# Objectivity in continuum mechanics, an introduction

## Motions, Eulerian and Lagrangian variables and functions, deformation gradient, Lie derivatives, velocity-addition formula, Coriolis, and mathematical foundations (appendix).

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In classical mechanics, there are two objectivities: 1- The covariant objectivity concerns the universal laws of physics required to be observer independent (true in any reference frame); This is a main topic in this manuscript. 2- The isometric objectivity concerns the constitutive laws of materials once expressed in a reference frame.

Covariant objectivity in continuum mechanics follows Maxwell's requirements, cf. [18] page 1: "2. (...) The formula at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result. (...) 10. (...) The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in mathematical progress, for it reduced the methods of geometry to calculations performed on numerical quantities. The position of a point is made to depend on the length of three lines which are always drawn in determinate directions (...) But for many purposes in physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other,..."

And see the (short) historical note given in the introduction of Abraham and Marsden book "Foundations of Mechanics" [1], about qualitative versus quantitative theory: "Mechanics begins with a long tradition of qualitative investigation culminating with Kepler and Galileo. Following this is the period of quantitative theory (1687-1889) characterized by concomitant developments in mechanics, mathematics, and the philosophy of science that are epitomized by the works of Newton, Euler, Lagrange, Laplace, Hamilton, and Jacobi. (...) For celestial mechanics (...) resolution we owe to the genius of Poincaré, who resurrected the qualitative point of view (...) One advantage (...) is that by suppressing unnecessary coordinates the full generality of the theory becomes evident."

After having (unambiguously) defined motions, Eulerian and Lagrangian variables and functions, we give the definition of the deformation gradient as a function. We then obtain a simple understanding of the Lie derivatives of vector fields which meet the needs of engineers. Then we get the velocity addition formula and verify that the Lie derivatives are (covariant) objective. (Maybe Cauchy would have used Lie derivatives if they had existed during his lifetime: To get a stress, Cauchy compares two vectors thanks to a Euclidean dot product, whereas one vector is enough when using Lie derivatives.) And a fairly long appendix tries to give in one manuscript the usual simple and precise definitions, properties and interpretations, usually scattered across several books (and not always that easy to find).

We systematically start with qualitative definitions (observer independent), then quantify with bases and/or Euclidean dot products (observer dependent).

# Contents

A quantity $f$ being given then: $g$ defined by « $g$ equals $f$ » is noted $g := f$.

# Part I
# Motions, Eulerian and Lagrangian descriptions, flows

## 1 Motions

The framework is classical mechanics, time being decoupled from space. $\mathbb{R}^3$ is the classical geometric affine space (the space we live in), and $(\vec{\mathbb{R}^3},+,.) = \{\vec{pq} : p,q \in \mathbb{R}^3\} =^{\text{written}} \vec{\mathbb{R}^3}$ is the associated vector space of bipoint vectors equipped with its usual rules. We also consider $\mathbb{R}$ and $\mathbb{R}^2$ as subspaces of $\mathbb{R}^3$, i.e. we consider $\mathbb{R}^n$ and $\vec{\mathbb{R}^n}$, $n = 1,2,3$.

### 1.1 Referential

**Origin:** An observer chooses an origin $\mathcal{O} \in \mathbb{R}^n$; Thus a point $p \in \mathbb{R}^n$ can be located by the observer thanks to the bipoint vector $\overrightarrow{\mathcal{O}p} = \vec{x} \in \vec{\mathbb{R}^n}$, and we write $p = \mathcal{O} + \vec{x}$, and $\vec{x} = p - \mathcal{O}$ (drawing).

**Cartesian coordinate system:** At $p \in \mathbb{R}^n$, an observer chooses a basis $(\vec{e}_i(p)) := (\vec{e}_i(p))_{i=1,...,n}$ in $\vec{\mathbb{R}^n}$. The basis is Cartesian if it is independent of $p$ (possible in classical mechanics), and we write $(\vec{e}_i(p)) = (\vec{e}_i)$. With a chosen origin $O$, the observer has built the "Cartesian coordinate system" $\mathcal{R}_{\text{Cart}} = (\mathcal{O}, (\vec{e}_i)_{i=1,...,n})$. His location of a point $p \in \mathbb{R}^n$:

$$p = \mathcal{O} + \vec{x} \quad \text{with} \quad \vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i, \quad \text{i.e.} \quad [\overrightarrow{\mathcal{O}p}]_{|\vec{e}} = [\vec{x}]_{|\vec{e}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \tag{1.1}$$

$[\vec{x}]_{|\vec{e}} = [\overrightarrow{\mathcal{O}p}]_{|\vec{e}}$ being the column matrix containing the components of $\overrightarrow{\mathcal{O}p} = \vec{x}$ in the basis $(\vec{e}_i)$.

**Chronology:** A chronology (or temporal coordinate system) is a set $\mathcal{R}_{\text{time}} = (t_0, (\Delta t))$ chosen by an observer, where $t_0 \in \mathbb{R}$ is the time origin, and $(\Delta t)$ is the time unit (a basis in $\vec{\mathbb{R}}$).

**Referentiel:** A referential $\mathcal{R}$ is the set

$$\mathcal{R} = (\mathcal{R}_{\text{time}}, \mathcal{R}_{\text{Cart}}) = (t_0, (\Delta t), \mathcal{O}, (\vec{e}_i)_{i=1,...,n}) = (\text{“chronologie”,“Cartesian coordinate system”}), \tag{1.2}$$

made of a chronology and a Cartesian coordinate system, chosen by an observer.

In the following, to simplify the writings, the same implicit chronology is used by all observers, and a referential will simply be written $\mathcal{R} = (\mathcal{O}, (\vec{e}_i))$.

**Remark 1.1** Another observer chooses an origin $\widetilde{O}$ and a Cartesian basis $(\vec{b}_i)$ to build $\widetilde{\mathcal{R}_{\text{Cart,b}}} = (\widetilde{\mathcal{O}}, (\vec{b}_i)_{i=1,...,n})$. His location of a point $p \in \mathbb{R}^n$ is $p = \widetilde{\mathcal{O}} + \vec{y}$. Thus $\vec{y} = \overrightarrow{\widetilde{\mathcal{O}}\mathcal{O}} + \vec{x}$ gives the relation between the stored vectors $\vec{x}$ and $\vec{y}$ (drawing). ▪

### 1.2 Einstein's convention (duality notation)

Starting point: The classical notation $x_i$ for the components of a vector $\vec{x}$ relative to a basis, cf. (1.1). Then the duality notion is introduced: $x_i =^{\text{written}} x^i$ (enables to see the difference between a vector and a function when using components). So

$$\vec{x} = \underbrace{\sum_{i=1}^{n} x_i \vec{e}_i}_{\text{classic not.}} = \underbrace{\sum_{i=1}^{n} x^i \vec{e}_i}_{\text{duality not.}}, \quad \text{and} \quad [\vec{x}]_{|\vec{e}} \overset{\text{clas.}}{=} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \overset{\text{dual}}{=} \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}. \tag{1.3}$$

The duality notation is part of the Einstein's convention; Moreover Einstein's convention uses the notation $\sum_{i=1}^{n} x^i \vec{e}_i =^{\text{written}} x^i \vec{e}_i$, i.e. the sum sign $\sum_{i=1}^{n}$ can be omitted when an index ($i$ here) is used twice, once up and once down, details at § A.5. However this omission of the sum sign $\sum$ will not be made in this manuscript (to avoid ambiguities): The TEX-LATEX program makes it easy to print $\sum_{i=1}^n$.

**Example 1.2** The height of a child is represented on a wall by a vertical bipoint vector $\vec{x}$ starting from the ground up to a pencil line. Question: What is the size of the child ?

Answer: It depends... on the observer (quantitative value = subjective result). E.g., an English observer chooses a vertical basis vector $\vec{a}_1$ which length is one English foot (ft). So he writes $\vec{x} = x_1\vec{a}_1$, and for him the size of the child (size of $\vec{x}$) is $x_1$ in foot. E.g. $x_1 = 4$ means the child is 4 ft tall. A French observer chooses a vertical basis vector $\vec{b}_1$ which length is one metre (m). So he writes $\vec{x} = y_1\vec{b}_1$, and for him the size of the child (size of $\vec{x}$) is $y^1$ metre. E.g., if $x_1 = 4$ then $y_1 \simeq 1.22$, since 1 ft := 0.3048 $m$: The child is both 4 and 1.22 tall... in foot or metre. This quantification is written $\vec{x} = 4$ ft = 1.22 m, where ft means $\vec{a}_1$ and m means $\vec{b}_1$ here. NB: The qualitative vector $\vec{x}$ is the same vector for all observers, not the quantitative values 4 or 1.22 (depends on a choice of a unit of measurement).

With duality notation: $\vec{x} = x^1\vec{a}_1 = y^1\vec{b}_1$, so if $x^1 = 4$ then $y^1 \simeq 1.22$. ▪

This manuscript insists on covariant objectivity; Thus an English engineer (and his foot) and a French engineer (and his metre) will be able to work together ... and be able to avoid crashes like that of the Mars Climate Orbiter probe, see remark A.17. And they will be able to use the results of Galileo, Descartes, Newton, Euler... who used their own unit of length, and knew nothing about the metre defined in 1793 and adopted in 1799 in France (after 6 years of measurements), and considered by the scientific community at the end of the ninetieth century... and couldn't explicitly use the "Euclidean dot products" either (which seems to have been defined mathematically by Grassmann around 1844).

## 1.3 Motion of an object

Let *Obj* be a "real object", or "material object", made of particles (e.g., the Moon: Exists independently of an observer). Let $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$.

**Definition 1.3** The motion of *Obj* in $\mathbb{R}^n$ is the map

$$\widetilde{\Phi} : \begin{cases} [t_1, t_2] \times Obj & \to \mathbb{R}^n \\ (t, \underbrace{P_{Obj}}_{\text{particle}}) & \to \underbrace{p = \widetilde{\Phi}(t, P_{Obj})}_{\text{its position at } t \text{ in the Universe}} \end{cases}. \tag{1.4}$$

And $t$ is the time variable, $p$ is the space variable, and $(t, p) \in \mathbb{R} \times \mathbb{R}^n$ is the time-space variable. And $\widetilde{\Phi}$ is supposed to be $C^2$ in time.

With an origin $\mathcal{O}$ (observer dependent), the motion can be described with the bi-point vector

$$\vec{x} = \overrightarrow{\mathcal{O}\widetilde{\Phi}(t, P_{Obj})} = \overrightarrow{\mathcal{O}p} \overset{\text{written}}{=} \widetilde{\vec{\varphi}}(t, P_{Obj}). \tag{1.5}$$

But then, two observers with different origins $\mathcal{O}$ and $\mathcal{O}_b$ have different description of the motion. Therefore, in the following we won't use $\widetilde{\vec{\varphi}}$. Then (quantification) with a Cartesian basis $(\vec{e}_i)$ to make a referential $\mathcal{R}$, we get (1.1).

## 1.4 Virtual and real motion

**Definition 1.4** A virtual (or possible) motion of *Obj* is a function $\widetilde{\Phi}$ "regular enough for the calculations to be meaningful". Among all the virtual motions, the observed motion is called the real motion.

## 1.5 Hypotheses (Newton and Einstein)

**Hypotheses of Newtonian mechanics (Galileo relativity) and general relativity (Einstein)**:

1- You can describe a phenomenon only at the actual time $t$ and from the location $p$ you are at (you have **no** gift of ubiquity in time or space);

2- You don't know the future;

3- You can use your memory, so use some past time $t_0$ and some past position $p_{t_0}$;

4- You can use someone else memory (results of measurements) if you can communicate objectively.

## 1.6 Configurations

If $t$ is fixed then (1.3) defines

$$\widetilde{\Phi}_t : \left\{ \begin{array}{ll} Obj & \to \mathbb{R}^n \\ P_{Obj} & \mapsto p = \widetilde{\Phi}_t(P_{Obj}) := \widetilde{\Phi}(t, P_{Obj}) \end{array} \right\} \quad \text{and} \quad \Omega_t := \widetilde{\Phi}_t(Obj). \tag{1.6}$$

**Definition 1.5** $\Omega_t := \widetilde{\Phi}_t(Obj)$ is the "configuration at $t$" (photo at $t$) of $Obj$ (range = image of $\widetilde{\Phi}_t$):

$$\Omega_t := \{p \in \mathbb{R}^n : \exists P_{Obj} \in Obj \text{ s.t. } p = \widetilde{\Phi}_t(P_{Obj})\} = \widetilde{\Phi}_t(Obj) \text{ (affine subset)}. \tag{1.7}$$

If $t$ is the actual time then $\Omega_t$ is the actual (or current or Eulerian) configuration.
If $t_0$ is a time in the past then $\Omega_{t_0}$ is the past (or initial or Lagrangian) configuration.

**Hypothesis:** At any time $t$, $\Omega_t$ is supposed to be a "smooth domain" in $\mathbb{R}^n$, and the map $\widetilde{\Phi}_t$ is assumed to be one-to-one (= injective): $Obj$ does not crash onto itself.

## 1.7 Definition of the Eulerian and Lagrangian variables

• If $t$ is the actual time, then $p_t = \widetilde{\Phi}_t(P_{Obj}) \in \Omega_t$ is called the Eulerian variable relative to $P_{Obj}$ and $t$.

• If $t_0$ is a time in the past, then $p_{t_0} = \widetilde{\Phi}_{t_0}(P_{Obj}) \in \Omega_{t_0}$ is called the Lagrangian variable relative to $P_{Obj}$ and $t_0$. (A Lagrangian variable is a "past Eulerian variable").

NB: Two observers with two different time origins $t_0$ and $t_0{}'$ get two different Lagrangian variables $p_{t_0}$ and $p_{t_0{}'}$ while they have the same Eulerian variable $p_t$.

## 1.8 Trajectories

Let $\widetilde{\Phi}$ be a motion of $Obj$, cf. (1.4), and $P_{Obj} \in Obj$ (a particle in $Obj$ = e.g. the Moon).

**Definition 1.6** The (parametric) trajectory of $P_{Obj}$ is the function

$$\widetilde{\Phi}_{P_{Obj}} : \left\{ \begin{array}{ll} [t_1, t_2] & \to \mathbb{R}^n, \\ t & \mapsto p(t) = \widetilde{\Phi}_{P_{Obj}}(t) := \widetilde{\Phi}(t, P_{Obj}) \quad \text{(position of } P_{Obj} \text{ at } t \text{ in the Universe).} \end{array} \right. \tag{1.8}$$

Its geometric trajectory is the range (image) of $\widetilde{\Phi}_{P_{Obj}}$, i.e.

$$\text{geometric trajectory of } P_{Obj} := \{q \in \mathbb{R}^n : \exists t \in [t_1, t_2] \text{ s.t. } q = \widetilde{\Phi}_{P_{Obj}}(t)\} = \text{Im}(\widetilde{\Phi}_{P_{Obj}}) = \widetilde{\Phi}_{P_{Obj}}([t_1, t_2]). \tag{1.9}$$

## 1.9 Pointed vector, tangent space, fiber, vector field, bundle

(See e.g. Abraham–Marsden [1].) In particular to deal with surfaces $S$ in $\mathbb{R}^3$ (e.g. $S$ a sphere), a tangent vector to $S$ isn't simply a "bi-point vector connecting two points of $S$" (would get "through the surface"). To define a tangent vector to $S$, or on $S$, let $p \in S$, consider a regular curve $c : s \in ]-\varepsilon, \varepsilon[ \to c(s) \in S$ s.t. $c(0) = p$, and let $\vec{w}(p) := \vec{c}\,'(0) = \lim_{h\to 0} \frac{c(h)-c(0)}{h}$: This vector is tangent ot $S$ at $p$. With all possible curves, you get all the tangent vectors ot $S$. And, for all $p \in S$, the tangent space to $S$ at $p$ is

$$T_pS := \{\text{set of tangent vectors to } S \text{ at } p\} \quad \text{(it is a vector space).} \tag{1.10}$$

E.g., if $S$ is a sphere in $\mathbb{R}^3$ and $p \in S$, then $T_pS$ is its usual tangent plane to $S$ at $p$.
E.g., particular case: If $S = \Omega$ is an open set in $\mathbb{R}^n$, then $T_pS = T_p\Omega = \vec{\mathbb{R}^n}$ is independent of $p$.

**Definition 1.7**

$$\text{The fiber at } p := \{p\} \times T_pS = \{ \underbrace{(p, \vec{w}_p)}_{\text{pointed vector}} \in \{p\} \times T_pS\}, \tag{1.11}$$

i.e., the fiber at $p$ is the set of "pointed vectors at $p$", a pointed vector being the couple $(p, \vec{w}_p)$ made of the "base point" $p$ and the vector $\vec{w}_p$ defined at $p$.

Calculation rules in a fiber: if $(p,\vec{u}_p),(p,\vec{w}_p) \in \{p\} \times T_pS$ and if $\lambda \in \mathbb{R}$ then

$$(p,\vec{u}_p+\lambda\vec{w}_p) := (p,\vec{u}_p)+(p,\lambda\vec{w}_p). \tag{1.12}$$

(You can only add vectors at one point $p$.)

Drawing: A "pointed vectors $(p,\vec{w}_p)$ at $p$" has to be drawn at the point $p$ in $\mathbb{R}^n$. (While a vector $\vec{w}$ can be drawn anywhere and is called a free-vector.)

If the context is clear, a pointed vector is simply noted $\widetilde{\vec{w}}(p) =^{\text{written}} \vec{w}(p)$ (lighten the writing).

Particular case: If $S=\Omega$ is an open set in $\mathbb{R}^n$, then $T_p\Omega = \vec{\mathbb{R}^n}$ and the fiber at $p$ is $\{p\} \times \vec{\mathbb{R}^n}$.

**Definition 1.8**

$$\text{The tangent bundle } TS := \bigcup_{p\in S} (\{p\} \times T_pS), \tag{1.13}$$

that is, is the union of the fibers.

**Definition 1.9** A vector field $\widetilde{\vec{w}}$ in $S$ is a regular function (at least $C^2$ in the following) of pointed vectors:

$$\widetilde{\vec{w}} : \begin{cases} S & \to TS \\ p & \to \widetilde{\vec{w}}(p) = (p, \vec{w}(p)). \end{cases} \tag{1.14}$$

If the context is clear, a vector field is simply noted $\widetilde{\vec{w}} =^{\text{written}} \vec{w}$ (lighten the writing).

# 2 Eulerian description (spatial description at actual time $t$)

## 2.1 The set of configurations

Let $\widetilde{\Phi}$ be a motion of *Obj*, cf. (1.4), and $\Omega_t = \widetilde{\Phi}_t(Obj) \subset \mathbb{R}^n$ be the configuration at $t$, cf. (1.7). The set of configurations is the subset $\mathcal{C} \subset \mathbb{R} \times \mathbb{R}^n$ (the "time-space sub-set") defined by

$$\begin{aligned} \mathcal{C} &:= \bigcup_{t\in[t_1,t_2]} (\{t\} \times \Omega_t) \quad (= \text{set of "time-space positions"}) \\ &= \{(t,p) \in \mathbb{R} \times \mathbb{R}^n : \exists (t, P_{Obj}) \in [t_1,t_2] \times Obj,\ p = \widetilde{\Phi}(t, P_{Obj})\}, \end{aligned} \tag{2.1}$$

Question: Why don't we simply use $\bigcup_{t\in[t_1,t_2]} \Omega_t$ instead of $\mathcal{C} = \bigcup_{t\in[t_1,t_2]}(\{t\}\times\Omega_t)$?

Answer: $\bigcup_{t\in[t_1,t_2]} \Omega_t$ is the superposition of all the photos on one image; While $\mathcal{C}$ gives the film of the life of *Obj* = the succession of the photos $\Omega_t$ taken at each $t$.

## 2.2 Eulerian variables and functions

**Definition 2.1** In short: A Eulerian function relative to *Obj* is a function, with $m \in \mathbb{N}^*$,

$$\mathcal{E}ul : \begin{cases} \mathcal{C} & \to \text{ some tensorial set } S \\ (t,p) & \to \mathcal{E}ul(t,p), \end{cases} \tag{2.2}$$

the spatial variable $p$ being the Eulerian variable. Precisely: A function $\mathcal{E}ul$ being given as in (2.2), the associated Eulerian function $\widehat{\mathcal{E}ul}$ is the "pointed time-space" function

$$\widehat{\mathcal{E}ul} : \begin{cases} \mathcal{C} & \to \mathcal{C} \times S \\ (t,p) & \to \widehat{\mathcal{E}ul}(t,p) = ((t,p), \mathcal{E}ul(t,p)) = (\text{time-space position , value}), \end{cases} \tag{2.3}$$

and is called "a field of functions". So $\widehat{\mathcal{E}ul}(t,p)$ is the "pointed $\mathcal{E}ul(t,p)$ at $(t,p)$" (in time-space).

NB: the range $\mathrm{Im}(\widehat{\mathcal{E}ul}) = \widehat{\mathcal{E}ul}(\mathcal{C})$ of an Eulerian function $\widehat{\mathcal{E}ul}$ is the graph of $\mathcal{E}ul$. (Recall: The graph of a function $f : x \in A \to f(x) \in B$ is the subset $\{(x,f(x)) \in A \times B\} \subset A \times B$: gives the "drawing of $f$"). And the Eulerian vector field at $t$ is

$$\widehat{\mathcal{E}ul}_t : \begin{cases} \Omega_t & \to \Omega_t \times S \\ p & \to \widehat{\mathcal{E}ul}_t(p) := (p, \mathcal{E}ul_t(p)) = (\text{position , value) at t}. \end{cases} \tag{2.4}$$

If there is no ambiguity, $\widehat{\mathcal{E}ul} =^{\text{written}} \mathcal{E}ul$ for short.

**Example 2.2** $\mathcal{E}ul(t,p) = \theta(t,p) \in \mathbb{R}$ = temperature of the particle $P_{Obj}$ which is at $t$ at $p = \widetilde{\Phi}(t, P_{Obj})$; Shortened notation of $\widehat{\mathcal{E}ul}(t,p) = ((t,p), \theta(t,p))$. ▪

**Example 2.3** $\mathcal{E}ul(t,p) = \vec{u}(t,p) \in \vec{\mathbb{R}^n}$ = force applied on the particle $P_{Obj}$ which is at $t$ at $p$. ▪

**Example 2.4** $\mathcal{E}ul(t,p) = d\vec{u}(t,p) \in \mathcal{L}(\vec{\mathbb{R}^n} : \vec{\mathbb{R}^n})$ = the differential at $t$ at $p$ of a Eulerian function $\vec{u}$. ▪

**Question:** Why introduce $\widehat{\mathcal{E}ul}$? Isn't $\mathcal{E}ul$ sufficient?

**Answer:** No: E.g., if $\vec{v}(t,p) \in \vec{\mathbb{R}^3}$ it the "velocity at $t$ at $p$" then the function $\widehat{\vec{v}} : (t,p) \to ((t,p); \vec{v}(t,p))$ is a "vector field" and $\vec{v}(t,p)$ must be drawn at $(t,p)$; While $\vec{v}(t,p)$ (alone) is a "free vector" (can be drawn anywhere). Moreover (2.3) emphasizes the difference between a Eulerian vector field and a Lagrangian vector function, see (3.12).

**Remark 2.5** The initial framework of Cauchy for his description of forces is Eulerian. ▪

## 2.3 Eulerian velocity (spatial velocity) and speed

Consider a particle $P_{Obj}$ and its (regular) trajectory $\widetilde{\Phi}_{P_{Obj}} : t \to p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$, cf. (1.8).

**Definition 2.6** In short: Its Eulerian velocity at $t$ at $p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$ is

$$\vec{v}(t,p(t)) := \widetilde{\Phi}_{P_{Obj}}{}'(t) \overset{\text{written}}{=} \frac{\partial\widetilde{\Phi}}{\partial t}(t, P_{Obj}) \quad (= \lim_{h\to 0} \frac{\widetilde{\Phi}(t+h, P_{Obj}) - \widetilde{\Phi}(t, P_{Obj})}{h}). \tag{2.5}$$

So $\vec{v}(t,p(t))$ is the tangent vector at $t$ at $p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$ to the trajectory $\widetilde{\Phi}_{P_{Obj}}$. This defines the Eulerian vector field $\widehat{\vec{v}} : \left\{ \begin{array}{ll} \mathcal{C} & \to \mathcal{C} \times \vec{\mathbb{R}^n} \\ (t,p) & \to \widehat{\vec{v}}(t,p) = ((t,p), \vec{v}(t,p)) \end{array} \right\}$, written (for short) $\vec{v} : \left\{ \begin{array}{ll} \mathcal{C} & \to \vec{\mathbb{R}^n} \\ (t,p_t) & \to \vec{v}(t,p_t) \end{array} \right\}$.

**Remark 2.7** $\frac{d\widetilde{\Phi}_{P_{Obj}}}{dt}(t) = \vec{v}(t, \widetilde{\Phi}_{P_{Obj}}(t))$, with $p(t) = \widetilde{\Phi}_{P_{Obj}}(t)$, is often written

$$\frac{dp}{dt}(t) = \vec{v}(t,p(t)), \quad \text{or} \quad \frac{d\vec{x}}{dt}(t) = \vec{v}(t,\vec{x}(t)), \quad \text{or} \quad \frac{d\vec{x}}{dt} = \vec{v}(t,\vec{x}), \tag{2.6}$$

the two last notations when an origin $O$ is chosen and $\vec{x}(t) = \overrightarrow{Op(t)}$. Such an equation is the prototype of an ODE (ordinary differential equation) solved with the Cauchy–Lipschitz theorem, see § 5. (A Lagrangian velocity does not produce an ODE, see (3.21).) ▪

**Definition 2.8** If an observer chooses a Euclidean dot product $(\cdot,\cdot)_g$ (e.g. foot or metre built) with its associated norm $||.||_g$, then the length $||\vec{v}(t,p)||_g$ is the speed (or scalar velocity) of $P_{Obj}$ (e.g. in ft/s or m/s). And the context must remove the ambiguities: "The velocity" is either the vector velocity $\vec{v}(t,p) = \widetilde{\Phi}_{P_{Obj}}{}'(t)$ or the speed (the scalar velocity) $||\vec{v}(t,p)||_g$.

**Exercice 2.9** Euclidean dot product $(\cdot,\cdot)_g$, $\vec{x}(t) = \overrightarrow{Op(t)}$, $\vec{T}(t) = \frac{\vec{x}\,'(t)}{||\vec{x}\,'(t)||_g}$ (unit tangent vector), $f(t) = ||\vec{x}\,'(t)||_g$ (speed). Prove : $\frac{df}{dt}(t) = (\vec{x}\,''(t), \vec{T}(t))_g =^{\text{written}} \vec{x}\,''(t) \bullet \vec{T}(t)$ (= tangential acceleration).

**Answer**. E.g. 2-D and Euclidean basis: $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ gives $f(t) = (x'(t)^2 + y'(t)^2)^{\frac{1}{2}}$, thus $f'(t) = \frac{x'(t)x''(t)+y'(t)y''(t)}{f(t)} = \frac{\vec{x}\,'(t) \bullet \vec{x}\,''(t)}{||\vec{x}\,'(t)||_g}$. Similar in $n$-D. ▪

## 2.4 Spatial derivative of the Eulerian velocity

### 2.4.1 Definition

For all observers (English, French... (there is no inner dot product here). $\mathcal{E}ul$ is supposed $C^1$.

**Definition 2.10** The space derivative of $\mathcal{E}ul$ at $(t,p) \in [t_1,t_2] \times \Omega_t$ is the differential $d\mathcal{E}ul_t$ at $p$: For all $\vec{w}_p \in \vec{\mathbb{R}}^n_t$ (vector at $p$),

$$(d\mathcal{E}ul_t(p).\vec{w}_p =) \quad \boxed{d\mathcal{E}ul(t,p).\vec{w}_p = \lim_{h\to 0} \frac{\mathcal{E}ul(t,p+h\vec{w}_p) - \mathcal{E}ul(t,p)}{h}} \overset{\text{written}}{=} \frac{\partial\mathcal{E}ul}{\partial p}(t,p).\vec{w}_p. \tag{2.7}$$

So in $\Omega_t$ (the photo at $t$), $d\mathcal{E}ul(t,p).\vec{w}_p$ gives "the rate of variations of $\mathcal{E}ul_t$ at $p$ in the direction $\vec{w}_p$".

E.g., at $t$, the space derivative $d\vec{v}$ of the Eulerian velocity field is defined by

$$d\vec{v}(t,p).\vec{w}_p = \lim_{h\to 0}\frac{\vec{v}(t,p+h\vec{w}_p)-\vec{v}(t,p)}{h} \quad (= d\vec{v}_t(p).\vec{w}_p). \tag{2.8}$$

**Remark 2.11** In differential geometry (a vector is defined to be a derivation) and $f = \mathcal{E}ul_t$, (2.7) is written $\vec{u}(f)(p) = \frac{d}{dh}f(p+h\vec{u})_{|h=0}$: That is $\vec{u}(f) = df.\vec{u}$ .

### 2.4.2 The convective derivative $d\mathcal{E}ul.\vec{v}$

**Definition 2.12** If $\vec{v}$ is the Eulerian velocity field, then $d\mathcal{E}ul.\vec{v}$ is called the convective derivative of $\mathcal{E}ul$.

### 2.4.3 Quantification in a basis: Jacobian matrices

Let $f\in C^1(\Omega;\mathbb{R})$ (scalar valued function), so $df\in C^0(\Omega;\mathbb{R}^{n*})$ (differential form) and $df(p)\in\mathbb{R}^{n*}$ (linear form). Let $\vec{u}\in C^0(\Omega;\mathbb{R})$ (vector field). Let $(df.\vec{u})(p) := df(p).\vec{u}(p)$ (objective value).

**Quantification:** $(\vec{e}_i)$ being a basis in $\vec{\mathbb{R}^n}$ (eventually dependent on $p$), let (usual definition)

$$\frac{\partial f}{\partial x_i}(p) := df(p).\vec{e}_i, \quad\text{and}\quad [df(p)]_{|\vec{e}} = (\ \tfrac{\partial f}{\partial x_1}(p) \quad \dots \quad \tfrac{\partial f}{\partial x_n}(p)\ ) = \text{ Jacobian matrix of } f \text{ at } p \tag{2.9}$$

(line matrix because it represents a linear form). A Jacobian matrix is subjective (depends on $(\vec{e}_i)$). This defines the function $\frac{\partial f}{\partial x_i} := df.\vec{e}_i\in C^0(\Omega;\mathbb{R})$ and $[df]_{|\vec{e}}\in C^0(\Omega;\mathcal{M}_{n1})$.

Thus, with $\vec{u}=\sum_{i=1}^n u_i\vec{e}_i$ vector at $p$ and with the usual matrix multiplication rule,

$$df(p).\vec{u}(p) = [df(p)]_{|\vec{e}}.[\vec{u}(p)]_{|\vec{e}} = \sum_{i=1}^n\frac{\partial f}{\partial x_i}(p)u_i(p) = \sum_{i=1}^n u_i(p)\frac{\partial f}{\partial x_i}(p) \overset{\text{written}}{=} (\vec{u}.\vec{\text{grad}})f(p), \tag{2.10}$$

written $df.\vec{u} = [df]_{|\vec{e}}.[\vec{u}]_{|\vec{e}} = \sum_{i=1}^n\frac{\partial f}{\partial x_i}u_i = \sum_{i=1}^n u_i\frac{\partial f}{\partial x_i} =^{\text{written}} (\vec{u}.\vec{\text{grad}})f \ \in C^0(\Omega;\mathbb{R})$, where $(\vec{u}.\vec{\text{grad}})$ is the differential operator $(\vec{u}.\vec{\text{grad}}) : \begin{Bmatrix} C^1(\Omega;\mathbb{R}) & \to C^0(\Omega;\mathbb{R}) \\ f & \to (\vec{u}.\vec{\text{grad}})f := df.\vec{u}\end{Bmatrix}$.

**Remark:** $df.\vec{u}$ is objective (value independent of $(\vec{e}_i)$), so $(\vec{u}.\vec{\text{grad}})(f) = df.\vec{u}$ is objective; However $(\vec{u}.\vec{\text{grad}})(f)$ is usually used for computational purposes $(= [df]_{|\vec{e}}.[\vec{u}]_{|\vec{e}})$ which requires a basis $(\vec{e}_i)$. Warning: Moreover the use of the gradient $\vec{\text{grad}}$ in mechanics implicitly means the use of a Euclidean basis.

For vector valued functions $\vec{f}:\Omega\to\vec{\mathbb{R}^m}$, the definition of the Jacobian requires the choice of a basis $(\vec{b}_i)$ in $\vec{\mathbb{R}^m}$ (subjective): If $\vec{f}=\sum_{i=1}^m f_i\vec{b}_i$, i.e. $\vec{f}(p)=\sum_{i=1}^m f_i(p)\vec{b}_i$ then (above steps with the $f_i$)

$$\begin{cases} [df(p)]_{|\vec{e},\vec{b}} = [\dfrac{\partial f_i}{\partial x_j}(p)]_{\substack{i=1,\dots,m\\ j=1,\dots,n}} = \text{ Jacobian matrix of } f \text{ at } p, \quad\text{and} \\ (\vec{u}.\vec{\text{grad}})_{|\vec{b}}(\vec{f}) := [d\vec{f}.\vec{u}]_{|\vec{b}} = \displaystyle\sum_{i=1}^m (df_i.\vec{u})\vec{b}_i = \sum_{i=1}^m((\vec{u}.\vec{\text{grad}})f_i)\vec{b}_i = \sum_{i=1}^m\sum_{j=1}^n(u_j\frac{\partial f_i}{\partial x_j})\vec{b}_i. \end{cases} \tag{2.11}$$

### 2.4.4 $\vec{\text{grad}}f$ = representation relative to a Euclidean dot product (subjective)

An observer chooses a Euclidean basis $(\vec{e}_i)$ (foot, metre...) and the associated Euclidean dot product $(\cdot,\cdot)_g$.

Let $\Omega$ be an open set in $\mathbb{R}^n$, $f\in C^1(\Omega;\mathbb{R})$ (scalar valued function), and $p\in\Omega$. Then the $(\cdot,\cdot)_g$-Riesz representation vector of the differential form $df(p)$ is called the gradient of $f$ at $p$ relative to $(\cdot,\cdot)_g$, and named $\vec{\text{grad}}_g f(p)\in\vec{\mathbb{R}^n}$: It is defined by

$$\forall\vec{u}\in\vec{\mathbb{R}^n}, \quad (\vec{\text{grad}}_g f(p),\vec{u})_g = df(p).\vec{u}, \quad\text{written}\quad \vec{\text{grad}}f\bullet\vec{u} = df.\vec{u}, \tag{2.12}$$

the last notation iff $(\cdot,\cdot)_g$ is implicit = imposed to all observer (subjective: foot, metre ?).

**Quantification:** With $\vec{u}=\sum_{i=1}^n u_i\vec{e}_i$ and (2.9), (2.12) gives $[\vec{\text{grad}}f]^T.[g].[\vec{u}] = [df].[\vec{u}]$ (more precisely $[df]_{|\vec{e}}.[\vec{u}]_{|\vec{e}} = [\vec{\text{grad}}_g f]^T_{|\vec{e}}.[g]_{|\vec{e}}.[\vec{u}]_{|\vec{e}}$), thus $[df]_{|\vec{e}} = [\vec{\text{grad}}_g f]^T_{|\vec{e}}.$, thus, $[g]_{|\vec{e}}$ being symmetric, $[g]_{|\vec{e}}.[\vec{\text{grad}}_g f]_{|\vec{e}} =$

$[df]^T_{|\vec{e}}$, thus $[\vec{\mathrm{grad}}_g f]_{|\vec{e}} = [g]^{-1}_{|\vec{e}}.[df]^T_{|\vec{e}}$, written (subjective)

$$[\vec{\mathrm{grad}} f] = [g]^{-1}.[df]^T = [g_{ij}]^{-1}. \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \quad \text{(column matrix)}, \tag{2.13}$$

where $[g] = [gij]$. That is, if $\vec{\mathrm{grad}} f = \sum_{i=1}^n a_i \vec{e}_i$ then $a_i = \sum_{j=1}^n g_{ij} \frac{\partial f}{\partial x_j}$.

NB: With duality notations, $\vec{\mathrm{grad}} f = \sum_{i=1}^n a^i \vec{e}_i$ and (2.13) gives $a^i = \sum_{j=1}^n g_{ij} \frac{\partial f}{\partial x^j}$: The Einstein convention is ***not*** satisfied (the index $j$ is twice bottom), which is expected since the definition of $\vec{\mathrm{grad}}_g f$ depends on a subjective choice (unit of length). In comparison, $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$ satisfies the Einstein convention (a differential is objective).

Particular case: If and only if $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis then $[\vec{\mathrm{grad}} f] = [df]^T$.

Application: The objective first order Taylor expansion $f(p{+}h\vec{u}) = f(p){+}h\, df(p).\vec{u}{+}o(h)$ can therefore be subjectively written:

$$f(p{+}h\vec{u}) = f(p) + h\, \vec{\mathrm{grad}}_g f(p) \bullet \vec{u} + o(h) \quad (= f(p) + h\,([g]^{-1}.[df]^T) \bullet \vec{u} + o(h)). \tag{2.14}$$

**Mind the notations:** The gradient $\vec{\mathrm{grad}}_g f =^{\text{written}} \vec{\mathrm{grad}} f$ depends on $(\cdot,\cdot)_g$, cf. (2.12)-(2.13); While $(\vec{u}.\vec{\mathrm{grad}})f$ depends on a basis: Historical gradients notations...

### 2.4.5 Vector valued functions

For vector valued functions $\vec{f} : \Omega \to \vec{\mathbb{R}^m}$, the above steps apply to the components $f_i$ of $\vec{f}$ relative to a basis $(\vec{b}_i)$ in $\vec{\mathbb{R}^m}$... But, depending on the book you read:

• The differential $d\vec{f}$ is unfortunately also sometimes ambiguously called the "gradient matrix" (although no Euclidean dot product is required to define $df$): It could mean the differential... or the Jacobian matrix... or its transposed...!

• In the objective framework of this manuscript, we will use the differential $d\vec{f}$ (objective); And for quantitative purposes, i.e. after an explicit choice of bases $(\vec{e}_i)$ and $(\vec{b}_i)$, only the Jacobian matrix $[d\vec{f}]_{|\vec{e}} = [\frac{\partial f_i}{\partial x_j}]$ will be used (non ambiguous).

**Exercice 2.13** Imposed Euclidean framework. Prove $(\vec{v}.\vec{\mathrm{grad}})\vec{v} = \frac{1}{2}\vec{\mathrm{grad}}(||\vec{v}||^2) + \vec{\mathrm{curl}}\vec{v} \wedge \vec{v}$.

**Answer**. Euclidean basis $(\vec{e}_i)$, associated Euclidean dot product $(\cdot,\cdot)_g =^{\text{written}} (\cdot,\cdot)$ and norm $||.||_g =^{\text{written}} ||.||$. $\vec{v} = \sum_{i=1}^n v_i \vec{e}_i$ gives $||\vec{v}||^2 = \sum_i v_i^2$, thus $\dfrac{\partial ||\vec{v}||^2}{\partial x_k} = \sum_i 2v_i \dfrac{\partial v_i}{\partial x_k}$, for any $k = 1,2,3$. And, the first component of $\vec{\mathrm{curl}}\vec{v}$ is $(\vec{\mathrm{curl}}\vec{v})_1 = \dfrac{\partial v_3}{\partial x_2} - \dfrac{\partial v_2}{\partial x_3}$, idem for $(\vec{\mathrm{curl}}\vec{v})_2$ and $(\vec{\mathrm{curl}}\vec{v})_3$ (circular permutation). Thus (first component) $(\vec{\mathrm{curl}}\vec{v} \wedge \vec{v})_1 = (\dfrac{\partial v_1}{\partial x_3} - \dfrac{\partial v_3}{\partial x_1})v_3 - (\dfrac{\partial v_2}{\partial x_1} - \dfrac{\partial v_1}{\partial x_2})v_2$, idem for $(\vec{\mathrm{curl}}\vec{v} \wedge \vec{v})_2$ and $(\vec{\mathrm{curl}}\vec{v} \wedge \vec{v})_2$. Thus $(\frac{1}{2}\vec{\mathrm{grad}}(||\vec{v}||^2) + \vec{\mathrm{curl}}\vec{v} \wedge \vec{v})_1 = v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_1} + v_3 \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} v_3 - \frac{\partial v_3}{\partial x_1} v_3 - \frac{\partial v_2}{\partial x_1} v_2 + \frac{\partial v_1}{\partial x_2} v_2 = v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} = (\vec{v}.\vec{\mathrm{grad}})v_1$. Idem for the other components. ∎

## 2.5 Streamline (current line)

Fix $t \in \mathbb{R}$, and consider the photo $\Omega_t = \widetilde{\Phi}_t(Obj)$. Let $p_t \in \Omega_t$, $\varepsilon > 0$, and consider a spatial curve in $\Omega_t$ at $p_t$, i.e. s.t.

$$c_{p_t} : \left\{ \begin{array}{rl} ]-\varepsilon,\varepsilon[ & \to \Omega_t \\ s & \to q = c_{p_t}(s) \end{array} \right\} \quad \text{and} \quad c_{p_t}(0) = p_t. \tag{2.15}$$

So $s$ is a curvilinear spatial coordinate (dimension = length), and $c_{p_t}$ is drawn in the photo $\Omega_t$ at $t$.

**Definition 2.14** $\vec{v} : (t,p) \to \vec{v}(t,p)$ being the Eulerian velocity field of $Obj$, a streamline through a point $p_t \in \Omega_t$ is a (parametric) spatial curve $c_{p_t}$ solution of the differential equation

$$\frac{dc_{p_t}}{ds}(s) = \vec{v}_t(c_{p_t}(s)) \quad \text{with} \quad c_{p_t}(0) = p_t. \tag{2.16}$$

And $\mathrm{Im}(c_{p_t})$ is the geometric associated streamline ($\subset \Omega_t$ the photo at $t$).

NB: (2.16) cannot be confused with (2.6): In (2.6) the variable is the time variable $t$, while in (2.16) the variable is the space variable $s$.

**Usual notation:** If an origin $\mathcal{O}$ is chosen at $t$ by an observer and $\vec{x}(s) := \overrightarrow{\mathcal{O}c_{p_t}(s)}$ , then (2.16) is written

$$\frac{d\vec{x}}{ds}(s) = \vec{v}_t(\vec{x}(s)) \quad \text{with} \quad \vec{x}(0) = \overrightarrow{Op_t}. \tag{2.17}$$

Moreover, with a Cartesian basis $(\vec{e}_i))$ chosen at $t$ by the observer and $\vec{v}_t(\vec{x}(s)) = \sum_{i=1}^n v_i(s)\vec{e}_i$ and $\vec{x}(s) = \sum_{i=1}^n x_i(s)\vec{e}_i$, we get the differential system of $n$ equations in $\vec{\mathbb{R}^n}$ (the unknowns are the functions $x_i$)

$$\forall i = 1, ..., n, \quad \frac{dx_i}{ds}(s) = v_i(x_1(s), ..., x_n(s)). \tag{2.18}$$

Also written

$$\frac{dx_1}{v_1} = ... = \frac{dx_n}{v_n} = ds \tag{2.19}$$

(meaning: It is the differential system (2.18) of $n$ equations and $n$ unknowns which must be solved.)

## 2.6 Material time derivative (dérivées particulaires)

### 2.6.1 Usual definition

Goal: To compute the variations of a Eulerian function $\mathcal{E}ul \in C^1(\mathcal{C}; S)$ along the trajectory $\widetilde{\Phi}_{P_{Obj}}$ of a particle $P_{Obj}$ (e.g. the temperature, the velocity, ..., of a particle along its trajectory). Consider the function $g_{P_{Obj}} : [t_1, t_2] \to F$ (gives the values of $\mathcal{E}ul$ relative to a $P_{Obj}$ along its trajectory): For all $t \in [t_1, t_2]$,

$$\boxed{g_{P_{Obj}}(t) := \mathcal{E}ul(t, \widetilde{\Phi}_{P_{Obj}}(t))}, \quad \text{written} \quad g_{P_{Obj}}(t) = \mathcal{E}ul(t, p(t)) \quad \text{when} \quad p(t) := \widetilde{\Phi}_{P_{Obj}}(t). \tag{2.20}$$

**Definition 2.15** At $t$ at $p(t) := \widetilde{\Phi}_{P_{Obj}}(t)$, the material time derivative of $\mathcal{E}ul$ is

$$\boxed{\frac{D\mathcal{E}ul}{Dt}(t, p(t)) := g_{P_{Obj}}{}'(t)} \quad (= \lim_{h\to 0} \frac{\mathcal{E}ul(t+h, p(t+h)) - \mathcal{E}ul(t, p(t))}{h}). \tag{2.21}$$

i.e. with $\widetilde{\Phi}'_{P_{Obj}}(t) = \vec{v}(t, p(t))$ (Eulerian velocity), $\frac{D\mathcal{E}ul}{Dt}(t, p(t)) := \frac{\partial \mathcal{E}ul}{\partial t}(t, p(t)) + d\mathcal{E}ul(t, p(t)).\vec{v}(t, p(t))$ $(= \lim_{h\to 0} \frac{\mathcal{E}ul(t+h,p(t+h))-\mathcal{E}ul(t,p(t))}{h})$, i.e.

$$\boxed{\frac{D\mathcal{E}ul}{Dt} := \frac{\partial \mathcal{E}ul}{\partial t} + d\mathcal{E}ul.\vec{v}}. \tag{2.22}$$

**Example 2.16** $f \in C^1(\mathcal{C}; \mathbb{R})$ (scalar valued function), thus $\frac{Df}{Dt}$ is the scalar valued function given by

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + df.\vec{v}, \tag{2.23}$$

i.e. $\frac{Df}{Dt}(t, p) = \frac{\partial f}{\partial t}(t, p) + df(t, p).\vec{v}(t, p)$ for all $(t, p) \in \mathcal{C}$. With a basis $(\vec{e}_i)$ and $\vec{v} = \sum_i v_i\vec{e}_i$ we get $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial x_i} v_i =^{\text{written}} \frac{\partial f}{\partial t} + (\vec{v}.\vec{\text{grad}})f$. ▪

**Example 2.17** $\vec{w} \in C^1(\mathcal{C}; \vec{\mathbb{R}^n})$ (vector valued):

$$\frac{D\vec{w}}{Dt} = \frac{\partial \vec{w}}{\partial t} + d\vec{w}.\vec{v}. \tag{2.24}$$

With a basis $(\vec{e}_i)$ we get $\frac{Dw_i}{Dt} = \frac{\partial w_i}{\partial t} + \sum_j \frac{\partial w_i}{\partial x_j} v_j$ for all $i = 1, ..., n$. ▪

**Exercice 2.18** Let $\alpha$ be a $C^1$ differential form: We have $\frac{D\alpha}{Dt} =^{(2.22)} \frac{\partial \alpha}{\partial t} + d\alpha.\vec{v}$; Check it with a Cartesian basis. And prove, for all $\vec{w}$,

$$\frac{D\alpha}{Dt}.\vec{w} = \frac{D(\alpha.\vec{w})}{Dt} - \alpha.\frac{D\vec{w}}{Dt}, \tag{2.25}$$

**Answer**. Cartesian basis $(\vec{e}_i)$, dual basis $(\pi_{ei})$, $\vec{v} = \sum_j v_j\vec{e}_j$ and $\alpha = \sum_i \alpha_i\pi_{ei}$ we have $\frac{D\alpha}{Dt} = \sum_i \frac{D\alpha_i}{Dt}\pi_{ei}$ with $\frac{D\alpha_i}{Dt} = \frac{\partial \alpha_i}{\partial t} + d\alpha_i.\vec{v} = \frac{\partial \alpha_i}{\partial t} + \sum_j \frac{\partial \alpha_i}{\partial x_j}.\vec{v}_j$, thus $\frac{D\alpha}{Dt} = \sum_i \frac{\partial \alpha_i}{\partial t}\pi_{ei} + \sum_{ij} \frac{\partial \alpha_i}{\partial x_j}.v_j\pi_{ei} = \frac{\partial \sum_i \alpha_i\pi_{ei}}{\partial t} + \sum_j \frac{\partial \sum_i \alpha_i\pi_{ei}}{\partial x_j}.\vec{v}_j = \frac{\partial \alpha}{\partial t} + d\alpha.\vec{v}$.

$\vec{w}$ vector field and $\alpha$ differential form, thus $\alpha.\vec{w}$ is a scalar valued function. Thus $\frac{D(\alpha.\vec{w})}{Dt} = \frac{\partial(\alpha.\vec{w})}{\partial t} + d(\alpha.\vec{w}).\vec{v} = \frac{\partial \alpha}{\partial t}.\vec{w} + \alpha.\frac{\partial \vec{w}}{\partial t} + (d\alpha.\vec{v}).\vec{w} + \alpha.(d\vec{w}.\vec{v}) = \frac{\partial \alpha}{\partial t}.\vec{w} + (d\alpha.\vec{v}).\vec{w} + \alpha.\frac{\partial \vec{w}}{\partial t} + \alpha.(d\vec{w}.\vec{v}) = \frac{D\alpha}{Dt}.\vec{w} + \alpha.\frac{D\vec{w}}{Dt}$. Thus $\frac{D\alpha}{Dt}.\vec{w} = \frac{D(\alpha.\vec{w})}{Dt} - \alpha.\frac{D\vec{w}}{Dt}$, i.e. (2.25). ▪

**Proposition 2.19** *$\frac{D}{Dt}$ is a derivation: All the functions being Eulerian and $C^1$,*

- *Linearity:*
$$\frac{D(\mathcal{E}ul_1+\lambda\mathcal{E}ul_2)}{Dt}=\frac{D\mathcal{E}ul_1}{Dt}+\lambda\frac{D\mathcal{E}ul_2}{Dt}. \tag{2.26}$$
- *Product rules: If $\mathcal{E}ul_1,\mathcal{E}ul_2$ are scalar valued functions then*
$$\frac{D(\mathcal{E}ul_1\mathcal{E}ul_2)}{Dt}=\frac{D\mathcal{E}ul_1}{Dt}\,\mathcal{E}ul_2+\mathcal{E}ul_1\,\frac{D\mathcal{E}ul_2}{Dt}. \tag{2.27}$$
- *If paticular $\vec w$ is a vector field and $T$ a compatible tensor (so $T.\vec w$ is meaningful) then*
$$\frac{D(T.\vec w)}{Dt}=\frac{DT}{Dt}.\vec w+T.\frac{D\vec w}{Dt}. \tag{2.28}$$

**Proof.** Let $i=1,2$, and $g_i$ defined by $g_i(t):=\mathcal{E}ul_i(t,p(t))$ where $p(t)=\widetilde{\Phi}_{P_{Obj}}(t)$.

- $(g_1+\lambda g_2)'=g_1'+\lambda g_2'$ gives (2.26).
- On the one hand $\frac{D(T.\vec w)}{Dt}=\frac{\partial(T.\vec w)}{\partial t}+d(T.\vec w).\vec v=\frac{\partial T}{\partial t}.\vec w+T.\frac{\partial\vec w}{\partial t}+(dT.\vec v).\vec w+T.(d\vec w.\vec v)$, and on the other hand $\frac{DT}{Dt}.\vec w+T.\frac{D\vec w}{Dt}=(\frac{\partial T}{\partial t}+dT.\vec v).\vec w+T.(\frac{\partial\vec w}{\partial t}+d\vec w.\vec v)$. Thus (2.27)-(2.28). ∎

### 2.6.2 Commutativity issue

Let $\mathcal{E}ul$ be $C^2$. The Schwarz theorem tells: $d(\frac{\partial\mathcal{E}ul}{\partial t})=\frac{\partial(d\mathcal{E}ul)}{\partial t}$: The derivatives commute.

**Proposition 2.20**

$$\frac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt}\neq\frac{\partial(\frac{D\mathcal{E}ul}{Dt})}{\partial t}\quad\text{and}\quad\frac{D(d\mathcal{E}ul)}{Dt}\neq d(\frac{D\mathcal{E}ul}{Dt})\quad\text{in general,} \tag{2.29}$$

*i.e. the material time derivative $\frac{D}{Dt}$ does not commute with the partial derivation $\frac{\partial}{\partial t}$ or with the spatial derivative $d$ (because the variables $t$ and $p$ are not independent along a trajectory). We have:*

$$\left.\begin{aligned}\frac{\partial(\frac{D\mathcal{E}ul}{Dt})}{\partial t}&=\frac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt}+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}\\&=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+d\frac{\partial\mathcal{E}ul}{\partial t}.\vec v+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t},\end{aligned}\right\}\quad\text{and}\quad\left\{\begin{aligned}d(\frac{D\mathcal{E}ul}{Dt})&=\frac{D(d\mathcal{E}ul)}{Dt}+d\mathcal{E}ul.d\vec v\\&=\frac{\partial(d\mathcal{E}ul)}{\partial t}+d^2\mathcal{E}ul.\vec v+d\mathcal{E}ul.d\vec v.\end{aligned}\right. \tag{2.30}$$

**Proof.** $\frac{\partial\frac{D\mathcal{E}ul}{Dt}}{\partial t}=\frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec v)}{\partial t}=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec v+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}$. And $d\frac{D\mathcal{E}ul}{Dt}=d(\frac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec v)=\frac{\partial(d\mathcal{E}ul)}{\partial t}+d(d\mathcal{E}ul).\vec v+d\mathcal{E}ul.d\vec v=\frac{D(d\mathcal{E}ul)}{Dt}+d\mathcal{E}ul.d\vec v$, thus (2.30). ∎

**Exercice 2.21** If $\mathcal{E}ul$ is $C^2$ and $\vec w$ is $C^1$, check $\frac{D(d\mathcal{E}ul.\vec w)}{Dt}=\frac{D(d\mathcal{E}ul)}{Dt}.\vec w+d\mathcal{E}ul.\frac{D\vec w}{Dt}$ (i.e. $\frac{D}{Dt}$ is a derivation), and

$$\begin{aligned}\frac{D(d\mathcal{E}ul.\vec w)}{Dt}&=d\frac{\partial\mathcal{E}ul}{\partial t}.\vec w+d\mathcal{E}ul.\frac{\partial\vec w}{\partial t}+(d(d\mathcal{E}ul).\vec v).\vec w+d\mathcal{E}ul.d\vec w.\vec v\\&=d\mathcal{E}ul.\frac{D\vec w}{Dt}+\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec w+d^2\mathcal{E}ul(\vec v,\vec w),\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}\frac{D^2\mathcal{E}ul}{Dt^2}&=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+2d\frac{\partial\mathcal{E}ul}{\partial t}.\vec v+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}+(d(d\mathcal{E}ul).\vec v).\vec v+d\mathcal{E}ul.d\vec v.\vec v\\&=d\mathcal{E}ul.\frac{D\vec v}{Dt}+\frac{\partial^2\mathcal{E}ul}{\partial t^2}+d\frac{\partial\mathcal{E}ul}{\partial t}.\vec v+\frac{D(d\mathcal{E}ul)}{Dt}.\vec v.\end{aligned} \tag{2.32}$$

**Answer**. $\frac{D(d\mathcal{E}ul.\vec w)}{Dt}=\frac{\partial(d\mathcal{E}ul.\vec w)}{\partial t}+d(d\mathcal{E}ul.\vec w).\vec v=\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec w+d\mathcal{E}ul.\frac{\partial\vec w}{\partial t}+(d(d\mathcal{E}ul).\vec v).\vec w+d\mathcal{E}ul.d\vec w.\vec v=\frac{D(d\mathcal{E}ul)}{Dt}.\vec w+d\mathcal{E}ul.\frac{D\vec w}{Dt}$. And $\mathcal{E}ul\in C^2$ and Schwarz give $\frac{\partial(d\mathcal{E}ul)}{\partial t}=d(\frac{\partial\mathcal{E}ul}{\partial t})$ and $(d^2\mathcal{E}ul.\vec v).\vec w=d^2\mathcal{E}ul(\vec v,\vec w)$, hence (2.31). And

$$\begin{aligned}\frac{D^2\mathcal{E}ul}{Dt^2}=g''_{P_{Obj}}(t)=\frac{D\frac{D\mathcal{E}ul}{Dt}}{Dt}&=\frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec v)}{\partial t}+d(\frac{\partial\mathcal{E}ul}{\partial t}+d\mathcal{E}ul.\vec v).\vec v\\&=\frac{\partial^2\mathcal{E}ul}{\partial t^2}+\frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec v+d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}+d\frac{\partial\mathcal{E}ul}{\partial t}.\vec v+(d^2\mathcal{E}ul.\vec v).\vec v+d\mathcal{E}ul.d\vec v.\vec v,\end{aligned}$$

with $\frac{\partial}{\partial t}\circ d=d\circ\frac{\partial}{\partial t}$ (Schwarz), $\frac{D(d\mathcal{E}ul)}{Dt}=\frac{\partial(d\mathcal{E}ul)}{\partial t}+d^2\mathcal{E}ul.\vec v$ and $d\mathcal{E}ul.\frac{D\vec v}{Dt}=d\mathcal{E}ul.\frac{\partial\vec v}{\partial t}+d\mathcal{E}ul.d\vec v.\vec v$, hence (2.32). ∎

**Exercice 2.22** Prove (2.30) with a Cartesian basis $(\vec{e}_i)$.

**Answer.** $\frac{\partial \frac{D\mathcal{E}ul}{Dt}}{\partial t} = \frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+\sum_i \frac{\partial\mathcal{E}ul}{\partial x^i}.v^i)}{\partial t} = \frac{\partial^2\mathcal{E}ul}{\partial t^2} + \sum_i \frac{\partial^2\mathcal{E}ul}{\partial t\partial x^i}.v^i + \sum_i \frac{\partial\mathcal{E}ul}{\partial x^i}.\frac{\partial v^i}{\partial t} = \frac{\partial^2\mathcal{E}ul}{\partial t^2} + \sum_i \frac{\partial^2\mathcal{E}ul}{\partial t\partial x^i}.v^i + d\mathcal{E}ul.\frac{\partial\vec{v}}{\partial t}$. And $\frac{D(\frac{\partial\mathcal{E}ul}{\partial t})}{Dt} = \frac{\partial^2\mathcal{E}ul}{\partial t^2} + \sum_i \frac{\partial\frac{\partial\mathcal{E}ul}{\partial t}}{\partial x^i}.v^i = \frac{\partial^2\mathcal{E}ul}{\partial t^2} + \sum_i \frac{\partial^2\mathcal{E}ul}{\partial t\partial x^i}.v^i$.

And $d(\frac{D\mathcal{E}ul}{Dt}).\vec{w} = \sum_j \frac{\partial\frac{D\mathcal{E}ul}{Dt}}{\partial x^j}w^j = \sum_j \frac{\partial(\frac{\partial\mathcal{E}ul}{\partial t}+\sum_i\frac{\partial\mathcal{E}ul}{\partial x^i}v^i)}{\partial x^j}w^j = \sum_j \frac{\partial^2\mathcal{E}ul}{\partial t\partial x^j}w^j + \sum_{ij}\frac{\partial^2\mathcal{E}ul}{\partial x^i\partial x^j}v^iw^j + \sum_{ij}\frac{\partial\mathcal{E}ul}{\partial x^i}\frac{\partial v^i}{\partial x^j}w^j = \sum_j \frac{\partial^2\mathcal{E}ul}{\partial t\partial x^j}w^j + d^2\mathcal{E}ul(\vec{v},\vec{w}) + d\mathcal{E}ul.d\vec{v}.\vec{w}$. And $\frac{D(d\mathcal{E}ul)}{Dt}.\vec{w} = (\frac{\partial(d\mathcal{E}ul)}{\partial t} + d(d\mathcal{E}ul).\vec{v}).\vec{w} = \frac{\partial(d\mathcal{E}ul)}{\partial t}.\vec{w} + d^2\mathcal{E}ul(\vec{v},\vec{w}) = \sum_i \frac{\partial^2\mathcal{E}ul}{\partial x^i\partial t}w^i + d^2\mathcal{E}ul(\vec{v},\vec{w})$. Thus $d(\frac{D\mathcal{E}ul}{Dt}).\vec{w} = \frac{D(d\mathcal{E}ul)}{Dt}.\vec{w} + d\mathcal{E}ul.d\vec{v}.\vec{w}$ for all $\vec{w}$. ▪

### 2.6.3 Remark: About notations

• The notation $\frac{d}{dt}$ (lowercase letters) concerns a function of one variable, e.g. $\frac{dg}{dt}(t) := g'(t) := \lim_{h\to 0}\frac{g(t+h))-g(t)}{h}$;

• The notation $\frac{\partial}{\partial t}$ concerns a function with more than one variable, e.g. $\frac{\partial\mathcal{E}ul}{\partial t}(t,p) = \lim_{h\to 0}\frac{\mathcal{E}ul(t+h,p)-\mathcal{E}ul(t,p)}{h}$;

• The notation $\frac{D}{Dt}$ (capital letters) concerns a Eulerian function differentiated along a motion.

• Other notations, often practical which are ambiguous if composed functions are considered:

$$\frac{d\mathcal{E}ul(t,p(t))}{dt} := g_{P_{Obj}}{}'(t) = \frac{D\mathcal{E}ul}{Dt}(t,p(t)), \quad\text{and}\quad \frac{d\mathcal{E}ul(t,p(t))}{dt}_{|t=t_0} := g_{P_{Obj}}{}'(t_0) = \frac{D\mathcal{E}ul}{Dt}(t_0,p(t_0)). \tag{2.33}$$

### 2.6.4 Definition bis: Time-space definition

Consider the affine time-space $\mathbb{R}\times\mathbb{R}^n$ and a $C^1$ function $f : (t,p) \in \mathbb{R}\times\mathbb{R}^n \to f(t,p)$.

**Definition 2.23** The differential of $f$ is called the "total differential", or "total derivative", and noted $Df$.

So, with $\vec{\mathbb{R}}\times\vec{\mathbb{R}^n}$ the associated time-space vector space, if $p_+ = (t,p) \in \mathbb{R}\times\mathbb{R}^n$ and $\vec{w}_+ = (w_0,\vec{w}) \in \vec{\mathbb{R}}\times\vec{\mathbb{R}^n}$ then we have (definition of a differential) $Df(p_+).\vec{w}_+ := \lim_{h\to 0}\frac{f(p_++h\vec{w}_+)-f(p_+)}{h}$, i.e.

$$Df(t,p).(w_0,\vec{w}) := \lim_{h\to 0}\frac{f(t+hw_0,p+h\vec{w}) - f(t,p)}{h}. \tag{2.34}$$

Thus

$$Df(t,p) = \frac{\partial f}{\partial t}(t,p)\,dt + df(t,p). \tag{2.35}$$

(Recall: $df$ is the space differentiation, so if $(\vec{e}_i)$ is a Cartesian basis then $df(t,p) = \frac{\partial f}{\partial x_1}(t,p)dx_1 + ... + \frac{\partial f}{\partial x_n}(t,p)dx_n$ and $\vec{w} = \sum_i w_i\vec{e}_i$ gives $Df(t,p).\vec{w} = \frac{\partial f}{\partial t}(t,p)w_0 + \frac{\partial f}{\partial x_1}(t,p)w_1 + ... + \frac{\partial f}{\partial x_n}(t,p)w_n$).

Then consider the time-space trajectory

$$\widetilde{\Psi}_{P_{Obj}} : \begin{cases} [t_1,t_2] & \to \mathbb{R}\times\mathbb{R}^n \\ \quad t & \to \widetilde{\Psi}_{P_{Obj}}(t) := (t,\widetilde{\Phi}_{P_{Obj}}(t)) \quad (= (t,p(t))). \end{cases} \tag{2.36}$$

(So $\mathrm{Im}(\widetilde{\Psi}_{P_{Obj}}) = \mathrm{graph}(\widetilde{\Phi}_{P_{Obj}})$.) The tangent vector to this curve at $t$ is

$$\widetilde{\Psi}_{P_{Obj}}{}'(t) = (1,\widetilde{\Phi}_{P_{Obj}}{}'(t)) = (1,\vec{v}(t,p(t)) \in \vec{\mathbb{R}}\times\vec{\mathbb{R}^n} \tag{2.37}$$

where $\vec{v}(t,p(t)) = \frac{d\widetilde{\Phi}_{P_{Obj}}}{dt}(t)$ is the Eulerian velocity at $(t,p(t))$. And (2.20) reads

$$g_{P_{Obj}}(t) = (\mathcal{E}ul\circ\widetilde{\Psi}_{P_{Obj}})(t) = \mathcal{E}ul(\widetilde{\Psi}_{P_{Obj}}(t)), \tag{2.38}$$

thus

$$g'_{P_{Obj}}(t) = D\mathcal{E}ul(\widetilde{\Psi}(t)).\widetilde{\Psi}_{P_{Obj}}{}'(t) = \frac{\partial\mathcal{E}ul}{\partial t}(t,p(t)).1 + d\mathcal{E}ul(t,p(t)).\vec{v}(t,p(t)) \overset{\text{written}}{=} \frac{D\mathcal{E}ul}{Dt}(t,p(t)), \tag{2.39}$$

i.e. (2.22): The material time derivative is the "total derivative" $D\mathcal{E}ul$ along the time-space trajectory $\widetilde{\Psi}_{P_{Obj}}$.

## 2.7 Eulerian acceleration

**Definition 2.24** In short: If $\widetilde{\Phi}_{P_{Obj}}$ is $C^2$, then the Eulerian acceleration of the particle $P_{Obj}$ which is at $t$ at $p_t = \widetilde{\Phi}(t, P_{Obj})$ is

$$\vec{\gamma}(t,p_t) := \widetilde{\Phi}_{P_{Obj}}{}''(t) \overset{\text{written}}{=} \frac{\partial^2\widetilde{\Phi}}{\partial t^2}(t, P_{Obj}). \tag{2.40}$$

In details: as in (2.3), the Eulerian acceleration (vector) field $\widehat{\vec{\gamma}}$ is defined with (2.40) by

$$\widehat{\vec{\gamma}}(t,p_t) = ((t,p_t), \vec{\gamma}(t,p_t)) \in \mathcal{C} \times \vec{\mathbb{R}}^n_t \quad \text{(pointed vector)}. \tag{2.41}$$

**Proposition 2.25**

$$\boxed{\vec{\gamma} = \frac{D\vec{v}}{Dt} = \frac{\partial\vec{v}}{\partial t} + d\vec{v}.\vec{v}}. \tag{2.42}$$

*And if* $\vec{v}$ *is* $C^2$ *then*

$$d\vec{\gamma} = \frac{\partial(d\vec{v})}{\partial t} + d^2\vec{v}.\vec{v} + d\vec{v}.d\vec{v} = \frac{D(d\vec{v})}{Dt} + d\vec{v}.d\vec{v}. \tag{2.43}$$

**Proof.** With $g(t) = \vec{v}(t,p(t)) = \widetilde{\Phi}_{P_{Obj}}{}'(t)$ and (2.22) we get $\vec{\gamma}(t,p(t)) = g'(t) = \frac{D\vec{v}}{Dt}(t,p(t))$. And $\vec{v}$ being $C^2$, the Schwarz theorem gives $d\frac{\partial\vec{v}}{\partial t} = \frac{\partial(d\vec{v})}{\partial t}$. ∎

**Definition 2.26** If an observer chooses a Euclidean dot product $(\cdot,\cdot)_g$ (based on a foot, a metre...), the associated norm being $||.||_g$, then the length $||\vec{\gamma}(t,p_t)||_g$ is the (scalar) acceleration of $P_{Obj}$.

## 2.8 Time Taylor expansion of $\widetilde{\Phi}$

Let $P_{Obj} \in Obj$ and $t \in ]t_1,t_2[$. Suppose $\widetilde{\Phi}_{P_{Obj}} \in C^2(]t_1,t_2[;\mathbb{R}^n)$. Its second-order (time) Taylor expansion of $\widetilde{\Phi}_{P_{Obj}}$ is, in the vicinity of a $t \in ]t_1,t_2[$,

$$\widetilde{\Phi}_{P_{Obj}}(\tau) = \widetilde{\Phi}_{P_{Obj}}(t) + (\tau{-}t)\widetilde{\Phi}'_{P_{Obj}}(t) + \frac{(\tau{-}t)^2}{2}\widetilde{\Phi}''_{P_{Obj}}(t) + o((\tau{-}t)^2), \tag{2.44}$$

i.e.

$$p(\tau) = p(t) + (\tau{-}t)\vec{v}(t,p(t)) + \frac{(\tau{-}t)^2}{2}\vec{\gamma}(t,p(t)) + o((\tau{-}t)^2). \tag{2.45}$$

# 3 Lagrangian description = Motion from an initial configuration

Instead of working on $Obj$, an observer may prefer to work with an initial configuration $\Omega_{t_0} = \widetilde{\Phi}(t_0, Obj)$ of $Obj$ (cf. elasticity): This is the "Lagrangian approach". This approach is not objective: Two observers may choose two different initial times (and configurations).

## 3.1 Initial configuration, Lagrangian "motion", Lagrangian variables

### 3.1.1 Definitions

$Obj$ is a material object, $\widetilde{\Phi} : [t_1,t_2[\times Obj \to \mathbb{R}^n$ is its motion, $\Omega_t = \widetilde{\Phi}(t, Obj)$ is its configuration at $t$. An observer chooses an "initial time" $t_0 \in ]t_1,t_2[$, hence $\Omega_{t_0}$ is his initial configuration.

**Definition 3.1** The motion of $Obj$ relative to the initial configuration $\Omega_{t_0} = \widetilde{\Phi}(t_0, Obj)$ is

$$\Phi^{t_0} : \begin{cases} [t_1,t_2] \times \Omega_{t_0} & \to \mathbb{R}^n \\ \quad (t,p_{t_0}) & \mapsto p_t = \Phi^{t_0}(t,p_{t_0}) := \widetilde{\Phi}(t,P_{Obj}) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}(t_0,P_{Obj}) : \end{cases} \tag{3.1}$$

$p_t = \Phi^{t_0}(t,p_{t_0})$ is the position at $t$ of the particle $P_{Obj}$ which was at $p_{t_0}$ at $t_0$, and $p_{t_0} = \Phi^{t_0}(t_0,p_{t_0})$ is its initial position.

**Definition 3.2** $t_0$, $p_{t_0}$ and $t$ are called the Lagrangian variables relative to the (subjective) choice $t_0$.

