Part IV

Velocity-addition formula and Objectivity

17 Change of referential

17.1 Introduction and problem

Let *Obj* be a material object. Let $P_{Obj} \in Obj$ be a particle, and consider its Eulerian velocities $\vec{v}_A$ in a referential $\mathcal{R}_A$ and $\vec{v}_B$ in a referential $\mathcal{R}_B$, and let $\vec{v}_D$ be the velocity of $\mathcal{R}_B$ in $\mathcal{R}_A$ (the drive speed = vitesse d'entrainement). The classical expression of the velocity-addition formula is:

$$(\vec{v}_A \text{ the absolute velocity}) = (\vec{v}_B \text{ the relative velocity}) + (\vec{v}_D \text{ the drive speed of } \mathcal{R}_B \text{ in } \mathcal{R}_A), \tag{17.1}$$

But (17.1) is problematic (self contradictory):

1- The velocities $\vec{v}_A$ and $\vec{v}_D$ are velocities in $\mathcal{R}_A$: E.g., expressed in foot/s by the absolute observer.

2- The velocity $\vec{v}_B$ is a velocity in $\mathcal{R}_B$: E.g., expressed in meter/s by the relative observer.

3- Thus (17.1) seems to indicate that the right hand side adds meter/s with foot/s, which is absurd.

Thus $\vec{v}_B$ cannot just be the velocity in the relative referential: It has to be the "translated velocity for observer A". Details and explanations:

17.2 Framework

Classical mechanics. The same universe for all observers, modeled as an affine space $\mathbb{R}^n$ with the associated vector space $\vec{\mathbb{R}}^n$ (made of bipoint vectors), with $n = 3$, or with $n = 1$ or 2 if 1-D or 2-D problems are studied. To lighten the writing, the observers use the same time scale and same time origin (otherwise we need to define referentials in $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^n$ = time × space).

An observer A chooses a point $O_A \in \mathbb{R}^n$ (origin), a basis $(\vec{A}_i) \in \vec{\mathbb{R}}^n$ to build his referential $\mathcal{R}_A = (O_A, (\vec{A}_i))$, the same at all time. And an observer B chooses a point $O_B \in \mathbb{R}^n$ (origin), a basis $(\vec{B}_i) \in \vec{\mathbb{R}}^n$ to build his referential $\mathcal{R}_B = (O_B, (\vec{B}_i))$, the same at all time. Illustration: They both live on "rigid like bodies", e.g., with O_A the center of the sun and $(\vec{A}_i)$ fixed relative to stars, and O_B some point on Earth and $(\vec{B}_i)$ fixed on Earth.

Observers A and B observe the same "deformable material object" *Obj*. The position of a particle $P_{Obj} \in Obj$ in the affine space $\mathbb{R}^n$ is given thanks to motions, cf. (1.3): The motion of *Obj* is described by observer A as being $\widetilde{\Phi}_A : [t_1, t_2] \times Obj \to \mathcal{R}_A$, and is called the absolute motion, and the associated Eulerian velocity field $\vec{v}_A$ is called the absolute velocity field. And the motion of *Obj* is described by observer B as being $\widetilde{\Phi}_B : [t_1, t_2] \times Obj \to \mathcal{R}_B$, and is called the relative motion, and the associated Eulerian velocity field $\vec{v}_B$ is called the relative velocity field. So, (1.3) and (2.5) give, for any particle $P_{Obj} \in Obj$ and any $t \in [t_1, t_2]$,

$$\begin{cases} p_{At} = \widetilde{\Phi}_A(t, P_{Obj}) = O_A + \sum_{i=1}^n x^i_{At} \vec{A}_i = \text{position of } P_{Obj} \text{ at } t \text{ located by } A \text{ in } \mathcal{R}_A, \\ \vec{v}_A(t, p_{At}) = \dfrac{\partial \widetilde{\Phi}_A}{\partial t}(t, P_{Obj}) = \text{velocity of } P_{Obj} \text{ at } t \text{ at } p_{At} \text{ in } \mathcal{R}_A, \\ p_{Bt} = \widetilde{\Phi}_B(t, P_{Obj}) = O_B + \sum_{i=1}^n x^i_{Bt} \vec{B}_i = \text{position of } P_{Obj} \text{ at } t \text{ located by } B \text{ in } \mathcal{R}_B, \\ \vec{v}_B(t, p_{Bt}) = \dfrac{\partial \widetilde{\Phi}_B}{\partial t}(t, P_{Obj}) = \text{velocity of } P_{Obj} \text{ at } t \text{ at } p_{Bt} \text{ in } \mathcal{R}_B. \end{cases} \tag{17.2}$$

Let *ObjB* be a material rigid body on which $\mathcal{R}_B$ is fixed (e.g. *ObjB* = the Earth). The rigid body motion of *ObjB* is described by observer A as being $\widetilde{\Phi}_D : [t_1, t_2] \times ObjB \to \mathcal{R}_A$, and is called the drive motion (mouvement d'entrainement), and the associated Eulerian velocity field $\vec{v}_D$ is called the drive velocity field. The rigid body motion of *ObjB* is described by observer B as being $\widetilde{\Psi} : ObjB \to \mathcal{R}_B$ (static function which locates the particles $Q_{ObjB} \in ObjB$ in the referential chosen by B), and the associated

Eulerian velocity field vanishes. So, (1.3) and (2.5) give, for any particle $Q_{ObjB} \in ObjB$ and any $t \in [t_1, t_2]$,

$$\begin{cases} q_{At} = \widetilde{\Phi}_D(t, Q_{ObjB}) = O_A + \sum_{i=1}^n y^i_{At} \vec{A}_i = \text{position of } Q_{ObjB} \text{ at } t \text{ located by } A \text{ in } \mathcal{R}_A, \\ \vec{v}_D(t, q_{At}) = \frac{\partial \widetilde{\Phi}_D}{\partial t}(t, Q_{ObjB}) = \text{velocity of } Q_{ObjB} \text{ at } t \text{ at } q_{At} \text{ in } \mathcal{R}_A, \\ q_B = \widetilde{\Psi}_B(Q_{ObjB}) = O_B + \sum_{i=1}^n y^i_B \vec{B}_i = \text{(static) position of } Q_{ObjB} \text{ located by } B \text{ in } \mathcal{R}_B, \\ \vec{0} = \frac{\partial \widetilde{\Psi}_B}{\partial t}(Q_{ObjB}) = \text{nul velocity since } Q_{ObjB} \text{ does not move in } \mathcal{R}_B. \end{cases} \tag{17.3}$$

If t is fixed, then let $\widetilde{\Phi}_{Zt}(P_{Obj}) := \widetilde{\Phi}_Z(t, P_{Obj})$ and $\vec{v}_{Zt}(p_{Zt}) := \vec{v}_Z(t, p_{Zt})$ and $\Omega_{Zt} := \widetilde{\Phi}_Z(t, P_{Obj})$, for $Z = A, B, D$.

