition 22.7 c being a regular curve in S , we have, for all $t, t_1 \in]a, b[$,

$$J_{c,t_1}^t \circ J_{c,t}^{t_0} = J_{c,t_1}^{t_0}, \quad \text{i.e.} \quad J_c(t_1, t) \circ J_c(t, t_0) = J_c(t_1, t_0). \quad (22.10)$$

(Linearity notation: $J_{c,t_1}^t \cdot J_{c,t}^{t_0} = J_{c,t_1}^{t_0}$, or $J_c(t_1, t) \cdot J_c(t, t_0) = J_c(t_1, t_0)$.) In particular, $(J_{c,t}^{t_0})^{-1} = J_{c,t_0}^t$.

Proof. The existence and uniqueness (Cauchy–Lipschitz theorem) gives (22.10). \blacksquare

22.5 Interpretation

Reminder: In \mathbb{R}^n , with $p_t = c(t)$ and $\vec{v}(p_t) = \vec{c}'(t)$, cf. (6.1),

$$d\vec{w}(p_{t_0}) \cdot \vec{v}(p_{t_0}) = \frac{d(\vec{w} \circ c)}{dt}(t_0) = \lim_{t \rightarrow t_0} \frac{\vec{w}(c(t)) - \vec{w}(c(t_0))}{t - t_0} = \lim_{t \rightarrow t_0} \frac{\vec{w}(p_t) - \vec{w}(p_{t_0})}{t - t_0}. \quad (22.11)$$

But the rate $\frac{\vec{w}(p_t) - \vec{w}(p_{t_0})}{t - t_0}$ is meaningless with the mathematical definition of a vector field, cf. (1.16), since the difference $\vec{w}(p_t) - \vec{w}(p_{t_0})$ is a nonsense: Not only $T_{p_t}S \neq T_{p_{t_0}}S$ in general, but with the full notation of a vector field, cf. (1.16), (22.11) reads

$$(p_{t_0}, \frac{D\vec{w}}{dt}(p_{t_0})) = \lim_{t \rightarrow t_0} \frac{(p_t, \vec{w}(p_t)) - (p_{t_0}, \vec{w}(p_{t_0}))}{t - t_0}, \quad (22.12)$$

and the difference in the limit is meaningless since the base points p_t and p_{t_0} are different. In \mathbb{R}^n , (22.12) is made meaningful thanks to the shifter $J_t^{t_0}$, cf. (22.2):

$$\begin{aligned} (p_{t_0}, \frac{D\vec{w}}{dt}(p_{t_0})) &= \lim_{t \rightarrow t_0} \frac{(p_{t_0}, J_t^{t_0}{}^{-1}(\vec{w}(p_t)) - (p_{t_0}, \vec{w}(p_{t_0})))}{t - t_0} \\ &= \lim_{t \rightarrow t_0} \frac{(p_t, \vec{w}(p_t)) - (p_t, J_t^{t_0}(\vec{w}(p_{t_0})))}{t - t_0}, \end{aligned} \quad (22.13)$$

simply written as (22.11).

Proposition 22.8 In a regular surface $S \subset \mathbb{R}^n$, we have

$$\begin{aligned} (p_{t_0}, \nabla_{\vec{v}} \vec{w}(p_{t_0})) &= (p_{t_0}, \frac{D\vec{w}}{dt}(p_{t_0})) = \text{Proj}_{T_{p_{t_0}}S} \left(\lim_{t \rightarrow t_0} \frac{(p_{t_0}, J_{c,t}^{t_0}{}^{-1} \cdot \vec{w}(p_t)) - (p_{t_0}, \vec{w}(p_{t_0}))}{t - t_0} \right) \\ &= \text{Proj}_{T_{p_{t_0}}S} \left(\lim_{t \rightarrow t_0} \frac{(p_t, \vec{w}(p_t)) - (p_t, J_{c,t}^{t_0} \cdot \vec{w}(p_{t_0}))}{t - t_0} \right). \end{aligned} \quad (22.14)$$

(The last equality is not used in a surface since $\vec{w}(p_t) - J_{c,t}^{t_0} \cdot \vec{w}(p_{t_0}) \in T_{p_t}S$ and $T_{p_t}S$ varies as $t \rightarrow t_0$.) Simply written:

$$\nabla_{\vec{v}} \vec{w}(p_{t_0}) = \frac{D\vec{w}}{dt}(p_{t_0}) = \text{Proj}_{T_{p_{t_0}}S} \left(\lim_{t \rightarrow t_0} \frac{J_{c,t}^{t_0}{}^{-1} \cdot \vec{w}(p_t) - \vec{w}(p_{t_0})}{t - t_0} \right). \quad (22.15)$$

Proof. $\nabla_{\vec{v}} \vec{w}(p_{t_0}) = \text{Proj}_{T_{p_{t_0}}S}(d\vec{w}(p_{t_0}) \cdot \vec{v}(p_{t_0})) = \text{Proj}_{T_{p_{t_0}}S}(\lim_{t \rightarrow t_0} \frac{\vec{w}(\vec{c}(t)) - \vec{w}(c(t_0))}{t - t_0})$. Let $\vec{w}_c(p_t) = J_{c,t}^{t_0}(\vec{w}(p_{t_0}))$. Then $\frac{\vec{w}(\vec{c}(t)) - \vec{w}(c(t_0))}{t - t_0} = \frac{\vec{w}(\vec{c}(t)) - \vec{w}_c(\vec{c}(t))}{t - t_0} + \frac{\vec{w}_c(\vec{c}(t)) - \vec{w}_c(c(t_0))}{t - t_0}$, with $\lim_{t \rightarrow t_0} \frac{\vec{w}_c(\vec{c}(t)) - \vec{w}_c(c(t_0))}{t - t_0} = \vec{0}$, cf. (22.4). Thus $\nabla_{\vec{v}} \vec{w}(p_{t_0}) = \frac{D\vec{w}}{dt}(p_{t_0}) = \text{Proj}_{T_{p_{t_0}}S}(\lim_{t \rightarrow t_0} \frac{\vec{w}(p_t) - \vec{w}_c(p_{t_0})}{t - t_0})$. And $J_{c,t}^{t_0} = I$ gives $\lim_{t \rightarrow t_0} \frac{\vec{w}(p_t) - \vec{w}_c(p_{t_0})}{t - t_0} = \lim_{t \rightarrow t_0} \frac{(J_{c,t}^{t_0})^{-1} \cdot \vec{w}(p_t) - (J_{c,t}^{t_0})^{-1} \cdot \vec{w}_c(p_{t_0})}{t - t_0}$ \blacksquare

22.6 Examples

22.6.1 Parallel transport in a curve in \mathbb{R}^2

Let S be a one dimensional regular manifold in \mathbb{R}^2 parametrized as a regular curve $c : t \in]a, b[\rightarrow c(t)$, $\text{Im}c = S$.

Proposition 22.9 *Let \vec{w} be a vector field in S , that is, $\vec{w}(p) = \alpha(t)\vec{c}'(t)$ at $p = c(t)$ with $\alpha \in \mathcal{F}(]a, b[; \mathbb{R})$. If $\|\vec{c}'(t)\| = 1$ for all t , then \vec{w} is parallel transported along $\text{Im}(c)$ iff α is constant. (In particular, the velocity field $\vec{v}(c(t)) := \vec{c}'(t)$ is parallel transported along c , and a geodesic in a curve is a part of this curve).*

Proof. $\|\vec{c}'(t)\|_{\mathbb{R}^2}^2 = 1$ gives $\frac{d}{dt}(\vec{c}'(t), \vec{c}'(t))_{\mathbb{R}^2} = 0 = 2(\vec{c}''(t), \vec{c}'(t))$, thus $\vec{c}''(t) \perp \vec{c}'(t)$ for all t . And $(\vec{w} \circ c)'(t) = \alpha'(t)\vec{c}'(t) + \alpha(t)\vec{c}''(t)$ gives $\frac{D\vec{w}}{dt}(t) = \text{Proj}_{T_p S}((\vec{w} \circ c)'(t)) = \alpha'(t)\vec{c}'(t)$. Since $\vec{c}'(t) \neq \vec{0}$ (its norm equals 1), $\frac{D\vec{w}}{dt}(t) = 0$ iff $\alpha' = 0$. \blacksquare

Example 22.10 (Calculations.) Polar coordinates, $S = C(\vec{0}, R) = \text{Im}c$ with $p = c(\theta) = \begin{pmatrix} R \cos \theta \\ R \sin \theta \end{pmatrix}$. So $\|\vec{v}(p)\| = \|\vec{c}'(t)\| = R = \|\vec{e}_2(p)\| = \text{constant}$. Let $\vec{w} \in \Gamma(S)$ given by, with $p = c(\theta)$,

$$\vec{w}(p) = \beta(p)\vec{e}_2(p), \quad \text{so} \quad d\vec{w} = d\beta \vec{e}_2 + \beta d\vec{e}_2.$$

Thus

$$d\vec{w} \cdot \vec{e}_2 = (d\beta \cdot \vec{e}_2) \cdot \vec{e}_2 + \beta(d\vec{e}_2 \cdot \vec{e}_2) = \frac{\partial \beta}{\partial \theta} \vec{e}_2 + \beta(\gamma_{22}^1 \vec{e}_1 + \gamma_{22}^2 \vec{e}_2).$$

Thus

$$\nabla_{\vec{e}_2} \vec{w} = \text{Proj}_{T_p S}(d\vec{w} \cdot \vec{e}_2) = \left(\frac{\partial \beta}{\partial \theta} + \beta \gamma_{22}^2 \right) \vec{e}_2. \quad (22.16)$$

With $\gamma_{22}^2 = 0$, cf. (3.9), thus $\frac{D\vec{w}}{dt} = 0$ iff $\frac{\partial \beta}{\partial \theta} = 0$, i.e., iff $\beta = \beta_0$ is constant. In particular, $\vec{v} = \vec{e}_2$ is parallel transported along itself. \blacksquare

22.6.2 Parallel transport in a cylinder in \mathbb{R}^3

Surface $\Phi(\theta, z) = R \cos \theta \vec{E}_1 + R \sin \theta \vec{E}_2 + z \vec{E}_3 = \text{written} \begin{pmatrix} R \cos \theta \\ R \sin \theta \\ z \end{pmatrix}$, and $S = \text{Im}\Phi$ (cylinder). Ba-

sis at $p = \Phi(\theta, z)$ in $T_p S$: $\vec{e}_1(p) = \frac{\partial \Phi}{\partial \theta}(\theta, z) = \begin{pmatrix} -R \sin \theta \\ R \cos \theta \\ 0 \end{pmatrix}$ and $\vec{e}_2(p) = \vec{E}_3 = \text{written} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Let

$a \neq 0$ and consider $c(t) = \Phi(t, at) = \begin{pmatrix} R \cos t \\ R \sin t \\ at \end{pmatrix} \in S$ (spiral). At $p = c(t)$, $\vec{e}_1(p) = \begin{pmatrix} -R \sin t \\ R \cos t \\ 0 \end{pmatrix}$

and $\vec{e}_2(p) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Consider the vector fields \vec{e}_1 and \vec{e}_2 in S ; $\frac{D\vec{e}_1}{dt}(p) = \text{Proj}_{T_p S}(\frac{d(\vec{e}_1 \circ c)}{dt}(t)) =$

$\text{Proj}_{T_p S} \begin{pmatrix} -R \cos t \\ -R \sin t \\ 0 \end{pmatrix} = \vec{0}$ et $\frac{D\vec{e}_2}{dt}(p) = \text{Proj}_{T_p S}(\vec{0}) = \vec{0}$. Thus any vector field $\vec{w} = \alpha \vec{e}_1(p) + \beta \vec{e}_2(p)$

with α and β constants is parallel transported along $\text{Im}c$.

And a vector field in S reads $\vec{w}(p) = \alpha(t)\vec{e}_1(p) + \beta(t)\vec{e}_2(p)$, thus $\frac{D\vec{w}}{dt}(t) = \alpha'(t)\vec{e}_1(p) + \beta'(t)\vec{e}_2(p)$, since $\frac{D\vec{e}_1}{dt}(t) = \frac{D\vec{e}_2}{dt}(t) = 0$. Thus \vec{w} is parallel transported along $\text{Im}c$ iff α and β are constant.

Exercise 22.11 Compute $\nabla_{\vec{v}} \vec{e}_1$ and $\nabla_{\vec{v}} \vec{e}_2$ from $d\vec{e}_1$ and $d\vec{e}_2$.

Answer. With (3.14) and the shift of index $2 \rightarrow 1$ and $3 \rightarrow 2$ we get $d\vec{e}_1(p) = -R \cos \theta \vec{E}_1 \otimes d\theta - R \sin \theta \vec{E}_2 \otimes d\theta$, and $d\vec{e}_2(p) = 0$.

And $\vec{v}(p) = \vec{c}'(t) = -R \sin t \vec{E}_1 + R \cos t \vec{E}_2 + a \vec{E}_3 = \vec{e}_1(p) + a \vec{e}_2(p) \in T_p S$.

Thus $d\vec{e}_1(p) \cdot \vec{v}(p) = (-R \cos t \vec{E}_1 \otimes d\theta - R \sin t \vec{E}_2 \otimes d\theta) \cdot (\vec{e}_1(p) + a \vec{e}_2(p)) = a(-R \cos t \vec{E}_1 - R \sin t \vec{E}_2) \perp \vec{e}_1(p)$ and $\perp \vec{e}_2(p)$, thus $\frac{D\vec{e}_1}{dt}(p) = \text{Proj}_{T_p S}(d\vec{e}_1(p) \cdot \vec{v}(p)) = \vec{0}$. And $\frac{D\vec{e}_2}{dt}(p) = \text{Proj}_{T_p S}(\vec{0}) = \vec{0}$. \blacksquare

22.6.3 Parallel transport in a sphere in \mathbb{R}^3

Example 22.12 c is a meridian $\varphi \rightarrow c(\varphi) = \begin{pmatrix} R \cos \theta_0 \cos \varphi \\ R \sin \theta_0 \cos \varphi \\ R \sin \varphi \end{pmatrix}$. Consider the vector field $\vec{w}(p) =$

$\frac{\vec{e}_2(p)}{\|\vec{e}_2(p)\|} = \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}$, so if $p = c(\varphi)$ then $\vec{w}(p)$ is a unit vector orthogonal to the meridian. And $\frac{d(\vec{w} \circ c)}{d\varphi}(\varphi) = \vec{0}$, thus $\text{Proj}_{TS}(\frac{d\vec{w} \circ c}{d\varphi}(\varphi)) = \vec{0}$, thus \vec{w} is parallel transported along the meridian.

And any vector field $\vec{w} = \alpha \vec{e}_3(p) + \beta \frac{\vec{e}_2(p)}{\|\vec{e}_2(p)\|}$ with α and β constant is parallel transported along the meridian. Here $\|\vec{w}\| = \sqrt{R^2 \alpha^2 + \beta^2}$ constant, and “turns with the meridian (parallel transport). E.g.: If a plane flies along a meridian, then its wings are parallel transported along this meridian.

Consider $\vec{w}(p) = \vec{e}_2(p)$: parallel to a meridian, but its length $\|\vec{e}_2(p)\| = R \cos \varphi$ varies with φ . So $\vec{w}(p) = (\vec{w} \circ c)(\varphi) = R \begin{pmatrix} -\sin \theta_0 \cos \varphi \\ \cos \theta_0 \cos \varphi \\ 0 \end{pmatrix}$, and $\frac{d(\vec{w} \circ c)}{d\varphi}(\varphi) = R \begin{pmatrix} \sin \theta_0 \sin \varphi \\ -\cos \theta_0 \sin \varphi \\ 0 \end{pmatrix} = -\frac{\cos \varphi}{\sin \varphi} \vec{e}_2(p)$ (every where but on the equator), thus $\text{Proj}_{TS}(\frac{d\vec{w} \circ c}{d\varphi}(\varphi)) = -\frac{\cos \varphi}{\sin \varphi} \vec{e}_2(p) \neq \vec{0}$ (every where but on the equator and at the poles). Thus \vec{e}_2 is not parallel transported along the meridian. \blacksquare

22.6.4 Parallel transport along a parallel

Continuing proposition 20.4 and remark 21.3. $\Phi(\theta, \varphi) = \begin{pmatrix} R \cos \theta \cos \varphi \\ R \sin \theta \cos \varphi \\ R \sin \varphi \end{pmatrix}$, $\vec{e}_2(p) = \frac{\partial \Phi}{\partial \theta}(\theta, \varphi) = \begin{pmatrix} -R \sin \theta \cos \varphi \\ R \cos \theta \cos \varphi \\ 0 \end{pmatrix}$ and $\vec{e}_3(p) = \frac{\partial \Phi}{\partial \varphi}(\theta, \varphi) = \begin{pmatrix} -R \cos \theta \sin \varphi \\ -R \sin \theta \sin \varphi \\ R \cos \varphi \end{pmatrix}$, cf. (3.29). And parallel $c_{\varphi_0}(\theta) = \begin{pmatrix} R \cos \theta \cos \varphi_0 \\ R \sin \theta \cos \varphi_0 \\ R \sin \varphi_0 \end{pmatrix}$. Let $p_{t_0} = c_{\varphi_0}(\theta_0)$, and let $\vec{u}_{p_{t_0}} \in T_{p_{t_0}}S$ be given, $\vec{u}_{p_{t_0}} \neq \vec{0}$. Consider $p = c_{\varphi_0}(\theta)$ and the parallel transported vector

$$\vec{u}_c(p) = (J_{c,t}^{t_0} \cdot \vec{u}_{p_{t_0}})(p) = \alpha(\theta) \vec{e}_2(p) + \beta(\theta) \vec{e}_3(p), \quad (22.17)$$

where $\alpha(\theta) = \alpha_{\varphi_0}(\theta)$ and $\beta(\theta) = \beta_{\varphi_0}(\theta)$ depends on φ_0 (the curve=parallel).