If $t$ is fixed then (3.1) defines

$$\Phi_t^{t_0} : \begin{cases} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} & \to p_t = \Phi_t^{t_0}(p_{t_0}) := \Phi^{t_0}(t, p_{t_0}). \end{cases} \tag{3.2}$$

And (3.1) gives $\Phi_t^{t_0}(\widetilde{\Phi}_{t_0}(P_{Obj})) = \widetilde{\Phi}_t(P_{Obj})$, for all $P_{Obj} \in Obj$, thus $\Phi_t^{t_0} \circ \widetilde{\Phi}_{t_0} = \widetilde{\Phi}_t$, thus $\Phi_t^{t_0}$ is defined by

$$\boxed{\Phi_t^{t_0} := \widetilde{\Phi}_t \circ (\widetilde{\Phi}_{t_0})^{-1}}. \quad \text{In particular} \quad \Phi_{t_0}^t = (\Phi_t^{t_0})^{-1}. \tag{3.3}$$

because $\Phi_{t_0}^t \circ \Phi_t^{t_0} = (\widetilde{\Phi}_t \circ (\widetilde{\Phi}_{t_0})^{-1}) \circ (\widetilde{\Phi}_{t_0} \circ (\widetilde{\Phi}_t)^{-1}) = I$.

**Hypothesis:** For all $t_0, t \in ]t_1, t_2[$, the map $\Phi_t^{t_0} : \Omega_{t_0} \to \Omega_t$ is a $C^k$ diffeomorphism (a $C^k$ invertible function whose inverse is $C^k$), where $k \in \mathbb{N}^*$ depends on the required regularity.

Then $\Phi_{t_0}^t(\Phi_t^{t_0}(p_{t_0})) = p_{t_0}$ gives $d\Phi_{t_0}^t(p_t).d\Phi_t^{t_0}(p_{t_0}) = I$, i.e.

$$d\Phi_{t_0}^t(p_t) = d\Phi_t^{t_0}(p_{t_0})^{-1} \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \tag{3.4}$$

**Marsden and Hughes notations:** Once an initial time $t_0$ has been chosen by an observer, then this observer writes $\Phi^{t_0} =^{\text{written}} \Phi$, then $p_{t_0} =^{\text{written}} P \in \Omega_{t_0}$ (capital letter for positions at $t_0$) and $p_t =^{\text{written}} p \in \Omega_t$ (lowercase letter for positions at $t$):

$$p = \Phi(t, P) = \Phi_t(P) \quad \text{when} \quad P = \Phi(t_0, P) = \Phi_{t_0}(P). \tag{3.5}$$

NB: • Talking about the motion of a position $p_{t_0}$ is absurd: A position in $\mathbb{R}^n$ does not refer to a motion. Thus $\Phi^{t_0}$ has no existence without the definition, at first, of the motion $\widetilde{\Phi}$ of particles.

• The definition domain $\Omega_{t_0}$ of $\Phi^{t_0}$ depends on $t_0$: The superscript $^{t_0}$ recalls it. E.g. a late observer with initial time $t_0{}' > t_0$ defines $\Phi^{t_0{}'}$ which definition domain is $[t_1, t_2] \times \Omega_{t_0{}'}$, thus $\Phi^{t_0{}'} \neq \Phi^{t_0}$ in general.

• The following notation is also used:

$$\Phi^{t_0}(t, p_{t_0}) = \Phi(t; t_0, p_{t_0}). \tag{3.6}$$

The couple $(t_0, p_{t_0})$ is "the initial condition", or $t_0$ and $p_{t_0}$ are the initial conditions, see § 5 (flows).

• If an observer chooses a origin $\mathcal{O} \in \mathbb{R}^n$ then with (1.5) he can also use

$$\vec{x}_{t_0} = \overrightarrow{\mathcal{O}p_{t_0}} = \vec{\varphi}^{t_0}(t_0, \vec{x}_{t_0}) = \vec{X} = \overrightarrow{\mathcal{O}P} \quad \text{and} \quad \vec{x}_t = \overrightarrow{\mathcal{O}p_t} = \vec{\varphi}^{t_0}(t, \vec{x}_{t_0}) = \vec{x} = \overrightarrow{\mathcal{O}p}. \tag{3.7}$$

### 3.1.2 Trajectories

Let $(t_0, p_{t_0}) \in [t_1, t_2] \times \Omega_{t_0}$ (initial conditions); Then (3.1) defines

$$\Phi_{p_{t_0}}^{t_0} : \begin{cases} [t_1, t_2] & \to \mathbb{R}^n \\ \quad t & \mapsto p(t) = \Phi_{p_{t_0}}^{t_0}(t) := \widetilde{\Phi}_{P_{Obj}}(t) = \Phi^{t_0}(t, p_{t_0}) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}_{P_{Obj}}(t_0). \end{cases} \tag{3.8}$$

**Definition 3.3** $\Phi_{p_{t_0}}^{t_0}$ is called the (parametric) "trajectory of $p_{t_0}$", which means: $\Phi_{p_{t_0}}^{t_0}$ is the trajectory of the particle $P_{Obj}$ that is located at $p_{t_0} = \widetilde{\Phi}(t, P_{Obj})$ at $t_0$. And the geometric "trajectory of $p_{t_0}$" is

$$\text{Im}(\Phi_{p_{t_0}}^{t_0}) = \Phi_{p_{t_0}}^{t_0}([t_1, t_2]) = \bigcup_{t \in [t_1, t_2]} \{\Phi_{p_{t_0}}^{t_0}(t)\} \quad (= \text{Im}(\widetilde{\Phi}_{P_{Obj}})). \tag{3.9}$$

NB: The terminology "trajectory of $p_{t_0}$" is awkward, since a position $p_{t_0}$ does not move: It is indeed the trajectory $\widetilde{\Phi}_{P_{Obj}}$ of a particle $P_{Obj}$ which is at $p_{t_0}$ at $t_0$ that must be understood.

### 3.1.3 Streaklines (lignes d'émission)

Take a film between $t_0$ and $T$ (start and end).

**Definition 3.4** Let $Q$ be a fixed point in $\mathbb{R}^n$ (you see the point $Q$ on each photo that make up the film). The streakline through $Q$ is the set

$$\begin{aligned} E_{t_0,T}(Q) &= \{p \in \Omega : \exists \tau \in [t_0,T] : p = \Phi_T^\tau(Q) = (\Phi_\tau^T)^{-1}(Q)\} \\ &= \{p \in \Omega : \exists u \in [0,T{-}t_0] : p = \Phi_T^{T-u}(Q) = (\Phi_{T-u}^T)^{-1}(Q)\} \\ &= \bigcup_{\tau\in[t_0,T]} \{\Phi_Q^\tau(T)\} = \bigcup_{u\in[0,T-t_0]} \{\Phi_Q^{T-u}(T)\}. \end{aligned} \tag{3.10}$$

= the set of the positions (a curve in $\mathbb{R}^n$) of all the particles which were at $Q$ at a $\tau \in [t_0,T]$.

**Example 3.5** Smoke comes out of a chimney. Fix a camera nearby, choose a point $Q$ at the top of the chimney where the particles are colored. At $t_0$ start a film and at $T$ stop filming. Then superimpose the photos of the film: The obtained colored curve is the streakline. ▞

## 3.2 Lagrangian functions: Two point tensors

Consider a motion $\widetilde{\Phi}$, choose (subjective) a $t_0 \in [t_1,t_2]$, let $\Omega_{t_0} = \widetilde{\Phi}(t_0, \mathit{Obj})$ (initial configuration).

**Definition 3.6** In short: A Lagrangian function is a function defined on the set of Lagrangian variables: It is a function

$$\mathcal{L}ag^{t_0} : \begin{cases} [t_1,t_2] \times \Omega_{t_0} & \to \text{ some tensorial set } S \\ (t,p_{t_0}) & \to \mathcal{L}ag^{t_0}(t,p_{t_0}), \end{cases} \tag{3.11}$$

(A Eulerian function does not depend on any $t_0$ cf. (2.2).)

**Precise definition:** $\mathcal{L}ag^{t_0}$ being defined in (3.11), a Lagrangian function is a function

$$\widetilde{\mathcal{L}ag}^{t_0} : \begin{cases} [t_1,t_2] \times \Omega_{t_0} & \to \mathcal{C} \times S \\ (t,p_{t_0}) & \to \widetilde{\mathcal{L}ag}^{t_0}(t,p_{t_0}) = ((t,p_t), \mathcal{L}ag^{t_0}(t,p_{t_0})) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \end{cases} \tag{3.12}$$

And $\widetilde{\mathcal{L}ag}^{t_0}(t,p_{t_0})$ has to be drawn at $(t,p_t)$ (not at $(t_0,p_{t_0})$).

**Interpretation:** (3.12) tells that $\mathcal{L}ag^{t_0}(t,p_{t_0})$ is ***not*** represented at $(t,p_{t_0})$ but at $(t,p_t)$:

$$\mathrm{Im}(\widetilde{\mathcal{L}ag}^{t_0}) = \{((t,p_t), \mathcal{L}ag^{t_0}(t,p_{t_0}))\} \quad \text{while} \quad \mathrm{graph}(\mathcal{L}ag^{t_0}) = \{((t,p_{t_0}), \mathcal{L}ag^{t_0}(t,p_{t_0})), \tag{3.13}$$

thus

$$\mathrm{Im}(\widetilde{\mathcal{L}ag}^{t_0}) \neq \mathrm{graph}(\mathcal{L}ag^{t_0}) : \tag{3.14}$$

So a Lagrangian function does ***not*** define a tensor in the usual sense. To compare with the Eulerian function $\mathcal{E}ul$ which defines a tensor (in particular $\mathrm{Im}(\widehat{\mathcal{E}ul}) = \mathrm{graph}(\mathcal{E}ul)$ cf. (2.3)).

**Definition 3.7** (Marsden and Hughes [17].) A Lagrangian function is a "two point tensor" in reference to the points $p_{t_0} \in \Omega_{t_0}$ (departure set) and $p_t \in \Omega_t$ (arrival set) where the value $\mathcal{L}ag^{t_0}(t,p_{t_0})$ is considered (the value $\mathcal{L}ag^{t_0}(t,p_{t_0})$ is not considered at $(t,p_{t_0})$).

**Example 3.8** Scalar values: $\mathcal{L}ag^{t_0}(t,p_{t_0}) = \Theta^{t_0}(t,p_{t_0}) =$ temperature at $t$ at $p_t = \Phi_t^{t_0}(p_{t_0}) = \widetilde{\Phi}(t,P_{\mathit{Obj}})$ of the particle $P_{\mathit{Obj}}$ that was at $p_{t_0}$ at $t_0$. (So, continuing example 2.2, $\Theta^{t_0}(t,p_{t_0}) = \theta(t,p_t)$.) ▞

**Example 3.9** Vectorial values: $\mathcal{L}ag^{t_0}(t,p_{t_0}) = \vec{U}^{t_0}(t,p_{t_0}) =$ force at $t$ at $p_t = \Phi_t^{t_0}(p_{t_0}) = \widetilde{\Phi}(t,P_{\mathit{Obj}})$ acting on the particle $P_{\mathit{Obj}}$ that was at $p_{t_0}$ at $t_0$. (So, continuing example 2.3, $\vec{U}^{t_0}(t,p_{t_0}) = \vec{u}(t,p_t)$.) ▞

If $t$ is fixed or if $p_{t_0} \in \Omega_{t_0}$ is fixed, then we define (in short)

$$\mathcal{L}ag_t^{t_0} : \begin{cases} \Omega_{t_0} & \to S \\ p_{t_0} & \to \mathcal{L}ag_t^{t_0}(p_{t_0}) := \mathcal{L}ag^{t_0}(t,p_{t_0}), \end{cases} \tag{3.15}$$

$$\mathcal{L}ag_{p_{t_0}}^{t_0} : \begin{cases} [t_1,t_2] & \to S \\ t & \to \mathcal{L}ag_{p_{t_0}}^{t_0}(t) := \mathcal{L}ag^{t_0}(t,p_{t_0}). \end{cases} \tag{3.16}$$

**Remark 3.10** The position $p_{t_0}$ is also sometimes called a "material point", which is counter intuitive: $P_{\mathit{Obj}}$ (objective) is the material point, and $p_{t_0}$ is just its spatial position at $t_0$ (subjective); And a Eulerian variable $p_t$ is not called a "material point" at $t$...

By the way, the variable $p_t$ is also called the "updated Lagrangian variable"... ▞

## 3.3 Lagrangian function associated with a Eulerian function

### 3.3.1 Definition

Let $\widetilde{\Phi}$ be a motion, cf. (1.4). Let $\mathcal{E}ul$ be a Eulerian function, cf. (2.3). Let $t_0 \in [t_1, t_2]$.

**Definition 3.11** The Lagrangian function $\mathcal{L}ag^{t_0}$ associated with the Eulerian function $\mathcal{E}ul$ is defined by

$$\boxed{\mathcal{L}ag_t^{t_0} := \mathcal{E}ul_t \circ \Phi_t^{t_0}} \tag{3.17}$$

for all $t \in [t_1, t_2]$. I.e., for all $(t, p_{t_0}) \in [t_1, t_2] \times \Omega_{t_0}$,

$$\mathcal{L}ag^{t_0}(t, p_{t_0}) := \mathcal{E}ul(t, p_t), \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \tag{3.18}$$

I.e., for all $(t, P_{Obj}) \in [t_1, t_2] \times Obj$,

$$\mathcal{L}ag_t^{t_0}(\widetilde{\Phi}(t_0, P_{Obj})) := \mathcal{E}ul_t(\widetilde{\Phi}(t, P_{Obj})). \tag{3.19}$$

### 3.3.2 Remarks

• For one motion, there is only one Eulerian function $\mathcal{E}ul$, while there are as many Lagrangian function $\mathcal{L}ag^{t_0}$ as they are $t_0$ (as many as observers): The Lagrangian function $\mathcal{L}ag^{t_0'}$ of a late observer ($t_0' > t_0$) is different from $\mathcal{L}ag^{t_0}$ since the domains of definition $\Omega_{t_0}$ and $\Omega_{t_0'}$ are different (in general).

• If you have a Lagrangian function, then you can associate the function (similar to (3.17))

$$\mathcal{E}ul_t^{t_0} := \mathcal{L}ag_t^{t_0} \circ (\Phi_t^{t_0})^{-1}, \tag{3.20}$$

but this function depends on $t_0$ (a priori).

## 3.4 Lagrangian velocity

### 3.4.1 Definition

**Definition 3.12** In short: The Lagrangian velocity at $t$ at $p_t = \widetilde{\Phi}(t, P_{Obj})$ of the particle $P_{Obj}$ is the function

$$\vec{V}^{t_0} : \begin{cases} \mathbb{R} \times \Omega_{t_0} & \to \vec{\mathbb{R}}_t^n \\ (t, p_{t_0}) & \to \vec{V}^{t_0}(t, p_{t_0}) := \widetilde{\Phi}_{P_{Obj}}{}'(t) \quad (= \lim_{h \to 0} \dfrac{\widetilde{\Phi}_{P_{Obj}}(t+h) - \widetilde{\Phi}_{P_{Obj}}(t)}{h}) \end{cases} \tag{3.21}$$

when $p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj})$. Thus $\vec{V}^{t_0}(t, p_{t_0}) = \widetilde{\Phi}_{P_{Obj}}{}'(t) = \vec{v}(t, p_t) \in \vec{\mathbb{R}}_t^n$ is the velocity at $t$ at $p_t = \widetilde{\Phi}(t, P_{Obj})$ of the particle $P_{Obj}$ which was at $p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj})$ at $t_0$, tangent to graph($\vec{v}$) at $(t, p_t)$: Drawn at $(t, p_t)$.

Precisely: The Lagrangian velocity is the two point vector field given by

$$\widehat{\vec{V}^{t_0}}(t, p_{t_0}) : \begin{cases} \mathbb{R} \times \Omega_{t_0} & \to \mathcal{C} \times \vec{\mathbb{R}}_t^n \\ (t, p_{t_0}) & \to \widehat{\vec{V}^{t_0}}(t, p_{t_0}) := ((t, p_t), \vec{V}^{t_0}(t, p_{t_0})), \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}). \end{cases} \tag{3.22}$$

**Remark:** A usual definition is given without explicit reference to a particle...: Instead of (3.21),

$$\vec{V}^{t_0}(t, p_{t_0}) := \frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0}), \quad \forall (t, p_{t_0}) \in \mathbb{R} \times \Omega_{t_0}. \tag{3.23}$$

### 3.4.2 Lagrangian velocity versus Eulerian velocity

Let

$$\vec{V}_t^{t_0}(p_{t_0}) := \vec{V}^{t_0}(t, p_{t_0}), \quad \text{and} \quad \vec{V}_{p_{t_0}}^{t_0}(t) := \vec{V}^{t_0}(t, p_{t_0}). \tag{3.24}$$

Then (3.21) and (2.5) give, alternative definition, with $p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj})$ and $p_t = \widetilde{\Phi}(t, P_{Obj})$,

$$\vec{V}^{t_0}(t, p_{t_0}) := \vec{v}(t, p_t) \quad (= \frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0}) = \widetilde{\Phi}_{P_{Obj}}{}'(t) = \text{velocity of } P_{Obj} \text{ at } t \text{ at } p_t). \tag{3.25}$$

Hence

$$\boxed{\vec{V}_t^{t_0} = \vec{v}_t \circ \Phi_t^{t_0}} : \Omega_{t_0} \to \vec{\mathbb{R}}_t^n. \tag{3.26}$$

### 3.4.3 Relation between differentials

For $C^2$ motions (3.26) gives, with $p_t = \Phi_t^{t_0}(p_{t_0})$,

$$d\vec{V}_t^{t_0}(p_{t_0}) = d\vec{v}_t(p_t).d\Phi_t^{t_0}(p_{t_0}) : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t. \tag{3.27}$$

(The differential $d\vec{V}_t^{t_0}$ is a two-point tensor.) I.e., with

$$F_t^{t_0} = d\Phi_t^{t_0} \stackrel{\text{written}}{=} \text{the deformation gradient relative to } t_0 \text{ and } t, \tag{3.28}$$

$$\boxed{d\vec{V}_t^{t_0}(p_{t_0}) = d\vec{v}_t(p_t).F_t^{t_0}(p_{t_0})} : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t. \tag{3.29}$$

Abusively written (dangerous notation: At what points, relative to what times?)

$$d\vec{V} = d\vec{v}.F. \tag{3.30}$$

### 3.4.4 Computation of $d\vec{v} =^{\text{written}} L = \dot{F}.F^{-1}$ with Lagrangian variables

Start with a Lagrangian velocity $\vec{V}^{t_0}$, then define the so-called Eulerian velocity by, with $p_t = \Phi_t^{t_0}(p_{t_0})$,

$$\vec{v}^{t_0}(t, p_t) := \vec{V}^{t_0}(t, p_{t_0}) \tag{3.31}$$

(so-called Eulerian despite its dependence on $t_0$ a priori), i.e. $\vec{v}^{t_0}(t, \Phi_t^{t_0}(p_{t_0})) := \frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0})$. Thus

$$d\vec{v}^{t_0}(t, p_t).d\Phi^{t_0}(t, p_{t_0}) = d(\frac{\partial \Phi^{t_0}}{\partial t})(t, p_{t_0}) = \frac{\partial (d\Phi^{t_0})}{\partial t}(t, p_{t_0}) = \frac{\partial F^{t_0}}{\partial t}(t, p_{t_0}), \tag{3.32}$$

with $\Phi^{t_0}$ $C^2$ for the second equality (Schwarz' theorem). Thus

$$d\vec{v}^{t_0}(t, p_t) = \frac{\partial F^{t_0}}{\partial t}(t, p_{t_0}).F^{t_0}(t, p_{t_0})^{-1}, \quad \text{written in short} \quad L := d\vec{v} = \dot{F}.F^{-1}, \tag{3.33}$$

but $L$ thus “defined” is defined at what points? What times? Eulerian? Lagrangian?

NB: Start with Eulerian quantities and use Eulerian quantities as long as possible[1], which in particular say that $L = d\vec{v}$ doesn't depend on $t_0$: It is Eulerian.

## 3.5 Lagrangian acceleration

Let $P_{Obj} \in Obj$, $t_0, t \in \mathbb{R}$, $p_{t_0} = \widetilde{\Phi}_{P_{Obj}}(t_0)$ and $p_t = \widetilde{\Phi}_{P_{Obj}}(t)$ (positions of $P_{Obj}$ at $t_0$ and $t$).

**Definition 3.13** In short, the Lagrangian acceleration at $t$ at $p_t$ of the particle $P_{Obj}$ is

$$\vec{\Gamma}^{t_0}(t, p_{t_0}) := \widetilde{\Phi}_{P_{Obj}}''(t) \quad \text{when} \quad p_{t_0} = \widetilde{\Phi}_{P_{Obj}}(t_0). \tag{3.34}$$

In other words

$$\vec{\Gamma}^{t_0}(t, p_{t_0}) := \vec{\gamma}(t, p_t) \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}), \tag{3.35}$$

where $\vec{\gamma}(t, p_t)$ is the Eulerian acceleration at $t$ at $p_t = \widetilde{\Phi}(t, P_{Obj})$, cf. (2.40).

Precisely: The Lagrangian acceleration is the “two point vector field” defined on $\mathbb{R} \times \Omega_{t_0}$ by

$$\widetilde{\vec{\Gamma}}^{t_0}(t, p_{t_0}) = ((t, p_t), \widetilde{\Phi}_{P_{Obj}}''(t)), \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}). \tag{3.36}$$

In particular $\vec{\Gamma}^{t_0}(t, p_{t_0})$ is not drawn on the graph of $\vec{\Gamma}^{t_0}$ at $(t, p_{t_0})$, but on the graph of $\vec{\gamma}$ at $(t, p_t)$.

[1] To get Eulerian results from Lagrangian computations can make the understanding of a Lie derivative quite difficult: To introduce the “so-called” Lie derivatives in classical mechanics you can find the following steps: 1- At $t$ consider the Cauchy stress vector $\vec{t}$ (Eulerian), 2- then with a unit normal vector $\vec{n}$, define the associated Cauchy stress tensor $\underline{\underline{\sigma}}$ (satisfying $\vec{t} = \underline{\underline{\sigma}}.\vec{n}$), 3- then use the virtual power and the change of variables in integrals to be back into $\Omega_{t_0}$ to be able to work with Lagrangian variables, 4- then introduce the first Piola–Kirchhoff (two point) tensor $\mathbb{K}$, 5- then introduce the second Piola–Kirchhoff tensor $\mathbb{S}$ (endomorphism in $\Omega_{t_0}$), 6- then differentiate $\mathbb{S}$ in $\Omega_{t_0}$ (in the Lagrangian variables although the initials variables are the Eulerian variables in $\Omega_t$), 7- then back in $\Omega_t$ to get back to Eulerian functions (change of variables in integrals), 8- then you get some Jaumann or Truesdell or other so called Lie derivatives type terms, the appropriate choice among all these derivatives being quite obscure because the covariant objectivity has been forgotten en route... While, with simple Eulerian considerations, it requires a few lines to understand the (real) Lie derivative (Eulerian concept) and its simplicity, see § 9, and deduce second order covariant objective results.

If $t$ is fixed, or if $p_{t_0} \in \Omega_{t_0}$ is fixed, then define

$$\vec{\Gamma}_t^{t_0}(p_{t_0}) := \vec{\Gamma}^{t_0}(t, p_{t_0}), \quad \text{and} \quad \vec{\Gamma}_{p_{t_0}}^{t_0}(t) := \vec{\Gamma}^{t_0}(t, p_{t_0}). \tag{3.37}$$

Thus

$$\vec{\Gamma}_t^{t_0} = \vec{\gamma}_t \circ \Phi_t^{t_0}, \quad \text{and} \quad d\vec{\Gamma}_t^{t_0}(p_{t_0}) = d\vec{\gamma}_t(p_t).F_t^{t_0}(p_{t_0}), \tag{3.38}$$

when $p_t = \Phi_t^{t_0}(p_{t_0})$ and $F_t^{t_0} := d\Phi_t^{t_0}$ (the deformation gradient).

Dangerous notation: $d\vec{\Gamma} = d\vec{\gamma}.F$ (points? times?).

## 3.6 Time Taylor expansion of $\Phi^{t_0}$

Let $p_{t_0} \in \Omega_{t_0}$. Then, at second order,

$$\Phi_{p_{t_0}}^{t_0}(\tau) = \Phi_{p_{t_0}}^{t_0}(t) + (\tau-t)\Phi_{p_{t_0}}^{t_0}{}'(t) + \frac{(\tau-t)^2}{2}\Phi_{p_{t_0}}^{t_0}{}''(t) + o((\tau-t)^2), \tag{3.39}$$

i.e.

$$p(\tau) = p(t) + (\tau-t)\vec{V}^{t_0}(t, p_{t_0}) + \frac{(\tau-t)^2}{2}\vec{\Gamma}^{t_0}(t, p_{t_0}) + o((\tau-t)^2) \quad \text{when} \quad p(\tau)\Phi_\tau^{t_0}(p_{t_0}). \tag{3.40}$$

NB: There are ***three*** times involved: $t_0$ (observer dependent), $t$ and $\tau$ (for the Taylor expansion). To compare with (2.44)-(2.45): $p(\tau) = p(t) + (\tau-t)\vec{v}(t, p(t)) + \frac{(\tau-t)^2}{2}\vec{\gamma}(t, p(t)) + o((\tau-t)^2)$, independent of $t_0$.

## 3.7 A vector field that let itself be deformed by a motion ($\to$ Lie)

Fix $t_0$ and let $\vec{w}_{t_0} : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \vec{\mathbb{R}}^n_{t_0} \\ p_{t_0} & \to \vec{w}_{t_0}(p_{t_0}) := \vec{w}(t_0, p_{t_0}) \end{array} \right\}$ (vector field in $\Omega_{t_0}$), and define the (virtual) vector field called the push-forward of $\vec{w}t_0$ by $\Phi_t^{t_0}$ (esult of the deformation of $\vec{w}_{t_0}$ by the motion, see figure 4.1):

$$\vec{w}_{t_0*} : \left\{ \begin{array}{ll} \mathcal{C} & \to \vec{\mathbb{R}}^n_t \\ (t, p_t) & \to \vec{w}_{t_0*}(t, p_t) := d\Phi^{t_0}(t, p_{t_0}).\vec{w}_{t_0}(p_{t_0}), \quad \text{when} \quad p(t) = \Phi^{t_0}(t, p_{t_0}). \end{array} \right. \tag{3.41}$$

**Proposition 3.14** *For $C^2$ motions, we have (time variation rate along a virtual trajectory)*

$$\frac{D\vec{w}_{t_0*}}{Dt} = d\vec{v}.\vec{w}_{t_0*}, \tag{3.42}$$

*i.e. $\mathcal{L}_{\vec{v}}\vec{w}_{t_0*} = \vec{0}$, where $\mathcal{L}_{\vec{v}}\vec{u} := \frac{D\vec{u}}{Dt} - d\vec{v}.\vec{u}$ $(= \frac{\partial\vec{u}}{\partial t} + d\vec{u}.\vec{v} - d\vec{v}.\vec{u})$ is the Lie derivative of a (unsteady) vector field $\vec{u} : \mathcal{C} \to \vec{\mathbb{R}}^n$ along $\vec{v}$.*

**Proof.** $p_{t_0}$ being fixed and $d\Phi^{t_0}(t, p_{t_0}) =^{\text{written}} F(t)$, (3.41) gives $\vec{w}_{t_0*}(t, p(t)) = F(t).\vec{w}_{t_0}(p_{t_0})$, thus $\frac{D\vec{w}_{t_0*}}{Dt}(t, p(t)) = F'(t).\vec{w}_{t_0}(p_{t_0}) = F'(t).F(t)^{-1}.\vec{w}_{t_0*}(t, p(t)) =^{(3.33)} d\vec{v}(t, p(t)).\vec{w}_{t_0*}(t, p(t))$, i.e. (3.42). ∎

**Interpretation:** Let $\vec{w} : \left\{ \begin{array}{ll} \mathcal{C} & \to \vec{\mathbb{R}}^n \\ (t, p_t) & \to \vec{w}(t, p_t) \end{array} \right\}$ be a $C^0$ Eulerian vector field, and $\vec{w}_{t_0}(p_{t_0}) := \vec{w}(t_0, p_{t_0})$. We will see that $\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) = \lim_{t\to t_0} \frac{\vec{w}(t,p(t))-\vec{w}_{t_0*}(t,p(t))}{h}$ measures the "resistance of $\vec{w}$ to a motion", see § 9.3.2; In particular the result $\mathcal{L}_{\vec{v}}\vec{w}_{t_0*}(t_0, p_{t_0}) = \vec{0}$ is "obvious" $(= \lim_{t\to t_0} \frac{\vec{w}_{t_0*}(t,p(t))-\vec{w}_{t_0*}(t,p(t))}{h})$, and tells that $\vec{w}$ does not oppose any resistance to the flow.

## 3.8 Examples

Let $\widetilde{\Phi} : [t_1, t_2] \times Obj \to \mathbb{R}^n$ be a $C^1$ motion, $t_0 \in ]t_1, t_2[$, $\Phi^{t_0}$ be the associated Lagrangian motion.

### 3.8.1 Rectilinear motion

The motion of $P_{Obj}$ is rectilinear iff, for all $\forall t_0, t \in [t_1, t_2]$, $\exists \alpha(t_0, t) \in \mathbb{R}$,

$$\widetilde{\Phi}_{P_{Obj}}(t) = \widetilde{\Phi}_{P_{Obj}}(t_0) + \alpha(t_0, t)\,\widetilde{\Phi}_{P_{Obj}}{}'(t_0), \quad \text{i.e.} \quad \widetilde{\Phi}_{P_{Obj}}(t) - \widetilde{\Phi}_{P_{Obj}}(t_0) \parallel \widetilde{\Phi}_{P_{Obj}}{}'(t_0). \tag{3.43}$$

And it is rectilinear uniform iff the rectilinear trajectory is traveled at constant velocity, i.e., $\forall t_0, t \in [t_1, t_2]$,

$$\widetilde{\Phi}_{P_{Obj}}(t) = \widetilde{\Phi}_{P_{Obj}}(t_0) + (t-t_0)\,\widetilde{\Phi}_{P_{Obj}}{}'(t_0), \tag{3.44}$$

i.e. $p(t) = p(t_0) + (t-t_0)\,\vec{V}^{t_0}(t_0, p(t_0))$ where $p(t) = \widetilde{\Phi}(t, P_{Obj})$ and $\vec{V}^{t_0}(t, p_{t_0}) = \widetilde{\Phi}_{P_{Obj}}{}'(t_0)$.

### 3.8.2 Circular motion

$P = \widetilde{\Phi}(t_0, P_{Obj}) \in \Omega_{t_0}$ and $\Phi_P^{t_0}(t) = \Phi^{t_0}(t, P)$. Choose an origin $O$ in $\mathbb{R}^2$ and a unique Euclidean basis $(\vec{E}_1, \vec{E}_2)$ at all time (in $\mathbb{R}^2$), and let $\vec{\varphi}_P^{t_0}(t) = \overrightarrow{\mathcal{O}\Phi_P^{t_0}(t)} = x(t)\vec{E}_1 + y(t)\vec{E}_2$. The motion $\Phi_P^{t_0}$ is a circular motion iff, for all $t$, $\vec{\varphi}_P^{t_0}(t) = (a+R\cos(\theta(t))\vec{E}_1 + (b+R\sin(\theta(t)))\vec{E}_2$, i.e.

$$[\vec{\varphi}_P^{t_0}(t)]_{|\vec{E}} = \begin{pmatrix} x(t) = a + R\cos(\theta(t)) \\ y(t) = b + R\sin(\theta(t)) \end{pmatrix}, \tag{3.45}$$

for some $R > 0$, some $a, b \in \mathbb{R}$, and some function $\theta : \mathbb{R} \to \mathbb{R}$: The particle $P_{Obj}$ stays on the circle of radius $R$ centered at $\mathcal{O}_C = \begin{pmatrix} a \\ b \end{pmatrix}$. The circular motion is uniform iff $\exists \omega_0 \in \mathbb{R}$, $\forall t \in [t_1, t_2]$, $\theta(t) = \omega_0 t$ (i.e. $\theta''(t) = 0$).

Thus the Lagrangian velocity of a circular motion is $\vec{V}_P^{t_0}(t) = (\Phi_P^{t_0})'(t) = (\vec{\varphi}_P^{t_0})'(t)$, i.e. given by

$$[\vec{V}_P^{t_0}(t)]_{|\vec{E}} = R\theta'(t) \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix} \tag{3.46}$$

(orthogonal to the radius vector $\vec{V}_P^{t_0}(t)$ is to $\vec{\varphi}_P^{t_0}(t)$). And the Lagrangian acceleration $\vec{\Gamma}_P^{t_0}(t)$ is given by

$$[\vec{\Gamma}_P^{t_0}(t)]_{|\vec{E}} = R\theta''(t) \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix} + R(\theta'(t))^2 \begin{pmatrix} -\cos(\theta(t)) \\ -\sin(\theta(t)) \end{pmatrix}. \tag{3.47}$$

Then consider the orthonormal basis $(\vec{e}_r(t), \vec{e}_\theta(t))$ given by (normalized polar basis)

$$[\vec{e}_r(t)]_{|\vec{E}} = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}, \quad \text{and} \quad [\vec{e}_\theta(t)]_{|\vec{E}} = \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix}. \tag{3.48}$$

We get

$$\vec{V}_P^{t_0} = R\theta' \, \vec{e}_\theta \quad \text{and} \quad \vec{\Gamma}_P^{t_0} = R\big(\theta'' \, \vec{e}_\theta - (\theta')^2 \, \vec{e}_r\big). \tag{3.49}$$

Immersed in $\mathbb{R}^3$, with $\vec{E}_3 = \vec{E}_1 \times \vec{E}_2$ and $\omega(t) = \theta'(t)$ and $\vec{\omega}(t) = \omega(t)\vec{E}_3$,

$$\vec{V}_P^{t_0} = \vec{\omega} \times (\vec{\varphi}_P^{t_0} - \overrightarrow{OO_C}), \quad \text{and} \quad \vec{\Gamma}_P^{t_0} = R\big(\frac{d\omega}{dt} \, \vec{e}_\theta - \omega^2 \, \vec{e}_r\big). \tag{3.50}$$

### 3.8.3 Motion of a planet (centripetal acceleration)

Illustration: *Obj* is a planet from the solar system. $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is a Euclidean basis fixed relative to stars where $(\vec{e}_1, \vec{e}_2)$ define the ecliptic plane, $(\cdot,\cdot)_g$ is the associated Euclidean dot product, $||.||$ the Euclidean associated norm, $\mathcal{O}$ the center of the Sun is the origin in $\mathbb{R}^3$, $\mathcal{R} = (\mathcal{O}, (\vec{e}_i))$, $\vec{\varphi}_P^{t_0}(t) = \overrightarrow{\mathcal{O}\Phi_P^{t_0}(t)}$. So the Lagrangian velocities and accelerations are given by

$$\vec{V}_P(t) = \frac{d\Phi_P}{dt}(t) = \frac{d\vec{\varphi}_P}{dt}(t), \quad \text{and} \quad \vec{A}_P(t) = \frac{d^2\Phi_P}{dt^2}(t) = \frac{d^2\vec{\varphi}_P}{dt^2}(t). \tag{3.51}$$

**Definition 3.15** The motion of a particle $P_{Obj}$ is a centripetal acceleration motion iff the motion is not rectilinear and, at all time, the acceleration vector points to a fixed point $F \in \mathbb{R}^3$ (focus).

**Example 3.16** The motion of a planet from the solar system is an elliptical motion, is a centripetal acceleration motion, one of its focus being at the center of the Sun. ▪

Consider a centripetal motion and choose $\mathcal{O} := F$ the focus: We have $\overrightarrow{\mathcal{O}\Phi_P(t)} \parallel \vec{A}_P(t)$ for all $t$, i.e.

$$\vec{\varphi}_P(t) \times \vec{A}_P(t) = \vec{0}, \quad \forall t. \tag{3.52}$$

**Definition 3.17** The areolar velocity at $t$ is the vector

$$\vec{Z}(t) = \frac{1}{2}\vec{\varphi}_P(t) \times \vec{V}_P(t). \tag{3.53}$$

**Proposition 3.18** *If $\Phi$ is a centripetal acceleration motion, then the velocity never vanishes, the areolar velocity is constant, i.e.*

$$\vec{Z}(t) = \vec{Z}(t_0), \quad \forall t, \tag{3.54}$$

*and the motion takes place in the affine plane orthogonal to $\vec{Z}(t_0)$ passing through $F$. And the position vectors sweep equal areas in equal times.*

**Proof.** $2\frac{d\vec{Z}}{dt}(t) =^{(3.53)} \frac{d\vec{\varphi}_P}{dt}(t)\times\vec{V}_P(t)+\vec{\varphi}(t)\times\frac{d\vec{V}_P}{dt}(t) = \vec{V}_P(t)\times\vec{V}_P(t)+\vec{\varphi}(t)\times\vec{A}_P(t) = \vec{0}+\vec{0}$ thanks to (3.52). Thus $\vec{Z}$ is constant, $\vec{Z}(t)=\vec{Z}(t_0)$ for all $t$. Then (3.53) gives: $\vec{\varphi}_P(t)$ and $\vec{V}_P(t)$ are $\perp \vec{Z}(t_0)$ for all $t$.

Thus if $\vec{V}_P(\tau)=\vec{0}$ for some $\tau$ then $\vec{Z}(\tau)=\vec{0}$, thus $\vec{Z}(t)=\vec{0}$ for all $t$, thus $\vec{\varphi}_P(t) \parallel \vec{V}_P(t)$ for all $t$, thus $\vec{\varphi}_P(t) \parallel \vec{\varphi}_P{}'(t)$ for all $t$, i.e. $\vec{\varphi}_P(t)=f(t)\vec{\varphi}_P{}'(t)$ for all $t$, thus $\vec{\varphi}_P(t)=\vec{\varphi}_P(t_0)e^{F(t)}$ where $F$ is a primitive of $f$ s.t. $F(t_0)=0$, thus $\vec{\varphi}_P(t) \parallel \vec{\varphi}_P(t_0)$, so $\overrightarrow{\mathcal{O}\Phi_P(t)} \parallel \overrightarrow{\mathcal{O}\Phi_P(t_0)}$, for all $t$: The motion is rectilinear, which is excluded in the definition of a centripetal acceleration motion. Thus the velocity never vanishes.

And $\vec{\varphi}_P\times\vec{A}_P=\vec{0}$ gives $\vec{V}_P \bullet (\vec{\varphi}_P\times\vec{A}_P)=0=\det(\vec{V}_P,\vec{\varphi}_P,\vec{A}_P)=(\vec{V}_P\times\vec{\varphi}_P)\bullet\vec{A}_P=\vec{Z}\bullet\vec{A}_P$, thus $\vec{A}_P(t)\perp\vec{Z}(t_0)$ for all $t$. And the Taylor expansion gives $\vec{\varphi}_P(t)-\vec{\varphi}_P(t_0)=\vec{V}_P(t_0)(t-t_0)+\int_{\tau=t_0}^t\vec{A}_P(\tau)(t-\tau)^2\,d\tau \perp \vec{Z}(t_0)$ for all $t$, thus the motion lives in the plane $\vec{\varphi}_P(t_0)+\mathrm{Vect}\{\vec{Z}(t_0)\}^T$. And $\vec{A}_P(t)$ is a vector at $\Phi_P(t)$ and points towards the focus $F$, thus $\overrightarrow{F\Phi_P(t)}\perp\vec{Z}(t_0)$, thus the affine plane passes through $F$.

The area $S(t,h)$ swept by $\vec{\varphi}_P$ between $t$ and $t{+}h$ is, at first order, the area of the triangle whose sides are $\vec{\varphi}_P(t)$ and $\vec{\varphi}_P(t{+}h)$ ("angular sector"):

$$S(t,h)=||\vec{S}(t,h)|| \quad\text{with}\quad \vec{S}(t,h)=\frac{1}{2}\vec{\varphi}_P(t)\times\vec{\varphi}_P(t{+}h)+o(h). \tag{3.55}$$

We want $\frac{\partial S}{\partial t}(t,h)=0$ for any (admissible) fixed $h$, i.e. $S(t,h)=S(t_0,h)$ for all $h$.

We have $\vec{\varphi}_P(t{+}h)=\vec{\varphi}_P(t)+\vec{\varphi}_P{}'(t)h+o(h)=\vec{\varphi}_P(t)+\vec{V}_P(t)h+o(h)$, thus

$$2\vec{S}(t,h)=\vec{\varphi}_P(t)\times(\vec{\varphi}_P(t)+\vec{V}_P(t)h+o(h))+o(h)=\vec{\varphi}_P(t)\times\vec{V}_P(t)h+o(h). \tag{3.56}$$

And $\vec{S}(t,0)=0$, thus $\frac{\vec{S}(t,h)-\vec{S}(t,0)}{h}=\frac{1}{2}\vec{\varphi}_P(t)\times\vec{V}_P(t)+o(1)$, thus

$$\frac{\partial\vec{S}}{\partial h}(t,0)=\frac{1}{2}\vec{\varphi}_P(t)\times\vec{V}_P(t)=\vec{Z}(t)\overset{(3.54)}{=}\vec{Z}(t_0)=\frac{\partial\vec{S}}{\partial h}(t_0,0). \tag{3.57}$$

And $S^2(t,h)=S(t,h)^2=||\vec{S}(t,h)||^2=\vec{S}(t,h)\bullet\vec{S}(t,h)$ gives

$$\frac{\partial(S)^2}{\partial h}(t,h)=2\frac{\partial\vec{S}}{\partial h}(t,h)\bullet\vec{S}(t,h), \quad\text{thus}\quad \frac{\partial(S)^2}{\partial h}(t,0)=0 \tag{3.58}$$

because $S(t,0)=0$. Thus the function $t\to S(t,0)$ is independent of $t$: The position vectors sweep equal areas in equal times. ∎

**Interpretation.** (Non rectilinear motion.) The area swept by $\vec{\varphi}_P(t)$ is, at first order, the area of the triangle whose sides are $\vec{\varphi}_P(t)$ and $\vec{\varphi}_P(t+\tau)$ ("angular sector"). So, with $\tau$ close to 0, let

$$\vec{S}_t(\tau)=\frac{1}{2}\vec{\varphi}_P(t)\times\vec{\varphi}_P(t+\tau), \quad\text{and}\quad S_t(\tau)=||\vec{S}_t(\tau)||, \tag{3.59}$$

the vectorial and scalar areas. With $\vec{\varphi}_P(t{+}\tau)=\vec{\varphi}_P(t)+\vec{V}_P(t)\tau+o(\tau)$ (Taylor) we get

$$\vec{S}_t(\tau)=\frac{1}{2}\vec{\varphi}_P(t)\times(\vec{V}_P(t)\tau+o(\tau)), \tag{3.60}$$

Since $\vec{S}_t(0)=0$ we get $\frac{\vec{S}_t(\tau)-\vec{S}(0)}{\tau}=\frac{1}{2}\vec{\varphi}_P(t)\times\vec{V}_P(t)+o(1)$, then

$$\frac{d\vec{S}_t}{d\tau}(0)=\frac{1}{2}\vec{\varphi}_P(t)\times\vec{V}_P(t)=\vec{Z}(t)=\vec{Z}(t_0), \tag{3.61}$$

thanks to (3.54), thus

$$\frac{d\vec{S}_t}{d\tau}(0)=\frac{d\vec{S}_{t_0}}{d\tau}(0), \qquad \forall t\in[t_0,T], \tag{3.62}$$

that is, the rate of variation of $\vec{S}_t$ is constant. And with $||\vec{S}_t(\Delta\tau)||^2=(\vec{S}_t(\Delta\tau),\vec{S}_t(\Delta\tau))$ we get

$$\frac{d||\vec{S}_t||^2}{d\tau}(\Delta\tau)=2(\frac{d\vec{S}_t}{d\tau}(\Delta\tau),\vec{S}_t(\Delta\tau)), \tag{3.63}$$

so, since $\vec{S}_t(0)=0$,

$$\frac{d||\vec{S}_t||^2}{d\tau}(0)=0. \tag{3.64}$$

So the function $t\to||\vec{S}_t(0)||^2=S_t(0)^2$ is constant, thus $t\to S_t(0)$ est constant, and $\frac{dS_t}{d\tau}(0)$ is constant.

**Exercice 3.19** Give a parametrization of the swept area, and redo the calculations.

**Answer**. Let
$$r(t) = ||\vec{\varphi}_P(t)||, \quad \theta(t) = \widehat{p(t)OP} \quad \text{(angle)}, \tag{3.65}$$
then
$$\vec{\varphi}_P(t) = \begin{pmatrix} r(t)\cos(\theta(t)) \\ r(t)\sin(\theta(t)) \\ 0 \end{pmatrix}. \tag{3.66}$$
Thus
$$\vec{V}_P(t) = \begin{pmatrix} r'(t)\cos(\theta(t) - r(t))\theta'(t)\sin(\theta(t)) \\ r'(t)\sin(\theta(t) + r(t))\theta'(t)\cos(\theta(t)) \\ 0 \end{pmatrix}. \tag{3.67}$$
With (3.53) we get
$$\vec{Z}(t) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ r^2(t)\theta'(t) \end{pmatrix}, \quad \text{with} \quad r^2(t)\theta'(t) = r^2(t_0)\theta'(t_0) \quad \text{(constant)}, \tag{3.68}$$
cf. (3.54). A parametrization of the swept area is then
$$\vec{\mathcal{A}} : \begin{cases} [0,1] \times [t_0, T] & \to \mathbb{R}^3 \\ (\rho, t) & \to \vec{\mathcal{A}}(\rho, t) \end{cases}, \qquad \vec{\mathcal{A}}(\rho, t) = \begin{pmatrix} \rho\, r(t)\cos(\theta(t)) \\ \rho\, r(t)\sin(\theta(t)) \\ 0 \end{pmatrix}. \tag{3.69}$$
Therefore, the tangent associated vectors are
$$\frac{\partial \vec{\mathcal{A}}}{\partial \rho}(\rho, t) = \begin{pmatrix} r(t)\cos(\theta(t)) \\ r(t)\sin(\theta(t)) \\ 0 \end{pmatrix}, \qquad \frac{\partial \vec{\mathcal{A}}}{\partial t}(\rho, t) = \begin{pmatrix} \rho r'(t)\cos(\theta(t) - \rho r(t))\theta'(t)\sin(\theta(t)) \\ \rho r'(t)\sin(\theta(t) + \rho r(t))\theta'(t)\cos(\theta(t)) \\ 0 \end{pmatrix}, \tag{3.70}$$
hence the vectorial and scalare element areas are
$$d\vec{\sigma} = (\frac{\partial \vec{\mathcal{A}}}{\partial \rho} \times \frac{\partial \vec{\mathcal{A}}}{\partial t}) d\rho dt = \begin{pmatrix} 0 \\ 0 \\ \rho r^2\theta'\, d\rho dt \end{pmatrix}, \qquad d\sigma = \rho r^2 \theta'\, d\rho d\theta. \tag{3.71}$$
Therefore the area between $t_0$ and $t$ is
$$\mathcal{A}(t) = \mathcal{A}(t_0) + \int_{\rho=0}^{1} \int_{\tau=t_0}^{t} \rho r^2(\tau)\theta'(\tau)\, d\rho d\tau = \frac{1}{2}\int_{\tau=t_0}^{t} r(\tau)^2\theta'(\tau)\, d\tau. \tag{3.72}$$
Hence
$$\mathcal{A}'(t) = r(t)^2\theta'(t) = r(t_0)^2\theta'(t_0) \quad (= \text{constant} = ||\vec{Z}(t_0)||), \tag{3.73}$$
cf. (3.68). ▪

**Exercice 3.20** Prove the Binet formulas (non rectilinear central motion):
$$V_P(t)^2 = Z_0^2\Big(\frac{1}{r^2} + (\frac{d\frac{1}{r}}{d\theta})^2\Big)(t), \qquad \vec{\Gamma}_P(t) = -\frac{Z_0^2}{r^2}\Big(\frac{1}{r} + \frac{d^2\frac{1}{r}}{d\theta^2}\Big)(t)\, \vec{e}_r(t), \tag{3.74}$$
for the energy and the acceleration.

**Answer**. Proposition 3.18 tells that $\Phi$ is a planar motion. With (3.65) and $\vec{e}_r(t) = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix}$ we have $\vec{\varphi}(t) = r(t)\vec{e}_r(t)$ (in the plane). Let $\vec{e}_\theta(t) = \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix}$, thus
$$\vec{V}(t) = \frac{dr}{dt}(t)\vec{e}_r(t) + r(t)\frac{d\vec{e}_r}{dt}(t) = r'(t)\vec{e}_r(t) + r(t)\theta'(t)\vec{e}_\theta(t).$$
And $\vec{e}_r(t) \perp \vec{e}_\theta(t)$ gives
$$V^2(t) = (r'(t))^2 + (r(t)\theta'(t))^2.$$
Since $\theta'(t) \neq 0$ for all $t$ (non rectilinear central motion) Let $s(\theta(t)) = r(t)$. Let us suppose that $\theta$ is $C^1$, thus

$\theta' > 0$ or $\theta' < 0$, and $\theta : t \to \theta(t)$ defines a change of variable. And

$$r'(t) = s'(\theta(t))\theta'(t).$$

And (3.73) and $\theta'(t) = \frac{Z_0}{r^2(t)}$ give

$$V^2(t(\theta)) = (s'(\theta))^2 \frac{Z_0^2}{r^4(t)} + r^2(t)\frac{Z_0^2}{r^4(t)} = Z_0^2(\frac{(s'(\theta))^2}{s^4(\theta)} + \frac{1}{s^2(\theta)}) = Z_0^2[\Big(\frac{d\frac{1}{s}}{d\theta}(\theta)\Big)^2 + \frac{1}{s^2(\theta)}].$$

Thus $r(t) = s(\theta)$ and $\frac{dr}{d\theta} := \frac{ds}{d\theta}$ give the first Binet formula. Then

$$\vec\Gamma(t) = r''(t)\vec e_r(t) + r'(t)\frac{d\vec e_r}{dt}(t) + (r'(t)\theta'(t) + r(t)\theta''(t))\vec e_\theta(t) + r(t)\theta'(t)\frac{d\vec e_\theta}{dt}(t),$$

with $\dfrac{d\vec e_r}{dt} \parallel \vec e_\theta$, and $\dfrac{d\vec e_\theta}{dt}(t) = -\theta'(t)\vec e_r(t)$, and $\vec e_\theta \perp \vec\Gamma$ (central motion), we get

$$\vec\Gamma(t) = (r''(t) - r(t)(\theta'(t))^2)\vec e_r(t).$$

And

$$r'(t) = s'(\theta)\theta'(t) = s'(\theta)\frac{Z_0}{r^2(t)} = Z_0\frac{s'(\theta)}{s^2(\theta)} = -Z_0\frac{d\frac{1}{s}}{d\theta}(\theta),$$

thus

$$r''(t) = -Z_0\frac{d^2\frac{1}{s}}{d\theta^2}(\theta)\,\theta'(t) = -\frac{Z_0^2}{r^2(t)}\frac{d^2\frac{1}{s}}{d\theta^2}(\theta),$$

which is the second Binet formula. ∎

### 3.8.4 Screw theory (= torsors, distributors)

See https://perso.isima.fr/leborgne/IsimathMeca/torseur.pdf

# 4 Deformation gradient $F := d\Phi$

Motion $\widetilde\Phi : \left\{\begin{array}{ll} \mathbb{R}\times Obj & \to \mathbb{R}^n \\ (t, P_{Obj}) & \to p_t = \widetilde\Phi(t, P_{Obj}) \end{array}\right\}$, $\Omega_t := \widetilde\Phi(t, Obj)$ configuration at $t$. Fix $t_0, t$ in $\mathbb{R}$, and let $\Phi_t^{t_0} : \left\{\begin{array}{ll} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} = \widetilde\Phi(t_0, P_{Obj}) & \to p_t = \Phi_t^{t_0}(p_{t_0}) := \widetilde\Phi(t, P_{Obj}) \end{array}\right\}$ supposed to be a $C^1$ diffeomorphism. Notations for calculations (quantification) to comply with practices:

1- Classical (unambiguous) notations as in Arnold, Germain: E.g., $(\vec a_i)$ and $(\vec b_i)$ are Cartesian bases resp. in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$, $\vec w_{t_0}(p_{t_0}) = \sum_i w_{t_0,i}(p_{t_0})\vec a_i \in \vec{\mathbb{R}}^n_{t_0}$, $\vec w_t(p_t) = \sum_i w_{t,i}(p_t)\vec b_i \in \vec{\mathbb{R}}^n_t$. And

2- Marsden–Hughes duality notations: Capital letters at $t_0$, lower case letters at $t$, $(\vec E_I)$ and $(\vec e_i)$ are Cartesian bases resp. in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$, $\vec W(P) = \sum_I W^I(P)\vec E_I \in \vec{\mathbb{R}}^n_{t_0}$, $\vec w(p) = \sum_i w^i(p)\vec e_i \in \vec{\mathbb{R}}^n_t$.

## 4.1 Definitions

### 4.1.1 Deformation gradient $F$

**Definition 4.1** The differential $d\Phi_t^{t_0} =^{\text{written}} F_t^{t_0} : \left\{\begin{array}{ll} \Omega_{t_0} & \to \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \\ p_{t_0} & \to F_t^{t_0}(p_{t_0}) := d\Phi_t^{t_0}(p_{t_0}) \end{array}\right\}$ is called "the covariant deformation gradient between $t_0$ and $t$", or simply the "deformation gradient".

The "covariant deformation gradient at $p_{t_0}$ between $t_0$ and $t$", or in short "the deformation gradient at $p_{t_0}$", is the linear map $F_t^{t_0}(p_{t_0}) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$. So, for all $\vec w_{t_0}(p_{t_0}) \in \vec{\mathbb{R}}^n_{t_0}$ (vector at $p_{t_0}$),

$$\boxed{F_t^{t_0}(p_{t_0}).\vec w_{t_0}(p_{t_0}) := \lim_{h\to 0}\frac{\Phi_t^{t_0}(p_{t_0}+h\vec w_{t_0}(p_{t_0})) - \Phi_t^{t_0}(p_{t_0})}{h}} \overset{\text{written}}{=} (\Phi_t^{t_0})_*(\vec w_{t_0})(p_t) \overset{\text{written}}{=} \vec w_{t_0*}(t, p_t) \tag{4.1}$$

vector at $p_t = \Phi_t^{t_0}(p_{t_0})$. See figure 4.1. Marsden–Hughes notations: $\Phi := \Phi_t^{t_0}$, $F := d\Phi$, $P := p_{t_0}$, $\vec W(P) := \vec w_{t_0}(p_{t_0})$, $p = \Phi(P)$, thus

$$\boxed{F(P).\vec W(P) := \lim_{h\to 0}\frac{\Phi(P+h\vec W(P)) - \Phi(P)}{h}} \overset{\text{written}}{=} \Phi_*\vec W(p) \overset{\text{written}}{=} \vec w_*(p). \tag{4.2}$$

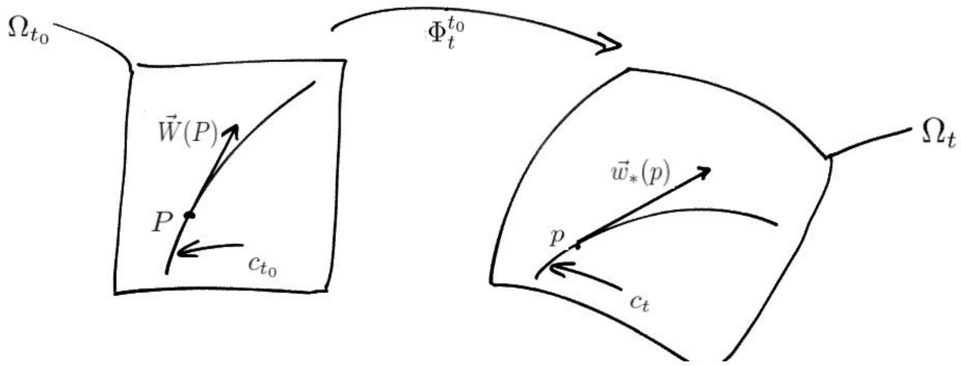


Figure 4.1: Cf. (4.1). $\vec{w}_{t_0} = \vec{W}$ is a vector field in $\Omega_{t_0}$. Consider the integral curve of $\vec{W} = \vec{w}_{t_0}$ in $\Omega_{t_0}$, i.e. the (spatial) curve $c_{t_0} : s \to p_{t_0} = c_{t_0}(s)$ in $\Omega_{t_0}$ s.t. $c_{t_0}{}'(s) = \vec{w}_{t_0}(c_{t_0}(s))$. It is transformed by $\Phi_t^{t_0}$ into the (spatial) curve $c_t = \Phi_t^{t_0} \circ c_{t_0} : s \to p_t = c_t(s) = \Phi_t^{t_0}(c_{t_0}(s))$ in $\Omega_t$; Thus the tangent vector at $c_t$ at $p_t$ is $c_t{}'(s) = d\Phi_t^{t_0}(p_{t_0}).c_{t_0}{}'(s) = d\Phi_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0}) =^{(4.1)} \vec{w}_{t_0*}(t, p_t)$ (push-forward of $\vec{w}_{t_0}$ by $\Phi_t^{t_0}$).

**NB:** The "deformation gradient" $F_t^{t_0} = d\Phi_t^{t_0}$ is ***not*** a "gradient" (its definition does ***not*** need a Euclidean dot product); This leads to confusions when covariance-contravariance and objectivity are at stake. It would be simpler to stick to the name "$F_t^{t_0}$ = the differential of $\Phi_t^{t_0}$", but it is not the standard usage, except in thermodynamics: E.g., the differential $dU$ of the internal energy $U$ is not called "the gradient of $U$" (there is no meaningful Euclidean dot product): It is just called "the differential of $U$"...

### 4.1.2 Push-forward

**Definition 4.2** Let $\vec{w}_{t_0} : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \vec{\mathbb{R}}^n_{t_0} \\ p_{t_0} & \to \vec{w}_{t_0}(p_{t_0}) \end{array} \right\}$ be a vector field in $\Omega_{t_0}$. Its push-forward by $\Phi_t^{t_0}$ is the vector field $(\Phi_t^{t_0})_*(\vec{w}_{t_0})$ in $\Omega_t$ defined with $p_t = \Phi_t^{t_0}(p_{t_0})$ by

$$(\vec{w}_{t_0,t*}(p_t) =) \quad (\Phi_t^{t_0})_* \vec{w}_{t_0}(p_t) := F_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0}), \quad \text{i.e.} \quad (\vec{w}_*(p) =) \quad \Phi_* \vec{W}(p) := F(P).\vec{W}(P) \tag{4.3}$$

with Marsden notations and $p = \Phi(P)$. See figure 4.1. That is

$$\vec{w}_{t_0,t*} = (\Phi_t^{t_0})_* \vec{w}_{t_0} := (F_t^{t_0}.\vec{w}_{t_0}) \circ (\Phi_t^{t_0})^{-1}, \quad \text{i.e.} \quad \vec{w}_* = \Phi_* \vec{W} := (F.\vec{W}) \circ \Phi^{-1}. \tag{4.4}$$

We have thus defined $\vec{w}_{t_0*}$ by

$$\vec{w}_{t_0*}(t, p(t)) := \vec{w}_{t_0,t*}(p_t) = F^{t_0}(t, p_{t_0}).\vec{w}_{t_0}(p_{t_0}) \quad \text{when} \quad p(t) = \Phi^{t_0}(t, p_{t_0}). \tag{4.5}$$

### 4.1.3 $F$ is a two point tensors

With (4.1), "the tangent map" is defined by

$$\widehat{F_t^{t_0}} : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \Omega_t \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \\ p_{t_0} & \to \widehat{F_t^{t_0}}(p_{t_0}) = (p_t, F_t^{t_0}(p_{t_0})) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \end{array} \right. \tag{4.6}$$

**Definition 4.3** (Marsden–Hughes [17].) $\widehat{F_t^{t_0}}$ is the two point tensor deformation gradient, referring to the points $p_{t_0} \in \Omega_{t_0}$ (departure set) and $p_t = \Phi_t^{t_0}(p_{t_0}) \in \Omega_t$ (arrival set where $\vec{w}_{t_0*}(t, p_t) = F_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0})$ is drawn). And in short $\widehat{F_t^{t_0}} =^{\text{written}} F_t^{t_0}$ is said to be a two point tensor.

**Remark 4.4** The name "two point tensor" is a shortcut than can create confusions and errors when dealing with the transposed: $F_t^{t_0}$ is not immediately a "tensor": A tensor is a multilinear form, so gives ***scalar*** results ($\in \mathbb{R}$), while $F(P) := F_t^{t_0}(P) =^{\text{written}} F_P \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ gives ***vector*** results (in $\vec{\mathbb{R}}^n_t$). However $F_P$ can be naturally and canonically associated with the bilinear form $\widetilde{F}_P \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t, \vec{\mathbb{R}}^n_{t_0}; \mathbb{R})$ defined by, for all $\vec{u}_P \in \vec{\mathbb{R}}^n_{t_0}$ and $\ell_p \in \vec{\mathbb{R}}^{n*}_t$, with $p = \Phi_t^{t_0}(P)$,

$$\widetilde{F}_P(\ell_p, \vec{u}_P) := \ell_p.F_P.\vec{u}_P \ (\in \mathbb{R}), \tag{4.7}$$

see § A.14, and it is $\widetilde{F}_P$ which defines the so-called "two point tensor".

NB: The confusion between the linear function $F_t^{t_0}(p_{t_0}) = F_P$ and the bilinear form $\widetilde{F}_{p_{t_0}} = \widetilde{F}_P$ produces errors: E.g. a transposed of a linear form ($F_P$ is ***not*** (directly deduced from) the transposed of the associated bilinear form $\widetilde{F}_P$ ! So be careful with the word "transposed" and its two distinct definitions: The transposed of a bilinear form $b(\cdot,\cdot)$ is intrinsic to $b(\cdot,\cdot)$ (is objective), given by $b^T(\vec{u},\vec{w}) = b(\vec{w},\vec{u})$, while the transposed of a linear function $L$ is not intrinsic to $L$ (is subjective), given by $(L^T.\vec{u},\vec{w})_g = (L.\vec{w},\vec{u})_h$ where $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are inner dot products (additional tools) chosen by Human beings ($L^T$ should be written $L^T_{gh}$). (Details in § A.9.2 and § A.12.1).

**Remark 4.5** More generally for manifolds, the differential of $\Phi := \Phi_t^{t_0}$ at $P \in \Omega_{t_0}$ is $F(P) := d\Phi(P)$ : $\left\{\begin{array}{ll} T_P\Omega_{t_0} & \to T_p\Omega_t \\ \vec{W}(P) & \to \vec{w}_*(p) := d\Phi(P).\vec{W}(P) \end{array}\right\}$ with $p = \Phi_t^{t_0}(P)$. And the tangent map is

$$T\Phi : \left\{\begin{array}{ll} T\Omega_{t_0} & \to T\Omega_t \\ (P,\vec{W}(P)) & \to T\Phi(P,\vec{W}(P)) := (p, d\Phi(P).\vec{W}(P)) = (p,\vec{w}_*(p)), \quad \text{where} \quad p = \Phi_t^{t_0}(P). \end{array}\right. \tag{4.8}$$

### 4.1.4 Evolution: Toward Lie's derivative

Consider a Eulerian vector field $\vec{w} : \left\{\begin{array}{ll} \mathcal{C} = \bigcup_t(\{t\}\times\Omega_t) & \to \vec{\mathbb{R}}^n \\ (t,p) & \to \vec{w}(t,p) \end{array}\right\}$, e.g. a "force field". Then, at $t_0$ consider $\vec{w}_{t_0} : \left\{\begin{array}{ll} \Omega_{t_0} & \to \vec{\mathbb{R}}^n_{t_0} \\ p_{t_0} & \to \vec{w}_{t_0}(p_{t_0}) := \vec{w}(t_0,p_{t_0}) \end{array}\right\}$. The push-forward of $\vec{w}_{t_0}$ by $\Phi_t^{t_0}$ is, cf. (4.3)-(4.5),

$$\vec{w}_{t_0*}(t,p(t)) = F_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0}), \quad \text{where} \quad p(t) = \Phi^{t_0}(t,p_{t_0}). \tag{4.9}$$

See figure 4.1. Then at $t$ at $p(t)$ we can compare $\vec{w}(t,p(t))$ (real value of $\vec{w}$ at $t$ at $p(t)$) with $\vec{w}_{t_0*}(t,p(t))$ (transported memory along the trajectory). Thus the rate, without any ubiquity gift,

$$\frac{\vec{w}(t,p(t)) - \vec{w}_{t_0*}(t,p(t))}{t-t_0} = \frac{\text{actual}(t,p(t)) - \text{transp. mem.}(t,p(t))}{t-t_0} \quad \text{is meaningful at } (t,p(t)). \tag{4.10}$$

When $t \to t_0$ this rate gives the Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ (the rate of stress); We will see at § 9.3 that $\mathcal{L}_{\vec{v}}\vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}$, the $d\vec{v}$ term telling that a "non-uniform flow" ($d\vec{v} \neq 0$) acts on the stress.

### 4.1.5 Pull-back

Formally the pull-back is the push-forward with $(\Phi_t^{t_0})^{-1}$.

**Definition 4.6** The pull-back $(\Phi_t^{t_0})^*\vec{w}_t$ of a vector field $\vec{w}_t$ defined on $\Omega_t$ is the vector field defined on $\Omega_{t_0}$ by, with $p_{t_0} = (\Phi_t^{t_0})^{-1}(p_t)$,

$$\vec{w}^*_{t_0,t}(p_{t_0}) = (\Phi_t^{t_0})^*\vec{w}_t(p_{t_0}) := (F_t^{t_0})^{-1}(p_t).\vec{w}_t(p_t), \quad \text{written} \quad \vec{W}^*(P) = F^{-1}(p).\vec{w}(p) \tag{4.11}$$

by Marsden. Which defines $\vec{w}^*_t$ by $\vec{w}^*_t(t_0,p_{t_0}) := \vec{w}^*_{t_0,t}(p_{t_0}) = (F_t^{t_0})^{-1}(p_t).\vec{w}_t(p_t)$.

We however need to give full explanations:
$\Phi_t^{t_0} : \left\{\begin{array}{ll} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} & \to p_t = \Phi_t^{t_0}(p_{t_0}) \end{array}\right\}$ gives $(\Phi_t^{t_0})^{-1} : \left\{\begin{array}{ll} \Omega_t & \to \Omega_{t_0} \\ p_t & \to p_{t_0} = (\Phi_t^{t_0})^{-1}(p_t) \end{array}\right\}$;

And $p_{t_0} = (\Phi_t^{t_0})^{-1}(p_t) = (\Phi_t^{t_0})^{-1}(\Phi_t^{t_0}(p_{t_0}))$ gives $I = d(\Phi_t^{t_0})^{-1}(p_t).d\Phi_t^{t_0}(p_{t_0}) = d(\Phi_t^{t_0})^{-1}(p_t).F_t^{t_0}(p_{t_0})$, which defines

$$(F_t^{t_0})^{-1} := d(\Phi_t^{t_0})^{-1} : \left\{\begin{array}{ll} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0}) \\ p_t & \to (F_t^{t_0})^{-1}(p_t) := F_t^{t_0}(p_{t_0})^{-1}. \end{array}\right. \tag{4.12}$$

Marsden notation:

$$F^{-1}(p) = F(P)^{-1} \quad \text{when} \quad p = \Phi(P). \tag{4.13}$$

## 4.2 Quantification with bases

(Simple Cartesian framework.) With Marsden notations: $(\vec{E}_i)$ is a Cartesian basis in $\vec{\mathbb{R}^n_{t_0}}$, $(\vec{e}_i)$ is a Cartesian basis in $\vec{\mathbb{R}^n_t}$, $o$ is an origin in $\mathbb{R}^n$ at $t$, $\Phi_t^{t_0} =^{\text{written}} \Phi$ supposed $C^1$ and $\varphi^i : \Omega_{t_0} \to \mathbb{R}$ is its $i$-th-component in the referential $(o,(\vec{e}_i))$:

$$p = \Phi(P) = o + \sum_{i=1}^n \varphi^i(P)\vec{e}_i. \tag{4.14}$$

Let $\frac{\partial \varphi^i}{\partial X^J}(P) := d\varphi^i(P).\vec{E}_J = F^i_J(P)$; Then, $(\vec{e}_i)$ being a Cartesian basis, with $F = d\Phi$ we get

$$F(P).\vec{E}_J = \sum_{i=1}^n (d\varphi^i(P).\vec{E}_J)\vec{e}_i = \sum_{i=1}^n \frac{\partial \varphi^i}{\partial X^J}(P)\vec{e}_i,$$

In short:

$$F.\vec{E}_J = \sum_{i=1}^n \frac{\partial \varphi^i}{\partial X^J}\vec{e}_i, \quad \text{i.e.} \quad [F]_{[\vec{E},\vec{e}]} = [\frac{\partial \varphi^i}{\partial X^J}] = [F]_{[\vec{E},\vec{e}]} \quad \text{(Jacobian matrix).} \tag{4.15}$$

We recover: If $\vec{W} = \sum_{J=1}^n W^J \vec{E}_J \in \vec{\mathbb{R}^n_{t_0}}$ is a vector at $P$ then, by linearity of differentials,

$$d\Phi.\vec{W} = F.\vec{W} = \sum_{i=1}^n F^i_J W^J \vec{e}_i, \quad \text{i.e.} \quad [F.\vec{W}]_{|\vec{e}} = [F]_{|\vec{E},\vec{e}}.[\vec{W}]_{|\vec{E}}, \tag{4.16}$$

more precisely: $F_t^{t_0}(P).\vec{W}(P) = \sum_{i=1}^n F^i_J(P)W^J(P)\vec{e}_i$.

Similarly, for the second order derivative $d^2\Phi = dF$ (when $\Phi$ is $C^2$): With $\vec{U} = \sum_{J=1}^n U^J \vec{E}_J$ we get

$$dF(\vec{U},\vec{W}) = \sum_{i=1}^n d^2\varphi^i(\vec{U},\vec{W})\vec{e}_i = \sum_{i,J,K=1}^n \frac{\partial^2 \varphi^i}{\partial X^J \partial X^K} U^J W^K \vec{e}_i = \sum_{i=1}^n \left([\vec{U}]^T_{|\vec{E}}.[d^2\varphi^i]_{|\vec{E}}.[\vec{W}]_{|\vec{E}}\right)\vec{e}_i, \tag{4.17}$$

$[d^2\varphi^i(P)]_{|\vec{E}} = [\frac{\partial^2 \varphi^i}{\partial X^J \partial X^K}(P)]_{\substack{j=1,\dots,n \\ k=1,\dots,n}}$ being the Hessian matrix of $\varphi^i$ at $P$ relative to the basis $(\vec{E}_i)$.