The drive motion $\widetilde{\Phi}_D$ being a rigid body motion, we have, for all $q_{Ait} = \widetilde{\Phi}_{Dt}(Q_{ObjB_i})$,

$$\vec{v}_{Dt}(q_{A2t}) = \vec{v}_{Dt}(q_{A1t}) + d\vec{v}_{Dt}.\overrightarrow{q_{A1t}, q_{A2t}} = \vec{v}_{Dt}(q_{A1t}) + \vec{\omega}_t \wedge \overrightarrow{q_{A1t}q_{A2t}} \in \mathcal{R}_A, \tag{17.4}$$

cf. (10.21).

Remark 17.1 Let P_{Obj} be a particle in *Obj*, followed by the observers A and B. Let $p_t \in \mathbb{R}^n$ be its position at t: In the Universe, a particle has just one position since there is no ubiquity in classical mechanics. E.g., P_{Obj} is the Eiffel tower and its position at t is p_t (the Earth moves in the Universe), regardless of the observer (qualitative approach). And, for quantification purposes, the position p_t is called $p_{At} \in \mathcal{R}_A$ by observer A and is called $p_{Bt} \in \mathcal{R}_B$ by observer B,

$$p_t = p_{At} = p_{Bt} \in \mathbb{R}^n, \tag{17.5}$$

cf. $(17.2)_{1,3}$ (quantification: Observer dependent). Thus, if you consider two particles $P_{Obj\,1}, P_{Obj\,2}$, then

$$\overrightarrow{p_{1t}p_{2t}} = \overrightarrow{p_{A1t}p_{A2t}} = \overrightarrow{p_{B1t}p_{B2t}} \in \vec{\mathbb{R}}^n \tag{17.6}$$

is the same bipoint vector for all observers (qualitative approach, see e.g. example 1.2), but its description in a referential (= quantification) depends on the observer, cf. $(17.2)_{1,3}$.

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17.3 The translator Θ_t for positions

17.3.1 Defintion

A motion is an "intra-referential" mapping connecting a particle and a position, cf. (1.3): $\widetilde{\Phi}_A$ and $\widetilde{\Phi}_D$ are defined by the observer A in his referential, $\widetilde{\Phi}_D$ and $\widetilde{\Psi}_B$ are defined by the observer B in his referential.

Definition 17.2 At any t, the translation operator (the translator) Θ_t from B to A is the "inter-referential" diffeomorphism which links at t the positions of particles of *ObjB* as referred to by observers A and B: That is, Θ_t is defined by, for any $Q_{ObjB} \in ObjB$,

$$\left\{ \begin{array}{l} q_B = \widetilde{\Psi}_B(Q_{ObjB}) \text{ (static) position of } Q_{ObjB} \text{ as described by } B \\ q_{At} = \widetilde{\Phi}_{Dt}(Q_{ObjB}) \text{ position of } Q_{ObjB} \text{ at } t \text{ as described by } A \end{array} \right\} \implies \Theta_t(q_B) = q_{At}. \tag{17.7}$$

That is, at any t, the translator Θ_t is defined by

$$\Theta_t := \widetilde{\Phi}_{Dt} \circ \widetilde{\Psi}_B^{-1} : \begin{cases} \mathcal{R}_B \to \mathcal{R}_A \\ \underbrace{q_B}_{\text{position}} \to \underbrace{q_{At}}_{\text{position}} = \Theta_t(q_B) := \widetilde{\Phi}_{Dt}(\widetilde{\Psi}_B^{-1}(q_B)) \overset{\text{named}}{=} q_{Bt*}. \end{cases} \tag{17.8}$$

the notation $q_{Bt*} := \Theta_t(q_B)$ being the notation of the push-forward by Θ_t, cf. (11.2).

So, at t, the link between the positions q_{At} and q_B of a particle $Q_{ObjB} \in ObjB$ is (translation for A):

$$q_{At} = \Theta_t(q_B) = \Theta_t(\widetilde{\Psi}_B(Q_{ObjB})) := \widetilde{\Phi}_{Dt}(\widetilde{\Psi}_B^{-1}(q_B)) = \widetilde{\Phi}_{Dt}(Q_{ObjB}) \overset{\text{named}}{=} q_{Bt*} \in \mathcal{R}_A. \tag{17.9}$$

In other words, Θ_t is characterized by,

$$\Theta_t \circ \widetilde{\Psi}_B = \widetilde{\Phi}_{Dt}, \quad \text{i.e.} \quad \Theta_t(\underbrace{\widetilde{\Psi}_B(Q_{ObjB})}_{\text{position } q_B \in \mathcal{R}_B}) = \underbrace{\widetilde{\Phi}_{Dt}(Q_{ObjB})}_{\text{position } q_{At} \in \mathcal{R}_A} \quad : \text{of a particle } Q_{ObjB} \text{ at } t. \tag{17.10}$$

With (17.8) we have defined

$$\Theta : \begin{cases} [t_1, t_2] \times \mathcal{R}_B & \to \mathcal{R}_A \\ \qquad (t, q_B) & \to q_A(t) = \Theta(t, q_B) := \Theta_t(q_B) = \widetilde{\Phi}_D(t, \widetilde{\Psi}_B^{-1}(q_B)) \overset{\text{named}}{=} q_{B*}(t), \end{cases} \tag{17.11}$$

called the translator from B and A.

Remark 17.3 **NB:** The translator Θ is not a motion: A motion is defined by only one observer and connects a particle and a position, cf. (1.3), when the translator connects two positions as spotted by two observers relative to their referentials, cf. (17.10). ▪

Exercice 17.4 $\widetilde{\Phi}_D$ being a rigid body motion, prove that a straight line in $\mathcal{R}_B$ is seen at t as a straight line in $\mathcal{R}_A$. Deduce that Θ_t is affine, that is, for all q_{B1}, q_{B2}

$$d\Theta_t(q_{B1}) = d\Theta_t(q_{B2}) \overset{\text{named}}{=} d\Theta_t, \quad \text{and} \quad \Theta_t(q_{B2}) = \Theta_t(q_{B1}) + d\Theta_t.\overrightarrow{q_{B1}q_{B2}}. \tag{17.12}$$

Answer. Straight line as described by observer B: Let $p \in \mathbb{R}^n$, $\vec{w} \in \vec{\mathbb{R}}^n$, and $c : s \in \mathbb{R} \to c(s) = p + s\vec{w} \in \mathbb{R}^n$ (straight line in $\mathcal{R}_B$, in particular $c(0) = p$). Then (17.6) and (17.10) give $\overrightarrow{\Theta_t(c(0))\Theta_t(c(s))} = \overrightarrow{c(0)c(s)} = s\vec{w} \in \mathbb{R}^n$, thus $\Theta_t(c(s)) = \Theta_t(c(0)) + s\vec{w}$, a straight line in $\mathcal{R}_A$. True for all $\vec{w}$. Thus Θ_t is affine. ▪