Remark 22.13 1- If $\vec{u}_c(p) = \alpha(\theta) \vec{e}_2(p)$ (tangent to the parallel), then, cf. (3.33) :

$$\frac{D\vec{u}_c}{d\theta}(p) = \alpha'(\theta) \vec{e}_2(p) + \alpha(\theta) \nabla_{\vec{e}_2} \vec{e}_2(p) = \alpha'(\theta) \vec{e}_2(p) + \alpha(\theta) \cos \varphi_0 \sin \varphi_0 \vec{e}_3(p).$$

So $\frac{D\vec{u}_c}{d\theta}(p) = 0$ along c_{φ_0} iff $\alpha' = 0$ and $\alpha(\theta) \cos \varphi_0 \sin \varphi_0 = 0$, i.e., iff α is constant and $\varphi_0 = 0$: A parallel transport is only possible along the equator. In particular, $\vec{e}_2(p)$ is not parallel transported along a parallel which is not the equator.

2- If $\vec{u}_c(p) = \beta(\theta) \vec{e}_3(p)$ (tangent to a meridian), then, cf. (3.34) :

$$\frac{D\vec{u}_c}{d\theta}(p) = \beta'(\theta) \vec{e}_3(p) + \beta(\theta) \nabla_{\vec{e}_2} \vec{e}_3(p) = \beta'(\theta) \vec{e}_3(p) - \beta(\theta) \tan \varphi_0 \vec{e}_2(p).$$

Thus $\frac{D\vec{u}_c}{d\theta}(p) = 0$ along c_{φ_0} iff $\beta' = 0$ and $\beta(\theta) \tan \varphi_0 = 0$, i.e., iff β is constant and $\varphi_0 = 0$: A parallel transport is only possible along the equator. In particular, $\vec{e}_3(p)$ is not parallel transported along a parallel which is not the equator. \blacksquare

Proposition 22.14 Let $s_0 = |\sin \varphi_0|$. The parallel transported vector $\vec{u}_c(p)$, cf. (22.17), is of the type

$$\begin{cases} \alpha(\theta) = c_1 \cos(s_0 \theta) + c_2 \sin(s_0 \theta), \\ \beta(\theta) = (-c_1 \sin(s_0 \theta) + c_2 \cos(s_0 \theta)) \cos \varphi_0, \end{cases} \quad (22.18)$$

where c_1 and c_2 are constants depending on $\vec{u}_{p_{t_0}} = U^2 \vec{e}_2 + U^3 \vec{e}_3$ given by

$$\begin{cases} c_1 = \cos(s_0 \theta_0) U^2 - \sin(s_0 \theta_0) \frac{U^3}{\cos \varphi_0}, \\ c_2 = \sin(s_0 \theta_0) U^2 + \cos(s_0 \theta_0) \frac{U^3}{\cos \varphi_0}. \end{cases} \quad (22.19)$$

Proof. Let $p = c_{\varphi_0}(\theta)$. (22.17) gives, with (3.43),

$$\frac{D\vec{u}_c}{d\theta}(p) = (\alpha'(\theta) - \beta(\theta) \tan \varphi_0) \vec{e}_2(p) + (\beta'(\theta) + \alpha(\theta) \cos \varphi_0 \sin \varphi_0) \vec{e}_3(p). \quad (22.20)$$

Thus $\frac{D\vec{u}_c}{d\theta}(p) = 0$ gives

$$\begin{cases} \alpha'(\theta) - \beta(\theta) \tan \varphi_0 = 0, \\ \beta'(\theta) + \alpha(\theta) \cos \varphi_0 \sin \varphi_0 = 0. \end{cases} \quad (22.21)$$

($\alpha = \beta = 0$ is not possible since $\vec{u}_0 \neq \vec{0}$, and we get the results in remark (22.13).)

Thus $\alpha'' = \beta' \tan \varphi_0$ and $\beta'' = -\alpha' \cos \varphi_0 \sin \varphi_0$, thus

$$\begin{cases} \alpha'' + \alpha \tan \varphi_0 \cos \varphi_0 \sin \varphi_0 = \alpha'' + \alpha \sin^2 \varphi_0 = 0, \\ \beta'' + \beta \tan \varphi_0 \cos \varphi_0 \sin \varphi_0 = \beta'' + \beta \sin^2 \varphi_0 = 0. \end{cases}$$

With $s_0 = |\sin \varphi_0|$ we get: there exists $c_1, c_2, c_3, c_4 \in \mathbb{R}$ s.t.

$$\begin{cases} \alpha(\theta) = c_1 \cos(s_0\theta) + c_2 \sin(s_0\theta), \\ \beta(\theta) = c_3 \cos(s_0\theta) + c_4 \sin(s_0\theta). \end{cases}$$

Thus $\begin{cases} \alpha'(\theta) = -c_1 s_0 \sin(s_0\theta) + c_2 s_0 \cos(s_0\theta), \\ \beta'(\theta) = -c_3 s_0 \sin(s_0\theta) + c_4 s_0 \cos(s_0\theta). \end{cases}$ Thus, with (22.21) (the sin and cos functions are independent),

$$-c_1 s_0 = c_4 \tan \varphi_0, \quad \text{and} \quad c_2 s_0 = c_3 \tan \varphi_0,$$

that is, $-c_1 \cos \varphi_0 = c_4$ and $c_2 \cos \varphi_0 = c_3$, thus (22.18). Thus

$$\beta(\theta) = c_2 \cos \varphi_0 \cos(s_0\theta) - c_1 \cos \varphi_0 \sin(s_0\theta).$$

Thus:

$$\vec{u}_c(p) = (c_1 \cos(s_0\theta) + c_2 \sin(s_0\theta)) \vec{e}_2(p) + (c_2 \cos(s_0\theta) - c_1 \sin(s_0\theta)) (\cos \varphi_0 \vec{e}_3(p)).$$

With $\|\vec{e}_2(p)\| = R \cos \varphi_0$, $\|\vec{e}_3(p)\| = R$ and $\vec{e}_2(p) \perp \vec{e}_3(p)$, we have $\|\vec{u}_c\| = (c_1^2 + c_2^2)^{\frac{1}{2}} = \text{constant}$ (the parallel transport keeps the metric). And the initial condition $\vec{u}_c(p_{t_0}) = \vec{U} = U^2 \vec{e}_2 + U^3 \vec{e}_3$ gives

$$\begin{pmatrix} \alpha(\theta_0) \\ \frac{\beta(\theta_0)}{\cos \varphi_0} \end{pmatrix} = \begin{pmatrix} \cos(s_0\theta_0) & \sin(s_0\theta_0) \\ -\sin(s_0\theta_0) & \cos(s_0\theta_0) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} U^2 \\ \frac{U^3}{\cos \varphi_0} \end{pmatrix}.$$

With $\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}^{-1} = \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix}$, we get (22.19). ▀

Example 22.15 On the equator: $\varphi_0 = 0$, $s_0 = 0$, thus $c_1 = U^2$, $c_2 = U^3$, so $\alpha(\theta) = U^2$ and $\beta(\theta) = U^3$, thus $\vec{u}_c(p) = U^2 \vec{e}_2(p) + U^3 \vec{e}_3(p)$, and the equator is a geodesic: Already known. ▀

Example 22.16 See figure 22.1. Parallel $\varphi_0 = \frac{\pi}{4}$ (radian, that is 45 degrees) north. So $s_0 = |\sin \varphi_0| = \frac{\sqrt{2}}{2}$. Choose $\theta_0 = 0$, $p_{t_0} = c_{\varphi_0}(0) = \begin{pmatrix} R \cos \varphi_0 \\ 0 \\ R \sin \varphi_0 \end{pmatrix}$ (on the Greenwich meridian), and travel along the 45-th parallel.

1- Initial vector $\vec{U} = \vec{e}_2(p_{t_0})$ (parallel to the parallel toward East). See figure 22.1. Thus (22.19) and (22.18) give

$$\begin{cases} c_1 = U^2 = 1, \\ c_2 = \frac{U^3}{\cos \varphi_0} = 0, \end{cases} \quad \text{and} \quad \begin{cases} \alpha(\theta) = \cos(s_0\theta), \\ \beta(\theta) = -\sin(s_0\theta) \cos \varphi_0, \end{cases}$$

and

$$\vec{u}_c(\theta) = \cos(s_0\theta) \vec{e}_2(p) - \sin(s_0\theta) \cos \varphi_0 \vec{e}_3(p). \quad (22.22)$$

Since $\|\vec{e}_2(p)\| = \|\cos \varphi_0 \vec{e}_3(p)\| = R \cos \varphi_0$, $\vec{u}_c(\theta)$ keeps its length and “turns”, see figure 22.1.

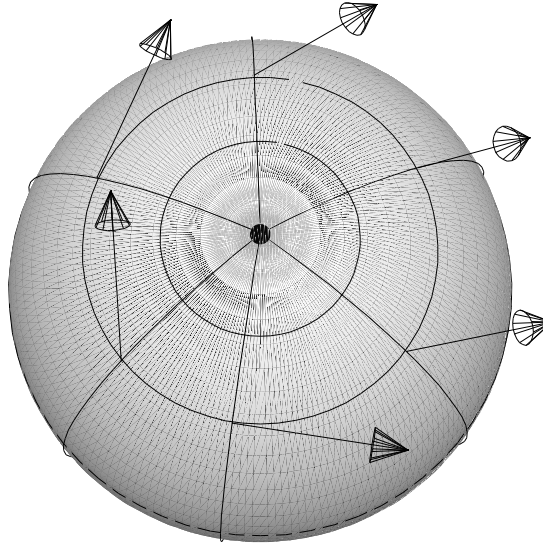


Figure 22.1: Parallel transport along the 45-th parallel, with, at $\theta = 0$, \vec{w} oriented to the East: The parallel transported vector turned south at $\theta \simeq 127$ degrés, see example 22.16. (Parallel transport along the 30-th parallel, with, at $\theta = 0$, \vec{w} oriented to the East: The parallel transported vector turned south at $\theta \simeq 180$ degrés. On the equator (geodesic) \vec{w} stays parallel to the equator.)

- If θ is s.t. $s_0\theta = \frac{\pi}{2}$, i.e. $\theta = \frac{\pi}{2s_0} = \frac{\pi}{\sqrt{2}}$ (toward East and $\frac{\pi}{\sqrt{2}} \frac{180}{\pi} \simeq 127$ degrees longitude), then $\vec{u}_c(\theta)$ is now oriented South. The orientation toward the South was expected: Indeed, the geodesic (great circle) through p_{t_0} which is parallel to $\vec{e}_2(p_{t_0})$ “goes toward the equator” (a geodesic looks like a straight line in S); Thus $\vec{u}_c(\theta)$ goes toward the South.

- If θ is s.t. $s_0\theta = \pi$, i.e. $\theta = \frac{\pi}{s_0} = \frac{2\pi}{\sqrt{2}}$ (toward East and $\frac{2\pi}{\sqrt{2}} \frac{180}{\pi} \simeq 154$ degrees longitude), then $\vec{u}_c(\theta) = -\vec{e}_2(p)$.

- If θ is s.t. $s_0\theta = 2\pi$, i.e. $\theta = \frac{2\pi}{s_0} = \frac{4\pi}{\sqrt{2}}$ (toward East and $\frac{4\pi}{\sqrt{2}} \frac{180}{\pi} \simeq 308$ degrees longitude) then $\vec{u}_c(\theta) = \vec{e}_2(p)$.

2- Initial vector $\vec{U} = \vec{e}_3(p_{t_0})$ (parallel to the meridian toward North). Then

$$c_1 = U^2 = 0, \quad c_2 = U^3 = 1, \quad \alpha(\theta) = \sin(s_0\theta), \quad \beta(\theta) = \frac{\sqrt{2}}{2} \cos(s_0\theta),$$

and

$$\vec{u}_c(\theta) = \sin(s_0\theta)\vec{e}_2(p) + \cos(s_0\theta)(\cos\varphi_0\vec{e}_3(p)), \quad (22.23)$$

and $\vec{u}_c(\theta)$ turns, see figure 22.1 with initial vector $-\vec{e}_3(p)$. ▀

22.7 The shifter preserves the Riemannian metric

Proposition 22.17 *With a usual Euclidean metric $(\cdot, \cdot)_g = (\cdot, \cdot)_{\mathbb{R}^n}$ and the associated Riemannian connection (given by $\nabla_{\vec{v}}\vec{w} = \text{Proj}_{TS}(d\vec{w} \cdot \vec{v}) \in \Gamma(S)$), the shifter $J_{c,t}^{t_0}$ in S along a curve c satisfies*

$$(J_{c,t}^{t_0} \cdot \vec{u}_{p_{t_0}}, J_{c,t}^{t_0} \cdot \vec{w}_{p_{t_0}})_{\mathbb{R}^n} = (\vec{u}_{p_{t_0}}, \vec{w}_{p_{t_0}})_{\mathbb{R}^n}. \quad (22.24)$$

(The shifter is an isometry: keeps lengths and angles). I.e.,

$$\frac{d}{dt} \left(J_{c,t}^{t_0} \cdot \vec{u}_{p_{t_0}}, J_{c,t}^{t_0} \cdot \vec{w}_{p_{t_0}} \right)_{\mathbb{R}^n} = 0. \quad (22.25)$$

Proof. Let $J_{c,t}^{t_0}(\vec{w}_{p_{t_0}}) = \vec{w}_c(c(t))$, cf. (22.6), so $\frac{D\vec{w}_c}{dt}(p_t) = 0$, idem with $\vec{u}_{p_{t_0}}$. Thus

$$\begin{aligned} \frac{d}{dt} (J_c^{t_0}(t, \vec{u}_{p_{t_0}}), J_c^{t_0}(t, \vec{w}_{p_{t_0}}))_{\mathbb{R}^n} &= \left(\frac{\partial J_c^{t_0}}{\partial t}(t, \vec{u}_{p_{t_0}}), J_c^{t_0}(t, \vec{w}_{p_{t_0}}) \right)_{\mathbb{R}^n} + \left(J_c^{t_0}(t, \vec{u}_{p_{t_0}}), \frac{\partial J_c^{t_0}}{\partial t}(t, \vec{w}_{p_{t_0}}) \right)_{\mathbb{R}^n} \\ &= \left(\text{Proj}_{T_p S} \frac{\partial J_c^{t_0}}{\partial t}(t, \vec{u}_{p_{t_0}}), J_c^{t_0}(t, \vec{w}_{p_{t_0}}) \right)_{\mathbb{R}^n} + \left(J_c^{t_0}(t, \vec{u}_{p_{t_0}}), \text{Proj}_{T_p S} \frac{\partial J_c^{t_0}}{\partial t}(t, \vec{w}_{p_{t_0}}) \right)_{\mathbb{R}^n}, \end{aligned}$$

since $J_c^{t_0}(t, \vec{u}_{p_{t_0}}), J_c^{t_0}(t, \vec{w}_{p_{t_0}}) \in T_{p_t}S$. That is,

$$\frac{d}{dt}(J_c^{t_0}(t, \vec{u}_{p_{t_0}}), J_c^{t_0}(t, \vec{w}_{p_{t_0}}))_{\mathbb{R}^n} = \left(\frac{D\vec{u}_c}{dt}(c(t)), \vec{u}_c(c(t))\right)_{\mathbb{R}^n} + (\vec{u}_c(c(t)), \frac{D\vec{w}_c}{dt}(c(t)))_{\mathbb{R}^n} = 0 + 0 = 0,$$

since $\frac{D\vec{u}_c}{dt} = 0 = \frac{D\vec{w}_c}{dt}$. \blacksquare

Proposition 22.18 Let ∂_i be the derivation relative to the i -th variable. For all $t_0, t, t_1 \in]a, b[$, we have

$$\partial_2 J_c(t_1, t) \cdot J_c(t, t_0) + J_c(t_1, t) \cdot \partial_1 J_c(t, t_0) = 0, \quad (22.26)$$

written

$$\frac{dJ_{c,t_1}^t}{dt} \cdot J_{c,t}^{t_0} + J_{c,t_1}^t \cdot \frac{dJ_{c,t}^{t_0}}{dt} = 0, \quad (22.27)$$

or, $\frac{dJ_c(t_1, t)}{dt} \cdot J_c(t, t_0) + J_c(t_1, t) \cdot \frac{dJ_c(t, t_0)}{dt} = 0$. In particular,

$$\partial_2 J_c(t_0, t_0) + \partial_1 J_c(t_0, t_0) = 0, \quad \text{i.e.} \quad \frac{dJ_{c,t_0}^t}{dt} \Big|_{t=t_0} = -\frac{dJ_{c,t}^{t_0}}{dt} \Big|_{t=t_0}, \quad (22.28)$$

or, $\frac{dJ_{c,t,t_0}}{dt} \Big|_{t=t_0} = -\frac{dJ_{c,t_0,t}}{dt} \Big|_{t=t_0}$.