**Remark 4.7** $J, j$ are dummy variables when used in a summation: E.g., $df.\vec{W} = \sum_{j=1}^n \frac{\partial f}{\partial X^j} W^j = \sum_{J=1}^n \frac{\partial f}{\partial X^J} W^J = \sum_{\alpha=1}^n \frac{\partial f}{\partial X^\alpha} W^\alpha = \frac{\partial f}{\partial X^1} W^1 + \frac{\partial f}{\partial X^2} W^2 + ...$ (there is no uppercase for 1, 2...). And Marsden–Hughes notations (capital letters for the past) are not at all compulsory, classical notations being just as good and often preferable (because they are not misleading). See § A. ▟

## 4.3 The unfortunate notation $d\vec{x} = F.d\vec{X}$

### 4.3.1 Issue

(4.3), i.e. $\vec{w}_*(p) := F(P).\vec{W}(P)$, is sometimes written

$$d\vec{x} = F.d\vec{X} \text{ : “a very unfortunate and misleading notation”} \tag{4.18}$$

which amounts to “confuse a length and a speed”... E.g. you see: “(4.18) is still true if $||d\vec{X}|| = 1$”... while $d\vec{X}$ is supposed to be small...

### 4.3.2 Where does this unfortunate notation come from?

The notation (4.18) comes from the first order Taylor expansion $\Phi(Q) = \Phi(P) + d\Phi_t^{t_0}(P).(Q-P) + o(||Q-P||)$, where $P, Q \in \Omega_{t_0}$, i.e., with $p = \Phi_t^{t_0}(P)$ and $q = \Phi_t^{t_0}(Q)$ and $h = ||Q-P||$,

$$q - p = F(P).(Q-P) + o(h), \quad \text{written} \quad \delta\vec{x} = F.\delta\vec{X} + o(\delta\vec{X}), \tag{4.19}$$

or $\vec{pq} = F(P).\vec{PQ} + o(h)$. So as $Q \to P$ we get $0 = 0$... Quite useless, isn't it?

While

$$\frac{q-p}{h} = F(P).\frac{Q-P}{h} + o(1) \quad \text{is useful:} \tag{4.20}$$

As $Q \to P$ we get $\vec{w}_* = F(P).\vec{W}$ which relates tangent vectors, cf. (4.3) and figure 4.1. Details:

### 4.3.3 Interpretation: Vector approach

Consider a spatial curve $c_{t_0} : \left\{ \begin{array}{rl} [s_1, s_2] & \to \Omega_{t_0} \\ s & \to P := c_{t_0}(s) \end{array} \right\}$ in $\Omega_{t_0}$, cf. figure 4.1. It is deformed by $\Phi_t^{t_0}$ to become the spatial curve defined by $c_t := \Phi_t^{t_0} \circ c_{t_0} : \left\{ \begin{array}{rl} [s_1, s_2] & \to \Omega_t \\ s & \to p := c_t(s) = \Phi_t^{t_0}(c_{t_0}(s)). \end{array} \right.$ in $\Omega_t$. Hence, relation between tangent vectors:

$$\frac{dc_t}{ds}(s) = d\Phi_t^{t_0}(c_{t_0}(s)).\frac{dc_{t_0}}{ds}(s), \quad \text{written} \quad \frac{d\vec{x}}{ds}(s) = F(X(s)).\frac{d\vec{X}}{ds}(s), \quad \text{written} \quad \frac{d\vec{x}}{ds} = F.\frac{d\vec{X}}{ds}, \tag{4.21}$$

But you ***can't*** simplify by $ds$ to get $d\vec{x} = F.d\vec{X}$: It is absurd to confuse "a slope $\frac{d\vec{X}}{ds}(s)$" and "a length $\delta\vec{X} = p - q$".

NB: $||\frac{dc_{t_0}}{ds}(s)|| = ||\frac{d\vec{X}}{ds}(s)|| = 1$ is meaningful in (4.21): It means that the parametrization of the spatial curve $c_{t_0}$ in $\Omega_{t_0}$ uses a curvilinear parameter $s$ such that $||c_{t_0}{}'(s)|| = 1$ for all $s$, i.e. s.t. $||\vec{W}_P|| = 1$ in figure 4.1. You ***cannot*** simplify by $ds$: $||d\vec{X}|| = 1$ is absurd together with $d\vec{X}$ "small" cf. (4.19).

### 4.3.4 Interpretation: Differential approach

In fact (4.18) is a relation between differentials... if you adopt the correct notations: With (4.14),

$$\vec{x} = \vec{op} = \overrightarrow{o\Phi(P)} = \sum_{i=1}^{n} \varphi^i(P)\vec{e}_i \overset{\text{written}}{=} \sum_{i=1}^{n} x^i(P)\vec{e}_i, \quad \text{with} \quad \varphi^i \overset{\text{written}}{=} x^i \quad (\text{function of } P). \tag{4.22}$$

Thus, with $(dX^I)$ the (covariant) dual basis of $(\vec{E}_I)$ we get the system of $n$ equations (functions):

$$d\Phi = F, \quad \text{i.e.} \quad \left\{ \begin{array}{l} d\varphi^1(P) = \sum_{J=1}^n \frac{\partial \varphi^1}{\partial X^J}(P)\, dX^J \\ \ldots \\ d\varphi^n(P) = \sum_{J=1}^n \frac{\partial \varphi^n}{\partial X^J}(P)\, dX^J \end{array} \right\}, \quad \text{written} \quad d\vec{x} = F.d\vec{X}, \tag{4.23}$$

this last notation being often misunderstood[2]: It is nothing more than $d\Phi = F$ (coordinate free notation).

### 4.3.5 The ambiguous notation $\dot{d\vec{x}} = \dot{F}.d\vec{X}$

The tricky notation $d\vec{x} = F.d\vec{X}$ gives the unfortunate (misunderstood) notation $\dot{d\vec{x}} = \dot{F}.d\vec{X}$, and then

$$\dot{d\vec{x}} = L.d\vec{x} \quad \text{where} \quad L = \dot{F}.F^{-1}. \tag{4.24}$$

**Question**: What is the meaning (and legitimate notation) of (4.24)?
**Answer**: $\dot{d\vec{x}} = L.d\vec{x}$ means

$$\boxed{\frac{D\vec{w}_{t_0*}}{Dt} = d\vec{v}.\vec{w}_{t_0*}} \quad = \text{evolution rate of tangent vectors along a trajectory} \tag{4.25}$$

see figure 4.1. Indeed, $\vec{w}_{t_0*}(t, p(t)) =^{(4.5)} F^{t_0}(t, p_{t_0}).\vec{w}_{t_0}(p_{t_0}) = F_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0})$ gives

$$\begin{aligned} \frac{D\vec{w}_{t_0*}}{Dt}(t, p(t)) &= \frac{\partial F^{t_0}}{\partial t}(t, p_{t_0}).\vec{w}_{t_0}(p_{t_0}) = \frac{\partial F^{t_0}}{\partial t}(t, p_{t_0}).F_t^{t_0}(p_{t_0})^{-1}.\vec{w}_{t_0*}(t, p(t)) \\ &= (d\vec{v}.\vec{w}_{t_0*})(t, p(t)) \end{aligned} \tag{4.26}$$

cf. (3.33). In particular $t = t_0$ gives $\frac{D\vec{w}_{t_0*}}{Dt}(t_0, p_{t_0}) = d\vec{v}(t_0, p_{t_0}).\vec{w}_{t_0}(p_{t_0}) =$ the evolution rate of the tangent vectors $\vec{w}_{t_0}(p_{t_0}) \in \vec{\mathbb{R}}^n_{t_0}$ at $p_{t_0}$ along "the trajectory of $p_{t_0}$".

[2] Spivak [23] chapter 4: Classical differential geometers (and classical analysts) did not hesitate to talk about "infinitely small" changes $dx^i$ of the coordinates $x^i$, just as Leibnitz had. No one wanted to admit that this was nonsense, because true results were obtained when these infinitely small quantities were divided into each other (provided one did it in the right way). Eventually it was realized that the closest one can come to describing an infinitely small change is to describe a direction in which this change is supposed to occur, i.e., a tangent vector. Since *df* is supposed to be the infinitesimal change of $f$ under an infinitesimal change of the point, *df* must be a function of this change, which means that *df* should be a function on tangent vectors. The $dX_i$ themselves then metamorphosed into functions, and it became clear that they must be distinguished from the tangent vectors $\partial/\partial X_i$. Once this realization came, it was only a matter of making new definitions, which preserved the old notation, and waiting for everybody to catch up.

## 4.4 Change of coordinate system at $t$ for $F$

$p_{t_0} \in \Omega_{t_0}$, $p_t = \Phi_t^{t_0}(p_{t_0}) \in \Omega_t$, $\vec{W}(p_{t_0}) \in \vec{\mathbb{R}}^n_{t_0}$, $\vec{w}(p_t) = F_t^{t_0}(p_{t_0}).\vec{W}(p_{t_0}) \in \vec{\mathbb{R}}^n_t$,

### 4.4.1 Change of basis system at $t$ for $F$

At $t_0$ in $\vec{\mathbb{R}}^n_{t_0}$ a past observer used a basis $(\vec{a}_i)$. At $t$ in $\vec{\mathbb{R}}^n_t$, a first observer chooses a Cartesian basis $(\vec{b}_{old,i})$ and a second observer chooses a Cartesian basis $(\vec{b}_{new,i})$, and $P = [P_{ij}]$ is the transition matrix from $(\vec{b}_{old,i})$ to $(\vec{b}_{new,i})$, i.e. $\vec{b}_{new,j} = \sum_{i=1}^n P_{ij}\vec{b}_{old,i}$ for all $j$. Let $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ and $\vec{w} = F.\vec{W} \in \vec{\mathbb{R}}^n_t$. Change of basis formula:

$$[\vec{w}]_{|\vec{b}_{new}} = P^{-1}.[\vec{w}]_{|\vec{b}_{old}}, \quad \text{thus} \quad [F.\vec{W}]_{|\vec{b}_{new}} = P^{-1}.[F.\vec{W}]_{|\vec{b}_{old}}, \tag{4.27}$$

thus $[F]_{|\vec{a},\vec{b}_{new}}.[\vec{W}]_{|\vec{a}} = P^{-1}.[F]_{|\vec{a},\vec{b}_{old}}.[\vec{W}]_{|\vec{a}}$. True for all $\vec{W}$, thus

$$\boxed{[F]_{|\vec{a},\vec{b}_{new}} = P^{-1}.[F]_{|\vec{a},\vec{b}_{old}}}. \tag{4.28}$$

**Remark 4.8** (4.28) is ***not*** $[L]_{|new} = P^{-1}.[L]_{|old}.P$, the change of basis formula for endomorphisms, which would be nonsense since $F := F_t^{t_0}(p_{t_0}) : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ is not an endomorphism; (4.28) is just the usual change of basis formula $[\vec{w}]_{|\vec{b}_{new}} = P^{-1}.[\vec{w}]_{|\vec{b}_{old}}$ for vectors $\vec{w}$ in $\vec{\mathbb{R}}^n_t$ (contravariant vectors). ∴

### 4.4.2 Change of basis system at $t_0$ for $F$

At $t$ in $\vec{\mathbb{R}}^n_t$ an actual observer used a basis $(\vec{b}_i)$. He wants to compare results of two past observers at $t_0$: The first used a Cartesian basis $(\vec{a}_{old,i})$ and the second used a Cartesian basis $(\vec{a}_{new,i})$. $P = [P_{ij}]$ being the transition matrix from $(\vec{a}_{old,i})$ to $(\vec{a}_{new,i})$, for any $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$,

$$[\vec{W}]_{|\vec{a}_{new}} = P^{-1}.[\vec{W}]_{|\vec{a}_{old}}. \tag{4.29}$$

And $F.\vec{W} = F.\vec{W}$ gives $[F.\vec{W}]_{|\vec{b}} = [F.\vec{W}]_{|\vec{b}}$, thus $[F]_{|\vec{a}_{new},\vec{b}}.[\vec{W}]_{|\vec{a}_{new}} = [F]_{|\vec{a}_{old},\vec{b}}.[\vec{W}]_{|\vec{a}_{old}}$, hence $[F]_{|\vec{a}_{new},\vec{b}}.P^{-1}.[\vec{W}]_{|\vec{a}_{old}} = [F]_{|\vec{a}_{old},\vec{b}}.[\vec{W}]_{|\vec{a}_{old}}$, for all $\vec{W}$. Thus $[F]_{|\vec{a}_{new},\vec{b}}.P^{-1} = [F]_{|\vec{a}_{old},\vec{b}}$, thus

$$\boxed{[F]_{|\vec{a}_{new},\vec{b}} = [F]_{|\vec{a}_{old},\vec{b}}.P}. \tag{4.30}$$

This is the change of basis formula for linear forms (covariant vectors), which is expected since here $F$ is considered to be a linear function that acts on vectors in $\vec{\mathbb{R}}^n_{t_0}$.

**Exercice 4.9** Detail the matrix calculation which gave (4.30) with Marsden's notations.

**Answer**. Let $F.\vec{E}_{old,J} = \sum_i F^i_{o,J}\vec{e}_i$ and $F.\vec{E}_{new,J} = \sum_i F^i_{n,J}\vec{e}_i$, and $\vec{W} = \sum_J W_o^J \vec{E}_{old,J} = \sum_J W_n^J \vec{E}_{new,J}$, and $Q = [Q^I_J] := P^{-1}$, so $[\vec{W}]_{\vec{E}_{new}} = Q.[\vec{W}]_{\vec{E}_{old}}$, i.e. $W_n^J = \sum_K Q^J_K W_o^K$ for all $J$. Thus $F.\vec{W} = \sum_{iJ} F^i_{n,J} W_n^J \vec{e}_i = \sum_{iJK} F^i_{n,J} Q^J_K W_o^K \vec{e}_i$ together with $F.\vec{W} = \sum_{iK} F^i_{o,K} W_o^K \vec{e}_i$, for all $\vec{W}$, thus $\sum_J F^i_{n,J} Q^J_K = F^i_{o,K}$ for all $i, K$, thus $[F]_{|\vec{E}_{new},\vec{e}}.Q = [F]_{|\vec{E}_{old},\vec{e}}$. ∴

## 4.5 Tensor notations: Warnings

As already noted, cf. (4.7), the linear map $F = d\Phi := d\Phi_t^{t_0}(p_{t_0}) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ is naturally canonically associated with the bipoint tensor $\widetilde{F} \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t, \vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ defined by, for all $(\ell, \vec{W}) \in \vec{\mathbb{R}}^{n*}_t \times \vec{\mathbb{R}}^n_{t_0}$,

$$\widetilde{F}(\ell, \vec{W}) := \ell.(F.\vec{W}). \tag{4.31}$$

Quantification: With § 4.2 (Marsden notations), with $(dX^I)$ in $\vec{\mathbb{R}}^{n*}_{t_0}$ the covariant) dual basis of $(\vec{E}_i)$ and with $F.\vec{E}_J = \sum_{i=1}^n \frac{\partial\varphi^i}{\partial X^J}\vec{e}_i$ cf. (4.15), we immediately get

$$\widetilde{F} = \sum_{i,J=1}^n \frac{\partial\varphi^i}{\partial X^J}\, \vec{e}_i \otimes dX^J. \tag{4.32}$$

Indeed, with $(dx^i)$ in $\vec{\mathbb{R}}^{n*}_t$ the covariant) dual basis of $(\vec{e}_i)$, $dx^k.(F.\vec{E}_\ell) = dx^k.(\sum_i \frac{\partial\varphi^i}{\partial X^\ell}\vec{e}_i) = \sum_i \frac{\partial\varphi^i}{\partial X^\ell} dx^k.\vec{e}_i = \sum_i \frac{\partial\varphi^i}{\partial X^\ell}\delta^k_i = \frac{\partial\varphi^k}{\partial X_\ell}$, and $(\sum_{iJ} \frac{\partial\varphi^i}{\partial X^J}\, \vec{e}_i \otimes dX^j)(dx^k, \vec{E}_\ell) = \sum_{iJ} \frac{\partial\varphi^i}{\partial X^J}\, (\vec{e}_i.dx^k)(dX^J.\vec{E}_\ell) = \sum_{iJ} \frac{\partial\varphi^i}{\partial X^J}\, \delta^k_i\delta^J_\ell = \frac{\partial\varphi^k}{\partial X_\ell}$: Equality for all $k, \ell$.

Similarly, $d^2\varphi^i(\vec{E}_J,\vec{E}_K)=\frac{\partial^2\varphi^i}{\partial X^J\partial X^K}$ gives $d^2\varphi^i=\sum_{JK}\frac{\partial^2\varphi^i}{\partial X^J\partial X^K}\,dX^J\otimes dX^K$, and

$$\begin{aligned}&dF(.,.)=\sum_{i=1}^n d^2\varphi_i(.,.)\vec{e}_i \overset{\text{written}}{=} d\widetilde{F}=\sum_{i=1}^n \vec{e}_i\otimes d^2\varphi^i=\sum_{i,J,K=1}^n\frac{\partial^2\varphi^i}{\partial X^J\partial X^K}\,\vec{e}_i\otimes(dX^J\otimes dX^K),\\ &\text{i.e. } dF(\vec{U},\vec{W})=d\widetilde{F}(\vec{U},\vec{W})=\sum_{i=1}^n d^2\varphi^i(\vec{U},\vec{W})\vec{b}_i=\sum_{i,J,K=1}^n\frac{\partial^2\varphi^i}{\partial X^J\partial X^K}U_JW_K\vec{e}_i.\end{aligned}\tag{4.33}$$

**Warning 1:** The tensor notation can be misleading, e.g. if you use the transposed, see remark 4.4. So, you should always use the standard $F.\vec{E}_J=\sum_{i,j=1}^n F^i{}_J\vec{e}_i$ notation (vector value); And avoid the use of $\widetilde{F}$, i.e. of $\widetilde{F}(\ell,\vec{W})$ (scalar value).

**Warning 2:** You can't use $\vec{E}_J$ instead of $dX^J$ in (4.32), i.e. you can't use $\widehat{F}=\sum_{iJ}\frac{\partial\varphi^i}{\partial X^J}\,\vec{e}_i\otimes\vec{E}_J$ instead of $\widetilde{F}$ in (4.32), because there is no canonical natural isomorphism between $\vec{\mathbb{R}}^n$ and $\mathbb{R}^{n*}$ (if you do then you contradict the change of basis formulas).

**Warning 3:** In some manuscripts you find the notation $F=d\Phi=^{\text{written}}\Phi\otimes\nabla_X$. It does not help to understand what $F$ is (it is the differential $d\Phi$), and must be avoided as far as objectivity is concerned:

- It could be misinterpreted because in mechanics $\nabla f$ is often understood to be a vector (contravariant) while the differential $df$ is covariant (unmissable in thermodynamics because you can't use gradients).
- A differentiation is ***not*** a tensor operation, see the fundamental example S.1; So you should use the usual notation $d\Phi$ (i.e. $d\Phi(.)=\sum_i d\varphi^i(.)\vec{e}_i$ with a basis $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_t$), and never use $\Phi\otimes\nabla_X$.
- Similarly you should use the usual notation $d^2\Phi$ (or $d^2\Phi(.,.)=\sum_{i=1}^n d^2\varphi^i(.,.)\vec{e}_i$ with a basis $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_t$), and never use $\Phi\otimes\nabla_X\otimes\nabla_X$.

## 4.6 Spatial Taylor expansion of $F$

$\Phi:=\Phi^{t_0}_t$ is $C^3$ for all $t_0,t$, and $F=d\Phi$. Then, in $\Omega_t$, with $P\in\Omega_{t_0}$ and $\vec{W}\in\vec{\mathbb{R}}^n_{t_0}$ vector at $P$, $\Phi(P{+}h\vec{W})=\Phi(P)+h\,F(P).\vec{W}+\frac{h^2}{2}\,dF(P)(\vec{W},\vec{W})+o(h)$, and

$$F(P{+}h\vec{W})=F(P)+h\,dF(P).\vec{W}+\frac{h^2}{2}\,d^2F(P)(\vec{W},\vec{W})+o(h^2).\tag{4.34}$$

## 4.7 Time Taylor expansion of $F$

$\Phi^{t_0}$ is $C^3$, $p_t=p(t)=\Phi^{t_0}(t,p_{t_0})=\Phi^{t_0}_{p_{t_0}}(t)$, $\vec{V}^{t_0}(t,p_{t_0})=\frac{\partial\Phi^{t_0}}{\partial t}(t,p_{t_0})=\vec{v}(t,p_t)=\vec{v}(t,\Phi^{t_0}(t,p_{t_0}))$ (Lagrangian and Eulerian velocities), $\vec{A}^{t_0}(t,p_{t_0})=\frac{\partial^2\Phi^{t_0}}{\partial t^2}(t,p_{t_0})=\vec{\gamma}(t,p(t))=\vec{\gamma}(t,\Phi^{t_0}(t,p_{t_0}))$ (Lagrangian and Eulerian accelerations), and $F^{t_0}(t,p_{t_0})=d\Phi^{t_0}(t,p_{t_0})=F^{t_0}_{p_{t_0}}(t)$. Hence

$$F^{t_0}_{p_{t_0}}{}'(t)=\frac{\partial F^{t_0}}{\partial t}(t,p_{t_0})=\frac{\partial(d\Phi^{t_0})}{\partial t}(t,p_{t_0})=d(\frac{\partial\Phi^{t_0}}{\partial t})(t,p_{t_0})=d\vec{V}^{t_0}(t,p_{t_0})=d\vec{v}(t,p_t).F^{t_0}_{p_{t_0}}(t),\tag{4.35}$$

$$F^{t_0}_{p_{t_0}}{}''(t)=\frac{\partial^2F^{t_0}}{\partial t^2}(t,p_{t_0})=\frac{\partial^2(d\Phi^{t_0})}{\partial t^2}(t,p_{t_0})=d(\frac{\partial^2\Phi^{t_0}}{\partial t^2})(t,p_{t_0})=d\vec{A}^{t_0}(t,p_{t_0})=d\vec{\gamma}(t,p_t).F(t).\tag{4.36}$$

(In short $\dot{F}=d\vec{V}=d\vec{v}.F$ and $\ddot{F}=d\vec{A}=d\vec{\gamma}.F$). Thus

$$\begin{aligned}F^{t_0}_{p_{t_0}}(t{+}h)&=F^{t_0}_{p_{t_0}}(t)+h\,d\vec{V}^{t_0}_{p_{t_0}}(t)+\frac{h^2}{2}\,d\vec{A}^{t_0}_{p_{t_0}}(t)+o(h^2)\\&=\Big(I+h\,d\vec{v}(t,p(t))+\frac{h^2}{2}\,d\vec{\gamma}(t,p(t))\Big).F^{t_0}_{p_{t_0}}(t)+o(h^2).\end{aligned}\tag{4.37}$$

**NB:** They are ***three*** times involved: $t$ and $t{+}h$ as usual, and $t_0$ (observer dependent) through $F^{t_0}$ and $\vec{V}^{t_0}$, as in (3.39).

Particular case: $t=t_0$ then $F^{t_0}_{p_{t_0}}(t_0)=I$ and

$$\begin{aligned}F^{t_0}_{p_{t_0}}(t_0{+}h)&=I+h\,d\vec{V}^{t_0}_{p_{t_0}}(t_0)+\frac{h^2}{2}\,d\vec{A}^{t_0}_{p_{t_0}}(t_0)+o(h^2)\\&=\Big(I+h\,d\vec{v}(t_0,p_{t_0})+\frac{h^2}{2}\,d\vec{\gamma}(t_0,p_{t_0})\Big)+o(h^2).\end{aligned}\tag{4.38}$$

**Exercice 4.10** Directly check that (short notation) $F' = d\vec{v}.F$ gives $F'' = d\vec{\gamma}.F$.

**Answer**. $F'(t) = d\vec{v}(t,p(t)).F(t)$ gives $F''(t) = \frac{D(d\vec{v})}{Dt}(t,p(t)).F(t) + d\vec{v}(t,p(t)).F'(t)$ with $\frac{D(d\vec{v})}{Dt} = d\vec{\gamma} - d\vec{v}.\vec{v}$, cf. (4.39), thus $F''(t) = (d\vec{\gamma} - d\vec{v}.d\vec{v})(t,p(t)).F(t) + d\vec{v}(t,p(t)).d\vec{v}(t,p(t)).F(t) = d\vec{\gamma}(t,p(t)).F(t)$. ∴

**Remark 4.11** $\gamma = \frac{\partial\vec{v}}{\partial t} + d\vec{v}.\vec{v}$ is not linear in $\vec{v}$. Idem,

$$d\vec{\gamma} = d(\frac{D\vec{v}}{Dt}) = d(\frac{\partial\vec{v}}{\partial t} + d\vec{v}.\vec{v}) = d\frac{\partial\vec{v}}{\partial t} + d^2\vec{v}.\vec{v} + d\vec{v}.d\vec{v} \quad (= \frac{D(d\vec{v})}{Dt} + d\vec{v}.d\vec{v}) \tag{4.39}$$

is non linear in $\vec{v}$, and gives $F^{t_0}_{p_{t_0}}{}''(t) = (d\frac{\partial\vec{v}}{\partial t} + d^2\vec{v}.\vec{v} + d\vec{v}.d\vec{v})(t,p_t).F^{t_0}_{p_{t_0}}(t)$, non linear in $\vec{v}$. ∴

## 4.8 Homogeneous and isotropic material

Let $P \in \Omega_{t_0}$ and $F(P) := d\Phi^{t_0}_t(P)$; Suppose that the "Cauchy stress vector" $\vec{f}_t(p_t)$ à $t$ at $p_t = \Phi^{t_0}_t(P)$ only depends on $P$ and on $F(P)$ (the first gradient at $P$), i.e. there exists a function $\vec{\text{fun}}$ such that

$$\vec{f}_t(p_t) = \vec{\text{fun}}(P, F(P)). \tag{4.40}$$

**Definition 4.12** A material is homogeneous iff $\vec{\text{fun}}$ doesn't depend on the first variable $P$ of $\vec{\text{fun}}$, i.e., iff, for all $P \in \Omega_{t_0}$,

$$\vec{\text{fun}}(P, F(P)) = \vec{\text{fun}}(F(P)) \quad (= \vec{f}_t(p_t)). \tag{4.41}$$

(Same mechanical property at any point.)

**Definition 4.13** Choose a Euclidean dot product, the same at all time. A material is isotropic at $P \in \Omega_{t_0}$ iff $\vec{\text{fun}}$ is independent of the direction you consider, i.e., iff, for any rotation $R_{t_0}(P)$ in $\vec{\mathbb{R}}^n_{t_0}$,

$$\vec{\text{fun}}(P, F(P) = \vec{\text{fun}}(P, F(P).R_{t_0}(P)) \quad (= \vec{f}_t(p_t)). \tag{4.42}$$

(Mechanical property unchanged when rotating the material first.)

**Definition 4.14** A material is isotropic homogeneous iff it is isotropic and homogeneous.

## 4.9 $H = F^{-1}$ the inverse of the deformation gradient

Let $\Phi = \Phi^{t_0}_t$. We have $\Phi : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \Omega_t \\ P & \to p = \Phi(P) \end{array} \right\}$, $\Phi^{-1} : \left\{ \begin{array}{ll} \Omega_t & \to \Omega_{t_0} \\ p & \to P = \Phi^{-1}(p) \end{array} \right\}$, and $(\Phi^{-1} \circ \Phi)(P) = P$. Thus $d\Phi^{-1}(p).d\Phi(P) = I_{t_0}$ where $p = \Phi(P)$, and, with $F^{t_0}_t = F = d\Phi$ is the deformation gradient,

$$F^{-1}[p) = d\Phi^{-1}(p) = d\Phi(P)^{-1} = F(P)^{-1}. \tag{4.43}$$

This define the two point tensor

$$H^{t_0}_t := (F^{t_0}_t)^{-1} \overset{\text{written}}{=} H := F^{-1} : \left\{ \begin{array}{ll} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0}) \\ p & \to \boxed{H(p) = F^{-1}(p) := (F(P))^{-1}} \quad \text{when} \quad p = \Phi(P), \end{array} \right. \tag{4.44}$$

i.e. $H^{t_0}_t(p) = (F^{t_0}_t)^{-1}(p) := (F^{t_0}_t(P))^{-1}$ when $P = \Phi^{t_0}_t(p)$. So, for all $\vec{w}(p) \in \vec{\mathbb{R}}^n_t$ vector at $p \in \Omega_t$:

$$H(p).\vec{w}(p) = F^{-1}(p).\vec{w}(p) = F(P)^{-1}.\vec{w}(p), \quad \text{in short} \quad H.\vec{w} = F^{-1}.\vec{w}, \tag{4.45}$$

This defines

$$H^{t_0} : \left\{ \begin{array}{ll} \mathcal{C} = \bigcup_t (\{t\} \times \Omega_t) & \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0}) \\ (t, p_t) & \to H^{t_0}(t, p_t) := H^{t_0}_t(p_t) = (F^{t_0}(t,P))^{-1} \quad \text{when} \quad p_t = \Phi^{t_0}(t,P). \end{array} \right. \tag{4.46}$$

NB: $H^{t_0}$ looks like a Eulerian map, but isn't: $H^{t_0}$ depends on a initial time $t_0$ and $H^{t_0}(t,p_t)$ is a two point tensor (starts in $\vec{\mathbb{R}}^n_t$ and finishes in $\vec{\mathbb{R}}^n_{t_0}$). We will however use the material time derivative $\frac{D}{Dt}$ notation in this case, that is, we define, along a trajectory $t \to p(t) = \Phi^{t_0}(t,P)$,

$$\frac{DH^{t_0}}{Dt}(t,p(t)) := \frac{\partial H^{t_0}}{\partial t}(t,p(t)) + dH^{t_0}(t,p(t)).\vec{v}(t,p(t)), \quad \text{i.e.} \quad \frac{DH^{t_0}}{Dt} = \frac{\partial H^{t_0}}{\partial t} + dH^{t_0}.\vec{v}, \tag{4.47}$$

which is the time derivative $g'(t)$ of the function $g : t \to g(t) = H^{t_0}(t,p(t)) = H^{t_0}(t, \Phi^{t_0}(t,P))$.

Then $H^{t_0}(t,p(t)).F^{t_0}(t,P)=I_{t_0}$ gives $\frac{DH^{t_0}}{Dt}(t,p(t)).F^{t_0}(t,P)+H^{t_0}(t,p(t)).\frac{\partial F^{t_0}}{\partial t}(t,P)=0$, thus with $\frac{\partial F^{t_0}}{\partial t}(t,P)=^{(4.35)}d\vec{v}(t,p(t)).F^{t_0}(t,P)$ and $F_t^{t_0}$ invertible we get

$$\frac{DH^{t_0}}{Dt}(t,p(t))=-H(t,p(t)).d\vec{v}(t,p(t))\quad\text{written}\quad\boxed{\frac{DH}{Dt}=-H.d\vec{v}}. \tag{4.48}$$

**Exercice 4.15** With $\vec{w}_{t_0*}(t,p(t))=F^{t_0}(t,P).\vec{W}(P)$, i.e. $H^{t_0}(t,p(t)).\vec{w}_{t_0*}(t,p(t))=\vec{W}(P)$, when $p(t)=\Phi^{t_0}(t,P)$, prove (4.48).

**Answer**. $\frac{D\vec{w}_{t_0*}}{Dt}(t,p(t))=d\vec{v}(t,p(t)).\vec{w}_{t_0*}(t,p(t))$, cf. (4.25); And $(H^{t_0}.\vec{w}_{t_0*})(t,p(t))=\vec{W}(P)$ gives $\frac{DH^{t_0}}{Dt}.\vec{w}_{t_0*}+H^{t_0}.\frac{D\vec{w}_{t_0*}}{Dt}=0$; Thus $\frac{DH^{t_0}}{Dt}.\vec{w}_{t_0*}+H^{t_0}.d\vec{v}.\vec{w}_{t_0*}=0$, thus $\frac{DH}{Dt}=-H.d\vec{v}$. ∎

**Exercice 4.16** Prove: $H_t^{t_0}=H_{t_1}^{t_0}\circ H_t^{t_1}$ and $\frac{DH^{t_0}}{Dt}(t,p(t))=H_{t_1}^{t_0}(p_{t_1}).\frac{DH^{t_1}}{Dt}(t,p(t))$ for all $t_0,t_1$ with $p_{t_1}=\Phi_{t_1}^{t_0}(p_{t_0})$.

**Answer**. We have $\Phi_t^{t_0}(p_{t_0})=\Phi_t^{t_1}(\Phi_{t_1}^{t_0}(p_{t_0}))$, cf. (5.17), hence $F_t^{t_0}(p_{t_0})=F_t^{t_1}(p_{t_1}).F_{t_1}^{t_0}(p_{t_0})$, thus $F_t^{t_0}(p_{t_0})^{-1}=F_{t_1}^{t_0}(p_{t_0})^{-1}.F_t^{t_1}(p_{t_1})^{-1}$, i.e. $H_t^{t_0}(p_t)=H_{t_1}^{t_0}(p_{t_1}).H_t^{t_1}(p(t))$, thus, $H^{t_0}(t,p(t))=H_{t_1}^{t_0}(p_{t_1}).H^{t_1}(t,p(t))$, thus $\frac{DH^{t_0}}{Dt}(t,p(t))=H_{t_1}^{t_0}(p_{t_1}).\frac{DH^{t_1}}{Dt}(t,p(t))$. ∎

**Exercice 4.17** Prove 1-

$$H''=H.(2d\vec{v}.d\vec{v}-d\vec{\gamma}) \tag{4.49}$$

(means?), i.e. 2- $(F^{-1})''=F^{-1}.(2d\vec{v}.d\vec{v}-d\vec{\gamma})$ (means?).

**Answer**. 1- Meaning: $\frac{D^2H^{t_0}}{Dt^2}(t,p(t))=H^{t_0}(t,p(t)).(2d\vec{v}(t,p(t)).d\vec{v}(t,p(t))-d\vec{\gamma}(t,p(t)))$. With $\frac{DH^{t_0}}{Dt}=^{(4.48)}-H^{t_0}.d\vec{v}$ we get $\frac{D^2H^{t_0}}{Dt^2}=-\frac{DH^{t_0}}{Dt}.d\vec{v}-H^{t_0}.\frac{D(d\vec{v})}{Dt}$ with $\frac{D(d\vec{v})}{Dt}=^{(2.30)}d\frac{D\vec{v}}{Dt}-d\vec{v}.d\vec{v}$, thus $\frac{D^2H^{t_0}}{Dt^2}=(H^{t_0}.d\vec{v}).d\vec{v}-H^{t_0}.d\vec{\gamma}+H^{t_0}.d\vec{v}.d\vec{v})$. 2- Meaning: With $g(t):=F^{t_0}(t,p_{t_0})^{-1}$ we want $g''(t)=g'(t).(2d\vec{v}(t,p(t)).d\vec{v}(t,p(t))-d\vec{\gamma}(t,p(t)))$, abusively written $\frac{\partial^2F^{t_0}(t,p_{t_0})^{-1}}{\partial t^2}=F^{t_0}(t,p_{t_0})^{-1}.(2d\vec{v}(t,p(t)).d\vec{v}(t,p(t))-d\vec{\gamma}(t,p(t)))$, abusively written $(F^{-1})''=F^{-1}.(2d\vec{v}.d\vec{v}-d\vec{\gamma})$. Having $g(t)=H^{t_0}(t,p(t)$, point 1- gives the result. ∎

**Exercice 4.18** Give the order 2 Taylor expansion of $H$ (why is it formal?), then verify $H.F=I$ (means?) is recovered up to the second order.

**Answer**. With $g(t-h)=g(t)-hg'(t)+\frac{h^2}{2}g''(t)+o(h^2)$ and $g(t)=H^{t_0}(t,p(t))$ and (4.48)-(4.49),

$$H^{t_0}(t-h,p(t-h))=H^{t_0}(t,p_t).(I-h\,d\vec{v}+\frac{h^2}{2}\,(2d\vec{v}.d\vec{v}-d\vec{\gamma}))(t,p_t)+o(h^2). \tag{4.50}$$

This is formal because $H_{t-h}^{t_0}:\vec{\mathbb{R}}_{t-h}^n\to\vec{\mathbb{R}}_{t_0}^n$ and $H_t^{t_0}:\vec{\mathbb{R}}_t^n\to\vec{\mathbb{R}}_{t_0}^n$, so this calculation requires ubiguity gift and a Cartesian structure to have $\vec{\mathbb{R}}_{t-h}^n=\vec{\mathbb{R}}_t^n=^{\text{written}}\vec{\mathbb{R}}^n$. Then $H.F=I$ means $F^{-1}.F=I$ i.e. $H_t^{t_0}(p_t).F_t^{t_0}(p_{t_0})=I$ when $p_t=\Phi_t^{t_0}(p_{t_0})$, and with (4.50) and (4.37) we have (shortened notations)

$$\begin{aligned} H.F= &= H.(I-h\,d\vec{v}+\frac{h^2}{2}\,(2d\vec{v}.d\vec{v}-d\vec{\gamma}))+o(h^2)].[\Big(I+h\,d\vec{v}+\frac{h^2}{2}\,d\vec{\gamma}\Big).F+o(h^2)]\\ &= F^{-1}.(I-h\,d\vec{v}+h\,d\vec{v}-h^2d\vec{v}.d\vec{v}+\frac{h^2}{2}(2d\vec{v}.d\vec{v}-d\vec{\gamma})+\frac{h^2}{2}d\vec{\gamma})+o(h^2)).F=I+o(h^2). \end{aligned}$$

Check done. ∎

# 5 Flow

## 5.1 Introduction: Motion versus flow

• A motion $\widetilde{\Phi}:(t,P_{Obj})\to p_t=\widetilde{\Phi}(t,P_{Obj})$ locates at $t$ a particle $P_{Obj}$ in the affine space $\mathbb{R}^n$, cf. (1.4), and the Eulerian velocity field $\vec{v}$ is deduced: $\vec{v}(t,p_t):=\frac{d\widetilde{\Phi}_{P_{Obj}}}{dt}(t)$, cf. (2.5).

• A flow starts with a Eulerian velocity field $\vec{v}$, and the motion is deduced by solving the ODE (ordinary differential equation) $\frac{d\Phi}{dt}(t)=\vec{v}(t,\Phi(t))$ with initial conditions.

## 5.2 Definition

Let $\vec{v}:\left\{\begin{array}{ll}\mathbb{R}\times\mathbb{R}^n & \to \vec{\mathbb{R}^n}\\ (t,p) & \to \vec{v}(t,p)\end{array}\right\}$ be a unstationary vector field, e.g., a Eulerian velocity field which definition domain is $\mathcal{C}\subset\mathbb{R}\times\mathbb{R}^n$. We look for maps $\Phi:\left\{\begin{array}{ll}\mathbb{R} & \to \mathbb{R}^n\\ t & \to p=\Phi(t)\end{array}\right\}$ which are locally (i.e. in the vicinity of some $t_0$) solutions of the ODE (ordinary differential equation)

$$\frac{d\Phi}{dt}(t)=\vec{v}(t,\Phi(t)),\quad \text{also written}\quad \frac{dp}{dt}(t)=\vec{v}(t,p(t)),\quad \text{or}\quad \frac{d\vec{x}}{dt}(t)=\vec{v}(t,\vec{x}(t)) \tag{5.1}$$

where $\vec{x}(t)=\overrightarrow{\mathcal{O}p(t)}$ after a choice of an origin. I.e. $\Phi'(t)=\vec{v}(t,\Phi(t))$, or $p'(t)=\vec{v}(t,p(t))$, or $\vec{x}\,'(t)=\vec{v}(t,\vec{x}(t))$. Also abusively written $\dfrac{dp}{dt}=\vec{v}(t,p)$ or $\dfrac{d\vec{x}}{dt}=\vec{v}(t,\vec{x})$.

**Definition 5.1** A solution $\Phi$ of (5.1) is a flow of $\vec{v}$; Also called an integral curve of $\vec{v}$ since (5.1) also reads $\Phi(t)=\int_{\tau=t_1}^{t}\vec{v}(\tau,\Phi(\tau))\,d\tau+\Phi(t_1)$.

**Remark 5.2** Improper notation for (5.1):

$$\frac{dp}{dt}(t)\overset{\text{written}}{=}\frac{dp(t)}{dt}\qquad (=\vec{v}(t,p(t))). \tag{5.2}$$

Question: If the notation $\frac{dp(t)}{dt}$ is used, then what is the meaning of $\frac{dp(f(t))}{dt}$?

Answer: It means, either $\frac{dp}{dt}(f(t))$, or $\frac{d(p\circ f)}{dt}(t)=\frac{dp}{dt}(f(t))\frac{df}{dt}(t)$: Ambiguous. So it is better to use $\frac{dp}{dt}(t)$ and to avoid $\frac{dp(t)}{dt}$. ∎

## 5.3 Cauchy–Lipschitz theorem

Let $(t_0,p_{t_0})$ be in the definition domain of $\vec{v}$. Purpose: Find $\Phi$ solution of "the ODE with initial condition $(t_0,p_{t_0})$", i.e. s.t.

$$\frac{d\Phi}{dt}(t)=\vec{v}(t,\Phi(t))\quad\text{and}\quad\Phi(t_0)=p_{t_0},\quad\text{in a vicinity of } t_0. \tag{5.3}$$

The couple $(t_0,p_{t_0})$ is the initial condition, and the values $t_0$ and $p_{t_0}$ are the initial conditions.

$\Omega$ is an open set in $\mathbb{R}^n$ s.t. its closure $\overline{\Omega}$ is a regular domain, and $||.||$ is a norm in $\vec{\mathbb{R}^n}$.

**Definition 5.3** Let $t_1,t_2\in\mathbb{R}$, $t_1<t_2$. A continuous map $\vec{v}:[t_1,t_2]\times\overline{\Omega}\to\vec{\mathbb{R}^n}$ is Lipschitzian iff it is "space Lipschitzian uniformly in time", i.e. iff

$$\exists k>0,\ \forall t\in[t_1,t_2],\ \forall p,q\in\overline{\Omega},\ ||\vec{v}(t,q)-\vec{v}(t,p)||\le k||q-p||, \tag{5.4}$$

i.e. $||\vec{v}_t(q)-\vec{v}_t(p)||\le k||q-p||$. So, $\frac{||\vec{v}_t(q)-\vec{v}_t(p)||}{||q-p||}\le k$, for all $t$ and all $p\ne q$: The variations of $\vec{v}$ are bounded in space, uniformly in time. (In particular implies that $\vec{v}$ is continuous.)

**Theorem 5.4 (and definition (Cauchy–Lipschitz))** *If $\vec{v}:[t_1,t_2]\times\overline{\Omega}\to\vec{\mathbb{R}^n}$ is Lipschitzian and $(t_0,p_{t_0})\in]t_1,t_2[\times\Omega$ then there exists $\varepsilon=\varepsilon_{t_0,p_{t_0}}>0$ s.t. (5.3) has a unique solution $\Phi:]t_0-\varepsilon,t_0+\varepsilon[\to\mathbb{R}^n$:*

$$\frac{d\Phi}{dt}(t)=\vec{v}(t,\Phi(t))\quad\textit{and}\quad\Phi(t_0)=p_{t_0},\quad\textit{and}\quad\Phi\overset{\textit{written}}{=}\Phi^{t_0}_{p_{t_0}}. \tag{5.5}$$

*Moreover, if $\vec{v}$ is $C^k$ then $\Phi^{t_0}_{p_{t_0}}$ is $C^{k+1}$.*

**Proof.** See e.g. Arnold [2]. In particular $||\vec{v}||_\infty:=\sup_{t\in]t_0-\varepsilon,t_0+\varepsilon[,\ p\in\Omega}||\vec{v}(t,p)||_{\mathbb{R}^n}$ (maximum speed) exists since $\vec{v}\in C^0$ on the compact $[t_1,t_2]\times\overline{\Omega})$, see definition 5.3, hence we can choose $\varepsilon=\min(t_0-t_1,t_2-t_0,\frac{d(p_{t_0},\partial\Omega)}{||\vec{v}||_\infty})$ (the time needed to reach the border $\partial\Omega$ from $p_{t_0}$). ∎

We have thus defined the function, also called "a flow",

$$\Phi : \begin{cases} ]t_1,t_2[\times]t_1,t_2[\times\Omega_{t_0} & \to \Omega \\ (t,t_0,p_{t_0}) & \to p = \Phi(t,t_0,p_{t_0}) := \Phi^{t_0}_{p_{t_0}}(t) \overset{\text{written}}{=} \Phi(t;t_0,p_{t_0}). \end{cases} \tag{5.6}$$

And (5.5) reads

$$\frac{\partial\Phi}{\partial t}(t;t_0,p_{t_0}) = \vec{v}(t,\Phi(t;t_0,p_{t_0})), \quad \text{with} \quad \Phi(t_0;t_0,p_{t_0}) = p_{t_0}. \tag{5.7}$$

And we have defined the function, also called "a flow",

$$\Phi^{t_0} : \begin{cases} [t_0-\varepsilon,t_0+\varepsilon]\times\Omega_{t_0} & \to \mathbb{R}^n \\ (t,p_{t_0}) & \to p = \Phi^{t_0}(t,p_{t_0}) := \Phi^{t_0}_{p_{t_0}}(t). \end{cases} \tag{5.8}$$

And (5.5) reads

$$\frac{\partial\Phi^{t_0}}{\partial t}(t,p_{t_0}) = \vec{v}(t,\Phi^{t_0}(t,p_{t_0})), \quad \text{and} \quad \Phi^{t_0}(t_0,p_{t_0}) = p_{t_0}. \tag{5.9}$$

Other notation: $\Phi_{t;t_0} := \Phi^{t_0}_t$, i.e. $\Phi_{t;t_0}(p_{t_0}) := \Phi^{t_0}_t(p_{t_0})$.

**Corollary 5.5** *Let $\Omega_{t_0}$ be an open set s.t. $\Omega_{t_0} \subset\subset \Omega$ (i.e. there exists a compact set $K \in \mathbb{R}^n$ s.t. $\Omega_{t_0} \subset K \subset \Omega$). Then there exists $\varepsilon > 0$ s.t. a flow $\Phi^{t_0}$ exists on $]t_0-\varepsilon,t_0+\varepsilon[\times\Omega_{t_0}$.*

**Proof.** Let $d = d(K,\mathbb{R}^n-\Omega)$ (la distance of $K$ to the border of $\Omega$.

Let $||\vec{v}||_\infty := \sup_{t\in[t_1,t_2],p\in\overline{\Omega}} ||\vec{v}(t,p)||_{\mathbb{R}^n}$ (exists since $\vec{v} \in C^0$ on the compact $[t_1,t_2]\times\overline{\Omega}$).

Let $\varepsilon = \min(t_0-t_1, t_2-t_0, \frac{d}{||\vec{v}||_\infty})$ (less that the minimum time to reach the border from $K$ at maximum speed $||v||_\infty$).

Let $p_{t_0} \in K$ and $t \in ]t_0-\varepsilon,t_0+\varepsilon[$. Then $\Phi^{t_0}_{p_{t_0}}$ exists, cf.theorem 5.4, and $||\Phi^{t_0}_{p_{t_0}}(t) - \Phi^{t_0}_{p_{t_0}}(t_0)||_{\mathbb{R}^n} \leq [t-t_0| \sup_{\tau\in]t_0-\varepsilon,t_0+\varepsilon[}(||(\Phi^{t_0}_{p_{t_0}})'(\tau)||_{\mathbb{R}^n})$ (mean value theorem since, $\vec{v}$ being $C^0$, $\Phi$ is $C^1$). Thus $||\Phi^{t_0}_{p_{t_0}}(t) - \Phi^{t_0}_{p_{t_0}}(t_0)||_{\mathbb{R}^n} \leq [t-t_0|\,||v||_\infty$, thus $\Phi^{t_0}_{p_{t_0}}(t) \in \Omega$. Thus $\Phi^{t_0}_{p_{t_0}}$ exists on $]t_0-\varepsilon,t_0+\varepsilon[$, for all $p_{t_0} \in K$. ∎

**Remark 5.6** The definition of a flow starts with a Eulerian velocity (independent of any initial time), and then, due to the introduction of initial conditions, leads to the Lagrangian functions $\Phi^{t_0}$, cf. (5.8). Once again, a Lagrangian function is the result of an Eulerian function. ∎

## 5.4 Examples

**Example 1** $\mathbb{R}^2$ with an origin $\mathcal{O}$, Euclidean basis $(\vec{e}_1,\vec{e}_2)$, $\Omega = [0,2]\times[0,1]$ (observation window), $p \in \mathbb{R}^2$, $\overrightarrow{\mathcal{O}p} =^{\text{written}} \vec{x} = x\vec{e}_1 + y\vec{e}_2 =^{\text{written}} (x,y)$, $t_1 = -1$, $t_2 = 1$, $t_0 \in ]t_1,t_2[$, $a,b \in \mathbb{R}$, $a \neq 0$, and

$$\vec{v}(t,p) = \begin{cases} v^1(t,x,y) = ay, \\ v^2(t,x,y) = b\sin(t-t_0). \end{cases} \tag{5.10}$$

($b = 0$ = stationary case = shear flow.) $\vec{x}(t_0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \overrightarrow{\mathcal{O}\Phi^{t_0}_{p_{t_0}}(t)}$ and (5.9) give

$$\begin{cases} \dfrac{dx}{dt}(t) = v^1(t,x(t),y(t)) = ay(t), \\ \dfrac{dy}{dt}(t) = v^2(t,x(t),y(t)) = b\sin(t-t_0), \end{cases} \quad \text{with} \quad \begin{cases} x(t_0) = x_0, \\ y(t_0) = y_0. \end{cases} \tag{5.11}$$

Thus

$$\vec{x}(t) = \overrightarrow{\mathcal{O}p(t)} = \overrightarrow{\mathcal{O}\Phi^{t_0}_{p_{t_0}}(t)} = \begin{pmatrix} x(t) = x_0 + a(y_0+b)(t-t_0) - ab\sin(t-t_0) \\ y(t) = y_0 + b - b\cos(t-t_0) \end{pmatrix}. \tag{5.12}$$

**Example 2** Similar framework. Let $\omega > 0$ and consider (spin vector field)

$$\vec{v}(t,x,y) = \begin{pmatrix} -\omega y \\ \omega x \end{pmatrix} = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \overset{\text{written}}{=} \vec{v}(x,y). \tag{5.13}$$

With $r_0 = \sqrt{x_0^2 + y_0^2}$ and $\theta_0$ s.t. $\vec{x}_0 = \begin{pmatrix} x_0 = r_0 \cos(\omega t_0) \\ y_0 = r_0 \sin(\omega t_0) \end{pmatrix}$, the solution $\Phi^{t_0}_{p_{t_0}}$ of (5.9) is

$$\vec{x}(t) = \overrightarrow{\mathcal{O}p(t)} = \overrightarrow{\mathcal{O}\Phi^{t_0}_{p_{t_0}}(t)} = \begin{pmatrix} x(t) = r_{t_0} \cos(\omega t) \\ y(t) = r_{t_0} \sin(\omega t) \end{pmatrix}. \tag{5.14}$$

Indeed, $\begin{pmatrix} \frac{\partial x}{\partial t}(t,\vec{x}_0) \\ \frac{\partial y}{\partial t}(t,\vec{x}_0) \end{pmatrix} = \begin{pmatrix} v^1(t,x(t,\vec{x}_0),y(t,\vec{x}_0)) \\ v^2(t,x(t,\vec{x}_0),y(t,\vec{x}_0)) \end{pmatrix} = \begin{pmatrix} -\omega y(t,\vec{x}_0) \\ \omega x(t,\vec{x}_0) \end{pmatrix}$, thus $\frac{\partial x}{\partial t}(t,\vec{x}_0) = -\omega y(t,\vec{x}_0)$ and $\frac{\partial y}{\partial t}(t,\vec{x}_0) = \omega x(t,\vec{x}_0)$, thus $\frac{\partial^2 y}{\partial t^2}(t,\vec{x}_0) = -\omega^2 y(t,\vec{x}_0)$, hence $y$; Idem for $x$. Here $d\vec{v}(t,x,y) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \omega \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}$ is the $\pi/2$-rotation composed with the homothety with ratio $\omega$.

## 5.5 Composition of flows

Let $\vec{v}$ be a vector field on $\mathbb{R} \times \Omega$ and $\Phi^{t_0}_{p_{t_0}}$ solution of (5.5). We use the notations

$$p_t = \Phi^{t_0}_{p_{t_0}}(t) = \Phi_{t_0,p_{t_0}}(t) = \Phi^{t_0}_t(p_{t_0}) = \Phi_{t;t_0}(p_{t_0}) = \Phi^{t_0}(t,p_{t_0}) = \Phi(t;t_0,p_{t_0}). \tag{5.15}$$

### 5.5.1 Law of composition of flows (determinism)

**Proposition 5.7** *For all $t_0, t_1, t_2 \in \mathbb{R}$, we have (determinism)*

$$\Phi^{t_1}_{t_2} \circ \Phi^{t_0}_{t_1} = \Phi^{t_0}_{t_2}, \quad \textit{i.e.} \quad \Phi_{t_2;t_1} \circ \Phi_{t_1;t_0} = \Phi_{t_2;t_0}. \tag{5.16}$$

*("The composition of the photos gives the film"). So, with $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0}) = \Phi_{t_1;t_0}(p_{t_0})$,*

$$p_{t_2} = \Phi^{t_1}_{t_2}(p_{t_1}) = \Phi^{t_0}_{t_2}(p_{t_0}), \quad \textit{i.e.} \quad p_{t_2} = \Phi_{t_2;t_1}(p_{t_1}) = \Phi_{t_2;t_0}(p_{t_0}). \tag{5.17}$$

*Thus*

$$d\Phi^{t_1}_{t_2}(p_{t_1}).d\Phi^{t_0}_{t_1}(p_{t_0}) = d\Phi^{t_0}_{t_2}(p_{t_0}), \quad \textit{i.e.} \quad d\Phi_{t_2;t_1}(p_{t_1}).d\Phi_{t_1;t_0}(p_{t_0}) = d\Phi_{t_2;t_0}(p_{t_0}). \tag{5.18}$$

*Summary: The following diagram commutes:*

$$\begin{tikzcd} & p_{t_1} \arrow[dr, "\Phi^{t_1}_{t_2}"] & \\ p_{t_0} \arrow[ur, "\Phi^{t_0}_{t_1}"] \arrow[rr, "\Phi^{t_0}_{t_2}"'] & & p_{t_2} \end{tikzcd} \quad \textit{i.e.} \quad \begin{tikzcd} & p_{t_1} \arrow[dr, "\Phi_{t_2;t_1}"] & \\ p_{t_0} \arrow[ur, "\Phi_{t_1;t_0}"] \arrow[rr, "\Phi_{t_2;t_0}"'] & & p_{t_2} \end{tikzcd}$$

**Proof.** Let $p_{t_1} = \Phi^{t_0}_{p_{t_0}}(t_1)$. (5.9) gives

$$\left\{ \begin{aligned} &\frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t) = \vec{v}(t,\Phi^{t_0}_{p_{t_0}}(t)), \\ &\frac{d\Phi^{t_1}_{p_{t_1}}}{dt}(t) = \vec{v}(t,\Phi^{t_1}_{p_{t_1}}(t)), \end{aligned} \right\} \quad \text{with} \quad p_{t_1} = \Phi^{t_0}_{p_{t_0}}(t_1) = \Phi^{t_1}_{p_{t_1}}(t_1).$$

Thus $\Phi^{t_0}_{p_{t_0}}$ and $\Phi^{t_1}_{p_{t_1}}$ satisfy the same ODE with the same value at $t_1$; Thus they are equal (uniqueness: Cauchy–Lipschitz theorem), thus $\Phi^{t_1}_{p_{t_1}}(t) = \Phi^{t_0}_{p_{t_0}}(t)$ i.e. $\Phi^{t_1}_t(p_{t_1}) = \Phi^{t_0}_t(p_{t_0})$ when $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0}) = \Phi^{t_0}_{t_1}(p_{t_0})$, which is (5.16) for any $t = t_2$. Thus $d\Phi^{t_1}_{t_2}(\Phi^{t_0}_{t_1}(p_{t_0})).d\Phi^{t_0}_{t_1}(p_{t_0}) = d\Phi^{t_0}_{t_2}(p_{t_0})$, i.e. (5.18). ∎

**Corollary 5.8** *A flow is compatible with the motion $\widetilde{\Phi}$ of an object Obj: (3.3) gives $\Phi^{t_1}_{t_2} \circ \Phi^{t_0}_{t_1} = (\widetilde{\Phi}_{t_2} \circ (\widetilde{\Phi}_{t_1})^{-1}) \circ (\widetilde{\Phi}_{t_1} \circ (\widetilde{\Phi}_{t_0})^{-1}) = \widetilde{\Phi}_{t_2} \circ (\widetilde{\Phi}_{t_0})^{-1} = \Phi^{t_0}_{t_2}$, that is (5.16).*

### 5.5.2 Stationnary case

**Definition 5.9** $\vec{v}$ is a stationary vector field iff $\frac{\partial \vec{v}}{\partial t} = 0$. Hence $\vec{v}(t,p) =^{\text{written}} \vec{v}(p)$, and the associated flow $\Phi^{t_0}$, which satisfies

$$\frac{\partial \Phi^{t_0}}{\partial t}(t, p_{t_0}) = \vec{v}(\Phi^{t_0}(t, p_{t_0})) = \vec{v}(p_t) \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}), \tag{5.19}$$

is said to be stationary.

**Proposition 5.10** *If $\vec{v}$ is a stationary vector field then, for all $t_0, t_1, h$ when meaningful (i.e. $t_1$ close enough to $t_0$ and $h$ small enough),*

$$\Phi^{t_1}_{t_1+h} = \Phi^{t_0}_{t_0+h}, \quad \text{i.e.} \quad \Phi_{t_1+h;t_1} = \Phi_{t_0+h;t_0}, \tag{5.20}$$

*i.e. $\Phi^{t_1}_{t_1+h}(q) = \Phi^{t_0}_{t_0+h}(q)$, i.e. $\Phi(t_1+h; t_1, q) = \Phi(t_0+h; t_0, q)$ for all $q \in \Omega_{t_0}$ (see corollary 5.5). In other words,*

$$\Phi^{t_0+h}_{t_1+h} = \Phi^{t_0}_{t_1}, \quad \text{i.e.} \quad \Phi_{t_1+h;t_0+h} = \Phi_{t_1;t_0}, \tag{5.21}$$

*i.e. $\Phi^{t_0+h}_{t_1+h}(q) = \Phi^{t_0}_{t_1}(q)$, i.e. $\Phi(t_1+h; t_0+h, q) = \Phi(t_1; t_0, q)$ for all $q \in \Omega_{t_0}$.*

**Proof.** Let $q \in \Omega_{t_0}$, $\alpha(h) = \Phi^{t_0}_{t_0+h}(q) = \Phi^{t_0}_q(t_0+h)$ and $\beta(h) = \Phi^{t_1}_{t_1+h}(q) = \Phi^{t_1}_q(t_1+h)$.

Thus $\alpha'(h) = \frac{d\Phi^{t_0}_q}{dt}(t_0+h) = \vec{v}(t_0+h, \Phi^{t_0}_q(t_0+h)) = \vec{v}(\Phi^{t_0}_q(t_0+h)) = \vec{v}(\alpha(h))$ (stationary flow), and $\beta'(h) = \frac{d\Phi_{t_1 q}}{dt}(t_1+h) = \vec{v}(t_1+h, \Phi^{t_1}_q(t_1+h)) = \vec{v}(\Phi^{t_1}_q(t_1+h)) = \vec{v}(\beta(h))$ (stationary flow).

Thus $\alpha$ and $\beta$ satisfy the same ODE with the same initial condition $\alpha(0) = \beta(0) = q$. Thus $\alpha = \beta$. Hence (5.20). Thus, with $h = t_1 - t_0$, i.e. with $t_1 = t_0+h$ and $t_0+h = t_1$, we get (5.21). ∎

**Corollary 5.11** *If $\vec{v}$ is a stationary vector field, cf. (5.19), then*

$$d\Phi^{t_0}_t(p_{t_0}).\vec{v}(p_{t_0}) = \vec{v}(p_t) \quad \textit{when} \quad p_t = \Phi^{t_0}_t(p_{t_0}), \tag{5.22}$$

*that is, if $\vec{v}$ is stationary, then $\vec{v}$ is transported (push-forwarded by $\Phi^{t_0}_t$) along itself.*

**Proof.** (5.17), $t_2 = t_1+s$ and $t_1 = t_0+s$ give $\Phi^{t_0+s}_{t_1+s}(\Phi^{t_0}_{t_0+s}(p_{t_0})) = \Phi^{t_0}_{t_1+s}(p_{t_0})$, and $\vec{v}$ is stationary, thus $\Phi^{t_0}_{t_1}(\Phi^{t_0}_{t_0+s}(p_{t_0})) = \Phi^{t_0}_{t_1+s}(p_{t_0})$, i.e. $\Phi(t_1; t_0, \Phi_{t_0,p_{t_0}}(t_0+s)) = \Phi_{t_0,p_{t_0}}(t_1+s)$, thus ($s$ derivative)

$$d\Phi(t_1; t_0, \Phi(t_0+s; t_0, p_{t_0})).\Phi_{t_0,p_{t_0}}{}'(t_0+s) = \Phi_{t_0,p_{t_0}}{}'(t_1+s),$$

thus $d\Phi^{t_0}_{t_1}(\Phi(t_0+s; t_0, p_{t_0})).\vec{v}(t_0+s, \Phi_{t_0,p_{t_0}}(t_0+s)) = \vec{v}(t_1+s, \Phi_{t_0,p_{t_0}}(t_1+s))$. Thus with $s = 0$, and $\vec{v}$ being stationary, $d\Phi^{t_0}_{t_1}(\Phi(t_0; t_0, p_{t_0})).\vec{v}(\Phi_{t_0,p_{t_0}}(t_0)) = \vec{v}(\Phi_{t_0,p_{t_0}}(t_1))$, thus (5.22). ∎

## 5.6 Velocity on the trajectory traveled in the opposite direction

Let $t_0, t_1 \in \mathbb{R}$, $t_1 > t_0$, and $p_{t_0} \in \mathbb{R}^n$. Consider the trajectory $\Phi^{t_0}_{p_{t_0}} : \left\{ \begin{array}{ll} [t_0, t_1] & \to \mathbb{R}^n \\ t & \to p(t) = \Phi^{t_0}_{p_{t_0}}(t) \end{array} \right\}$. So $p_{t_0}$ is the beginning of the trajectory, $p_{t_1} = \Phi^{t_0}_{t_1}(p_{t_0})$ the end, $\vec{v}(t, p(t)) = \frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t)$ being the velocity.

Define the trajectory traveled in the opposite direction, i.e. define

$$\Psi^{t_1}_{p_{t_1}} : \left\{ \begin{array}{ll} [t_0, t_1] & \to \mathbb{R}^n \\ u & \to q(u) = \Psi^{t_1}_{p_{t_1}}(u) := \Phi^{t_0}_{p_{t_0}}(t_0+t_1-u) = \Phi^{t_0}_{p_{t_0}}(t) = p(t) \quad \text{when} \quad t = t_0+t_1-u. \end{array} \right. \tag{5.23}$$

In particular $q(t_0) = \Psi^{t_1}_{p_{t_1}}(t_0) = \Phi^{t_0}_{p_{t_0}}(t_1) = p(t_1)$ and $q(t_1) = \Psi^{t_1}_{p_{t_1}}(t_1) = \Phi^{t_0}_{p_{t_0}}(t_0) = p(t_0)$.

**Proposition 5.12** *The velocity on the trajectory traveled in the opposite direction is the opposite of the velocity on the initial trajectory:*

$$\frac{d\Psi^{t_1}_{p_{t_1}}}{du}(u) = q'(u) = -p'(t) = -\vec{v}(t, p(t)) \quad \textit{when} \quad t = t_0+t_1-u, \tag{5.24}$$

**Proof.** $\Psi^{t_1}_{p_{t_1}}(u) = \Phi^{t_0}_{p_{t_0}}(t_0+t_1-u)$ gives $\frac{d\Psi^{t_1}_{p_{t_1}}}{du}(u) = -\frac{d\Phi^{t_0}_{p_{t_0}}}{dt}(t_0+t_1-u) = -\vec{v}(t_0+t_1-u, \Phi^{t_0}_{p_{t_0}}(t_0+t_1-u)) = -\vec{v}(t, \Phi^{t_0}_{p_{t_0}}(t))$ when $t = t_0+t_1-u$. ∎

## 5.7 Variation of the flow as a function of the initial time

### 5.7.1 Ambiguous and non ambiguous notations

Let $\Phi : (t,u,p) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \to \Phi(t,u,p) \in \mathbb{R}^n$ be a $C^1$ function. The "numbered partial derivatives" are

$$\partial_1\Phi(t,u,p) = \lim_{h\to 0} \frac{\Phi(t{+}h,u,p) - \Phi(t,u,p)}{h}, \tag{5.25}$$

$$\partial_2\Phi(t,u,p) = \lim_{h\to 0} \frac{\Phi(t,u{+}h,p) - \Phi(t,u,p)}{h}, \tag{5.26}$$

$$\partial_3\Phi(t,u,p).\vec{w} := d\Phi(t,u,p).\vec{w} = \lim_{h\to 0} \frac{\Phi(t,u,p{+}h\vec{w}) - \Phi(t,u,p)}{h} \tag{5.27}$$

for all $\vec{w} \in \vec{\mathbb{R}^n}$ vectors at $p$ (space differentiation).

When the name of the first variable is systematically noted $t$, then

$$\partial_1\Phi(t,u,p) \overset{\text{written}}{=} \frac{\partial\Phi}{\partial t}(t,u,p) \underset{\text{writing}}{\overset{\text{ambiguous}}{=}} \frac{\partial\Phi(t,u,p)}{\partial t}. \tag{5.28}$$

NB: This notation can be ambiguous: What is the meaning of $\frac{\partial\Phi}{\partial t}(t,t,p)$? In ambiguous situations, use the notation $\partial_1\Phi$, or (if no composed functions inside) use $\frac{\partial\Phi(t,u,p)}{\partial t}_{|u=t}$ (so $t$ is the derivation variable, and after the calculation you take $u=t$).

When the name of the second variable is systematically noted $u$, then

$$\partial_2\Phi(t,u,p) \overset{\text{written}}{=} \frac{\partial\Phi}{\partial u}(t,u,p) \underset{\text{writing}}{\overset{\text{ambiguous}}{=}} \frac{\partial\Phi(t,u,p)}{\partial u}. \tag{5.29}$$

NB: Idem this notation can be ambiguous: What is the meaning of $\frac{\partial\Phi}{\partial u}(u,u,p)$? In ambiguous situations, use the notation $\partial_2\Phi$, or use $\frac{\partial\Phi(t,u,p)}{\partial u}_{|t=u}$.

When the name of the third variable is systematically a space variable noted $p$, then

$$\partial_3\Phi(t,u,p) \overset{\text{written}}{=} d\Phi(t,u,p) \overset{\text{written}}{=} \frac{\partial\Phi}{\partial p}(t,u,p) \underset{\text{writing}}{\overset{\text{ambiguous}}{=}} \frac{\partial\Phi(t,u,p)}{\partial p}. \tag{5.30}$$

### 5.7.2 Variation of the flow as a function of the initial time

The law of composition of the flows (5.17) gives $g(u) := \Phi(t,u,\Phi(u,t_0,p_0)) = \Phi(t,t_0,p_0)$, thus $g'(u)=0$, thus

$$\begin{aligned} &\partial_2\Phi(t,u,\Phi(u,t_0,p_0)) + \partial_3\Phi(t,u,\Phi(u,t_0,p_0)).\partial_1\Phi(u,t_0,p_0) = 0,\\ \text{i.e. } &\partial_2\Phi(t,u,p(u)) = -d\Phi(t,u,p(u)).\vec{v}(u,p(u)) \quad \text{when} \quad p(u) = \Phi(u,t_0,p_0). \end{aligned} \tag{5.31}$$

In particular $u = t_0$ gives, for all $(t,t_0,p_0) \in \mathbb{R}^2 \times \Omega_{t_0}$,

$$\Big(\frac{\partial\Phi(t,t_0,p_0)}{\partial t_0} :=\Big) \quad \partial_2\Phi(t,t_0,p_0) = -d\Phi(t,t_0,p_0).\vec{v}(t_0,p_0). \tag{5.32}$$

In particular $\Big(\frac{d\Phi(t,t_0,p_0)}{dt_0}_{|t=t_0} :=\Big) \quad \partial_2\Phi(t_0,t_0,p_0) = -\vec{v}(t_0,p_0).$

# Part II
# Push-forward

## 6 Push-forward

The general tool to describe "transport" is "push-forward by a motion" (the "take with you" operator), cf. § 4.1 and figure 4.1. The push-forward also gives the tool needed to understand the velocity addition formula: In that case, the push-forward is the translator between observers. The push-forward can also be used to write coordinate systems. As usual, we start with qualitative results (observer independent results), then quantitative results are deduced.

### 6.1 Definition

(Simplified framework.) $\mathcal{E}$ and $\mathcal{F}$ are affine spaces, $E$ and $F$ are the associated vector spaces equipped with norms $||.||_E$ and $||.||_F$, $\dim E = \dim F = n \in \mathbb{N}^*$ (finite dimension), $\mathcal{U}_\mathcal{E}$ and $\mathcal{U}_\mathcal{F}$ are open sets in the affine space $\mathcal{E}$ and $\mathcal{F}$, or possibly in the vector spaces $E$ and $F$, and

$$\Psi : \begin{Bmatrix} \mathcal{U}_\mathcal{E} & \to \mathcal{U}_\mathcal{F} \\ p_\mathcal{E} & \to p_\mathcal{F} = \Psi(p_\mathcal{E}) \end{Bmatrix} \quad \text{is a diffeomorphism,} \tag{6.1}$$

i.e. a $C^1$ invertible map which inverse is $C^1$.

**Definition 6.1** $\Psi$ is called a push-forward, and $\Psi^{-1}$ the pull-back (push-forward with $\Psi^{-1}$). See fig. 6.1.

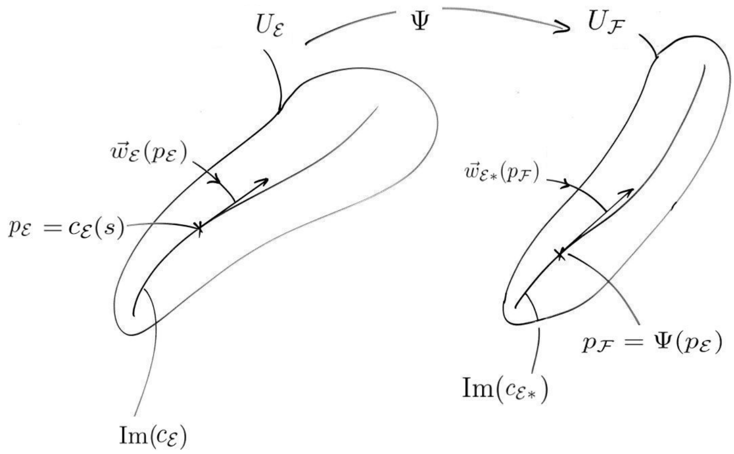


Figure 6.1: $c_\mathcal{E} : s \to p_\mathcal{E} = c_\mathcal{E}(s)$ is a curve in $\mathcal{U}_\mathcal{E}$. Push-forwarded by $\Psi$ it becomes the curve $c_{\mathcal{E}*} := \Psi \circ c_\mathcal{E}$ in $\mathcal{U}_\mathcal{F}$. The tangent vector at $p_\mathcal{E} = c_\mathcal{E}(s)$ is $\vec{w}_\mathcal{E}(p_\mathcal{E}) = c_\mathcal{E}{}'(s)$, and the tangent vector at $p_\mathcal{F} = c_\mathcal{F}(s) = \Psi(c_\mathcal{E}(s))$ is $\vec{w}_{\mathcal{E}*}(p_\mathcal{F}) = c_\mathcal{F}{}'(s) = d\Psi(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E})$. Other illustation: See figure 4.1.