Exercice 17.5 Setting of exercise 17.4. Observer B chooses a Euclidean basis $(\vec{B}_i)$ at t at O_B, and calls $(\cdot,\cdot)_b$ the associated Euclidean dot product. Let q_{Bi} be the points defined by $\vec{B}_i = \overrightarrow{O_B q_{Bi}}$ as seen by B. Then let $O_{Bt*} := \Theta_t(O_B)$ and $q_{Bit*} := \Theta_t(q_{Bi})$ and $\vec{B}_{it*} := \overrightarrow{O_{Bt*}q_{Bit*}}$, the positions and bipoint vectors as seen by A. Observer A chooses a Euclidean dot product $(\cdot,\cdot)_a$ at t. Let $\lambda > 0$ such that $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$. Deduce from (17.6) and (17.12) that

$$(\vec{B}_{it*}, \vec{B}_{jt*})_a = \lambda^2\delta_{ij}, \quad \text{and} \quad \vec{B}_{it*} = d\Theta_t.\vec{B}_i, \quad \text{and} \quad (d\Theta_t)^T_{ab}.d\Theta_t = \lambda^2 I. \tag{17.13}$$

(In particular, $(\frac{\vec{B}_{it*}}{\lambda})$ is a $(\cdot,\cdot)_a$-Euclidean basis at t.)

Answer. (17.6) gives $\overrightarrow{\Theta_t(O_B)\Theta_t(q_{Bi})} = \overrightarrow{O_{Bt*}q_{Bit*}} = \overrightarrow{O_B q_{Bi}}$, thus $\vec{B}_{it*} = \vec{B}_i$, thus $(\vec{B}_{it*}, \vec{B}_{jt*})_a = (\vec{B}_i, \vec{B}_j)_a = \lambda^2(\vec{B}_i, \vec{B}_j)_b = \lambda^2\delta_{ij}$. Thus $(17.13)_1$.

Then (17.12) gives $\vec{B}_{it*} = d\Theta_t.\vec{B}_i$. Thus $(17.13)_2$, since Θ_t is affine, cf. (17.12).

Then, $d\Theta_t(q_B) = d\Theta_t \in \mathcal{L}(\mathcal{R}_A; \mathcal{R}_B)$ being a linear map, its transposed $(d\Theta_t)^T_{ab}$ relative to the chosen dot products is given by $((d\Theta_t)^T_{ab}.\vec{x}_A, \vec{B}_j)_b = (d\Theta_t.\vec{B}_j, \vec{x}_A)_a$ for all $\vec{x}_A \in \mathcal{R}_A$ and all j, cf. (A.57). Thus $(17.13)_1$ gives $\lambda^2\delta_{ij} = (d\Theta_t.\vec{B}_i, d\Theta_t.\vec{B}_j)_a = ((d\Theta_t)^T_{ab}.d\Theta_t.\vec{B}_i, \vec{B}_j)_b$ for all i, j, with $\delta_{ij} = (\vec{B}_i, \vec{B}_j)_b = (I.\vec{B}_i, \vec{B}_j)_b$. Thus $((d\Theta_t)^T_{ab}.d\Theta_t.\vec{B}_i, \vec{B}_j)_b = \lambda^2(I.\vec{B}_i, \vec{B}_j)_b$ for all i, j, thus $(17.13)_3$. ▪

17.3.2 With an initial time

Consider a $t_0 \in [t_1, t_2]$ (e.g. if you want to consider Lagrangian variables). The motion $\Phi^{t_0}_D$ associated to $\widetilde{\Phi}_D$ is defined by, cf. (3.5),

$$q_{At_0} = \widetilde{\Phi}_D(t_0, Q_{ObjB}) \text{ and } q_{At} = \widetilde{\Phi}_D(t, Q_{ObjB}) \Longrightarrow \Phi^{t_0}_{Dt}(q_{At_0}) := q_{At}. \tag{17.14}$$

Thus (17.7) gives $\Theta_t(q_B) = \Phi^{t_0}_D(q_{At_0}) = \Phi^{t_0}_D(\Theta_{t_0}(q_B))$, so

$$\Theta_t = \Phi^{t_0}_{Dt} \circ \Theta_{t_0}. \tag{17.15}$$

Interpretation: The motion $\widetilde{\Phi}_D$ being known by A, if Θ_{t_0} is known (there has been an initial communication between the observers B and A), then Θ_t is known at all t (e.g., if B does the measurements for A, and if B gives to A its referential at t_0, then A knows the referential of B at all times).

Remark 17.6 (17.7) and (17.14) look alike, but (17.7) has no t_0 (recall: Θ is not a motion). And a space differentiation $d\Theta$ of Θ is meaningful for the translator Θ which acts in $\mathbb{R}^n$ (acts on positions); When a space differentiation for a motion $\widetilde{\Phi}$ cannot be done directly, since $\widetilde{\Phi}$ acts on a material object *Obj* (acts on particles), and there is no differential structure in *Obj*, cf. remark 1.5. In fact, we use a motion to associate to *Obj* a differential structure at any t: in $\Omega_t = \widetilde{\Phi}(t, Obj)$. ▪

17.4 The translator for vector fields

Usual steps with push-forwards. We recall the steps (a vector field is a field of tangent vectors).

Let t be fixed, let c_{Bt} be a curve in $\mathbb{R}^n$ as described by B at t, and let $\vec{w}_{Bt}$ the vector field of tangent vectors to $\mathrm{Im}(c_{Bt})$:

$$c_{Bt}: \left\{\begin{array}{ll}\mathbb{R} & \to \mathcal{R}_B \\ s & \to q_{Bt} = c_{Bt}(s)\end{array}\right\} \quad \text{and} \quad \vec{w}_{Bt}(q_{Bt}) = \vec{w}_{Bt}(c_{Bt}(s)) := \frac{dc_{Bt}}{ds}(s). \tag{17.16}$$

See figure 11.1. (The variable s is a spatial coordinate in the photo taken by B at t.)

Still at t, the curve c_{Bt} is spotted in $\mathcal{R}_A$, by the observer A, as being the (translated) curve

$$c_{Bt*} = \Theta_t \circ c_{Bt}: \left\{\begin{array}{ll}\mathbb{R} & \to \mathcal{R}_A \\ s & \to q_{Bt*} = c_{Bt*}(s) := \Theta_t(q_{Bt}), \quad \text{when} \quad q_{Bt} = c_{Bt}(s),\end{array}\right. \tag{17.17}$$

thus its tangent vectors at $q_{Bt*} = c_{Bt*}(s) = \Theta_t(c_{Bt}(s)) = \Theta_t(q_{Bt})$ are given by

$$\vec{w}_{Bt*}(q_{Bt*}) = \vec{w}_{Bt*}(c_{Bt*}(s)) = \frac{dc_{Bt*}}{ds}(s) = d\Theta_t(c_{Bt}(s)).\frac{dc_{Bt}}{ds}(s) = d\Theta_t(q_{Bt}).\vec{w}_{Bt}(q_{Bt}). \tag{17.18}$$

So (push-forward by Θ_t for vector fields formula (11.25), see figure 11.1),

$$\boxed{\vec{w}_{Bt*}(q_{Bt*}) = d\Theta_t(q_{Bt}).\vec{w}_{Bt}(q_{Bt})} \in \mathcal{R}_A \quad \text{when} \quad q_{Bt*} = \Theta_t(q_B). \tag{17.19}$$

That is, the vector $\vec{w}_{Bt}(q_{Bt})$ seen by B is seen by A as being the vector $\vec{w}_{Bt*}(q_{Bt*})$.