Proof. (22.10) gives $J(t_1, t) \cdot J(t, t_0) \cdot \vec{w}_{p_{t_0}} = J(t_1, t_0) \cdot \vec{w}_{p_{t_0}}$, thus

$$\partial_2 J(t_1, t) \cdot J(t, t_0) \cdot \vec{w}_{p_{t_0}} + J(t_1, t) \cdot \partial_1 J(t, t_0) \cdot \vec{w}_{p_{t_0}} = \vec{0}, \quad (22.29)$$

for all $\vec{w}_{p_{t_0}} \in T_{p_{t_0}}S$, i.e. (22.26). In particular, $\partial_2 J(t_0, t_0) \cdot J(t_0, t_0) \cdot \vec{w}_{p_{t_0}} + J(t_0, t_0) \cdot \partial_1 J(t_0, t_0) \cdot \vec{w}_{p_{t_0}} = \vec{0}$, thus $\partial_2 J(t_0, t_0) + \partial_1 J(t_0, t_0) = \vec{0}$, since $J_c(t_0, t_0) = I$. \blacksquare

Corollary 22.19 Let $p_t = c(t)$, $\vec{c}'(t) = \vec{v}(p_t) = \sum_{k=1}^m v^k(p_t) \vec{e}_k(p_t)$, and $J_{c,t}^{t_0} = \sum_{i,j=1}^m (J_{c,t}^{t_0})_j^i \vec{e}_i(p_t) \otimes e^j(p_{t_0})$, cf. (22.8), Then, for all i, j ,

$$\frac{d(J_{c,t}^{t_0})_j^i}{dt} = -\sum_{k,\ell=1}^m (J_{c,t}^{t_0})_j^\ell v^k \gamma_{k\ell}^i, \quad \text{thus} \quad \frac{d(J_{c,t}^{t_0})_j^i}{dt} \Big|_{t=t_0} = -\sum_{k=1}^m \gamma_{kj}^i v^k. \quad (22.30)$$

Proof. Let $\vec{w}_{p_{t_0}} = \vec{e}_j(p_{t_0})$ and $\vec{w}_c(p_t) = J_c(t, t_0) \cdot \vec{e}_j(p_t) = \sum_{i=1}^m J_c(t, t_0)_j^i \vec{e}_i(c(t))$, cf. (22.8). Thus $\nabla_{\vec{v}} \vec{w}_c = \frac{D\vec{w}_c}{dt} = \vec{0}$, cf. (22.4), that is $\text{Proj}_{T_{p_t}S} \frac{d(\vec{w}_c \circ c)}{dt} = \vec{0}$, gives

$$\sum_{i=1}^m \frac{d(J_{c,t}^{t_0})_j^i}{dt} \vec{e}_i(p_t) + \sum_{i=1}^m (J_{c,t}^{t_0})_j^i \text{Proj}_{T_{p_t}S} (d\vec{e}_i(p_t) \cdot \vec{v}(p_t)) = \vec{0}.$$

With $\text{Proj}_{T_{p_t}S} (d\vec{e}_i \cdot \vec{v}) = \nabla_{\vec{v}} \vec{e}_i = \sum_{k=1}^m v^k \nabla_{\vec{e}_k} \vec{e}_i = \sum_{k,\ell=1}^m v^k \gamma_{ki}^\ell \vec{e}_\ell$, thus

$$\sum_{i=1}^m \frac{d(J_{c,t}^{t_0})_j^i}{dt} \vec{e}_i(p_t) + \sum_{i,k,\ell=1}^m (J_{c,t}^{t_0})_j^i v^k \gamma_{ki}^\ell \vec{e}_\ell = \vec{0}.$$

And $\sum_{i,k,\ell=1}^m (J_{c,t}^{t_0})_j^i v^k \gamma_{ki}^\ell \vec{e}_\ell = \sum_{i,k,\ell=1}^m (J_{c,t}^{t_0})_j^\ell v^k \gamma_{k\ell}^i \vec{e}_i$ gives (22.30). \blacksquare

22.8 Theorem 2 of Levi-Civita

Manifold S with a connection ∇ .

Definition 22.20 A metric $g(\cdot, \cdot)$ in S is parallel transported in S relative to the connection ∇ iff (22.25) is satisfied for all curve c in S , all $p_{t_0} \in S$ and all $\vec{u}_{t_0}, \vec{w}_{t_0} \in T_{p_{t_0}}S$, that is,

$$((J_{c,t}^{t_0} \cdot \vec{u}_{t_0})(p_t), (J_{c,t}^{t_0} \cdot \vec{w}_{t_0})(p_t))_{g_{p_t}} = (\vec{u}_{t_0}(p_{t_0}), \vec{w}_{t_0}(p_{t_0}))_{g_{p_{t_0}}}, \quad (22.31)$$

when $p_{t_0} = c(t_0)$, $p_t = c(t)$, that is,

$$(\vec{u}_c(p_t), \vec{w}_c(p_t))_{g_{p_t}} = (\vec{u}_{t_0}(p_{t_0}), \vec{w}_{t_0}(p_{t_0}))_{g_{p_{t_0}}}. \quad (22.32)$$

That is, iff

$$\frac{d}{dt} \left(((J_{c,t}^{t_0} \cdot \vec{u}_{t_0})(c(t)), (J_{c,t}^{t_0} \cdot \vec{w}_{t_0})(c(t)))_{g_{c(t)}} \right) = 0, \quad (22.33)$$

i.e., $\frac{d}{dt} \left((\vec{u}_c(c(t)), \vec{w}_c(c(t)))_{g_{c(t)}} \right) = 0$.

Proposition 22.21 *If $g(\cdot, \cdot)$ is parallel transported relative to ∇ , then $g(\cdot, \cdot)$ is a metric of Killing relative to ∇ .*

Proof. $\frac{Dg}{dt} = 0 = \nabla_{\vec{v}}g$ along any curves tells that \vec{v} is a vector of Killing for all $\vec{v} \in \Gamma(S)$, thus $g(\cdot, \cdot)$ is a metric of Killing. \blacksquare

Theorem 22.22 (Levi–Civita, and definition.) *Let $(S, g(\cdot, \cdot))$ be a Riemannian manifold. Then there exists a unique connection ∇ in S such that:*

- 1- ∇ is torsion free, cf. (11.11), and
- 2- ∇ parallel transport $g(\cdot, \cdot)$, cf. (22.31).

This connection is called the “metric connection” or the “Levi–Civita connection”.

(In other words: there exists a unique torsion free connection ∇ in S such that the shifter preserves the Riemannian metric in S .)

Using a coordinate system, the connection is given by, for all i, j, k ,

$$\gamma_{jk}^i = \frac{1}{2} \sum_{\ell} g^{i\ell} \left(\frac{\partial g_{j\ell}}{\partial q^k} - \frac{\partial g_{kj}}{\partial q^\ell} + \frac{\partial g_{\ell k}}{\partial q^j} \right), \quad (22.34)$$

where $[g^{ij}] := [g_{ij}]^{-1}$.

Proof. Apply the previous proposition and theorem 17.8. \blacksquare

(If S is a surface in \mathbb{R}^n , and $g(\cdot, \cdot)$ a usual Euclidean metric, then the Levi-Civita connection is the usual Riemannian connection.)

Part VII

Normal and second fundamental form

23 Metric and volume in \mathbb{R}^n

23.1 Expression in a coordinate system basis in \mathbb{R}^n

Let (\vec{E}_i) be a Euclidean basis in \mathbb{R}^n , (dx^i) the dual basis, and $g(\cdot, \cdot)$ the associated Euclidean dot product, that is,

$$g(\cdot, \cdot) = \sum_i dx^i \otimes dx^i \stackrel{\text{written}}{=} (\cdot, \cdot)_{\mathbb{R}^n}, \quad [g(\cdot, \cdot)]_{|\vec{E}} = I. \quad (23.1)$$

$\Phi_+ : \vec{q} \in \mathbb{R}^n \rightarrow p = \Phi(\vec{q}) \in \mathbb{R}^n$ is a coordinate system in \mathbb{R}^n , cf. (2.1), $(\vec{e}_i(p) = d\Phi(\vec{q}) \cdot \vec{A}_i)_{i=1, \dots, n}$ is the basis of the system at $p = \Phi_+(\vec{q})$, $(e^i(p) = dq^i(p))_{i=1, \dots, n}$ is the dual basis, and

$$g(p) = \sum_{i,j=1}^n g_{ij}(p) dq^i(p) \otimes dq^j(p), \quad g_{ij}(p) = (\vec{e}_i(p), \vec{e}_j(p))_{\mathbb{R}^n}, \quad [g(p)]_{|\vec{e}} = [g_{ij}(p)]. \quad (23.2)$$

Proposition 23.1 *If $J(\vec{q}) = \det(d\Phi_+(\vec{q})) > 0$, i.e. $\det(\vec{e}_1(p), \dots, \vec{e}_n(p)) > 0$ at $p = \Phi_+(\vec{q})$ (direct basis), then*

$$J(\vec{q}) = \sqrt{\det([g(p)]_{|\vec{e}})}, \quad \text{and} \quad \det_{|\vec{E}}(p) = \sqrt{\det([g(p)]_{|\vec{e}})} dq^1(p) \wedge \dots \wedge dq^n(p). \quad (23.3)$$

So,

$$\vec{w}_j(p) = \sum_i w_j^i(p) \vec{e}_i(p) \Rightarrow \det_{|\vec{E}}(p)(\vec{w}_1(p), \dots, \vec{w}_n(p)) = \sqrt{\det([g(p)]_{|\vec{e}})} \det[w_j^i(p)]. \quad (23.4)$$

Proof. $d\Phi_+(\vec{q}) = \sum_{i,j=1}^n \frac{\partial \Phi_{\pm}^i}{\partial q^j}(\vec{q}) \vec{E}_i \otimes A^j$ gives $\det_{\vec{A}, \vec{E}}(d\Phi_+(\vec{q})) = \det([\frac{\partial \Phi_{\pm}^i}{\partial q^j}(\vec{q})])$, and $\det([\frac{\partial \Phi_{\pm}^i}{\partial q^j}(\vec{q})]^2) = \det([\frac{\partial \Phi_{\pm}^i}{\partial q^j}(\vec{q})]^T) \det([\frac{\partial \Phi_{\pm}^i}{\partial q^j}(\vec{q})]) = \det([\frac{\partial \Phi_{\pm}^i}{\partial q^j}(\vec{q})]^T \cdot [\frac{\partial \Phi_{\pm}^i}{\partial q^j}(\vec{q})]) = \det([\vec{e}_i(p)]_{|\vec{E}}^T \cdot [\vec{e}_j(p)]_{|\vec{E}}) = \det([\vec{e}_i(p), \vec{e}_j(p)]_g) = \det([g(p)]_{|\vec{E}})$. Thus (23.3) and (23.4). \blacksquare

Example 23.2 Polar coordinates, $r = r(p) = \sqrt{x^2 + y^2}$; $\det_{\vec{E}}(p)(\vec{e}_1(p), \vec{e}_2(p)) = \sqrt{\det[g_p]_{|\vec{E}}} = r$. And $[g(p)]_{|\vec{E}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$, $J_{\Phi}(\vec{q}) = \sqrt{\det[g(p)]_{|\vec{E}}} = r$, $\det_{\vec{E}}(p) = r dr(p) \wedge d\theta(p)$ (algebraic volume at p in the polar system), and $|\det_{\vec{E}}(p)| = \text{written } r dr d\theta$ (positive volume at p). \blacksquare

23.2 Volume in a surface

Let $\Phi : \vec{q} \in \mathbb{R}^m \rightarrow p = \Phi(\vec{q}) \in S \subset \mathbb{R}^n$ a coordinate system in S , cf. (1.1), $(\vec{e}_i(p))$ the system basis at p , cf. (1.10), and $(dq^i(p))$ the dual basis at p , cf. (1.20). Consider a Euclidean metric in S at $p \in S$ written as

$$g(p)(\cdot, \cdot) = \sum_{i,j=1}^m g_{ij}(p) dq^i(p) \otimes dq^j(p), \quad g_{ij} = g(\vec{e}_i, \vec{e}_j), \quad [g(p)]_{|\vec{E}} = [g_{ij}(p)]_{\substack{i=1, \dots, m \\ j=1, \dots, m}}. \quad (23.5)$$

Definition 23.3 The algebraic volume in S (the algebraic area element) at p is

$$\det_{\vec{E}}(p) = \sqrt{\det([g(p)]_{|\vec{E}})} dq^1(p) \wedge \dots \wedge dq^m(p). \quad (23.6)$$

The algebraic volume in $T_p S$ limited by m vectors $\vec{w}_1(p), \dots, \vec{w}_m(p) \in T_p S$ is

$$\det_{\vec{E}}(p)(\vec{w}_1(p), \dots, \vec{w}_m(p)) = \sqrt{\det([g(p)]_{|\vec{E}})} \det([w_j^i(p)]), \quad \text{when } \vec{w}_j = \sum_{i=1}^m w_j^i(p) \vec{e}_i(p), \quad (23.7)$$

and the non negative positive volume is $|\det_{\vec{E}}(p)(\vec{w}_1(p), \dots, \vec{w}_m(p))|$.

23.3 Unit normal form

Let $m = n-1$, so S is a regular hyper-surface in \mathbb{R}^n . Let $\vec{n}(p) = \text{written } \vec{e}_n(p)$ be one (of the two) unit normal vector at S at p , so

$$\forall i = 1, \dots, n-1, (\vec{e}_i(p), \vec{n}(p))_{\mathbb{R}^n} = 0, \quad \text{and} \quad \|\vec{n}(p)\|_{\mathbb{R}^n} = 1. \quad (23.8)$$

Thus $(\vec{e}_i(p))_{i=1, \dots, n}$ is a basis at p in \mathbb{R}^n .

Let $(e^i(p))_{i=1, \dots, n}$ be the dual basis, and let

$$e^n(p) \stackrel{\text{written}}{=} n^b(p). \quad (23.9)$$

Definition 23.4 The linear form $n^b(p)$ is called the unit normal form at S at p .

With $\vec{w}(p) = \vec{w}_{\parallel}(p) + \vec{w}_{\perp}(p) \in T_p S \oplus^{\perp} \text{Vect}\{\vec{n}(p)\}$, we get, for all $\vec{w} \in \Gamma(S)$,

$$n^b \cdot \vec{w} = (\vec{n}, \vec{w})_g, \quad (23.10)$$

that is, \vec{n} is the $(\cdot, \cdot)_g$ -Riesz representation vector of n^b . And $n^b(p) \cdot \vec{w}_p$ is the normal component at S at p of \vec{w}_p .

Example 23.5 Polar coordinate:

- $p = \Phi_R(\theta) = \begin{pmatrix} R \cos \theta \\ R \sin \theta \end{pmatrix}$, then $\vec{n}(p) = \pm \vec{e}_1(p)$ and $n^b(p) = \pm dr(p)$.
- $p = \Phi_{\theta_0}(r) = \begin{pmatrix} r \cos \theta_0 \\ r \sin \theta_0 \end{pmatrix}$, then $\vec{n}(p) = \pm \frac{\vec{e}_2(p)}{r}$ and $n^b(p) = \pm r d\theta(p)$. \blacksquare

Example 23.6 Spherical coordinates, $p = \Phi_R(\theta, \varphi) = \begin{pmatrix} R \cos \theta \cos \varphi \\ R \sin \theta \cos \varphi \\ R \sin \varphi \end{pmatrix}$, tangent vectors $\vec{f}_1(p) = \frac{\partial \Phi_R}{\partial \theta}(\theta, \varphi) = \vec{e}_2(p)$ and $\vec{f}_2(p) = \frac{\partial \Phi_R}{\partial \varphi}(\theta, \varphi) = \vec{e}_3(p)$, thus

$$\vec{n}(p) = \pm \frac{\vec{f}_1 \wedge \vec{f}_2}{\|\vec{f}_1 \wedge \vec{f}_2\|}(p) = \pm \vec{e}_1(p) \quad \text{and} \quad n^b(p) = \pm dr(p). \quad (23.11)$$

And the endomorphism $\vec{n}_p \otimes n_p^b : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the projection on $T_p S$:

$$(\vec{n}_p \otimes n_p^b) \cdot \vec{v} = \vec{v}_\perp = v_\perp \vec{n}_p, \quad (23.12)$$

when $\vec{v} = \vec{v}_\parallel + \vec{v}_\perp = \vec{v}_\parallel + v_\perp \vec{n}_p \in T_p S \oplus^\perp \text{Vect}\{\vec{n}_p\}$. Indeed, $(\vec{n}_p \otimes n_p^b) \cdot \vec{v} = \vec{n}_p (n_p^b \cdot \vec{v}) = \vec{v}_\perp$. \blacksquare

23.4 * Notation * ℓ de Hodge

23.4.1 Definition

The Hodge operator could be defined on vectors: With $V = \mathbb{R}^n$, It would be the operator $*$: $V^{n-1} \rightarrow V$ defined by, for all orthonormal direct basis $(\vec{v}_1, \dots, \vec{v}_n)$,

$$*(\vec{v}_1, \dots, \vec{v}_{n-1}) = \vec{v}_n.$$

However, to be generalized (in particular for Maxwell equations to exchange the roles of \vec{E} and \vec{B}), let Ω^k be the set of k -alternate multilinear forms in \mathbb{R}^n , $1 \leq k \leq n-1$. Since $\dim \Omega^k = \dim \Omega^{n-k} = \binom{n}{k}$ there is an isomorphism between Ω^k and Ω^{n-k} .