Example: $\Psi = \Phi_t^{t_0} : \Omega_{t_0} \to \Omega_t$, the motion that transforms $\Omega_{t_0}$ into $\Omega_t$, cf. (3.2).
Example: $\Psi : U_E \to U_F$ a coordinate system, see example 6.12.
Example: $\Psi = \Theta_t : \mathcal{R}_B \to \mathcal{R}_A$, a change of referential at $t$ (change of observer), see § 10.

NB: $\Psi$ being a diffeomorphism, $\Psi^{-1}(\Psi(p_\mathcal{E})) = p_\mathcal{E}$ and $p_\mathcal{F} = \Psi(p_\mathcal{E})$ give $d\Psi^{-1}(p_\mathcal{F}).d\Psi(p_\mathcal{E}) = I$.

### 6.2 Push-forward and pull-back of points

**Definition 6.2** If $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$ (a point in $\mathcal{U}_\mathcal{E}$) then its push-forward by $\Psi$ is the point

$$p_\mathcal{F} = \boxed{\Psi_* p_\mathcal{E} := \Psi(p_\mathcal{E})} = p_{\mathcal{E}*} \ \in \mathcal{U}_\mathcal{F}, \tag{6.2}$$

see figure 6.1, the last notation if $\Psi$ is implicit. And if $p_\mathcal{F} \in \mathcal{U}_\mathcal{F}$ then its pull-back by $\Psi$ is the point

$$p_\mathcal{E} = \boxed{\Psi^* p_\mathcal{F} := \Psi^{-1}(p_\mathcal{F})} = p_\mathcal{F}{}^* \ \in \mathcal{U}_\mathcal{E}. \tag{6.3}$$

We immediately have $\Psi^* \circ \Psi_* = I$.

The notations $_*$ for push-forward and $^*$ for pull-back have been proposed by Spivak; Also see Abraham and Marsden [1] second edition who adopt this notation.

## 6.3 Push-forward and pull-back of curves

Let $c_{\mathcal{E}} : \left\{ \begin{array}{ll} ]-\varepsilon,\varepsilon[ & \to \mathcal{U}_{\mathcal{E}} \\ s & \to p_{\mathcal{E}} = c_{\mathcal{E}}(s) \end{array} \right\}$ be a curve in $\mathcal{U}_{\mathcal{E}}$.

**Definition 6.3** Its push-forward by $\Psi$ is the curve

$$\Psi_* c_{\mathcal{E}} := \Psi \circ c_{\mathcal{E}} : \left\{ \begin{array}{ll} ]-\varepsilon,\varepsilon[ & \to \mathcal{U}_{\mathcal{F}} \\ s & \to p_{\mathcal{F}} = \Psi_* c_{\mathcal{E}}(s) := \Psi(c_{\mathcal{E}}(s)) \overset{\text{written}}{=} c_{\mathcal{E}*}(s) \quad (= \Psi(p_{\mathcal{E}})), \end{array} \right. \tag{6.4}$$

where $\Psi_* c_{\mathcal{E}} =^{\text{written}} c_{\mathcal{E}*}$ when $\Psi$ is implicit. See figure 6.1. This defines

$$\Psi_* : \left\{ \begin{array}{ll} \mathcal{F}(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{E}}) & \to \mathcal{F}(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{F}}) \\ c_{\mathcal{E}} & \to \Psi_*(c_{\mathcal{E}}) := \Psi \circ c_{\mathcal{E}} \overset{\text{written}}{=} \Psi_* c_{\mathcal{E}} = c_{\mathcal{E}*}. \end{array} \right. \tag{6.5}$$

Let $c_{\mathcal{F}} : \left\{ \begin{array}{ll} ]-\varepsilon,\varepsilon[ & \to \mathcal{U}_{\mathcal{F}} \\ s & \to p_{\mathcal{F}} = c_{\mathcal{F}}(s) \end{array} \right\}$ is a curve in $\mathcal{U}_{\mathcal{F}}$.

**Definition 6.4** Its pull-back by $\Psi$ is

$$\Psi^* c_{\mathcal{F}} := \Psi^{-1} \circ c_{\mathcal{E}} \left\{ \begin{array}{ll} ]-\varepsilon,\varepsilon[ & \to \mathcal{U}_{\mathcal{E}} \\ s & \to p_{\mathcal{E}} = \Psi^* c_{\mathcal{F}}(s) := \Psi^{-1}(c_{\mathcal{F}}(s)) \overset{\text{written}}{=} {c_{\mathcal{F}}}^*(s) \quad (= \Psi^{-1}(p_{\mathcal{F}})). \end{array} \right. \tag{6.6}$$

We have thus defined

$$\Psi^* : \left\{ \begin{array}{ll} \mathcal{F}(C^1(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{F}}) & \to \mathcal{F}(C^1(]-\varepsilon,\varepsilon[;\mathcal{U}_{\mathcal{E}}) \\ c_{\mathcal{F}} & \to \Psi^*(c_{\mathcal{F}}) := \Psi^{-1} \circ c_{\mathcal{F}} \overset{\text{written}}{=} \Psi^* c_{\mathcal{F}} = {c_{\mathcal{F}}}^*. \end{array} \right. \tag{6.7}$$

## 6.4 Push-forward and pull-back of scalar functions

### 6.4.1 Definitions

Let $f_{\mathcal{E}} : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{E}} & \to \mathbb{R} \\ p_{\mathcal{E}} & \to f_{\mathcal{E}}(p_{\mathcal{E}}) \end{array} \right\}$ (scalar valued function).

**Definition 6.5** Its push-forward by $\Psi$ is the (scalar valued) function

$$\Psi_* f_{\mathcal{E}} := f_{\mathcal{E}} \circ \Psi^{-1} : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{F}} & \to \mathbb{R} \\ p_{\mathcal{F}} & \to \Psi_* f_{\mathcal{E}}(p_{\mathcal{F}}) := f_{\mathcal{E}}(p_{\mathcal{E}}) \overset{\text{written}}{=} f_{\mathcal{E}*}(p_{\mathcal{F}}) \quad \text{when} \quad p_{\mathcal{E}} = \Psi^{-1}(p_{\mathcal{F}}), \end{array} \right. \tag{6.8}$$

(noted $f_{\mathcal{E}*}$ when $\Psi$ is implicit), i.e. $\Psi_* f_{\mathcal{E}}(\Psi_* p_{\mathcal{E}}) := f_{\mathcal{E}}(p_{\mathcal{E}})$, or $f_{\mathcal{E}*}(p_{\mathcal{E}*}) := f_{\mathcal{E}}(p_{\mathcal{E}})$ when $p_{\mathcal{E}*} = \Psi(p_{\mathcal{E}})$. We have thus defined

$$\Psi_* : \left\{ \begin{array}{ll} \mathcal{F}(\mathcal{U}_{\mathcal{E}};\mathbb{R}) & \to \mathcal{F}(\mathcal{U}_{\mathcal{F}};\mathbb{R}) \\ f_{\mathcal{E}} & \to f_{\mathcal{F}} := \Psi_*(f_{\mathcal{E}}) = f_{\mathcal{E}} \circ \Psi^{-1} \overset{\text{written}}{=} \Psi_* f_{\mathcal{E}}. \end{array} \right. \tag{6.9}$$

Notation $\Psi_*(f_{\mathcal{E}}) = \Psi_* f_{\mathcal{E}}$ because $\Psi_*$ is linear: $((f_{\mathcal{E}} + \lambda g_{\mathcal{E}}) \circ \Psi^{-1})(p_{\mathcal{F}}) = (f_{\mathcal{E}} + \lambda g_{\mathcal{E}})(p_{\mathcal{E}}) = f_{\mathcal{E}}(p_{\mathcal{E}}) + \lambda g_{\mathcal{E}}(p_{\mathcal{E}}) = (f_{\mathcal{E}} \circ \Psi^{-1})(p_{\mathcal{F}}) + \lambda(g_{\mathcal{E}} \circ \Psi^{-1})(p_{\mathcal{F}})$ gives $\Psi_*(f_{\mathcal{E}} + \lambda g_{\mathcal{E}}) = \Psi_*(f_{\mathcal{E}}) + \lambda \Psi_*(g_{\mathcal{E}})$.

Let $f_{\mathcal{F}} : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{F}} & \to \mathbb{R} \\ p_{\mathcal{F}} & \to f_{\mathcal{F}}(p_{\mathcal{F}}) \end{array} \right\}$ (scalar valued function).

**Definition 6.6** Its pull-back by $\Psi$ is the push-forward by $\Psi^{-1}$, i.e. is

$$\Psi^* f_{\mathcal{F}} := f_{\mathcal{F}} \circ \Psi : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{E}} & \to \mathbb{R} \\ p_{\mathcal{E}} & \to \Psi^* f_{\mathcal{F}}(p_{\mathcal{E}}) := f_{\mathcal{F}}(p_{\mathcal{F}}) \overset{\text{written}}{=} {f_{\mathcal{F}}}^*(p_{\mathcal{E}}) \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}), \end{array} \right. \tag{6.10}$$

i.e. $\Psi^* f_{\mathcal{F}}(\Psi^* p_{\mathcal{F}}) := f_{\mathcal{F}}(p_{\mathcal{F}})$, i.e. ${f_{\mathcal{F}}}^*({p_{\mathcal{F}}}^*) := f_{\mathcal{F}}(p_{\mathcal{F}})$ when $p_{\mathcal{F}} = \Psi^*(p_{\mathcal{F}})$. We have thus defined

$$\Psi^* : \left\{ \begin{array}{ll} \mathcal{F}(\mathcal{U}_{\mathcal{F}};\mathbb{R}) & \to \mathcal{F}(\mathcal{U}_{\mathcal{E}};\mathbb{R}) \\ f_{\mathcal{F}} & \to \Psi^*(f_{\mathcal{F}}) = {f_{\mathcal{F}}}^* := f_{\mathcal{F}} \circ \Psi \overset{\text{written}}{=} \Psi^* f_{\mathcal{F}}. \end{array} \right. \tag{6.11}$$

We immediately have $\Psi^* \circ \Psi_* = I$ and $\Psi_* \circ \Psi^* = I$ (the first $I$ is the identity in $\mathcal{F}(\mathcal{U}_{\mathcal{E}};\mathbb{R})$, the second $I$ is the identity in $\mathcal{F}(\mathcal{U}_{\mathcal{F}};\mathbb{R})$).

Warning: We used the same notations $\Psi_*$ and $\Psi^*$ for the push-forward and pull-backs of points, of curves and of functions: The context removes ambiguities.

### 6.4.2 Interpretation: Why is it useful?

E.g.: Let $\widetilde{\Phi} : \mathbb{R} \times Obj \to \mathbb{R}^n$ be a motion of an object $Obj$. An observer records the temperature $\theta$ at all $t \in [t_0, T]$ and all $p \in \Omega_t = \widetilde{\Phi}(t, Obj)$; He gets $\theta : \left\{ \begin{array}{ll} \mathcal{C} = \bigcup_t (\{t\} \times \Omega_t) & \to \mathbb{R} \\ (t,p) & \to \theta(t,p) \end{array} \right\}$ a Eulerian scalar valued function, cf. (2.2). Then he chooses an initial time $t_0$, considers the associated motion $\Phi^{t_0}$ cf. (3.1), and considers $\theta_{t_0} : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \mathbb{R} \\ p_{t_0} & \to \theta_{t_0}(p_{t_0}) := \theta(t_0, p_{t_0}) \end{array} \right\}$ (snapshot of the temperatures at $t_0$ in $\Omega_{t_0}$). The push-forward of $\theta_{t_0}$ by $\Phi_t^{t_0}$ is $(\Phi_t^{t_0})_*\theta_{t_0} := \theta_{t_0} \circ (\Phi_t^{t_0})^{-1}$ defines the "memory function"

$$(\Phi_t^{t_0})_*\theta_{t_0} : \left\{ \begin{array}{ll} \Omega_t & \to \mathbb{R} \\ p_t & \to (\Phi_t^{t_0})_*\theta_{t_0}(p_t) := \theta_{t_0}(p_{t_0}) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}), \end{array} \right. \tag{6.12}$$

And he writes $(\Phi_t^{t_0})_*\theta_{t_0}(p_t) =^{\text{written}} \theta_{t_0*}(t, p_t)$, so the memory transported is at $t$ at $p_t$ (along a trajectory) by

$$\theta_{t_0*}(t, p(t)) = \theta_{t_0}(p_{t_0}). \tag{6.13}$$

**Question:** Why do we introduce $\theta_{t_0*}$ since we have $\theta_{t_0}$?

**Answer:** An observer does not have the gift of temporal and/or spatial ubiquity; He has to do with values at the actual time $t$ and position $p_t$ where he is (Newton and Einstein's point of view). So, when he was at $t_0$ at $p_{t_0}$ the observer wrote the value $\theta_{t_0}(p_{t_0})$ on a piece of paper (for memory), puts the piece of paper is his pocket, then once at $t$ at $p(t) = \Phi^{t_0}(t, p_{t_0})$, he takes the paper out of his pocket, and renames the value he reads as $\theta_{t_0*}(t, p_t)$ because he is now at $t$ at $p_t$. And, now at $t$ at $p_t$, he can compare the past and present value. In particular the rate

$$\frac{\theta(t, p(t)) - \theta_{t_0*}(t, p(t))}{t - t_0} = \frac{\text{actual}(t, p(t)) - \text{memory}_*(t, p(t))}{t - t_0} \tag{6.14}$$

is physically meaningful for one observer at $t$ at $p_t$ (no ubiquity gift required). For scalar value functions, we get the usual rate $\frac{\theta(t,p(t)) - \theta(t_0, p(t_0))}{t - t_0} \longrightarrow_{t \to t_0} \frac{D\theta}{Dt}(t_0, p_{t_0})$. It it isn't that simple for vector valued functions (the limit $t \to t_0$ defines the Lie derivative).

## 6.5 Push-forward and pull-back of vector fields

This is one of the most important concept for mechanical engineers.

### 6.5.1 An elementary introduction (approximations)

Consider two points $p_{\mathcal{E}}, q_{\mathcal{E}} \in \mathcal{U}_{\mathcal{E}}$ and their push-forwards by $\Psi$ cf. (6.2): $p_{\mathcal{F}} = p_{\mathcal{E}*} = \Psi(p_{\mathcal{E}})$ and $q_{\mathcal{F}} = q_{\mathcal{E}*} = \Psi(q_{\mathcal{E}})$ in $\mathcal{U}_{\mathcal{F}}$. The first order Taylor expansion gives

$$(\Psi(q_{\mathcal{E}}) - \Psi(p_{\mathcal{E}}) =) \quad q_{\mathcal{F}} - p_{\mathcal{F}} = d\Psi(p_{\mathcal{E}}).(q_{\mathcal{E}} - p_{\mathcal{E}}) + o(||q_{\mathcal{E}} - p_{\mathcal{E}}||_E), \tag{6.15}$$

i.e. $\overrightarrow{p_{\mathcal{F}} q_{\mathcal{F}}} = d\Psi(p_{\mathcal{E}}).\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}} + o(||\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}||_E)$, i.e.

$$\frac{\overrightarrow{p_{\mathcal{F}} q_{\mathcal{F}}}}{||\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}||_E} = d\Psi(p_{\mathcal{E}}).\frac{\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}}{||\overrightarrow{p_{\mathcal{E}} q_{\mathcal{E}}}||_E} + o(1). \tag{6.16}$$

The definition of the push-forward fo vectors is obtained by "neglecting" the $o(1)$ (limit as $q_{\mathcal{E}} \to p_{\mathcal{E}}$):

**Definition 6.7** If $\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \in E$ is a vector at $p_{\mathcal{E}} \in U$ then its push-forward by $\Psi$ is the vector $\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) =^{\text{written}} \vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) =^{\text{written}} \Psi_*\vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) \in F$ defined at $p_{\mathcal{F}} = p_{\mathcal{E}*} = \Psi(p_{\mathcal{E}}) \in \mathcal{U}_{\mathcal{F}}$ by

$$\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) = \Psi_*\vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) = \boxed{\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) := d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})}. \tag{6.17}$$

### 6.5.2 Definition of the push-forward of a vector field

To fully grasp the definition and to avoid interpretation errors as in § 4.3 (the unfortunate notation $d\vec{x} = F.d\vec{X}$), we use the definition: "A vector" is a "tangent vector to a curve" (needed for surfaces):

• Let $c_{\mathcal{E}} : \left\{ \begin{array}{rl} ]-\varepsilon,\varepsilon[ & \to \mathcal{U}_{\mathcal{E}} \\ s & \to p_{\mathcal{E}} = c_{\mathcal{E}}(s) \end{array} \right\}$ be a $C^1$ curve in $\mathcal{U}_{\mathcal{E}}$. Its tangent vector at $p_{\mathcal{E}} = c_{\mathcal{E}}(s)$ is

$$\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) := c_{\mathcal{E}}{}'(s) \quad (= \lim_{h\to 0} \frac{c_{\mathcal{E}}(s+h) - c_{\mathcal{E}}(s)}{h}), \tag{6.18}$$

see fig 6.1, which defines the function $\vec{w}_{\mathcal{E}} : \left\{ \begin{array}{rl} \mathrm{Im}(c_{\mathcal{E}}) & \to E \\ p_{\mathcal{E}} & \to \vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \end{array} \right\}$ called a vector field along $\mathrm{Im}(c_{\mathcal{E}}) \subset \mathcal{U}_{\mathcal{E}}$.

• The push-forward of $c_{\mathcal{E}}$ by $\Psi$ is the image curve $c_{\mathcal{E}*} = \Psi \circ c_{\mathcal{E}}$ (the curve transformed by $\Psi$) cf. (6.4); Its tangent vector at $p_{\mathcal{F}} = c_{\mathcal{E}*}(s)$ is

$$\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) := c_{\mathcal{E}*}{}'(s) = d\Psi(c_{\mathcal{E}}(s)).c_{\mathcal{E}}{}'(s) = d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \quad \text{where} \quad p_{\mathcal{E}} = c_{\mathcal{E}}(s). \tag{6.19}$$

Thus we have defined the vector field $\vec{w}_{\mathcal{E}*}$ along $\mathrm{Im}(c_{\mathcal{E}*})$ called the push-forward of $\vec{w}_{\mathcal{E}}$ by $\Psi$:

**Definition 6.8** The push-forward by $\Psi$ of a $C^0$ vector field $\vec{w}_{\mathcal{E}} : \left\{ \begin{array}{rl} \mathcal{U}_{\mathcal{E}} & \to E \\ p_{\mathcal{E}} & \to \vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \end{array} \right\}$ is the vector field

$$\Psi_*\vec{w}_{\mathcal{E}} = \vec{w}_{\mathcal{E}*} : \left\{ \begin{array}{rl} \mathcal{U}_{\mathcal{F}} & \to F \\ p_{\mathcal{F}} & \to \boxed{\Psi_*\vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) := d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})} \overset{\text{written}}{=} \vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}), \end{array} \right. \tag{6.20}$$

see figu. 6.1, the notation $\vec{w}_{\mathcal{E}*}$ when $\Psi$ is implicit. In other words,

$$\Psi_*\vec{w}_{\mathcal{E}} := (d\Psi.\vec{w}_{\mathcal{E}}) \circ \Psi^{-1}. \tag{6.21}$$

This defines the map $\Psi_* : \left\{ \begin{array}{rl} C^\infty(\mathcal{U}_{\mathcal{E}}; E) & \to C^\infty(\mathcal{U}_{\mathcal{F}}; F) \\ \vec{w}_{\mathcal{E}} & \to \Psi_*(\vec{w}_{\mathcal{E}}) := \Psi_*\vec{w}_{\mathcal{E}} = \vec{w}_{\mathcal{E}*} \end{array} \right\}$.

Warning: Same notation $\Psi_*$ as in definition 6.5: The context removes ambiguities.

**Remark 6.9** Unlike scalar functions, cf. § 6.4.2: At $t_0$ at $p_{t_0}$ you cannot just draw a vector $\vec{w}_{t_0}(p_{t_0})$ on a piece of paper, put the paper in your pocket, then let yourself be carried by the flow $\Psi = \Phi_t^{t_0}$ (push-forward), then, once arrived at $t$ at $p_t$, take the paper out of your pocket and read it to get the push-forward: The direction and length of the vector $\vec{w}_{t_0*}(t, p_t)$ are modified by the flow (a vector is not just a collection of scalar components). ∴

**Exercice 6.10** Prove:

$$\vec{c_{\mathcal{E}}}''(s) = d\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}), \tag{6.22}$$

and

$$d\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}) = d\Psi(p_{\mathcal{E}}).d\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) + d^2\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}), \tag{6.23}$$

and

$$c_{\mathcal{E}*}{}''(s) = d\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}).\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) \quad (= d\Psi(p_{\mathcal{E}}).\vec{c_{\mathcal{E}}}''(s) + d^2\Psi(p_{\mathcal{E}}).\vec{c_{\mathcal{E}}}'(s).\vec{c_{\mathcal{E}}}'(s)). \tag{6.24}$$

**Answer**. $\vec{c_{\mathcal{E}}}'(s) = \vec{w}_{\mathcal{E}}(c_{\mathcal{E}}(s))$ gives $\vec{c_{\mathcal{E}}}''(s) = d\vec{w}_{\mathcal{E}}(c_{\mathcal{E}}(s)).\vec{c_{\mathcal{E}}}'(s)$, hence (6.22).

$\vec{w}_{\mathcal{E}*}(\Psi(p_{\mathcal{E}})) = d\Phi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})$ by definition of $\vec{w}_{\mathcal{E}*}$, hence (6.23).

$c_{\mathcal{F}}(s) = \Psi(c_{\mathcal{E}}(s))$ gives $\vec{c_{\mathcal{F}}}'(s) = d\Psi(c_{\mathcal{E}}(s)).\vec{c_{\mathcal{E}}}'(s) = d\Psi(c_{\mathcal{E}}(s)).\vec{w}_{\mathcal{E}}(c_{\mathcal{E}}(s)) = \vec{w}_{\mathcal{E}*}(c_{\mathcal{F}}(s))$. Thus $\vec{c_{\mathcal{F}}}''(s) = (d^2\Psi(c_{\mathcal{E}}(s)).\vec{c_{\mathcal{E}}}'(s)).\vec{c_{\mathcal{E}}}'(s) + d\Psi(c_{\mathcal{E}}(s)).\vec{c_{\mathcal{E}}}''(s) = d\vec{w}_{\mathcal{E}*}(c_{\mathcal{F}}(s)).\vec{c_{\mathcal{F}}}'(s)$, hence (6.24). ∴

### 6.5.3 Pull-back of a vector field

**Definition 6.11** If $\vec{w}_{\mathcal{F}} : \left\{ \begin{array}{rl} \mathcal{U}_{\mathcal{F}} & \to F \\ p_{\mathcal{F}} & \to \vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) \end{array} \right\}$ is a vector field on $\mathcal{U}_{\mathcal{F}}$, then its pull-back by $\Psi$ is the push-forward by $\Psi^{-1}$, i.e. is the vector field on $\mathcal{U}_{\mathcal{E}}$ defined by

$$\Psi^*\vec{w}_{\mathcal{F}} : \left\{ \begin{array}{rl} \mathcal{U}_{\mathcal{E}} & \to E \\ p_{\mathcal{E}} & \to \boxed{\Psi^*\vec{w}_{\mathcal{F}}(p_{\mathcal{E}}) := d\Psi^{-1}(p_{\mathcal{F}}).\vec{w}_{\mathcal{F}}(p_{\mathcal{F}})} \overset{\text{written}}{=} \vec{w}_{\mathcal{F}}{}^*(p_{\mathcal{E}}), \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}). \end{array} \right. \tag{6.25}$$

In other words,

$$\Psi^*\vec{w}_{\mathcal{F}} := (d\Psi^{-1}.\vec{w}_{\mathcal{F}}) \circ \Psi \overset{\text{written}}{=} \vec{w}_{\mathcal{F}}{}^*. \tag{6.26}$$

And we immediately get

$$\Psi^* \circ \Psi_* = I \quad \text{and} \quad \Psi_* \circ \Psi^* = I, \tag{6.27}$$

because $\Psi^*(\Psi_*\vec{w}_{\mathcal{E}})(p_{\mathcal{E}}) = d\Psi^{-1}(p_{\mathcal{F}}).\Psi_*\vec{w}_{\mathcal{E}}(p_{\mathcal{F}}) = d\Psi^{-1}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) = \vec{w}_{\mathcal{E}}(p_{\mathcal{E}})$. Idem for $\Psi_* \circ \Psi^*$.

## 6.6 Quantification with bases

### 6.6.1 Usual result

$(\vec{a}_i)$ is a Cartesian basis in $E$, $O_\mathcal{F}$ and $(\vec{b}_i)$ are an origin in $\mathcal{F}$ and a Cartesian basis in $F$, $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$,

$$p_\mathcal{F} = \Psi(p_\mathcal{E}) = O_\mathcal{F} + \sum_{i=1}^n \psi_i(p_\mathcal{E})\,\vec{b}_i, \quad \text{i.e.} \quad [\overrightarrow{O_\mathcal{F} p_\mathcal{F}}]_{|\vec{b}} = \begin{pmatrix} \psi_1(p_\mathcal{E}) \\ \vdots \\ \psi_n(p_\mathcal{E}) \end{pmatrix}. \tag{6.28}$$

And $d\Psi(p_\mathcal{E}).() = \sum_i (d\psi_i(p_\mathcal{E}).(.))\vec{b}_i$, i.e. $d\Psi(p_\mathcal{E}).\vec{w}_E = \sum_i (d\psi_i(p_\mathcal{E}).\vec{w}_E)\vec{b}_i$ for all $\vec{w}_E \in E$. Thus, if $\vec{w}_\mathcal{E}$ is a vector field in $\mathcal{U}_\mathcal{E}$ and $\vec{w}_\mathcal{E} = \sum_i w_j \vec{a}_i$, we get $\Psi_*\vec{w}_\mathcal{E}(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E}) = \sum_{i=1}^n (d\psi_i(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E}))\,\vec{b}_i = \sum_{i,j=1}^n w_j(p_\mathcal{E})(d\psi_i(p_\mathcal{E}).\vec{a}_j)\,\vec{b}_i = \sum_{i,j=1}^n \frac{\partial\psi_i}{\partial x_j}(p_\mathcal{E}) w_j(p_\mathcal{E})\,\vec{b}_i$, so (matrix calculation)

$$[\Psi_*\vec{w}_\mathcal{E}(p_\mathcal{F})]_{|\vec{b}} = [d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}}.[\vec{w}_\mathcal{E}(p_\mathcal{E})]_{|\vec{a}}, \tag{6.29}$$

where $[d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}} = [d\psi_i(p_\mathcal{E}).\vec{a}_j] =^{\text{written}} [\frac{\partial\psi_i}{\partial x_j}(p_\mathcal{E})]$ is the Jacobian matrix.

### 6.6.2 Example: Polar coordinate system

**Example 6.12** Change of coordinate system interpreted as a push-forward: Paradigmatic example of the polar coordinate system (model generalized for the parametrization of any manifold).

Parametric Cartesian vector space $\mathbb{R} \times \mathbb{R} =^{\text{written}} \vec{\mathbb{R}}^2_p = \{\vec{q} = (r,\theta)\}$, with its canonical basis $(\vec{a}_1, \vec{a}_2)$, and $\vec{q} = r\vec{a}_1 + \theta\vec{a}_2 =^{\text{written}} (r,\theta)$, so $[\vec{q}]_{|\vec{a}} = \begin{pmatrix} r \\ \theta \end{pmatrix}$. Geometric affine space $\mathbb{R}^2$ (of positions), $p \in \mathbb{R}^2$, associated vector space $\vec{\mathbb{R}^2}$, $O \in \mathbb{R}^2$ (origin), $\vec{x} = \vec{Op}$, and a Euclidean basis $(\vec{b}_1, \vec{b}_2)$ in $\vec{\mathbb{R}^2}$. The "polar coordinate system" is the associated map $\Psi : \left\{ \begin{array}{ll} \vec{\mathbb{R}}^*_+ \times \mathbb{R} \subset \vec{\mathbb{R}}^2_p & \to \vec{\mathbb{R}^2} \\ \vec{q} = (r,\theta) & \to \vec{x} = \Psi(\vec{q}) = \Psi(r,\theta), \end{array} \right\}$ defined by

$$\vec{x} = \Psi(\vec{q}) := r\cos\theta\,\vec{b}_1 + r\sin\theta\,\vec{b}_2, \quad \text{i.e.} \quad [\vec{x}]_{|\vec{b}} = \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}. \tag{6.30}$$

The $i$-th coordinate line at $\vec{q}$ in $\vec{\mathbb{R}}^2_p$ (parametric space) is the straight line $\vec{c}_{\vec{q},i} : \left\{ \begin{array}{ll} \mathbb{R} & \to \vec{\mathbb{R}}^2_p \\ s & \to \vec{c}_{\vec{q},i}(s) = \vec{q} + s\vec{a}_i \end{array} \right\}$, and its tangent vector at $\vec{c}_{\vec{q},i}(s)$ is $\vec{c}_{\vec{q},i}{}'(s) = \vec{a}_i$ for all $s$. This line is transformed by $\Psi$ into the curve $\Psi_*(c_{q,i}) = \Psi \circ \vec{c}_{\vec{q},i} =^{\text{written}} c_{\vec{x},i} : \left\{ \begin{array}{ll} \mathbb{R} & \to \mathbb{R}^2 \\ s & \to c_{\vec{x},i}(s) = \Psi(\vec{q} + s\vec{a}_i) \end{array} \right\}$ (in particular $c_{\vec{x},i}(0) = \vec{x}$). So

$$[\overrightarrow{Oc_{\vec{x},1}(s)}]_{|\vec{b}} = \begin{pmatrix} (r{+}s)\cos\theta \\ (r{+}s)\sin\theta \end{pmatrix} \quad \text{(straight line)}, \quad \text{and} \quad [\overrightarrow{Oc_{\vec{x},2}(s)}]_{|\vec{b}} = \begin{pmatrix} r\cos(\theta{+}s) \\ r\sin(\theta{+}s) \end{pmatrix} \quad \text{(circle)}, \tag{6.31}$$

and the tangent vector at $c_{\vec{x},i}(s)$ is $c_{\vec{x},i}{}'(s) =^{\text{written}} \vec{a}_{i*}(\vec{x})$ (push-forward by $\Psi$), so

$$\begin{aligned} \vec{a}_{1*}(\vec{x}) &:= \Psi_*\vec{a}_1(\vec{x}) = d\Psi(\vec{q}).\vec{a}_1 = \lim_{h\to 0} \frac{\Psi(\vec{q}{+}h\vec{a}_1) - \Psi(\vec{q})}{h} = \lim_{h\to 0} \frac{\Psi(r{+}h,\theta) - \Psi(r,\theta)}{h} = \frac{\partial\Psi}{\partial r}(\vec{q}), \\ \vec{a}_{2*}(\vec{x}) &:= \Psi_*\vec{a}_2(\vec{x}) = d\Psi(\vec{q}).\vec{a}_2 = \lim_{h\to 0} \frac{\Psi(\vec{q}{+}h\vec{a}_2) - \Psi(\vec{q})}{h} = \lim_{h\to 0} \frac{\Psi(r,\theta{+}h) - \Psi(r,\theta)}{h} = \frac{\partial\Psi}{\partial\theta}(\vec{q}). \end{aligned} \tag{6.32}$$

So

$$\vec{a}_{1*}(\vec{x}) = \cos\theta\vec{b}_1 + \sin\theta\vec{b}_2 \quad \text{and} \quad \vec{a}_{2*}(\vec{x}) = -r\sin\theta\vec{b}_1 + r\cos\theta\vec{b}_2 \tag{6.33}$$

vectors at $\vec{x} = \Psi(\vec{q})$, i.e.

$$[\vec{a}_{1*}(\vec{x})]_{|\vec{b}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad [\vec{a}_{2*}(\vec{x})]_{|\vec{b}} = \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix}. \tag{6.34}$$

The basis $(\vec{a}_{1*}(\vec{x}), \vec{a}_{2*}(\vec{x}))$ is called the basis of the polar coordinate system at $\vec{x}$ (it is orthogonal but not orthonormal since $||\vec{a}_{2*}(\vec{x})|| = r \neq 1$ in general); And $[d\Psi(\vec{q})]_{|\vec{a},\vec{b}} = [\frac{\partial\Psi^i}{\partial q^j}(\vec{q})] = \big([\frac{\partial\Psi}{\partial r}(\vec{q})]_{|\vec{b}} \;\; [\frac{\partial\Psi}{\partial\theta}(\vec{q})]_{|\vec{b}}\big) = \big([\vec{a}_{1*}(\vec{x})]_{|\vec{b}} \;\; [\vec{a}_{2*}(\vec{x})]_{|\vec{b}}\big) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ is the Jacobian matrix of $\Psi$ at $\vec{q}$ considered at $\vec{x} = \Psi(\vec{q})$.

And the dual basis of the polar system basis $(\vec{a}_{1*}(\vec{x}), \vec{a}_{2*}(\vec{x}))$ is called $(dq_1(\vec{x}), dq_2(\vec{x}))$ (defined by $dq_i(\vec{x}).\vec{a}_{j*}(\vec{x}) = \delta_{ij}$), so

$$dq_1(\vec{x}) = \cos\theta\, dx_1 + \sin\theta\, dx_2 \quad \text{and} \quad dq_2(\vec{x}) = -\frac{1}{r}\sin\theta\, dx_1 + \frac{1}{r}\cos\theta\, dx_2, \tag{6.35}$$

i.e. $[dq_1(\vec{x})]_{|\vec{b}} = (\cos\theta \quad \sin\theta)$ and $[dq_2(\vec{x})]_{|\vec{b}} = -\frac{1}{r}(\sin\theta \quad \cos\theta)$ (row matrices: rows of $[d\Psi(\vec{q})]^{-1}_{|\vec{a},\vec{b}}$).

**Remark 6.13** The components $\gamma_{ij}^k(\vec{x})$ of the vector $d\vec{a}_{j*}(\vec{x}).\vec{a}_{i*}(\vec{x}) \in \vec{\mathbb{R}^2}$ in the basis $(\vec{a}_{i*}(\vec{x}))$ are the Christoffel symbols of the polar coordinate system (with duality notations as it is usually presented):

$$d\vec{a}_{j*}(\vec{x}).\vec{a}_{i*}(\vec{x}) = \sum_{k=1}^{n} \gamma_{ij}^k(\vec{x})\vec{a}_{k*}(\vec{x}). \tag{6.36}$$

At $\vec{x} = \Psi(\vec{q})$, with $\vec{a}_{j*}(\vec{x}) = d\Psi(\vec{q}).\vec{a}_j$, i.e. $(\vec{a}_{j*} \circ \Psi)(\vec{q}) = \frac{\partial \Psi}{\partial q^j}$, we get

$$d\vec{a}_{j*}(\vec{x}).\vec{a}_{i*}(\vec{x}) = \frac{\partial^2 \Psi}{\partial q^i \partial q^j}(\vec{q}) = d\vec{a}_{i*}(\vec{x}).\vec{a}_{j*}(\vec{x}), \quad \text{so} \quad \gamma_{ij}^k = \gamma_{ji}^k \tag{6.37}$$

for all $i, j$ (symmetry of the bottom indices as soon as $\Psi$ is $C^2$).

Here for the polar coordinates, $\frac{\partial \Psi}{\partial r}(\vec{q}) = \cos\theta \vec{b}_1 + \sin\theta \vec{b}_2$ gives $\frac{\partial^2 \Psi}{\partial r^2}(\vec{q}) = \vec{0}$, thus $\gamma_{11}^1 = \gamma_{11}^2 = 0$, and $\frac{\partial^2 \Psi}{\partial \theta \partial r}(\vec{q}) = -\sin\theta \vec{b}_1 + \cos\theta \vec{b}_2 = \frac{1}{r}\vec{a}_{2*}(\vec{x})$, thus $\gamma_{12}^1 = 0 = \gamma_{21}^1$ and $\gamma_{12}^2 = \frac{1}{r} = \gamma_{21}^2$. And $\frac{\partial \Psi}{\partial \theta}(\vec{q}) = -r\sin\theta \vec{b}_1 + r\cos\theta \vec{b}_2$ gives $\frac{\partial^2 \Psi}{\partial \theta^2}(\vec{q}) = -r\cos\theta \vec{b}_1 - r\sin\theta \vec{b}_2 = -r\vec{a}_{1*}(\vec{x})$, thus $\gamma_{22}^1 = -r$ and $\gamma_{22}^2 = 0$.

**Remark 6.14** The (widely used) normalized polar coordinate basis $(\vec{n}_1(\vec{x}), \vec{n}_2(\vec{x})) = (\vec{a}_{1*}(\vec{x}), \frac{1}{r}\vec{a}_{2*}(\vec{x}))$ is not holonomic, i.e. is not the basis of a coordinate system (and its use makes higher derivation formulas complicated). Indeed $\vec{n}_2(\vec{x}) = \frac{1}{r}\vec{a}_{2*}(\vec{x})$ gives $d\vec{n}_2(\vec{x}).\vec{n}_1(\vec{x}) = (d(\frac{1}{r})(\vec{x}).\vec{n}_1(\vec{x}))\vec{a}_{2*}(\vec{x}) + \frac{1}{r}d\vec{a}_{2*}(\vec{x}).\vec{n}_1(\vec{x})$, and $\vec{n}_1(\vec{x}) = \vec{a}_{1*}(\vec{x})$ gives $d\vec{n}_1(\vec{x}).\vec{n}_2(\vec{x}) = d\vec{a}_{1*}(\vec{x}).(\frac{1}{r}\vec{a}_{2*})$, thus $d\vec{n}_2(\vec{x}).\vec{n}_1(\vec{x}) - d\vec{n}_1(\vec{x}).\vec{n}_2(\vec{x}) = (d(\frac{1}{r})(\vec{x}).\vec{n}_1(\vec{x}))\vec{a}_{2*}(\vec{x}) \neq \vec{0}$, since $\frac{1}{r} = (x^2+y^2)^{-\frac{1}{2}}$ gives $d(\frac{1}{r})(\vec{x}).\vec{n}_1(\vec{x}) = (\,-x(x^2+y^2)^{-\frac{3}{2}} \quad -y(x^2+y^2)^{-\frac{3}{2}}\,).\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = \frac{1}{r^3}(-r\cos^2\theta - r\sin^2\theta) = \frac{-1}{r^2} \neq 0$.

**Remark 6.15** (Pay attention to the notations.) Let $f : \vec{q} \in \vec{\mathbb{R}}^2_p \to f(\vec{q}) \in \mathbb{R}$ be $C^2$. Call $g$ its push-forward by $\Psi$, i.e. $g : \vec{x} \in \mathbb{R}^2 \to g(\vec{x}) = f(\vec{q}) \in \mathbb{R}$ when $\vec{x} = \Psi(\vec{q})$. So $f(\vec{q}) = (g \circ \Psi)(\vec{q})$ and

$$df(\vec{q}).\vec{a}_j = dg(\Psi(\vec{q})).d\Psi(\vec{q}).\vec{a}_j = dg(\vec{x}).\vec{a}_{j*}(\vec{x}). \tag{6.38}$$

With $df(\vec{q}).\vec{a}_j =^{\text{written}} \frac{\partial f}{\partial q^j}(\vec{q})$ and $dg(\vec{x}).\vec{b}_j =^{\text{written}} \frac{\partial g}{\partial x^j}(\vec{x})$ and $\vec{a}_{j*}(\vec{x}) = d\Psi(\vec{q}).\vec{a}_j = \sum_i \frac{\partial \Psi^i}{\partial q^j}(\vec{q})\vec{a}_j$, we get

$$\frac{\partial f}{\partial q^j}(\vec{q}) = \sum_i \frac{\partial g}{\partial x^i}(\vec{x})\frac{\partial \Psi^i}{\partial q^j}(\vec{q}) \overset{\text{written}}{=} \frac{\partial g}{\partial q^j}(\vec{x}) \;\; ... \text{(!!)} \tag{6.39}$$

Mind this notation!! $g$ is a function of $\vec{x}$, not of $\vec{q}$, so $\frac{\partial g}{\partial q^i}(\vec{x}) \overset{\text{means}}{=} \frac{\partial f}{\partial q^i}(\vec{q})$, i.e. $\frac{\partial g}{\partial q^i}(\vec{x}) \overset{\text{means}}{=} \frac{\partial (g\circ\Psi)}{\partial q^i}(\vec{q})$... which is $[df(\vec{q})] = [dg(\vec{x})].[d\Psi(\vec{q})$...

**Remark 6.16** Then (with $f$ and $\Psi$ $C^2$)

$$\begin{aligned}\frac{\partial \frac{\partial g}{\partial q^i}}{\partial q^j}(\vec{x}) &\overset{\text{means}}{=} \frac{\partial \frac{\partial (g\circ\Psi)}{\partial q^i}}{\partial q^j}(\vec{q}) = d(dg.\vec{a}_{i*})(\vec{x}).d\Psi(\vec{q}).\vec{a}_j = d(dg.\vec{a}_{i*})(\vec{x}).\vec{a}_{j^*}(\vec{x}) \\ &= d((dg(\vec{x}).\vec{a}_{j^*}(\vec{x})).\vec{a}_{i*}(\vec{x}) + dg(\vec{x}).(d\vec{a}_{i*}(\vec{x}).\vec{a}_j(\vec{x})) \overset{\text{written}}{=} \frac{\partial^2 g}{\partial q^i \partial q^j}(\vec{x}).\end{aligned} \tag{6.40}$$

So

$$\frac{\partial^2 g}{\partial q^i \partial q^j}(\vec{x}) \overset{\text{means}}{=} d^2g(\vec{x})(\vec{a}_{i*}(\vec{x}), \vec{a}_{j*}(\vec{x})) + \sum_{k=1}^{n} \frac{\partial g}{\partial x^k}(\vec{x})\gamma_{ij}^k(\vec{x})\vec{a}_k(\vec{x}), \tag{6.41}$$

and $\frac{\partial^2 g}{\partial q^i \partial q^j}(\vec{x})$ is ***not*** reduced to $d^2g(\vec{x})(\vec{a}_{i*}(\vec{x}), \vec{a}_{j*}(\vec{x}))$ (the Christoffel symbols have appeared), first order derivatives $\frac{\partial g}{\partial x^k}$ being still alive (contrary to $\frac{\partial^2 g}{\partial x^i \partial x^j}(\vec{x}) = d^2g(\vec{x})(\vec{b}_i, \vec{b}_j)$ with a Cartesian basis $(\vec{b}_i)$).

NB: The independent variables $r$ and $\theta$ don't have the same dimension (a length and an angle): There is no physical meaningful inner dot product in the parameter space $\vec{\mathbb{R}}^2_p = \mathbb{R}\times\mathbb{R} = \{(r,\theta)\}$, but this space is very useful... (As in thermodynamics: No meaningful inner dot product in the $(T, P)$ space.)

# 7 Push-forward and pull-back of differential forms

Setting of § 6.1. The set of vector fields on $\mathcal{U}_{\mathcal{E}}$ is called $\Gamma(\mathcal{U}_{\mathcal{E}})$.

## 7.1 Definition

**Definition 7.1** In short: A differential form on $\mathcal{U}_{\mathcal{E}}$ is a function $\alpha_{\mathcal{E}} : \begin{Bmatrix} \mathcal{U}_{\mathcal{E}} & \to E^* = \mathcal{L}(E;\mathbb{R}) \\ p_{\mathcal{E}} & \to \alpha_{\mathcal{E}}(p_{\mathcal{E}}) \end{Bmatrix}$, i.e. s.t. $\alpha_{\mathcal{E}}(p_{\mathcal{E}})$ is a linear form on $E$. Precisely: A differential form on $\mathcal{U}_{\mathcal{E}}$ is a field of linear forms, i.e. a function $\alpha_{\mathcal{E}} : \begin{Bmatrix} \mathcal{U}_{\mathcal{E}} & \to \mathcal{U}_{\mathcal{E}} \times E^* \\ p_{\mathcal{E}} & \to (p_{\mathcal{E}}, \alpha_{\mathcal{E}}(p_{\mathcal{E}})) \end{Bmatrix}$ (looked at at the point $p_{\mathcal{E}}$). And

$$\Omega^1(\mathcal{U}_{\mathcal{E}}) = \text{the set of differentials forms.} \tag{7.1}$$

Consider a differential form $\alpha_{\mathcal{E}} \in \Omega^1(\mathcal{U}_{\mathcal{E}})$ and a vector field $\vec{w}_{\mathcal{E}} : \begin{Bmatrix} \mathcal{U}_{\mathcal{E}} & \to E \\ p_{\mathcal{E}} & \to \vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \end{Bmatrix}$. The push-forward by $\Psi$ of the scalar valued function

$$f_{\mathcal{E}} = \alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}} : \begin{cases} \mathcal{U}_{\mathcal{E}} & \to \mathbb{R} \\ p_{\mathcal{E}} & \to f_{\mathcal{E}}(p_{\mathcal{E}}) = (\alpha.\vec{w}_{\mathcal{E}})(p_{\mathcal{E}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}). \end{cases}$$

is, cf. (6.8) with $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$,

$$\begin{aligned} \Psi_*(\alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}})(p_{\mathcal{F}}) = (\alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}})(p_{\mathcal{E}}) &= \alpha_{\mathcal{E}}(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \\ &= \underbrace{\alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi(p_{\mathcal{E}})^{-1}}_{=\text{written } \Psi_*\alpha_{\mathcal{E}}(p_{\mathcal{F}})} . \underbrace{d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})}_{=\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}})} . \end{aligned} \tag{7.2}$$

And $\Psi^{-1}(\Psi(p_{\mathcal{E}})) = p_{\mathcal{E}}$ gives $d\Psi^{-1}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}) = I$, i.e. $d\Psi(p_{\mathcal{E}})^{-1} = d\Psi^{-1}(p_{\mathcal{F}})$. Hence (compatibility):

**Definition 7.2** The push-forward of a differential form $\alpha_{\mathcal{E}} \in \Omega^1(\mathcal{U}_{\mathcal{E}})$ is the differential form in $\Omega^1(\mathcal{U}_{\mathcal{F}})$, when $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$,

$$\Psi_*\alpha_{\mathcal{E}} \overset{\text{written}}{=} \alpha_{\mathcal{E}*} : \begin{Bmatrix} \mathcal{U}_{\mathcal{F}} & \to F^* = \mathcal{L}(F;\mathbb{R}) \\ p_{\mathcal{F}} & \to \boxed{\Psi_*\alpha_{\mathcal{E}}(p_{\mathcal{F}}) := \alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}})} \end{Bmatrix} \in \Omega^1(\mathcal{U}_{\mathcal{F}}). \tag{7.3}$$

(If you prefer, $\Psi_*\alpha_{\mathcal{E}}(p_{\mathcal{F}}) := \alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi(p_{\mathcal{E}})^{-1}$.) (And $\Psi_*\alpha_{\mathcal{E}} =^{\text{written}} \alpha_{\mathcal{E}*}$ when $\Psi$ is implicit.) In other words, $\Psi_*\alpha_{\mathcal{E}}(p_{\mathcal{F}}) = \alpha_{\mathcal{E}}(\Psi^{-1}(p_{\mathcal{F}})).d\Psi^{-1}(p_{\mathcal{F}})$, i.e.

$$\Psi_*\alpha_{\mathcal{E}} := (\alpha_{\mathcal{E}} \circ \Psi^{-1}).d\Psi^{-1}. \tag{7.4}$$

(Warning: Once again, we used the same notation $\Psi_*$ as for the push-forward of vector fields and functions: The context removes ambiguities.)

Hence, for all $\vec{w}_{\mathcal{F}} : \mathcal{U}_{\mathcal{F}} \to \vec{\mathbb{R}^n}$, when $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$ and $\vec{w}_{\mathcal{F}} = \vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) = d\Psi(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}})$,

$$((\alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}})_*(p_{\mathcal{F}}) =) \quad \alpha_{\mathcal{E}*}(p_{\mathcal{F}}).\vec{w}_{\mathcal{E}*}(p_{\mathcal{F}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).\vec{w}_{\mathcal{E}}(p_{\mathcal{E}}) \quad (= (\alpha_{\mathcal{E}}.\vec{w}_{\mathcal{E}})(p_{\mathcal{E}})), \tag{7.5}$$

or

$$(\Psi_*\alpha_{\mathcal{E}})(p_{\mathcal{F}}).\vec{w}_{\mathcal{F}}(p_{\mathcal{F}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).(\Psi_*\vec{w}_{\mathcal{F}})(p_{\mathcal{E}}). \tag{7.6}$$

In particular if $\alpha_{\mathcal{E}} = df$ (exact differential form) where $f \in C^1(\mathcal{U}_{\mathcal{E}};\mathbb{R})$, then

$$d(\Psi_*f) = \Psi_*(df). \tag{7.7}$$

(This commutativity result is very particular to the case $\alpha = df$: In general $d(\Psi_*T) \neq \Psi_*(dT)$ for a tensor of order $\geq 2$, see e.g. (8.20)).

**Remark 7.3** We cannot always see a vector field (e.g. we can't see an internal force field): To "see" it we need to measure it with a well defined tool, the tool being here a differential form; And the definition 7.2 is a compatbility definition so that we can recover the push-forward of the vector field. ▪

**Definition 7.4** The pull-back of a differential form $\alpha_{\mathcal{F}} \in \Omega^1(\mathcal{U}_{\mathcal{F}})$ is, with $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$, the differential form

$$\Psi^*\alpha_{\mathcal{F}} : \left\{ \begin{array}{ll} \mathcal{U}_{\mathcal{E}} & \to \mathcal{L}(E;\mathbb{R}) \\ p_{\mathcal{E}} & \to \Psi^*\alpha_{\mathcal{F}}(p_{\mathcal{E}}) := \alpha_{\mathcal{F}}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}) \end{array} \right\} \in \Omega^1(\mathcal{U}_{\mathcal{E}}). \tag{7.8}$$

In other words,

$$\Psi^*\alpha_{\mathcal{F}} := (\alpha_{\mathcal{F}} \circ \Psi).d\Psi. \tag{7.9}$$

(For an alternative definition, see remark 7.5.)

And we have

$$\Psi^* \circ \Psi_* = I \quad \text{and} \quad \Psi_* \circ \Psi^* = I. \tag{7.10}$$

Indeed $\Psi^*(\Psi_*\alpha_{\mathcal{E}})(p_{\mathcal{E}}) = \Psi_*\alpha_{\mathcal{E}}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi^{-1}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}}) = \alpha_{\mathcal{E}}(p_{\mathcal{E}})$. Idem for $\Psi_* \circ \Psi^* = I$.

**Remark 7.5** The pull-back $\alpha_{\mathcal{F}}{}^*$ can also be defined thanks to the natural canonical isomorphism $\left\{ \begin{array}{ll} \mathcal{L}(E;F) & \to \mathcal{L}(F^*;E^*) \\ L & \to L^* \end{array} \right\}$ given by $L^*(\ell_F).\vec{u}_E = \ell_F.(L.\vec{u}_E)$ for all $(\vec{u}_E, \ell_F) \in E \times F^*$, and $L^*(\ell_F) = \ell_F.L$ is called the pull-back of $\ell_F$ by $L$. In particular with $\ell_F = \alpha_{\mathcal{F}}(p_{\mathcal{F}})$ and $L = d\Psi(p_{\mathcal{E}})$ we get $d\Psi(p_{\mathcal{E}})^*(\alpha_{\mathcal{F}}(p_{\mathcal{F}})) = \alpha_{\mathcal{F}}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}})$, i.e. (7.8). ∎

## 7.2 Incompatibility: Riesz representation and push-forward

A push-forward is independent of any inner dot product: It is objective. Subjectivity: Here we introduce inner dot products $(\cdot,\cdot)_g$ in $E$ and $(\cdot,\cdot)_h$ in $F$, e.g. Euclidean dot products in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$ (foot? metre?), because some can't begin with their beloved Euclidean dot products.

Let $\alpha_{\mathcal{E}} \in \Omega^1(\mathcal{U}_{\mathcal{E}})$ and call $\beta_{\mathcal{F}} := \Psi_*\alpha_{\mathcal{E}}$ its push-forward by $\Psi$:

$$\beta_{\mathcal{F}}(p_{\mathcal{F}}) := \alpha_{\mathcal{E}}(p_{\mathcal{E}}).d\Psi(p_{\mathcal{E}})^{-1} \quad \text{when} \quad p_{\mathcal{F}} = \Psi(p_{\mathcal{E}}). \tag{7.11}$$

Then call $\vec{a}_g(p_{\mathcal{E}}) \in E$ and $\vec{b}_h(p_{\mathcal{F}}) \in F$ the $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$-Riesz representation vectors of $\alpha_{\mathcal{E}}$ and $\beta_{\mathcal{F}}$: For all $\vec{u}_{\mathcal{E}} \in \Gamma(\mathcal{U}_{\mathcal{E}})$ and all $\vec{w}_{\mathcal{F}} \in \Gamma(\mathcal{U}_{\mathcal{F}})$,

$$\alpha_{\mathcal{E}}.\vec{u}_{\mathcal{E}} = (\vec{a}_g, \vec{u}_{\mathcal{E}})_g \quad \text{and} \quad \beta_{\mathcal{F}}.\vec{w}_{\mathcal{F}} = (\vec{b}_h, \vec{w}_{\mathcal{F}})_h. \tag{7.12}$$

This defines the vector fields $\vec{a}_g \in \Gamma(\mathcal{U}_{\mathcal{E}})$ and $\vec{b}_h \in \Gamma(\mathcal{U}_{\mathcal{F}})$.

**Proposition 7.6** *Although $\beta_{\mathcal{F}} = \Psi_*\alpha_{\mathcal{E}}$, we have $\vec{b}_h \neq \Psi_*\vec{a}_g$ in general: Indeed we have*

$$\begin{aligned} \vec{b}_h(p_{\mathcal{F}}) &= d\Psi(p_{\mathcal{E}})^{-T}.\vec{a}_g(p_{\mathcal{E}}) \\ &\neq d\Psi(p_{\mathcal{E}}).\vec{a}_g(p_{\mathcal{E}}) \ \text{ in general} \end{aligned} \tag{7.13}$$

*(unless $d\Psi(p_{\mathcal{E}})^{-T} = d\Psi(p_{\mathcal{E}})$, i.e. $d\Psi(p_{\mathcal{E}})^T.d\Psi(p_{\mathcal{E}})^{-1} = I$, i.e. unless $\Psi$ is "a rigid body motion").*

*So the Riesz representation vector of the push-forwarded linear form is not the push-forwarded representation vector of the linear form push-forwarded.*

*This is not a surprise: A push-forward is independent of any inner dot product, while a Riesz representation vector depends on a chosen inner dot product.*

*So, as long as possible (i.e. not before you need to quantify), you should avoid using a Riesz representation vector, i.e. you should use the original (the qualitative differential form) and delay the use of a representative (quantification with which dot product?) as late as possible.*

**Proof.** Recall cf. (A.47): The transposed of the linear map $d\Psi(p_{\mathcal{E}}) \in \mathcal{L}(E;F)$ relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ is the linear map $d\Psi(p_{\mathcal{E}})^T_{gh} \in \mathcal{L}(F;E)$ defined by, for all $\vec{u}_{\mathcal{E}} \in E$ and $\vec{w}_{\mathcal{F}} \in F$ vectors at $p_{\mathcal{E}}$ and $p_{\mathcal{F}}$,

$$(d\Psi(p_{\mathcal{E}})^T_{gh}.\vec{w}_{\mathcal{F}}, \vec{u}_{\mathcal{E}})_g = (\vec{w}_{\mathcal{F}}, d\Psi(p_{\mathcal{E}}).\vec{u}_{\mathcal{E}})_h. \tag{7.14}$$

If $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ is imposed to all observers, then $d\Psi(p_{\mathcal{E}})^T_{gh} =^{\text{written}} d\Psi(p_{\mathcal{E}})^T$. It is the case here. (7.12) gives, with $p_{\mathcal{F}} = \Psi(p_{\mathcal{E}})$,

$$\begin{aligned} (\vec{a}_g(p_{\mathcal{E}}), \vec{u}_{\mathcal{E}})_g &= \alpha_{\mathcal{E}}(p_{\mathcal{E}}).\vec{u}_{\mathcal{E}} = \big(\beta_{\mathcal{F}}(p_{\mathcal{F}}).d\Psi(p_{\mathcal{E}})\big).\vec{u}_{\mathcal{E}} = \beta_{\mathcal{F}}(p_{\mathcal{F}}).\big(d\Psi(p_{\mathcal{E}}).\vec{u}_{\mathcal{E}}\big) \\ &= (\vec{b}_h(p_{\mathcal{F}}), d\Psi(p_{\mathcal{E}}).\vec{u}_{\mathcal{E}})_h = (d\Psi(p_{\mathcal{E}})^T.\vec{b}_h(p_{\mathcal{F}}), \vec{u}_{\mathcal{E}})_g, \end{aligned} \tag{7.15}$$

true for all $\vec{u}_{\mathcal{E}}$, thus $\vec{a}_g(p_{\mathcal{E}}) = d\Psi(p_{\mathcal{E}})^T.\vec{b}_h(p_{\mathcal{F}})$, thus (7.13). ∎

# 8 Push-forward and pull-back of tensors

To lighten the presentation, we only deal with order 1 and 2 tensors. Similar approach for any tensor.

## 8.1 Push-forward and pull-back of order 1 tensors

**Proposition 8.1** *If $T$ is either a vector field or a differential form, then its push-forward satisfies, for all $\xi$ vector field or differential form (when required) in $\mathcal{U}_\mathcal{F}$,*

$$(\Psi_*T)(\xi) = T(\Psi^*\xi), \quad \textit{written} \quad \Psi_*T(.) = T(\Psi^*.), \tag{8.1}$$

*i.e. $(\Psi_*T)(p_\mathcal{F}).\xi(p_\mathcal{F}) = T(p_\mathcal{E}).\Psi^*\xi(p_\mathcal{E})$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$. Similarly:*

$$(\Psi^*T)(\xi) = T(\Psi_*\xi), \quad \textit{written} \quad \Psi^*T(.) = T(\Psi_*.), \tag{8.2}$$

*i.e. $(\Psi^*T)(p_\mathcal{E}).\xi(p_\mathcal{E}) = T(p_\mathcal{F}).\Psi_*\xi(p_\mathcal{F})$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.*

**Proof.** • Case $T = \alpha_\mathcal{E} \in \Omega^1(\mathcal{U}_\mathcal{E})$ (differential form $=$ a $\binom{0}{1}$ tensor), then here $\xi = \vec{w}_\mathcal{F} \in \Gamma(\mathcal{U}_\mathcal{F})$ and we have to check: $(\Psi_*\alpha_\mathcal{E})(p_\mathcal{F}).\vec{w}_\mathcal{F}(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).\Psi^*\vec{w}_\mathcal{F}(p_\mathcal{E})$, i.e. $(\alpha_\mathcal{E}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{E})).\vec{w}_\mathcal{F}(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).(d\Psi^{-1}(p_\mathcal{E}).\vec{w}_\mathcal{F}(p_\mathcal{F}))$: True.

• Case $T = \vec{w}_\mathcal{E} \in \Gamma(\mathcal{U}_\mathcal{E})$ (vector field $\simeq$ a $\binom{1}{0}$ tensor), then here $\xi = \alpha_\mathcal{F} \in \Omega^1(\mathcal{U}_\mathcal{F})$ we have to check: $(\Psi_*\vec{w}_\mathcal{E})(p_\mathcal{F}).\alpha_\mathcal{F}(p_\mathcal{F}) = \vec{w}_\mathcal{E}(p_\mathcal{E}).\Psi^*(\alpha_\mathcal{F})(p_\mathcal{E})$, where we implicitly use to the natural canonical isomorphism $\mathcal{J} : \left\{ \begin{array}{l} E \to E^{**} \\ \vec{w} \to w \overset{\text{written}}{=} \vec{w} \end{array} \right\}$ defined by $w(\ell) = \ell.\vec{w}$ for all $\ell \in E^*$. So we have to check: $\alpha_\mathcal{F}(p_\mathcal{F}).(\Psi_*\vec{w}_\mathcal{E})(p_\mathcal{F}) = \Psi^*(\alpha_\mathcal{F})(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E})$, i.e. $\alpha_\mathcal{F}(p_\mathcal{F}).(d\Psi(p_\mathcal{E}).\vec{w}_\mathcal{E}(p_\mathcal{E})) = (\alpha_\mathcal{F}(p_\mathcal{F}).d\Psi(p_\mathcal{E})^{-1}).\vec{w}_\mathcal{E})(p_\mathcal{E})$ : True.

For (8.2), use $\Psi^{-1}$ instead of $\Psi$. ∎

## 8.2 Push-forward and pull-back of order 2 tensors

**Definition 8.2** Let $T$ be an order 2 tensor in $\mathcal{U}_\mathcal{E}$. Its push-forward by $\Psi$ is the order 2 tensor $\Psi_*T$ in $\mathcal{U}_\mathcal{F}$ defined by, for all $\xi_1, \xi_2$ vector field or differential form (when required) in $\mathcal{U}_\mathcal{F}$,

$$\Psi_*T(\xi_1, \xi_2) := T(\Psi^*\xi_1, \Psi^*\xi_2) \quad \text{written} \quad \Psi_*T(\cdot,\cdot) := T(\Psi^*\cdot, \Psi^*\cdot), \tag{8.3}$$

i.e. $\Psi_*T(p_\mathcal{F})(\xi_1(p_\mathcal{F}), \xi_2(p_\mathcal{F})) := T(p_\mathcal{E})(\Psi^*\xi_1(p_\mathcal{E}), \Psi^*\xi_2(p_\mathcal{E}))$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Let $T$ be an order 2 tensor in $\mathcal{U}_\mathcal{F}$. Its pull-back by $\Psi$ is the order 2 tensor $\Psi^*T$ in $\mathcal{U}_\mathcal{E}$ defined by, for all $\xi_1, \xi_2$ vector field or differential form (when required) in $\mathcal{U}_\mathcal{E}$,

$$\Psi^*T(\xi_1, \xi_2) := T(\Psi_*\xi_1, \Psi_*\xi_2) \quad \text{written} \quad \Psi^*T(\cdot,\cdot) := T(\Psi_*\cdot, \Psi_*\cdot), \tag{8.4}$$

i.e., $\Psi^*T(p_\mathcal{E})(\xi_1(p_\mathcal{E}), \xi_2(p_\mathcal{E})) := T(p_\mathcal{F})(\Psi_*\xi_1(p_\mathcal{F}), \Psi_*\xi_2(p_\mathcal{F}))$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

**Example 8.3** If $T \in T^0_2(\mathcal{U}_\mathcal{E})$ (e.g., a metric) then, for all vector fields $\vec{w}_1, \vec{w}_2$ in $\mathcal{U}_\mathcal{F}$,

$$T_*(\vec{w}_1, \vec{w}_2) \overset{(8.3)}{=} T(\vec{w}_1{}^*, \vec{w}_2{}^*) = T(d\Psi^{-1}.\vec{w}_1, d\Psi^{-1}.\vec{w}_2), \tag{8.5}$$

i.e., $T_*(p_\mathcal{F})(\vec{w}_1(p_\mathcal{F}), \vec{w}_2(p_\mathcal{F})) = T(p_\mathcal{E})(d\Psi^{-1}(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F}), d\Psi^{-1}(p_\mathcal{F}).\vec{w}_2(p_\mathcal{F}))$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Expression with bases $(\vec{a}_i)$ in $E$ and $(\vec{b}_i)$ in $F$: In short we have $(T_*)_{ij} = T_*(\vec{b}_i, \vec{b}_j) = T(\vec{b}_i{}^*, \vec{b}_j{}^*) = [\vec{b}_i^*]^T_{|\vec{a}}.[T]_{|\vec{a}}.[\vec{b}_j^*]_{|\vec{a}} = ([\vec{b}_i]^T_{|\vec{b}}.[d\Psi]^{-T}_{|\vec{a},\vec{b}}).[T]_{|\vec{a}}.([d\Psi]^{-1}_{|\vec{a},\vec{b}}.[\vec{b}_j]_{|\vec{b}}) = ([d\Psi]^{-T}_{|\vec{a},\vec{b}}.[T]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}})_{ij}$, thus

$$[T_*]_{|\vec{b}} = [d\Psi]^{-T}_{|\vec{a},\vec{b}}.[T]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}}, \tag{8.6}$$

which means $[(\Psi_*T)(p_\mathcal{F})]_{|\vec{b}} = ([d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}})^{-T}.[T(p_\mathcal{E})]_{|\vec{a}}.([d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}})^{-1}$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Particular case of an elementary tensor $T = \alpha_1 \otimes \alpha_2 \in T^0_2(\mathcal{U}_\mathcal{E})$, where $\alpha_1, \alpha_2 \in \Omega^1(\mathcal{U}_\mathcal{E})$, so $T(\vec{u}_1, \vec{u}_2) = (\alpha_1 \otimes \alpha_2)(\vec{u}_1, \vec{u}_2) = (\alpha_1.\vec{u}_1)(\alpha_2.\vec{u}_2)$: For all $\vec{w}_1, \vec{w}_2 \in \Gamma(\mathcal{U}_\mathcal{F})$,

$$(\alpha_1 \otimes \alpha_2)_*(\vec{w}_1, \vec{w}_2) \overset{(8.3)}{=} (\alpha_1 \otimes \alpha_2)(\vec{w}_1^*, \vec{w}_2^*) = (\alpha_1.\vec{w}_1^*)(\alpha_2.\vec{w}_2^*) \overset{(7.6)}{=} (\alpha_{1*}.\vec{w}_1)(\alpha_{2*}.\vec{w}_2), \tag{8.7}$$

thus

$$(\alpha_1 \otimes \alpha_2)_* = \alpha_{1*} \otimes \alpha_{2*}. \tag{8.8}$$

(And any tensor is a finite sum of elementary tensors.) ∎

And for the pull-back: For all vector fields $\vec{u}_1, \vec{u}_2$ in $\mathcal{U}_\mathcal{E}$,

$$T^*(\vec{u}_1, \vec{u}_2) \overset{(8.3)}{=} T(\vec{u}_{1*}, \vec{u}_{2*}) = T(d\Psi.\vec{u}_1, d\Psi.\vec{u}_2). \tag{8.9}$$

Thus

$$\Psi_* \circ \Psi^* = I \quad \text{and} \quad \Psi^* \circ \Psi_* = I. \tag{8.10}$$

Indeed $(\Psi_* \circ \Psi^*)(\xi) = \xi$ and $(\Psi^* \circ \Psi_*)(\xi) = \xi$ give $(\Psi_* \circ \Psi^*)(T)(\xi_1, \xi_2) = \Psi_*(\Psi^*(T))(\xi_1, \xi_2) =^{(8.3)} \Psi^*(T)(\Psi^*\xi_1, \Psi^*\xi_2) =^{(8.9)} T(\Psi_*.\Psi^*\xi_1, \Psi_*.\Psi^*\xi_2) = T(\xi_1, \xi_2)$. Idem with $\Psi^* \circ \Psi_*$.

**Example 8.4** If $T \in T^1_1(\mathcal{U}_\mathcal{E})$ then for all vector fields $\vec{w} \in \Gamma(\mathcal{U}_\mathcal{F})$ and differential forms $\beta \in \Omega^1(\mathcal{U}_\mathcal{F})$,

$$T_*(\beta, \vec{w}) = T(\beta^*, \vec{w}^*) = T(\beta.d\Psi, d\Psi^{-1}.\vec{w}), \tag{8.11}$$

i.e., $T_*(p_\mathcal{F})(\beta(p_\mathcal{F}), \vec{w}(p_\mathcal{F})) = T(p_\mathcal{E})(\beta(p_\mathcal{F}).d\Psi(p_\mathcal{E}), d\Psi^{-1}(p_\mathcal{F}).\vec{w}(p_\mathcal{F}))$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

For the elementary tensor $T = \vec{u} \otimes \alpha \in T^1_1(\mathcal{U}_\mathcal{E})$, made of the vector field $\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$ and of the differential form $\alpha \in \Omega^1(\mathcal{U}_\mathcal{E})$: For all $\beta, \vec{w} \in \Omega^1(\mathcal{U}_\mathcal{F}) \times \Gamma(\mathcal{U}_\mathcal{F})$, in short,

$$(\vec{u} \otimes \alpha)_*(\beta, \vec{w}) \overset{(8.3)}{=} (\vec{u} \otimes \alpha)(\beta^*, \vec{w}^*) = (\vec{u}.\beta^*)(\alpha.\vec{w}^*) \overset{(7.6)}{=} (\vec{u}_*.\beta)(\alpha_*.\vec{w}) = (\vec{u}_* \otimes \alpha_*)(\beta, \vec{w}), \tag{8.12}$$

thus

$$(\vec{u} \otimes \alpha)_* = \vec{u}_* \otimes \alpha_*. \tag{8.13}$$

Expression with bases $(\vec{a}_i)$ in $E$ and $(\vec{b}_i)$ in $F$: In short we have $(T_*)_{ij} = T_*(b^i, \vec{b}_j) = T(\Psi^*(b^i), \Psi^*(\vec{b}_j)) = [\Psi^*(b^i)].[T].[\Psi^*(\vec{b}_j)] = [b^i].[d\Psi].[T].[d\Psi^{-1}].[\vec{b}_j] = ([d\Psi].[T].[d\Psi^{-1}])_{ij}$, thus

$$[T_*]_{|\vec{b}} = [d\Psi]_{|\vec{a},\vec{b}}.[T]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}}, \tag{8.14}$$

which means $[(\Psi_*T)(p_\mathcal{F})]_{|\vec{b}} = [d\Psi(p_\mathcal{E})]_{|\vec{a},\vec{b}}.[T(p_\mathcal{E})]_{|\vec{a}}.[d\Psi(p_\mathcal{E})]^{-1}_{|\vec{a},\vec{b}}$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$. ▪

## 8.3 Push-forward and pull-back of endomorphisms

We have the natural canonical isomorphism

$$\mathcal{J}_2 : \begin{cases} \mathcal{L}(E;E) & \to \mathcal{L}(E^*, E; \mathbb{R}) \\ L & \to T_L = \mathcal{J}_2(L) \quad \text{where} \quad T_L(\alpha, \vec{u}) := \alpha.L.\vec{u}, \quad \forall(\alpha, \vec{u}) \in E^* \times E. \end{cases} \tag{8.15}$$

Thus $\Psi_*T_L(m, \vec{w}) = T_L(\Psi^*m, \Psi^*\vec{w}) = (\Psi^*m).L.(\Psi^*\vec{w}) = m.d\Psi.L.d\Psi^{-1}.\vec{w}$, thus:

**Definition 8.5** The push-forward by $\Psi$ of a field of endomorphisms $L$ on $\mathcal{U}_\mathcal{E}$ is the field of endomorphisms $\Psi_*L = L_*$ on $\mathcal{U}_\mathcal{F}$ defined by

$$\Psi_*L = \boxed{L_* = d\Psi.L.d\Psi^{-1}}, \tag{8.16}$$

i.e., $L_*(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).L(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F})$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

Thus with bases we get $[L_*]_{|\vec{b}} = [d\Psi]_{|\vec{a},\vec{b}}.[L]_{|\vec{a}}.[d\Psi]^{-1}_{|\vec{a},\vec{b}}$, "as in (8.14)".