And the pull-back of a vector field $\vec{w}_{At}$ by Θ_t (= the push-forward by ${\Theta_t}^{-1}$) is the translation at t from A to B: It is the vector field ${\vec{w}_{At}}^*$ defined in $\mathcal{R}_B$ by

$${\vec{w}_{At}}^*({q_{At}}^*) = d{\Theta_t}^{-1}(q_{At}).\vec{w}_{At}(q_{At}) \in \mathcal{R}_B, \quad \text{when} \quad {q_{At}}^* = {\Theta_t}^{-1}(q_{At}). \tag{17.20}$$

Exercice 17.7 Prove that, Θ_t being affine, if $p_{At} = \Theta_t(p_{Bt})$ then

$$d\vec{w}_{Bt*}(p_{At}) = d\Theta_t.d\vec{w}_{Bt}(p_{Bt}).d{\Theta_t}^{-1} = (d\vec{w}_{Bt})_*(p_{At}), \tag{17.21}$$

push-forward of the endomorphism $d\vec{w}_{Bt}(p_{Bt})$ by Θ_t, cf. (15.19).

Answer. $\vec{w}_{Bt*}(\Theta_t(p_{Bt})) = d\Theta_t(p_{Bt}).\vec{w}_{Bt}(p_{Bt})$ gives $d\vec{w}_{Bt*}(p_{At}).d\Theta_t(p_{Bt}) = d^2\Theta_t(p_{Bt}).\vec{w}_{Bt}(p_{Bt}) + d\Theta_t(p_{Bt}).d\vec{w}_{Bt}(p_{Bt})$. And Θ_t affine gives $d^2\Theta_t = 0$ and $d\vec{w}_{Bt*}(p_{At}) = d\Theta_t.d\vec{w}_{Bt}(p_{Bt}).d{\Theta_t}^{-1}$. ▪

17.5 The "Θ-velocity" = the drive velocity

Definition 17.8 The vector defined at t at $q_{At} \in \mathcal{R}_A$ by:

$$\vec{v}_\Theta(t, q_{At}) = \frac{\partial\Theta}{\partial t}(t, q_B), \quad \text{when} \quad q_{At} = \Theta_t(q_B), \tag{17.22}$$

is called the "Θ-velocity" at t at q_{At} in $\mathcal{R}_A$.

NB: $\vec{v}_\Theta$ looks like a Eulerian velocity, since $\vec{v}_\Theta(t, \Theta(t, q_B)) = \frac{\partial\Theta}{\partial t}(t, q_B)$, but is not, because Θ is not a motion: Θ is an inter-referential mapping, see remark 17.3.

With (17.11) we get, in $\mathcal{R}_A$,

$$\frac{\partial\Theta}{\partial t}(t, q_B) = \frac{\partial\widetilde{\Phi}_D}{\partial t}(t, Q_{ObjB}) \quad \text{when} \quad q_B = \widetilde{\Psi}_B(Q_{ObjB}), \tag{17.23}$$

that is, with (17.22) and (17.3),

$$\vec{v}_\Theta(t, q_{At}) = \vec{v}_D(t, q_{At}), \quad \text{so} \quad \boxed{\vec{v}_\Theta = \vec{v}_D}. \tag{17.24}$$

Thus the "Θ-velocity" $\vec{v}_\Theta(t, q_{At}) \in \mathcal{R}_A$ is equal to the (Eulerian) drive velocity $\vec{v}_D(t, q_{At}) \in \mathcal{R}_A$, which is the velocity in $\mathcal{R}_A$ of the particle $Q_{ObjB} \in ObjB$ which is at t at q_{At}.

E.g., if $\widetilde{\Phi}_D$ is a rigid body motion, then $\vec{v}_{\Theta t}(q_{At}) = \vec{v}_{Dt}(q_{At}) = \vec{v}_{Dt}(O_A) + \vec{\omega}(t) \wedge \overrightarrow{O_A q_{At}}$, cf. (17.4).

Remark 17.9 We could formally define $\vec{V}_\Theta(t, q_B) := \frac{\partial\Theta}{\partial t}(t, q_B)$ which looks like a "Lagrangian velocity", but is not: A Lagrangian function depends on some motion and on some initial time t_0, and $\vec{V}_\Theta := \frac{\partial\Theta}{\partial t}$ does not. See remark 17.3. ▪

17.6 The velocity-addition formula

Let $P_{Obj} \in Obj$ (a particle of Obj), spotted at t by A at $p_A(t) = \widetilde{\Phi}_A(t, P_{Obj}) \in \mathcal{R}_A$, and by B at $p_B(t) = \widetilde{\Phi}_B(t, P_{Obj}) \in \mathcal{R}_B$. We have $p_A(t) = \Theta(t, p_B(t))$, cf. (17.8), that is,

$$\widetilde{\Phi}_A(t, P_{Obj}) = \Theta(t, \widetilde{\Phi}_B(t, P_{Obj})). \tag{17.25}$$

Thus

$$\underbrace{\frac{\partial \widetilde{\Phi}_A}{\partial t}(t, P_{Obj})}_{=\vec{v}_{At}(p_{At})} = \underbrace{\frac{\partial \Theta}{\partial t}(t, \widetilde{\Phi}_B(t, P_{Obj}))}_{=\vec{v}_{\Theta t}(p_{At})} + \underbrace{d\Theta(t, \widetilde{\Phi}_B(t, P_{Obj})).\frac{\partial \widetilde{\Phi}_B}{\partial t}(t, P_{Obj})}_{=d\Theta_t(p_{Bt}).\vec{v}_{Bt}(p_{Bt})=\vec{v}_{Bt*}(p_{At})}, \tag{17.26}$$

and $\vec{v}_{Bt*}(p_{At}) = d\Theta_t(p_{Bt}).\vec{v}_{Bt}(p_{Bt})$ is the translated velocity at t for observer A at $p_{At} = \Theta_t(p_{Bt})$, cf. (17.19) (the push-forward vector by Θ_t). Thus, for all t and all $p_{At} \in \widetilde{\Phi}_{At}(Obj)$,

$$\vec{v}_{At}(p_{At}) = \vec{v}_{Bt*}(p_{At}) + \vec{v}_{\Theta t}(p_{At}). \tag{17.27}$$

This is "the velocity-addition formula" in $\mathcal{R}_A$, which reads with (17.24),

$$\boxed{\vec{v}_{At} = \vec{v}_{Bt*} + \vec{v}_{Dt}}, \quad \text{where} \quad \vec{v}_{Bt*}(p_{At}) = d\Theta_t(p_{Bt}).\vec{v}_{Bt}(p_{Bt}), \tag{17.28}$$

when $p_{At} = \Theta_t(p_{Bt})$.