Definition 23.7 The Hodge operator $*$: $\Omega^{n-k} \rightarrow \Omega^k$ is defined by, for all orthonormal direct basis $(\vec{v}_1, \dots, \vec{v}_n)$,

$$(*\ell)(\vec{v}_1, \dots, \vec{v}_{n-k}) = \ell(\vec{v}_{n-k+1}, \dots, \vec{v}_n). \quad (23.13)$$

E.g., $*$: $\Omega^1 \rightarrow \Omega^{n-1}$ is defined by, for all linear form ℓ by, for all orthonormal direct basis $(\vec{v}_1, \dots, \vec{v}_n)$,

$$(*\ell)(\vec{v}_1, \dots, \vec{v}_{n-1}) = \ell(\vec{v}_n), \quad (23.14)$$

that is, $(*\ell)(\vec{v}_1, \dots, \vec{v}_{n-1}) \cdot \vec{w} = \ell(\vec{v}_n) \cdot \vec{w}$ for all $\vec{w} \in \Gamma(S)$.

23.4.2 Example: Faraday electromagnetic field

Consider a particle with mass m , electrical charge e , velocity \vec{v} , and momentum $\vec{p} = m\vec{v}$. Let \vec{E} and \vec{B} be the ambient electric and magnetic fields. The energy $\mathcal{E} = mc^2 + \frac{1}{2}mv^2$ and the electromagnetic force $e(\vec{E} + \vec{v} \wedge \vec{B})$ satisfy

$$\frac{d\mathcal{E}}{dt} = e \vec{E} \cdot \vec{v} \quad \text{and} \quad \frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v} \wedge \vec{B}). \quad (23.15)$$

Choose a basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3) \in \mathbb{R}^3$. With $\vec{B} = B_x \vec{e}_1 + B_y \vec{e}_2 + B_z \vec{e}_3$ and $\vec{E} = E_x \vec{e}_1 + E_y \vec{e}_2 + E_z \vec{e}_3$, write $\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$ and $\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$, and $R_{\vec{B}} = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}$ and $R_{\vec{E}} = \begin{pmatrix} 0 & E_z & -E_y \\ -E_z & 0 & E_x \\ E_y & -E_x & 0 \end{pmatrix}$

Thus:

$$\vec{v} \wedge \vec{B} = R_{\vec{B}} \cdot \vec{v}, \quad \text{and} \quad \frac{d\vec{p}}{dt} = e(\vec{E} + \vec{v} \wedge \vec{B}) = e\vec{E} + eR_{\vec{B}} \cdot \vec{v}. \quad (23.16)$$

($R_{\vec{B}}$ is the antisymmetric endomorphism characterizing the magnetic field, and \vec{B} is the associated field, relative to the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, characterized by $R_{\vec{B}} \cdot \vec{w} = -\vec{B} \wedge \vec{w}$ for all \vec{w} .) Then consider the position, velocity and momentum quadri-vectors

$$\vec{x} := \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \text{ written } \begin{pmatrix} t \\ \vec{x} \end{pmatrix}, \quad \vec{v} := \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix} \text{ written } \begin{pmatrix} 1 \\ \vec{v} \end{pmatrix}, \quad \vec{p} := \begin{pmatrix} \mathcal{E} \\ p_x \\ p_y \\ p_z \end{pmatrix} \text{ written } \begin{pmatrix} \mathcal{E} \\ \vec{p} \end{pmatrix}, \quad (23.17)$$

Then the Faraday tensor F is defined by its matrix relative to the extended basis $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3) \in \mathbb{R}^4$ (with $\vec{e}_0 = (1, 0, 0, 0)$, $\vec{e}_1 = (0, \vec{e}_1, \dots)$)

$$[F] = [F_j^i] = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & [\vec{E}]^T \\ [\vec{E}] & R_{\vec{B}} \end{pmatrix}. \quad (23.18)$$

So (23.15) reads

$$\frac{d\vec{p}}{dt} = e[F] \cdot \vec{v}. \quad (23.19)$$

Then consider the Minkowski pseudo-metric given by $\eta = \sum_{i,j=0}^4 \eta_{ij} e^i \otimes e^j = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3$, that is,

$$[\eta] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (23.20)$$

and the associated Faraday tensor $F^\flat \in \mathcal{A}_2(\mathbb{R}^4)$

$$F^\flat = \eta \cdot F, \quad [F^\flat] = [F_{ij}] = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -[\vec{E}]^T \\ [\vec{E}] & R_{\vec{B}} \end{pmatrix}. \quad (23.21)$$

(Given by the contraction $\sum_{i,j=1}^4 F_{ij} e^i \otimes e^j = (\sum_{i,k=1}^4 \eta_{ik} e^i \otimes e^k)(\sum_{k,j=1}^4 F_j^k \vec{e}_k \otimes e^j)$.) That is,

$$F^\flat = -E_x dt \wedge dx - E_y dt \wedge dy - E_z dt \wedge dz + B_z dx \wedge dy + B_y dz \wedge dx + B_x dy \wedge dz. \quad (23.22)$$

Thus the Hodge operator gives

$$(*F)(\vec{e}_0, \vec{e}_1) = F^\flat(\vec{e}_2, \vec{e}_3), \quad (*F)(\vec{e}_0, \vec{e}_2) = -F^\flat(\vec{e}_1, \vec{e}_3),$$

the second equality because of the opposite orientation of $(\vec{e}_0, \vec{e}_2, \vec{e}_1, \vec{e}_3)$ relative to $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$, and

$$(*F)(\vec{e}_1, \vec{e}_2) = F^\flat(\vec{e}_0, \vec{e}_3), \quad (*F)(\vec{e}_1, \vec{e}_3) = -F^\flat(\vec{e}_0, \vec{e}_2), \quad (*F)(\vec{e}_2, \vec{e}_3) = F^\flat(\vec{e}_0, \vec{e}_1).$$

Thus,

$$[*F] = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & [\vec{B}]^T \\ -[\vec{B}] & -R_{\vec{E}} \end{pmatrix}. \quad (23.23)$$

The Hodge operator $*$ has switched the roles of \vec{E} and \vec{B} (and the sign), cf. (23.21).

We will also need the action of the exterior differential: With (23.22) we get

$$\begin{aligned} d_{\text{ext}} F^\flat &= \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right) dx \wedge dy \wedge dz \\ &+ \left(\frac{\partial B_x}{\partial t} + \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) dt \wedge dy \wedge dz \\ &+ \left(\frac{\partial B_y}{\partial t} + \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) dt \wedge dz \wedge dx \\ &+ \left(\frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) dt \wedge dx \wedge dy, \end{aligned} \quad (23.24)$$

and

$$\begin{aligned} d_{\text{ext}} (*F) &= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) dx \wedge dy \wedge dz \\ &+ \left(\frac{\partial E_x}{\partial t} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} \right) dt \wedge dy \wedge dz \\ &+ \left(\frac{\partial E_y}{\partial t} - \frac{\partial B_x}{\partial z} + \frac{\partial B_z}{\partial x} \right) dt \wedge dz \wedge dx \\ &+ \left(\frac{\partial E_z}{\partial t} - \frac{\partial B_y}{\partial x} + \frac{\partial B_x}{\partial y} \right) dt \wedge dx \wedge dy. \end{aligned} \quad (23.25)$$

With $F^\sharp = \eta^{-1} \cdot F$ and $[F^\sharp] = [F^{ij}]$, since $[\eta]^{-1} = [\eta]$, we get :

$$F^\sharp = -E_x d\vec{e}_0 \wedge d\vec{e}_1 - E_y d\vec{e}_0 \wedge d\vec{e}_2 - E_z d\vec{e}_0 \wedge d\vec{e}_3 + B_z d\vec{e}_1 \wedge d\vec{e}_2 + B_y d\vec{e}_3 \wedge d\vec{e}_1 + B_x d\vec{e}_2 \wedge d\vec{e}_3. \quad (23.26)$$

And the divergence $\nabla \cdot F^\sharp$ is (derivation line by line)

$$\begin{aligned} \nabla \cdot F^\sharp &= \sum_{ij} F^{ij}{}_{,j} \vec{e}_i \\ &= -\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \vec{e}_0 \\ &\quad + \left(\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) \vec{e}_1 + \left(\frac{\partial E_y}{\partial t} - \frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z}\right) \vec{e}_2 + \left(\frac{\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) \vec{e}_3 \end{aligned} \quad (23.27)$$

23.4.3 Maxwell equations

Maxwell equations:

$$\left\{ \begin{array}{l} \operatorname{div} \vec{B} = 0, \\ \frac{\partial \vec{B}}{\partial t} + \nabla \wedge \vec{E} = \vec{0} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \operatorname{div} \vec{E} = 4\pi\rho, \\ \frac{\partial \vec{E}}{\partial t} - \nabla \wedge \vec{B} = -4\pi\vec{J}. \end{array} \right. \quad (23.28)$$

that is,

$$d_{\text{ext}}(*F) = 0, \quad \text{i.e.} \quad \nabla \cdot F^\sharp = 4\pi\vec{J}, \quad (23.29)$$

where $\vec{J} = (\rho, \vec{J}) \in \mathbb{R}^4$. And

$$d_{\text{ext}}(*F) = 4\pi(*\vec{J}), \quad (23.30)$$

where $*\vec{J} \in \mathcal{A}_3(\mathbb{R}^4)$ is given by,

$$*\vec{J} = J^0 dx \wedge dy \wedge dz - J^1 dy \wedge dz \wedge dt + J^2 dz \wedge dt \wedge dx - J^3 dt \wedge dx \wedge dy \quad (23.31)$$

24 Second fundamental form: The curvature tensor $k \in T_2^0(S)$

24.1 Curvature

24.1.1 Positive curvature of a curve in $S \subset \mathbb{R}^n$

Let $c : s \in [0, L] \rightarrow p = c(s) \in \mathbb{R}^n$ be a regular curve in \mathbb{R}^n parametrized with an intrinsic parameter s , that is, such that $\|\vec{c}'(s)\| = 1$ for all s . With $p = c(s)$, let $\vec{v}(p) = \vec{c}'(s)$. The non negative curvature of c at $p = c(s)$ is

$$k(p) := \|\vec{c}''(s)\|. \quad (24.1)$$

And $\vec{v}(\vec{c}(s)) = \vec{c}'(s)$ gives

$$d\vec{v}(p) \cdot \vec{v}(p) = \vec{c}''(s) \quad \text{when} \quad p = \vec{c}(s). \quad (24.2)$$

So,

$$k(p) = \|\vec{c}''(s)\| = \|d\vec{v}(p) \cdot \vec{v}(p)\|. \quad (24.3)$$

24.1.2 Algebraic curvature of a curve in $S \subset \mathbb{R}^n$

Let S be a regular hyper-surface in \mathbb{R}^n . Let $c : s \in [0, L] \rightarrow p = c(s) \in S$ be a regular curve in S parametrized with an intrinsic parameter s , that is, such that $\|\vec{c}'(s)\| = 1$ for all s . With $p = c(s)$, let $\vec{v}(p) = \vec{c}'(s) = \vec{v}_p$.

Recall that if c is a geodesic with an intrinsic parametrization, then, for all $p \in \operatorname{Im}c$, $d\vec{v}(p) \cdot \vec{v}(p) \perp T_p S$, cf. (20.1), that is $d\vec{v}(p) \cdot \vec{v}(p) \parallel \vec{n}(p)$, so

$$\exists \kappa(p) \in \mathbb{R}, \quad d\vec{v}(p) \cdot \vec{v}(p) = -\kappa(p) \vec{n}(p). \quad (24.4)$$

Definition 24.1 The orientation of S being given by the sign of \vec{n} , the algebraic curvature of c at $p \in \text{Im}c$ is the scalar $\kappa(p) = \overset{\text{written}}{=} k(p)(\vec{v}(p), \vec{v}(p)) \in \mathbb{R}$ (depends on the curve) defined by (24.4), that is, along $\text{Im}c$,

$$\kappa = k(\vec{v}, \vec{v}) = -(d\vec{v}.\vec{v}, \vec{n})_g. \quad (24.5)$$

(And $|k(\vec{v}, \vec{v})| = \|\vec{c}''\|$ is the non negative curvature along $\text{Im}c$, cf. (24.1).) In other words, along $\text{Im}c$,

$$k(\vec{v}, \vec{v}) := -n^b.(d\vec{v}.\vec{v}) \quad (= -(d\vec{v}.\vec{v}, \vec{n})_g). \quad (24.6)$$

(Recall that $\|\vec{v}\| = 1$ here.)

Interpretation: We will see that $k = dn^b$, cf. (24.10): so, if n^b varies slowly along c , then the curvature is small (and the radius of curvature is large).

Corollary 24.2 The geodesic equation then reads in \mathbb{R}^n (with intrinsic coordinates),

$$\frac{d\vec{v}}{ds} = -k(\vec{v}, \vec{v})\vec{n}. \quad (24.7)$$

Proof. In \mathbb{R}^n , with $p = \vec{c}(s)$ we have $\frac{d\vec{v}}{ds}(p) := \frac{d(\vec{v} \circ \vec{c})}{ds}(s) = d\vec{v}(p).\vec{v}(p)$, and (24.5) gives (24.7). \blacksquare

24.1.3 Curvature tensor for $S \subset \mathbb{R}^n$

Definition 24.3 The curvature tensor $k \in T_2^0(S)$ is defined (in S) by, for all $\vec{v}, \vec{w} \in \Gamma(S)$:

$$k(\vec{v}, \vec{w}) = -n^b.(d\vec{w}.\vec{v}) \quad (= -(\vec{n}, d\vec{w}.\vec{v})_g). \quad (24.8)$$

That is, $-k(\vec{w}, \vec{v})(p)$ gives the normal component of $d\vec{w}(p).\vec{v}(p)$ at S at p .

Proposition 24.4 k is a symmetric tensor: For all $\vec{v}, \vec{w} \in \Gamma(S)$,

$$k(\vec{v}, \vec{w}) = k(\vec{w}, \vec{v}), \quad \text{i.e.} \quad n^b.(d\vec{w}.\vec{v}) = n^b.(d\vec{v}.\vec{w}). \quad (24.9)$$

Proof. $[\vec{v}, \vec{w}] = d\vec{w}.\vec{v} - d\vec{v}.\vec{w} \in \Gamma(S)$, thus $n^b.[\vec{v}, \vec{w}] = 0 = n^b.(d\vec{w}.\vec{v}) - n^b.(d\vec{v}.\vec{w})$. \blacksquare

24.2 $k = dn^b \in T_2^0(S)$

Proposition 24.5

$$k = dn^b, \quad \text{i.e.} \quad k(\vec{v}, \vec{w}) = dn^b(\vec{v}, \vec{w}), \quad \forall \vec{v}, \vec{w} \in \Gamma(S). \quad (24.10)$$

and k is a $\binom{0}{2}$ tensor in S (measures the variations of n^b on S). In particular, dn^b is symmetric.

Proof. For all $\vec{w} \in \Gamma(S)$, $n^b.\vec{w} = 0$, thus $d(n^b.\vec{w}) = 0$, thus, for all $\vec{v} \in \Gamma(S)$,

$$(dn^b.\vec{v}).\vec{w} + n^b.(d\vec{w}.\vec{v}) = 0 = (dn^b.\vec{v}).\vec{w} - k(\vec{v}, \vec{w}),$$

thus $k(\vec{v}, \vec{w}) = (dn^b.\vec{v}).\vec{w}$. And (24.9) gives $(dn^b.\vec{v}).\vec{w} = (dn^b.\vec{w}).\vec{v}$, thus $(dn^b.\vec{v}).\vec{w} = \overset{\text{written}}{=} dn^b(\vec{v}, \vec{w})$ with dn^b symmetric, thus (24.10). Hence $k = dn^b$ is $\mathcal{F}(S)$ -multilinear: it is a tensor. \blacksquare

24.3 Components $k_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^n}$

Let Φ be a coordinate system in S , $(\vec{e}_i)_{i=1, \dots, n-1}$ the coordinate basis, and

$$k = \sum_{i,j=1}^{n-1} k_{ij} e^i \otimes e^j. \quad (24.11)$$

So $k_{ij} = k(\vec{e}_i, \vec{e}_j)$ and $[k]_{|\vec{e}} = [k_{ij}]$ the $(n-1) * (n-1)$ matrix of k relative to (\vec{e}_i) .

Consider $\vec{n}(p) = \overset{\text{written}}{=} \vec{e}_n(p)$ and $(\vec{e}_i(p))_{i=1, \dots, n}$. The Euclidean metric reads

$$g = \sum_{i,j=1}^n g_{ij} e^i \otimes e^j, \quad g_{ij}(p) = (\vec{e}_i(p), \vec{e}_j(p))_g, \quad \forall p \in S. \quad (24.12)$$

(In particular $g_{in} = 0 = g_{ni}$ for all $i = 1, \dots, n-1$. And the Christoffel symbols are given by $d\vec{e}_j.\vec{e}_i = \sum_{k=1}^n \gamma_{ij}^k \vec{e}_k$.)

Corollary 24.6 For all $i, j = 1, \dots, n-1$,

$$k_{ij} = -\gamma_{ij}^n, \quad \text{and} \quad k_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial q^n} \quad (= \gamma_{ij}^n), \quad (24.13)$$

so, the curvature tensor is given thanks to the Euclidean metric.