**Example 8.6** Elementary field of endomorphisms $L = (\mathcal{J}_2)^{-1}(\vec{u} \otimes \alpha)$, where $\vec{u} \in \Gamma(E)$ and $\alpha \in \Omega^1(E)$: So $T_L = \vec{u} \otimes \alpha$ and $L.\vec{u}_2 = (\alpha.\vec{u}_2)\vec{u}$ for all $\vec{u}_2 \in \Gamma(\mathcal{U}_\mathcal{E})$). Thus $L_*.\vec{w}_2 = d\Psi.L.d\Psi^{-1}.\vec{w}_2 = d\Psi.L.\vec{w}_2{}^* = (\alpha.\vec{w}_2{}^*)d\Psi.\vec{u} = (\alpha_*.\vec{w}_2)\vec{u}_*$ for all $\vec{w}_2 \in \Gamma(E)$, thus $(T_L)_* = \vec{u}_* \otimes \alpha_*$. ▪

**Definition 8.7** Let $L$ be a field of endomorphisms on $\mathcal{U}_\mathcal{F}$. Its pull-back by $\Psi$ is the field of endomorphisms $\Psi^*L = L^*$ on $\mathcal{U}_\mathcal{E}$ defined by

$$\Psi^*L = \boxed{L^* = d\Psi^{-1}.L.d\Psi}, \tag{8.17}$$

i.e., $L^*(p_\mathcal{E}) = d\Psi^{-1}(p_\mathcal{F}).L(p_\mathcal{F}).d\Psi(p_\mathcal{E})$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

## 8.4 Derivatives of vector fields

$\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$ is a $C^1$ vector field in $\mathcal{U}_\mathcal{E}$), $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$, so $d\vec{u} : \mathcal{U}_\mathcal{E} \to \mathcal{L}(E;E)$ (given by $d\vec{u}(p_\mathcal{E}).\vec{w}(p_\mathcal{E}) = \lim_{h\to 0} \frac{\vec{u}(p_\mathcal{E}+h\vec{w}(p_\mathcal{E}))-\vec{u}(p_\mathcal{E})}{h}$ for all $\vec{w} \in \Gamma(\mathcal{U}_\mathcal{E})$). Thus its push-forward:

$$((d\vec{u})_* =) \quad \Psi_*(d\vec{u}) = d\Psi.d\vec{u}.d\Psi^{-1} \tag{8.18}$$

i.e. $(d\vec{u})_*(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).d\vec{u}(p_\mathcal{E}).d\Psi(p_\mathcal{E})^{-1}$ when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

## 8.5 $\Psi_*(d\vec{u}) \neq d(\Psi_*\vec{u})$ in general: No commutativity

Here $\Psi$ is $C^2$, $\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$, $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$, $p_\mathcal{F} = \Psi(p_\mathcal{E})$, so $\Psi_*\vec{u}(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).\vec{u}(p_\mathcal{E}) = (d\Psi(\Psi^{-1}(p_\mathcal{F})).(\vec{u}(\Psi^{-1}(p_\mathcal{F}))$, and, for all $\vec{w} \in \Gamma(\mathcal{U}_\mathcal{F})$,

$$d(\Psi_*\vec{u})(p_\mathcal{F}).\vec{w}(p_\mathcal{F}) = (d^2\Psi(p_\mathcal{E}).(d\Psi^{-1}(p_\mathcal{F}).\vec{w}(p_\mathcal{F}))).\vec{u}(p_\mathcal{E}) + d\Psi(p_\mathcal{E}).d\vec{u}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}).\vec{w}(p_\mathcal{F}), \tag{8.19}$$

with $\Psi_*(d\vec{u})(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).d\vec{u}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F})$, thus, in short,

$$d(\Psi_*\vec{u}).\vec{w} = \Psi_*(d\vec{u}).\vec{w} + d^2\Psi(\Psi^*\vec{w}, \vec{u}) \neq \Psi_*(d\vec{u}) \quad \text{in general.} \tag{8.20}$$

So the differentiation $d$ and the push-forward $_*$ do not commute, unless $\Psi$ is affine.

## 8.6 Derivative of differential forms

Let $\alpha \in \Omega^1(\mathcal{U}_\mathcal{E})$ (a differential form on $\mathcal{U}_\mathcal{E}$). Its derivative $d\alpha : \mathcal{U}_\mathcal{E} \to \mathcal{L}(E; E^*)$ is given by $d\alpha(p_\mathcal{E}).\vec{u}(p_\mathcal{E}) = \lim_{h\to 0} \frac{\alpha(p_\mathcal{E}+h\vec{u}(p_\mathcal{E}))-\alpha(p_\mathcal{E})}{h} \in E^*$, for all $\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$, i.e., for all $\vec{u}_1, \vec{u}_2 \in \Gamma(\mathcal{U}_\mathcal{E})$,

$$(d\alpha(p_\mathcal{E}).\vec{u}_1(p_\mathcal{E})).\vec{u}_2(p_\mathcal{E}) = \lim_{h\to 0} \frac{\alpha(p_\mathcal{E} + h\vec{u}_1(p_\mathcal{E})).\vec{u}_2(p_\mathcal{E}) - (\alpha(p_\mathcal{E}).\vec{u}_1(p_\mathcal{E})).\vec{u}_2(p_\mathcal{E})}{h} \quad \in \mathbb{R}. \tag{8.21}$$

With the natural canonical isomorphism $\mathcal{L}(E; E^*) \simeq \mathcal{L}(E, E; \mathbb{R})$, cf. (U.17) with $E^{**} \simeq E$, we can write $d\alpha(p_\mathcal{E})(\vec{u}_1(p_\mathcal{E})).\vec{u}_2(p_\mathcal{E}) = d\alpha(p_\mathcal{E})(\vec{u}_1(p_\mathcal{E}), \vec{u}_2(p_\mathcal{E}))$, i.e.

$$d\alpha(\vec{u}_1).\vec{u}_2 = d\alpha(\vec{u}_1, \vec{u}_2). \tag{8.22}$$

Thus the push-forward $\Psi_*(d\alpha) =^{\text{written}} (d\alpha)_*$ of $d\alpha$, is given by, for all $\vec{w}_1, \vec{w}_2 \in \Gamma(\mathcal{U}_\mathcal{F})$, in short,

$$(d\alpha)_*(\vec{w}_1, \vec{w}_2) = d\alpha(\vec{w}_1^*, \vec{w}_2^*), \tag{8.23}$$

i.e., with $p_\mathcal{F} = \Psi(p_\mathcal{E})$, $(d\alpha)_*(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F})).\vec{w}_2(p_\mathcal{F}) = (d\alpha(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F})).d\Psi^{-1}(p_\mathcal{F}).\vec{w}_2(p_\mathcal{F})$.

In particular, $(d^2f)_*(\vec{w}_1, \vec{w}_2) = d^2f(d\Psi^{-1}.\vec{w}_1, d\Psi^{-1}.\vec{w}_2)$ $(= d^2f(\vec{w}_1^*, \vec{w}_2^*))$.

## 8.7 $\Psi_*(d\alpha) \neq d(\Psi_*\alpha)$ in general: No commutativity

Here $\Psi$ is $C^2$, $\vec{u} \in \Gamma(\mathcal{U}_\mathcal{E})$, $p_\mathcal{E} \in \mathcal{U}_\mathcal{E}$ and $p_\mathcal{F} = \Psi(p_\mathcal{E})$. We have $\Psi_*\alpha(p_\mathcal{F}) = \alpha(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}) = \alpha(\Psi^{-1}(p_\mathcal{F})).d\Psi^{-1}(p_\mathcal{F})$, thus, for all $\vec{w}_1 \in \Gamma(\mathcal{U}_\mathcal{F})$,

$$d(\psi_*\alpha)(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F}) = (d\alpha(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F})).d\Psi^{-1}(p_\mathcal{F}) + \alpha(p_\mathcal{E}).d^2\Psi^{-1}(p_\mathcal{F}).\vec{w}_1(p_\mathcal{F}) \ \in F^*, \tag{8.24}$$

thus, for all $\vec{w}_1, \vec{w}_2 \in \Gamma(\mathcal{U}_\mathcal{F})$, in short

$$d(\psi_*\alpha)(\vec{w}_1, \vec{w}_2) = d\alpha(d\Psi^{-1}.\vec{w}_1, d\Psi^{-1}.\vec{w}_2) + \alpha.d^2\Psi^{-1}(\vec{w}_1, \vec{w}_2) \neq d\alpha(\vec{w}_1^*, \vec{w}_2^*) \quad \text{in general.} \tag{8.25}$$

So the differentiation $d$ and the push-forward $_*$ do not commute, unless $\Psi$ is affine.

# Part III
# Lie derivative

## 9 Lie derivative

### 9.0 Purpose and first results

#### 9.0.1 Purpose?

Cauchy's approach may be insufficient, e.g.:

• Cauchy's approach: - Needs to compare **two** vectors deformed by a motion, thanks to a Euclidean dot product $(\cdot,\cdot)_g$ and the deformation gradient $F$. It uses the Cauchy deformation tensor $C = F^T.F$ obtained by comparing $(\vec{u},\vec{w})_g$ and $(\vec{u}_*,\vec{w}_*)_g$ where $\vec{u}_* = F.\vec{u}$ and $\vec{w}_* = F.\vec{w}$ are the deformed vectors by the motion (the push-forwards): We have $(\vec{u}_*,\vec{w}_*)_g - (\vec{u},\vec{w})_g = ((F^T.F - I).\vec{u},\vec{w})_g$. It is a quantitative approach (needs a chosen Euclidean dot product; built with foot, metre...).

- It is a first order method (dedicated to linear material): Only the first order Taylor expansion of the motion is used: Only $d\Phi = F$ is used (the "slope"), not $d^2\Phi = dF$ (the "curvature") or higher derivatives. Remark: The use of $F^T$ is an obstacle to get simple second (or higher) order methods.

While:

• Lie's derivative approach: - uses $\mathcal{L}_{\vec{v}}\vec{u}$ of a vector field $\vec{u}$ measures the resistance of **one** vector field $\vec{u}$ submitted to a motion.

- It can "naturally" used to get non-linear materials thanks to second order Lie derivatives which uses the second order Taylor expansion of the motion (no $F^T$).

- It is qualitative. So no Euclidean dot product are required to begin with. (Be reassured: The quantification in a Galilean Euclidean framework for the first order approximation will give the usual results of Cauchy's approach.)

- In a non planar surface $S$, you need the Lie derivative if you want to derive along a trajectory.

(Cauchy died in 1857, and Lie was born in 1842.)

#### 9.0.2 Basic results

In $\mathbb{R}^2$ or $\mathbb{R}^3$. With $\vec{v}$ the Eulerian velocity of the motion:

The Lie derivative $\mathcal{L}_{\vec{v}}f$ of a Eulerian scalar valued function $f$ is the material derivative

$$\mathcal{L}_{\vec{v}}f = \frac{Df}{Dt}. \tag{9.1}$$

The Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ of a Eulerian vector field $\vec{w}$ is more than just the material derivative $\frac{D\vec{w}}{Dt}$:

$$\mathcal{L}_{\vec{v}}\vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}, \tag{9.2}$$

the $-d\vec{v}.\vec{w}$ term telling that the spatial variations $d\vec{v}$ of $\vec{v}$ act on the evolution of the stress.

(9.1)-(9.2) enable to define the Lie derivatives of tensors of any type and order (consistency results).

### 9.1 Definition

#### 9.1.1 Issue (ubiquity gift)...

The motion $\widetilde{\Phi} : [t_1,t_2] \times \mathit{Obj} \to \mathbb{R}^n$ is supposed to be $C^1$, $\vec{v}(t,p(t)) = \frac{\partial\widetilde{\Phi}}{\partial t}(t,P_{\mathit{Obj}})$ is the Eulerian velocity at $t$ at $p(t) = \widetilde{\Phi}(t,P_{\mathit{Obj}})$. Recall: If $\mathcal{E}ul$ is a Eulerian function then its material time derivative is

$$\frac{D\mathcal{E}ul}{Dt}(t,p(t)) = \lim_{\tau\to t}\frac{\mathcal{E}ul(\tau,p(\tau)) - \mathcal{E}ul(t,p(t))}{\tau - t} \quad (= \lim_{h\to 0}\frac{\mathcal{E}ul(t{+}h,p(t{+}h)) - \mathcal{E}ul(t,p(t))}{h}). \tag{9.3}$$

Issue: The difference $\mathcal{E}ul(\tau,p(\tau)) - \mathcal{E}ul(t,p(t))$ requires the time and space ubiquity gift to be computed (two distinct times $t$ and $\tau$ and positions $p(t)$ and $p(\tau)$).

### 9.1.2 ... circumvented

To compare $\mathcal{E}ul(\tau,p(\tau))$ and $\mathcal{E}ul(t,p(t))$ along a trajectory, you need the duration $h=\tau-t$ to get from $t$ to $\tau$ and to move from $p(t)$ to $p(\tau)$. So, for a $P_{Obj}$, you must:

- At $t$ at $p_t=\widetilde{\Phi}_{P_{Obj}}(t)$, take the value $\mathcal{E}ul(t,p_t)$ with you (for memory),
- move along the trajectory $\widetilde{\Phi}_{P_{Obj}}$ from $(t,p_t$ to $(\tau,p_\tau)$; Doing so the value $\mathcal{E}ul(t,p_t)$ (you carry with you) has been transported (forwarded by the flow), so has become

$$((\Phi^t_\tau)_*\mathcal{E}ul_t)(p(\tau)) \overset{\text{written}}{=} \mathcal{E}ul_{t*}(\tau,p(\tau)) \quad \text{(push-forward by the flow).} \tag{9.4}$$

- Now that you are at $(\tau,p(\tau))$, you can compare the actual value $\mathcal{E}ul(\tau,p(\tau))$ with the value $\mathcal{E}ul_{t*}(\tau,p(\tau))$ you arrived with (the transported memory) see fig. 9.1, and the difference

$$\mathcal{E}ul(\tau,p(\tau))-\mathcal{E}ul_{t*}(\tau,p(\tau)) \tag{9.5}$$

is meaningful for a human being because no gift of ubiquity required. This gives an initial definition:

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t):=\lim_{\tau\to t}\frac{\mathcal{E}ul(\tau,p(\tau))-\mathcal{E}ul_{t*}(\tau,p(\tau))}{\tau-t}. \tag{9.6}$$

E.g. with $\mathcal{E}ul=\vec{w}$ a vector field,

$$\mathcal{L}_{\vec{v}}\vec{w}(t,p_t)=\lim_{\tau\to t}\frac{\vec{w}_\tau(p_\tau)-d\Phi^t_\tau(p_t).\vec{w}_t(p_t)}{\tau-t}. \tag{9.7}$$

But this definition has a draw-back: $\mathcal{E}ul(\tau,p(\tau))$ and $\mathcal{E}ul_{t*}(\tau,p(\tau))$ are in the tangent plane $T_{p(\tau)}\Omega_\tau)$ which varies with $\tau$ (on a non planar surface). Thus we prefer one the the two following definitions.

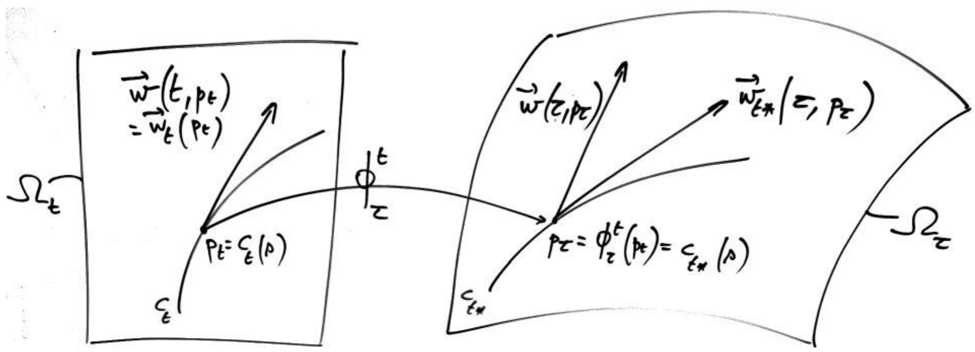

Figure 9.1: (9.5) with $\mathcal{E}ul=\vec{w}$ a (Eulerian) vector field. At $t$, let $w_t: p_t\in\Omega_t\to\vec{w}_t(p_t):=\vec{w}(t,p_t)\in\vec{\mathbb{R}^n}$, and consider its integral (spatial) curve $c_t: s\to p_t=c_t(s)\in\Omega_t$, i.e. s.t. $c_t{}'(s)=\vec{w}_t(c_t(s))$. This curve $c_t$ is transported by $\Phi^t_\tau$ into the (spatial) curve $c_\tau=c_{t*}=\Phi^t_\tau\circ c_t : s\to p_\tau=\Phi^t_\tau(c_t(s))\in\Omega_\tau$; And $c_{t*}{}'(s)=d\Phi^t_\tau(p_t).c'(s)=d\Phi^t_\tau(p_t).\vec{w}_t(p_t)=\vec{w}_{t*}(\tau,p_\tau)$ is the tangent vector at $c_\tau$ at $p_\tau$ (push-forward). And the difference $\vec{w}(\tau,p_\tau)-\vec{w}_{t*}(\tau,p_\tau)$ can be computed by a human being, i.e. without ubiquity gift.

### 9.1.3 First definition

The "natural" definition is given when you arrive with your memory:

- At $\tau<t$ at $p(\tau)=p_\tau=\widetilde{\Phi}_{P_{Obj}}(\tau)$, take the past value $\mathcal{E}ul(\tau,p(\tau))$ (memory);
- Then consider the (virtual) value $\mathcal{E}ul_{\tau*}(t,p(t))$ which is the push-forward by the flow (deformed with no resistance by the flow);
- Now, without any ubiquity gift, you can compare this value with the actual (real) value $\mathcal{E}ul(t,p_t)$:

**Definition 9.1** The Lie derivative $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ of an Eulerian function $\mathcal{E}ul$ along $\vec{v}$ is the Eulerian function $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ defined by, at $t$ at $p_t=\widetilde{\Phi}_{P_{Obj}}(t)$,

$$\begin{aligned}\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) &:= \lim_{h\to 0}\frac{\mathcal{E}ul_t(p_t)-(\Phi^{t-h}_t)_*\mathcal{E}ul_{t-h}(p_t)}{t-\tau}=\lim_{h\to 0}\frac{\text{present} - \text{memory transported}}{t-\tau}\\ &= \lim_{\tau\to t}\frac{\mathcal{E}ul_t(p_t)-(\Phi^\tau_t)_*\mathcal{E}ul_\tau(p_t)}{t-\tau}=\lim_{\tau\to t}\frac{(\mathcal{E}ul-\mathcal{E}ul_{\tau*})(t,p_t)}{t-\tau}.\end{aligned} \tag{9.8}$$

E.g. with $\mathcal{E}ul = \vec{w}$ a vector field,

$$\mathcal{L}_{\vec{v}}\vec{w}(t,p_t) = \lim_{\tau\to t} \frac{\vec{w}_t(p_t) - d\Phi_t^\tau(p_\tau).\vec{w}_\tau(p_\tau)}{t-\tau}. \tag{9.9}$$

**Remark 9.2** In classical mechanics (9.5) and (9.8) are equivalent. E.g. for vector fields, (9.9) gives $\mathcal{L}_{\vec{v}}\vec{w}(t,p_t) = \lim_{\tau\to t} d\Phi_t^\tau(p_\tau).\frac{d\Phi_t^\tau(p_\tau)^{-1}.\vec{w}_t(p_t)-\vec{w}_\tau(p_\tau)}{t-\tau} = \lim_{\tau\to t} d\Phi_t^\tau(p_\tau).\lim_{\tau\to t}\frac{d\Phi_\tau^t(p_\tau).\vec{w}_t(p_t)-\vec{w}_\tau(p_\tau)}{t-\tau} = I.\lim_{\tau\to t}\frac{\vec{w}_\tau(p_\tau)-d\Phi_\tau^t(p_t).\vec{w}_t(p_t)}{\tau-t} = (9.7)$. ▪

**Remark 9.3** Precise definition (as in (2.3)): With $\mathcal{C} = \bigcup_{t\in[t_1,t_2]}([t_1,t_2]\times\Omega_t)$ and (9.8), the Lie derivative of the Eulerian field of functions $\widehat{\mathcal{E}ul} : \left\{\begin{array}{ll} \mathcal{C} & \to \mathcal{C}\times S \\ (t,p_t) & \to ((t,p_t),\mathcal{E}ul(t,p_t)) \end{array}\right\}$ is

$$\widetilde{\mathcal{L}}_{\vec{v}}\widehat{\mathcal{E}ul} : \left\{\begin{array}{ll} \mathcal{C} & \to \mathcal{C}\times S \\ (t,p_t) & \to \widetilde{\mathcal{L}}_{\vec{v}}\widehat{\mathcal{E}ul}(t,p_t) := ((t,p_t),\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t)). \end{array}\right. \tag{9.10}$$

Shortened (lighened) notation: We write $\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t)$ instead of $\widetilde{\mathcal{L}}_{\vec{v}}\widehat{\mathcal{E}ul}(t,p_t)$. ▪

### 9.1.4 Second (equivalent) definition

Differential geometry course: The Lie derivative is defined with pull-backs:

- At $t{+}h$ at $p(t{+}h) = \widetilde{\Phi}(t{+}h, P_{Obj})$, take the value $\mathcal{E}ul(t{+}h, p(t{+}h))$;
- Go back in the past along the trajectory $\widetilde{\Phi}_{P_{Obj}}$ and consider the (virtual) value $\mathcal{E}ul^*_{t+h}(t,p_t)$ which is the pull-back by the flow;
- Then compared it with $\mathcal{E}ul(t,p_t)$ (no ubiquity gift required):

**Definition 9.4** The Lie derivative of a Eulerian function $\mathcal{E}ul$ along a flow of Eulerian velocity $\vec{v}$ is the Eulerian function $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ defined at $(t,p_t)$ by

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) := \lim_{\tau\to t}\frac{(\Phi_\tau^t)^*\mathcal{E}ul_\tau(p_t) - \mathcal{E}ul_t(p_t)}{\tau-t} \quad (= \lim_{h\to 0}\frac{(\mathcal{E}ul^*_{t+h} - \mathcal{E}ul)(t,p_t)}{h}). \tag{9.11}$$

In other words, let

$$g(\tau) = (\Phi_\tau^t)^*\mathcal{E}ul_\tau(p_t) \tag{9.12}$$

(function defined along a trajectory which satisfies $g(t) = \mathcal{E}ul_t(p_t)$). Then $\mathcal{L}_{\vec{v}}\mathcal{E}ul$ is defined by

$$\mathcal{L}_{\vec{v}}\mathcal{E}ul(t,p_t) := g'(t) \quad (= \lim_{\tau\to t}\frac{g(\tau)-g(t)}{\tau-t} \overset{\text{written}}{=} \frac{(\Phi_\tau^t)^*\mathcal{E}ul_\tau(p_t)}{d\tau}_{|\tau=t}). \tag{9.13}$$

E.g. with $\mathcal{E}ul = \vec{w}$ a vector field,

$$\mathcal{L}_{\vec{v}}\vec{w}(t,p_t) = \lim_{\tau\to t}\frac{d\Phi_\tau^t(p_t)^{-1}.\vec{w}_\tau(p_\tau) - \vec{w}_t(p_t)}{\tau-t}. \tag{9.14}$$

**Proposition 9.5** *(9.8) and (9.11) are equivalent.*

**Proof.** 1- Vector fields. From (9.9): $\frac{\vec{w}_t(p_t)-d\Phi_t^\tau(p_\tau).\vec{w}_\tau(p_\tau)}{t-\tau} = \frac{d\Phi_t^\tau(p_\tau).\vec{w}_\tau(p_\tau)-\vec{w}_t(p_t)}{\tau-t} = d\Phi_t^\tau(p_\tau).\frac{\vec{w}_\tau(p_\tau)-d\Phi_\tau^t(p_t).\vec{w}_t(p_t)}{\tau-t}$, because $p_t = \Phi_t^\tau(p_\tau) = \Phi_t^\tau(\Phi_\tau^t(p_t))$ gives $I = d\Phi_t^\tau(p_\tau).d\Phi_\tau^t(p_t)$. And "Product of limits = limit of products", thus $\lim_{\tau\to t}\frac{\vec{w}_t(p_t)-d\Phi_t^\tau(p_\tau).\vec{w}_\tau(p_\tau)}{t-\tau} = I.\lim_{\tau\to t}\frac{\vec{w}_\tau(p_\tau)-d\Phi_\tau^t(p_t).\vec{w}_t(p_t)}{\tau-t}$.

From (9.14): $\frac{d\Phi_\tau^t(p_t)^{-1}.\vec{w}_\tau(p_\tau)-\vec{w}_t(p_t)}{\tau-t} = d\Phi_\tau^t(p_t)^{-1}.\frac{\vec{w}_\tau(p_\tau)-d\Phi_\tau^t(p_t).\vec{w}_t(p_t)}{\tau-t}$. Thus $\lim_{\tau\to t}\frac{(d\Phi_\tau^t(p_t)^{-1}.\vec{w}_\tau(p_\tau)-\vec{w}_t(p_t)}{\tau-t} = I.\lim_{\tau\to t}\frac{\vec{w}_\tau(p_\tau)-d\Phi_\tau^t(p_t).\vec{w}_t(p_t)}{\tau-t}$. Same result. Thus (9.9) $\Leftrightarrow$ (9.14).

2- Similar for any tensor: (9.8) and (9.11) are equivalent. ▪

## 9.2 Lie derivative of a scalar function

Let $f$ be a $C^1$ Eulerian scalar valued function. With $(\Phi_t^{t-h})_* f_{t-h}(p_t) =^{(6.10)} f_{t-h}(p(t-h))$, (9.8) gives

$$\mathcal{L}_{\vec{v}} f(t,p_t) = \lim_{h\to 0} \frac{f(t,p_t) - f(t-h, p(t-h))}{h}, \quad \text{i.e.} \quad \boxed{\mathcal{L}_{\vec{v}} f = \frac{Df}{Dt}} = \frac{\partial f}{\partial t} + df.\vec{v}. \tag{9.15}$$

So, for scalar valued functions, the Lie derivative is the material derivative. (Full notations: $\mathcal{L}_{\vec{v}} f(t,p_t) = \frac{Df}{Dt}(t,p_t) = \frac{\partial f}{\partial t}(t,p_t) + df(t,p_t).\vec{v}(t,p_t)$.)

**Proposition 9.6** *$\mathcal{L}_{\vec{v}} f = 0$ iff $f$ is constant along any trajectory (at $t$ at $p_t$ the real value = the memory value), i.e. iff $f(t,p(t)) = f(t_0, p_{t_0})$ when $p(t) = \Phi^{t_0}(t, p_{t_0})$ (i.e. iff $f$ is unchanged along the flow):*

$$\mathcal{L}_{\vec{v}} f = 0 \iff \forall t,\tau \in [t_0,T],\ (\Phi^t_{\tau *}) f_t(p_\tau) = f(t,p(t)) \ \text{ when } \ p_\tau = \Phi^t_\tau(p_t). \tag{9.16}$$

**Proof.** Let $p(t) = \widetilde{\Phi}(t, P_{Obj}) = p_t$ for all $t$, so $p(\tau) = \widetilde{\Phi}(\tau, P_{Obj}) = p_\tau = \Phi^t_{t+h}(p_t) = \Phi^t(\tau, p_t)$.

$\Leftarrow$: If $f_\tau = (\Phi^t_{t+h})_* f_t$, then $f_\tau(p_\tau) = f_t(p_t)$, thus $\lim_{\tau\to t} \frac{f(\tau,p(\tau)) - f(t,p(t))}{\tau - t} = 0$, i.e. $\frac{Df}{Dt} = 0$.

$\Rightarrow$: If $\frac{Df}{Dt} = 0$ then $f(t,p(t))$ is a constant function on the trajectory $t \to \widetilde{\Phi}(t, P_{Obj})$, for any particle $P_{Obj}$, so $f(\tau, p(\tau)) = f(t,p_t)$ when $p(\tau) = \Phi^t_{t+h}(p_t)$, i.e. $f(\tau,p_\tau) = (\Phi^t_{t+h})_* f_t(p_\tau)$. ∎

**Proposition 9.7** *If $f$ is $C^2$ then (commutativity)*

$$\mathcal{L}_{\vec{v}}(df) = d(\mathcal{L}_{\vec{v}} f). \tag{9.17}$$

*(Commutativity only true for scalar valued functions, see e.g. (9.22) and (9.57).)*

**Proof.** $df$ is a $C^1$ thus $d^2 f$ symmetric (Schwarz' theorem), thus with a little advance see (9.52), $\mathcal{L}_{\vec{v}}(df) = \frac{D(df)}{Dt} + df.\vec{v} = \frac{\partial(df)}{\partial t} + d^2 f(\vec{v},.) + df.d\vec{v}$ and $d(\mathcal{L}_{\vec{v}} f) = d(\frac{\partial f}{\partial t} + df.\vec{v}) = \frac{\partial(df)}{\partial t} + d^2 f(\vec{v},.) + df.d\vec{v}$ give (9.17). ∎

**Exercice 9.8** Prove: $\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}} f) = \frac{D^2 f}{Dt^2} = \frac{\partial^2 f}{\partial t^2} + 2d(\frac{\partial f}{\partial t}).\vec{v} + d^2 f(\vec{v},\vec{v}) + df.(\frac{\partial \vec{v}}{\partial t} + d\vec{v})$.

**Answer**. See (2.32). ∎

## 9.3 Lie derivative of a vector field

### 9.3.1 Formula

**Proposition 9.9** *The motion is supposed $C^2$ and $\vec{w}$ is a $C^1$ (Eulerian) vector field. We have*

$$\boxed{\mathcal{L}_{\vec{v}} \vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}} = \frac{\partial \vec{w}}{\partial t} + d\vec{w}.\vec{v} - d\vec{v}.\vec{w}. \tag{9.18}$$

*So the Lie derivative is not reduced to the material derivative $\frac{D\vec{w}}{Dt}$ (unless $d\vec{v} = 0$, i.e. unless $\vec{v}$ is uniform): The spatial variations $d\vec{v}$ of $\vec{v}$ influences the rate of stress (e.g. $\vec{v}$ tries to bend $\vec{w}$ which is expected).*

**Proof.** Let $\vec{g}(\tau) =^{(9.12)} d\Phi^t_\tau(p_t)^{-1}.\vec{w}(\tau, p(\tau))$. Thus thus $\vec{w}(\tau,p(\tau)) = d\Phi^t(\tau,p_t).\vec{g}(\tau)$ and (9.13) gives

$$\frac{D\vec{w}}{D\tau}(\tau,p(\tau)) = \vec{g}\,'(\tau) = \underbrace{\frac{\partial(d\Phi^t)}{\partial \tau}(\tau,p_t)}_{d\vec{v}(\tau,p(\tau)).d\Phi^t(\tau,p_t)} . \underbrace{\vec{g}(\tau)}_{d\Phi^t(\tau,p_t)^{-1}.\vec{w}(\tau,p(\tau))} + \underbrace{d\Phi^t(\tau,p_t)}_{F^t_\tau(p_t)} . \underbrace{\vec{g}\,'(\tau)}_{\mathcal{L}_{\vec{v}}\vec{w}(\tau,p(\tau))} \tag{9.19}$$

Thus $\frac{D\vec{w}}{Dt}(t,p_t) = d\vec{v}(t,p_t).\vec{w}(t,p_t) + I.\mathcal{L}_{\vec{v}}\vec{w}(t,p_t)$, thus (9.18). ∎

**Quantification:** Basis $(\vec{e}_i)$, $\vec{v} = \sum_i v_i \vec{e}_i$, $\vec{w} = \sum_i w_i \vec{e}_i$, $d\vec{v}.\vec{e}_j = \sum_{ij} v_{i|j} \vec{e}_i$, $d\vec{w}.\vec{e}_j = \sum_{ij} w_{i|j} \vec{e}_i$; Then

$$\mathcal{L}_{\vec{v}} \vec{w} = \sum_{i=1}^n \frac{\partial w_i}{\partial t} \vec{e}_i + \sum_{i,j=1}^n w_{i|j} v_j \vec{e}_i - \sum_{i,j=1}^n v_{i|j} w_j \vec{e}_i. \tag{9.20}$$

So, with $[\cdot] := [\cdot]_{|\vec{e}}$,

$$\boxed{[\mathcal{L}_{\vec{v}} \vec{w}] = [\frac{D\vec{w}}{Dt}] - [d\vec{v}].[\vec{w}]} \quad (= [\frac{\partial \vec{w}}{\partial t}] + [d\vec{w}.\vec{v}] - [d\vec{v}].[\vec{w}]). \tag{9.21}$$

(And $[d\vec{w}.\vec{v}] = [d\vec{w}].[\vec{v}]$.) Duality notations: $\mathcal{L}_{\vec{v}} \vec{w} = \sum_i \frac{\partial w^i}{\partial t} \vec{e}_i + \sum_{ij} w^i_{|j} v^j \vec{e}_i - \sum_{ij} v^i_{|j} w^j \vec{e}_i$.

**Proposition 9.10** *For $C^2$ vector fields (no commutativity in general):*

$$d(\mathcal{L}_{\vec{v}}\vec{w}) = \mathcal{L}_{\vec{v}}(d\vec{w}) + d^2\vec{v}(.,\vec{w}) \quad (\neq \mathcal{L}_{\vec{v}}(d\vec{w}) \text{ in general}). \tag{9.22}$$

**Proof.** $d(\mathcal{L}_{\vec{v}}\vec{w}) = d(\frac{\partial\vec{w}}{\partial t} + d\vec{w}.\vec{v} - d\vec{v}.\vec{w}) = d\frac{\partial\vec{w}}{\partial t} + d^2\vec{w}(.,\vec{v}) + d\vec{w}.d\vec{v} - d^2\vec{v}(.,\vec{w}) - d\vec{v}.d\vec{w}$. And $d\vec{w}$ being a endomorphism, (9.66) gives $\mathcal{L}_{\vec{v}}(d\vec{w}) = \frac{\partial(d\vec{w})}{\partial t} + d^2\vec{w}(.,\vec{v}) - d\vec{v}.d\vec{w} + d\vec{w}.d\vec{v} \neq d(\mathcal{L}_{\vec{v}}\vec{w})$. ∎

### 9.3.2 Interpretation: Flow resistance measurement

**Proposition 9.11** *$\Phi^{t_0}$ is regular motion and $\vec{w}$ is a vector field.*

$$\mathcal{L}_{\vec{v}}\vec{w} = 0 \quad \Longleftrightarrow \quad \forall t \in [t_0, T], \quad \vec{w}_t = (\Phi_t^{t_0})_*\vec{w}_{t_0}. \tag{9.23}$$

*That is: $\frac{D\vec{w}}{Dt} = d\vec{v}.\vec{w} \Leftrightarrow$ the actual vector $\vec{w}(t,p(t))$ is equal to $F_t^{t_0}(p_{t_0}).\vec{w}_{t_0}(p_{t_0}) = \vec{w}_{t_0*}(t,p(t))$ the deformed vector by the flow. See figure 9.1. So: The Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ vanishes iff $\vec{w}$ does not resist the flow (let itself be deformed by the flow), i.e. iff $\vec{w}(t,p_t) = \vec{w}_{t_0*}(t,p_t)$ for all $t$ and all $p_t \in \Omega_t$.*

**Proof.** We have $\mathcal{L}_{\vec{v}}\vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}$ and $\frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}) = d\vec{v}(t,p(t)).F_t^{t_0}(p_{t_0})$, cf. (3.33). Let $p(t) = \Phi_t^{t_0}(p_{t_0})$.

$\Leftarrow$ Suppose $\vec{w}(t,p(t)) = F^{t_0}(t,p_{t_0}).\vec{w}(t_0,p_{t_0})$. Then $\frac{D\vec{w}}{Dt}(t,p(t)) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).\vec{w}(t_0,p_{t_0}) = (d\vec{v}(t,p(t)).F_t^{t_0}(p_{t_0})).(F_t^{t_0}(p_{t_0})^{-1}.\vec{w}(t,p(t))) = d\vec{v}(t,p(t)).\vec{w}(t,p(t))$, thus $\frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w} = 0$, i.e. $\mathcal{L}_{\vec{v}}\vec{w} = 0$.

$\Rightarrow$ Suppose $\frac{D\vec{w}}{Dt} = d\vec{v}.\vec{w}$. Let $\vec{f}(t) = (F_t^{t_0}(p_{t_0}))^{-1}.\vec{w}(t,p(t))$ (pull-back); So $\vec{w}(t,p(t)) = F^{t_0}(t,p_{t_0}).\vec{f}(t)$ and $\frac{D\vec{w}}{Dt}(t,p(t)) = \frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}).\vec{f}(t) + F_t^{t_0}(p_{t_0}).\vec{f}\,'(t) = d\vec{v}(t,p(t)).F_t^{t_0}(p_{t_0}).\vec{f}(t) + F_t^{t_0}(p_{t_0}).\vec{f}\,'(t) = d\vec{v}(t,p(t)).\vec{w}(t,p(t)) + F_t^{t_0}(p_{t_0}).\vec{f}\,'(t) =^{hyp.} \frac{D\vec{w}}{Dt}(t,p(t)) + F_t^{t_0}(p_{t_0}).\vec{f}\,'(t)$ for all $t$; Thus $F_t^{t_0}(p_{t_0}).\vec{f}\,'(t) = \vec{0}$, thus $\vec{f}\,'(t) = \vec{0}$ (because $\Phi_t^{t_0}$ is a diffeomorphism), thus $\vec{f}(t) = \vec{f}(t_0)$, i.e. $\vec{w}_t = (\Phi_t^{t_0})_*\vec{w}_{t_0}$, for all $t$. ∎

### 9.3.3 Autonomous Lie derivative and Lie bracket

The Lie bracket of two vector fields $\vec{v}$ and $\vec{w}$ is

$$[\vec{v},\vec{w}] := d\vec{w}.\vec{v} - d\vec{v}.\vec{w} \overset{\text{written}}{=} \mathcal{L}^0_{\vec{v}}\vec{w}. \tag{9.24}$$

And $\mathcal{L}^0_{\vec{v}}\vec{w} = [\vec{v},\vec{w}]$ is called the autonomous Lie derivative of $\vec{w}$ along $\vec{v}$. Thus

$$\mathcal{L}_{\vec{v}}\vec{w} = \frac{\partial\vec{w}}{\partial t} + [\vec{v},\vec{w}] = \frac{\partial\vec{w}}{\partial t} + \mathcal{L}^0_{\vec{v}}\vec{w}. \tag{9.25}$$

Remark: $\mathcal{L}^0_{\vec{v}}\vec{w}$ is generally used when $\vec{v}$ et $\vec{w}$ are stationary vector fields, thus does not concern objectivity: A stationary vector field in a referential is not necessary stationary in another (moving) referential.

## 9.4 Examples

### 9.4.1 Lie Derivative of a vector field along itself

(9.18) gives

$$\mathcal{L}_{\vec{v}}\vec{v} = \frac{\partial\vec{v}}{\partial t}. \tag{9.26}$$

In particular, if $\vec{v}$ is a stationary vector field then $\mathcal{L}_{\vec{v}}\vec{v} = \vec{0}$ $(= [\vec{v},\vec{v}])$.

### 9.4.2 Lie derivative along a uniform flow

$$d\vec{v} = 0 \quad \Longrightarrow \quad \mathcal{L}_{\vec{v}}\vec{w} = \frac{D\vec{w}}{Dt} \quad (= \frac{\partial\vec{w}}{\partial t} + d\vec{w}.\vec{v}). \tag{9.27}$$

Here the flow is rectilinear ($d\vec{v} = 0$): there is no curvature (of the flow) to influence the stress on $\vec{w}$.

Moreover, if $\vec{w}$ is stationary then $\mathcal{L}_{\vec{v}}\vec{w} = d\vec{w}.\vec{v} =$ directional derivative of $\vec{w}$ in the direction $\vec{v}$.

### 9.4.3 Lie derivative of a uniform vector field

$$d\vec{w} = 0 \quad \Longrightarrow \quad \mathcal{L}_{\vec{v}}\vec{w} = \frac{\partial\vec{w}}{\partial t} - d\vec{v}.\vec{w}, \tag{9.28}$$

thus the stress on $\vec{w}$ is due to the space variations of $\vec{v}$. E.g. is $\vec{w}$ is stationary then $\mathcal{L}_{\vec{v}}\vec{w} = -d\vec{v}.\vec{w}$.

### 9.4.4 1-D uniaxial stretch of an elastic material

• Motion. With $X = [\vec{X}]_{|\vec{e}} = [\overrightarrow{OP}]_{|\vec{e}} \in [0, L]$, $x = [\vec{\varphi}^{t_0}(t, X)]_{|\vec{e}} = [\overrightarrow{O\Phi^{t_0}(t, P)}]_{|\vec{e}} = [\overrightarrow{Op_t}]_{|\vec{e}}$, $\xi > 0$, $t \geq t_0$:

$$x = X + \xi(t-t_0)X = X(1 + \xi(t-t_0)). \tag{9.29}$$

Lagrangian and Eulerian velocity $V(t,P) = \xi X = v(t,p) = \frac{\xi}{1+\xi(t-t_0)}x$, and $d\vec{v}(t,p) = \frac{\xi}{1+\xi(t-t_0)}$ (independent of $p$). And $\Delta x = x - X = \xi(t-t_0)X$, restoring force $f(t,x) = -k\Delta X = -k\xi(t-t_0)X$.
• Deformation gradient (independent of $P$):

$$[d\Phi_t^{t_0}(P)]_{|\vec{e}} = [F_t^{t_0}(P)]_{|\vec{e}} = 1 + \xi(t-t_0) \overset{\text{written}}{=} F_t^{t_0}. \tag{9.30}$$

Infinitesimal strain tensor (independent of $P$):

$$\underline{\underline{\varepsilon}}_t^{t_0}(P) = F_t^{t_0} - I = \xi(t-t_0) = \underline{\underline{\varepsilon}}_t^{t_0}. \tag{9.31}$$

• Stress density, linear elasticity (independent of $p_t$):

$$[\underline{\underline{\sigma}}_t(p_t)]_{|\vec{e}} = \lambda \underline{\underline{\varepsilon}}_t^{t_0} = \lambda\xi(t-t_0) \overset{\text{written}}{=} \underline{\underline{\sigma}}_t. \tag{9.32}$$

Cauchy stress vector $T$ on a surface at $p_t$ (normal $n_t(p_t) = +1$ "on the right side") (independent of $p_t$):

$$[\vec{T}_t(p_t)]_{|\vec{e}} = \underline{\underline{\sigma}}_t(p_t) = \lambda\xi(t-t_0) \overset{\text{written}}{=} \vec{T}_t. \tag{9.33}$$

• Lie derivative (rate of stress) $[\mathcal{L}_{\vec{v}}\vec{T}]_{|\vec{e}} =^{\text{written}} \mathcal{L}_{\vec{v}}\vec{T}$ at $(t,p)$ (independent of $p_t$):

$$\mathcal{L}_{\vec{v}}\vec{T}(t,p) = (\frac{\partial \vec{T}}{\partial t} + d\vec{T}.\vec{v} - d\vec{v}.\vec{T}(t,p) = \lambda\xi + 0 - \frac{\lambda\xi^2(t-t_0)}{1+\xi(t-t_0)} = \lambda\xi(1 - \frac{\xi(t-t_0)}{1+\xi(t-t_0)}) = \frac{\lambda\xi}{1+\xi(t-t_0)}. \tag{9.34}$$

• Work along a trajectory: $W_{t_1}^{t_0}(P) = \int_{\Phi_P^{t_0}} \mathcal{L}_{\vec{v}}\vec{T} = \int_{t_0}^{t_1} (\mathcal{L}_{\vec{v}}\vec{T}.\vec{v})(t,p(t))\,dt = \int_{t_0}^{t_1} \mathcal{L}_{\vec{v}}\vec{T}(t,p(t)).\vec{V}(t,P)\,dt = \int_{t_0}^{t_1} \frac{\lambda\xi^2 X}{1+\xi(t-t_0)}\,dt = \lambda\xi X \log(1 + \xi(t_1-t_0))$.
Energy: $E_{t_1}^{t_0}(\Omega) = \int_{X=0}^{L} \int_{t_0}^{t_1} \mathcal{L}_{\vec{v}}\vec{T}.\vec{v}(t,p(t))\,dt\,dX = \lambda\xi\frac{L^2}{2}\log(1 + \xi(t_1-t_0))$.

### 9.4.5 2-D uniaxial stretch of an elastic material

• Strain. With $[\overrightarrow{OP}]_{|\vec{e}} = [\vec{X}]_{|\vec{e}} = \begin{pmatrix} X \\ Y \end{pmatrix}$, with $\xi > 0$, $t \geq t_0$, $p(t) = \Phi^{t_0}(t,P)$ and $[\vec{x}]_{|\vec{e}} = [\overrightarrow{Op(t)}]_{|\vec{e}}$:

$$[\vec{x}]_{|\vec{e}} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} + \xi(t-t_0)\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X(1+\xi(t-t_0)) \\ Y \end{pmatrix}. \tag{9.35}$$

• Deformation gradient (independent of $P$):

$$F_t^{t_0} = d\Phi_t^{t_0}(P) = \begin{pmatrix} 1+\kappa_t^{t_0} & 0 \\ 0 & 1 \end{pmatrix} = I + \kappa_t^{t_0}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad \kappa_t^{t_0} = \xi(t-t_0). \tag{9.36}$$

Infinitesimal strain tensor: $F^T = F$ gives

$$\underline{\underline{\varepsilon}}_t^{t_0}(P) = F_t^{t_0} - I = \kappa_t^{t_0}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \underline{\underline{\varepsilon}}_t^{t_0} \quad \text{(independent of } P\text{)}. \tag{9.37}$$

• Stress, linear isotropic elasticity:

$$\underline{\underline{\sigma}}_t(p_t) = \lambda \mathrm{Tr}(\underline{\underline{\varepsilon}}_t^{t_0})I + 2\mu\underline{\underline{\varepsilon}}_t^{t_0} = \kappa_t^{t_0}\begin{pmatrix} \lambda+2\mu & 0 \\ 0 & \lambda \end{pmatrix} = \underline{\underline{\sigma}}_t^{t_0} \quad \text{(independent of } p_t\text{)}. \tag{9.38}$$

Cauchy stress vector $\vec{T}$ on a surface at $p_t$ with normal $\vec{n}_t(p_t) = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \vec{n}$:

$$\vec{T}_t(p_t) = \underline{\underline{\sigma}}_t.\vec{n} = \kappa_t^{t_0}\begin{pmatrix} (\lambda+2\mu)n_1 \\ \lambda n_2 \end{pmatrix} = \xi(t-t_0)\begin{pmatrix} (\lambda+2\mu)n_1 \\ \lambda n_2 \end{pmatrix} = \vec{T}_t. \tag{9.39}$$

• Push-forwards: $\vec{T}_{t_0}(p_{t_0}) = 0$, thus $F_{t_0+h}^{t_0}(p_{t_0}).\vec{T}_{t_0}(p_{t_0}) = \vec{0}$.

• Eulerian velocity $\vec{v}(t,p) = \begin{pmatrix} \xi X \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\xi}{1+\xi(t-t_0)} x \\ 0 \end{pmatrix}$, $d\vec{v}(t,p) = \begin{pmatrix} \frac{\xi}{1+\xi(t-t_0)} & 0 \\ 0 & 0 \end{pmatrix}$ (independent of $p$).
• Lie derivative (rate of stress at $(t_0, p_{t_0})$):

$$\mathcal{L}_{\vec{v}}\vec{T}(t_0,p_{t_0}) = \lim_{t\to t_0} \frac{\vec{T}_t(p_t) - F_t^{t_0}(p_{t_0}).\vec{T}_{t_0}(p_{t_0})}{t-t_0} = \xi \begin{pmatrix} (\lambda{+}2\mu)n_1 \\ \lambda n_2 \end{pmatrix}. \tag{9.40}$$

Remark, generic computation with $\mathcal{L}_{\vec{v}}\vec{T} = \frac{\partial \vec{T}}{\partial t} + d\vec{T}.\vec{v} - d\vec{v}.\vec{T}$: (9.39) gives $\frac{\partial \vec{T}}{\partial t}(t) = \xi \begin{pmatrix} (\lambda{+}2\mu)\, n_1 \\ \lambda\, n_2 \end{pmatrix}$ and $d\vec{T} = 0$ and $d\vec{v}_t.\vec{T}_t = \begin{pmatrix} \frac{\xi}{1+\xi(t-t_0)} & 0 \\ 0 & 0 \end{pmatrix} .\xi(t{-}t_0) \begin{pmatrix} (\lambda{+}2\mu)\, n_1 \\ \lambda\, n_2 \end{pmatrix} = \frac{\xi^2(t-t_0)}{1+\xi(t-t_0)} \begin{pmatrix} (\lambda{+}2\mu)\, n_1 \\ 0 \end{pmatrix}$. In particular, $d\vec{v}(t_0,p_{t_0}).\vec{T}(t_0,p_{t_0}) = \vec{0}$. Thus $\mathcal{L}_{\vec{v}}\vec{T}(t_0,p_{t_0}) = \xi \begin{pmatrix} (\lambda{+}2\mu)\, n_1 \\ \lambda\, n_2 \end{pmatrix}$ = rate of stress at $(t_0,p_{t_0})$.

### 9.4.6 Simple shear of an elastic material

Fixed Euclidean basis $(\vec{e}_1,\vec{e}_2)$ in $\mathbb{R}^2$ at all time. Initial configuration $\Omega_{t_0} = [0,L_1]\otimes[0,L_2]$. Initial position: $[\overrightarrow{OP}]_{\vec{e}} = [\overrightarrow{Op_{t_0}}]_{\vec{e}} = [\vec{X}]_{\vec{e}} = \begin{pmatrix} X \\ Y \end{pmatrix} =^{\text{written}} \vec{X}$. Position at $t$: $p_t = \Phi_t^{t_0}(p_{t_0})$, $[\vec{x}]_{|\vec{e}} = [\overrightarrow{Op(t)}]_{|\vec{e}} =^{\text{written}} \vec{x}$. Let $\xi \in \mathbb{R}^*$, and

$$\vec{x} = \begin{pmatrix} x = \varphi^1(t,X,Y) \\ y = \varphi^2(t,X,Y) \end{pmatrix} = \begin{pmatrix} X + \xi(t{-}t_0)Y \\ Y \end{pmatrix} = \begin{pmatrix} X + \kappa_t^{t_0} Y \\ Y \end{pmatrix} \quad \text{where} \quad \kappa_t^{t_0} = \xi(t{-}t_0). \tag{9.41}$$

• Deformation gradient (not diagonalizable):

$$d\Phi_t^{t_0}(P) = \begin{pmatrix} 1 & \kappa_t^{t_0} \\ 0 & 1 \end{pmatrix} = F_t^{t_0}, \quad \text{thus} \quad F_t^{t_0} - I = \kappa_t^{t_0} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9.42}$$

• Lagrangian velocity $\vec{V}_t^{t_0}(p_{t_0}) = \begin{pmatrix} \xi Y \\ 0 \end{pmatrix} = \vec{V}(p_{t_0})$. Thus $d\vec{V}_t(p_{t_0}) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = d\vec{V}$.
• Eulerian velocity: $\vec{v}_t(p_t) = \vec{V}_t^{t_0}(p_{t_0}) = \begin{pmatrix} \xi y \\ 0 \end{pmatrix} = \vec{v}(p_t)$. Thus $d\vec{v}_t(p_t) = \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} = d\vec{v}$.
• Infinitesimal strain tensor:

$$\underline{\underline{\varepsilon}}_t^{t_0}(P) = \frac{F_t^{t_0}(P){-}I + (F_t^{t_0}(P){-}I)^T}{2} = \frac{\kappa_t^{t_0}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{\varepsilon}}_t^{t_0}. \tag{9.43}$$

• Stress. Constitutive law, usual linear isotropic elasticity (requires a Euclidean dot product):

$$\underline{\underline{\sigma}}^{t_0}(t,p_t) = \lambda \mathrm{Tr}(\underline{\underline{\varepsilon}}_t^{t_0})I + 2\mu \underline{\underline{\varepsilon}}_t^{t_0} = \mu\kappa_t^{t_0} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{\sigma}}_t^{t_0}. \tag{9.44}$$

Cauchy stress vector $\vec{T}(t,p_t)$ (at $t$ at $p_t$) on a surface at $p$ with normal $\vec{n}_t(p) = \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} = \vec{n}$:

$$\vec{T}_t(p_t) = \underline{\underline{\sigma}}_t^{t_0}.\vec{n} = \mu\kappa_t^{t_0} \begin{pmatrix} n^2 \\ n^1 \end{pmatrix} = \mu\xi(t{-}t_0) \begin{pmatrix} n^2 \\ n^1 \end{pmatrix} = \vec{T}(t) \quad \text{(stress independent of } p_t\text{)}. \tag{9.45}$$

• Lie derivative, with $\vec{T}_{t_0} = \vec{0}$:

$$\mathcal{L}_{\vec{v}}\vec{T}(t_0,p_{t_0}) = \lim_{t\to t_0} \frac{\vec{T}_t(p_t) - F_t^{t_0}(p_{t_0}).\vec{T}_{t_0}(p_{t_0})}{t-t_0} = \mu\xi \begin{pmatrix} n^2 \\ n^1 \end{pmatrix} \quad \text{(rate of stress at } (t_0,p_{t_0})\text{)}. \tag{9.46}$$

• Generic computation: $\mathcal{L}_{\vec{v}}\vec{T} = \frac{\partial \vec{T}}{\partial t} + d\vec{T}.\vec{v} - d\vec{v}.\vec{T}$. (9.45) gives $\frac{\partial \vec{T}}{\partial t}(t) = \mu\xi \begin{pmatrix} n^2 \\ n^1 \end{pmatrix}$ and $d\vec{T} = 0$ and $d\vec{v}.\vec{T}(t_0) = \vec{0}$. Thus $\mathcal{L}_{\vec{v}}\vec{T}(t_0,p_{t_0}) = \mu\xi \begin{pmatrix} n^2 \\ n^1 \end{pmatrix}$.

### 9.4.7 Shear flow

Stationary shear field, see (5.10) with $\alpha = 0$ and $t_0 = 0$ (or see (9.41) with $\xi = \lambda$):

$$\vec{v}(x,y) = \begin{cases} v^1(x,y) = \lambda y, \\ v^2(x,y) = 0, \end{cases} \qquad d\vec{v}(x,y) = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}. \tag{9.47}$$

Let $\vec{w}(t,p) = \begin{pmatrix} 0 \\ b \end{pmatrix} = \vec{w}(t_0, p_{t_0})$ (constant in time and uniform in space). Then $\mathcal{L}_{\vec{v}}\vec{w} = -d\vec{v}.\vec{w} = \begin{pmatrix} -\lambda b \\ 0 \end{pmatrix}$ measures "the resistance to deformation due to the flow". See figure 9.2, the virtual vector $\vec{w}_*(t,p) = d\Phi(t_0, p_{t_0}).\vec{w}(t_0, p_{t_0})$ being the vector that would have let itself be carried by the flow (the push-forward).

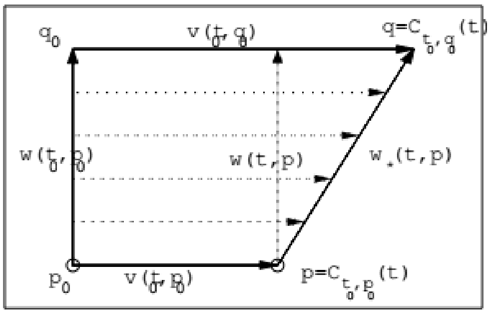


Figure 9.2: Shear flow, cf. (9.47), with $\vec{w}$ constant and uniform. $\mathcal{L}_{\vec{v}}\vec{w}$ measures the resistance to the deformation.

### 9.4.8 Spin

Rotating flow: Continuing (5.13):

$$\vec{v}(x,y) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad d\vec{v}(x,y) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \omega\, Rot(\pi/2). \tag{9.48}$$

In particular $d^2\vec{v} = 0$. With $\vec{w} = \vec{w}_0$ constant and uniform we get

$$\mathcal{L}_{\vec{v}}\vec{w}_0 = -d\vec{v}(p).\vec{w}_0 = -\omega\, Rot(\pi/2).\vec{w}_0 \qquad (\perp \begin{pmatrix} a \\ b \end{pmatrix} = \vec{w}_0). \tag{9.49}$$

gives "the force at which $\vec{w}$ refuses to turn with the flow".

### 9.4.9 Second order Lie derivative

**Exercice 9.12** Let $\vec{v}, \vec{w}$ be $C^2$ and $\vec{g}(t) = (\Phi_\tau^{t*}\vec{w})(t, p_t) = d\Phi_\tau^t(p_t)^{-1}.\vec{w}(\tau, p(\tau))$ when $p(\tau) = \Phi^t(\tau, p_t)$. We have $\mathcal{L}_{\vec{v}}\vec{w}(t, p(t)) =^{(9.11)} \vec{g}\,'(t)$. Prove $\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w})(t, p(t)) = \vec{g}\,''(t)$, i.e.:

$$\begin{aligned} \mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w}) &= \frac{D^2\vec{w}}{Dt^2} - 2d\vec{v}.\frac{D\vec{w}}{Dt} - \frac{D(d\vec{v})}{Dt}.\vec{w} + d\vec{v}.d\vec{v}.\vec{w} \\ &= \frac{\partial^2\vec{w}}{\partial t^2} + 2d\frac{\partial\vec{w}}{\partial t}.\vec{v} - 2d\vec{v}.\frac{\partial\vec{w}}{\partial t} + d\vec{w}.\frac{\partial\vec{v}}{\partial t} - d\frac{\partial\vec{v}}{\partial t}.\vec{w} \\ &\quad + (d^2\vec{w}.\vec{v}).\vec{v} + d\vec{w}.d\vec{v}.\vec{v} - 2d\vec{v}.d\vec{w}.\vec{v} - (d^2\vec{v}.\vec{v}).\vec{w} + d\vec{v}.d\vec{v}.\vec{w} \end{aligned} \tag{9.50}$$

**Answer.**

$$\begin{aligned} \mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w}) &= \frac{D(\mathcal{L}_{\vec{v}}\vec{w})}{Dt} - d\vec{v}.(\mathcal{L}_{\vec{v}}\vec{w}) = \frac{D(\frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w})}{Dt} - d\vec{v}.(\frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}) \\ &= \frac{D^2\vec{w}}{Dt^2} - \frac{D(d\vec{v})}{Dt}.\vec{w} - d\vec{v}.\frac{D\vec{w}}{Dt} - d\vec{v}.\frac{D\vec{w}}{Dt} + d\vec{v}.d\vec{v}.\vec{w}, \end{aligned}$$

with (2.30)-(2.31)-(2.32). ▪

## 9.5 Lie derivative of a differential form

When the Lie derivative of a vector field $\vec{w}$ cannot be obtained by direct measurements, you need to use a "measuring device" (Germain: To know the weight of a suitcase you have to lift it: You use work).

Here the measuring device is a differential form $\alpha$. With $\vec{w}$ is a vector field $f = \alpha.\vec{v}$ is a scalar function, thus $\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) =^{(9.15)} \frac{D(\alpha.\vec{w})}{Dt} = \frac{D\alpha}{Dt}.\vec{w} + \alpha.\frac{D\vec{w}}{Dt}$, thus

$$\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) = \underbrace{\frac{D\alpha}{Dt}.\vec{w} + \alpha.d\vec{v}.\vec{w}}_{\to(\mathcal{L}_{\vec{v}}\alpha).\vec{w}} + \underbrace{\alpha.\frac{D\vec{w}}{Dt} - \alpha.d\vec{v}.\vec{w}}_{=\alpha.\mathcal{L}_{\vec{v}}\vec{w}} : \tag{9.51}$$

**Definition 9.13** The Lie derivative of a $C^1$ differential form $\alpha$ along $\vec{v}$ is the differential form $\mathcal{L}_{\vec{v}}\alpha$ defined by

$$\boxed{\mathcal{L}_{\vec{v}}\alpha := \frac{D\alpha}{Dt} + \alpha.d\vec{v}} = \frac{\partial\alpha}{\partial t} + d\alpha.\vec{v} + \alpha.d\vec{v}, \tag{9.52}$$

i.e., for all vector field $\vec{w}$,

$$\mathcal{L}_{\vec{v}}\alpha.\vec{w} := \frac{D\alpha}{Dt}.\vec{w} + \alpha.d\vec{v}.\vec{w} = \frac{\partial\alpha}{\partial t}.\vec{w} + (d\alpha.\vec{v}).\vec{w} + \alpha.d\vec{v}.\vec{w}. \tag{9.53}$$

This definition immediately gives (9.51) (i.e. (9.52) is a compatibility definition):

**Corollary 9.14** *$\mathcal{L}_{\vec{v}}$ satisfies the derivation property:*

$$\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) = (\mathcal{L}_{\vec{v}}\alpha).\vec{w} + \alpha.(\mathcal{L}_{\vec{v}}\vec{w}). \tag{9.54}$$

**Remark 9.15** Equivalent definitions: With (9.11), $g(\tau) = (\Phi^t_\tau)^*\alpha_\tau(p_t) = \alpha_\tau(p_\tau).d\Phi^t_\tau(p_t) = \alpha(\tau, p(\tau)).d\Phi^t(\tau, p_t)$ and

$$\mathcal{L}_{\vec{v}}\alpha(t, p_t) := g'(t) = \lim_{\tau\to t}\frac{g(\tau)-g(t)}{\tau - t} = \lim_{\tau\to t}\frac{(\Phi^t_\tau)^*\alpha_\tau(p_t) - \alpha_t(p_t)}{\tau - t} \overset{\text{written}}{=} \frac{d(\Phi^{t*}_\tau\alpha_\tau(p_t))}{d\tau}_{|\tau=t}. \tag{9.55}$$

Indeed $g(\tau) = \alpha(\tau, p(\tau)).d\Phi^t(\tau, p_t)$ gives $g'(\tau) = \frac{D\alpha}{D\tau}(\tau, p(\tau)).d\Phi^t_\tau(p_t) + \alpha(\tau, p(\tau)).d\vec{V}^t(\tau, p_t)$, hence $g'(t) = \frac{D\alpha}{Dt}(t, p(t)).I + \alpha(t, p(t)).d\vec{v}(t, p_t).I$, and (9.52) is recovered. ▪

**Exercice 9.16** Prove: If $f$ is $C^2$ (so $\alpha = df$ is exact and $C^1$), then

$$\mathcal{L}_{\vec{v}}(df) = \frac{\partial(df)}{\partial t} + d(df.\vec{v}), \tag{9.56}$$

i.e. $\mathcal{L}_{\vec{v}}(df).\vec{w} = \frac{\partial(df)}{\partial t}.\vec{w} + d(df.\vec{v}).\vec{w} = \frac{\partial(df)}{\partial t}.\vec{w} + (d(df).\vec{w}).\vec{v} + df.(d\vec{v}.\vec{w})$, for all $\vec{w}$.

**Answer**. $d(df) = d^2f$ is symmetric (Schwarz), $(d(df).\vec{w}).\vec{v} = (d(df).\vec{v}).\vec{w}$. Thus $\mathcal{L}_{\vec{v}}(df).\vec{w} =^{(9.53)} \frac{\partial(df)}{\partial t}.\vec{w} + (d(df).\vec{v}).\vec{w} + df.d\vec{v}.\vec{w} =^{Schwarz} \frac{\partial(df)}{\partial t}.\vec{w} + (d(df).\vec{w}).\vec{v} + df.(d\vec{v}.\vec{w}) = \frac{\partial(df)}{\partial t}.\vec{w} + d(df.\vec{v}).\vec{w}$. ▪

**Exercice 9.17** Prove: If $\alpha$ is $C^2$ then

$$\mathcal{L}_{\vec{v}}(d\alpha) \neq d(\mathcal{L}_{\vec{v}}\alpha) \quad \text{(no commutativity)}. \tag{9.57}$$

**Answer**. $d(\mathcal{L}_{\vec{v}}\alpha) = d(\frac{\partial\alpha}{\partial t} + d\alpha.\vec{v} + \alpha.d\vec{v}) = \frac{\partial(d\alpha)}{\partial t} + d^2\alpha.\vec{v} + \alpha.d\vec{v} + d\alpha.d\vec{v} + \alpha.d^2\vec{v}$.

And with a little advance see (9.71), $\mathcal{L}_{\vec{v}}d\alpha = \frac{\partial d\alpha}{\partial t} + d^2\alpha.\vec{v} + d\alpha.d\vec{v} + d\vec{v}^*.d\alpha$. ▪

**Quantification:** Relative to a basis $(\vec{e}_i)$ and with $[\cdot] := [\cdot]_{|\vec{e}}$,

$$[\mathcal{L}_{\vec{v}}\alpha] = [\frac{D\alpha}{Dt}] + [\alpha].[d\vec{v}] = [\frac{\partial\alpha}{\partial t}] + [d\alpha.\vec{v}] + [\alpha].[d\vec{v}] \quad \text{(row matrix)}. \tag{9.58}$$

Thus

$$[\mathcal{L}_{\vec{v}}\alpha.\vec{w}] = [\mathcal{L}_{\vec{v}}\alpha].[\vec{w}] = [\frac{\partial\alpha}{\partial t}].[\vec{w}] + [d\alpha.\vec{v}].[\vec{w}] + [\alpha].[d\vec{v}].[\vec{w}]. \tag{9.59}$$

**Exercice 9.18** Prove (9.58) with components. And prove $[d\alpha.\vec{v}] = [\vec{v}]^T.[d\alpha]^T$ (row matrix), thus $[d\alpha.\vec{v}].[\vec{w}] = [\vec{v}]^T.[d\alpha]^T.[\vec{w}] = [\vec{w}]^T.[d\alpha].[\vec{v}]$.

**Answer**. Basis $(\vec{e}_i)$, dual basis $(\pi_{ei})$, thus (9.52) gives $[\mathcal{L}_{\vec{v}}\alpha] = [\frac{D\alpha}{Dt}] + [\alpha.d\vec{v}]$. Let $\alpha = \sum_i \alpha_i\pi_{ei}$, $\vec{v} = \sum_i v_i\vec{e}_i$, $d\vec{v} = \sum_{ij} v_{i|j}\vec{e}_i\otimes\pi_{ej}$ (tensorial writing convenient for calculations), i.e. $[d\vec{v}]_{|\vec{e}} = [v_{i|j}]$, thus $\alpha.d\vec{v} = \sum_{ij}\alpha_i v_{i|j}\pi_{ej}$, thus $[\alpha.d\vec{v}]_{|\pi_e} = [\alpha]_{|\pi_e}.[d\vec{v}]_{|\vec{e}}$ (row matrix). And $d\alpha = \sum_{ij}\alpha_{i|j}\pi_{ei}\otimes\pi_{ej}$, i.e. $[d\alpha]_{|\pi_e} = [\alpha_{i|j}]$, gives $d\alpha.\vec{v} = \sum_{ij}\alpha_{i|j}v_j\pi_{ei} = \sum_{ij} v_i\alpha_{j|i}\pi_{ej}$, and $[d\alpha.\vec{v}]_{|\pi_e}$ is a row matrix ($d\alpha.\vec{v}$ is a differential form), thus $[d\alpha.\vec{v}]_{|\pi_e} = [\vec{v}]^T_{|\vec{e}}.[d\alpha]^T_{|\pi_e}$. (Or compute $(d\alpha.\vec{v}).\vec{w} = \sum_{ij}\alpha_{i|j}v_jw_i = [\vec{w}]^T_{|\vec{e}}.[d\alpha]_{|\vec{e}}.[\vec{v}]_{|\vec{e}} = [\vec{v}]^T_{|\vec{e}}.[d\alpha]^T_{|\pi_e}.[\vec{w}]_{|\vec{e}}$.) ∎

**Exercice 9.19** Let $\alpha$ be a differential form, and let $\alpha_t(p) := \alpha(t,p)$. Prove, when $\Phi_t^{t_0}$ is a diffeomorphism,

$$\mathcal{L}_{\vec{v}}\alpha = 0 \iff \forall t\in[t_0,T],\ \alpha_t = (\Phi_t^{t_0})_*\alpha_{t_0}. \tag{9.60}$$

I.e.: $\frac{D\alpha}{Dt} = -\alpha.d\vec{v} \iff \alpha_t(p_t) = \alpha_{t_0}(p_{t_0}).F_t^{t_0}(p_{t_0})^{-1}$ for all $t$, when $p_t = \Phi_t^{t_0}(p_{t_0})$.

**Answer**. $\Leftarrow$: If $\alpha_t(p(t)) = \alpha_{t_0}(p_{t_0}).F_t^{t_0}(p_{t_0})^{-1}$, then $\alpha(t,p(t)).F^{t_0}(t,p_{t_0}) = \alpha_{t_0}(p_{t_0})$, thus $\frac{D\alpha}{Dt}(t,p_t).F_t^{t_0}(p_{t_0}) + \alpha_t(p_t).\frac{\partial F^{t_0}}{\partial t}(t,p_{t_0}) = 0$, thus $\frac{D\alpha}{Dt}(t,p(t)).F_t^{t_0}(p_{t_0}) + \alpha_t(p_t).d\vec{v}(t,p_t).F_t^{t_0}(p_{t_0}) = 0$, thus $\mathcal{L}_{\vec{v}}\alpha = 0$, since $\Phi_t^{t_0}$ is a diffeomorphism.