Reading: If A is the "absolute observer" and if B is the "relative observer", then in $\mathcal{R}_A$:

$$\begin{aligned}(\vec{v}_{At} \text{ the absolute velocity}) = (&\vec{v}_{Bt*} \text{ the relative velocity translated for } A)\\ &+ (\vec{v}_{Dt} \text{ the drive velocity}).\end{aligned} \tag{17.29}$$

17.7 The Acceleration-addition formula

At t, with $p_{At} = \widetilde{\Phi}_A(t, P_{Obj}) = \Theta_t(p_{Bt}) = \widetilde{\Phi}_{Dt}(Q_{ObjB}) = \Theta_t(q_B)$, define

$$\begin{cases} \vec{\gamma}_A(t, p_{At}) := \dfrac{\partial^2 \widetilde{\Phi}_A}{\partial t^2}(t, P_{Obj}) \quad (\text{acceleration of } P_{Obj} \text{ in } \mathcal{R}_A \text{ at } t),\\ \vec{\gamma}_B(t, p_{Bt}) := \dfrac{\partial^2 \widetilde{\Phi}_B}{\partial t^2}(t, P_{Obj}) \quad (\text{acceleration of } P_{Obj} \text{ in } \mathcal{R}_B \text{ at } t),\\ \vec{\gamma}_\Theta(t, q_{At}) := \dfrac{\partial^2 \Theta}{\partial t^2}(t, q_B) = \dfrac{\partial^2 \widetilde{\Phi}_D}{\partial t^2}(t, Q_{ObjB}) = \vec{\gamma}_D(t, p_{At}) \quad (\text{acceleration of } Q_{ObjB} \text{ in } \mathcal{R}_A \text{ at } t), \end{cases} \tag{17.30}$$

the last equation thanks to (17.23). Then (17.26) gives, with $p_{Bt} = \widetilde{\Phi}_B(t, P_{Obj})$,

$$\begin{aligned}\frac{\partial^2 \widetilde{\Phi}_A}{\partial t^2}(t, P_{Obj}) = &\frac{\partial^2 \Theta}{\partial t^2}(t, p_{Bt}) + d\frac{\partial \Theta}{\partial t}(t, p_{Bt}).\frac{\partial \widetilde{\Phi}_B}{\partial t}(t, P_{Obj})\\ &+ (\frac{\partial d\Theta}{\partial t}(t, p_{Bt}) + d^2\Theta(t, p_{Bt}).\frac{\partial \widetilde{\Phi}_B}{\partial t}(t, P_{Obj})).\frac{\partial \widetilde{\Phi}_B}{\partial t}(t, P_{Obj}) + d\Theta(t, p_{Bt}).\frac{\partial^2 \widetilde{\Phi}_B}{\partial t^2}(t, P_{Obj}).\end{aligned} \tag{17.31}$$

That is, with (17.22), that is $\frac{\partial \Theta}{\partial t}(t, q_B) = \vec{v}_\Theta(t, \Theta_t(q_B))$,

$$\begin{aligned}\vec{\gamma}_{At}(p_{At}) = &\vec{\gamma}_{\Theta t}(p_{At}) + (d\vec{v}_{\Theta t}(p_{At}).d\Theta_t(p_{Bt})).\vec{v}_{Bt}(p_{Bt})\\ &+ ((d\vec{v}_{\Theta t}(p_{At}).d\Theta_t(p_{Bt})) + d^2\Theta_t(p_{Bt}).\vec{v}_{Bt}(p_{Bt})).\vec{v}_{Bt}(p_{Bt}) + d\Theta_t(p_{Bt}).\vec{\gamma}_{Bt}(p_{Bt}).\end{aligned} \tag{17.32}$$

(With the classical setting, Θ_t is affine thus $d^2\Theta_t = 0$.) Thus, with (push-forward by the translator Θ)

$$\begin{cases} \vec{v}_{Bt*}(p_{At}) = d\Theta_t(p_{Bt}).\vec{v}_{Bt}(p_{Bt}) \quad (\text{the velocity } \vec{v}_{Bt}(p_{Bt}) \text{ translated for A}),\\ \vec{\gamma}_{Bt*}(p_{At}) = d\Theta_t(p_{Bt}).\vec{\gamma}_{Bt}(p_{Bt}) \quad (\text{the acceleration } \vec{\gamma}_{Bt}(p_{Bt}) \text{ translated for A}), \end{cases} \tag{17.33}$$

we get

$$\vec{\gamma}_{At}(p_{At}) = \vec{\gamma}_{Bt*}(p_{At}) + \underbrace{\vec{\gamma}_{\Theta t}(p_{At})}_{=\vec{\gamma}_{Dt}(p_{At})} + \underbrace{2((d\vec{v}_{\Theta t}.\vec{v}_{Bt*})(p_{At}) + (d^2\Theta_t(p_{Bt}).\vec{v}_{Bt}(p_{Bt})).\vec{v}_{Bt}(p_{Bt}))}_{=\vec{\gamma}_C(t,p_{At})}. \tag{17.34}$$

Definition 17.10 The Coriolis acceleration $\vec{\gamma}_C$ at t and $p_{At} = \Theta_t(p_{Bt})$ is the Eulerian vector field defined by

$$\vec{\gamma}_C(t, p_{At}) = 2d\vec{v}_{\Theta t}(p_{At}).\vec{v}_{Bt*}(p_{At}) + (d^2\Theta_t(p_{Bt}).\vec{v}_{Bt}(p_{Bt})).\vec{v}_{Bt}(p_{Bt}). \tag{17.35}$$

(With $d\vec{v}_{\Theta t} = d\vec{v}_{Dt}$, cf. (17.24), and with $d^2\Theta_t = 0$ if $\widetilde{\Phi}_D$ is a rigid body motion, cf. (17.12).)