Proof. (24.10) gives $dn^b(\vec{e}_j, \vec{e}_i) = -n^b(d\vec{e}_i \cdot \vec{e}_j) = -\gamma_{ij}^n$. And (17.15) gives $\gamma_{jk}^n = \frac{1}{2} \sum_{\ell} g^{n\ell} (\frac{\partial g_{\ell j}}{\partial q^k} + \frac{\partial g_{\ell k}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell}) = \frac{1}{2} (\frac{\partial g_{nj}}{\partial q^k} + \frac{\partial g_{nk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^n}) = \frac{1}{2} (0 + 0 - \frac{\partial g_{jk}}{\partial q^n})$, thus (24.13). \blacksquare

24.4 Example: Sphere

Example 24.7 Let $S = S(\vec{0}, R)$ in \mathbb{R}^3 . GPS coordinates $p = \Phi(r, \theta, \varphi) = \begin{pmatrix} x = r \cos \theta \cos \varphi \\ y = r \sin \theta \cos \varphi \\ z = r \sin \varphi \end{pmatrix}$, cf. (3.27). Basis $\vec{e}_j(p) = \frac{\partial \Phi}{\partial q^j}(r, \theta, \varphi)$ at $p = \Phi(r, \theta, \varphi)$ in \mathbb{R}^3 . Basis $(\vec{f}_1(p), \vec{f}_2(p), \vec{f}_3(p)) = (\vec{e}_2(p), \vec{e}_3(p), \vec{e}_1(p))$ at $p = \Phi(r, \theta, \varphi)$. Thus

$$[g(p)]_{|\vec{f}} = [g_{ij}(p)] = \begin{pmatrix} r^2 \cos^2 \varphi & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (24.14)$$

Choose $\vec{n}(p) = +\vec{e}_1(p) = \vec{f}_3(p)$ at $p \in S$, thus $n^b(p) = dr(p)$. Thus $k_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial q^n} = \frac{1}{2} \frac{\partial g_{ij}}{\partial r} = -\gamma_{ij}^n$ for $j = 1, 2$ and with the basis $(\vec{f}_1(p), \vec{f}_2(p))$ in $T_p S$:

$$[k(p)]_{|\vec{f}} = \begin{pmatrix} R \cos^2 \varphi & 0 \\ 0 & R \end{pmatrix} = [dn^b(p)]_{|\vec{f}}, \quad (24.15)$$

i.e., $k(p) = R \cos^2 \varphi d\theta(p) \otimes d\theta(p) + R d\varphi(p) \otimes d\varphi(p) = dn^b(p)$ (or see (3.42)-(3.43)).

E.g., the curvature of the sphere along a meridian (a geodesic in S) is

$$k\left(\frac{\vec{f}_2}{\|\vec{f}_2\|}, \frac{\vec{f}_2}{\|\vec{f}_2\|}\right) = k\left(\frac{\vec{e}_3}{\|\vec{e}_3\|}, \frac{\vec{e}_3}{\|\vec{e}_3\|}\right) = R \frac{1}{R^2} = \frac{1}{R},$$

thus the radius of curvature is $-R$, and the center of the sphere is $p - R \cdot \vec{n}(p) = \vec{0}$. E.g., the curvature of the sphere along a parallel is

$$k\left(\frac{\vec{f}_1}{\|\vec{f}_1\|}, \frac{\vec{f}_1}{\|\vec{f}_1\|}\right) = k\left(\frac{\vec{e}_2}{\|\vec{e}_2\|}, \frac{\vec{e}_2}{\|\vec{e}_2\|}\right) = R \cos^2 \varphi \frac{1}{R^2 \cos^2 \varphi} = \frac{1}{R},$$

which is the curvature of the sphere.

NB: This is not the curvature of a parallel: This is the curvature of the sphere at a point p which belong to many curves, in particular belongs to a geodesic tangent to $\vec{f}_1(p) = \vec{e}_2(p)$ at p (the only information used is the vector $\vec{f}_1(p)$ which is tangent to many curves at p).

Reminder: At a point p of the parallele is the curve $c(\theta) = \Phi(R, \theta, \varphi_0)$ we have $\vec{c}'(\theta) = R \cos \varphi \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}$ and $\vec{c}''(\theta) = -R \cos \varphi \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}$. And the osculating plane is the "horizontal affine plane" ($p, \text{Vect}\{\vec{c}'(\theta), \vec{c}''(\theta)\}$) (contains the parallel, not $\vec{n}(p)$ the normal vector to the sphere); And the curvature of the parallel is $\|\vec{c}''(s)\|$ where $\tilde{c}(s)$ is the parallel with an intrinsic parameter, that is $\tilde{c}(s) = \begin{pmatrix} x = R \cos \frac{s}{R \cos \varphi_0} \cos \varphi_0 \\ y = R \sin \frac{s}{R \cos \varphi_0} \cos \varphi_0 \\ z = R \sin \varphi_0 \end{pmatrix}$; So $\tilde{c}'(s) = \begin{pmatrix} -\sin \frac{s}{R \cos \varphi_0} \\ \cos \frac{s}{R \cos \varphi_0} \\ 0 \end{pmatrix}$, and $\tilde{c}''(s) = \begin{pmatrix} x = -\frac{1}{R \cos \varphi_0} \cos \frac{s}{R \cos \varphi_0} \\ y = -\frac{1}{R \cos \varphi_0} \sin \frac{s}{R \cos \varphi_0} \\ z = 0 \end{pmatrix}$, and $\|\tilde{c}''(s)\| = \frac{1}{R \cos \varphi_0}$, so the radius of curvature of the parallel is $R \cos \varphi_0$. \blacksquare

24.5 The associated curvature tensor $K \in T_1^1(S)$

With the Euclidean dot product $(\cdot, \cdot)_g$, the Riesz representation theorem and $k \in T_2^0(S)$, let $K \in T_1^1(S)$ be defined by, for all $\vec{v}, \vec{w} \in TS$,

$$(K.\vec{v}, \vec{w})_g = k(\vec{v}, \vec{w}). \quad (24.16)$$

that is,

$$g.K = k, \quad \text{i.e.} \quad K^\flat := k, \quad (24.17)$$

(In a basis: $k = \sum_{i,j=1}^n k_{ij} e^i \otimes e^j$, $g = \sum_{i,j=1}^n g_{ij} e^i \otimes e^j$, $K = \sum_{i,j=1}^n K_j^i \vec{e}_i \otimes e^j$, $g.K = \sum_{ij\ell=1}^n g_{i\ell} K_j^\ell \vec{e}_i \otimes e^j = k$, $[k] = [g].[K]$, and $[K] = [g]^{-1}.[k]$.)

k being symmetric, K is symmetric, that is $K^T = K$ (relative to $(\cdot, \cdot)_g$): For all \vec{v}, \vec{w} ,

$$(K.\vec{v}, \vec{w})_g = k(\vec{v}, \vec{w}) = k(\vec{w}, \vec{v}) = (K.\vec{w}, \vec{v})_g. \quad (24.18)$$

Proposition 24.8 For all $\vec{v}, \vec{w} \in \Gamma(S)$,

$$(d\vec{n}.\vec{v}, \vec{w})_g = (d\vec{n}^\flat.\vec{v}).\vec{w}. \quad (24.19)$$

And

$$K = d\vec{n}, \quad (24.20)$$

i.e., K gives the variations of \vec{n} in S .

Proof. $n^\flat.\vec{w} = (\vec{n}, \vec{w})_g = g(\vec{n}, \vec{w})$ gives

$$(dn^\flat.\vec{v}).\vec{w} + n^\flat.(d\vec{w}.\vec{v}) = (dg.\vec{v})(\vec{n}, \vec{w}) + g(d\vec{n}.\vec{v}, \vec{w}) + g(\vec{n}, d\vec{w}.\vec{v}).$$

And $dg = 0$ (Euclidean metric), and $n^\flat.(d\vec{w}.\vec{v}) = (\vec{n}, d\vec{w}.\vec{v})_g$, cf. (23.10), give (24.19). So, with (24.10),

$$(K.\vec{v}, \vec{w})_{\mathbb{R}^n} = k(\vec{w}, \vec{v}) = (dn^\flat.\vec{v}).\vec{w} = (d\vec{n}.\vec{v}, \vec{w})_g, \quad (24.21)$$

i.e. (24.20). ▀

Example 24.9 Continuing example 24.7. $[k(p)]_{\mathcal{F}} = \begin{pmatrix} -R \cos^2 \varphi & 0 \\ 0 & -R \end{pmatrix}$, cf. (24.15), $[g(p)]_{|\mathcal{F}} = [g_{ij}(p)] = \begin{pmatrix} R^2 \cos^2 \varphi & 0 \\ 0 & R^2 \end{pmatrix}$, thus $[g]^{-1} = \begin{pmatrix} \frac{1}{R^2 \cos^2 \varphi} & 0 \\ 0 & \frac{1}{R^2} \end{pmatrix} = [g^{ij}(p)]$, thus

$$[K_p]_{|\mathcal{F}} = \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix} = \frac{1}{R} I. \quad (24.22)$$

And the curvature of the sphere is $\frac{1}{R}$ (the radius of curvature is R). ▀

24.5.1 Principals, mean and Gaussian curvatures

Definition 24.10 Let $p \in S \subset \mathbb{R}^3$. The eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$ of the endomorphism $K(p)$ are the principal curvatures at $p \in S$, and their inverse $\frac{1}{\kappa_1}$ and $\frac{1}{\kappa_2}$ are the principal radius of curvatures.

The trace (first invariant)

$$\text{Tr}(K_p) = \kappa_1 + \kappa_2 \quad (24.23)$$

is the mean curvature. The determinant (second invariant)

$$\det(K_p) = \kappa_1 \kappa_2 \quad (24.24)$$

is the Gaussian curvature.

Remark 24.11 An alternative definition of the mean curvature is $\frac{1}{n-1} |\text{Tr}(K)|$, which for the sphere gives $\frac{1}{R}$ (expected), cf. (24.22).

The non negative Gaussian curvature can be defined as $(|\det K|)^{\frac{1}{n-1}}$, which for the sphere gives $\frac{1}{R}$ (expected), cf. (24.22). ▀

Part VIII

Riemann curvature tensor

25 Geodesic deviation

Geodesics which are “parallel” do not remain “parallel” in a non-planar surface: E.g., two nearby meridians are parallel on the equator, but meet at the poles, see figure 21.1. This loss of the parallelism gives a way to measure the curvature of a surface “from inside the surface” (Gauss).

25.1 Family of geodesics and separation vector

Let S be a surface in \mathbb{R}^n , and ∇ be a usual Riemannian connection in \mathbb{R}^n (the usual differential operator $\nabla = d$ in \mathbb{R}^n). Let $\varepsilon > 0$, $a < b$, and consider a regular geodesic family $(c_u)_{u \in [-\varepsilon, \varepsilon]}$ in S ,

$$u \in [-\varepsilon, \varepsilon], \quad c_u : \left\{ \begin{array}{l} [a, b] \rightarrow S \\ s \rightarrow p = c_u(s), \end{array} \right\}, \quad \|\vec{c}_u'(s)\| = 1, \forall s \in [a, b]. \quad (25.1)$$

Thus

$$c : \left\{ \begin{array}{l} [a, b] \times [-\varepsilon, \varepsilon] \rightarrow S \\ (s, u) \rightarrow p = c(s, u) := c_u(s), \end{array} \right. \quad (25.2)$$

is a 2-D parametrized (sub-)surface in S , supposed regular.

The transverse displacement at $s \in [a, b]$ is the curve $c_s : \left\{ \begin{array}{l} [a, b] \rightarrow S \\ u \rightarrow c_s(u) := c_u(s) = c(s, u) \end{array} \right\}$.

The velocity at $p = c_u(s)$ along c_u is

$$\vec{v}(p) = c_u'(s) = \frac{\partial c}{\partial s}(s, u) \quad (\text{and } \|\vec{v}(p)\| = 1). \quad (25.3)$$

Definition 25.1 And the transverse velocity at $p = c(s, u)$, also called the separation vector at p , is the velocity along c_s :

$$\vec{w}(p) = c_s'(u) = \frac{\partial c}{\partial u}(s, u). \quad (25.4)$$

Thus, the surface c being supposed regular,

$$\vec{f}_1(p) = \vec{v}(p) = \frac{\partial c}{\partial s}(s, u) \quad \text{and} \quad \vec{f}_2(p) = \vec{w}(p) = \frac{\partial c}{\partial u}(s, u) \quad (25.5)$$

are the basis tangent vectors to the 2-D surface c .

Example 25.2 \mathbb{R}^2 , polar curves $\vec{c}_\theta(r) = \overrightarrow{Oc_\theta(r)} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$ (straight lines); Here $\vec{c}_\theta'(r) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ is unitary, so r is an intrinsic curvilinear coordinate. So $\vec{v}(p) = \vec{c}_\theta'(r) = \vec{e}_1(p)$, and $\vec{w}(p) = \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix} = \vec{e}_2(p)$ (the geodesics separate at 90°). \blacksquare

Example 25.3 Family of meridians on the 2-D sphere in \mathbb{R}^3 : Intrinsic parametrization $p = \vec{c}_\theta(s) = \vec{c}(s, \theta) = R \begin{pmatrix} \cos \theta \cos(\frac{s}{R}) \\ \sin \theta \cos(\frac{s}{R}) \\ \sin(\frac{s}{R}) \end{pmatrix}$ (indeed, $\|\vec{c}_\theta'(s)\| = 1$ for all s), with $s \in]-R\frac{\pi}{2}, R\frac{\pi}{2}[$. So $\vec{v}(p) = \vec{c}_\theta'(s) = \begin{pmatrix} -\cos \theta \sin(\frac{s}{R}) \\ -\sin \theta \sin(\frac{s}{R}) \\ \cos(\frac{s}{R}) \end{pmatrix} = \frac{\vec{e}_3(p)}{R}$ (unit velocity along the meridian). In particular for $s = 0$ (on the equator), $\vec{v}(p) = \vec{c}_\theta'(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for all θ . And $\vec{w}(p) = \frac{\partial \vec{c}}{\partial \theta}(s, \theta) = R \cos(\frac{s}{R}) \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix} = \vec{e}_2(p)$ (the geodesics separate at 90°). \blacksquare

25.2 Separation velocity

Definition 25.4 The separation velocity at $p = c(s, u)$ is the vector

$$\frac{\partial \frac{\partial \vec{c}}{\partial u}}{\partial s}(s, u) = d\vec{w}(p) \cdot \vec{v}(p) \stackrel{\text{written}}{=} \nabla_{\vec{v}} \vec{w}(p) = \frac{D\vec{w}}{ds}(p) \quad (= \frac{\partial^2 \vec{c}}{\partial s \partial u}). \quad (25.6)$$

(Measures the variations of \vec{w} along a geodesic $c_u : s \rightarrow c_u(s)$.)

c being regular, we also have

$$d\vec{w} \cdot \vec{v} = \frac{\partial \frac{\partial \vec{c}}{\partial u}}{\partial s}(s, u) = \frac{\partial \frac{\partial \vec{c}}{\partial s}}{\partial u}(s, u) = d\vec{v} \cdot \vec{w}. \quad (25.7)$$

(Also obtained with (25.5): basis of a coordinate system.)

Example 25.5 Continuing example 25.2: $d\vec{e}_2 \cdot \vec{e}_1(p) = (\nabla_{\vec{v}} \vec{w})(p) = \frac{\partial \frac{\partial \vec{c}}{\partial \theta}}{\partial r}(s, \theta) = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} = \frac{\vec{e}_2}{r}$, and the separation velocity is orthogonal to the radial geodesic c_θ . \blacksquare

Example 25.6 Continuing example 25.3: $d\vec{e}_3(p) \cdot \frac{\vec{e}_2(p)}{R} = (\nabla_{\vec{v}} \vec{w})(p) = \frac{\partial \frac{\partial \vec{c}}{\partial \theta}}{\partial s}(s, \theta) = -\sin(\frac{s}{R}) \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}$, which varies along the meridian θ (and vanishes at the equator). \blacksquare

Exercise 25.7 Consider a family $(\vec{c}_u)_{u \in [-\varepsilon, \varepsilon]}$ of parallel straight lines (intrinsic coordinate) in a plane surface. Prove

$$\frac{D\vec{w}}{ds}(p_f) = 0 \quad (25.8)$$

i.e., the separation velocity vanishes (parallel straight lines remain parallel).