$\Rightarrow$: If $\beta(t) := (\Phi_t^{t_0})_*\alpha_{t_0}(p_{t_0}) = \alpha_t(p(t)).F_t^{t_0}(p_{t_0})$ (pull-back at $(t_0,p_{t_0})$), then $\beta(t) = \alpha(t,p(t)).F^{t_0}(t,p_{t_0})$, thus $\beta'(t) = \frac{D\alpha}{Dt}(t,p_t).F_t^{t_0}(p_{t_0}) + \alpha(t,p_t).d\vec{v}(t,p_t).F_t^{t_0}(p_{t_0}) = 0$ (hypothesis $\mathcal{L}_{\vec{v}}\alpha = 0$), thus $\beta(t) = \beta(t_0) = \alpha_{t_0}(p_{t_0})$. ∎

**Exercice 9.20** $\vec{v}$ and $\alpha$ being $C^2$, prove:

$$\begin{aligned}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\alpha) = {}& \frac{\partial^2\alpha}{\partial t^2} + 2d\frac{\partial\alpha}{\partial t}.\vec{v} + 2\frac{\partial\alpha}{\partial t}.d\vec{v} + d\alpha.\frac{\partial\vec{v}}{\partial t} + \alpha.\frac{\partial d\vec{v}}{\partial t}\\ &+ (d^2\alpha.\vec{v}).\vec{v} + d\alpha.(d\vec{v}.\vec{v}) + 2(d\alpha.\vec{v}).d\vec{v} + \alpha.(d^2\vec{v}.\vec{v}) + (\alpha.d\vec{v}).d\vec{v}.\end{aligned} \tag{9.61}$$

**Answer**.

$$\begin{aligned}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\alpha) &= \mathcal{L}_{\vec{v}}(\frac{\partial\alpha}{\partial t}) + \mathcal{L}_{\vec{v}}(d\alpha.\vec{v}) + \mathcal{L}_{\vec{v}}(\alpha.d\vec{v})\\
&= \frac{\partial^2\alpha}{\partial t^2} + d\frac{\partial\alpha}{\partial t}.\vec{v} + \frac{\partial\alpha}{\partial t}.d\vec{v} + \frac{\partial(d\alpha.\vec{v})}{\partial t} + d(d\alpha.\vec{v}).\vec{v} + (d\alpha.\vec{v}).d\vec{v} + \frac{\partial(\alpha.d\vec{v})}{\partial t} + d(\alpha.d\vec{v}).\vec{v} + (\alpha.d\vec{v}).d\vec{v}\\
&= \frac{\partial^2\alpha}{\partial t^2} + d\frac{\partial\alpha}{\partial t}.\vec{v} + \frac{\partial\alpha}{\partial t}.d\vec{v} + \frac{\partial d\alpha}{\partial t}.\vec{v} + d\alpha.\frac{\partial\vec{v}}{\partial t} + (d^2\alpha.\vec{v}).\vec{v} + d\alpha.(d\vec{v}.\vec{v}) + (d\alpha.\vec{v}).d\vec{v}\\
&\quad + \frac{\partial\alpha}{\partial t}.d\vec{v} + \alpha.\frac{\partial d\vec{v}}{\partial t} + (d\alpha.\vec{v}).d\vec{v} + \alpha.d^2\vec{v}.\vec{v} + (\alpha.d\vec{v}).d\vec{v}\\
&= \frac{\partial^2\alpha}{\partial t^2} + 2d\frac{\partial\alpha}{\partial t}.\vec{v} + 2\frac{\partial\alpha}{\partial t}.d\vec{v} + d\alpha.\frac{\partial\vec{v}}{\partial t} + (d^2\alpha.\vec{v}).\vec{v} + d\alpha.(d\vec{v}.\vec{v}) + 2(d\alpha.\vec{v}).d\vec{v} + \alpha.\frac{\partial d\vec{v}}{\partial t}\\
&\quad + \alpha.(d^2\vec{v}.\vec{v}) + (\alpha.d\vec{v}).d\vec{v}.\end{aligned}$$

∎

## 9.6 Incompatibility with Riesz representation vectors

The Lie derivative has nothing to do with any inner dot product (the Lie derivative does not compare two vectors, contrary to a Cauchy type approach).

Here we introduce a Euclidean dot product $(\cdot,\cdot)_g$ and show that the Lie derivative of a linear form $\alpha$ is not trivially deduced from the Lie derivative of a Riesz representation vector of $\alpha$ (which one?). (Same issue as at § 7.2.)

Let $\alpha$ be a Eulerian differential form. So $\alpha(t,p)\in\mathbb{R}^{n*}$ (linear form); Call $\vec{a}_g(t,p)\in\vec{\mathbb{R}^n}$ its $(\cdot,\cdot)_g$-Riesz representation vector:

$$\forall\vec{w},\quad \alpha.\vec{w} = (\vec{a}_g,\vec{w})_g \quad (= \vec{a}_g \bullet_g \vec{w}), \tag{9.62}$$

(The Eulerian vector field $\vec{a}_g$ is not intrinsic to $\alpha$: $\vec{a}_g$ depends on the choice of $(\cdot,\cdot)_g$ cf. (F.12)).

**Proposition 9.21** *For all $\vec{v},\vec{w}\in\vec{\mathbb{R}^n}$,*

$$\frac{\partial\alpha}{\partial t}.\vec{w} = (\frac{\partial\vec{a}_g}{\partial t},\vec{w})_g,\quad (d\alpha.\vec{v}).\vec{w} = (d\vec{a}_g.\vec{v},\vec{w})_g,\quad \frac{D\alpha}{Dt}.\vec{w} = (\frac{D\vec{a}_g}{Dt},\vec{w})_g. \tag{9.63}$$

*Thus*

$$\mathcal{L}_{\vec{v}}\alpha.\vec{w} = (\mathcal{L}_{\vec{v}}\vec{a}_g,\vec{w})_g + (\vec{a}_g,(d\vec{v}+d\vec{v}^T).\vec{w})_g,\quad \textit{thus}\quad \boxed{\mathcal{L}_{\vec{v}}\alpha.\vec{w} \neq (\mathcal{L}_{\vec{v}}\vec{a}_g,\vec{w})_g}\quad \textit{in general.} \tag{9.64}$$

*So $\mathcal{L}_{\vec{v}}\vec{a}_g$ is **not** the Riesz representation vector of $\mathcal{L}_{\vec{v}}\alpha$ (but for solid body motions). (Expected: A Lie derivative is covariant objective, see § 11.4, and the use of an inner dot product ruins this objectivity.)*

**Proof.** A Euclidean dot product $g(\cdot,\cdot)$ is constant and uniform, thus $\alpha.\vec{w} = (\vec{a}_g, \vec{w})_g$ gives :

1- $\frac{\partial\alpha}{\partial t}.\vec{w}+\alpha.\frac{\partial\vec{w}}{\partial t} = (\frac{\partial\vec{a}_g}{\partial t}, \vec{w})_g+(\vec{a}_g, \frac{\partial\vec{w}}{\partial t})_g$, with $\alpha.\frac{\partial\vec{w}}{\partial t} = (\vec{a}_g, \frac{\partial\vec{w}}{\partial t})_g$, thus we are left with $\frac{\partial\alpha}{\partial t}.\vec{w} = (\frac{\partial\vec{a}_g}{\partial t}, \vec{w})_g$, for all $\vec{w}$. And

2- $d(\alpha.\vec{w}).\vec{v} = d(\vec{a}_g, \vec{w})_g.\vec{v}$ for all $\vec{v}, \vec{w}$, thus $(d\alpha.\vec{v}).\vec{w} + \alpha.(d\vec{w}.\vec{v}) = (d\vec{a}_g.\vec{v}, \vec{w})_g + (\vec{a}_g, d\vec{w}.\vec{v})_g$, with $\alpha.(d\vec{w}.\vec{v}) = (\vec{a}_g, d\vec{w}.\vec{v})_g$, thus we are left with $(d\alpha.\vec{v}).\vec{w} = (d\vec{a}_g.\vec{v}, \vec{w})_g$.

Thus $\frac{D\alpha}{Dt}.\vec{w} = (\frac{D\vec{a}_g}{Dt}, \vec{w})_g$.

Thus $(\mathcal{L}_{\vec{v}}\alpha).\vec{w} = \frac{D\alpha}{Dt}.\vec{w}+\alpha.d\vec{v}.\vec{w} = (\frac{D\vec{a}_g}{Dt}, \vec{w})_g+(\vec{a}_g, d\vec{v}.\vec{w})_g = (\frac{D\vec{a}_g}{Dt} - d\vec{v}.\vec{a}_g+d\vec{v}.\vec{a}_g, \vec{w})_g+(d\vec{v}^T.\vec{a}_g, \vec{w})_g = (\mathcal{L}_{\vec{v}}\vec{a}_g + d\vec{v}.\vec{a}_g, \vec{w})_g + (d\vec{v}_g^T.\vec{a}_g, \vec{w})_g$. ▪

**Remark 9.22** Chorus: a "differential form" (measuring instrument, covariant) should not be confused with a "vector field" (object to be measured, contravariant); Thus, the use of a dot product (which one?) and the Riesz representation theorem should be restricted for computational purposes, after an objective equation has been established. See also remark F.12. ▪

## 9.7 Lie derivative of a tensor

The Lie derivative of any tensor of order $\geq 2$ is defined thanks to

$$\mathcal{L}_{\vec{v}}(T\otimes S) = (\mathcal{L}_{\vec{v}}T)\otimes S + T\otimes(\mathcal{L}_{\vec{v}}S) \quad \text{(derivation formula).} \tag{9.65}$$

(Or direct definition: $\mathcal{L}_{\vec{v}}T(t_0, p_{t_0}) = \frac{D((\Phi_t^{t_0})^*T_t)(p_{t_0})}{Dt}_{|t=t_0}$).

### 9.7.1 Lie derivative of a mixed tensor

Let $T_m \in T_1^1(\Omega)$, and $T_m$ is called a mixed tensor; Its Lie derivative, called the Jaumann derivative, is

$$\boxed{\mathcal{L}_{\vec{v}}T_m = \frac{DT_m}{Dt} - d\vec{v}.T_m + T_m.d\vec{v}} = \frac{\partial T_m}{\partial t} + dT_m.\vec{v} - d\vec{v}.T_m + T_m.d\vec{v}. \tag{9.66}$$

Can be checked with an elementary tensor $T = \vec{w}\otimes\alpha$: we have $d(\vec{w}\otimes\alpha).\vec{v} = (d\vec{w}.\vec{v})\otimes\alpha + \vec{w}\otimes(d\alpha.\vec{v})$ and $(d\vec{v}.\vec{w})\otimes\alpha = d\vec{v}.(\vec{w}\otimes\alpha)$, and $\vec{w}\otimes(\alpha.d\vec{v}) = (\vec{w}\otimes\alpha).d\vec{v}$ , thus (9.65) gives $\mathcal{L}_{\vec{v}}(\vec{w}\otimes\alpha) = (\mathcal{L}_{\vec{v}}\vec{w})\otimes\alpha+\vec{w}\otimes(\mathcal{L}_{\vec{v}}\alpha)$
$= \frac{\partial\vec{w}}{\partial t}\otimes\alpha + (d\vec{w}.\vec{v})\otimes\alpha - (d\vec{v}.\vec{w})\otimes\alpha + \vec{w}\otimes\frac{\partial\alpha}{\partial t} + \vec{w}\otimes(d\alpha.\vec{v}) + \vec{w}\otimes(\alpha.d\vec{v})$
$= \frac{\partial\vec{w}\otimes\alpha}{\partial t} + d(\vec{w}\otimes\alpha).\vec{v} - d\vec{v}.(\vec{w}\otimes\alpha) + (\vec{w}\otimes\alpha).d\vec{v}$.

**Quantification.** Relative to a basis $(\vec{e}_i)$:

$$[\mathcal{L}_{\vec{v}}T_m] = [\frac{DT_m}{Dt}] - [d\vec{v}].[T_m] + [T_m].[d\vec{v}] = [\frac{\partial T_m}{\partial t}] + [dT_m].[\vec{v}] - [d\vec{v}].[T_m] + [T_m].[d\vec{v}]. \tag{9.67}$$

The signs $\mp$ are mixed because of the covariant and the contravariant constitution of $T_m$. "Mixed" also refers to the up and down positions of indices with duality notations: $T_m = \sum_{i,j=1}^n T^i{}_j\vec{e}_i\otimes e^j$.

**Exercice 9.23** Prove (9.67) with $T_m = \sum_{i,j=1}^n T^i{}_j\vec{e}_i\otimes e^j$.

**Answer.** $dT_m = \sum_{ijk} T^i{}_{j|k}\vec{e}_i\otimes e^j\otimes e^k$, $\vec{v} = \sum_i v^i\vec{e}_i$, $d\vec{v} = \sum_{ij} v^i_{|j}\vec{e}_i\otimes e^j$, thus $dT_m.\vec{v} = \sum_{ijk} T^i{}_{j|k}v^k\vec{e}_i\otimes e^j$, $d\vec{v}.T_m = \sum_{ijk} v^i_{|k}T^k{}_j\vec{e}_i\otimes e^j$, $T_m.d\vec{v} = \sum_{ijk} T^i{}_k v^k_{|j}\vec{e}_i\otimes e^j$. And $\frac{\partial T_m}{\partial t} = \sum_{ij}\frac{\partial T^i{}_j}{\partial t}\vec{e}_i\otimes e^j$,. ▪

### 9.7.2 Lie derivative of a up-tensor

If $L \in \mathcal{L}(E;F)$ (a linear map) then its adjoint $L^* \in \mathcal{L}(F^*;E^*)$ is defined by, cf. § A.13,

$$\forall m \in F^*, \quad \boxed{L^*.m := m.L}, \quad \text{i.e.,} \quad \forall m, \vec{u} \in (F^*\times E), \quad (L^*.m).\vec{u} = m.L.\vec{u}. \tag{9.68}$$

(There is no inner dot product involved here.) In particular, $d\vec{v}^*.m := m.d\vec{v}$.

Let $T_u \in T_0^2(\Omega)$, and $T_u$ is called a up tensor; Its Lie derivative, called the upper-convected (Maxwell) derivative or the Oldroyd derivative, is

$$\boxed{\mathcal{L}_{\vec{v}}T_u = \frac{DT_u}{Dt} - d\vec{v}.T_u - T_u.d\vec{v}^*} = \frac{\partial T_u}{\partial t} + dT_u.\vec{v} - d\vec{v}.T_u - T_u.d\vec{v}^*. \tag{9.69}$$

Can be checked with an elementary tensor $T = \vec{u}\otimes\vec{w}$ and $\mathcal{L}_{\vec{v}}(\vec{u}\otimes\vec{w}) = (\mathcal{L}_{\vec{v}}\vec{u})\otimes\vec{w} + \vec{u}\otimes(\mathcal{L}_{\vec{v}}\vec{w})$.

**Quantification.** Relative to a basis $(\vec{e}_i)$:

$$[\mathcal{L}_{\vec{v}}T_u] = [\frac{DT_u}{Dt}] - [d\vec{v}].[T_u] - [T_u].[d\vec{v}]^T. \tag{9.70}$$

"up" refers to the up positions of indices with duality notations: $T_u = \sum_{i,j=1}^n T^{ij}\vec{e}_i \otimes \vec{e}_j$.

**Exercice 9.24** With components, prove (9.69).

**Answer**. $\frac{\partial T_u}{\partial t} = \sum_{ij} \frac{\partial T^{ij}}{\partial t}\vec{e}_i\otimes\vec{e}_j$, $dT_u = \sum_{ijk} T^{ij}_{|k}\vec{e}_i\otimes\vec{e}_j\otimes e^k$, $\vec{v} = \sum_i v^i\vec{e}_i$, $d\vec{v} = \sum_{ij} v^i_{|j}\vec{e}_i\otimes e^j$, $d\vec{v}^* = \sum_{ij} v^j_{|i}e^i\otimes\vec{e}_j$, thus $dT_u.\vec{v} = \sum_{ijk} T^{ij}_{|k}v^k\vec{e}_i \otimes e^j$, $d\vec{v}.T_u = \sum_{ijk} v^i_{|k}T^{kj}\vec{e}_i \otimes \vec{e}_j$, $T_u.d\vec{v}^* = \sum_{ijk} T^{ik}v^j_{|k}e^i \otimes \vec{e}_j$. ▪

### 9.7.3 Lie derivative of a down-tensor

Let $T_d \in T^0_2(\Omega)$, and $T_d$ is called a down tensor; its Lie derivative, called the lower-convected Maxwell derivative, is

$$\boxed{\mathcal{L}_{\vec{v}}T_d = \frac{DT_d}{Dt} + T_d.d\vec{v} + d\vec{v}^*.T_d} = \frac{\partial T_d}{\partial t} + dT_d.\vec{v} + T_d.d\vec{v} + d\vec{v}^*.T_d. \tag{9.71}$$

Can be checked with an elementary tensor $T = \ell \otimes m$ and $\mathcal{L}_{\vec{v}}(\ell \otimes m) = (\mathcal{L}_{\vec{v}}\ell) \otimes m + \ell \otimes (\mathcal{L}_{\vec{v}}m)$.

**Quantification.** Relative to a basis $(\vec{e}_i)$:

$$[\mathcal{L}_{\vec{v}}T_d] = [\frac{DT_d}{Dt}] + [T_d].[d\vec{v}] + [d\vec{v}]^T.[T_d]. \tag{9.72}$$

"down" refers to the down positions of indices with duality notations: $T_d = \sum_{i,j=1}^n T_{ij}e^i \otimes e^j$.

**Exercice 9.25** With components, prove (9.72).

**Answer**. $\frac{\partial T_d}{\partial t} = \sum_{ij} \frac{\partial T_{ij}}{\partial t}e^i\otimes e^j$, $dT_d = \sum_{ijk} T_{ij|k}e^i\otimes e^j\otimes e^k$, $\vec{v} = \sum_i v^i\vec{e}_i$, $d\vec{v} = \sum_{ij} v^i_{|j}\vec{e}_i\otimes e^j$, $d\vec{v}^* = \sum_{ij} v^j_{|i}e^i\otimes\vec{e}_j$, thus $dT_d.\vec{v} = \sum_{ijk} T_{ij|k}v^k e^i \otimes e^j$, $T_d.d\vec{v} = \sum_{ijk} T_{ik}v^k_{|j}e^i \otimes \vec{e}_j$, $d\vec{v}^*.T_d = \sum_{ijk} v^k_{|i}T_{kj}e^i \otimes \vec{e}_j$. ▪

**Example 9.26** Let $g = (\cdot,\cdot)_g \in T^0_2(\Omega)$ be a constant and uniform metric (e.g. a unique Euclidean dot product at all $t$). Then $\frac{Dg}{Dt} = 0$, thus $\mathcal{L}_{\vec{v}}g = 0 + g.d\vec{v} + d\vec{v}^*.g$, thus $[\mathcal{L}_{\vec{v}}g] = [g].[d\vec{v}] + [d\vec{v}]^T.[g]$. ▪

# Part IV
# Velocity-addition formula

## 10 Change of referential and velocity-addition formula

$f(t,x)$ will be written $f_t(x)$ when $t$ is fixed. $\mathcal{M}_{n1}$ is the space of $n*1$ matrices (column matrices).

### 10.0 Issue and result (summary)

#### 10.0.1 Issue

**Issue:** The velocity-addition formula is usually written (classical mechanics)

$$\vec{v}_A = \vec{v}_D + \vec{v}_B, \quad \text{i.e.} \quad \text{absolute velocity} = (\text{drive+relative}) \text{ velocities}, \tag{10.1}$$

$\vec{v}_A$ and $\vec{v}_D$ being measured by an observer A in his referential $\mathcal{R}_A = (O_A, (\vec{A}_i))$ and $\vec{v}_B$ being measured by an observer B in his referential $\mathcal{R}_B = (O_B, (\vec{B}_i))$. This "obvious" relation (10.1) is problematic (inconsistent) in general, e.g. it caused the crash of the Mars climate orbiter probe. E.g.

- $\vec{v}_A$ and $\vec{v}_D$ are given relative to the basis $(\vec{A}_i)$, e.g. in foot/s, chosen by the "absolute" observer,
- $\vec{v}_B$ is given relative to an another basis $(\vec{B}_i)$, e.g. in metre/s, chosen by the "relative" observer;

Thus, in (10.1), $\vec{v}_B + \vec{v}_D$ adds metre/s and foot/s... relative to different bases..., Absurd. (If you prefer, $\vec{v}_A - \vec{v}_D = \vec{v}_B$ with $\vec{v}_A - \vec{v}_D$ and $\vec{v}_B$ given in two different referentials.)

Issue: An explicit link is missing between $\mathcal{R}_A$ and $\mathcal{R}_B$ (the "obvious" implicit relation).

#### 10.0.2 Summary: Absolute and relative motion...

An object $Obj$ is made of particles $P_{Obj}$. Its motion in "our classic affine Universe", independent of the observers, is

$$\widetilde{\Phi} : (t, P_{Obj}) \in [t_1, t_2] \times Obj \to \text{position } p_t = p(t) = \widetilde{\Phi}(t, P_{Obj}) \in \mathbb{R}^n. \tag{10.2}$$

At $t$ at $p_t = \widetilde{\Phi}(t, P_{Obj})$, the (Eulerian) velocities and accelerations are

$$\vec{v}(t,p_t) = \frac{\partial \widetilde{\Phi}}{\partial t}(t, P_{Obj}) \quad \text{and} \quad \vec{\gamma}(t,p_t) = \frac{\partial^2 \widetilde{\Phi}}{\partial t^2}(t, P_{Obj}) \ \in \vec{\mathbb{R}^n}. \tag{10.3}$$

Two observers A and B quantify the motion in their referentials:

$$\begin{aligned} &\text{Absolute motion: } \vec{\varphi}_A : (t, P_{Obj}) \to \vec{x}_A(t) = \vec{\varphi}_A(t, P_{Obj}) := [\overrightarrow{O_A \widetilde{\Phi}(t, P_{Obj})}]_{|\vec{A}} = \vec{x}_{At} \ \in \mathcal{M}_{n1}, \\ &\text{Relative motion: } \vec{\varphi}_B : (t, P_{Obj}) \to \vec{x}_B(t) = \vec{\varphi}_B(t, P_{Obj}) := [\overrightarrow{O_B \widetilde{\Phi}(t, P_{Obj})}]_{|\vec{B}} = \vec{x}_{Bt} \ \in \mathcal{M}_{n1}, \end{aligned} \tag{10.4}$$

where $\vec{x}_A(t) = \vec{x}_{At} = \begin{pmatrix} x_{A1t} \\ \vdots \\ x_{Ant} \end{pmatrix}$ and $\vec{x}_B(t) = \vec{x}_{Bt} = \begin{pmatrix} x_{B1t} \\ \vdots \\ x_{Bnt} \end{pmatrix}$ are the column matrices in $\mathcal{M}_{n1}$ defined by $\overrightarrow{O_A \widetilde{\Phi}(t, P_{Obj})} = \sum_{i=1}^n x_{Ait} \vec{A}_i$ (for A) and $\overrightarrow{O_B \widetilde{\Phi}(t, P_{Obj})} = \sum_{i=1}^n x_{Bit} \vec{B}_i$ (for B). The absolute and relative velocities and accelerations are (Eulerian type matrices)

$$\begin{aligned} \vec{v}_A(t, \vec{x}_{At}) &:= \frac{\partial \vec{\varphi}_A}{\partial t}(t, P_{Obj}) = [\vec{v}(t,p_t)]_{|\vec{A}}, \quad \vec{\gamma}_A(t, \vec{x}_{At}) := \frac{\partial^2 \vec{\varphi}_A}{\partial t^2}(t, P_{Obj}) = [\vec{\gamma}(t,p_t)]_{|\vec{A}}, \\ \vec{v}_B(t, \vec{x}_{Bt}) &:= \frac{\partial \vec{\varphi}_B}{\partial t}(t, P_{Obj}) = [\vec{v}(t,p_t)]_{|\vec{B}}, \quad \vec{\gamma}_B(t, \vec{x}_{Bt}) := \frac{\partial^2 \vec{\varphi}_B}{\partial t^2}(t, P_{Obj}) = [\vec{\gamma}(t,p_t)]_{|\vec{B}}. \end{aligned} \tag{10.5}$$

#### 10.0.3 ... The translator $\Theta$ and the "good" velocity addition formula...

At $t$ the translator $\Theta_t : \mathcal{M}_{n1} \to \mathcal{M}_{n1}$ links the quantified positions by A and B:

$$\vec{x}_{At} = \Theta_t(\vec{x}_{Bt}) \quad \text{when} \quad \vec{x}_{At} = \vec{\varphi}_A(t, P_{Obj}) \quad \text{and} \quad \vec{x}_{Bt} = \vec{\varphi}_B(t, P_{Obj}). \tag{10.6}$$

Which defines $\Theta : [t_1, t_2] \times \mathcal{M}_{n1} \to \mathcal{M}_{n1}$ by $\Theta(t, \vec{x}_B(t)) = \vec{x}_A(t)$, i.e.

$$\boxed{\vec{\varphi}_A(t, P_{Obj}) = \Theta(t, \vec{\varphi}_B(t, P_{Obj}))}. \tag{10.7}$$

Thus

$$\underbrace{\frac{\partial\vec{\varphi}_A}{\partial t}(t,P_{Obj})}_{\text{absolute velocity } \vec{v}_A(t,\vec{x}_{At})} = \underbrace{\frac{\partial\Theta}{\partial t}(t,\vec{\varphi}_B(t,P_{Obj}))}_{\text{drive velocity } \vec{v}_D(t,\vec{x}_{At})} + \underbrace{d\Theta(t,\vec{\varphi}_B(t,P_{Obj})).\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj})}_{\text{relative velocity translated for } A:\ \vec{v}_{B*}(t,\vec{x}_{At})}. \tag{10.8}$$

This is "the good velocity-addition formula $\vec{v}_A = \vec{v}_D + \vec{v}_{B*}$ to be used by A":

$$\boxed{\vec{v}_A(t,\vec{x}_{At}) = \vec{v}_{Dt}(t,\vec{x}_{At}) + \vec{v}_{B*}(t,\vec{x}_{At})}, \quad \text{where} \quad \vec{v}_{B*}(t,\vec{x}_{At}) = d\Theta(t,\vec{x}_{Bt}).\vec{v}_B(t,\vec{x}_{Bt}), \tag{10.9}$$

i.e. Absolute velocity = Drive velocity + Translated relative velocity.

### 10.0.4 ... and the "good" acceleration formula

(10.8) gives,

$$\begin{aligned}\vec{\gamma}_A(t,\vec{x}_{At}) = {} & \frac{\partial^2\Theta}{\partial t^2}(t,\vec{x}_{Bt}) + d(\frac{\partial\Theta}{\partial t})(t,\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) \\ & + (\frac{\partial(d\Theta)}{\partial t}(t,\vec{x}_{Bt}) + d^2\Theta(t,\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})).\vec{v}_{Bt}(\vec{x}_{Bt}) + d\Theta(t,\vec{x}_{Bt}).\vec{\gamma}_{Bt}(\vec{x}_{Bt})\end{aligned} \tag{10.10}$$

which is "the good acceleration-addition formula to be used by A": At $t$ at $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$,

$$\boxed{\vec{\gamma}_{At} = \vec{\gamma}_{Dt} + \vec{\gamma}_{Ct} + \vec{\gamma}_{Bt*}}, \quad \text{i.e.} \tag{10.11}$$

$$\text{Absolute acceleration} = (\text{Drive} + \text{Coriolis} + \text{Translated relative}) \text{ accelerations}, \tag{10.12}$$

where

$$\begin{cases} \vec{\gamma}_{Dt}(\vec{x}_{At}) := \dfrac{\partial^2\Theta}{\partial t^2}(t,\vec{x}_{Bt}) = \text{drive acceleration}, \\ \vec{\gamma}_{Bt*}(\vec{x}_{At}) := d\Theta_t(\vec{x}_{Bt}).\vec{\gamma}_B(\vec{x}_{Bt}) = \text{relative acceleration translated for } A, \\ \vec{\gamma}_{Ct}(\vec{x}_{At}) := 2d\dfrac{\partial\Theta}{\partial t}(t,\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) + d^2\Theta_t(\vec{x}_{Bt})(\vec{v}_{Bt}(\vec{x}_{Bt}),\vec{v}_{Bt}(\vec{x}_{Bt})) = \text{Coriolis acceleration} \end{cases} \tag{10.13}$$

(in fact called the Coriolis acceleration when $\mathcal{R}_A$ is Galilean). And $\frac{\partial\Theta}{\partial t}(t,\vec{x}_{Bt}) = \vec{v}_D(t,\Theta_t(\vec{x}_{Bt}))$ gives $d\frac{\partial\Theta}{\partial t}(t,\vec{x}_{Bt}) = d\vec{v}_D(t,\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt})$. In classical mechanics: $\Theta_t$ is affine thus $d^2\Theta_t = 0$ and $d\frac{\partial\Theta}{\partial t}(t) = d\vec{v}_{Dt}.d\Theta_t$ and

$$\text{Coriolis acceleration:} \ \boxed{\vec{\gamma}_{Ct}(\vec{x}_{At}) := 2\, d\vec{v}_{Dt}.\vec{v}_{Bt*}(\vec{x}_{At})}. \tag{10.14}$$

## 10.1 Absolute and relative referentials

Classical mechanics: Time and space are decoupled, observers A and B use the same time unit and origin (to simplify the notations).

• The "absolute" observer A chooses at $t$ four positions $O_{At}, p_{A1t}, p_{A2t}, p_{A3t}$ in the Universe $\mathbb{R}^n$ s.t. the bi-point vectors $\vec{A}_{it} := \overrightarrow{O_{At}p_{Ait}}$ make a Euclidean basis in $\mathbb{R}^3$. He has built at $t$ his absolute referential $\mathcal{R}_{At} = (O_{At},(\vec{A}_{it}))$. And A is "static in his referential", so he writes $\mathcal{R}_{At} = \mathcal{R}_A = (O_A,(\vec{A}_i))$.

E.g., at all $t$, $O_{At}$ is the position of the center of the Sun in the Universe, $(\vec{A}_{it}) = (\overrightarrow{O_{At}p_{Ait}})$ is a Euclidean basis fixed relative to stars and built with the foot.

• The "relative" observer B chooses at $t$ four positions $O_{Bt}, p_{B1t}, p_{B2t}, p_{B3t}$ in the Universe $\mathbb{R}^n$ s.t. the bi-point vectors $\vec{B}_{it} := \overrightarrow{O_{Bt}p_{Bit}}$ make a Euclidean basis in $\mathbb{R}^3$. He has built at $t$ his relative referential $\mathcal{R}_{Bt} = (O_{Bt},(\vec{B}_{it}))$. And $\mathcal{R}_{Bt}$ is seen as a "rigid object extended to infinity". And B is "static in his referential", so he writes $\mathcal{R}_{Bt} = \mathcal{R}_B = (O_B,(\vec{B}_i))$

E.g., at all $t$, $O_{Bt}$ is the position of the center of the Earth in the Universe, $(\vec{B}_{it}) = (\overrightarrow{O_{Bt}p_{Bit}})$ is a Euclidean basis fixed relative to the Earth and built with the metre.

• $\mathcal{M}_{n1}$ is the vector space of $n*1$ real column matrices. $\vec{E}_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$, ..., $\vec{E}_n = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$ make its canonical basis $(\vec{E}_i)$. So $[\vec{A}_{it}]_{|\vec{A}} = [\vec{A}_i]_{|\vec{A}} = \vec{E}_i = [\vec{B}_i]_{|\vec{B}} = [\vec{B}_{it}]_{|\vec{B}}$ in $\mathcal{M}_{31}$.

## 10.2 Motions of *Obj* and $\mathcal{R}_B$ in our classical Universe

The motions are $C^2$, *Obj* is an object made of particles $P_{Obj}$, and $\mathcal{R}_B$ is (assimilated to) an object made of particles $Q_{\mathcal{R}_B}$. Their motions in the Universe are

$$\widetilde{\Phi}:\begin{cases}[t_1,t_2]\times Obj & \to \mathbb{R}^n\\ (t,P_{Obj}) & \to p(t)=\widetilde{\Phi}(t,P_{Obj})=p_t=\text{position at } t \text{ of the particle } P_{Obj},\end{cases}\tag{10.15}$$

$$\widetilde{\Phi}_{\mathcal{R}_B}:\begin{cases}[t_1,t_2]\times \mathcal{R}_B & \to \mathbb{R}^n\\ (t,Q_{\mathcal{R}_B}) & \to q(t)=\widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})=q_t=\text{position at } t \text{ of the particle } Q_{\mathcal{R}_B}.\end{cases}\tag{10.16}$$

The associated Eulerian velocities and accelerations $[t_1,t_2]\times\mathbb{R}^n\to\vec{\mathbb{R}^n}$ are given by

$$\vec{v}(t,p_t)=\frac{\partial\widetilde{\Phi}}{\partial t}(t,P_{Obj})\quad\text{and}\quad\vec{\gamma}(t,p_t)=\frac{\partial^2\widetilde{\Phi}}{\partial t^2}(t,P_{Obj}),\tag{10.17}$$

$$\vec{v}_{\mathcal{R}_B}(t,q_t)=\frac{\partial\widetilde{\Phi}_{\mathcal{R}_B}}{\partial t}(t,Q_{\mathcal{R}_B})\quad\text{and}\quad\vec{\gamma}_{\mathcal{R}_B}(t,q_t)=\frac{\partial^2\widetilde{\Phi}_{\mathcal{R}_B}}{\partial t^2}(t,Q_{\mathcal{R}_B}).\tag{10.18}$$

## 10.3 Absolute and relative motions

### 10.3.1 Absolute and relative "motions" of *Obj*

(10.15) stored by A and B gives the "absolute" and "relative" motions of *Obj* (matrix valued)

$$\vec{\varphi}_A:\begin{cases}[t_1,t_2]\times Obj & \to \mathcal{M}_{n1}\\ (t,P_{Obj}) & \to \vec{x}_A(t)=\boxed{\vec{\varphi}_A(t,P_{Obj}):=[\overrightarrow{O_A\widetilde{\Phi}(t,P_{Obj})}]_{|\vec{A}}}=\vec{x}_{At},\end{cases}\tag{10.19}$$

$$\vec{\varphi}_B:\begin{cases}[t_1,t_2]\times Obj & \to \mathcal{M}_{n1}\\ (t,P_{Obj}) & \to \vec{x}_B(t)=\boxed{\vec{\varphi}_B(t,P_{Obj}):=[\overrightarrow{O_B\widetilde{\Phi}(t,P_{Obj})}]_{|\vec{B}}}=\vec{x}_{Bt}.\end{cases}\tag{10.20}$$

The associated Eulerian "absolute" and "relative" velocities and accelerations $[t_1,t_2]\times\mathcal{M}_{n1}\to\mathcal{M}_{n1}$ are given by

$$\vec{v}_A(t,\vec{x}_{At}):=\frac{\partial\vec{\varphi}_A}{\partial t}(t,P_{Obj})\quad\text{and}\quad\vec{\gamma}_A(t,\vec{x}_{At}):=\frac{\partial^2\vec{\varphi}_A}{\partial t^2}(t,P_{Obj}),\tag{10.21}$$

$$\vec{v}_B(t,\vec{x}_{Bt}):=\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj})\quad\text{and}\quad\vec{\gamma}_B(t,\vec{x}_{Bt}):=\frac{\partial^2\vec{\varphi}_B}{\partial t^2}(t,P_{Obj}).\tag{10.22}$$

**Exercice 10.1** Prove: $\vec{v}_A(t,\vec{x}_{At})=[\vec{v}(t,p_t)]_{|\vec{A}}$ and $\vec{v}_B(t,\vec{x}_{Bt})=[\vec{v}(t,p_t)]_{|\vec{B}}$.
And $\vec{\gamma}_A(t,\vec{x}_{At})=[\vec{\gamma}(t,p_t)]_{|\vec{A}}$ and $\vec{\gamma}_B(t,\vec{x}_{Bt})=[\vec{\gamma}(t,p_t)]_{|\vec{B}}$.

**Answer**. $\vec{v}(t,p_t)=\frac{\partial\widetilde{\Phi}}{\partial t}(t,P_{Obj})=\lim_{h\to0}\frac{\widetilde{\Phi}(t+h,P_{Obj})-\widetilde{\Phi}(t,P_{Obj})}{h}=\lim_{h\to0}\frac{\overrightarrow{\widetilde{\Phi}(t,P_{Obj})\widetilde{\Phi}(t+h,P_{Obj})}}{h}$.
And $\vec{\varphi}_A(t,P_{Obj})=[\overrightarrow{O_A\widetilde{\Phi}(t,P_{Obj})}]_{|\vec{A}}$ gives $\frac{\partial\vec{\varphi}_A}{\partial t}(t,P_{Obj})=\lim_{h\to0}\frac{[\overrightarrow{O_A\widetilde{\Phi}(t+h,P_{Obj})}]_{|\vec{A}}-[\overrightarrow{O_A\widetilde{\Phi}(t,P_{Obj})}]_{|\vec{A}}}{h}=\lim_{h\to0}[\frac{\overrightarrow{\widetilde{\Phi}(t,P_{Obj})\widetilde{\Phi}(t+h,P_{Obj})}}{h}]_{|\vec{A}}=[\frac{\partial\widetilde{\Phi}}{\partial t}(t,P_{Obj})]_{|\vec{A}}=[\vec{v}(t,p_t)]_{|\vec{A}}$ as wanted. Idem for $\vec{\gamma}_A$ and B. ∴

**Exercice 10.2** $t$ is fixed, $p\in\mathbb{R}^n$ (point), $\vec{x}_A:=[\overrightarrow{O_Ap}]_{|\vec{A}}\in\mathcal{M}_{n1}$, $\vec{u}\in C^1(\mathbb{R}^n;\vec{\mathbb{R}^n})$ (vector field), and $\vec{u}_A(\vec{x}_A):=[\vec{u}(p)]_{|\vec{A}}$. Prove: $[d\vec{u}(p)]_{|\vec{A}}=d\vec{u}_A(\vec{x}_A)$, i.e. $d\vec{u}_A(\vec{x}_A).[\vec{w}]_{|\vec{A}}=[d\vec{u}(p)]_{|\vec{A}}.[\vec{w}]_{|\vec{A}}$ for all $\vec{w}\in\vec{\mathbb{R}^n}$.

**Answer**. $p+h\vec{w}\in\mathbb{R}^n$ is stored by A as $[\overrightarrow{O_Ap}+h\vec{w}]_{|\vec{A}}=[\overrightarrow{O_Ap}]_{|\vec{A}}+h[\vec{w}]_{|\vec{A}}=\vec{x}_A+h[\vec{w}]_{|\vec{A}}\in\mathcal{M}_{n1}$. Thus $d\vec{u}_A(\vec{x}_A).[\vec{w}]_{|\vec{A}}=\lim_{h\to0}\frac{\vec{u}_A(\vec{x}_A+h[\vec{w}]_{|\vec{A}})-\vec{u}_A(\vec{x}_A)}{h}=\lim_{h\to0}\frac{\vec{u}_A([\overrightarrow{O_Ap}+h\vec{w}]_{|\vec{A}})-\vec{u}_A([\overrightarrow{O_Ap}]_{|\vec{A}})}{h}=\lim_{h\to0}\frac{[\vec{u}(p+h\vec{w})]_{|\vec{A}}-[\vec{u}(p)]_{|\vec{A}}}{h}=[\lim_{h\to0}\frac{\vec{u}(p+h\vec{w})-\vec{w}(p)}{h}]_{|\vec{A}}=[d\vec{u}(p).\vec{w}]_{|\vec{A}}=d\vec{u}(p)]_{|\vec{A}}.[\vec{w}]_{|\vec{A}}$, true for all $\vec{w}$. ∴

**Exercice 10.3** Call $Q_t$ the transition matrix from $(\vec{A}_{it})$ to $(\vec{B}_{it})$ at $t$. Prove $\vec{x}_{At}=[\overrightarrow{O_AO_{Bt}}]_{|\vec{A}}+Q_t.\vec{x}_{Bt}$.

**Answer**. The $\vec{A}_{it}$ and $\vec{B}_{it}$ are bipoint vectors in the same vector space $\vec{\mathbb{R}^3}$ cf. § 10.1, so it makes sense to speak of a transition matrix. Change of basis formula: $[\overrightarrow{O_{Bt}p_t}]_{|\vec{A}}=Q_t.[\overrightarrow{O_{Bt}p_t}]_{|\vec{B}}$, thus $\vec{x}_{At}=[\overrightarrow{O_Ap_t}]_{|\vec{A}}=[\overrightarrow{O_AO_{Bt}}+\overrightarrow{O_{Bt}p_t}]_{|\vec{A}}=[\overrightarrow{O_AO_{Bt}}]_{|\vec{A}}+[\overrightarrow{O_{Bt}p_t}]_{|\vec{A}}=[\overrightarrow{O_AO_{Bt}}]_{|\vec{A}}+Q_t.[\overrightarrow{O_{Bt}p_t}]_{|\vec{B}}=[\overrightarrow{O_AO_{Bt}}]_{|\vec{A}}+Q_t.\vec{x}_{Bt}$. ∴

### 10.3.2 Drive and static "motions" of $\mathcal{R}_B$

(10.16) stored by A and B gives the "absolute" and "relative" motions of $\mathcal{R}_B$ which are called the "drive" and the "static" motions (matrix valued):

$$\vec\varphi_D : \begin{cases} [t_1,t_2]\times\mathcal{R}_B & \to \mathcal{M}_{n1} \\ (t,Q_{\mathcal{R}_B}) & \to \vec y_D(t) = \boxed{\vec\varphi_D(t,Q_{\mathcal{R}_B}) := [\overrightarrow{O_A\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec A}} = \vec y_{Dt}, \end{cases} \tag{10.23}$$

$$\vec\varphi_S : \begin{cases} \mathcal{R}_B & \to \mathcal{M}_{n1} \\ Q_{\mathcal{R}_B} & \to \vec y_S = \boxed{\vec\varphi_S(Q_{\mathcal{R}_B}) := [\overrightarrow{O_B\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec B}} = \vec y_S \end{cases} \tag{10.24}$$

($\vec\varphi_S$ is independent of $t$ since $Q_{\mathcal{R}_B}$ is fixed in $\mathcal{R}_B$), and the associated Eulerian "drive" and "static" velocities and accelerations $[t_1,t_2]\times\mathcal{M}_{n1}\to\mathcal{M}_{n1}$:

$$\vec v_D(t,\vec y_{Dt}) = \frac{\partial\vec\varphi_D}{\partial t}(t,Q_{\mathcal{R}_B}) \quad\text{and}\quad \vec\gamma_D(t,\vec y_{Dt}) = \frac{\partial^2\vec\varphi_D}{\partial t^2}(t,Q_{\mathcal{R}_B}), \tag{10.25}$$

$$\vec v_S(t,\vec y_S) = \vec 0 \quad\text{and}\quad \vec\gamma_S(t,\vec y_S) = \vec 0. \tag{10.26}$$

**Exercice 10.4** Why introduce $\vec\varphi_S$ (static)?

**Answer**. You can't confuse a particle $Q_{\mathcal{R}_B}$ with its stored position $\vec y_S = \vec\varphi_S(Q_{\mathcal{R}_B}) = [\overrightarrow{O_B\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec B}$ = the matrix stored by B. In particular the stored position by A at $t$ is $\vec y_{Dt} = [\overrightarrow{O_A\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec A} \neq \vec y_S$ in general. ▪

## 10.4 The translator $\Theta_t$

### 10.4.1 Definition

**Definition 10.5** At $t$, $\Theta_t : \mathcal{M}_{n1}\to\mathcal{M}_{n1}$ is the inter-referential function from $\mathcal{R}_B$ to $\mathcal{R}_A$ which translates (which links) the positions stored by B into the corresponding positions stored by A: For all $Q_{\mathcal{R}_B}\in\mathcal{R}_B$,

$$\vec\varphi_{Dt}(Q_{\mathcal{R}_B}) = \Theta_t(\vec\varphi_S(Q_{\mathcal{R}_B})), \quad\text{i.e.}\quad \vec y_{Dt} = \Theta_t(\vec y_S), \tag{10.27}$$

i.e. $[\overrightarrow{O_A\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec A} = \Theta_t([\overrightarrow{O_B\widetilde\Phi_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec B})$. So

$$\boxed{\vec\varphi_{Dt} = \Theta_t\circ\vec\varphi_S}, \tag{10.28}$$

i.e.

$$\Theta_t := \vec\varphi_{Dt}\circ\vec\varphi_S^{-1} : \begin{cases} \mathcal{M}_{n1} & \to \mathcal{M}_{n1} \\ \vec y_S & \to \vec y_{Dt} = \Theta_t(\vec y_S) := \vec\varphi_{Dt}(\vec\varphi_S^{-1}(\vec y_S)). \end{cases} \tag{10.29}$$

In other words, at $t$, the translator $\Theta_t$ is defined such that the following diagram commutes:

$$\begin{tikzcd} & \vec y_S = \vec\varphi_S(Q_{\mathcal{R}_B}) = \text{the stored position of } Q_{\mathcal{R}_B} \text{ at } t \text{ in } \mathcal{R}_B \arrow[d, "\Theta_t"] \\ Q_{\mathcal{R}_B}\in\mathcal{R}_B \arrow[ur, "\vec\varphi_S"] \arrow[dr, "\vec\varphi_{Dt}"'] & \\ & \vec y_{Dt} = \vec\varphi_{Dt}(Q_{\mathcal{R}_B}) = \Theta_t(\vec y_S) = \text{the stored position of } Q_{\mathcal{R}_B} \text{ at } t \text{ in } \mathcal{R}_A. \end{tikzcd} \tag{10.30}$$

E.g., if $Q_{O_B}$ is the particle in $\mathcal{R}_B$ which is at $t$ at $O_{Bt}$ the origin chosen by B, i.e. s.t. $O_{Bt} = \widetilde\Phi_{\mathcal{R}_B}(t,Q_{O_B})$, then

$$\vec y_{Dt} = [\overrightarrow{O_AO_{Bt}}]_{|\vec A} = \vec\varphi_{Dt}(Q_{O_B}) = \Theta_t(\vec 0) = \text{ the position of } Q_{O_B} \text{ stored by } A \text{ at } t. \tag{10.31}$$

E.g., for a particle $P_{Obj}\in Obj$ which is at $t$ at $p_t = \widetilde\Phi(t,P_{Obj})$: With $\vec x_{At} = [\overrightarrow{O_Ap_t}]_{|\vec A}$ and $\vec x_{Bt} = [\overrightarrow{O_Bp_t}]_{|\vec B}$ (positions as stored by A and B), we have $\vec x_{At} = \Theta_t(\vec x_{Bt})$, i.e. $\vec\varphi_{At}(P_{Obj}) = \Theta_t(\vec\varphi_{Bt}(P_{Obj}))$. Thus

$$\boxed{\vec\varphi_{At} = \Theta_t\circ\vec\varphi_{Bt}}. \tag{10.32}$$

In other words: If $Q_{\mathcal{R}_B}\in\mathcal{R}_B$ is the particle in $\mathcal{R}_B$ which is at $t$ at $q_t = p_t$, then $\vec x_{At} = \vec\varphi_{Dt}(Q_{\mathcal{R}_B}) = \vec y_{Dt}$ and $\vec x_{Bt} = \vec\varphi_S(Q_{\mathcal{R}_B}) = \vec y_S$, and $\vec y_{Dt} =^{(10.27)} \Theta_t(\vec y_S)$ gives $\vec x_{At} = \Theta_t(\vec x_{Bt})$.

### 10.4.2 $\Theta_t$ is affine in classical mechanics

Classical mechanics: Each observer can choose the same "time independent Euclidean basis at all points".

**Proposition 10.6** *$\Theta_t$ is affine at all $t$, i.e., for all $\vec{y}_{S0}, \vec{y}_{S1} \in \mathcal{M}_{n1}$ and all $u \in \mathbb{R}$,*

$$\Theta_t((1-u)\vec{y}_{S0}+u\,\vec{y}_{S1}) = (1-u)\Theta_t(\vec{y}_{S0}) + u\,\Theta_t(\vec{y}_{S1}), \tag{10.33}$$

*i.e.*

$$\Theta_t(\vec{y}_{S0}+u\,(\vec{y}_{S1}-\vec{y}_{S0})) = \Theta_t(\vec{y}_{S0}) + u\,(\Theta_t(\vec{y}_{S1})-\Theta_t(\vec{y}_{S0})). \tag{10.34}$$

*I.e.*

$$d\Theta_t(\vec{y}_{S0}) = d\Theta_t(\vec{0}) \overset{written}{=} d\Theta_t \text{ is independent of } \vec{y}_{S0}. \tag{10.35}$$

*So, with $\vec{y}_{Dt0} = \Theta_t(\vec{y}_{S0})$ and $\vec{y}_{Dt1} = \Theta_t(\vec{y}_{S1})$,*

$$\vec{y}_{Dt1} = \vec{y}_{Dt0} + d\Theta_t.(\vec{y}_{S1}-\vec{y}_{S0}). \tag{10.36}$$

*I.e., at $t$, for all positions $q_{0t}, q_{1t} \in \mathbb{R}^3$ (our affine Universe),*

$$[\overrightarrow{q_{0t}q_{1t}}]_{|\vec{A}} = d\Theta_t.[\overrightarrow{q_{0t}q_{1t}}]_{|\vec{B}}. \tag{10.37}$$

**Proof.** Consider two particles $Q_{B0}, Q_{B1} \in \mathcal{R}_B$. Their positions at $t$ are $q_{0t} = \widetilde{\Phi}_{\mathcal{R}_{Bt}}(Q_{B0})$, $q_{1t} = \widetilde{\Phi}_{\mathcal{R}_{Bt}}(Q_{B1})$ in $\mathbb{R}^n$. Consider the straight line

$$q_t(u) = q_{0t} + u\,\overrightarrow{q_{0t}q_{1t}}. \tag{10.38}$$

For all $p \in \mathbb{R}^3$, $\overrightarrow{pq_t(u)} = \overrightarrow{pq_{0t}} + u\,\overrightarrow{q_{0t}q_{1t}} = \overrightarrow{pq_{0t}} + u\,\overrightarrow{pq_{1t}} - u\,\overrightarrow{pq_{0t}}$, thus

$$\left\{\begin{array}{l} [\overrightarrow{O_Aq_t(u)}]_{|\vec{A}} = (1-u)[\overrightarrow{O_Aq_{0t}}]_{|\vec{A}} + u\,[\overrightarrow{O_Aq_{1t}}]_{|\vec{A}}, \\ [\overrightarrow{O_Bq_t(u)}]_{|\vec{B}} = (1-u)[\overrightarrow{O_Bq_{0t}}]_{|\vec{B}} + u\,[\overrightarrow{O_Bq_{1t}}]_{|\vec{B}}, \end{array}\right\} \quad \text{with} \quad [\overrightarrow{O_Aq_t(u)}]_{|\vec{A}} \overset{(10.27)}{=} \Theta_t([\overrightarrow{O_Bq_t(u)}]_{|\vec{B}}), \tag{10.39}$$

thus

$$(1-u)\Theta_t([\overrightarrow{O_Bq_{0t}}]_{|\vec{B}}) + u\,\Theta_t([\overrightarrow{O_Bq_{1t}}]_{|\vec{B}}) = \Theta_t((1-u)[\overrightarrow{O_Bq_{0t}}]_{|\vec{B}} + u\,[\overrightarrow{O_Bq_{1t}}]_{|\vec{B}}), \tag{10.40}$$

thus $(1-u)\Theta_t(\vec{y}_{S0}) + u\,\Theta_t(\vec{y}_{S1}) = \Theta_t((1-u)\vec{y}_{S0} + u\,\vec{y}_{S1})$ for all $\vec{y}_{S0}, \vec{y}_{S1} \in \mathcal{M}_{n1}$, thus $\Theta_t$ is affine. Thus (10.36), thus $\vec{y}_{Dt1} - \vec{y}_{Dt0} = d\Theta_t.(\vec{y}_{S1}-\vec{y}_{S0})$ when $\vec{y}_{Dt0} = \Theta_t(\vec{y}_{S0})$ and $\vec{y}_{Dt1} = \Theta_t(\vec{y}_{S1})$, thus (10.37) with $\vec{y}_{S0} = [\overrightarrow{O_Bq_{0t}}]_{|\vec{B}}$ and $\vec{y}_{S1} = [\overrightarrow{O_Bq_{1t}}]_{|\vec{B}}$ which give $\vec{y}_{Dt0} = [\overrightarrow{O_Aq_{0t}}]_{|\vec{A}}$ and $\vec{y}_{Dt1} = [\overrightarrow{O_Aq_{1t}}]_{|\vec{A}}$. ∎

### 10.4.3 The differential $d\Theta_t$: Push-forward

**Definition 10.7** Let $\vec{y}_{Dt} = \Theta_t(\vec{y}_S) \in \mathcal{M}_{n1}$ and $\vec{w}_S(\vec{y}_S)$ a vector at $\vec{y}_S$ in $\mathcal{M}_{n1}$. The push-forward $\vec{w}_{St*}(\vec{y}_{Dt})$ of $\vec{w}_S(\vec{y}_S)$ by $\Theta_t$ is

$$\vec{w}_{St*}(\vec{y}_{Dt}) := d\Theta_t(\vec{y}_S).\vec{w}_S(\vec{y}_S), \quad \text{i.e.} \quad \vec{w}_{St*}([\overrightarrow{O_Aq_t}]_{|\vec{A}}) = d\Theta_t([\overrightarrow{O_Bq_t}]_{|\vec{B}}).\vec{w}_S([\overrightarrow{O_Bq_t}]_{|\vec{B}}) \tag{10.41}$$

for all $q_t \in \mathbb{R}^3$. (Recall: $d\Theta_t(\vec{y}_S).\vec{w}_S(\vec{y}_S) := \lim_{h\to 0} \frac{\Theta_t(\vec{y}_S+h\vec{w}_S(\vec{y}_S))-\Theta_t(\vec{y}_S)}{h}$ ).

In particular when $\Theta_t$ is affine:

$$\boxed{\vec{w}_{St*}(\vec{y}_{Dt}) = d\Theta_t.\vec{w}_S(\vec{y}_S)}, \quad \text{i.e.} \quad \vec{w}_{St*}([\overrightarrow{O_Aq_t}]_{|\vec{A}}) = d\Theta_t.\vec{w}_S([\overrightarrow{O_Bq_t}]_{|\vec{B}}). \tag{10.42}$$

### 10.4.4 Translated velocities for A

The translated relative velocities and accelerations at $t$ are the push-forwards of $\vec{v}_{Bt}$ and $\vec{\gamma}_{Bt}$ by $\Theta_t$:

$$\vec{v}_{Bt*}(\vec{x}_{At}) := d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) \quad \text{and} \quad \vec{\gamma}_{Bt*}(\vec{x}_{At}) := d\Theta_t(\vec{x}_{Bt}).\vec{\gamma}_{Bt}(\vec{x}_{Bt}) \tag{10.43}$$

when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$. In particular when $\Theta_t$ is affine:

$$\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t.\vec{v}_{Bt}(\vec{x}_{Bt}) \quad \text{and} \quad \vec{\gamma}_{Bt*}(\vec{x}_{At}) = d\Theta_t.\vec{\gamma}_{Bt}(\vec{x}_{Bt}). \tag{10.44}$$

#### 10.4.5 Translated basis for A: $[\vec{B}_{jt}]_{|\vec{A}} = d\Theta_t.[\vec{B}_j]_{|\vec{B}} = d\Theta_t.[\vec{A}_j]_{|\vec{A}}$

$$\vec{B}_{jt} = \overrightarrow{O_{Bt}p_{Bjt}} \text{ is stored } \left\{ \begin{array}{l} \text{by A as: } [\vec{B}_{jt}]_{|\vec{A}} = [\overrightarrow{O_{Bt}p_{Bjt}}]_{|\vec{A}}, \\ \text{by B as: } [\vec{B}_j]_{|\vec{B}} = [\overrightarrow{O_B p_{Bjt}}]_{|\vec{B}}. \end{array} \right\} \text{ Thus}$$

$$[\vec{B}_{jt}]_{|\vec{A}} \overset{(10.37)}{=} d\Theta_t(\vec{0}).[\vec{B}_{jt}]_{|\vec{B}} = \text{ gives “the basis } (\vec{B}_i) \text{ of B as stored by A at } t\text{”}. \tag{10.45}$$

(With the push-forward notation: $[\vec{B}_{jt}]_{\vec{A}} = ([\vec{B}_j]_{|B})_{t*}.)$ And $[\vec{B}_j]_{\vec{B}} = [\vec{A}_j]_{\vec{A}}$ (= $\vec{E}_j$ the $j$-th canonical basis vector in $\mathcal{M}_{n1}$), thus

$$[\vec{B}_{jt}]_{\vec{A}} = d\Theta_t(\vec{0}).[\vec{A}_j]_{|\vec{A}} : \text{ so } d\Theta_t(\vec{0}) \text{ is the transition matrix from } (\vec{A}_i) \text{ to } (\vec{B}_{it}) \text{ for A}, \tag{10.46}$$

i.e. the $j$-th column of $d\Theta_t(\vec{0})$ stores the components of $\vec{B}_{jt}$ in the basis $(\vec{A}_i)$ for A.

#### 10.4.6 $d\Theta_t{}^T.d\Theta_t = \lambda^2 I$

Recall: $(\cdot,\cdot)_A = \lambda^2(\cdot,\cdot)_B$ (Euclidean framework). $\lambda^2\delta_{ij} = \lambda^2(\vec{B}_{it},\vec{B}_{jt})_B = (\vec{B}_{it},\vec{B}_{jt})_A = [\vec{B}_{it}]^T_{|\vec{A}}.[\vec{B}_{jt}]_{|\vec{A}} =^{(10.45)} [\vec{B}_i]^T_{|\vec{B}}.d\Theta_t{}^T.d\Theta_t[\vec{B}_j]_{|\vec{B}}$, thus

$$d\Theta_t{}^T.d\Theta_t = \lambda^2 I, \quad \text{i.e.} \quad d\Theta_t{}^{-1} = \frac{1}{\lambda^2} d\Theta_t{}^T. \tag{10.47}$$

## 10.5 Definition of Θ

### 10.5.1 Definition

**Definition 10.8** The translator from B to A is the function $\Theta : [t_1,t_2]\times\mathcal{M}_{n1} \to \mathcal{M}_{n1}$ defined with (10.29) by $\Theta(t,\vec{y}_S) := \Theta_t(\vec{y}_S)$, i.e., for all $Q_{\mathcal{R}_B} \in \mathcal{R}_B$ and all $t$,

$$\Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B})) = \vec{\varphi}_D(t,Q_{\mathcal{R}_B}), \quad \text{i.e.} \quad \Theta(t,\vec{y}_S) = \vec{y}_D(t) \tag{10.48}$$

when $\vec{y}_S = [\overrightarrow{O_B\widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec{B}} = \vec{\varphi}_S(Q_{\mathcal{R}_B})$ and $\vec{y}_D(t) = [\overrightarrow{O_A\widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}]_{|\vec{A}} = \vec{\varphi}_D(t,Q_{\mathcal{R}_B})$.

E.g., $\Theta(t,\vec{0}) = [\overrightarrow{O_A O_B(t)}]_{|\vec{A}}$ gives at any $t$ the components of $O_B(t) := O_{Bt}$ stored by A.

**Remark 10.9** The translator Θ looks like a motion, but is not: A motion gives at $t$ the position of one particle in one referential; While Θ connects two referentials: It connects at $t$ the stored “matrix positions” of one particle by two observers: It is an “inter-referential” function. (It is a motion in $\mathcal{M}_{n1}$ but not a motion of physical particles.) ∎

### 10.5.2 The “Θ-velocity” = the drive velocity

**Definition 10.10** The “Θ-velocity” and “Θ-acceleration” $\vec{v}_\Theta$, $\vec{\gamma}_\Theta : [t_1,t_2]\times\mathcal{M}_{n1} \to \mathcal{M}_{n1}$ are defined by

$$\vec{v}_\Theta(t,\Theta(t,\vec{y}_S)) := \frac{\partial\Theta}{\partial t}(t,\vec{y}_S) \quad \text{and} \quad \vec{\gamma}_\Theta(t,\Theta(t,\vec{y}_S)) = \frac{\partial^2\Theta}{\partial t^2}(t,\vec{y}_S) \tag{10.49}$$

(Eulerian type), i.e. $\vec{v}_\Theta(t,\vec{y}_D(t)) := \frac{\partial\Theta}{\partial t}(t,\vec{y}_S)$ and $\vec{\gamma}_\Theta(t,\vec{y}_D(t)) = \frac{\partial^2\Theta}{\partial t^2}(t,\vec{y}_S)$ when $\vec{y}_D(t) = \Theta(t,\vec{y}_S)$.

**Proposition 10.11**

$$\begin{aligned} &\bullet\ \vec{v}_D(t,\vec{y}_{Dt}) = \vec{v}_\Theta(t,\vec{y}_{Dt}), \quad \text{so} \quad \boxed{\vec{v}_\Theta = \vec{v}_D}, \\ &\bullet\ \vec{\gamma}_D(t,\vec{y}_{Dt}) = \vec{\gamma}_\Theta(t,\vec{y}_{Dt}), \quad \text{so} \quad \boxed{\vec{\gamma}_\Theta = \vec{\gamma}_D}. \end{aligned} \tag{10.50}$$

**Proof.** $\vec{\varphi}_D(t,Q_{\mathcal{R}_B}) =^{(10.48)} \Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B}))$ gives $\frac{\partial\vec{\varphi}_D}{\partial t}(t,Q_{\mathcal{R}_B}) = \frac{\partial\Theta}{\partial t}(t,\vec{\varphi}_S(Q_{\mathcal{R}_B}))$, i.e. $\vec{v}_D(t,\vec{\varphi}_D(t,Q_{\mathcal{R}_B})) = \vec{v}_\Theta(t,\Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B})))$, i.e. $(10.50)_1$ when $\vec{y}_D(t) = \vec{\varphi}_D(t,Q_{\mathcal{R}_B}) = \Theta(t,\vec{\varphi}_S(Q_{\mathcal{R}_B}))$. Idem with $\frac{\partial^2}{\partial t^2}$. ∎

### 10.5.3 $d\frac{\partial\Theta}{\partial t}$ versus $d\vec{v}_D$

$\frac{\partial\Theta}{\partial t}(t,\vec{y}_S) =^{(10.50)} \vec{v}_{Dt}(\Theta_t(\vec{y}_S))$ gives $d\frac{\partial\Theta}{\partial t}(t,\vec{y}_S) = d\vec{v}_{Dt}(\vec{y}_{Dt}).d\Theta_t(\vec{y}_S)$ when $\vec{y}_{Dt} = \Theta_t(\vec{y}_S)$. And $\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})$ (translated velocity), thus

$$d\frac{\partial\Theta}{\partial t}(t,\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) = d\vec{v}_{Dt}(\vec{x}_{At}).\vec{v}_{Bt*}(\vec{x}_{At}) \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}). \tag{10.51}$$

In particular when $\Theta_t$ is affine, then $d\Theta_t(\vec{y}_S) = d\Theta_t$ and $d(\frac{\partial\Theta}{\partial t})(t,\vec{y}_S) = \frac{\partial(d\Theta)}{\partial t}(t)$ is independent of $\vec{y}_S$, thus $d\vec{v}_{Dt}(\vec{y}_{Dt}) =^{(10.51)} d\frac{\partial\Theta}{\partial t}(t).d\Theta_t{}^{-1}$ is independent of $\vec{y}_{Dt}$, thus

$$d\vec{v}_{Dt}(\vec{y}_{Dt}) = d\vec{v}_{Dt} \quad \text{when} \quad \Theta_t \text{ is affine,} \tag{10.52}$$

then, when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$,

$$d\frac{\partial\Theta}{\partial t}(t).\vec{v}_{Bt}(\vec{x}_{Bt}) = d\vec{v}_{Dt}.\vec{v}_{Bt*}(\vec{x}_{At}). \tag{10.53}$$

## 10.6 The velocity-addition formula

(10.32) gives

$$\vec{\varphi}_A(t,P_{Obj}) = \Theta(t,\vec{\varphi}_B(t,P_{Obj})). \tag{10.54}$$

Thus

$$\underbrace{\frac{\partial\vec{\varphi}_A}{\partial t}(t,P_{Obj})}_{\vec{v}_{At}(\vec{x}_{At})} = \underbrace{\frac{\partial\Theta}{\partial t}(t,\vec{\varphi}_B(t,P_{Obj}))}_{\overset{(10.50)}{=}\vec{v}_{Dt}(\vec{x}_{At})} + \underbrace{d\Theta(t,\vec{\varphi}_B(t,P_{Obj})).\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj})}_{= d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) \overset{(10.44)}{=} \vec{v}_{Bt*}(\vec{x}_{At})}, \tag{10.55}$$

i.e. $\vec{v}_{At}(\vec{x}_{At}) = \vec{v}_{Dt}(\vec{x}_{At}) + \vec{v}_{Bt*}(\vec{x}_{At})$,i.e.

$$\boxed{\vec{v}_{At} = \vec{v}_{Dt} + \vec{v}_{Bt*}} = \text{the velocity-addition formula for A,}$$
$$\text{i.e.:}\quad \text{absolute velocity} = (\text{drive} + \text{translated relative}) \text{ velocities.} \tag{10.56}$$

In particular when $\Theta_t$ is affine: For all $p_t$,

$$\boxed{[\vec{v}_t(p_t)]_{|\vec{A}} = [\vec{v}_{\mathcal{R}_{Bt}}(p_t)]_{|\vec{A}} + d\Theta_t.[\vec{v}_t(p_t)]_{\vec{B}}}. \tag{10.57}$$

## 10.7 Coriolis acceleration, and the acceleration-addition formula

(10.55) gives, when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$,

$$\begin{aligned}\underbrace{\frac{\partial^2\vec{\varphi}_A}{\partial t^2}(t,P_{Obj})}_{\vec{\gamma}_{At}(\vec{x}_{At})} = &\underbrace{\frac{\partial^2\Theta}{\partial t^2}(t,\vec{x}_{Bt})}_{\vec{\gamma}_{Dt}(\vec{x}_{At})} + d\frac{\partial\Theta}{\partial t}(t,\vec{x}_{Bt}).\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj})\\ &+ \Big(\frac{\partial(d\Theta)}{\partial t}(t,\vec{x}_{Bt}) + d^2\Theta_t(\vec{x}_{Bt}).\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj})\Big).\frac{\partial\vec{\varphi}_B}{\partial t}(t,P_{Obj}) + \underbrace{d\Theta_t(\vec{x}_{Bt}).\frac{\partial^2\vec{\varphi}_B}{\partial t^2}(t,P_{Obj})}_{\vec{\gamma}_{Bt*}(\vec{x}_{At})}.\end{aligned} \tag{10.58}$$

Thus, with the Coriolis acceleration at $t$ at $\vec{x}_{At}$ defined by

$$\vec{\gamma}_{Ct}(\vec{x}_{At}) := 2\,d\vec{v}_{Dt}(\vec{x}_{At}).\vec{v}_{Bt*}(\vec{x}_{At}) + d^2\Theta_t(\vec{x}_{Bt})(\vec{v}_{Bt}(\vec{x}_{Bt}),\vec{v}_{Bt}(\vec{x}_{Bt})), \tag{10.59}$$

we get

$$\vec{\gamma}_{At}(\vec{x}_{At}) = \vec{\gamma}_{Dt}(\vec{x}_{At}) + \vec{\gamma}_{Ct}(\vec{x}_{At}) + \vec{\gamma}_{Bt*}(\vec{x}_{At}), \tag{10.60}$$

i.e.

$$\boxed{\vec{\gamma}_{At} = \vec{\gamma}_{Dt} + \vec{\gamma}_{Ct} + \vec{\gamma}_{Bt*}} = \text{the acceleration-addition formula in } \mathcal{R}_A : \tag{10.61}$$

$$\text{absolute acceleration} = (\text{drive} + \text{Coriolis} + \text{translated relative}) \text{ accelerations.} \tag{10.62}$$

Particular case $\Theta_t$ affine: $d^2\Theta_t = 0$ and $d\vec{v}_{Dt}(\vec{x}_{At}) =^{(10.52)} d\vec{v}_{Dt}$, thus at $t$,

$$\vec{\gamma}_{Ct}(\vec{x}_{At}) = 2\,d\vec{v}_{Dt}.\vec{v}_{Bt*}(\vec{x}_{At}), \quad \text{and} \quad \boxed{\vec{\gamma}_{Ct} = 2\,d\vec{v}_{Dt}.\vec{v}_{Bt*}}. \tag{10.63}$$

## 10.8 With an initial time (Lagrangian variables)

Let $t_0, t \in \mathbb{R}$. Consider the Lagrangian associated function $\Phi_t^{t_0}$ with the motion $\widetilde{\Phi}$ of $Obj$:

$$\Phi_t^{t_0} : \left\{ \begin{array}{rl} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj}) & \to p_t = \Phi_t^{t_0}(p_{t_0}) := \widetilde{\Phi}(t, P_{Obj}). \end{array} \right. \tag{10.64}$$

And, with $\vec{x}_{At} = \vec{\varphi}_A(t, P_{Obj}) = [\overrightarrow{O_A p_t}]_{|\vec{A}}$ and $\vec{x}_{Bt} = \vec{\varphi}_B(t, P_{Obj}) = [\overrightarrow{O_B p_t}]_{|\vec{B}}$, define the "matrix motions" $\Phi_{At}^{t_0} : \mathcal{M}_{n1} \to \mathcal{M}_{n1}$ and $\Phi_{Bt}^{t_0} : \mathcal{M}_{n1} \to \mathcal{M}_{n1}$ by

$$\left\{ \begin{array}{l} \Phi_{At}^{t_0}(\vec{x}_{At_0}) := \vec{x}_{At} \quad (= [\overrightarrow{O_A \widetilde{\Phi}(t, P_{Obj})}]_{|\vec{A}} = [\overrightarrow{O_A \Phi_t^{t_0}(p_{t_0})}]_{|\vec{A}} = \vec{\varphi}_{At}(P_{Obj})), \\ \Phi_{Bt}^{t_0}(\vec{x}_{Bt_0}) := \vec{x}_{Bt} \quad (= [\overrightarrow{O_B \widetilde{\Phi}(t, P_{Obj})}]_{|\vec{B}} = [\overrightarrow{O_B \Phi_t^{t_0}(p_{t_0})}]_{|\vec{B}} = \vec{\varphi}_{Bt}(P_{Obj})). \end{array} \right. \tag{10.65}$$

And $\Theta_t(\vec{x}_{Bt}) = \vec{x}_{At}$, i.e. $\Theta_t(\Phi_{Bt}^{t_0}(\vec{x}_{Bt_0})) = \Phi_{At}^{t_0}(\vec{x}_{At_0})$ with $\vec{x}_{At_0} = \Theta_{t_0}(\vec{x}_{Bt_0})$, thus

$$\boxed{\Theta_t \circ \Phi_{Bt}^{t_0} = \Phi_{At}^{t_0} \circ \Theta_{t_0}} : \mathcal{M}_{n1} \to \mathcal{M}_{n1}. \tag{10.66}$$

In other words, the following diagram commutes:

$$\begin{tikzcd} & \vec{x}_{Bt_0} = \vec{\varphi}_B(t_0, P_{Obj}) \arrow[r, "\Phi_{Bt}^{t_0}"'] \arrow[d, "\Theta_{t_0}"'] & \vec{x}_{Bt} = \Phi_{Bt}^{t_0}(\vec{x}_{Bt_0}) \arrow[d, "\Theta_t"'] \\ P_{Obj} \in Obj \arrow[ur, "\vec{\varphi}_{Bt_0}"] \arrow[dr, "\vec{\varphi}_{At_0}"'] & & \\ & \vec{x}_{At_0} = \vec{\varphi}_A(t_0, P_{Obj}) = \Theta_{t_0}(\vec{x}_{Bt_0}) \arrow[r, "\Phi_{At}^{t_0}"] & \vec{x}_{At} = \Phi_{At}^{t_0}(\vec{x}_{At_0}) = \Theta_t(\vec{x}_{Bt}). \end{tikzcd} \tag{10.67}$$

Thus, for any vector field $\vec{u}_{Bt_0}$ in $\mathcal{R}_B$,

$$\underbrace{d\Theta_t(\vec{x}_{Bt})}_{\text{(translation at t)}} . \underbrace{d\Phi_{Bt}^{t_0}(\vec{x}_{Bt_0}).\vec{u}_{Bt_0}(\vec{x}_{Bt_0})}_{\text{(deformation from } t_0 \text{ to t)}} = \underbrace{d\Phi_{At}^{t_0}(\vec{x}_{At_0})}_{\text{(deformation from } t_0 \text{ to t)}} . \underbrace{d\Theta_{t_0}(\vec{x}_{Bt_0}).\vec{u}_{Bt_0}(\vec{x}_{Bt_0})}_{\text{(translation at } t_0\text{)}} . \tag{10.68}$$

**Exercice 10.12** Redo the above steps with $\mathcal{R}_B$ instead of $Obj$.

**Answer**. Consider the Lagrangian associated function $\Phi_{RBt}^{t_0}$ with the motion $\widetilde{\Phi}_{\mathcal{R}_B}$ of $\mathcal{R}_B$:

$$\Phi_{RBt}^{t_0} : \left\{ \begin{array}{rl} \Omega_{RBt_0} = \mathbb{R}^n & \to \Omega_{RBt} = \mathbb{R}^n \\ q_{t_0} = \widetilde{\Phi}_{\mathcal{R}_B}(t_0, Q_{\mathcal{R}_B}) & \to q_t = \Phi_{RBt}^{t_0}(q_{t_0}) := \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B}), \end{array} \right\} \tag{10.69}$$

then define the "matrix motions" $\Phi_{Dt}^{t_0} : \mathcal{M}_{n1} \to \mathcal{M}_{n1}$ and $\Phi_{St}^{t_0} : \mathcal{M}_{n1} \to \mathcal{M}_{n1}$ by

$$\left\{ \begin{array}{l} \Phi_{Dt}^{t_0}(\vec{y}_{Dt0}) := \vec{y}_{Dt} \quad (= [\overrightarrow{O_A \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}]_{|\vec{A}} = [\overrightarrow{O_A \Phi_{RBt}^{t_0}(p_{t_0})}]_{|\vec{A}} = \vec{\varphi}_{Dt}(Q_{\mathcal{R}_B})), \\ \Phi_{St}^{t_0}(\vec{y}_S) := \vec{y}_S \quad (= [\overrightarrow{O_B \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})}]_{|\vec{B}} = [\overrightarrow{O_B \Phi_{RBt}^{t_0}(q_{t_0})}]_{|\vec{B}} = \vec{\varphi}_S(Q_{\mathcal{R}_B})), \end{array} \right. \tag{10.70}$$

Thus $\vec{\varphi}_S$ is a time-shift, which is also abusively noted $\Phi_{St}^{t_0} = I$ (algebraic identity). So with $\Theta_t(\vec{y}_S) = \vec{y}_{Dt}$ we get $\Theta_t(\Phi_{Dt}^{t_0}(\vec{y}_S)) = \Phi_{Dt}^{t_0}(\vec{y}_{Dt0})$, with $\vec{y}_{Dt0} = \Theta_{t_0}(\vec{y}_S)$, thus

$$\boxed{\Theta_t \circ \Phi_{St}^{t_0} = \Phi_{Dt}^{t_0} \circ \Theta_{t_0}} : \mathcal{M}_{n1} \to \mathcal{M}_{n1} \tag{10.71}$$

(also abusively written $\Theta_t = \Phi_{Dt}^{t_0} \circ \Theta_{t_0}$). In other words, the following diagram commutes:

$$\begin{tikzcd} & \vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B}) \arrow[r, "\Phi_{St}^{t_0} = \text{time shift}"'] \arrow[d, "\Theta_{t_0}"] & \vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B}) \arrow[d, "\Theta_t"'] \\ Q_{\mathcal{R}_B} \in \mathcal{R}_B \arrow[ur, "\vec{\varphi}_S"] \arrow[dr, "\Phi_D^{t_0}"] & & \\ & \vec{y}_{Dt0} = \vec{\varphi}_{Dt_0}(Q_{\mathcal{R}_B}) = \Theta_{t_0}(\vec{y}_S) \arrow[r, "\Phi_{Dt}^{t_0}"] & \vec{y}_{Dt} = \vec{\varphi}_{Dt}(Q_{\mathcal{R}_B}) = \Phi_{Dt}^{t_0}(\vec{y}_{Dt0}) = \Theta_t(\vec{y}_S). \end{tikzcd} \tag{10.72}$$

And (10.71) gives, for any $\vec{y}_S = \vec{\varphi}_S(Q_{\mathcal{R}_B})$ and all vector field $\vec{u}_S$ (static in $\mathcal{R}_B$), with $\vec{y}_{Dt0} = \Theta_{t_0}(\vec{y}_S)$,

$$\underbrace{d\Theta_t(\vec{y}_S)}_{\text{(translation at t)}} . \underbrace{d\Phi_{St}^{t_0}(\vec{y}_S).\vec{u}_S(\vec{y}_S)}_{\text{(time shift from } t_0 \text{ to t)}} = \underbrace{d\Phi_{Dt}^{t_0}(\vec{y}_{Dt0})}_{\text{(Drive motion from } t_0 \text{ to t)}} . \underbrace{d\Theta_{t_0}(\vec{y}_S).\vec{u}_S(\vec{y}_S)}_{\text{(translation at } t_0\text{)}} . \tag{10.73}$$

## 10.9 Drive and Coriolis forces

### 10.9.1 Newton's fundamental principle: requires a Galilean referential

Second Newton's law of motion (fundamental principle of dynamics): In a Galilean referential, the sum $\vec{f}$ of the external forces is equal to the mass multiplied by the acceleration:

$$\vec{f} = m\vec{\gamma} \quad \text{(in a Galilean referential).} \tag{10.74}$$

Question: And in a Non Galilean referential?
Answer: You have to add "apparent forces" due to the motion of the non Galilean observer, because the motion of an object in our Universe does not care about the observer's motion.