Thus (17.34) reads

$$\boxed{\vec{\gamma}_{At}(p_{At}) = \vec{\gamma}_{Bt*}(p_{At}) + \vec{\gamma}_{Dt}(p_{At}) + \vec{\gamma}_{Ct}(p_{At})} \tag{17.36}$$

Reading: At t, if A is the "absolute observer" and if B is the "relative observer", then in $\mathcal{R}_A$:

$$\begin{aligned} \vec{\gamma}_{At} \text{ the absolute acceleration} = {} & \vec{\gamma}_{Bt*} \text{ the relative acceleration translated for } A \\ & + \vec{\gamma}_{Dt} \text{ the drive acceleration} \\ & + \vec{\gamma}_{Ct} \text{ the Coriolis acceleration.} \end{aligned} \tag{17.37}$$

Example 17.11 Classical setting: $\widetilde{\Phi}_D$ is a rigid body motion, thus $d\vec{v}_{Dt}(p_{At}) = d\vec{v}_{Dt} = \vec{\omega}(t)\wedge$, cf. (17.4), and Θ_t is affine, cf. (17.12), thus $d^2\Theta_t = 0$, thus

$$\vec{\gamma}_{Ct}(p_{At}) = 2\vec{\omega}_t \wedge \vec{v}_{Bt*}(p_{At}), \tag{17.38}$$

usual expression of the Coriolis expression on Earth. ▟

17.8 Inter-referential change of basis formula

$\widetilde{\Phi}_D$ is supposed to be a rigid body motion. So Θ_t is affine, cf. (17.12). (17.19) (and (17.13)) gives in $\mathcal{R}_A$ (inter-referential relation and classical setting)

$$\vec{B}_{jt*} := d\Theta_t.\vec{B}_j \quad \text{(the basis of } B \text{ at any } q_B \text{ as seen by } A \text{ at } t). \tag{17.39}$$

Let $\mathcal{P}_t \in \mathcal{L}(\mathcal{R}_A; \mathcal{R}_A)$ be the change of basis endomorphism from $(\vec{A}_i)$ to $(\vec{B}_{it*})$ in $\mathcal{R}_A$: For all j,

$$\mathcal{P}_t.\vec{A}_j := \vec{B}_{jt*} \overset{\text{named}}{=} \sum_{i=1}^{n} (P_t)^i_j \vec{A}_i, \quad \text{and} \quad [\mathcal{P}_t]_{|\vec{A}} = [(P_t)^i_j], \tag{17.40}$$

thus $(P_t)^i_j$ is the ith component of $\vec{B}_{jt*}$ in the basis $(\vec{A}_i)$. Let $\mathcal{Q}_t := \mathcal{P}_t^{-1}$ (inverse endomorphism in $\mathcal{R}_A$), thus

$$\vec{A}_j = \mathcal{Q}_t.\vec{B}_{jt*} = \mathcal{Q}_t.d\Theta_t.\vec{B}_j. \tag{17.41}$$

and $\mathcal{Q}_t.d\Theta_t \in \mathcal{L}(\mathcal{R}_B; \mathcal{R}_A)$ is the inter-referential change of basis linear map from $(\vec{B}_i)$ to $(\vec{A}_i)$.

Let $\vec{u}_B$ be a vector field in $\mathcal{R}_B$, and $\vec{u}_{Bt*}$ be its translation in $\mathcal{R}_A$, that is, with $q_{Bt*} = \Theta_t(q_B)$,

$$\vec{u}_B(q_B) \overset{\text{named}}{=} \sum_i u^i_B(q_B)\vec{B}_i, \quad \text{and} \quad \vec{u}_{Bt*}(q_{Bt*}) = d\Theta_t.\vec{u}_B(q_B) \overset{\text{named}}{=} \sum_i u^i_{Bt*}(q_{Bt*})\vec{A}_i. \tag{17.42}$$

Proposition 17.12 *Then,*

$$\sum_{i=1}^{n} u^i_{Bt*}(q_{Bt*})\vec{A}_i = \sum_{j=1}^{n} u^j_B(q_B)\vec{B}_{jt*} = \sum_{i,j=1}^{n} (P_t)^i_j u^j_B(q_B)\vec{A}_i, \tag{17.43}$$

thus,

$$[\vec{u}_{Bt*}(q_{Bt*})]_{|\vec{A}} = [\mathcal{P}_t]_{|\vec{A}}.[\vec{u}_B(q_B)]_{|\vec{B}}, \quad \textit{inter-referential change of basis at } t. \tag{17.44}$$

Proof. $\sum_{i=1}^n u^i_{Bt*}(q_{Bt*})\vec{A}_i = \vec{u}_{Bt*}(q_{Bt*}) = d\Theta_t.\vec{u}_B(q_B) = d\Theta_t.(\sum_{j=1}^n u^j_B(q_B)\vec{B}_j) = \sum_{j=1}^n u^j_B(q_B)d\Theta_t.\vec{B}_j = \sum_{j=1}^n u^j_B(q_B)\vec{B}_{jt*} = \sum_{i,j=1}^n u^j_B(q_B)(P_t)^i_j\vec{A}_i$, thus $u^i_{Bts} = \sum_{j=1}^n (P_t)^i_j u^j_B(q_B)$. ▟

Exercice 17.13 Suppose: $\widetilde{\Phi}_D$ is a rigid body motion, Θ_t is affine, and $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ are Euclidean dot products designed by A and B, and $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$. Prove: $\mathcal{P}_t^T.\mathcal{P}_t = \lambda^2 I$.

Answer. $(\mathcal{P}_t.\vec{A}_i, \mathcal{P}_t.\vec{A}_j)_a = (\vec{B}_{it*}, \vec{B}_{jt*})_a = (d\Theta_t.\vec{B}_i, d\Theta_t.\vec{B}_j)_a = \lambda^2(\vec{B}_i, \vec{B}_j)_a = \lambda^2\delta_{ij}$, cf. (17.13). ▟

17.9 A summary: Commutative diagrams

17.9.1 Motions and translator, Eulerian

(17.8) gives $\Theta_t \circ \widetilde{\Psi}_B = \widetilde{\Phi}_{Dt}$ for any t, thus the following diagram commutes:

$$\begin{array}{ccc} & & q_B = \widetilde{\Psi}_B(Q_{ObjB}) \in \mathcal{R}_B \\ & \nearrow^{\widetilde{\Psi}_B} & \\ Q_{ObjB} \in ObjB & & \downarrow \Theta_t \\ & \searrow_{\widetilde{\Phi}_{Dt}} & \\ & & q_{At} = \widetilde{\Phi}_D(t, Q_{ObjB}) = \Theta_t(q_B) \in \mathcal{R}_A. \end{array} \tag{17.45}$$

17.9.2 Motions and translator, Lagrangian

With an initial time t_0. Consider $ObjB$, its motion $\widetilde{\Psi}_B$ in $\mathcal{R}_B$ (fixed motion that gives the positions or the particles in $\mathcal{R}_B$) and its motion $\widetilde{\Phi}_D$ in $\mathcal{R}_A$. We have, cf. (17.15),

$$\Phi^{t_0}_{Dt} \circ \Theta_{t_0} = \Theta_t. \tag{17.46}$$

Slight correction: Observer B is not supposed to have a time ubiquity gift. Thus introduce the time-shift operator $S^{t_0}_t$ in $\mathcal{R}_B$ given by $S^{t_0}_t(q_B) = q_B$. Thus (17.46) in fact reads:

$$\Phi^{t_0}_{Dt} \circ \Theta_{t_0} = \Theta_t \circ S^{t_0}_t. \tag{17.47}$$

Thus the following diagram commutes:

$$\begin{array}{ccccc} & & q_B = \widetilde{\Psi}_B(Q_{ObjB}) & \xrightarrow[S^{t_0}_t]{} & q_B = \widetilde{\Psi}_B(Q_{ObjB}) \\ & \nearrow^{\widetilde{\Psi}_B} & & & \\ Q_{ObjB} \in ObjB & & \downarrow \Theta_{t_0} & & \downarrow \Theta_t \\ & \searrow_{\widetilde{\Phi}_{Dt_0}} & & & \\ & & q_{At_0} = \widetilde{\Phi}_{Dt_0}(Q_{ObjB}) = \Theta_{t_0}(q_B) & \xrightarrow{\Phi^{t_0}_{Dt}} & q_{At} = \widetilde{\Phi}_{Dt}(P_{Obj}) = \Phi^{t_0}_{Dt}(p_{At_0}) = \Theta_t(q_B) \end{array} \tag{17.48}$$

(Top line in $\mathcal{R}_B$ with $S^{t_0}_t$ the time shift, bottom line in $\mathcal{R}_A$, and translation between lines.)