Answer. Let $\vec{z}_0 = \vec{c}'_u(s) = \frac{\partial \vec{c}}{\partial s}(u, s)$ the (unitary) common tangent vector to all curves, so, for all $s \in [a, b]$,

$$(\vec{c}_u(s))' = \vec{c}'(u, s) = s\vec{z}_0 + \vec{c}'(u, 0). \quad (25.9)$$

Thus $\frac{\partial \vec{c}}{\partial u}(u, s) = \frac{\partial \vec{c}}{\partial u}(u, 0)$, and $\frac{\partial \frac{\partial \vec{c}}{\partial u}}{\partial s}(u, s) = \vec{0} = \frac{\partial \frac{\partial \vec{c}}{\partial s}}{\partial u}(u, s)$, thus $\frac{D\vec{w}}{ds}(p_f) = \frac{\partial \frac{\partial \vec{c}}{\partial s}}{\partial u}(0, s) = \vec{0}$. \blacksquare

25.3 Separation acceleration (relative-acceleration)

Definition 25.8 The separation acceleration field along the fiducial geodesic $c_0 : s \rightarrow c_0(s)$ is, at $p = \vec{c}(0, s)$,

$$\frac{\partial^2 \frac{\partial \vec{c}}{\partial u}}{\partial s^2}(0, s) \stackrel{\text{written}}{=} \nabla_{\vec{v}} (\nabla_{\vec{v}} \vec{w})(p) \stackrel{\text{written}}{=} \frac{D^2 \vec{w}}{ds^2}(p) = \text{called the relative-acceleration.} \quad (25.10)$$

Example 25.9 Continuing example 25.5: $\nabla_{\vec{v}} \nabla_{\vec{v}} \vec{w}(p) = \vec{0}$ (a vector space is flat). \blacksquare

Example 25.10 Continuing example 25.6: $(\nabla_{\vec{v}} \nabla_{\vec{v}} \vec{w})(p) = \frac{\partial^2 \frac{\partial \vec{c}}{\partial \theta}}{\partial s^2}(s, \theta) = -\frac{1}{R} \cos(\frac{s}{R}) \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}$,

which varies along the meridian θ (and its norm is $\frac{1}{R}$ at the equator where the meridians are parallel). \blacksquare

26 Riemann curvature tensor

Let S be a manifold, let ∇ be connection, cf. definition 11.1, and let $\tilde{\nabla}$ be the differential operator on scalar functions $f \in \mathcal{F}(S)$, that is, $\tilde{\nabla}_{\vec{v}} f = df \cdot \vec{v} = \tilde{\mathcal{L}}_{\vec{v}}^0(f)$, cf. (5.3).

We only consider torsion-free connections ∇ (like Riemannian connections), i.e. connections ∇ related to the derivation $\tilde{\nabla}$ by,

$$\nabla_{\vec{v}} \vec{w} - \nabla_{\vec{w}} \vec{v} = [\vec{v}, \vec{w}] \quad (= \tilde{\nabla}_{\vec{v}} \vec{w} - \tilde{\nabla}_{\vec{w}} \vec{v}), \quad (26.1)$$

cf. (11.11), that is, $(\nabla_{\vec{v}} \vec{w} - \nabla_{\vec{w}} \vec{v})(f) = d(df \cdot \vec{w}) \cdot \vec{v} - d(df \cdot \vec{v}) \cdot \vec{w}$ for all $f \in \mathcal{F}(S)$.

26.1 Lie bracket $[\nabla_{\vec{v}}, \nabla_{\vec{w}}]$ on $\Gamma(S)$

Definition 26.1 Let ∇ be a connection in S . The Lie bracket $[\nabla_{\vec{v}}, \nabla_{\vec{w}}] : \Gamma(S) \rightarrow \Gamma(S)$ is defined by

$$\begin{aligned} [\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) &:= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) \\ &= (\nabla_{\vec{v}} \circ \nabla_{\vec{w}} - \nabla_{\vec{w}} \circ \nabla_{\vec{v}})(\vec{u}) \stackrel{\text{written}}{=} (\nabla_{\vec{v}}\nabla_{\vec{w}} - \nabla_{\vec{w}}\nabla_{\vec{v}})(\vec{u}). \end{aligned} \quad (26.2)$$

Example 26.2 And in S , with the Riemannian connection $\nabla_{\vec{v}}\vec{w} = \text{Proj}_{TS}(d\vec{w}.\vec{v})$, we get, for all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$,

$$\begin{aligned} [\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) \\ &= \text{Proj}_{TS}\left(d(\text{Proj}_{TS}(d\vec{u}.\vec{w}).\vec{v}) - d(\text{Proj}_{TS}(d\vec{u}.\vec{v}).\vec{w})\right). \end{aligned} \quad (26.3)$$

(So the derivations of the projection $\text{Proj}_{TS}(d\vec{u}.\vec{w})$ and $\text{Proj}_{TS}(d\vec{u}.\vec{v})$ are considered before being differentiated.) E.g., on the 2-D sphere in \mathbb{R}^3 , with $\vec{u} = \vec{w} = \vec{e}_3$ and $\vec{v} = \vec{e}_2$, and with $d\vec{e}_3(p).\vec{e}_3(p) = -r\vec{e}_1(p)$ and $d\vec{e}_2(p).\vec{e}_3(p) = -\tan\varphi\vec{e}_2(p)$, cf. (3.34)-(3.35), we get $\nabla_{\vec{e}_3}\vec{e}_3 = \text{Proj}_{TS}(d\vec{e}_3.\vec{e}_3) = \vec{0}$, and $\nabla_{\vec{e}_2}\vec{e}_3 = \text{Proj}_{TS}(d\vec{e}_3.\vec{e}_2) = -\tan\varphi\vec{e}_2 = \nabla_{\vec{e}_3}\vec{e}_2$, thus, $\nabla_{\vec{e}_2}(\nabla_{\vec{e}_3}\vec{e}_3) = \nabla_{\vec{e}_2}(\vec{0}) = \vec{0}$, and $\nabla_{\vec{e}_3}(\nabla_{\vec{e}_2}\vec{e}_3) = \text{Proj}_{TS}(d(-\tan\varphi\vec{e}_2).\vec{e}_3) = -\text{Proj}_{TS}\left(\frac{\partial(\tan\varphi\vec{e}_2)}{\partial\varphi}\right) = -\text{Proj}_{TS}((1 + \tan^2\varphi)\vec{e}_2 - \tan\varphi\vec{e}_2) = -\vec{e}_2$. Hence,

$$[\nabla_{\vec{e}_2}, \nabla_{\vec{e}_3}](\vec{e}_3) = \vec{e}_2. \quad (26.4)$$

▀

Proposition 26.3 The Lie bracket operator $B : (\vec{v}, \vec{w}) \in \Gamma(S)^2 \rightarrow B(\vec{v}, \vec{w}) = [\nabla_{\vec{v}}, \nabla_{\vec{w}}] \in \mathcal{F}(\Gamma(S); \Gamma(S))$ is \mathbb{R} -bilinear and antisymmetric. But is not $\mathcal{F}(S)$ -bilinear (not tensorial because of a df term): For all $f \in \mathcal{F}(S)$ and all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$,

$$[\nabla_{f\vec{v}}, \nabla_{\vec{w}}](\vec{u}) = f[\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) - (df.\vec{w})\nabla_{\vec{v}}\vec{u}, \quad (26.5)$$

and,

$$[\nabla_{\vec{v}}, \nabla_{\vec{w}}](f\vec{u}) = f[\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) + ([\vec{v}, \vec{w}](f))\vec{u}. \quad (26.6)$$

Proof. For all $\vec{u}, \vec{v}, \vec{w}, \vec{z} \in \Gamma(S)$, $B(\vec{v}, \vec{w})(\vec{u}) = -B(\vec{w}, \vec{v})(\vec{u})$ (trivial). For all $\vec{u}, \vec{v}, \vec{w}, \vec{z} \in \Gamma(S)$:

$$\begin{aligned} B(\vec{v} + \vec{z}, \vec{w})(\vec{u}) &= \nabla_{\vec{v}+\vec{z}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}+\vec{z}}\vec{u}), \quad \text{cf. (26.1),} \\ &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) + \nabla_{\vec{z}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u} + \nabla_{\vec{z}}\vec{u}), \\ &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) + \nabla_{\vec{z}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{z}}\vec{u}), \\ &= B(\vec{v}, \vec{w})(\vec{u}) + B(\vec{z}, \vec{w})(\vec{u}). \end{aligned}$$

For all $f \in \mathcal{F}(S)$, $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$:

$$\begin{aligned} B(f\vec{v}, \vec{w})(\vec{u}) &= \nabla_{f\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{f\vec{v}}\vec{u}), \quad \text{cf. (26.1),} \\ &= f\nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(f\nabla_{\vec{v}}\vec{u}), \quad \text{cf. (11.2),} \\ &= f\nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - (df.\vec{v})\nabla_{\vec{w}}\vec{u} - f\nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}), \\ &= fB(\vec{v}, \vec{w})(\vec{u}) - (df.\vec{v})\nabla_{\vec{w}}\vec{u}. \end{aligned} \quad (26.7)$$

Thus B is \mathbb{R} -bilinear (take $f = \text{constant}$), but not $\mathcal{F}(S)$ -bilinear since $(df.\vec{v}) \neq 0$ in general.

And for all $f \in \mathcal{F}(S)$, $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$:

$$\begin{aligned} [\nabla_{\vec{v}}, \nabla_{\vec{w}}](f\vec{u}) &= \nabla_{\vec{v}}(\nabla_{\vec{w}}(f\vec{u})) - \nabla_{\vec{w}}(\nabla_{\vec{v}}(f\vec{u})) \\ &= \nabla_{\vec{v}}((df.\vec{w})\vec{u} + f\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}((df.\vec{v})\vec{u} + f\nabla_{\vec{v}}\vec{u}) \\ &= (d(df.\vec{w}).\vec{v})\vec{u} + (df.\vec{w})\nabla_{\vec{v}}\vec{u} + (df.\vec{v})\nabla_{\vec{w}}\vec{u} + f\nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) \\ &\quad - (d(df.\vec{v}).\vec{w})\vec{u} - (df.\vec{v})\nabla_{\vec{w}}\vec{u} - (df.\vec{w})\nabla_{\vec{v}}\vec{u} - f\nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) \\ &= ([\vec{v}, \vec{w}](f))\vec{u} + f[\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) \end{aligned} \quad (26.8)$$

▀

Example 26.4 Usual Riemannian connection ∇ on a surface S in \mathbb{R}^n , that is, $\nabla_{\vec{w}}\vec{u} = \text{Proj}_{TS}(d\vec{u}.\vec{w})$:
We get

$$\begin{aligned} [\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) \\ &= \text{Proj}_{TS}\left(d(\text{Proj}_{TS}(d\vec{u}.\vec{w})).\vec{v}\right) - \text{Proj}_{TS}\left(d(\text{Proj}_{TS}(d\vec{u}.\vec{v})).\vec{w}\right). \end{aligned} \quad (26.9)$$

To compare with

$$\nabla_{[\vec{v}, \vec{w}]} \vec{u} = \text{Proj}_{TS}(d\vec{u}.[\vec{v}, \vec{w}]) = \text{Proj}_{TS}(d\vec{u}.(d\vec{w}.\vec{v}) - d\vec{u}.(d\vec{v}.\vec{w})). \quad (26.10)$$

■

Proposition 26.5

$$[\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) \neq \nabla_{[\vec{v}, \vec{w}]} \vec{u} \quad \text{in general.} \quad (26.11)$$

In particular for the usual Riemann connection in S a 2-D surface in \mathbb{R}^3 , if (\vec{e}_i) is a coordinate basis, then $[\vec{e}_i, \vec{e}_j] = 0$ but

$$[\nabla_{\vec{e}_i}, \nabla_{\vec{e}_j}] \neq \vec{0} \quad \text{in general.} \quad (26.12)$$

Proof. On the 2-D sphere in \mathbb{R}^3 , and \vec{e}_2, \vec{e}_3 the coordinate basis, (26.4) gives $[\nabla_{\vec{e}_2}, \nabla_{\vec{e}_3}](\vec{e}_3) = \vec{e}_2$, whereas $[\vec{e}_2, \vec{e}_3] = \vec{0}$ (basis of a coordinate system) and then $\nabla_{[\vec{e}_2, \vec{e}_3]}(\vec{e}_3) = 0$. ■

26.2 Gravitation forces (tidal forces)

Proposition 26.6 If ∇ is a torsion-free connection, then the separation vector \vec{w} , cf. (25.4), satisfies

$$\left(\frac{D^2\vec{w}}{ds^2} = \underbrace{\nabla_{\vec{v}}\nabla_{\vec{v}}\vec{w}}_{\text{relative-acceleration}} = - \underbrace{[\nabla_{\vec{w}}, \nabla_{\vec{v}}]\vec{v}}_{\text{gravitation force}} \right) \quad (= -\nabla_{\vec{w}}\nabla_{\vec{v}}\vec{v} - \nabla_{\vec{v}}\nabla_{\vec{w}}\vec{v}). \quad (26.13)$$

(The “separation acceleration” = “relative-acceleration”: gives a measure of gravity.)

Proof. $\nabla_{\vec{v}}\vec{w} = \nabla_{\vec{w}}\vec{v} = \frac{\partial^2 c}{\partial s \partial u} = \frac{\partial^2 c}{\partial u \partial s}$, cf. (25.4), thus $\nabla_{\vec{v}}(\nabla_{\vec{v}}\vec{w}) = \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{v})$.

And $s \rightarrow c_u(s)$ being a geodesic, $\nabla_{\vec{v}}\vec{v} = \vec{0}$. Thus $\nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{v}) = 0$. Thus

$$\nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{v}) + [\nabla_{\vec{w}}, \nabla_{\vec{v}}](\vec{v}) = 0 \quad (= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{v}) + \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{v}) - \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{v}) = \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{v})),$$

thus $\nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{v}) + [\nabla_{\vec{w}}, \nabla_{\vec{v}}](\vec{v}) = 0$. ■

Remark 26.7 The Riemann curvature tensor is defined by $R(\vec{a}, \vec{b}, \vec{c}) := [\nabla_{\vec{a}}, \nabla_{\vec{b}}]\vec{c} - \nabla_{[\vec{a}, \vec{b}]} \vec{c}$, see (26.17). In particular, here $[\vec{w}, \vec{v}] = 0$, cf. (25.5), and $R(\vec{w}, \vec{v}, \vec{v}) = [\nabla_{\vec{w}}, \nabla_{\vec{v}}]\vec{v}$ describe the gravitation field. ■

26.3 Riemann curvature on $\Gamma(S)$

If $\vec{v}, \vec{w} \in \Gamma(S)$ then the Riemann curvature operation $\rho(\vec{v}, \vec{w})$ relative to \vec{v} and \vec{w} is the map $\rho(\vec{v}, \vec{w}) : \Gamma(S) \rightarrow \Gamma(S)$ defined by

$$\begin{aligned} \rho(\vec{v}, \vec{w})(\vec{u}) &:= [\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) - \nabla_{[\vec{v}, \vec{w}]} \vec{u} \\ &= \nabla_{\vec{v}}(\nabla_{\vec{w}}\vec{u}) - \nabla_{\vec{w}}(\nabla_{\vec{v}}\vec{u}) - \nabla_{[\vec{v}, \vec{w}]} \vec{u} \stackrel{\text{written}}{=} \nabla_{\vec{v}}\nabla_{\vec{w}}\vec{u} - \nabla_{\vec{w}}\nabla_{\vec{v}}\vec{u} - \nabla_{[\vec{v}, \vec{w}]} \vec{u}. \end{aligned} \quad (26.14)$$

The Riemann curvature operator is the map $\rho : \Gamma(S) \times \Gamma(S) \rightarrow \mathcal{F}(\Gamma(S); \Gamma(S))$ defined by

$$\begin{aligned} \rho(\vec{v}, \vec{w}) &:= [\nabla_{\vec{v}}, \nabla_{\vec{w}}] - \nabla_{[\vec{v}, \vec{w}]} \\ &= \nabla_{\vec{v}}\nabla_{\vec{w}} - \nabla_{\vec{w}}\nabla_{\vec{v}} - \nabla_{[\vec{v}, \vec{w}]} \end{aligned} \quad (26.15)$$

The Riemann curvature is the map $R : (\Gamma(S))^3 \rightarrow \Gamma(S)$ defined by

$$R(\vec{u}, \vec{v}, \vec{w}) := \rho(\vec{v}, \vec{w})(\vec{u}) = (\nabla_{\vec{v}}\nabla_{\vec{w}} - \nabla_{\vec{w}}\nabla_{\vec{v}} - \nabla_{[\vec{v}, \vec{w}]})\vec{u}. \quad (26.16)$$

It is also the name of the associated multilinear map

$$\mathcal{R} : \begin{cases} \Omega^1(S) \times (\Gamma(S))^3 & \rightarrow \mathbb{R} \\ (\alpha, \vec{u}, \vec{v}, \vec{w}) & \rightarrow \alpha.R(\vec{u}, \vec{v}, \vec{w}) := \alpha.\rho(\vec{v}, \vec{w}).\vec{u} \quad (= \alpha.[\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) - \alpha.\nabla_{[\vec{v}, \vec{w}]}(\vec{u})). \end{cases} \quad (26.17)$$

Remark 26.8 NB: (26.16) is the definition found in Misner–Thorne–Wheeler [15] (to get “usual” symmetries). \blacksquare

Proposition 26.9 \mathcal{R} defined in (26.17) is a $\binom{1}{3}$ tensor: $\mathcal{R} \in T_3^1(S)$. And R defined in (26.16) is also said to be a tensor, as well as ρ defined in (26.15). And ρ is antisymmetric, that is, for all $\vec{v}, \vec{w} \in \Gamma(S)$,

$$\rho(\vec{v}, \vec{w}) = -\rho(\vec{w}, \vec{v}), \quad (26.18)$$

that is, for all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$,

$$R(\vec{u}, \vec{v}, \vec{w}) = -R(\vec{u}, \vec{w}, \vec{v}) \quad (\text{antisymmetry for the last two slots}), \quad (26.19)$$

that is, for all $\alpha \in \Omega^1(S)$ and $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$,

$$\mathcal{R}(\alpha, \vec{u}, \vec{v}, \vec{w}) = -\mathcal{R}(\alpha, \vec{u}, \vec{w}, \vec{v}) \quad (\text{antisymmetry for the last two slots}). \quad (26.20)$$

Proof. We have to prove that \mathcal{R} is $\mathcal{F}(S)$ -multilinear. $\mathcal{R}(\dots, z_1 + z_2, \dots) = \mathcal{R}(\dots, z_1, \dots) + \mathcal{R}(\dots, z_2, \dots)$ is trivial, for all z_1, z_2 in $\Omega^1(S)$ or $\Gamma(S)$ where appropriate. Then for $f \in \mathcal{F}(S)$:

$\mathcal{R}(f\alpha, \vec{u}, \vec{v}, \vec{w}) = f\mathcal{R}(\alpha, \vec{u}, \vec{v}, \vec{w})$ is trivial.