### 10.9.2 Drive + Coriolis forces = the fictitious (inertial) force

The "absolute observer A" chooses a Galilean referential $\mathcal{R}_A$; Newton laws (10.74) is quantified by A at $t$ ss, with $p_t = \widetilde{\Phi}(t, P_{Obj}) \in \mathbb{R}^n$ the position in the Universe of a particle $P_{Obj}$,

$$[\vec{f}_t(p_t)]_{|\vec{A}} = m\,[\vec{\gamma}_t(p_t)]_{|\vec{A}}, \quad \text{written} \quad \boxed{\vec{f}_{At}(\vec{x}_{At}) = m\,\vec{\gamma}_{At}(\vec{x}_{At})} \quad (\in \mathcal{M}_{n1}) \tag{10.75}$$

when $\vec{x}_{At} := [\overrightarrow{O_A p_t}]_{|\vec{A}}$, $\vec{f}_{At}(\vec{x}_{At}) := [\vec{f}_t(p_t)]_{|\vec{A}}$ and $\vec{\gamma}_{At}(\vec{x}_{At}) = [\vec{\gamma}_t(p_t)]_{|\vec{A}}$.

For the "relative observer B" and is referential $\mathcal{R}_B$, with $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, the acceleration addition formula (10.61) gives $\vec{f}_{At}(\vec{x}_{At}) = m\,d\Theta_t.\vec{\gamma}_B(\vec{x}_{Bt}) + m\vec{\gamma}_{Dt}(\vec{x}_{At}) + m\vec{\gamma}_{Ct}(\vec{x}_{At})$ for A, thus for B:

$$\underbrace{d\Theta_t{}^{-1}.\vec{f}_{At}(\vec{x}_{At})}_{\vec{f}_{Bt}(\vec{x}_{Bt})} = m\,\vec{\gamma}_B(\vec{x}_{Bt}) + \underbrace{m\,d\Theta_t{}^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At})}_{m\,\vec{\gamma}_{Dt}{}^*(\vec{x}_{Bt})} + \underbrace{m\,d\Theta_t{}^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At})}_{m\,\vec{\gamma}_{Ct}{}^*(\vec{x}_{Bt})}. \tag{10.76}$$

(We use an affine $\Theta_t$ to lighten the notations.) Here $d\Theta_t{}^{-1}.\vec{f}_{At}(\vec{x}_{At}) = d\Theta_t{}^{-1}.[\vec{f}_t(p_t)]_{|\vec{A}} =^{(10.41)} [\vec{f}_t(p_t)]_{|\vec{B}} =^{\text{written}} \vec{f}_{Bt}(\vec{x}_{Bt})$ is $\vec{f}_t(p_t)$ as quantified by B at $t$ (= pull-back by $\Theta_t$).

**Definition 10.13** For B at $t$ at $p_t$, with $\vec{x}_{Bt} = [\overrightarrow{O_{Bt} p_t}]_{|\vec{B}}$:

- Quantification of $\vec{f}_t(p_t)$ by B: $\vec{f}_{Bt}(\vec{x}_{Bt}) := d\Theta_t{}^{-1}.\vec{f}_{At}(\vec{x}_{At}) \quad (= \vec{f}_{At}{}^*(\vec{x}_{Bt}))$.
- The drive force: $\vec{f}_{BDt}(\vec{x}_{Bt}) := -m\,d\Theta_t{}^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At}) \quad (= -m\,\vec{\gamma}_{Dt}{}^*(\vec{x}_{Bt}))$. (10.77)
- The Coriolis force: $\vec{f}_{BCt}(\vec{x}_{Bt}) := -m\,d\Theta_t{}^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At}) \quad (= -m\,\vec{\gamma}_{Ct}{}^*(\vec{x}_{Bt}))$.
- The fictitious force = the inertial force $:= (\vec{f}_{BDt} + \vec{f}_{BCt})(\vec{x}_{Bt})$.

(The pull-backs by $\Theta_t$.)

Then (10.76) is the fundamental principle quantified for B (living in a non Galilean referential):

$$\boxed{\vec{f}_{Bt}(\vec{x}_{Bt}) + \vec{f}_{BDt}(\vec{x}_{Bt}) + \vec{f}_{BCt}(\vec{x}_{Bt}) = m\,\vec{\gamma}_B(\vec{x}_{Bt})}, \tag{10.78}$$

i.e. for B at $t$: The (external + drive + Coriolis) forces = $m$ times the acceleration.

## 10.10 Summary for a "merry-go-round"

1. **Referentials.**

1.1. Galilean referential. Observer A chooses a referential $\mathcal{R}_A = (O_A, (\vec{A}_1, \vec{A}_2))$ where $O_A$ is the center of the merry-go-round and $(\vec{A}_1, \vec{A}_2)$ fixed on Earth is an horizontal Euclidean basis. And $(\cdot,\cdot)_A$ and $||.||_A$ are the associated Euclidean dot product and norm. $\mathcal{R}_A$ will be considered Galilean with an approximation "good enough" for a usual merry-go-round (for a more precise result, you can apply next §).

1.2. Relative referential. Observer B chooses a referential $\mathcal{R}_B = (O_B, (\vec{B}_1, \vec{B}_2)$ where $O_B = O_A$ and $(\vec{B}_1, \vec{B}_2)$ fixed on the merry-go-round is an horizontal Euclidean basis. And $(\cdot,\cdot)_B$ and $||.||_B$ are the associated Euclidean dot product and norm.

1.3. For A, $\mathcal{R}_B$ is $\mathcal{R}_{Bt} = (O_{Bt}, (\vec{B}_{1t}, \vec{B}_{2t}))$. For B, $\mathcal{R}_A$ is $\mathcal{R}_{At} = (O_{At}, (\vec{A}_{1t}, \vec{A}_{2t}))$. And the bases being Euclidean, $||.||_A = \lambda||.||_B$ and $(\cdot,\cdot)_A = \lambda^2(\cdot,\cdot)_B$ where $\lambda > 0$. E.g. $(\cdot,\cdot)_A$ is built with the foot and $(\cdot,\cdot)_B$ is built with the metre, $\lambda = \frac{1}{0.3048}$ (because 1 ft=0.3048 m).

2. **Motions, velocities, accelerations.** Given in (10.15)–(10.18), here with $[t_1,t_2]=[0,t_2]$.
3. **Absolute and relative motions, velocities, accelerations.** cf. (10.19)–(10.22).
4. **Drive and static motions, velocities, accelerations.**

4.1. Static motion (in $\mathcal{R}_B$), cf. (10.23)–(10.25): Consider a particle $Q_{\mathcal{R}_B}\in\mathcal{R}_B$ s.t. $\overrightarrow{O_B\widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}=R_B(Q_{\mathcal{R}_B})(\cos(\theta_{Q_{\mathcal{R}_B}})\vec{B}_1+\sin(\theta_{Q_{\mathcal{R}_B}})\vec{B}_2)$, the position of $Q_{\mathcal{R}_B}$ is stored by B as the matrix

$$\vec{\varphi}_S(Q_{\mathcal{R}_B})=[\overrightarrow{O_B\widetilde{\Phi}(t,Q_{\mathcal{R}_B})}]_B=R_B(Q_{\mathcal{R}_B})\begin{pmatrix}\cos(\theta_{Q_{\mathcal{R}_B}})\\ \sin(\theta_{Q_{\mathcal{R}_B}})\end{pmatrix}=\vec{y}_S=\begin{pmatrix}y_{S1}\\ y_{S2}\end{pmatrix},\tag{10.79}$$

so with $R_B(Q_{\mathcal{R}_B})=||\overrightarrow{O_B\widetilde{\Phi}_{\mathcal{R}_B}(t,Q_{\mathcal{R}_B})}||_B$ the distance from $O_B$ and $\theta_{Q_{\mathcal{R}_B}}\in]-\pi,\pi]$ the angular position in $\mathcal{R}_B$.

4.2. Drive motion (in $\mathcal{R}_A$): With $\omega$ the angular velocity of $\mathcal{R}_B$ in $\mathcal{R}_A$ supposed constant to simplify, we get $\overrightarrow{O_B\widetilde{\Phi}(t,Q_{\mathcal{R}_B})}=R_A(Q_{\mathcal{R}_B})(\cos(\omega t{+}\theta_{Q_{\mathcal{R}_B}})\vec{A}_1+\sin(\omega t{+}\theta_{Q_{\mathcal{R}_B}})\vec{A}_2)$, and the position of $Q_{\mathcal{R}_B}$ is stored by A as the matrix

$$\vec{\varphi}_D(t,Q_{\mathcal{R}_B})=[\overrightarrow{O_A\widetilde{\Phi}(t,Q_{\mathcal{R}_B})}]_{|\vec{A}}=R_A(Q_{\mathcal{R}_B})\begin{pmatrix}\cos(\omega t{+}\theta_{Q_{\mathcal{R}_B}})\\ \sin(\omega t{+}\theta_{Q_{\mathcal{R}_B}})\end{pmatrix}=\vec{y}_D(t)=\begin{pmatrix}y_{D1}(t)\\ y_{D2}(t)\end{pmatrix}.\tag{10.80}$$

(More generally, replace $t$ by $t{-}t_0$ and $\omega t$ by $\theta(t)$.) And (change of unit of measurement) $R_A(Q_{\mathcal{R}_B})=||\overrightarrow{O_A\widetilde{\Phi}(t,Q_{\mathcal{R}_B})}||_A=\lambda||\overrightarrow{O_B\widetilde{\Phi}(t,Q_{\mathcal{R}_B})}||_B=\lambda R_B(Q_{\mathcal{R}_B})$.

4.3. Drive velocity:

$$\begin{aligned}\vec{v}_D(t,\vec{y}_D(t))=\vec{y}_D{}'(t)&=\omega R_A(Q_{\mathcal{R}_B})\begin{pmatrix}-\sin(\omega t{+}\theta_{Q_{\mathcal{R}_B}})\\ \cos(\omega t{+}\theta_{Q_{\mathcal{R}_B}})\end{pmatrix}=\omega\begin{pmatrix}-y_{D2}(t)\\ y_{D1}(t)\end{pmatrix}\\ &=\omega\begin{pmatrix}0&-1\\1&0\end{pmatrix}.\vec{y}_D(t)=\begin{pmatrix}0&-\omega\\ \omega&0\end{pmatrix}.\vec{y}_D(t),\end{aligned}\tag{10.81}$$

so it is a rotation of angle $\frac{\pi}{2}$ times $\vec{y}_D(t)$ times $\omega$. So, with the chosen origin $O_A$ at the center of the merry-do-round, $\vec{v}_D(t,\vec{y}_D(t))\perp\vec{y}_D(t)$, and the velocity $\vec{v}(t,q_t)=\frac{\partial\widetilde{\Phi}_{\mathcal{R}_B}}{\partial t}(t,Q_{\mathcal{R}_B})$ of the particle $Q_{\mathcal{R}_B}$ is orthogonal to $\overrightarrow{O_Aq_t}$ the radial position vector. Immersed in $\mathbb{R}^3$, $\vec{v}_D(t,\vec{y}_D(t))=\vec{\omega}_D\times\vec{y}_D(t)$ where $\vec{\omega}_D=\omega\begin{pmatrix}0\\0\\1\end{pmatrix}$.

4.4. Differential of the drive velocity: $\vec{v}_{Dt}(\vec{y})=^{(10.81)}\begin{pmatrix}0&-\omega\\ \omega&0\end{pmatrix}.\vec{y}$ gives

$$(d\vec{v}_D(t,\vec{y})=)\quad d\vec{v}_{Dt}(\vec{y})=\begin{pmatrix}0&-\omega\\ \omega&0\end{pmatrix}=\omega\begin{pmatrix}0&-1\\1&0\end{pmatrix}\overset{\text{written}}{=}d\vec{v}_D\tag{10.82}$$

(time and space independent). Immersed in $\mathbb{R}^3$, $d\vec{v}_D.\vec{w}=\vec{\omega}\times\vec{w}$.

4.5. Drive acceleration = centripetal acceleration toward $O_A$:

$$\begin{aligned}\vec{\gamma}_{Dt}(\vec{y}_{Dt})=\vec{\gamma}_D(t,\vec{y}_D(t))\overset{(10.81)}{=}\vec{y}_D{}''(t)&=\begin{pmatrix}0&-\omega\\ \omega&0\end{pmatrix}.\vec{y}_D{}'(t)=-\omega^2\begin{pmatrix}y_{D1}(t)\\ y_{D2}(t)\end{pmatrix}=-\omega^2\vec{y}_D(t)\\ &=d\vec{v}_D.\vec{v}_D(t,\vec{y}_{Dt})=\omega\begin{pmatrix}-v_{D2}(t)\\ v_{D1}(t)\end{pmatrix}.\end{aligned}\tag{10.83}$$

Its magnitude is $\omega^2R_A(Q_{\mathcal{R}_B})$. And the minus sign in $(10.83)_1$ tells that it is centripetal (from $\vec{y}_{Dt}$ toward the center); Interpretation: A particle glued on $\mathcal{R}_B$ is not ejected from $\mathcal{R}_B$ despite rotation. Immersed in $\mathbb{R}^3$, $\vec{\gamma}_{Dt}(\vec{y}_{Dt})=\vec{\omega}\times\vec{v}_{Dt}(\vec{y}_{Dt})$.

4.6. Centrifugal force in $\mathcal{R}_A$ felt here by a particle of mass $m$ fixed on $\mathcal{R}_B$ at $t$ at $q_t$ s.t. $[\overrightarrow{O_Aq_t}]_{|\vec{A}}=\vec{y}_{Dt}$: The retaining force (because the particle is glued on $\mathcal{R}_B$) is, Newton's principle,

$$\text{retaining force}(t,\vec{y}_{Dt})=m\vec{\gamma}_{Dt}(\vec{y}_{Dt})=m\,d\vec{v}_D.\vec{v}_D(t,\vec{y}_{Dt})=m\omega\begin{pmatrix}v_{D2}(t)\\ -v_{D1}(t)\end{pmatrix}\perp\vec{v}_{Dt}(\vec{y}_{Dt}).\tag{10.84}$$

It is centripetal because $\vec{\gamma}_D(t,\vec{y}_{Dt})$ is. And the centrifugal force is the opposite (it is the "felt force"):

$$\text{centrifugal force}(t,\vec{y}_{Dt})=-m\,d\vec{v}_D.\vec{v}_D(t,\vec{y}_{Dt})=m\omega\begin{pmatrix}-v_{D2}(t)\\ v_{D1}(t)\end{pmatrix}.\tag{10.85}$$

Immersed in $\mathbb{R}^3$: $=-m\,\vec{\omega}\times\vec{v}_{Dt}(\vec{y}_{Dt})$.

5. **Translator from B to A.**

5.1. Defined by $\Theta_t(\vec{y}_S) =^{(10.27)} \vec{y}_{Dt}$. In particular $\Theta_t(\vec{0}) = \vec{0}$ (null matrix) because $O_A = O_B$. And $\Theta_t$ is affine (classical mechanics).

5.2. $d\Theta_t$: Characterized by $d\Theta_t.[\vec{B}_i]_{|\vec{B}} =^{(10.45)} [\vec{B}_{it}]_{|\vec{A}}$. With $R_B(Q_{\mathcal{R}_B}) = ||\vec{B}_i||_{\vec{B}} = 1$ we have $R_A(Q_{\mathcal{R}_B}) = ||\vec{B}_{it}||_{\vec{A}} = \lambda$ (change of unit of measurement) and

- $\theta_{Q_{\mathcal{R}_B}}=0$ gives $[\vec{B}_{1t}]_{|\vec{A}} = \lambda \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}$ and $[\vec{B}_{1t}]_{|\vec{A}} = d\Theta_t.[\vec{B}_1]_{|\vec{B}} = d\Theta_t.\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,
- $\theta_{Q_{\mathcal{R}_B}}=\frac{\pi}{2}$ gives $[\vec{B}_{2t}]_{|\vec{A}} = \lambda \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$ and $[\vec{B}_{2t}]_{|\vec{A}} = d\Theta_t.[\vec{B}_2]_{|\vec{B}} = d\Theta_t.\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus

$$d\Theta_t = \lambda \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} \tag{10.86}$$

$=$ the transition matrix from $([\vec{B}_{it}]_{|\vec{B}}) = ([\vec{B}_i]_{|\vec{B}}) = (\vec{E}_i) = ([\vec{A}_i]_{|\vec{A}})$ to $([\vec{B}_{it}]_{|\vec{A}})$ in $\mathcal{M}_{n1}$: The expected rotation matrix expanded by $\lambda$ (change of unit of measurement).

6. **Coriolis acceleration.**

$$\vec{\gamma}_{Ct}(\vec{x}_{At}) \overset{(10.63)}{=} 2\, d\vec{v}_{Dt}.\vec{v}_{Bt*}(\vec{x}_{At}) = 2\, d\vec{v}_{Dt}.d\Theta_t.\vec{v}_{Bt}(\vec{x}_{Bt}). \tag{10.87}$$

In particular, a particle fixed on Earth ($\vec{v}_{Bt} = 0$) is not subject to a Coriolis acceleration ($\vec{\gamma}_{Ct} = 0$), which is obvious since then $\vec{v}_A = \vec{v}_D$.

7. **Coriolis force:**

$$d\vec{v}_{Dt}.d\Theta_t = \lambda\omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.\begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix} = \lambda\omega \begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) \\ \cos(\omega t) & -\sin(\omega t) \end{pmatrix} = d\Theta_t.d\vec{v}_{Dt} \tag{10.88}$$

(the matricies commute: Composition of "rotations around $\vec{0}$", which read $\lambda\omega\, e^{i\frac{\pi}{2}} e^{i\omega t} = \lambda\omega\, e^{i(\frac{\pi}{2}+\omega t)} = \lambda\omega\, e^{i\omega t}.e^{i\frac{\pi}{2}}$). Thus $d\Theta_t^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At}) = 2\, d\vec{v}_{Dt}.\vec{v}_{Bt}(\vec{x}_{Bt})$, and (10.77) gives

$$\vec{f}_{BCt}(\vec{x}_{Bt}) = -2m\, d\vec{v}_{Dt}.\vec{v}_{Bt}(\vec{x}_{Bt}) = 2m\omega \begin{pmatrix} v_{B2}(t) \\ -v_{B1}(t) \end{pmatrix} \quad (\perp \vec{v}_{Bt}(\vec{x}_{Bt})), \tag{10.89}$$

pointed vector at $\vec{x}_{Bt}$ orthogonal to $\vec{v}_{Bt}(\vec{x}_{Bt})$. Immersed in $\mathbb{R}^3$, $\vec{f}_{BCt}(\vec{x}_{Bt}) = -2m\,\vec{\omega} \times \vec{v}_{Bt}(\vec{x}_{Bt})$.

8. **Drive force:**

$$\vec{f}_{BDt}(\vec{x}_{Bt}) \overset{(10.77)}{=} -m\, d\Theta_t^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At}) \overset{(10.83)}{=} -m\, d\Theta_t^{-1}.d\vec{v}_{Dt}.\vec{v}_{Dt}(\vec{x}_{At}), \tag{10.90}$$

with $d\Theta_t^{-1} =^{(10.86)} \frac{1}{\lambda} \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$. Immersed in $\mathbb{R}^3$, $\vec{f}_{BDt}(\vec{x}_{Bt}) = -2m\, d\Theta_t^{-1}.(\vec{\omega} \times \vec{v}_{Bt}(\vec{x}_{Bt}))$.

9. **Inertial force:**

$$(\vec{f}_{BCt} + \vec{f}_{BDt})(\vec{x}_{Bt}) = -m\, d\vec{v}_{Dt}.\Big(2\vec{v}_{Bt}(\vec{x}_{Bt}) + d\Theta_t^{-1}.\vec{v}_{Dt}(\vec{x}_{At})\Big). \tag{10.91}$$

Immersed in $\mathbb{R}^3$, $(\vec{f}_{BCt} + \vec{f}_{BDt})(\vec{x}_{Bt}) = -m\,\vec{\omega} \wedge \Big(2\,\vec{v}_{Bt}(\vec{x}_{Bt}) + d\Theta_t^{-1}.\vec{v}_{Dt}(\vec{x}_{At})\Big)$.

## 10.11 Summary for "Sun and Earth" (and Coriolis forces on the Earth)

Simplificaions: The Earth is a spherical rigid body $\mathcal{R}_B$ which rotates around its South-North axis fixed relative to stars (no precession nor nutation), and its center rotates around the Sun.

1. **Referentials.**

1.1. Absolute Galilean referential. Observer A first chooses a referential $\mathcal{R}_{AS} = (O_{AS}, (\vec{A}_1, \vec{A}_2, \vec{A}_3))$ where $O_{AS}$ is the center of the Sun and $(\vec{A}_1, \vec{A}_2, \vec{A}_3)$ a Euclidean basis fixed relative to the stars with $\vec{A}_3$ along the rotation axis of the Earth and oriented from the south pole to the north pole. And $(\cdot,\cdot)_A$ and $||.||_A$ are the associated Euclidean dot product and norm.

Because it takes more that 365 days for the center of the Earth to complete a rotation around the Sun, the motion $t \to O_A(t) = O_{At}$ of the center of the Earth will be considered "rectilinear at constant velocity in a short interval of time", "short enough" for the computation of the Coriolis acceleration to be "accurate enough" (simplifies the calculations). Hence Observer A writes $O_{At} = O_A$ and $\mathcal{R}_A = (O_A, (\vec{A}_1, \vec{A}_2, \vec{A}_3))$.

1.2. Relative referential. Observer B chooses a referential $\mathcal{R}_B = (O_B, (\vec{B}_1, \vec{B}_2, \vec{B}_3))$ with $O_B = O_A =$ the center of the Earth and $(\vec{B}_1, \vec{B}_2, \vec{B}_3)$ a Euclidean basis fixed on the Earth with $\vec{B}_3$ along the South-North axis and oriented from the south pole to the north pole. And $(\cdot,\cdot)_B$ and $||.||_B$ are the associated Euclidean dot product and norm.

1.3. For A, $\mathcal{R}_B$ is $\mathcal{R}_{Bt} = (O_{Bt}, (\vec{B}_{1t}, \vec{B}_{2t}))$. For B, $\mathcal{R}_A$ is $\mathcal{R}_{At} = (O_{At}, (\vec{A}_{1t}, \vec{A}_{2t}))$. And the bases being Euclidean, $||.||_A = \lambda ||.||_B$ and $(\cdot,\cdot)_A = \lambda^2 (\cdot,\cdot)_B$ where $\lambda > 0$. E.g. $(\cdot,\cdot)_A$ is built with the foot and $(\cdot,\cdot)_B$ is built with the metre, $\lambda = \frac{1}{0.3048}$ (because 1 ft=0.3048 m). In particular $\vec{B}_3 = \lambda \vec{A}_3$ because $||\vec{B}_3||_B = 1 = ||\vec{A}_3||_A = \lambda ||\vec{A}_3||_B = ||\lambda \vec{A}_3||_B$ and same direction and orientation.

2. **Motions, velocities, accelerations.** Given in (10.15)...(10.18).
3. **Absolute and relative motions, velocities, accelerations.** Given in (10.19)...(10.22).
4. **Drive and static motions, velocities, accelerations.**

4.1. Static motion (in $\mathcal{R}_B$). Consider a particle $Q_{\mathcal{R}_B} \in \mathcal{R}_B$ at distance $R_B(Q_{\mathcal{R}_B}) = ||\overrightarrow{O_B \widetilde{\Phi}(t, Q_{\mathcal{R}_B})}||_B$ from $O_B$, at longitude $\theta_{Q_{\mathcal{R}_B}} \in ]-\pi, \pi]$ and latitude $\varphi_{Q_{\mathcal{R}_B}} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$: Its position is stored by B as the matrix

$$\vec{y}_S = [\overrightarrow{O_B \widetilde{\Phi}(t, Q_{\mathcal{R}_B})}]_B = \vec{\varphi}_S(Q_{\mathcal{R}_B}) = R_B(Q_{\mathcal{R}_B}) \begin{pmatrix} \cos(\theta_{Q_{\mathcal{R}_B}}) \cos(\varphi_{Q_{\mathcal{R}_B}}) \\ \sin(\theta_{Q_{\mathcal{R}_B}}) \cos(\varphi_{Q_{\mathcal{R}_B}}) \\ \sin(\varphi_{Q_{\mathcal{R}_B}}) \end{pmatrix} = \begin{pmatrix} y_{S1} \\ y_{S2} \\ y_{S3} \end{pmatrix}. \tag{10.92}$$

(E.g. on Earth $R_B(Q_{\mathcal{R}_B}) \simeq 6\,371$ km.)

4.2. Drive motion (in $\mathcal{R}_A$). With $\omega$ the angular velocity of the Earth in $\mathcal{R}_A$ and $R_A(Q_{\mathcal{R}_B}) = ||\overrightarrow{O_A \widetilde{\Phi}(t, Q_{\mathcal{R}_B})}||_A$, the position of $Q_{\mathcal{R}_B}$ is stored by A as the matrix

$$\vec{y}_D(t) = [\overrightarrow{O_A \widetilde{\Phi}(t, Q_{\mathcal{R}_B})}]_{|\vec{A}} = \vec{\varphi}_D(t, Q_{\mathcal{R}_B}) = R_A(Q_{\mathcal{R}_B}) \begin{pmatrix} \cos(\omega t + \theta_{Q_{\mathcal{R}_B}}) \cos \varphi_{Q_{\mathcal{R}_B}} \\ \sin(\omega t + \theta_{Q_{\mathcal{R}_B}}) \cos \varphi_{Q_{\mathcal{R}_B}} \\ \sin \varphi_{Q_{\mathcal{R}_B}} \end{pmatrix} = \begin{pmatrix} y_{D1}(t) \\ y_{D2}(t) \\ y_{D3}(t) \end{pmatrix}. \tag{10.93}$$

(The latitude is constant and on Earth $R_A(Q_{\mathcal{R}_B}) = \lambda R_B(Q_{\mathcal{R}_B}) \simeq 20\,902\,231$ foot).

4.3. Drive velocity:

$$\begin{aligned} \vec{v}_D(t, \vec{y}_D(t)) = \vec{y}_D{}'(t) = \omega R_A(Q_{\mathcal{R}_B}) \begin{pmatrix} -\sin(\omega t + \theta_{Q_{\mathcal{R}_B}}) \cos \varphi_{Q_{\mathcal{R}_B}} \\ \cos(\omega t + \theta_{Q_{\mathcal{R}_B}}) \cos \varphi_{Q_{\mathcal{R}_B}} \\ 0 \end{pmatrix} &= \omega \begin{pmatrix} -y_{D2}(t) \\ y_{D1}(t) \\ 0 \end{pmatrix} \\ = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \vec{y}_D(t) = \vec{\omega}_D \times \vec{y}_D(t), \quad \text{where} \quad \vec{\omega}_D &= \omega \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{10.94}$$

So, with $q_t = \widetilde{\Phi}_{\mathcal{R}_B}(t, Q_{\mathcal{R}_B})$, the velocity $\vec{v}_{\mathcal{R}_B}(t, q_t) = \frac{\partial \widetilde{\Phi}_{\mathcal{R}_B}}{\partial t}(t, Q_{\mathcal{R}_B})$ of the particle $Q_{\mathcal{R}_B}$ is a pointed vector at $q_t$ orthogonal to $\vec{\omega} = \omega \vec{A}_3$, thus in the $(x, y)$-vectorial plane.

4.4. Differential of the drive velocity: (10.94) gives $\vec{v}_{Dt}(\vec{y}) = \vec{\omega}_D \times \vec{y}$, thus $d\vec{v}_{Dt}(\vec{y}) = \vec{\omega}_D \times . =^{\text{written}} d\vec{v}_D$, with

$$d\vec{v}_D = \vec{\omega}_D \times . = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(time and space independent)}. \tag{10.95}$$

4.5. Drive acceleration = centripetal acceleration of a fixed point on Earth: $\omega$ being constant,

$$\vec{\gamma}_D(t, \vec{y}_{Dt}) \overset{(10.94)}{=} \vec{y}_D{}''(t) = \vec{\omega}_D \times \vec{y}_D{}'(t) = \vec{\omega}_D \times \vec{v}_{Dt}(\vec{y}_{Dt}) = d\vec{v}_D . \vec{v}_{Dt}(\vec{y}_{Dt}) = -\omega^2 \begin{pmatrix} y_{D1}(t) \\ y_{D2}(t) \\ 0 \end{pmatrix}. \tag{10.96}$$

So its magnitude is $\omega^2 R_A(Q_{\mathcal{R}_B}) \cos(\varphi_{Q_{\mathcal{R}_B}})$. The minus sign tells that it is centripetal relative to the Earth circle parallel to the equator (a particle glued on the Earth is not ejected from the Earth despite rotation).

4.6. Centrifugal force felt by a particle of mass $m$ fixed on Earth: It is $-m$ times the centripetal acceleration $\vec{\gamma}_D(t, \vec{y}_{Dt})$ in $\mathcal{R}_A$:

$$\text{centrifugal force} = -m \vec{\gamma}_D(t, \vec{y}_{Dt}) = -m \vec{\omega}_D \times \vec{v}_{Dt}(\vec{y}_{Dt}) \quad (= -m\, d\vec{v}_D . \vec{v}_{Dt}(\vec{y}_{Dt})). \tag{10.97}$$

On the Earth, the gravity force $m\vec{g}$ directed toward the center of the Earth is large compared to $-m\vec{\gamma}_D(t, \vec{y}_{Dt})$ (we are not ejected from the Earth because $\omega$ is small enough and $\vec{g}$ strong enough).

5. **Translator.**

5.1. Defined by $\Theta_t(\vec{y}_S) =^{(10.27)} \vec{y}_{Dt}$. In particular $\Theta_t(\vec{0}) = \vec{0}$ (null matrix) because $O_{At} = O_{Bt}$, with $\Theta_t$ affine.

5.2. Calculation of $d\Theta_t$. Given by $d\Theta_t.[\vec{B}_i]_{|\vec{B}} =^{(10.45)} [\vec{B}_{it}]_{|\vec{A}}$. With $||\overrightarrow{O_B\widetilde{\Phi}(t,Q_{\mathcal{R}_B})}||_B = R_B(Q_{\mathcal{R}_B})$=1, so $||\overrightarrow{O_B\widetilde{\Phi}(t,Q_{\mathcal{R}_B})}||_A = R_A(Q_{\mathcal{R}_B})$=$\lambda$ (change of unit of measurement), and

- $\theta_{Q_{\mathcal{R}_B}}$=0 and $\varphi_{Q_{\mathcal{R}_B}}$=0 give $d\Theta_t.[\vec{B}_1]_{|\vec{B}} = [\vec{B}_{1t}]_{|\vec{A}} = \lambda \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{pmatrix}$,
- $\theta_{Q_{\mathcal{R}_B}}$=$\frac{\pi}{2}$ and $\varphi_{Q_{\mathcal{R}_B}}$=0 give $d\Theta_t.[\vec{B}_2]_{|\vec{B}} = [\vec{B}_{2t}]_{|\vec{A}} = \lambda \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \\ 0 \end{pmatrix}$,
- $\theta_{Q_{\mathcal{R}_B}} = 0$ and $\varphi_{Q_{\mathcal{R}_B}} = \frac{\pi}{2}$ give $d\Theta_t.[\vec{B}_3]_{|\vec{B}} = [\vec{B}_{3t}]_{|\vec{A}} = [\lambda\vec{A}_3]_{|\vec{A}} = \lambda[\vec{A}_3]_{|\vec{A}} = \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Thus

$$d\Theta_t = \lambda \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10.98}$$

It is the expected transition matrix from $([\vec{B}_i]_{|\vec{B}})$ to $([\vec{B}_{it}]_{|\vec{A}})$ (with the change of unit of measurement).

6. **Coriolis acceleration.**

$$\begin{aligned} \vec{\gamma}_{Ct}(\vec{x}_{At}) &= 2\, d\vec{v}_{Dt}.\vec{v}_{Bt*}(\vec{x}_{At}) = 2\, d\vec{v}_{Dt}.d\Theta_t.\vec{v}_{Bt}(\vec{x}_{Bt}) \\ &= 2\vec{\omega}_D \times \vec{v}_{Bt*}(\vec{x}_{At}) = 2\vec{\omega}_D \times (d\Theta_t.\vec{v}_{Bt}(\vec{x}_{Bt})) \end{aligned} \tag{10.99}$$

E.g. a particle fixed on Earth ($\vec{v}_{Bt}(\vec{x}_{Bt}) = 0$) is not subject to a Coriolis acceleration ($\vec{\gamma}_{Ct}(\vec{x}_{At}) = 0$), which is obvious since then $\vec{v}_A = \vec{v}_D$ and $\vec{\gamma}_A = \vec{\gamma}_D$.

7. **Coriolis force:** We have

$$d\vec{v}_{Dt}.d\Theta_t = d\Theta_t.d\vec{v}_{Dt} = \lambda\omega \begin{pmatrix} -\sin(\omega t) & -\cos(\omega t) & 0 \\ \cos(\omega t) & -\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(the matrices commute)}, \tag{10.100}$$

thus $\vec{\gamma}_{Ct}(\vec{x}_{At}) =^{(10.87)} 2\, d\Theta_t.d\vec{v}_{Dt}.\vec{v}_{Bt}(\vec{x}_{Bt})$; And $\vec{f}_{BCt}(\vec{x}_{Bt}) =^{(10.77)} -m\, d\Theta_t{}^{-1}.\vec{\gamma}_{Ct}(\vec{x}_{At})$, thus

$$\vec{f}_{BCt}(\vec{x}_{Bt}) = -2m\, d\vec{v}_{Dt}.\vec{v}_{Bt}(\vec{x}_{Bt}) = -2m\,\vec{\omega}_D \times \vec{v}_{Bt}(\vec{x}_{Bt}). \tag{10.101}$$

Thus $\vec{f}_{BCt}(\vec{x}_{Bt})$ is a pointed vector at $\vec{x}_{Bt}$ orthogonal to $\vec{v}_{Bt}(\vec{x}_{Bt})$ and $\vec{\omega}_D$ (so in particular in a plane parallel to the equatorial plane). And $\vec{f}_{BCt}(\vec{x}_{Bt})$ vanishes for a particle fixed on the Earth ($\vec{x}_{Bt} = \vec{0}$).

8. **Drive force:**

$$\vec{f}_{BDt}(\vec{x}_{Bt}) \overset{(10.77)}{=} -m\, d\Theta_t{}^{-1}.\vec{\gamma}_{Dt}(\vec{x}_{At}) = -m\, d\Theta_t{}^{-1}.d\vec{v}_D.\vec{v}_{Dt}(\vec{x}_{At}) \tag{10.102}$$

with $d\Theta_t{}^{-1} =^{(10.98)} \frac{1}{\lambda} \begin{pmatrix} \cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$. I.e. $\vec{f}_{BDt}(\vec{x}_{Bt}) = -m\,\vec{\omega}_D \times (d\Theta_t{}^{-1}.\vec{v}_{Dt}(\vec{x}_{At}))$, cf. (10.100).

9. **Inertial force:**

$$(\vec{f}_{BCt} + \vec{f}_{BDt})(\vec{x}_{Bt}) = -m\,\vec{\omega}_D \times \Big(2\vec{v}_{Bt}(\vec{x}_{Bt}) + d\Theta_t{}^{-1}.\vec{v}_{Dt}(\vec{x}_{At})\Big). \tag{10.103}$$

# 11 Objectivities

Goal: To give an objective expression of the laws of mechanics; As Maxwell [18] said: "The formula at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result".

Generic notation: if a function $z$ is given as $z(t,x)$, then $z_t(x) := z(t,x)$, and conversely.

## 11.1 "Isometric objectivity" and "Frame Invariance Principle"

This manuscript is not intended to describe "isometric objectivity":

"Isometric objectivity" is the framework in which the "principle of material frame-indifference" ("frame invariance principle") is settled, principle which states that "Rigid body motions should not affect the stress constitutive law of a material". E.g., Truesdell–Noll [26] p. 41:

« Constitutive equations must be invariant under changes of frame of reference. »

Or Germain [12] :

« Axiom of power of internal forces. The virtual power of the "internal forces" acting on a system S for a given virtual motion is an objective quantity; i.e., it has the same value whatever be the frame in which the motion is observed. »

**NB:** Both of these affirmations are limited to "isometric changes of frame" (the same metric for all), as Truesdell–Noll [26] page 42-43 explain: The "isometric objectivity" concern one observer who defines his Euclidean dot product and consider only orthonormal change of bases to validate a constitutive law.

If you want to interpret "isometric objectivity" in the "covariant objectivity" framework, then "isometric objectivity" corresponds to a dictatorial management: One observer with his Euclidean referential (e.g. based on the English foot), imposes his unit of length to all other users (isometry hypothesis). (Note: The metre was not adopted by the scientific community until after 1875.)

Moreover, isometric objectivity leads to despise the difference between covariance and contravariance, due to the uncontrolled use of the Riesz representation theorem.

**Remark 11.1** Marsden and Hughes [17] p. 8 use this isometric framework to begin with. But, pages 22 and 163, they write that a "good modelization" has to be "covariant objective" (observer independent) to begin with; And they propose a covariant modelization for elasticity at § 3.3. ▪

## 11.2 Definition and characterization of the covariant objectivity

Consider a regular motion $\widetilde{\Phi}$ of an object $Obj$, $p_t = \widetilde{\Phi}(t, P_{Obj}) \in \mathbb{R}^n$ the position at $t$ of a particle in our Universe, $\Omega_t = \widetilde{\Phi}(t, Obj)$ the configuration at $t$, and $\mathcal{C} = \bigcup_{t\in[a,b]}(\{t\}\times\Omega_t)$ the set of configurations.

Consider two observers A and B and their referentials $\mathcal{R}_A = (O_A, (\vec{A}_i))$ and $\mathcal{R}_B = (O_B, (\vec{B}_i))$. E.g., $(\vec{A}_i)$ and $(\vec{B}_i)$ are Euclidean bases in foot and metre, $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_B$ is their associated Euclidean dot products. And $\Theta$ is the translator, cf. (10.27).

Let $\vec{x}_{At} := [\overrightarrow{O_A p_t}]_{|\vec{A}} \in \mathcal{M}_{n1}$ and $\vec{x}_{Bt} := [\overrightarrow{O_B p_t}]_{|\vec{B}} \in \mathcal{M}_{n1}$, the stored components of $p_t$ relative to the chosen referentials, $\mathcal{M}_{n1}$ and $\mathcal{M}_{n1}$ being the spaces of $n*1$ matrices.

### 11.2.1 Covariant objectivity of a scalar function

Let $f : \left\{ \begin{array}{ll} \mathcal{C} & \to \mathbb{R} \\ (t,p_t) & \to f(t,p_t) \end{array} \right\}$ be a Eulerian scalar valued field of functions (e.g. temperature field). And $f_A : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1} & \to \mathbb{R} \\ (t,\vec{x}_{At}) & \to f_A(t,\vec{x}_{At}) := f(t,p_t) \end{array} \right\}$ and $f_B : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1} & \to \mathbb{R} \\ (t,\vec{x}_{Bt}) & \to f_B(t,\vec{x}_{Bt}) := f(t,p_t) \end{array} \right\}$

**Definition 11.2** $f$ is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all $t$,

$$f_{At}(\vec{x}_{At}) = f_{Bt}(\vec{x}_{Bt}) \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}), \tag{11.1}$$

i.e. $f_{At} = f_{Bt*}$ is the push-forward of $f_{Bt}$ by $\Theta_t$ cf. (6.8).

**Remark 11.3** NB: If e.g. $f$ gives temperatures, then we supposed that $f_{At}$ and $f_{Bt}$ gives values in the same unit, e.g. Celsius, because it is the covariant objectivity and the isometric objectivity which are at stake: We are only interested in the changes of referential characterized by the translator $\Theta : \mathcal{R}_B \to \mathcal{R}_A$ which links positions between two referentials cf. (10.27). ▪

### 11.2.2 Covariant objectivity of a vector field

Let $\vec{w} : \left\{ \begin{array}{ll} \mathcal{C} & \to \mathbb{R}^{\vec{n}} \\ (t,p_t) & \to \vec{w}(t,p_t) \end{array} \right\}$ be a Eulerian vector field (e.g. a force field). And $\vec{w}_A : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1} & \to \mathcal{M}_{n1} \\ (t,\vec{x}_{At}) & \to \vec{w}_A(t,\vec{x}_{At}) := [\vec{w}(t,p_t)]_{\vec{A}} \end{array} \right\}$ and $\vec{w}_B : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1} & \to \mathcal{M}_{n1} \\ (t,\vec{x}_{Bt}) & \to \vec{w}_B(t,\vec{x}_{Bt}) := [\vec{w}(t,p_t)]_{\vec{B}} \end{array} \right\}$ are the quantifications of $\vec{w}$ by A and B, $\vec{w}_A(t,\vec{x}_{At})$ and $\vec{w}_B(t,\vec{x}_{Bt})$ being the column matrices of the components of $\vec{w}(t,p_t)$ in $\mathcal{R}_A$ and $\mathcal{R}_B$.

**Definition 11.4** $\vec{w}$ is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all $t$,

$$\vec{w}_{At}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}) \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}), \tag{11.2}$$

i.e. $\vec{w}_{At} = \vec{w}_{Bt*}$ is the push-forward of $\vec{w}_{Bt}$ by $\Theta_t$ cf. (6.20).

**Example 11.5** Fundamental counter-example: A Eulerian velocity field is not objective, cf. (10.56), because of the drive velocity $\vec{v}_D \neq \vec{0}$ in general. ∴

**Example 11.6** The field of gravitational forces (external forces) is objective covariant. ∴

**Remark 11.7** Recall: "Isometric objective" implies
- The use of the same Euclidean metric in $\mathcal{R}_B$ and $\mathcal{R}_A$, i.e. $(\cdot,\cdot)_A = (\cdot,\cdot)_B$,
- The motion $\widetilde{\Phi}_{\mathcal{R}_B}$ of $\mathcal{R}_B$ in $\mathcal{R}_A$ is a solid body motion, and
- $\Theta_t$ is affine (so $d^2\Theta_t = 0$ for all $t$).
- Covariant objectivity implies isometric objectivity, the converse is false. ∴

### 11.2.3 Covariant objectivity of a differential form

Let $\alpha : \left\{ \begin{array}{ll} \mathcal{C} & \to \mathbb{R}^{n*} \\ (t,p_t) & \to \alpha(t,p_t) \end{array} \right\}$ be a Eulerian differential form (a measuring device). And $\alpha_A : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1} & \to \mathcal{M}_{n1} \\ (t,\vec{x}_{At}) & \to \alpha_A(t,\vec{x}_{At}) := [\alpha(t,p_t)]_{\vec{A}} \end{array} \right\}$ and $\alpha_B : \left\{ \begin{array}{ll} \mathbb{R}\times\mathcal{M}_{n1} & \to \mathcal{M}_{n1} \\ (t,\vec{x}_{Bt}) & \to \alpha_B(t,\vec{x}_{Bt}) := [\alpha(t,p_t)]_{\vec{B}} \end{array} \right\}$ are the quantifications of $\alpha$ by A and B, $\alpha_A(t,\vec{x}_{At})$ and $\alpha_B(t,\vec{x}_{Bt})$ being the row matrices of the components of $\alpha(t,p_t)$ in $\mathcal{R}_A$ and $\mathcal{R}_B$.

**Definition 11.8** $\alpha$ is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all $t$,

$$\alpha_{At}(\vec{x}_{At}) = \alpha_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}). \tag{11.3}$$

i.e. $\alpha_{At} = \alpha_{Bt*}$ is the push-forward of $\alpha_{Bt}$ by $\Theta_t$ cf. (7.3).

NB: (11.3) and (11.2) are compatible: If $\vec{w}$ is an objective vector field and if $\alpha$ is an objective differential form, then the scalar function $\alpha.\vec{w}$ is objective:

$$\alpha_{At}(\vec{x}_{At}).\vec{w}_{At}(\vec{x}_{At}) = \alpha_{Bt}(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}) \quad (= (\alpha(t,p_t).\vec{w}(t,p_t)), \tag{11.4}$$

since $\alpha_{At}(\vec{x}_{At}).\vec{w}_{At}(\vec{x}_{At}) = (\alpha_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1}).(d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})) = \alpha_{Bt}(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})$.

### 11.2.4 Covariant objectivity of tensors

A tensor acts on both vector fields and differential forms, and its objectivity is deduced from the previous §.

So, let $T$ be a (Eulerian) tensor corresponding to a "physical quantity". The observers $A$ and $B$ describe $T$ as being the functions $T_A$ and $T_B$.

**Definition 11.9** $T$ is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all $t$,

$$T_{At}(\vec{x}_{At}) = T_{Bt*}(\vec{x}_{At}) \tag{11.5}$$

i.e. $T_{At}$ is the push-forward of $T_{Bt}$ by $\Theta_t$.

(Recall: $T_{Bt*}(\vec{x}_{At})(\alpha_1(\vec{x}_{At}),...,\vec{w}_1(\vec{x}_{At})) := T_{Bt}(\vec{x}_{Bt})(\alpha_1{}^*(\vec{x}_{Bt}),...,\vec{w}_1{}^*(\vec{x}_{Bt})).$)

**Example 11.10 (Non covariant objectivity of a differential $d\vec{w}$)** Let $\vec{w}$ be an objective vector field, seen as $\vec{w}_A$ by A and $\vec{w}_B$ by B; So $\vec{w}_{At}(\vec{x}_{At}) =^{(11.2)} d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})$ when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, thus

$$d\vec{w}_{At}(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}) + (d^2\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})), \tag{11.6}$$

hence

$$\begin{aligned} d\vec{w}_{At}(\vec{x}_{At}) &= d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} + (d^2\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})).d\Theta_t(\vec{x}_{Bt})^{-1} \\ &\neq d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} \quad \text{when} \quad d^2\Theta_t \neq 0. \end{aligned} \tag{11.7}$$

Thus $d\vec{w}$ is not covariant objective in general. However in classical mechanics for "change of Cartesian referentials" $\Theta_t$ is affine, so $d^2\Theta_t = 0$, and in particular $d\vec{w}$ is objective when $\vec{w}$ is. Similarly

$$\begin{aligned} &(d^2\vec{w}_{At}(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt})).d\Theta_t(\vec{x}_{Bt}) + d\vec{w}_{At}(\vec{x}_{At}).d^2\Theta_t(\vec{x}_{Bt}) \\ &= d\Theta_t(\vec{x}_{Bt}).d^2\vec{w}_{Bt}(\vec{x}_{Bt}) + 2\, d^2\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}) + d^3\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}), \end{aligned} \tag{11.8}$$

thus $d^2\vec{w}$ is not covariant objective in general (but if $\Theta_t$ is affine then $d^2\vec{w}$ is objective if $\vec{w}$ is). ▪

## 11.3 Non objectivity of the velocities

### 11.3.1 Eulerian velocity $\vec{v}$: not covariant (and not isometric) objective

Velocity addition formula: With $\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{w}(\vec{x}_{Bt})$ when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$, cf. (10.56),

$$\begin{aligned} \vec{v}_{At}(\vec{x}_{At}) &= \vec{v}_{Bt*}(\vec{x}_{At}) + \vec{v}_{Dt}(\vec{x}_{At}) \\ &\neq \vec{v}_{Bt*}(\vec{x}_{At}) \quad \text{when} \quad \vec{v}_{Dt}(\vec{x}_{At}) \neq \vec{0}, \end{aligned} \tag{11.9}$$

thus a Eulerian velocity field is not covariant objective (and not isometric objective).

### 11.3.2 $d\vec{v}$ is not objective

The velocity addition formula $(\vec{v}_{At} - \vec{v}_{Dt})(\vec{x}_{At}) = \vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})$ when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$ gives

$$d(\vec{v}_{At} - \vec{v}_{Dt})(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d\Theta_t(\vec{x}_{Bt}).d\vec{v}_{Bt}(\vec{x}_{Bt}) + d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}), \tag{11.10}$$

thus $d\vec{v}$ is neither covariant objective nor isometric objective, especially because of $d\vec{v}_D$:

$$d\vec{v}_{At}(\vec{x}_{At}) = d\vec{v}_{Dt}(\vec{x}_{At}) + d\vec{v}_{Bt*}(\vec{x}_{At}) + d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}).d\Theta_t(\vec{x}_{Bt})^{-1} \neq d\vec{v}_{Bt*}(\vec{x}_{At}) \quad \text{in general.} \tag{11.11}$$

**Exercice 11.11** Prove that $d^2\vec{v}$ is "isometric objective" when $\widetilde{\Phi}_{\mathcal{R}_B}$ is a rigid body motion.

**Answer**. (11.8) with $\vec{v}_A - \vec{v}_D$ instead of $\vec{w}_A$, and $\vec{v}_B$ instead of $\vec{w}_B$ give, in an "isometric objective" framework,

$$d^2(\vec{v}_{At} - \vec{v}_{Dt})(\vec{x}_{At}).(\vec{u}_{Bt*}, \vec{w}_{Bt*}) = d\Theta_t(\vec{x}_{Bt}).d^2\vec{v}_{Bt}(\vec{x}_{Bt})(\vec{u}_B, \vec{w}_B). \tag{11.12}$$

Here $d^2\vec{v}_{Dt} = 0$ (rigid body motion), thus $d^2\vec{v}$ is "isometric objective". ▪

**Exercice 11.12** Isometric setting. Prove, with $Q_t$ the (orthonormal) transition matrix from $(\vec{A}_i)$ to $(\vec{B}_i)$:

$$[d\vec{v}_t]_{|\vec{B}} = Q_t.[d\vec{v}_t]_{|\vec{A}}.Q_t^{-1} + Q'(t).Q_t^{-1}, \quad \text{written} \quad [L]_{|\vec{B}} = Q.[L]_{|\vec{A}}.Q^T + \dot{Q}.Q^T. \tag{11.13}$$

(Used in classical mechanics courses, to prove that $d\vec{v}$ isn't "isometric objective" because of $\dot{Q}.Q^T$.)

**Answer**. $t_0, t \in \mathbb{R}$, $p_{t_0} = \widetilde{\Phi}(t_0, P_{Obj})$, $p_t = \widetilde{\Phi}(t, P_{Obj}) = \Phi_t^{t_0}(p_{t_0})$, $\vec{v}(t, p_t) = \frac{\partial\widetilde{\Phi}}{\partial t}(t, P_{Obj})$, and $F_t^{t_0}(p_{t_0}) = d\Phi_t^{t_0}(p_{t_0})$. So $\vec{v}(t, \Phi_t^{t_0}(p_{t_0})) = \frac{\partial\Phi_{p_{t_0}}^{t_0}}{\partial t}(t, p_{t_0})$, thus $d\vec{v}(t, p_t).F_{p_{t_0}}^{t_0}(t) = \frac{\partial F_{p_{t_0}}^{t_0}}{\partial t}(t)$. And (4.28), with $F_{p_{t_0}}^{t_0} =^{\text{written}} F$, gives $[F(t)]_{|\vec{a}_{t_0}, \vec{B}} = Q(t).[F(t)]_{|\vec{a}_{t_0}, \vec{A}}$, thus $[F'(t)]_{|\vec{a}_{t_0}, \vec{B}} = Q'(t).[F(t)]_{|\vec{a}_{t_0}, \vec{A}} + Q(t).[F'(t)]_{|\vec{a}_{t_0}, \vec{A}}$. Thus $[d\vec{v}(t, p_t)]_{|\vec{B}} = [F_{p_{t_0}}^{t_0}{}'(t).F_{p_{t_0}}^{t_0}(t)]_{|\vec{B}} = [F_{p_{t_0}}^{t_0}{}'(t)]_{|\vec{B}}.[F_{p_{t_0}}^{t_0}(t)]_{|\vec{B}} = (Q'(t).[F(t)]_{|\vec{a}_{t_0}, \vec{A}} + Q(t).[F'(t)]_{|\vec{a}_{t_0}, \vec{A}}).[F(t)]_{|\vec{a}_{t_0}, \vec{A}}^{-1}.Q(t)^{-1} = Q'(t).Q(t)^{-1} + Q(t).[F'(t)]_{|\vec{a}_{t_0}, \vec{A}}.[F(t)]_{|\vec{a}_{t_0}, \vec{A}}^{-1}.Q(t)^{-1} = Q'(t).Q(t)^{-1} + Q(t).[d\vec{v}(t, p_t)]_{|\vec{A}}.Q(t)^{-1}$. And cf. (3.33). ▪

### 11.3.3 $d\vec{v} + d\vec{v}^T$ is "isometric objective"

**Proposition 11.13** *If $\widetilde{\Phi}_{\mathcal{R}_B}$ is a rigid body motion then $d\vec{v}_t + d\vec{v}_t^T$ is "isometric objective"*

$$d\vec{v}_{At} + d\vec{v}_{At}^T = (d\vec{v}_{Bt} + d\vec{v}_{Bt}^T)_*. \tag{11.14}$$

*(Isometric framework: The rate of deformation tensor is independent of an added added rigid motion.)*

**Proof.** $Q.Q^T = I$ gives $\dot{Q}.Q^T + (\dot{Q}.Q^T)^T = 0$, then apply (11.13). ∎

**Exercice 11.14** Prove that $\Omega = \frac{d\vec{v}-d\vec{v}^T}{2}$ is not isometric objective.

**Answer**. (11.11) gives $d\vec{v}_{At}^T = d\vec{v}_{Bt*}^T + d\vec{v}_{Dt}^T$, thus $\frac{d\vec{v}_{At}-d\vec{v}_{At}^T}{2} = \frac{d\vec{v}_{Bt*}-d\vec{v}_{Bt*}^T}{2} + \frac{d\vec{v}_{Dt}-d\vec{v}_{Dt}^T}{2} \neq \frac{d\vec{v}_{Bt*}-d\vec{v}_{Bt*}^T}{2}$, even if $\widetilde{\Phi}_{\mathcal{R}_B}$ is a solid body motion (then $\frac{d\vec{v}_{Dt}-d\vec{v}_{Dt}^T}{2} = \vec{\omega}\wedge$ is a rotation times a dilation). ∎

### 11.3.4 Lagrangian velocities

The Lagrangian velocities do not define a vector field, cf. § 3.2. Thus asking about the objectivity of Lagrangian velocities is meaningless.

## 11.4 The Lie derivatives are covariant objective

Framework of § 10. In particular we have the velocity-addition formula $\vec{v}_{At} = \vec{v}_{Bt*} + \vec{v}_{Dt}$ in $\mathcal{R}_A$ where $\vec{v}_{Bt*}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})$ and $\vec{x}_{Bt} = \Theta_t(\vec{x}_{At})$, cf. (10.56).

The objectivity under concern is the covariant objectivity (no inner dot product or basis required). The Lie derivatives are also called "objective rates" because they are covariant objectives.

### 11.4.1 Scalar functions

**Proposition 11.15** *If $f$ be a covariant objective function, cf. (11.1), then its Lie derivative $\mathcal{L}_{\vec{v}}f$ is covariant objective:*

$$\mathcal{L}_{\vec{v}_A} f_A = \Theta_*(\mathcal{L}_{\vec{v}_B} f_B), \quad \text{i.e.} \quad \mathcal{L}_{\vec{v}_A} f_A(t,\vec{x}_{At}) = \mathcal{L}_{\vec{v}_B} f_B(t,\vec{x}_{Bt}) \quad \text{when} \quad \vec{x}_{At} = \Theta_t(\vec{x}_{Bt}), \tag{11.15}$$

*i.e.,* $\frac{Df_A}{Dt}(t,\vec{x}_{At}) = \frac{Df_B}{Dt}(t,\vec{x}_{Bt})$, i.e. $(\frac{\partial f_A}{\partial t} + df_A.\vec{v}_A)(t,\vec{x}_{At}) = (\frac{\partial f_B}{\partial t} + df_B.\vec{v}_B)(t,\vec{x}_{Bt})$.

**Proof.** Consider the motion $t \to p(t) = \widetilde{\Phi}(tP_{Obj})$ of a particle $P_{Obj}$, and $\vec{x}_A(t) = [\overrightarrow{O_A p(t)}]_{|\vec{A}}$ and $\vec{x}_B(t) = [\overrightarrow{O_B p(t)}]_{|\vec{B}}$. With $f$ objective, (11.1) gives $f_B(t,\vec{x}_B(t)) = f_A(t,\Theta(t,\vec{x}_B(t)))$ $(= f_A(t,\vec{x}_A(t)))$, thus

$$\begin{aligned}\frac{Df_B}{Dt}(t,\vec{x}_B(t)) &= \frac{\partial f_A}{\partial t}(t,\vec{x}_A(t)) + df_{At}(\vec{x}_A(t)).(\underbrace{\frac{\partial \Theta}{\partial t}(t,\vec{x}_B(t))}_{\vec{v}_{Dt}(\vec{x}_{At})} + \underbrace{d\Theta_t(\vec{x}_B(t)).\vec{v}_{Bt}(\vec{x}_B(t))}_{\vec{v}_{Bt*}(\vec{x}_{At})}) \\ &= \frac{\partial f_A}{\partial t}(t,\vec{x}_{At}) + df_{At}(\vec{x}_{At}).\vec{v}_{At}(\vec{x}_{At}) = \frac{Df_A}{Dt}(t,\vec{x}_{At}),\end{aligned} \tag{11.16}$$

thanks to velocity addiction formula $\vec{v}_{At} = \vec{v}_{Bt*} + \vec{v}_{Dt}$. ∎

### 11.4.2 Vector fields

**Proposition 11.16** *Let $\vec{w}$ be a covariant objective vector field, cf. (11.2). Then its Lie derivative $\mathcal{L}_{\vec{v}}\vec{w}$ is covariant objective:*

$$\mathcal{L}_{\vec{v}_A}\vec{w}_A = \Theta_*(\mathcal{L}_{\vec{v}_B}\vec{w}_B), \tag{11.17}$$

*i.e., when $\vec{x}_{At} = \Theta_t(\vec{x}_{Bt})$,*

$$\mathcal{L}_{\vec{v}_A}\vec{w}_A(t,\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\mathcal{L}_{\vec{v}_B}\vec{w}_B(t,\vec{x}_{Bt}), \tag{11.18}$$

*i.e.,*

$$(\frac{D\vec{w}_A}{Dt} - d\vec{v}_A.\vec{w}_A)(t,\vec{x}_{At}) = d\Theta(t,\vec{x}_{Bt}).(\frac{D\vec{w}_B}{Dt} - d\vec{v}_B.\vec{w}_B)(t,\vec{x}_{Bt}), \tag{11.19}$$

*i.e.,*

$$(\frac{\partial\vec{w}_A}{\partial t} + d\vec{w}_A.\vec{v}_A - d\vec{v}_A.\vec{w}_A)(t,\vec{x}_{At}) = d\Theta(t,\vec{x}_{Bt}).(\frac{\partial\vec{w}_B}{\partial t} + d\vec{w}_B.\vec{v}_B - d\vec{v}_B.\vec{w}_B)(t,\vec{x}_{Bt}). \tag{11.20}$$

*But the partial, convected, material, and Lie autonomous derivatives are not covariant objective (not* *even isometric objective because of the drive velocity $\vec{v}_D$): We have*

$$(d\vec{w}_{At}.(\vec{v}_{At}-\vec{v}_{Dt}))(\vec{x}_{At}) = (d\Theta_t.(d\vec{w}_{Bt}.\vec{v}_{Bt}) + (d^2\Theta_t.\vec{w}_{Bt}).\vec{v}_{Bt})(\vec{x}_{Bt}), \tag{11.21}$$

$$(d(\vec{v}_{At}-\vec{v}_{Dt}).\vec{w}_{At})(\vec{x}_{At}) = (d\Theta_t.(d\vec{v}_{Bt}.\vec{w}_{Bt}) + (d^2\Theta_t.\vec{v}_{Bt}).\vec{w}_{Bt})(\vec{x}_{Bt}), \tag{11.22}$$

$$(d(\vec{v}_{At}-\vec{v}_{Dt}).(\vec{v}_{At}-\vec{v}_{Dt}))(\vec{x}_{At}) = (d\Theta_t.(d\vec{v}_{Bt}.\vec{v}_{Bt}) + d^2\Theta_t(\vec{v}_{Bt},\vec{v}_{Bt}))(\vec{x}_{Bt}), \tag{11.23}$$

$$\mathcal{L}^0_{(\vec{v}_{At}-\vec{v}_{Dt})}\vec{w}_{At}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\mathcal{L}^0_{\vec{v}_{Bt}}\vec{w}_{Bt}(\vec{x}_{Bt}), \tag{11.24}$$

$$\frac{\partial\vec{w}_A}{\partial t}(t,\vec{x}_{At}) + \mathcal{L}^0_{\vec{v}_D}\vec{w}_{At}(\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\frac{\partial\vec{w}_B}{\partial t}(t,\vec{x}_{Bt}), \tag{11.25}$$

$$\frac{D\vec{w}_A}{Dt}(t,\vec{x}_{At}) - d\vec{v}_{Dt}.\vec{w}_{At}(\vec{x}_{At}) = d\Theta_t.(\vec{x}_{Bt}).\frac{D\vec{w}_B}{Dt}(t,\vec{x}_{Bt}) + d^2\Theta_t(\vec{v}_{Bt},\vec{w}_{Bt})(\vec{x}_{Bt}), \tag{11.26}$$

$$\frac{\partial(\vec{v}_A-\vec{v}_D)}{\partial t}(t,\vec{x}_{At}) + \mathcal{L}^0_{\vec{v}_D}(\vec{v}_A-\vec{v}_D)(t,\vec{x}_{At}) = d\Theta_t(\vec{x}_{Bt}).\frac{\partial\vec{v}_B}{\partial t}(t,\vec{x}_{Bt}). \tag{11.27}$$

**Proof.** • $\vec{w}_{At}(\Theta_t(\vec{x}_{Bt})) = d\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt})$ gives

$$d\vec{w}_{At}(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d^2\Theta_t(\vec{x}_{Bt}).\vec{w}_{Bt}(\vec{x}_{Bt}) + d\Theta_t(\vec{x}_{Bt}).d\vec{w}_B(\vec{x}_{Bt}), \tag{11.28}$$

thus, with $d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) = (\vec{v}_{At}-\vec{v}_{Dt})(\vec{x}_{At}) = \vec{v}_{Bt*}(\vec{x}_{At})$ (velocity-addition formula),

$$d\vec{w}_{At}(\vec{x}_{At}).(\vec{v}_{At}-\vec{v}_{Dt})(\vec{x}_{At}) = (d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})).\vec{w}_{Bt}(\vec{x}_{Bt}) + d\Theta_t(\vec{x}_{Bt}).d\vec{w}_{Bt}(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}),$$

hence (11.21). In particular $d\vec{w}_{At}(\vec{x}_{At}).\vec{v}_{At}(\vec{x}_{At}) \neq d\Theta_t(\vec{x}_{Bt}).(d\vec{w}_{Bt}(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}))$ (the vector field $d\vec{w}.\vec{v}$ is not objective).

• $(\vec{v}_{At}-\vec{v}_{Dt})(\Theta_t(\vec{x}_{Bt})) = d\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt})$ gives

$$d(\vec{v}_{At}-\vec{v}_{Dt})(\vec{x}_{At}).d\Theta_t(\vec{x}_{Bt}) = d^2\Theta_t(\vec{x}_{Bt}).\vec{v}_{Bt}(\vec{x}_{Bt}) + d\Theta_t(\vec{x}_{Bt}).d\vec{v}_{Bt}(\vec{x}_{Bt}),$$

so, applied to $\vec{w}_{Bt}$ (resp. $\vec{v}_{Bt}$), we get (11.22) (resp. (11.23)). Hence (11.24).