Consider Obj, its motion $\widetilde{\Phi}_B$ in $\mathcal{R}_B$ and its motion $\widetilde{\Phi}_A$ in $\mathcal{R}_A$. Let $\Phi^{t_0}_{Bt} : \mathcal{R}_B \to \mathcal{R}_B$ and $\Phi^{t_0}_{At} : \mathcal{R}_A \to \mathcal{R}_A$ be the associated motions: Defined by $\Phi^{t_0}_{Bt} \circ \widetilde{\Phi}_{Bt_0} = \widetilde{\Phi}_{Bt}$ and $\Phi^{t_0}_{At} \circ \widetilde{\Phi}_{At_0} = \widetilde{\Phi}_{At}$, cf. (3.6), that is, $p_{Bt} = \Phi^{t_0}_{Bt}(p_{Bt_0}) = \widetilde{\Phi}_{Bt}(P_{Obj})$ when $p_{Bt_0} = \widetilde{\Phi}_{Bt_0}(P_{Obj})$ and $p_{At} = \Phi^{t_0}_{At}(p_{At_0}) = \widetilde{\Phi}_{At}(P_{Obj})$ when $p_{At_0} = \widetilde{\Phi}_{At_0}(P_{Obj})$. So we have $p_{At_0} = \Theta_{t_0}(p_{Bt_0})$ and $p_{At} = \Theta_{t_0}(p_{Bt})$, thus $\Phi^{t_0}_{At}(\Theta_{t_0}(p_{Bt_0})) = \Theta_t(\Phi^{t_0}_{Bt}(p_{Bt_0}))$, so

$$\Phi^{t_0}_{At} \circ \Theta_{t_0} = \Theta_t \circ \Phi^{t_0}_{Bt}. \tag{17.49}$$

Thus the following diagram commutes:

$$\begin{array}{ccccc} & & p_{Bt_0} = \widetilde{\Phi}_{Bt_0}(P_{Obj}) & \xrightarrow[\Phi^{t_0}_{Bt}]{} & p_{Bt} = \Phi^{t_0}_{Bt}(p_{Bt_0}) = \widetilde{\Phi}_{Bt}(P_{Obj}) \\ & \nearrow^{\widetilde{\Phi}_{Bt_0}} & & & \\ P_{Obj} \in Obj & & \Theta_{t_0} \downarrow & & \Theta_t \downarrow \\ & \searrow_{\widetilde{\Phi}_{At_0}} & & & \\ & & p_{At_0} = \widetilde{\Phi}_{At_0}(P_{Obj}) = \Theta_{t_0}(p_{Bt_0}) & \xrightarrow{\Phi^{t_0}_{At}} & p_{At} = \widetilde{\Phi}_{At}(P_{Obj}) = \Phi^{t_0}_{At}(p_{At_0}) = \Theta_t(p_{Bt}) \end{array} \tag{17.50}$$

(Top line in $\mathcal{R}_B$, bottom line in $\mathcal{R}_A$, and translation between lines.)

17.9.3 Differentials

(17.46) gives, for $q_B \in \mathcal{R}_B$ and $q_{Bt_0*} = \Theta_{t_0}(q_B) \in \mathcal{R}_A$,

$$d\Phi^{t_0}_{Dt}(q_{Bt_0*}).d\Theta_{t_0}(q_B) = d\Theta_t(q_B). \tag{17.51}$$

Let $\vec{u}_B$ be a (stationary) vector field in $\mathcal{R}_B$. Its push-forwards by Θ_{t_0} and Θ_t (translations from B to A) are, cf. (17.19),

$$\begin{cases} \vec{u}_{Bt_0*}(q_{Bt_0*}) = d\Theta_{t_0}(q_B).\vec{u}_B(q_B), \quad \text{when} \quad q_{Bt_0*} = \Theta_{t_0}(q_B), \\ \vec{u}_{Bt*}(q_{Bt*}) = d\Theta_t(q_B).\vec{u}_B(q_B), \quad \text{when} \quad q_{Bt*} = \Theta_t(q_B). \end{cases} \tag{17.52}$$

Then (17.51) gives :

$$d\Phi^{t_0}_{Dt}(q_{Bt_0*}).\vec{u}_{Bt_0*}(q_{Bt_0*}) = \vec{u}_{Bt*}(q_{Bt*}), \tag{17.53}$$

and then $\vec{u}_{Bt*}$ is the push-forward of $\vec{u}_{Bt_0*}$ by $\Phi^{t_0}_{Dt}$. Thus the commutative diagram (17.51) reads

$$\begin{array}{ccc} & \overset{d\Theta_{t_0}(q_B)}{\nearrow} & \vec{u}_{Bt_0*}(q_{Bt_0*}) = (d\Theta_{t_0}.\vec{u}_B)(q_B) \in \mathcal{R}_A \\ \vec{u}_B(q_B) \in \mathcal{R}_B & & \downarrow d\Phi^{t_0}_{Dt}(q_{Bt_0*}) \\ & \underset{d\Theta_t(q_B)}{\searrow} & \vec{u}_{Bt*}(q_{Bt*}) = (d\Theta_t.\vec{u}_B)(q_B) = d\Phi^{t_0}_{Dt}(q_{Bt*}).\vec{u}_{Bt_0*}(q_{Bt_0*}) \in \mathcal{R}_A. \end{array} \tag{17.54}$$

18 Coriolis force

18.1 Fundamental principal: In a Galilean referential

The second Newton's law of motion (fundamental principle of dynamics) tells: If you are in a Galilean referential and you quantify vectors (you measure the components relative to your basis), then the sum of the external forces $\vec{f}$ on an object is equal to the mass of this object multiplied by its acceleration:

$$\sum \text{external} \vec{f} = m\vec{\gamma} \quad \text{(Galilean referential)}. \tag{18.1}$$

Remark 18.1 This result is observer dependent (quantification that requires a Galilean setup): The acceleration depends on the observer, cf. (17.36), while usually $\vec{f}$ can is an objective vector field (forces independent of an observer), see next § 18.2. (The explicit definition of objectivity is given at § 19).

In a Non Galilean referential? Then you have to add "observer dependent forces" = the inertial and Coriolis forces ("apparent forces"). The following § 18.2 details this.