Then (10.5) gives $[f\vec{v}, \vec{w}] = f([\vec{v}, \vec{w}]) - (df.\vec{w})\vec{v}$, and (26.7) gives $\rho(f\vec{v}, \vec{w})(\vec{u}) = B(f\vec{v}, \vec{w})\vec{u} - [f\vec{v}, \vec{w}](\vec{u}) = fB(\vec{v}, \vec{w})\vec{u} - (df.\vec{w})\nabla_{\vec{v}}\vec{u} - (f([\vec{v}, \vec{w}]) - (df.\vec{w})\vec{v}) = f(B(\vec{v}, \vec{w})\vec{u} - [\vec{v}, \vec{w}](\vec{u})) = f\rho(\vec{v}, \vec{w})(\vec{u})$.

Then ρ being antisymmetric, $\rho(\vec{v}, f\vec{w})(\vec{u}) = -\rho(f\vec{w}, \vec{v})(\vec{u}) = -f\rho(\vec{w}, \vec{v})(\vec{u}) = f\rho(\vec{v}, \vec{w})(\vec{u})$.

And, (26.6) and (11.3) give $\rho(\vec{v}, \vec{w})(f\vec{u}) = [\nabla_{\vec{v}}, \nabla_{\vec{w}}](f\vec{u}) - \nabla_{[\vec{v}, \vec{w}]}(f\vec{u}) = ([\vec{v}, \vec{w}](f))\vec{u} + f[\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) - ([\vec{v}, \vec{w}](f))\vec{u} - f\nabla_{[\vec{v}, \vec{w}]}(\vec{u}) = f[\nabla_{\vec{v}}, \nabla_{\vec{w}}](\vec{u}) - f\nabla_{[\vec{v}, \vec{w}]}(\vec{u}) = f\rho(\vec{v}, \vec{w})(\vec{u})$. \blacksquare

26.4 Expression in a coordinate system

With the coordinate basis (\vec{e}_i) , let R_{jkl}^i be the components of the tensor $\mathcal{R} \in T_3^1$:

$$\mathcal{R} = \sum_{i,j,k,\ell=1}^m R_{jkl}^i \vec{e}_i \otimes e^j \otimes e^k \otimes e^\ell, \quad \text{i.e.} \quad R_{jkl}^i = \mathcal{R}(e^i, \vec{e}_j, \vec{e}_k, \vec{e}_\ell). \quad (26.21)$$

Thus $R_{jkl}^i = e^i.R(\vec{e}_j, \vec{e}_k, \vec{e}_\ell) = e^i.\rho(\vec{e}_k, \vec{e}_\ell).\vec{e}_j$ and

$$R(\vec{e}_j, \vec{e}_k, \vec{e}_\ell) = \sum_{i=1}^m R_{jkl}^i \vec{e}_i, \quad \text{and} \quad \rho(\vec{e}_k, \vec{e}_\ell) = \sum_{i,j=1}^m R_{jkl}^i \vec{e}_i \otimes e^j. \quad (26.22)$$

In particular, (26.20) gives $\mathcal{R}(e^i, \vec{e}_j, \vec{e}_k, \vec{e}_\ell) = -\mathcal{R}(e^i, \vec{e}_j, \vec{e}_\ell, \vec{e}_k)$, that is

$$R_{jkl}^i = -R_{j\ell k}^i \quad (\text{last two indices antisymmetry}). \quad (26.23)$$

Proposition 26.10 With the Christoffel symbols $\gamma_{jk}^i = e^i.\nabla_{\vec{e}_j}\vec{e}_k$, we get

$$\nabla_{\vec{e}_k}\nabla_{\vec{e}_\ell}\vec{e}_j = \sum_i \left(\frac{\partial \gamma_{\ell j}^i}{\partial q^k} + \sum_\nu \gamma_{\ell j}^\nu \gamma_{k\nu}^i \right) \vec{e}_i. \quad (26.24)$$

Then,

$$R_{jkl}^i = \frac{\partial \gamma_{j\ell}^i}{\partial q^k} - \frac{\partial \gamma_{jk}^i}{\partial q^\ell} + \sum_\nu \gamma_{k\nu}^i \gamma_{j\ell}^\nu - \sum_\nu \gamma_{\ell\nu}^i \gamma_{jk}^\nu. \quad (26.25)$$

Proof. $\nabla_{\vec{e}_\ell}\vec{e}_j = \sum_i \gamma_{\ell j}^i \vec{e}_i$ gives

$$\nabla_{\vec{e}_k}(\nabla_{\vec{e}_\ell}\vec{e}_j) = \sum_i (d\gamma_{\ell j}^i.\vec{e}_k)\vec{e}_i + \sum_i \gamma_{\ell j}^i \nabla_{\vec{e}_k}\vec{e}_i = \sum_i \frac{\partial \gamma_{\ell j}^i}{\partial q^k} \vec{e}_i + \sum_{i\nu} \gamma_{\ell j}^i \gamma_{ki}^\nu \vec{e}_\nu$$

Thus $\nabla_{\vec{e}_k}\nabla_{\vec{e}_\ell}\vec{e}_j - \nabla_{\vec{e}_\ell}\nabla_{\vec{e}_k}\vec{e}_j$ and $[\vec{e}_k, \vec{e}_\ell] = 0$ and $\gamma_{bc}^a = \gamma_{cb}^a$ give (26.25). \blacksquare

Remark 26.11 $R(\vec{e}_j, \vec{e}_k, \vec{e}_\ell) = \sum_{i=1}^m R_{jkl}^i \vec{e}_i$ is also written $R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}) = \sum_{i=1}^m R_{jkl}^i \frac{\partial}{\partial x^i}$. \blacksquare

26.5 Flat space: $R = 0$

Corollary 26.12 Let ∇ be a usual Riemannian metric. If S is a flat surface in \mathbb{R}^n , then $R = 0$.

Proof. Choose a Cartesian basis: Thus $\gamma_{jk}^i = 0$, and (26.25) gives the result. \blacksquare

Definition 26.13 A flat manifold is a manifold such that $R = 0$.

26.6 Bianchi identities

Proposition 26.14 For all $\vec{u}, \vec{v}, \vec{w} \in \Gamma(S)$:

First Bianchi identity (cyclic permutation):

$$R(\vec{u}, \vec{v}, \vec{w}) + R(\vec{w}, \vec{u}, \vec{v}) + R(\vec{v}, \vec{w}, \vec{u}) = 0, \quad \text{i.e.} \quad R_{jkl}^i + R_{k\ell j}^i + R_{\ell jk}^i = 0 \stackrel{\text{written}}{=} R_{[jkl]}^i. \quad (26.26)$$

Second Bianchi identity: If ∇ is torsion free then, for all $\vec{z} \in \Gamma(S)$,

$$(\nabla_{\vec{z}}\rho)(\vec{v}, \vec{w}) + (\nabla_{\vec{w}}\rho)(\vec{z}, \vec{v}) + (\nabla_{\vec{v}}\rho)(\vec{w}, \vec{z}) = 0, \quad (26.27)$$

that is, $(\nabla_{\vec{z}}\rho)(\vec{v}, \vec{w}) \cdot \vec{u} + (\nabla_{\vec{w}}\rho)(\vec{z}, \vec{v}) \cdot \vec{u} + (\nabla_{\vec{v}}\rho)(\vec{w}, \vec{z}) \cdot \vec{u} = 0$. And

$$R_{jkl}^i{}_{|\nu} + R_{j\ell\nu|k}^i + R_{j\nu k|\ell}^i = 0 \stackrel{\text{written}}{=} R_{j[k\ell|\nu]}^i. \quad (26.28)$$

Proof. The Jacobi identity (10.10) reads $\nabla_{\vec{v}}([\vec{w}, \vec{u}]) - \nabla_{[\vec{v}, \vec{w}]} \vec{u} + \nabla_{\vec{w}}([\vec{u}, \vec{v}]) - \nabla_{[\vec{w}, \vec{u}]} \vec{v} + \nabla_{\vec{u}}([\vec{v}, \vec{w}]) - \nabla_{[\vec{u}, \vec{v}]} \vec{w} = 0$. Thus, with (26.16) we have $R(\vec{u}, \vec{v}, \vec{w}) + R(\vec{v}, \vec{w}, \vec{u}) + R(\vec{w}, \vec{u}, \vec{v})$
 $= \nabla_{\vec{v}} \nabla_{\vec{w}} \vec{u} - \nabla_{\vec{w}} \nabla_{\vec{v}} \vec{u} - \nabla_{[\vec{v}, \vec{w}]} \vec{u} + \nabla_{\vec{w}} \nabla_{\vec{u}} \vec{v} - \nabla_{\vec{u}} \nabla_{\vec{w}} \vec{v} - \nabla_{[\vec{w}, \vec{u}]} \vec{v} + \nabla_{\vec{u}} \nabla_{\vec{v}} \vec{w} - \nabla_{\vec{v}} \nabla_{\vec{u}} \vec{w} - \nabla_{[\vec{u}, \vec{v}]} \vec{w}$
 $= \nabla_{\vec{v}}(\nabla_{\vec{w}} \vec{u} - \nabla_{\vec{u}} \vec{w}) + \nabla_{\vec{w}}(\nabla_{\vec{v}} \vec{u} - \nabla_{\vec{u}} \vec{v}) + \nabla_{\vec{u}}(\nabla_{\vec{v}} \vec{w} - \nabla_{\vec{w}} \vec{v}) - \nabla_{[\vec{v}, \vec{w}]} \vec{u} - \nabla_{[\vec{w}, \vec{u}]} \vec{v} - \nabla_{[\vec{u}, \vec{v}]} \vec{w}$
 $= \nabla_{\vec{v}}([\vec{w}, \vec{u}]) + \nabla_{\vec{w}}([\vec{u}, \vec{v}]) + \nabla_{\vec{u}}([\vec{v}, \vec{w}]) - \nabla_{[\vec{v}, \vec{w}]} \vec{u} - \nabla_{[\vec{w}, \vec{u}]} \vec{v} - \nabla_{[\vec{u}, \vec{v}]} \vec{w} = 0$ (Jacobi identity), i.e. (26.26).

Then (See <https://math.stackexchange.com/questions/1494262/direct-proof-of-the-second-bianchi-identity>) $\nabla_{\vec{z}}(\rho(\vec{v}, \vec{w}) \cdot \vec{u}) = (\nabla_{\vec{z}}\rho)(\vec{v}, \vec{w}) \cdot \vec{u} + \rho(\nabla_{\vec{z}}\vec{v}, \vec{w}) \cdot \vec{u} + \rho(\vec{v}, \nabla_{\vec{z}}\vec{w}) \cdot \vec{u} + \rho(\vec{v}, \vec{w}) \cdot \nabla_{\vec{z}}\vec{u}$, i.e.,

$$\begin{aligned} (\nabla_{\vec{z}}\rho)(\vec{v}, \vec{w}) \cdot \vec{u} &= \nabla_{\vec{z}}(\rho(\vec{v}, \vec{w}) \cdot \vec{u}) - \rho(\nabla_{\vec{z}}\vec{v}, \vec{w}) \cdot \vec{u} - \rho(\vec{v}, \nabla_{\vec{z}}\vec{w}) \cdot \vec{u} - \rho(\vec{v}, \vec{w}) \cdot \nabla_{\vec{z}}\vec{u}, \\ &= \nabla_{\vec{z}}(\rho(\vec{v}, \vec{w}) \cdot \vec{u}) + \rho(\nabla_{\vec{z}}\vec{v}, \vec{w}) \cdot \vec{u} - \rho(\nabla_{\vec{z}}\vec{v}, \vec{w}) \cdot \vec{u} - \rho(\vec{v}, \vec{w}) \cdot \nabla_{\vec{z}}\vec{u}, \end{aligned} \quad (26.29)$$

ρ being antisymmetric. And for the first term in (26.29),

- $\rho(\vec{v}, \vec{w}) \cdot \vec{u} = \nabla_{\vec{v}} \nabla_{\vec{w}} \vec{u} - \nabla_{\vec{w}} \nabla_{\vec{v}} \vec{u} - \nabla_{[\vec{v}, \vec{w}]} \vec{u}$ gives

$$\nabla_{\vec{z}}(\rho(\vec{v}, \vec{w}) \cdot \vec{u}) = \nabla_{\vec{z}} \nabla_{\vec{v}} \nabla_{\vec{w}} \vec{u} - \nabla_{\vec{z}} \nabla_{\vec{w}} \nabla_{\vec{v}} \vec{u} - \nabla_{\vec{z}} \nabla_{[\vec{v}, \vec{w}]} \vec{u},$$

with $\nabla_{\vec{z}} \nabla_{[\vec{v}, \vec{w}]} \vec{u} - \nabla_{[\vec{v}, \vec{w}]} \nabla_{\vec{z}} \vec{u} - \nabla_{[\vec{z}, [\vec{v}, \vec{w}]]} \vec{u} = \rho(\vec{z}, [\vec{v}, \vec{w}]) \cdot \vec{u}$, so

$$\nabla_{\vec{z}}(\rho(\vec{v}, \vec{w}) \cdot \vec{u}) = \nabla_{\vec{z}} \nabla_{\vec{v}} \nabla_{\vec{w}} \vec{u} - \nabla_{\vec{z}} \nabla_{\vec{w}} \nabla_{\vec{v}} \vec{u} - \nabla_{[\vec{v}, \vec{w}]} \nabla_{\vec{z}} \vec{u} - \rho(\vec{z}, [\vec{v}, \vec{w}]) \cdot \vec{u}.$$

Thus (circular permutation),

$$\begin{aligned} &\nabla_{\vec{z}}(\rho(\vec{v}, \vec{w}) \cdot \vec{u}) + \nabla_{\vec{w}}(\rho(\vec{z}, \vec{v}) \cdot \vec{u}) + \nabla_{\vec{v}}(\rho(\vec{w}, \vec{z}) \cdot \vec{u}) \\ &= \nabla_{\vec{z}} \nabla_{\vec{v}} \nabla_{\vec{w}} \vec{u} - \nabla_{\vec{z}} \nabla_{\vec{w}} \nabla_{\vec{v}} \vec{u} + \nabla_{\vec{w}} \nabla_{\vec{z}} \nabla_{\vec{v}} \vec{u} - \nabla_{\vec{w}} \nabla_{\vec{v}} \nabla_{\vec{z}} \vec{u} + \nabla_{\vec{v}} \nabla_{\vec{w}} \nabla_{\vec{z}} \vec{u} - \nabla_{\vec{v}} \nabla_{\vec{z}} \nabla_{\vec{w}} \vec{u} \\ &\quad - \nabla_{[\vec{v}, \vec{w}]} \nabla_{\vec{z}} \vec{u} - \nabla_{[\vec{z}, \vec{v}]} \nabla_{\vec{w}} \vec{u} - \nabla_{[\vec{w}, \vec{z}]} \nabla_{\vec{v}} \vec{u} - \nabla_{[\vec{z}, [\vec{v}, \vec{w}]]} \vec{u} - \nabla_{[\vec{w}, [\vec{z}, \vec{v}]]} \vec{u} - \nabla_{[\vec{v}, [\vec{w}, \vec{z}]]} \vec{u} \\ &\quad - \rho(\vec{z}, [\vec{v}, \vec{w}]) \cdot \vec{u} - \rho(\vec{w}, [\vec{z}, \vec{v}]) \cdot \vec{u} - \rho(\vec{v}, [\vec{w}, \vec{z}]) \cdot \vec{u} \\ &= [\nabla_{\vec{z}}, \nabla_{\vec{v}}] \nabla_{\vec{w}} \vec{u} + [\nabla_{\vec{w}}, \nabla_{\vec{z}}] \nabla_{\vec{v}} \vec{u} + [\nabla_{\vec{v}}, \nabla_{\vec{w}}] \nabla_{\vec{z}} \vec{u} \\ &\quad - \nabla_{[\vec{v}, \vec{w}]} \nabla_{\vec{z}} \vec{u} - \nabla_{[\vec{z}, \vec{v}]} \nabla_{\vec{w}} \vec{u} - \nabla_{[\vec{w}, \vec{z}]} \nabla_{\vec{v}} \vec{u} - 0 \text{ (Jacobi identity)} \\ &\quad - \rho(\vec{z}, [\vec{v}, \vec{w}]) \cdot \vec{u} - \rho(\vec{w}, [\vec{z}, \vec{v}]) \cdot \vec{u} - \rho(\vec{v}, [\vec{w}, \vec{z}]) \cdot \vec{u} \\ &= \rho(\vec{z}, \vec{v}) \nabla_{\vec{w}} \vec{u} + \rho(\vec{w}, \vec{z}) \nabla_{\vec{v}} \vec{u} + \rho(\vec{v}, \vec{w}) \nabla_{\vec{z}} \vec{u} \\ &\quad - \rho(\vec{z}, [\vec{v}, \vec{w}]) \cdot \vec{u} - \rho(\vec{w}, [\vec{z}, \vec{v}]) \cdot \vec{u} - \rho(\vec{v}, [\vec{w}, \vec{z}]) \cdot \vec{u}. \end{aligned}$$