E.g., your referential $\mathcal{R}_B$ is fixed on a carousel (a spinning merry-go-round); If you sit still in $\mathcal{R}_B$ then $\vec{\gamma}_B = \vec{0}$: Your acceleration relative to $\mathcal{R}_B$ vanishes; But you feel an external force $\vec{f}_B \neq \vec{0}$ (a "centrifugal force"), hence $\vec{f}_B \neq m\vec{\gamma}_B$: (18.1) does not apply. But here $\mathcal{R}_B$ isn't Galilean. Whereas after a change toward a Galilean referential $\mathcal{R}_A$, you get $\vec{f}_A = m\vec{\gamma}_A$, that is $\vec{f}_A + \vec{f}_{\text{ref.dep.}} = m\vec{\gamma}_{B*} \in \mathcal{R}_A$ where $\vec{f}_{\text{ref.dep.}} = -m(\vec{\gamma}_\Theta + \vec{\gamma}_C)$ (referential dependent) and $\vec{\gamma}_{B*} = \Theta_{t*}\vec{\gamma}_B$ (the translated acceleration), cf. (17.36). ∴

18.2 Inertial and Coriolis forces, and Fundamental Principle

Let $\mathcal{R}_A$ be a Galilean referential and $\mathcal{R}_B$ be any referential.

Let $\vec{f}_{At}(p_{At})$ be the sum, as measured in $\mathcal{R}_A$, of the external forces acting on a particle $P_{Obj} \in Obj$. Thus (18.1) (Newton) gives, at all t,

$$\vec{f}_{At} = m\vec{\gamma}_{At} \in \mathcal{R}_A \quad \text{(Galilean referential)}. \tag{18.2}$$

Then let Θ_t be the translator from $\mathcal{R}_A$ to $\mathcal{R}_B$, cf. (17.7). The addition-acceleration formula (17.36) gives

$$\vec{f}_{At} = m(\vec{\gamma}_{Bt*} + \vec{\gamma}_{\Theta_t} + \vec{\gamma}_{Ct}) \in \mathcal{R}_A. \tag{18.3}$$

Notation: Then let $\vec{f}_B := \vec{f}_{At}{}^*$ be the pull-back of $\vec{f}_{At}$ by the translator Θ_t, cf. (17.20), that is, at any time t and with $p_{At} = \Theta_t(p_{Bt})$,

$$\vec{f}_{Bt}(p_{Bt}) := d\Theta_t(p_{Bt})^{-1}.\vec{f}_{At}(p_{At}) \ \in \mathcal{R}_B, \tag{18.4}$$

cf. (17.19)-(17.20). Thus (18.3) gives

$$\vec{f}_{Bt}(p_{Bt}) = m\, d\Theta_t(p_{Bt})^{-1}.(\vec{\gamma}_{Bt*}(q_{At}) + \vec{\gamma}_{\Theta_t}(q_{At}) + \vec{\gamma}_{Ct}(q_{At})) \tag{18.5}$$

with the pull-backs notation, that is, $\vec{\gamma}^*(p_{Bt}) = d\Theta_t(p_{Bt})^{-1}.\vec{\gamma}(q_{At})$ (translation from A to B for vector fields, cf. (17.20)). Thus

$$\vec{f}_{Bt}(p_{Bt}) - m\vec{\gamma}_{\Theta_t}{}^*(p_{Bt}) - m\vec{\gamma}_{Ct}{}^*(p_{Bt}) = m\vec{\gamma}_{Bt}(p_{Bt}) \ \in \mathcal{R}_B. \tag{18.6}$$

Definition 18.2

$$\begin{cases} \vec{f}_{it}(p_{Bt}) = -m\vec{\gamma}_{\Theta_t}{}^*(p_{Bt}) = \text{ inertial force in } \mathcal{R}_B \text{ at } t, \\ \vec{f}^*_{Ct}(p_{Bt}) = -m\vec{\gamma}_{Ct}{}^*(p_{Bt}) = \text{ Coriolis force in } \mathcal{R}_B \text{ at } t. \end{cases} \tag{18.7}$$

They are called "fictitious forces" (do not exist in a Galilean referential, are not objective).

Then (18.6) gives the fundamental principle in a non Galilean referential:

$$\vec{f}_{Bt}(p_{Bt}) + \vec{f}_{it}(p_{Bt}) + \vec{f}_{Ct}(p_{Bt}) = m\vec{\gamma}_{Bt}(p_{Bt}) \quad \text{(non Galilean referential).} \tag{18.8}$$

See e.g. villemin.gerard.free.fr/Scienmod/Coriolis.htm,
planet-terre.ens-lyon.fr/article/force-de-coriolis.xml.

19 Objectivities

Framework of § 17 (classical mechanics: The time scale is the same for all users).

The goal is to give an objective expression of the laws of mechanics (observer independent), so that two observers, quantifying relative to their referentials, will obtain results which they can communicate to each other (thanks to the translator).

Illustration: observer A chooses a referential $\mathcal{R}_A = (\mathcal{O}_A, (\vec{A}_i))$ fixed on the Sun with $(\vec{A}_i)$ a Euclidean basis in foot, observer B chooses a referential $\mathcal{R}_B = (\mathcal{O}_B, (\vec{B}_i))$ fixed on the Earth with $(\vec{B}_i)$ a Euclidean basis in meter, and $\Theta_t : \mathcal{R}_B \to \mathcal{R}_A$ is the translator cf. (17.8).

Remark: The functions involved are Eulerian functions (objectivity cannot depend on the choice of an initial time t_0 which depends on an observer); If Lagrangian associated expressions are needed, they are deduced a posteriori.

19.1 Covariant objectivity of a scalar function

Let f be a Eulerian scalar function corresponding to a "physical quantity" (e.g., temperature).

The observers A and B describe f in their referential as being the functions f_A : $\left\{\begin{array}{ll} \mathbb{R}\times\mathcal{R}_A & \to \mathbb{R} \\ (t,p_A) & \to f_A(t,p_A) \end{array}\right\}$ and f_B : $\left\{\begin{array}{ll} \mathbb{R}\times\mathcal{R}_B & \to \mathbb{R} \\ (t,p_B) & \to f_B(t,p_B) \end{array}\right\}$. At t fixed, let $f_{At}(p) := f_A(t,p)$ and $f_{Bt}(p) := f_B(t,p)$.

Definition 19.1 The "physical quantity f" is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all t,

$$f_{At} = \Theta_{t*} f_{Bt} = \text{push-forward by } \Theta_t, \text{ cf. (11.5)}, \tag{19.1}$$

that is,

$$f_{At}(p_{At}) = f_{Bt}(p_{Bt}) \quad \text{when} \quad p_{At} = \Theta_t(p_{Bt}). \tag{19.2}$$

Or iff f_{Bt} is the pull-back of f_{At} by Θ_t, cf. (11.10), that is, $f_{Bt} = \Theta_t{}^* f_{At}$.

And then f_A and f_B are denoted f.