- And for the last three terms in (26.29), (circular permutation),

$$\begin{aligned} &\rho(\nabla_{\vec{z}}\vec{v}, \vec{w}) \cdot \vec{u} - \rho(\nabla_{\vec{z}}\vec{v}, \vec{w}) \cdot \vec{u} - \rho(\vec{v}, \vec{w}) \cdot \nabla_{\vec{z}}\vec{u} + \rho(\nabla_{\vec{v}}\vec{z}, \vec{w}) \cdot \vec{u} - \rho(\nabla_{\vec{v}}\vec{w}, \vec{z}) \cdot \vec{u} - \rho(\vec{z}, \vec{v}) \cdot \nabla_{\vec{w}}\vec{u} \\ &\quad + \rho(\nabla_{\vec{w}}\vec{v}, \vec{z}) \cdot \vec{u} - \rho(\nabla_{\vec{w}}\vec{z}, \vec{v}) \cdot \vec{u} - \rho(\vec{w}, \vec{z}) \cdot \nabla_{\vec{v}}\vec{u} \\ &= \rho(\nabla_{\vec{z}}\vec{w} - \nabla_{\vec{w}}\vec{z}, \vec{v}) \cdot \vec{u} + \rho(\nabla_{\vec{v}}\vec{z} - \nabla_{\vec{z}}\vec{v}, \vec{w}) \cdot \vec{u} + \rho(\nabla_{\vec{w}}\vec{v} - \nabla_{\vec{v}}\vec{w}, \vec{z}) \cdot \vec{u} \\ &\quad - \rho(\vec{v}, \vec{w}) \cdot \nabla_{\vec{z}}\vec{u} - \rho(\vec{z}, \vec{v}) \cdot \nabla_{\vec{w}}\vec{u} - \rho(\vec{w}, \vec{z}) \cdot \nabla_{\vec{v}}\vec{u} \\ &= \rho([\vec{z}, \vec{w}]) \cdot \vec{u} + \rho([\vec{v}, \vec{z}], \vec{w}) \cdot \vec{u} + \rho([\vec{w}, \vec{v}], \vec{z}) \cdot \vec{u} \text{ (torsion-free)} \\ &\quad - \rho(\vec{v}, \vec{w}) \cdot \nabla_{\vec{z}}\vec{u} - \rho(\vec{z}, \vec{v}) \cdot \nabla_{\vec{w}}\vec{u} - \rho(\vec{w}, \vec{z}) \cdot \nabla_{\vec{v}}\vec{u}. \end{aligned}$$

Hence (26.27) (sum of the two above equations with ρ antisymmetric). And (26.25) gives (26.28). \blacksquare

26.6.1 Example with the sphere

2-D in \mathbb{R}^3 , usual Riemannian connection, and (\vec{e}_2, \vec{e}_3) renamed (\vec{e}_1, \vec{e}_2) . So, with (3.33)-(3.34)-(3.35) (and Proj_{TS}), $\nabla_{\vec{e}_1} \vec{e}_1 = \cos \varphi \sin \varphi \vec{e}_2$, $\nabla_{\vec{e}_1} \vec{e}_2 = -\tan \varphi \vec{e}_1 = \nabla_{\vec{e}_2} \vec{e}_1$, $\nabla_{\vec{e}_2} \vec{e}_2 = \vec{0}$. Thus:

$$\begin{aligned}\nabla_{\vec{e}_1}(\nabla_{\vec{e}_1} \vec{e}_1) &= \nabla_{\vec{e}_1}(\cos \varphi \sin \varphi \vec{e}_2) = \cos \varphi \sin \varphi \nabla_{\vec{e}_1} \vec{e}_2 = -\sin^2 \varphi \vec{e}_1, \\ \nabla_{\vec{e}_1}(\nabla_{\vec{e}_1} \vec{e}_2) &= \nabla_{\vec{e}_1}(\nabla_{\vec{e}_2} \vec{e}_1) = -\nabla_{\vec{e}_1}(\tan \varphi \vec{e}_1) = -\tan \varphi \nabla_{\vec{e}_1}(\vec{e}_1) = -\sin^2 \varphi \vec{e}_2, \\ \nabla_{\vec{e}_1}(\nabla_{\vec{e}_2} \vec{e}_2) &= \nabla_{\vec{e}_1}(\vec{0}) = \vec{0}, \\ \nabla_{\vec{e}_2}(\nabla_{\vec{e}_1} \vec{e}_1) &= \nabla_{\vec{e}_2}(\cos \varphi \sin \varphi \vec{e}_2) = (-\sin^2 \varphi + \cos^2 \varphi) \vec{e}_2 + \vec{0}, \\ \nabla_{\vec{e}_2}(\nabla_{\vec{e}_1} \vec{e}_2) &= \nabla_{\vec{e}_2}(\nabla_{\vec{e}_2} \vec{e}_1) = -\nabla_{\vec{e}_2}(\tan \varphi \vec{e}_1) = -(1 + \tan^2 \varphi) \vec{e}_1 + \tan^2 \varphi \vec{e}_1 = -\vec{e}_1, \\ \nabla_{\vec{e}_2}(\nabla_{\vec{e}_2} \vec{e}_2) &= \vec{0}.\end{aligned}$$

Thus $\rho(\vec{e}_1, \vec{e}_1) = \rho(\vec{e}_2, \vec{e}_2) = 0$, and with $[\vec{e}_i, \vec{e}_j] = 0$ for all i, j (basis of a coordinate system) we get

$$\begin{cases} R(\vec{e}_1, \vec{e}_2, \vec{e}_2) = \rho(\vec{e}_1, \vec{e}_2) \vec{e}_1 = \nabla_{\vec{e}_1}(\nabla_{\vec{e}_2} \vec{e}_1) - \nabla_{\vec{e}_2}(\nabla_{\vec{e}_1} \vec{e}_1) = -\cos^2 \varphi \vec{e}_2 = R_{122}^2 \vec{e}_2 \\ R(\vec{e}_2, \vec{e}_2, \vec{e}_2) = \rho(\vec{e}_1, \vec{e}_2) \vec{e}_2 = \nabla_{\vec{e}_1}(\nabla_{\vec{e}_2} \vec{e}_2) - \nabla_{\vec{e}_2}(\nabla_{\vec{e}_1} \vec{e}_2) = \vec{e}_1 = R_{222}^1 \vec{e}_1. \end{cases} \quad (26.30)$$

(The Riemann tensor does not vanish: The sphere is not flat!).

26.7 Remarkable Theorem of Gauss (theorem egregium)

With a Riemannian metric and the associated connection, cf. Lévi-Civita theorem 22.22:

Theorem 26.15 (theorem egregium) *The Riemann tensor only depends on the metric in the manifold (does not depend on the metric in a supposed affine space containing the manifold).*

Proof. (17.14) tells that the γ_{jk}^i are function of the metric, thus, so are the R_{jkl}^i , cf. (26.25). \blacksquare

26.8 Metric and R^b associated tensor

With a Riemannian metric $(\cdot, \cdot)_g$ and the associated connection ∇ , cf. Lévi-Civita theorem 22.22:

26.8.1 Associated covariant Riemann tensor

Definition 26.16 $\mathcal{R} \in T_3^1(S)$ being the Riemann tensor in S , the associated covariant Riemann tensor $\mathcal{R}^b \in T_4^0(S)$ is defined by

$$\mathcal{R}^b(\vec{a}, \vec{u}, \vec{v}, \vec{w}) = (\vec{a}, R(\vec{u}, \vec{v}, \vec{w}))_g = (\vec{a}, \rho(\vec{v}, \vec{w}) \cdot \vec{u})_g. \quad (26.31)$$

In other words, if $a^b \in \Omega^1(S)$ (differential form) and if $\vec{a} \in \Gamma(S)$ (vector field) is the $(\cdot, \cdot)_g$ -Riesz representation of α , that is,

$$a^b \cdot \vec{b} = (\vec{a}, \vec{b})_g, \quad (26.32)$$

for all $\vec{b} \in \Gamma(S)$, then

$$\mathcal{R}^b(\vec{a}, \vec{u}, \vec{v}, \vec{w}) := \mathcal{R}(a^b, \vec{u}, \vec{v}, \vec{w}) = (a^b \cdot R(\vec{u}, \vec{v}, \vec{w})), \quad (26.33)$$

Quantification: With a basis (\vec{e}_i) of a coordinate system, let $R_{ijkl} = \mathcal{R}^b(\vec{e}_i, \vec{e}_j, \vec{e}_k, \vec{e}_\ell)$ be the components of \mathcal{R}^b :

$$\mathcal{R}^b = \sum_{ijkl} R_{ijkl} e^i \otimes e^j \otimes e^k \otimes e^\ell. \quad (26.34)$$

Then, with $R(\vec{e}_j, \vec{e}_k, \vec{e}_\ell) = \sum_i R_{jkl}^i \vec{e}_i$ and $g = \sum_{\alpha, \beta} g_{\alpha\beta} e^\alpha \otimes e^\beta$, we get

$$R_{ijkl} = \sum_{\alpha} g_{i\alpha} R_{jkl}^{\alpha} = (g(\vec{e}_i, R(\vec{e}_j, \vec{e}_k, \vec{e}_\ell))), \quad (26.35)$$

since $R_{ijkl} = \mathcal{R}^b(\vec{e}_i, \vec{e}_j, \vec{e}_k, \vec{e}_\ell) = g(\vec{e}_i, R(\vec{e}_j, \vec{e}_k, \vec{e}_\ell)) = g(\vec{e}_i, \sum_{\alpha} R_{jkl}^{\alpha} \vec{e}_\alpha) = \sum_{\alpha} R_{jkl}^{\alpha} g_{i\alpha}$.

Thus, (26.25) gives

$$R_{ijkl} = \sum_{\alpha} g_{i\alpha} \left(\frac{\partial \gamma_{j\ell}^{\alpha}}{\partial q^k} - \frac{\partial \gamma_{jk}^{\alpha}}{\partial q^{\ell}} + \sum_{\nu} \gamma_{k\nu}^{\alpha} \gamma_{j\ell}^{\nu} - \sum_{\nu} \gamma_{\ell\nu}^{\alpha} \gamma_{jk}^{\nu} \right). \quad (26.36)$$

26.8.2 Anti-symmetries of \mathcal{R}^b

(26.19) gives $a^b \cdot R(\vec{u}, \vec{v}, \vec{w}) = -a^b \cdot R(\vec{u}, \vec{w}, \vec{v})$, thus $\mathcal{R}^b(\vec{a}, \vec{u}, \vec{v}, \vec{w}) = -\mathcal{R}^b(\vec{a}, \vec{u}, \vec{w}, \vec{v})$, and

$$R_{ijkl} = -R_{ijlk} \quad (\text{antisymmetry of the last 2 indices}). \quad (26.37)$$

Proposition 26.17 $a^b \cdot (\rho(\vec{v}, \vec{w}) \cdot \vec{u}) = -a^b \cdot (\rho(\vec{v}, \vec{w}) \cdot \vec{a})$, i.e. $(\vec{a}, \rho(\vec{v}, \vec{w}) \cdot \vec{u})_g = -(\vec{u}, \rho(\vec{v}, \vec{w}) \cdot \vec{a})_g$, that is

$$\mathcal{R}(a^b, \vec{u}, \vec{v}, \vec{w}) = -\mathcal{R}(a^b, \vec{a}, \vec{v}, \vec{w}), \quad \text{and} \quad R_{ijkl} = -R_{jikl} \quad (\text{antisymmetry of the first 2 indices}). \quad (26.38)$$

Proof. At a point $p \in S$, choose an orthonormal coordinate system Φ , that is, such that the $\vec{e}_i(p) = \frac{\partial \Phi}{\partial q^i}(\vec{q})$ satisfy $g(p)(\vec{e}_i(p), \vec{e}_j(p)) = \delta_j^i$ (the coordinate lines are orthonormal geodesics at p). Then, at p , the Christoffel symbols vanish (not their derivatives), and (26.36) gives

$$R_{ijkl} = \sum_{\alpha} g_{i\alpha} \left(\frac{\partial \gamma_{j\ell}^{\alpha}}{\partial q^k} - \frac{\partial \gamma_{jk}^{\alpha}}{\partial q^{\ell}} \right)$$

And (17.15), that is $\gamma_{jk}^{\alpha} = \frac{1}{2} \sum_{\beta} g^{\alpha\beta} \left(\frac{\partial g_{\beta j}}{\partial q^k} + \frac{\partial g_{\beta k}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^{\beta}} \right)$, give, at p ,

$$d\gamma_{jk}^{\alpha} \cdot \vec{e}_{\ell} = \frac{\partial \gamma_{jk}^{\alpha}}{\partial q^{\ell}} = \frac{1}{2} \sum_{\beta} g^{\alpha\beta} \left(\frac{\partial^2 g_{\beta j}}{\partial q^k \partial q^{\ell}} + \frac{\partial^2 g_{\beta k}}{\partial q^j \partial q^{\ell}} - \frac{\partial^2 g_{jk}}{\partial q^{\beta} \partial q^{\ell}} \right)$$

since $dg^{\alpha\beta} \cdot \vec{e}_{\ell} = 0$ (thanks to $\nabla g = 0$ at p). Thus

$$\frac{\partial \gamma_{j\ell}^{\alpha}}{\partial q^k} - \frac{\partial \gamma_{jk}^{\alpha}}{\partial q^{\ell}} = \frac{1}{2} \sum_{\beta} g^{\alpha\beta} \left(\frac{\partial^2 g_{\beta j}}{\partial q^{\ell} \partial q^k} + \frac{\partial^2 g_{\beta \ell}}{\partial q^j \partial q^k} - \frac{\partial^2 g_{j\ell}}{\partial q^{\beta} \partial q^k} - \frac{\partial^2 g_{\beta j}}{\partial q^k \partial q^{\ell}} - \frac{\partial^2 g_{\beta k}}{\partial q^j \partial q^{\ell}} + \frac{\partial^2 g_{jk}}{\partial q^{\beta} \partial q^{\ell}} \right).$$

And $\frac{\partial^2 g_{\beta j}}{\partial q^{\ell} \partial q^k} = \frac{\partial^2 g_{\beta j}}{\partial q^k \partial q^{\ell}}$ (Schwarz equality) and at p , $g_{ij} = \delta_{ij}$ (orthonormal system), thus thus

$$R_{ijkl} = \frac{\partial^2 g_{i\ell}}{\partial q^j \partial q^k} - \frac{\partial^2 g_{j\ell}}{\partial q^i \partial q^k} - \frac{\partial^2 g_{ik}}{\partial q^j \partial q^{\ell}} + \frac{\partial^2 g_{jk}}{\partial q^i \partial q^{\ell}}.$$

Thus, at p , $R_{ijkl} = -R_{jikl}$, thus $\mathcal{R}^b(\vec{a}, \vec{u}, \vec{v}, \vec{w}) = -\mathcal{R}^b(\vec{u}, \vec{a}, \vec{w}, \vec{v})$ for all $\vec{a}, \vec{u}, \vec{v}, \vec{w}$. \blacksquare

26.9 Example in \mathbb{R}^3

Let S be a 2-D surface in \mathbb{R}^3 . Then $[R_{ijkl}]_{i,j,k,\ell=1,2}$ is a set of $2^4 = 16$ scalars. The k, ℓ antisymmetry and the i, j antisymmetry gives $2 * 2 = 4$ independent scalars.

Proposition 26.18 Let K be the mean curvature, cf. (24.23). Then, for all $i, j, k, \ell = 1, 2$,

$$R_{ijkl} = K(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}). \quad (26.39)$$

26.10 Ricci tensor and scalar curvature

Let (\vec{e}_i) be a coordinate basis.

Definition 26.19 Ricci tensor $(Ric)^b \in T_2^0(S)$ is obtained by contracting the first and third index of R_{ijkl}^i :

$$(Ric)^b = \sum_{j,\ell=1}^m (Ric)_{j\ell}^b e^j \otimes e^{\ell} \quad \text{where} \quad (Ric)_{j\ell}^b = \sum_{i=1}^m R_{jil}^i \stackrel{\text{written}}{=} R_{j\ell}^b. \quad (26.40)$$

And the associated tensor $(Ric) \in T_1^1(S)$ is defined by

$$((Ric)\vec{a}, \vec{u})_g = (Ric)^b(\vec{a}, \vec{u}). \quad (26.41)$$

Thus

$$(Ric) = \sum_{j,\ell=1}^m (Ric)_{\ell}^j \vec{e}_j \otimes e^{\ell} \quad \text{where} \quad (Ric)_{\ell}^j = \sum_{\alpha} g^{j\alpha} R_{\alpha\ell}. \quad (26.42)$$

Definition 26.20 The scalar Riemannian curvature is $R = \text{Tr}(Ric) = \sum_{i=1}^m (Ric)_i^i$.

Thus,

$$R = \sum_j (Ric)_j^j = \sum_{j\alpha} g^{j\alpha} R_{\alpha j} = \sum_{ij\alpha} g^{j\alpha} R_{\alpha ij}. \quad (26.43)$$

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