• If $\vec{x}_{At} = \Theta_t(\vec{x}_B)$, then $\vec{w}_A(t,\Theta(t,\vec{x}_B)) = d\Theta(t,\vec{x}_B).\vec{w}_B(t,\vec{x}_B)$, so, with $\frac{\partial\Theta}{\partial t}(t,\vec{x}_B) = \vec{v}_{\Theta t}(\vec{x}_{At})$, we get

$$\begin{aligned}\frac{\partial\vec{w}_A}{\partial t}(t,\vec{x}_{At}) + d\vec{w}_{At}(\vec{x}_{At}).\vec{v}_{\Theta t}(\vec{x}_{At}) &= d\frac{\partial\Theta}{\partial t}(t,\vec{x}_B).\vec{w}_{Bt}(\vec{x}_B) + d\Theta_t(\vec{x}_B).\frac{\partial\vec{w}_B}{\partial t}(t,\vec{x}_B)\\ &= (d\vec{v}_{\Theta t}(\vec{x}_{At}).d\Theta_t(\vec{x}_B)).\vec{w}_{Bt}(\vec{x}_B) + d\Theta_t(\vec{x}_B).\frac{\partial\vec{w}_B}{\partial t}(t,\vec{x}_B),\end{aligned}$$

Thus (11.25) since $\vec{v}_\Theta = \vec{v}_D$; Then (11.21) gives (11.26).

• $\vec{v}_{B*}(t,\Theta(t,\vec{x}_B)) = d\Theta(t,\vec{x}_B).\vec{v}_B(t,\vec{x}_B)$ gives

$$\frac{\partial\vec{v}_{B*}}{\partial t}(t,\vec{x}_{At}) + d\vec{v}_{B*}(\vec{x}_{At}).\vec{v}_\Theta(t,\vec{x}_{At}) = \underbrace{\frac{\partial d\Theta}{\partial t}(t,\vec{x}_B)}_{d\vec{v}_{\Theta t}(\vec{x}_{At}).d\Theta_t(\vec{x}_B)}.\vec{v}_{Bt}(\vec{x}_B) + d\Theta(t,\vec{x}_B).\frac{\partial\vec{v}_B}{\partial t}(t,\vec{x}_B,)$$

since $\frac{\partial d\Theta}{\partial t}(t,\vec{x}_B) = d(\frac{\partial\Theta}{\partial t})(t,\vec{x}_B)$ and $\frac{\partial\Theta}{\partial t}(t,\vec{x}_B) = \vec{v}_\Theta(t,\vec{x}_{At}) = \vec{v}_{\Theta_t}(\Theta_t(\vec{x}_B))$; hence (11.27). ▪

### 11.4.3 Tensors

**Proposition 11.17** *It $T$ is a covariant objective tensor, then its Lie derivatives are covariant objectives:*

$$\mathcal{L}_{\vec{v}_A}T_A = \Theta_*(\mathcal{L}_{\vec{v}_B}T_B). \tag{11.29}$$

**Proof.** Corollary of (11.15) and (11.18) to get $\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) = (\mathcal{L}_{\vec{v}}\alpha).\vec{w} + \alpha.(\mathcal{L}_{\vec{v}}\vec{w})$; Then use $\mathcal{L}_{\vec{v}}(t_1\otimes t_2) = (\mathcal{L}_{\vec{v}}t_1)\otimes t_2 + t_1\otimes(\mathcal{L}_{\vec{v}}t_2)$. ▪

## 11.5 Taylor expansions and ubiquity gift

### 11.5.1 First order Taylor expansion and ubiquity issue

Let $\vec{w} : \mathbb{R}\times\mathbb{R}^n \to \vec{\mathbb{R}^n}$ be regular and $p(t) = \Phi^{t_0}(t, p_{t_0})$. With $\vec{f}(t) = \vec{w}(t, p(t))$, $\vec{f}(t) = \vec{f}(t_0)+(t-t_0)\,\vec{f}'(t_0)+o(t-t_0)$ (first order Taylor expansion), we get

$$\vec{w}(t, p(t)) = \vec{w}(t_0, p_{t_0}) + h\,\frac{D\vec{w}}{Dt}(t_0, p_{t_0}) + o(t-t_0). \tag{11.30}$$

**Issue:** The left hand side $\vec{w}(t, p(t))$ lives in $T_{p_t}(\Omega_t)$ while the right hand side (calculation) $\vec{w}(t_0, p_{t_0}) + h\,\frac{D\vec{w}}{Dt}(t_0, p_{t_0})$ lives in $T_{p_{t_0}}(\Omega_{t_0})$. Thus (11.30) is meaningless: To be meaningful, the $\vec{w}(t, p(t))$ term should first be pull-backed by $\Phi^{t_0}_t(p_{t_0})$ to be compared with $\vec{w}(t_0, p_{t_0})$ (or the $\vec{w}(t_0, p_{t_0})$ term should first be push-forwarded by $\Phi^{t_0}_t(p_{t_0})$ to be compared with $\vec{w}(t, p_t)$). E.g., in a non-planar manifold (e.g. in a surface in $\mathbb{R}^3$), $\vec{w}(t, p_t)$ and $\vec{w}(t_0, p_{t_0})$ don't belong to the same vector space (the "tangent spaces" $T_{p_t}(\Omega_t)$ and $T_{p_{t_0}}(\Omega_{t_0})$ are different in general).

**Ok with Lie:** With the Lie derivative defined with pull-backs, i.e.

$$\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) \overset{(9.11)}{=} \frac{d\Phi^{t_0}_t(p_{t_0})^{-1}.\vec{w}(t, p(t)) - \vec{w}(t_0, p_{t_0})}{t-t_0} + o(1) \quad \Big(= \frac{(\Phi^{t_0*}_t\vec{w} - \vec{w})(t_0, p_{t_0})}{t-t_0} + o(1)\Big); \tag{11.31}$$

It is an equation in $T_{p_{t_0}}(\Omega_{t_0})$ which gives the first order Taylor expansion in $T_{p_{t_0}}(\Omega_{t_0})$: With $h = t-t_0$:

$$d\Phi^{t_0}_t(p_{t_0})^{-1}.\vec{w}(t, p(t)) = \vec{w}(t_0, p_{t_0}) + h\,\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) + o(h) \quad (= \Phi^{t_0*}_t\vec{w}(t_0, p_{t_0})). \tag{11.32}$$

Or with push-forwards: We have obtained the first order Taylor expansion in $T_{p_t}(\Omega_t)$: With $h = t-t_0$:

$$\begin{aligned}\vec{w}(t, p(t)) &= d\Phi^{t_0}_t(p_{t_0}).(\vec{w}(t_0, p_{t_0}) + h\,\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) + o(h))\\ &= d\Phi^{t_0}_t(p_{t_0}).\vec{w}(t_0, p_{t_0}) + h\,d\Phi^{t_0}_t(p_{t_0}).\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}) + o(h)\\ &= (\Phi^{t_0}_{t*}\vec{w})(t, p(t)) + h\,\Phi^{t_0}_{t*}(\mathcal{L}_{\vec{v}}\vec{w})(t, p(t)) + o(h).\end{aligned} \tag{11.33}$$

**Proposition 11.18** *In $\mathbb{R}^n$, with the gift of ubiquity, (11.33) gives (11.30) (of course).*

*Interpretation: Because ubiquity gifts don't exist, (11.30) is meaningless while (11.33) is meaningful; Which tells that "The Lie derivative is the meaningful derivative in physical sciences".*

**Proof.** With $d\Phi^{t_0}(t_0{+}h, p_{t_0}) \overset{(4.38)}{=} I + h\,d\vec{v}(t_0, p_{t_0}) + o(h)$ and $\mathcal{L}_{\vec{v}}\vec{w} \overset{(9.18)}{=} \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}$, (11.33) gives

$$\begin{aligned}\vec{w}(t, p(t)) &= \underbrace{d\Phi^{t_0}_t(p_{t_0})}_{(I + h\,d\vec{v}(t_0, p_{t_0}) + o(h))} \;.\; \underbrace{(\vec{w}(t_0, p_{t_0}) + h\,\mathcal{L}_{\vec{v}}\vec{w}(t_0, p_{t_0}))}_{(\vec{w} + h\,(\frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}))(t_0, p_{t_0}) + o(h)} + o(h)\\ &= (\vec{w} + h\,(\frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w}) + h\,d\vec{v}.\vec{w})(t_0, p_{t_0}) + o(h),\end{aligned}$$

which is (11.30). ∎

### 11.5.2 Second order Taylor expansion

In $\mathbb{R}^n$, with $\vec{w} \in C^2$ let $\vec{f} : \left\{\begin{array}{rl} ]t_0{-}\varepsilon, t_0{+}\varepsilon[ & \to \vec{\mathbb{R}^n} \\ t & \to \vec{f}(t) := \vec{w}(t, p(t)) \end{array}\right\}$. Thus $\vec{f}$ is $C^2$, and $\vec{f}(t) = \vec{f}(t_0) + h\,\vec{f}'(t_0) + \frac{h^2}{2}\vec{f}''(t_0) + o(h^2)$ where $h = t-t_0$ (second order Taylor expansion). Thus, near $(t_0, p_{t_0})$,

$$\vec{w}(t, p(t)) = (\vec{w} + h\,\frac{D\vec{w}}{Dt} + \frac{h^2}{2}\,\frac{D^2\vec{w}}{Dt^2})(t_0, p(t_0)) + o(h^2). \tag{11.34}$$

Once again there is an ubiquity issue. Without ubiquity gifts, we have "the second order Taylor expansion:

$$\Phi^{t_0*}_t\vec{w}(t, p(t)) = \big(\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w})\big)(t_0, p_{t_0}) + o(h^2), \tag{11.35}$$

i.e. $d\Phi^{t_0}_t(p_{t_0})^{-1}.\vec{w}(t, p(t)) = \big(\vec{w} + h\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w})\big)(t_0, p_{t_0}) + o(h^2)$ (pull-back),
i.e. $\vec{w}(t, p(t)) = \Phi^{t_0}_{t*}(\vec{w} + h\,\mathcal{L}_{\vec{v}}\vec{w} + \frac{h^2}{2}\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{w}))(t, p(t)) + o(h^2)$ (push-forward). Indeed:

**Proposition 11.19** *In $\mathbb{R}^n$, with the gift of ubiquity, (11.35) gives (11.34).*

**Proof.** (4.37) gives $F^{t_0}_{p_{t_0}}(t) = I_{t_0} + h\, d\vec v(t_0,p_{t_0}) + \frac{h^2}{2}\, d\vec\gamma(t_0,p_{t_0}) + o(h^2)$. Thus, omitting the reference to $(t_0,p_{t_0})$ to lighten the writing, (11.35) gives

$$\begin{aligned} &d\Phi^{t_0}_t(p_{t_0}).(\vec w + h\mathcal{L}_{\vec v}\vec w + \frac{h^2}{2}\mathcal{L}_{\vec v}\mathcal{L}_{\vec v}\vec w + o(h^2)) \\ &= \Big(I + h\, d\vec v + \frac{h^2}{2}\, d(\frac{D\vec v}{Dt}) + o(h^2)\Big).\Big(\vec w + h\mathcal{L}_{\vec v}\vec w + \frac{h^2}{2}\mathcal{L}_{\vec v}\mathcal{L}_{\vec v}\vec w + o(h^2)\Big) \end{aligned} \tag{11.36}$$

The $h^0$ term is $I.\vec w = \vec w$. The $h$ term is $\mathcal{L}_{\vec v}\vec w + d\vec v.\vec w = \frac{D\vec w}{Dt}$. The $h^2$ term is the sum of

- $\frac12\mathcal{L}_{\vec v}\mathcal{L}_{\vec v}\vec w = \frac12(\frac{D^2\vec w}{Dt^2} - 2\, d\vec v.\frac{D\vec w}{Dt} - \frac{D(d\vec v)}{Dt}.\vec w + d\vec v.d\vec v.\vec w)$, cf.(9.50),
- $d\vec v.\mathcal{L}_{\vec v}\vec w = d\vec v.\frac{D\vec w}{Dt} - d\vec v.d\vec v.\vec w = \frac12(2d\vec v.\frac{D\vec w}{Dt} - 2d\vec v.d\vec v.\vec w)$,
- $\frac12 d(\frac{D\vec v}{Dt}).\vec w = \frac12(\frac{D(d\vec v)}{Dt}.\vec w + d\vec v.d\vec v.\vec w)$, cf.(2.30),

which indeed gives $\frac{D^2\vec w}{Dt^2}$. ▪

### 11.5.3 Higher order Taylor expansion

**Exercice 11.20** Let $\vec w \in C^n$ and $\mathcal{L}^{(n)}_{\vec v} = \mathcal{L}_{\vec v}\circ ...\circ\mathcal{L}_{\vec v}$ ($n$-times). For all $n \in \mathbb{N}^*$, prove (Taylor expansion)

$$\vec w(t,p(t)) = d\Phi^{t_0}_t(p_{t_0}).(\vec w + (t-t_0)\mathcal{L}_{\vec v}\vec w + ... + \frac{(t-t_0)^n}{n!}\mathcal{L}^{(n)}_{\vec v}\vec w)(t_0,p_{t_0}) + o((t-t_0)^n), \tag{11.37}$$

i.e. $F^{t_0}_t(p_{t_0})^{-1}.\vec w(t,p(t)) = \big(\sum_{k=0}^n \frac{(t-t_0)^k}{k!}(\mathcal{L}_{\vec v})^{(k)}\vec w\big)(t_0,p_{t_0}) + o((t-t_0)^n)$ in $T_{p_{t_0}}(\Omega_{t_0})$.

**Answer**. (Proof similar to one of the classical proof of Taylor's theorem.) $t_0$ and $p_{t_0}$ are fixed, $p(t) = \Phi^{t_0}(t,p_{t_0})$, and $H^{t_0}(t,p(t)) := H^{t_0}_t(p(t)) := F^{t_0}_t(p_{t_0})^{-1}$. With

$$\vec f_{\vec w,n}(t) = (H^{t_0}.\vec w)(t,p(t)) - (\vec w + (t-t_0)\mathcal{L}_{\vec v}\vec w + ... + \frac{(t-t_0)^n}{n!}\mathcal{L}^{(n)}_{\vec v}\vec w)(t_0,p_{t_0}), \tag{11.38}$$

we have to prove: $f_{\vec w,n}(t) = o((t-t_0)^n)$ (which means $\forall\varepsilon>0$, $\exists h>0$, $\forall t\in[t_0-h,t_0+h]$, $||\vec f_{\vec w,n}(t)||_g \le \varepsilon$).

Recurrence hypothesis: With $n\in\mathbb{N}^*$, for all $\vec w \in C^n$ , $||\vec f_{\vec w,n}(t)||_g = o((t-t_0)^n)$.

This is true for $n$=1, cf. (11.32). Suppose it is true for $n$.

Let $\vec w \in C^{n+1}$. With $\frac{DH^{t_0}}{Dt} = -H^{t_0}.d\vec v$, cf. (4.48), we get

$$\begin{aligned} \vec f_{\vec w,n+1}{}'(t) &= (-H^{t_0}.d\vec v.\vec w + H^{t_0}.\frac{D\vec w}{Dt})(t,p(t)) - (0 + \mathcal{L}_{\vec v}\vec w + ... + \frac{(t-t_0)^n}{n!}\mathcal{L}^{(n+1)}_{\vec v}\vec w)\Big)(t_0,p_{t_0}) \\ &= (H^{t_0}.\mathcal{L}_{\vec v}\vec w)(t,p(t)) - (\mathcal{L}_{\vec v}\vec w + ... + \frac{(t-t_0)^n}{n!}\mathcal{L}^n_{\vec v}.\mathcal{L}_{\vec v}\vec w)\Big)(t_0,p_{t_0}) = \vec f_{\mathcal{L}_{\vec v}\vec w,n}(t). \end{aligned} \tag{11.39}$$

And the mean value theorem tells $\frac{||\vec f_{\vec w,n+1}(t)-\vec f_{\vec w,n+1}(t_0)||_g}{|t-t_0|} \le \sup_{\tau\in[t_0-h,t_0+h]}||\vec f_{\vec w,n+1}{}'(\tau)||_g$; And $\vec f_{\vec w,n+1}(t_0) =^{(11.35)} \vec 0$, thus $\frac{||\vec f_{\vec w,n+1}(t)||_g}{|t-t_0|} \le \sup_{\tau\in[t_0-h,t_0+h]}||\vec f_{\mathcal{L}_{\vec v}\vec w,n}(\tau)||_g$. And, $\mathcal{L}_{\vec v}\vec w \in C^n$, hence the recurrence hypothesis tells: $||\vec f_{\mathcal{L}_{\vec v}\vec w,n}(t)||_g = o((t-t_0)^n)$. Thus $\frac{||\vec f_{\vec w,n+1}(t)||_g}{|t-t_0|} = o((t-t_0)^n$, thus $||\vec f_{\vec w,n+1}(t)||_g = o((t-t_0)^{n+1}$. ▪

# 12 The virtual work and power principles

## 12.1 Newton fundamental laws

(See e.g. Germain [11]). Consider $N \geq 1$ distinct particles $P_{Obj_i}$ of mass $m_i$ which make the "body" $Obj = \{P_{Obj_1}, ..., P_{Obj_N}\}$. In our universe, at $t$ call $p_{it} =^{\text{written}} p_i \in \mathbb{R}^n$ the position of $P_{Obj_j}$ and $\Omega_t := \{p_{1t}, ..., p_{Nt}\}$. Each $P_{Obj_i}$ is subject at $t$ to the acceleration $\vec{\gamma}_t(p_{it}) =^{\text{written}} \vec{\gamma}_i$, to the external force $\vec{f}_t(p_{it}) =^{\text{written}} \vec{f}_i$, and to the internal forces $\vec{f}_{t,p_{jt}}(p_{it}) =^{\text{written}} \vec{f}_{ji}$ due to the other $P_{Obj_j}$.

**Newton postulates**: There exists a Galilean referential $\mathcal{R}_a$ (called absolute) s.t. at any $t$:

- 1st law (Galileo law of inertia): "a body not acted upon remains at constant speed". (12.1)
- 2nd law (Newton): $\forall i = 1, ..., N: \; m_i\vec{\gamma}_i = \vec{f}_i + \sum_{j=1}^{N} \vec{f}_{ji}$. (12.2)
- 3rd law (of action and reaction): $\forall i, j = 1, ..., N: \; \vec{f}_{ji} = -\vec{f}_{ij}$ and $\vec{f}_{ij} \parallel \overrightarrow{p_ip_j}$. (12.3)

Remarks: - If $N = 1$ (one particle), then (12.2) reads $m\vec{\gamma} = \vec{f}$ and (12.3) is trivial.
- $\vec{f}_{ii} = \vec{0}$ for all $i$.
- The laws apply to any subset of $Obj$ (the other particles being considered external).

## 12.2 D'Alembert formulation

### 12.2.1 The virtual power formulation, discrete framework

At $t$, with the above discrete Eulerian vectors fields $\vec{\gamma}_t, \vec{f}_t, \vec{f}_{t,p_{jt}} : \Omega_t \to \vec{\mathbb{R}^3}$, consider any discrete Eulerian vector field $\vec{u}_t : p \in \Omega_t \to \vec{u}_t(p) \in \vec{\mathbb{R}^3}$ called virtual vector field, and let $\vec{u}_t(p_i) =^{\text{written}} \vec{u}_i$.

Then choose a Euclidean dot product $(\cdot, \cdot)_g =^{\text{written}} . \bullet .$ in $\vec{\mathbb{R}^3}$.

**Definition 12.1** At $t$, the acceleration virtual power, the external virtual power, the internal virtual power relative to $\vec{u}$ are the scalars

$$\mathcal{P}_a(\vec{u}) = \sum_{i=1}^{N} m_i\vec{\gamma}_i \bullet \vec{u}_i, \quad \mathcal{P}_e(\vec{u}) = \sum_{i=1}^{N} \vec{f}_i \bullet \vec{u}_i, \quad \mathcal{P}_{int}(\vec{u}) = \sum_{i=1}^{N}(\sum_{j=1}^{N} \vec{f}_{ji}) \bullet \vec{u}_i. \tag{12.4}$$

Remark: If $N = 1$ (one particle), then $\mathcal{P}_a(\vec{u}) = m\vec{\gamma} \bullet \vec{u}$, $\mathcal{P}_e(\vec{u}) = \vec{f} \bullet \vec{u}$, and $\mathcal{P}_{int}(\vec{u}) = 0$.

**D'Alembert virtual power formulation**[3] (variational formulation of 2nd and 3rd Newton's laws). There exists a Galilean referential $\mathcal{R}_a$ s.t. at any $t$, together with Galileo's law of inertia,

$$\forall \vec{u} \in \mathcal{F}(\Omega_t; \vec{\mathbb{R}^3}), \quad \mathcal{P}_a(\vec{u}) = \mathcal{P}_e(\vec{u}) + \mathcal{P}_{int}(\vec{u}). \tag{12.5}$$

**Interpretation (Germain):** To measure a force needed to move $Obj$, you need to move the $P_{Obj_i}$, i.e. you need to measure a work (subsequently a power), i.e. you need d'Alembert's formulation. Germain's words: "to know the weight of a suitcase you have to move it" (it is not enough to look at it).

**Proposition 12.2** *1- (12.2) is equivalent to (12.5).*

*2- (12.3) is equivalent to: $\mathcal{P}_{int}(\vec{u}) = 0$ for all discrete rigid body velocity field $\vec{u} \in \mathcal{F}(\{p_1, ..., p_N\}; \vec{\mathbb{R}^3})$.*

**Proof.** 1- (12.2) $\Leftrightarrow \Big(m_i\vec{\gamma}_i - \vec{f}_i - \sum_{j\neq i} \vec{f}_{ji} = \vec{0}$ for all $i\Big) \Leftrightarrow \Big((m_i\vec{\gamma}_i - \vec{f}_i - \sum_{j\neq i} \vec{f}_{ji}) \bullet \vec{u}_i = 0$ for all $\vec{u}_i\Big) \Leftrightarrow \Big(\sum_i (m_i\vec{\gamma}_i - \vec{f}_i - \sum_{j\neq i} \vec{f}_{ji}) \bullet \vec{u}_i = 0$ for all $(\vec{u}_i)_{i=1,...,N}\Big) \Leftrightarrow \Big(\mathcal{P}_a(\vec{u}) - \mathcal{P}_e(\vec{u}) - \mathcal{P}_{int}(\vec{u}) = 0$ for all $\vec{u} \in (\vec{\mathbb{R}^3})^N\Big)$.

2- Consider the body $B = \{P_{Obj_1}, P_{Obj_2}\}$ (the others particles being considered external). A rigid body motion of $B$ is characterized by $\vec{u}_2 = \vec{u}_1 + \vec{\omega} \times \overrightarrow{p_1p_2}$. Having $\vec{f}_{ii} = \vec{0}$, the internal virtual power is $\mathcal{P}_{int}(\vec{u}) = \vec{f}_{21} \bullet \vec{u}_1 + \vec{f}_{12} \bullet \vec{u}_2 = (\vec{f}_{21} + \vec{f}_{12}) \bullet \vec{u}_1 + \vec{f}_{12} \bullet (\vec{\omega} \times \overrightarrow{p_1p_2}) = (\vec{f}_{21} + \vec{f}_{12}) \bullet \vec{u}_1 + \vec{\omega} \bullet (\overrightarrow{p_1p_2} \times \vec{f}_{12})$.

21- Suppose (12.3), i.e. $\vec{f}_{21} + \vec{f}_{12} = \vec{0}$ and $\overrightarrow{p_1p_2} \times \vec{f}_{12} = \vec{0}$: A rigid body motion of $\{p_1, p_2\}$ gives $\mathcal{P}_{int}(\vec{u}) = 0 + 0$.

[3] Also called Lagrange, Euler, ... virtual power formulation

22- Suppose $\mathcal{P}_{int}(\vec{u}) = 0$ for all rigid body motion of $\{p_1, p_2\}$: $(\vec{f}_{21} + \vec{f}_{12}) \bullet \vec{u}_1 + \vec{f}_{12} \bullet (\vec{\omega} \times \overrightarrow{p_1 p_2}) = 0$ for all $\vec{u}_1, \vec{\omega}$. In particular $\vec{\omega} = \vec{0}$ (translation) gives $(\vec{f}_{21} + \vec{f}_{12}) \bullet \vec{u}_1 = 0$ for all $\vec{u}_1$, thus $\vec{f}_{21} + \vec{f}_{12} = \vec{0}$. We are left with $\vec{f}_{12} \bullet (\vec{\omega} \times \overrightarrow{p_1 p_2}) = \vec{0} = \vec{\omega} \bullet (\overrightarrow{p_1 p_2} \times \vec{f}_{12})$ for all $\vec{\omega}$, thus $\overrightarrow{p_1 p_2} \times \vec{f}_{12} = \vec{0}$.

23- Idem for any two particles at $P_{Obj_i}$ and $P_{Obj_j}$ for all $i, j$. And a rigid body motion of $\Omega_t = \{P_{Obj_1}, ..., P_{Obj_N}\}$ implies a rigid body motion of any $\{P_{Obj_i}, P_{Obj_j}\}$. ▪

### 12.2.2 Towards continuum: $L^2(\Omega)$ framework

$\Omega$ is an open set in $\mathbb{R}^n$. The space of finite energy scalar valued functions, its usual inner dot product and norm are:

$$\begin{aligned} & L^2(\Omega) := \{u : \Omega \to \mathbb{R} \text{ s.t. } \int_{p\in\Omega} u(p)^2 \, d\Omega < \infty\}, \\ & (u, w)_{L^2} := \int_{p\in\Omega} u(p) w(p) \, d\Omega \overset{\text{written}}{=} \int_{\Omega} uw \, d\Omega, \quad ||u||_{L^2} = \sqrt{(u,u)_{L^2}} = \Big(\int_{p\in\Omega} u(p)^2 \, d\Omega\Big)^{\frac{1}{2}}. \end{aligned} \tag{12.6}$$

Choose a Euclidean dot product $.\bullet.$ in $\vec{\mathbb{R}^n}$ with its associated norm $||.||_{\mathbb{R}^n} =^{\text{written}} ||.||$. The space of finite energy vector fields $\vec{u} \in T_0^1(\Omega)$, $\vec{u} : p \in \Omega \to \vec{u}(p) \in \vec{\mathbb{R}^n}$ (simplified notations for $\vec{u} : p \in \Omega \to (p, \vec{u}(p)) \in \Omega \times \vec{\mathbb{R}^n}$), its usual inner dot product and norm are:

$$\begin{aligned} & L^2(\Omega)^n = \{\vec{u} : \Omega \to \vec{\mathbb{R}^n} \text{ s.t. } \int_{p\in\Omega} ||\vec{u}(p)||^2 \, d\Omega < \infty\}, \\ & (\vec{u}, \vec{w})_{L^2} := \int_{p\in\Omega} \vec{u}(p) \bullet \vec{w}(p) \, d\Omega, \quad ||\vec{u}||_{L^2} = \sqrt{(\vec{u},\vec{u})_{L^2}} = \Big(\int_{p\in\Omega} ||\vec{u}(p)||^2 \, d\Omega\Big)^{\frac{1}{2}}. \end{aligned} \tag{12.7}$$

### 12.2.3 D'Alembert formulation, continuous framework

In (12.4), replace the sum sign $\sum$ by the sum sign $\int$: Consider a body $Obj$ made of particles $P_{Obj}$ and a motion $\widetilde{\Phi} : \left\{ \begin{array}{ll} [t_1, t_2] \times Obj & \to \mathbb{R}^n \\ (t, P_{Obj}) & \to p_t = \widetilde{\Phi}(t, P_{Obj}) \end{array} \right\}$ of $Obj$ where $\Omega_t := \widetilde{\Phi}(t, Obj)$ is an open subset in $\mathbb{R}^n$ at all $t$. Choose a Euclidean referential $\mathcal{R}$, and, at any $t$, call $\vec{\gamma}_t(p_t)$ the acceleration of $P_{Obj}$, $\vec{f}_t(p_t)$ the external force on $P_{Obj}$, $\rho_t(p_t)$ the mass density, and take a (so-called virtual) vector field $\vec{u}_t$ in $\Omega_t$.

**Definition 12.3** The acceleration, external and internal virtual powers relative to $\vec{u}$ at $t$ are

$$\begin{aligned} & \mathcal{P}_a(\vec{u}_t) := \int_{p\in\Omega_t} \rho(p) \vec{\gamma}_t(p) \bullet \vec{u}_t(p) \, d\Omega \overset{\text{written}}{=} \int_{\Omega_t} \rho \vec{\gamma}_t \bullet \vec{u}_t \, d\Omega \overset{\text{written}}{=} \int_{\Omega_t} p_a(t, \vec{u}_t) \, d\Omega, \\ & \mathcal{P}_e(\vec{u}_t) := \int_{p\in\Omega_t} \vec{f}_t(p) \bullet \vec{u}_t(p) \, d\Omega \overset{\text{written}}{=} \int_{\Omega_t} \vec{f}_t \bullet \vec{u}_t \, d\Omega \overset{\text{written}}{=} \int_{\Omega_t} p_e(t, \vec{u}_t) \, d\Omega, \\ & \mathcal{P}_{int}(\vec{u}_t) := \int_{p\in\Omega_t} p_{int}(t, \vec{u}_t)(p) \, d\Omega \overset{\text{written}}{=} \int_{\Omega_t} p_{int}(t, \vec{u}_t) \, d\Omega, \end{aligned} \tag{12.8}$$

where $p_a, p_e, p_{int}$ are the virtual "acceleration, external force and internal force densities".

**D'Alembert virtual power formulation**. There exists a Galilean Euclidean referential $\mathcal{R}_a$ s.t., together with Galileo's law of inertia, at any $t$ for all (regular enough) vector field $u$,

$$\mathcal{P}_a(\vec{u}_t) = \mathcal{P}_e(\vec{u}_t) + \mathcal{P}_{int}(\vec{u}_t). \tag{12.9}$$

### 12.2.4 Remark: Rigid body motion and Germain's notations

Choose a Euclidean basis $(\vec{e}_i)$, call $.\bullet.$ and $.\times.$ the associated Euclidean dot product and vector product, let

$$\begin{aligned} SC = \text{the screws} & := \{\vec{u} : \mathbb{R}^3 \to \vec{\mathbb{R}^3} \text{ s.t. } \exists \vec{\omega} \in \vec{\mathbb{R}^3}, \ \forall p, q \in \Omega, \ \vec{u}(q) = \vec{u}(p) + \vec{\omega} \times \overrightarrow{pq}\} \\ & = \{\vec{u} : \mathbb{R}^3 \to \vec{\mathbb{R}^3} \text{ s.t. } \exists \vec{\omega} \in \vec{\mathbb{R}^3}, \ \forall q \in \Omega, \ \vec{u}(q) = \vec{u}(O) + \vec{\omega} \times \overrightarrow{Oq}\} \end{aligned} \tag{12.10}$$

independent of a chosen "origin" $O \in \mathbb{R}^3$ (trivial check: $\vec{u}$ is affine). And $SC$ is a vector space (trivial check), and $\dim(SC) = 6$ because $\vec{u}(O)$ and $\vec{\omega}$ characterize a screw (6 degrees of freedom).

Recall:

- The velocity field of a rigid body motion is a screw called a twist or a kinematic screw or a distributor.
- A screw which is the moment of a force field is called a wrench.

Germain notations:

- A virtual twist is noted $\widehat{\vec{u}}$ (with a "hat"), $\widehat{\vec{u}}(q) = \widehat{\vec{u}}(p) + \widehat{\vec{\omega}} \times \vec{pq}$, and is represented by the $6*1$ matrix $\{\widehat{C}\} = \begin{pmatrix} [\widehat{\vec{u}}(p)]_{|\vec{e}} \\ [\widehat{\vec{\omega}}]_{|\vec{e}} \end{pmatrix} \overset{\text{written}}{=} \begin{pmatrix} \widehat{\vec{u}}(p) \\ \widehat{\vec{\omega}} \end{pmatrix}$ (reduction elements of $\widehat{\vec{u}}$ at $p$).
- A wrench is noted $\vec{m}$ (for moment), $\vec{m}(q) = \vec{m}(p) + \vec{F} \times \vec{pq}$, and is represented by the $6*1$ matrix $[\mathcal{F}] = \begin{pmatrix} [\vec{F}]_{|\vec{e}} \\ [\vec{m}(p)]_{|\vec{e}} \end{pmatrix} =^{\text{written}} \begin{pmatrix} \vec{F} \\ \vec{m}(p) \end{pmatrix}$ (reduction elements of $\vec{m}$ at $p$ in this order).
- $\mathcal{SC}'$ is the dual of $\mathcal{SC}$, i.e. the set of linear forms $\ell : \mathcal{SC} \to \mathbb{R}$.
- If $\mathcal{SC}$ = the twists, then $\mathcal{SC}'$ = the wrenches; If $\mathcal{SC}$ = the wrenches, then $\mathcal{SC}'$ = the twists.
- The dual product of an element $M \in \mathcal{SC}$ and an element $N \in \mathcal{SC}'$ is noted $M.N$.

**Proposition 12.4** *$\Omega$ is bounded, $p \in \Omega$, $\ell \in \mathcal{SC}'$ represented by $[\mathcal{F}] = \begin{pmatrix} \vec{F} \\ \vec{m}(p) \end{pmatrix}$ (so $\vec{m}(q) = \vec{m}(p) + \vec{F} \times \vec{pq}$). For all $\widehat{\vec{u}} \in \mathcal{SC}$ represented by $\{\widehat{C}\} = \begin{pmatrix} \widehat{\vec{u}}(p) \\ \widehat{\vec{\omega}} \end{pmatrix}$ (so $\widehat{\vec{u}}(q) = \widehat{\vec{u}}(p) + \widehat{\vec{\omega}} \times \vec{pq}$) we have*

$$\ell.\widehat{\vec{u}} = \vec{F} \bullet \widehat{\vec{u}}(p) + \vec{m}(p) \bullet \widehat{\vec{\omega}} \overset{\text{written}}{=} [\mathcal{F}].\{\widehat{C}\}. \tag{12.11}$$

*(In fact should be noted $[\mathcal{F}]^T.\{\widehat{C}\}$ if the matrix product is understood; The notation $[\mathcal{F}].\{\widehat{C}\}$ means that the canonical inner dot product in the vector space $M_{61}$ of $6*1$ matrices is implicit: $[\mathcal{F}]$ and $\{\widehat{C}\}$ do not belong to a same space, so $[\mathcal{F}].\{\widehat{C}\}$ can't be anything else.)*

**Proof.** $\Omega$ bounded implies $\mathcal{SC} \subset L^2(\Omega)$: Indeed, $\int_\Omega ||\vec{u}(p)||^2 \, d\Omega = \int_\Omega ||\vec{u}(O) + \vec{\omega} \times \vec{Op}||^2 \, d\Omega \le \int_\Omega 2||\vec{u}(O)||^2 + 2||\vec{\omega}|| \, ||\vec{Op}||^2 \, d\Omega < \infty$, since the volume of $\Omega$ is bounded.

And $\mathcal{SC}$ is a vector space (sub-vector space of $L^2(\Omega)$); Indeed, $\vec{u}(q) = \vec{u}(p) + \vec{\omega}_{\vec{u}} \times \vec{pq}$ and $\vec{v}(q) = \vec{v}(p) + \vec{\omega}_{\vec{v}} \times \vec{pq}$ give $(\vec{u}+\lambda\vec{v})(q) = \vec{u}(q) + \lambda\vec{v}(q) = (\vec{u}+\lambda\vec{v})(p) + (\vec{\omega}_{\vec{u}} + \lambda\vec{\omega}_{\vec{v}}) \times \vec{pq} = (\vec{u}+\lambda\vec{v})(p) + \vec{\omega}_{\vec{u}+\lambda\vec{v}} \times \vec{pq}$ where $\vec{\omega}_{\vec{u}+\lambda\vec{v}} := \vec{\omega}_{\vec{u}} + \lambda\vec{\omega}_{\vec{v}} \in \vec{\mathbb{R}^3}$; Thus $\vec{u}, \vec{v} \in \mathcal{SC}$ implies $\vec{u} + \lambda\vec{v} \in \mathcal{SC}$.

And $\mathcal{SC}$ being finite dimensional ($\dim \mathcal{SC} = 6$), $\mathcal{SC}$ is a closed sub-vector space in $L^2(\Omega)$, thus $(\mathcal{SC}, (\cdot,\cdot)_{L^2})$ is a Hilbert space, and any $\ell \in \mathcal{SC}'$ is (linear) continuous. Hence we can apply the $(\cdot,\cdot)_{L^2}$-Riesz representation theorem: If $\ell \in \mathcal{SC}'$, then $\exists \vec{\ell} \in \mathcal{SC}$, $\forall \vec{u} \in \mathcal{SC}$,

$$\begin{aligned} \ell(\vec{u}) &= (\vec{\ell}, \vec{u})_{L^2} = \int_{q\in\Omega} \vec{\ell}(q) \bullet \vec{u}(q) \, d\Omega = \int_{q\in\Omega} \vec{\ell}(q) \bullet (\vec{u}(p) + \vec{\omega} \times \vec{pq}) \, d\Omega \\ &= \vec{F} \bullet \vec{u}(p) + \vec{\omega} \bullet \vec{m}_e(p) \quad \text{where} \quad \vec{F} = \int_{q\in\Omega} \vec{\ell}(q) \, d\Omega \quad \text{and} \quad \vec{m}(p) = \int_{q\in\Omega} \vec{pq} \times \vec{\ell}(q) \, d\Omega, \end{aligned} \tag{12.12}$$

Thus (12.11). ∎

## 12.3 D'Alembert formulation and linear hypothesis

Setting: Geometric vector space $\vec{\mathbb{R}^n}$, $n = 1, 2, 3$, Euclidean basis $(\vec{e}_i)$ imposed by an observer, associated Euclidean dot product. In this § the "tensor writing" is in fact a matrix writing.

### 12.3.1 First order linear hypothesis

With $\frac{\partial u}{\partial x_j} := du.\vec{e}_j$ and $\vec{\nabla} u := \sum_j \frac{\partial u}{\partial x_j} \vec{e}_j$, let (Hilbert space of order 1 needed for "deformation gradients")

- $H^1(\Omega) := \{u \in L^2(\Omega) : \forall j = 1, ..., n, \ \dfrac{\partial u}{\partial x_j} \in L^2(\Omega)\} \overset{\text{written}}{=} \{u \in L^2(\Omega) : \vec{\nabla} u \in L^2(\Omega)^n\}$,
- $(u, v)_{H^1} = (u, v)_{L^2} + \displaystyle\sum_{j=1}^n \Big(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j}\Big)_{L^2} = \int_{\vec{x}\in\Omega} u(\vec{x}) v(\vec{x}) \, d\Omega + \int_\Omega \vec{\nabla} u(\vec{x}) \bullet \vec{\nabla} v(\vec{x}) \, d\Omega,$ (12.13)
- $||u||_{H^1} := \sqrt{(u, u)_{H^1}}$.

$(H^1(\Omega), (\cdot,\cdot)_{H^1})$ is a Hilbert space (Riesz–Fisher theorem). And $\int_\Omega \vec{\nabla} u(\vec{x}) \bullet \vec{\nabla} v(\vec{x}) \, d\Omega =^{\text{written}} (\vec{\nabla} u, \vec{\nabla} v)_{L^2}$, so $(u, v)_{H^1} =^{\text{written}} (u, v)_{L^2} + (\vec{\nabla} u, \vec{\nabla} v)_{L^2}$.

The dual space of $H^1(\Omega)$ is $H^1(\Omega)'$ the space of continuous linear forms $\ell : H^1(\Omega) \to \mathbb{R}$. We use theorem V.12: If $\ell \in H^1(\Omega)'$ then $\exists (f,\vec{g}) \in L^2(\Omega) \times L^2(\Omega)^n$ s.t., $\forall \psi \in H^1(\Omega)$,

$$\ell(\psi) = (f,\psi)_{L^2} + (\vec{g},\vec{\nabla}\psi)_{L^2} = \int_\Omega f\psi + \vec{g} \bullet \vec{\nabla}\psi \, d\Omega. \tag{12.14}$$

**Application to vector valued functions:** With $\nabla\vec{u} = [\frac{\partial u_i}{\partial x_j}]$ (matrix), let

$$\begin{aligned} &H^1(\Omega)^n = \{\vec{u} \in L^2(\Omega)^n : \nabla\vec{u} \in L^2(\Omega)^{n^2}\} = \{\vec{u} = \sum_i u_i\vec{e}_i \in L^2(\Omega)^n : \forall i,j,\ \frac{\partial u_i}{\partial x_j} \in L^2(\Omega)\}, \\ &(\vec{u},\vec{v})_{H^1} := (\vec{u},\vec{v})_{L^2} + (\nabla\vec{u},\nabla\vec{v})_{L^2}, \quad ||\vec{u}||_{H^1} = \sqrt{(\vec{u},\vec{u})_{H^1}} \end{aligned} \tag{12.15}$$

where $(\vec{u},\vec{v})_{L^2} = \int_\Omega \vec{u} \bullet \vec{v} \, d\Omega$ and $(\nabla\vec{u},\nabla\vec{v})_{L^2} := \sum_{i,j=1}^n \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j} \, d\Omega =^{\text{written}} \int_\Omega \nabla\vec{u} : \nabla\vec{v} \, d\Omega$ where the double matrix contraction $\nabla\vec{u} : \nabla\vec{v} = \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}$ is used.

The dual space of $H^1(\Omega)^n$ is $H^1(\Omega)^{n'} = \{\mathcal{P} : H^1(\Omega)^n \to \mathbb{R}$ linear and continuous$\}$. Thus (application of (12.14) component wise): If $\mathcal{P} \in H^1(\Omega)^{n'}$ then $\exists(\vec{f},\underline{\underline{\sigma}}) \in L^2(\Omega)^n \times L^2(\Omega)^{n^2}$ s.t., $\forall \vec{v} \in H^1(\Omega)^n$,

$$\mathcal{P}(\vec{v}) = (\vec{f},\vec{v})_{L^2} + (\underline{\underline{\sigma}},\nabla\vec{v})_{L^2} = \int_\Omega \vec{f} \bullet \vec{v} + \underline{\underline{\sigma}} : \nabla\vec{v} \, d\Omega \tag{12.16}$$

when $\vec{v} = \sum_i v_i\vec{e}_i$, $\vec{f} = \sum_i f_i\vec{e}_i$, $\underline{\underline{\sigma}} = [\sigma_{ij}]$, and $v_{i,j} = \frac{\partial v_i}{\partial x_j}$.

### 12.3.2 Application: Usual Cauchy's result

The (linear approximation of the) virtual internal power $\mathcal{P}_{int}$ is in $H^1(\Omega)^{n'}$, hence of the type (12.16), and in a Galilean Euclidean referential:

1) $\mathcal{P}_{int}(\vec{v}) = 0$ for all $\vec{v}$ uniform (i.e. $d\vec{v} = 0$). Thus (12.16) gives $(\vec{f},\vec{v})_{L^2} + 0 = 0$ for all $\vec{v}$ uniform, also true for any subset in $\Omega$. Thus $\vec{f} = \vec{0}$ and $\exists \underline{\underline{\sigma}} = [\sigma_{ij}] \in L^2(\Omega)^{n^2}$ s.t., $\forall \vec{v} \in H^1(\Omega)^n$,

$$\mathcal{P}_{int}(\vec{v}) = \int_\Omega \underline{\underline{\sigma}} : \nabla\vec{v} \, d\Omega = -\int_\Omega \text{div}\underline{\underline{\sigma}} \bullet \vec{v} \, d\Omega + \int_\Gamma (\underline{\underline{\sigma}}.\vec{n}) \bullet \vec{v} \, d\Omega \tag{12.17}$$

where $\text{div}\underline{\underline{\sigma}}$ is the matrix divergence cf. (T.88).

2) $\mathcal{P}_{int}(\vec{v}) = 0$ for any rigid body motion, i.e. s.t. $\nabla\vec{v} + \nabla\vec{v}^T = 0$. Thus $0 = \int_\Omega \underline{\underline{\sigma}} : \frac{\nabla\vec{v} - \nabla\vec{v}^T}{2} \, d\Omega$ for any $\vec{v}$ s.t. $\nabla\vec{v} + \nabla\vec{v}^T = 0$. Also true for all subset in $\Omega$, thus $\underline{\underline{\sigma}}(p) : \frac{\nabla\vec{v}(p) - \nabla\vec{v}(p)^T}{2} = 0$ at all $p$, thus $\underline{\underline{\sigma}}$ is symmetric:

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T, \quad \text{and} \quad \mathcal{P}_{int}(\vec{v}) = \int_\Omega \underline{\underline{\sigma}} : \frac{\nabla\vec{v} + \nabla\vec{v}^T}{2} \, d\Omega, \ \forall v \in H^1(\Omega)^n. \tag{12.18}$$

**Example 12.5** Pressure in a perfect fluid: $\vec{f} = \vec{0}$ and $\underline{\underline{\sigma}} = p_r I$ where $p_r \in L^2(\Omega)$ (pressure), thus

$$\mathcal{P}(\vec{v}) = \int_\Omega p_r \, \text{div}\vec{v} \, d\Omega = -\int_\Omega \vec{\text{grad}} p_r \bullet \vec{v} \, d\Omega + \int_\Gamma p_r \, \vec{v} \bullet \vec{n} \, d\Gamma. \tag{12.19}$$

(Germain's notations: $\mathcal{P}(\hat{\vec{v}}) = \int_\Omega p_r \, \text{div}\hat{\vec{v}} \, d\Omega$ with $p_r$ the pressure and $\hat{\vec{v}}$ a virtual velocity.) ∴

**Exercice 12.6** What is the correct notation for (12.17)?

**Answer.** $\mathcal{P}_{int}(\vec{v}) = \int_\Omega [\underline{\underline{\sigma}}]_{|\vec{e}} : [\nabla\vec{v}]_{|\vec{e}} \, d\Omega = -\int_\Omega \text{div}([\underline{\underline{\sigma}}]_{|\vec{e}}) \bullet [\vec{v}]_{|\vec{e}} \, d\Omega + \int_\Gamma ([\underline{\underline{\sigma}}]_{|\vec{e}}.[\vec{n}]_{|\vec{e}}) \bullet [\vec{v}]_{|\vec{e}} \, d\Omega$ where $(\vec{e}_i)$ is a chosen Euclidean basis, $[\underline{\underline{\sigma}}]_{|\vec{e}} = [\sigma_{ij}] \in \mathcal{M}_{nn}$, $[\vec{v}]_{|\vec{e}} = [v_i] \in \mathcal{M}_{n1}$, $[\nabla\vec{v}]_{|\vec{e}} = [\frac{\partial v_i}{\partial x_j}] \in \mathcal{M}_{nn}$, $[\vec{n}] = [n_i] \in \mathcal{M}_{n1}$, $\text{div}([\underline{\underline{\sigma}}]_{|\vec{e}}) \in \mathcal{M}_{n1}$ is the divergence of the matrix $[\underline{\underline{\sigma}}]_{|\vec{e}}$ cf. (T.88), and $. \bullet .$ is the canonical inner dot product in $\mathcal{M}_{n1}$. ∴

### 12.3.3 Second order linear hypothesis

Generalization to

$$H^2(\Omega) := \{u \in L^2(\Omega) : \vec{\text{grad}} u \in L^2(\Omega)^n, \ d^2u \in L^2(\Omega)^{n^2}\}. \tag{12.20}$$

with its inner dot product $(u,v)_{H^2} = (u,v)_{L^2} + (\vec{\text{grad}} u, \vec{\text{grad}} v)_{L^2} + (d^2u, d^2v)_{L^2}$ and associated norm $||u||_{H^2} = \sqrt{(u,u)_{H^2}}$. And (similar to the $H^1(\Omega)^n$ case): If $\mathcal{P} \in (H^2(\Omega)^n)'$ (i.e. linear and continuous on $H^2(\Omega)$) then $\exists(\vec{f},\underline{\underline{\sigma}},\underline{\underline{\underline{\chi}}}) \in L^2(\Omega)^n \times L^2(\Omega)^{n^2} \times L^2(\Omega)^{n^3}$ s.t., for all $\vec{u} \in H^2(\Omega)^n$,

$$\mathcal{P}(\vec{u}) = (\vec{f},\vec{u})_{L^2} + (\underline{\underline{\sigma}},\nabla\vec{u})_{L^2} + (\underline{\underline{\underline{\chi}}}, d^2\vec{u})_{L^2}. \tag{12.21}$$

Gives "micropolar materials". See e.g. Germain [12].

# 13 First order virtual power formulation with Lie derivatives

## 13.1 The weird classical justification of the linear approach

The classic approach for elastic materials is clever but weird. Clever because it uses the comparison between two vectors to measure a relative deformation. Weird because it starts by squaring the a linear motion and then... linearize it... Moreover it produces a spurious $F^T$. Details:

1. Take two vectors $\vec{W}_1$ and $\vec{W}_2$ at $t_0$; They become $\vec{w}_1 = F.\vec{W}_1$ and $\vec{w}_2 = F.\vec{W}_2$ at $t$ (linear hypothesis).
2. Compute $(\vec{w}_1, \vec{w}_2)_g = (F.\vec{W}_1, F.\vec{W}_2)_g = (F^T.F.\vec{W}_1, \vec{W}_2)_G$: doing so the motion has been "squared" (product of two deformed vectors), and you have built $C = F^T.F$, see (G.15).
3. Compare $(\vec{w}_1, \vec{w}_2)_g$ (at $t$) with $(\vec{W}_1, \vec{W}_2)_G$ (at $t_0$) thanks to $\frac{(\vec{w}_1,\vec{w}_2)_g-(\vec{W}_1,\vec{W}_2)_G}{2} = \frac{((F^T.F-I).\vec{W}_1,\vec{W}_2)_G}{2}$, the $\frac{1}{2}$ because it is a squared quantity (the linearisation of $f(x) = \frac{x^2}{2}$ gives $f'(x) = x$). You have obtained the Green–Lagrange tensor $E = \frac{1}{2}(F^T.F - I) = \frac{1}{2}(C - I)$.
4. Linearize $E$: $E$ is approximated by $\underline{\underline{\varepsilon}} = \frac{F+F^T}{2} - I$. In doing so, you have introduced the spurious $F^T$ (in $\underline{\underline{\varepsilon}}$), which does not exists in the first order Taylor expansion $\Phi(P+h\vec{W}) = \Phi(P) + h\,F.\vec{W} + o(h)$, which isn't $\Phi(P+h\vec{W}) = \Phi(P) + h\,\frac{F+F^T}{2}.\vec{W} + o(h)$.

This classic approach rises the questions:

1- Is it "normal" (convincing) to start from a constitutive law with $E$ that does not give good results, to deduce (legitimize) a linear constitutive law, moreover with a spurious $F^T$?

2- Can we get a linear law without $E$ (without squaring first), thus without the spurious $F^T$? Yes:

## 13.2 $\underline{\underline{\sigma}}$ with Lie derivatives of vector fields

The Lie derivative of a Cauchy stress vector $\vec{T}$ along $\vec{v}$ at $t$ at $p \in \Omega_t$ (rate of stress along $\vec{v}$) is, cf. (9.18),

$$\mathcal{L}_{\vec{v}}\vec{T} = \frac{D\vec{T}}{Dt} - d\vec{v}.\vec{T} \quad (= \frac{\partial \vec{T}}{\partial t} + d\vec{T}.\vec{v} - d\vec{v}.\vec{T}). \tag{13.1}$$

Choose a differential form $\alpha$ to measure $\mathcal{L}_{\vec{v}}\vec{T}$ to get the internal power density $\alpha.\mathcal{L}_{\vec{v}}\vec{T}$, i.e. the real objective values $\alpha(t,p_t).\mathcal{L}_{\vec{v}}\vec{T}(t,p_t)$ at $t$ at each $p_t \in \Omega_t$. You get the internal virtual power

$$\mathcal{P}_{int}(...,\alpha) := \int_{\Omega_t} \alpha.\mathcal{L}_{\vec{v}}\vec{T}\,d\Omega = \int_{\Omega_t} \alpha.\frac{D\vec{T}}{Dt} - (\vec{T}\otimes\alpha) \odot d\vec{v}\,d\Omega, \tag{13.2}$$

since $\alpha.d\vec{v}.\vec{T} = (\alpha \otimes \vec{T}) \odot d\vec{v}$ (objective double contraction between $\binom{1}{1}$ tensors). The use of the Cauchy stress vector field $\vec{T}$ (order 1 tensor) is explicit (the obtained order 2 tensor $\vec{u}\otimes T$ is obtained thanks to a choice of a direction of measurement $\alpha$). No initial time $t_0$ (Eulerian approach), and covariant objective approach (no basis and no inner dot product required).

Restrictions to get admissible constitutive laws:

- Galilean referential: $\mathcal{P}_{int}$ vanishes when $d\vec{v} = 0$, true for all subset of $\Omega_t$. We are left with

$$\mathcal{P}_{int}(...,\alpha) = -\int_{\Omega_t} \underline{\underline{\tau}}_\alpha \odot d\vec{v}\,d\Omega, \quad \text{where} \quad \underline{\underline{\tau}}_\alpha := \vec{T}\otimes\alpha \tag{13.3}$$

NB: The use of the Cauchy stress vector field $\vec{T}$ (tensor of order 1) is explicit, the obtained tensor $\underline{\underline{\tau}}_\alpha = \vec{T}\otimes\alpha$ of order 2 being obtained after a choice of a direction of measurement $\alpha$.

- Isometric framework: Choose a Euclidean basis $(\vec{e}_i)$ and its associated Euclidean dot product $(\cdot,\cdot)_g \overset{\text{written}}{=} .\bullet.$, and call $\vec{n}$ the exterior $(\cdot,\cdot)_g$-normal unit vector field on $\Gamma$. We get

$$\begin{aligned}\mathcal{P}_{int}(...,\alpha) &= \int_{\Omega_t} \widetilde{\text{div}}\underline{\underline{\tau}}_\alpha.\vec{v}\,d\Omega - \int_{\Gamma_t} (\underline{\underline{\tau}}_\alpha.\vec{v}) \bullet \vec{n}\,d\Gamma, \quad \text{where} \quad \underline{\underline{\tau}}_\alpha := \vec{T}\otimes\alpha \\ &= \int_{\Omega_t} (\text{div}\vec{T})\alpha.\vec{v} + (d\alpha.\vec{T}).\vec{v}\,d\Omega - \int_{\Gamma_t} (\alpha.\vec{v})(\vec{T}\bullet\vec{n})\,d\Gamma,\end{aligned} \tag{13.4}$$

where $\widetilde{\text{div}}(\underline{\underline{\tau}}_\alpha) = \widetilde{\text{div}}(\vec{T}\otimes\alpha) = (\text{div}\vec{T})\alpha + d\alpha.\vec{T}$ is the objective divergence of $\underline{\underline{\tau}}_\alpha$ see (T.75).

• Classical formulation recovered with representation matrices relative to $(\vec{e}_i)$: With $\vec{v} = \sum_i v^i \vec{e}_i$, $d\vec{v}.\vec{e}_j = \sum_i \frac{\partial v^i}{\partial x^j}\vec{e}_i$, $\vec{T} = \sum_i T^i \vec{e}_i$, $(e^i)$ the dual basis of $(\vec{e}_i)$, $\alpha = \sum_i \alpha_i e^i$, $[\underline{\underline{\tau}}_\alpha] = [\vec{T}\otimes\alpha] = [T^i\alpha_j]$, call

$$\underline{\underline{\sigma}}_\ell := [\underline{\underline{\tau}}_\alpha]^T = [\alpha_i T^j] = [\sigma_{\ell ij}] \ \ \text{(matrix)}. \tag{13.5}$$

With the canonical inner dot product in $\mathcal{M}_{n1}$ also called $\bullet$ . and with the matrix double contraction $[M_{ij}] : [N_{ij}] := \sum_{ij} M_{ij}N_{ij}$, (13.3) gives

$$\begin{aligned} \mathcal{P}_{int}(...,\alpha) = &-\int_{\Omega_t} \underline{\underline{\sigma}}_\ell : [d\vec{v}]\,d\Omega \quad \Big(= \int_{\Omega_t} \text{div}\underline{\underline{\sigma}}_\ell \bullet [\vec{v}] - \int_{\Gamma_t} (\underline{\underline{\sigma}}_\ell.[\vec{n}])\bullet[\vec{v}]\,d\Gamma\Big),\\ \overset{\text{written}}{=} &-\int_{\Omega_t} \underline{\underline{\sigma}}_\ell : d\vec{v}\,d\Omega \quad \Big(= \int_{\Omega_t} \text{div}\underline{\underline{\sigma}}_\ell \bullet \vec{v} - \int_{\Gamma_t} (\underline{\underline{\sigma}}_\ell.\vec{n})\bullet\vec{v}\,d\Gamma\Big). \end{aligned} \tag{13.6}$$

Moreover $\mathcal{P}_{int}$ must be independent of rigid body motions (frame invariance principle), thus

$$\begin{aligned} \mathcal{P}_{int}(...,\alpha) = &-\int_{\Omega_t} \underline{\underline{\sigma}}_\ell : \frac{d\vec{v}+d\vec{v}^T}{2}\,d\Omega = -\int_{\Omega_t} \frac{\underline{\underline{\sigma}}_\ell+\underline{\underline{\sigma}}_\ell^{\,T}}{2} : \frac{d\vec{v}+d\vec{v}^T}{2}\,d\Omega\\ = &-\int_{\Omega_t} \underline{\underline{\sigma}} : \frac{d\vec{v}+d\vec{v}^T}{2}\,d\Omega = -\int_{\Omega_t} \underline{\underline{\sigma}} : d\vec{v}\,d\Omega \quad \text{where} \quad \underline{\underline{\sigma}} = \frac{\underline{\underline{\sigma}}_\ell+\underline{\underline{\sigma}}_\ell^{\,T}}{2}. \end{aligned} \tag{13.7}$$

We have recovered the usual classical formulation: $\mathcal{P}_{int} = -\int_\Omega \underline{\underline{\sigma}} : d\vec{v}\,d\Omega$.

**Remark 13.1** $\vec{T}$ is the main unknown with its 3 components, not $\underline{\underline{\sigma}}$ with its 9 components (or 6 components thanks to symmetry). And for the oncoming second order theory with Lie, a second order term $\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}T_2)$ will be introduced (see § 13.5), which only introduces 3 more unknowns with $T_2$; To compare with the 81 unknowns (number that can be reduced with symmetry considerations) with the tensor $\underline{\underline{\underline{\underline{\chi}}}}$ which gives the linear model (12.21), cf. Germain [12].

## 13.3 $\underline{\underline{\sigma}}$ with Lie derivatives of differential forms

Germain [12] and others have proposed that the Cauchy stress is a (Eulerian) differential form $T$ (which objectively acts on vector fields). Its Lie derivative along a (Eulerian) vector field $\vec{v}$ is

$$\mathcal{L}_{\vec{v}}T = \frac{DT}{Dt} + T.d\vec{v} \quad \Big(= \frac{\partial T}{\partial t} + dT.\vec{v} + T.d\vec{v}\Big). \tag{13.8}$$

$\mathcal{L}_{\vec{v}}T$ acts on a vector fields $\vec{u}$ (measurement direction) to give the density of power $\pi_{int}$ and the power $\mathcal{P}_{int}$:

$$\pi_{int}(T,\vec{v},\vec{u}) := \mathcal{L}_{\vec{v}}T.\vec{u}, \quad \text{and} \quad \mathcal{P}_{int}(\Omega_t,T,\vec{v},\vec{u}) := \int_{\Omega_t} \mathcal{L}_{\vec{v}}T.\vec{u}\,d\Omega. \tag{13.9}$$

The use of the Cauchy stress differential form $T$ (order 1 tensor) is explicit, no initial time $t_0$ (Eulerian approach), and covariant objective approach (no basis and no inner dot product required). Notation for Germain's "distribution" (duality) approach: $\mathcal{P}_{int}(\vec{u}) = \langle \mathcal{L}_{\vec{v}}T, \vec{u}\rangle$.

• Galilean referential: $\mathcal{P}_{int}$ vanishes when $\vec{v}$ is uniform ($d\vec{v} = 0$), true for all subset in $\Omega_t$. Hence

$$\begin{aligned} \mathcal{P}_{int}(...,\vec{u}) = &\int_{\Omega_t} T.d\vec{v}.\vec{u}\,d\Omega = \int_{\Omega_t} (\vec{u}\otimes T) \odot d\vec{v}\,d\Omega\\ = &-\int_{\Omega_t} \underline{\underline{\tau}}_{\vec{u}} \odot d\vec{v}\,d\Omega, \quad \text{where} \quad \underline{\underline{\tau}}_{\vec{u}} := -\vec{u}\otimes T, \end{aligned} \tag{13.10}$$

since $(T.d\vec{v}).\vec{u} = (\vec{u}\otimes T) \odot d\vec{v}$ = objective double contraction between $\binom{1}{1}$ tensors. (The $-$ signs for comparison with the usual classic approach.)

• Euclidean isometric framework:

$$\begin{aligned} \mathcal{P}_{int}(...,\vec{u}) = &\int_{\Omega_t} \widetilde{\text{div}}\underline{\underline{\tau}}_{\vec{u}}.\vec{v}\,d\Omega - \int_{\Gamma_t} (\underline{\underline{\tau}}_{\vec{u}}.\vec{v})\bullet\vec{n}\,d\Gamma, \quad \text{where} \quad \underline{\underline{\tau}}_{\vec{u}} := -\vec{u}\otimes T,\\ = &-\int_{\Omega_t} (\text{div}\vec{u})T.\vec{v} + (dT.\vec{u}).\vec{v}\,d\Omega + \int_{\Gamma_t} (T.\vec{v})(\vec{u}\bullet\vec{n})\,d\Gamma. \end{aligned} \tag{13.11}$$

- Classical formulation recovered:

$$\underline{\underline{\sigma}}_\ell = [\sigma_{\ell ij}] := [\underline{\underline{\tau}}_{\vec{u}}]^T = [\vec{u} \otimes T]^T = [T_i u^j], \tag{13.12}$$

thus

$$\mathcal{P}_{int}(...) \overset{(13.10)}{=} -\int_\Omega \underline{\underline{\sigma}}_\ell : [d\vec{v}]\, d\Omega \overset{\text{written}}{=} -\int_\Omega \underline{\underline{\sigma}}_\ell : d\vec{v}\, d\Omega. \tag{13.13}$$

And $\mathcal{P}_{int}$ is independent of rigid body motions, thus

$$\mathcal{P}_{int}(...) = -\int_\Omega \underline{\underline{\sigma}}_\ell : \frac{d\vec{v} + d\vec{v}^T}{2}\, d\Omega = -\int_\Omega \underline{\underline{\sigma}} : d\vec{v}\, d\Omega, \quad \text{where} \quad \underline{\underline{\sigma}} = \frac{\underline{\underline{\sigma}}_\ell + \underline{\underline{\sigma}}_\ell{}^T}{2}. \tag{13.14}$$

which gives the classical formulation.

## 13.4 Non linear first order virtual power formulation with Lie derivatives

Add to (13.2) a differential form $\alpha_1$ (measuring tool) imbedded in the flow to measure some internal force $\vec{T}_1$ subject to the flow:

$$\mathcal{P}_{int}(\alpha, \alpha_1, \vec{v}, \vec{T}, \vec{T}_1) = \int_\Omega \alpha.\mathcal{L}_{\vec{v}}\vec{T} + \mathcal{L}_{\vec{v}}\alpha_1.(\mathcal{L}_{\vec{v}}\vec{T}_1)\, d\Omega. \tag{13.15}$$

(May be an alternative to Cosserat material or viscoelastic material.) A first choice is $\alpha_1 = \alpha$ and $\vec{T}_1 = \vec{T}$. ($\mathcal{L}_{\vec{v}}\alpha_1 = \frac{\partial \alpha_1}{\partial t} + d\alpha_1.\vec{v} + \alpha.d\vec{v}$ is the rate of deformation of $\alpha_1$ along $\vec{v}$).

Then choose $\alpha_1$ uniform and stationary, so $\mathcal{L}_{\vec{v}}\alpha_1 = \alpha_1.d\vec{v}$, and

$$\mathcal{P}_{int}(...) = \int_\Omega \alpha.(\frac{\partial \vec{T}}{\partial t} + d\vec{T}.\vec{v} - d\vec{v}.\vec{T}) + \alpha_1.d\vec{v}.(\frac{\partial \vec{T}_1}{\partial t} + d\vec{T}_1.\vec{v} - d\vec{v}.\vec{T}_1)\, d\Omega. \tag{13.16}$$

It is non linearin $\vec{v}$. The internal power has to vanish whenever $d\vec{v} = 0$, true for all subset of $\Omega$, hence the $\alpha.(\frac{\partial \vec{T}}{\partial t} + d\vec{T}.\vec{v})$ term vanishes, and, with $\underline{\underline{\tau}} := \vec{T} \otimes \alpha$ and $\underline{\underline{\tau}}_1 := \vec{T}_1 \otimes \alpha_1$, we are left with

$$\begin{aligned} \mathcal{P}_{int}(...) &= \int_\Omega -\alpha.d\vec{v}.\vec{T} + \alpha_1.d\vec{v}.(\frac{D\vec{T}_1}{Dt} - d\vec{v}.\vec{T}_1)\, d\Omega \\ &= \int_\Omega -\underline{\underline{\tau}} \odot d\vec{v} + \frac{D\underline{\underline{\tau}}_1}{Dt} \odot d\vec{v} - \underline{\underline{\tau}}_1 \odot (d\vec{v}.d\vec{v})\, d\Omega. \end{aligned} \tag{13.17}$$

Recall: Only Lie derivatives of the vector fields $\vec{T}$ and $\vec{T}_1$ are used (no derivative of order 2 tensors).

## 13.5 Second order virtual power formulation with Lie derivatives

We add the second order Lie derivative $\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\vec{T}_2) =^{\text{written}} \mathcal{L}_{\vec{v}}^{(2)}\vec{T}_2$ of a vector field $\vec{T}_2$ (not the first order Lie derivative $\mathcal{L}_{\vec{v}}\underline{\underline{\sigma}}$ of a tensor $\underline{\underline{\sigma}}$ cf. e.g. the Jaumann derivative). E.g.

$$\mathcal{P}_{int}(\alpha, \vec{v}, \vec{T}, \vec{T}_2) = \int_\Omega \alpha.(\mathcal{L}_{\vec{v}}\vec{T} + \mathcal{L}_{\vec{v}}^{(2)}\vec{T}_2)\, d\Omega, \tag{13.18}$$

A simple choice is $\vec{T}_2 = c\vec{T}$.

Galilean framework: $\mathcal{P}_{int}$ vanishes if $d\vec{v} = 0$, thus moreover choosing a stationary $\vec{v}$ (so $\frac{\partial \vec{v}}{\partial t} = \vec{0}$),

$$\begin{aligned} \mathcal{P}_{int}(...) &= \int_\Omega \alpha.(-d\vec{v}.\vec{T} - 2d\vec{v}.\frac{\partial \vec{T}_2}{\partial t} + d\vec{T}_2.d\vec{v}.\vec{v} - 2d\vec{v}.d\vec{T}_2.\vec{v} - (d^2\vec{v}.\vec{v}).\vec{T}_2 + d\vec{v}.d\vec{v}.\vec{T}_2)\, d\Omega \\ &= \int_\Omega \alpha.(-d\vec{v}.\vec{T} - 2d\vec{v}.\frac{D\vec{T}_2}{Dt} + d\vec{T}_2.d\vec{v}.\vec{v} - (d^2\vec{v}.\vec{v}).\vec{T}_2 + d\vec{v}.d\vec{v}.\vec{T}_2)\, d\Omega. \end{aligned} \tag{13.19}$$

Restrictions on $\vec{T}$ and $\vec{T}_2$ in a Galilean Euclidean framework: $\mathcal{P}_{int}$ vanishes when $d\vec{v} + d\vec{v}^T = 0$.

Then define $\underline{\underline{\tau}} := \vec{T} \otimes \alpha$ and $\underline{\underline{\tau}}_2 := \vec{T}_2 \otimes \alpha$ (for constitutive laws) and choose $\alpha$ uniform: We get

$$\mathcal{P}_{int}(...) = \int_\Omega -\underline{\underline{\tau}} \odot d\vec{v} - 2\frac{D\underline{\underline{\tau}}_2}{Dt} \odot d\vec{v} + d\underline{\underline{\tau}}_2 \odot (d\vec{v}.\vec{v})\, d\Omega + \underline{\underline{\tau}}_2 \odot (d\vec{v}.d\vec{v} - d^2\vec{v}.\vec{v}). \tag{13.20}$$

NB: The result (13.20) is given with tensors $\underline{\underline{\tau}}$ and $\underline{\underline{\tau}}_2$ to be able to compare classical results, e.g. with Jaumann derivatives (Lie derivative of $\binom{1}{1}$ tensor). But here we only used Lie derivatives of the vector fields: No derivative of order 2 tensors.

# Part V
# Appendix

Bertrand Russell (beginning of the 20th century):
*"Studying Mathematics I had hoped to penetrate the essence of truth...*
*... But all I was learning was cheap calculating tricks."*

And isn't this still too often the case in continuum mechanics: "Studying Continuum Mechanics I had hoped to penetrate the essence of truth... But all I was learning was cheap calculating tricks"?

It is... Mainly due to the lack of basic math definitions, e.g.:

What is a motion? A Eulerian variable? A Lagrangian variable?
Why domain and codomain of a function are rarely mentioned (hence errors and misunderstandings)?
What is a "canonical", a "Cartesian", a "Euclidean" basis?
What is a transposed (of what)?
What is pseudo-vector versus a vector?
What is covariant vector versus a contravariant vector?
Why a linear scalar valued function can't be identified with a vector?
What is the difference between a differential and a gradient?
What is a tensor?
Why the infinitesimal tensor $\underline{\underline{\varepsilon}}$ is not a tensor?
Why a endomorphism $E \to E$ can't be identified with a bilinear form $E \times E \to \mathbb{R}$?
What is the definition of Einstein's convention?
What is the Lie derivative? And why is it "The natural derivative in continuum mechanics"?
What is a distribution?
What does $\frac{\partial W}{\partial F_{ij}}$ mean (derivation relative to components?)?
$\vdots$

One of my teachers: *"This is the big advantage of not giving definitions: It allows you to say anything."*

In this appendix, we give standard simple definitions and results, useful in mechanics, often scattered in the existing literature, and sometimes difficult to find. Hence no ambiguity is possible. We avoid notations which are of no use or add to confusion, or come like a bull in a china-shop.

All the definitions apply to electromagnetism, chemistry, quantum mechanics, general relativity... and continuum mechanics (solids, fluids, thermodynamics...): Mathematics applies to everyone.

For simplicity, we mainly consider finite dimensional vector spaces.

## A Classical and duality notations

### A.1 Contravariant vectors, covariant vectors

Let $(E,+,.) =^{\text{written}} E$ be a finite dimension real vector space (= a linear space on the field $\mathbb{R}$).

**Definition A.1** An element $\vec{x} \in E$ is called a vector, and it is also called a "contravariant vector".

A vector is a vector... So why is it also called a "contravariant vector"?

Historical answer: Because of the change of basis formula $[\vec{x}]_{|new} = P^{-1}.[\vec{x}]_{|old}$, $P$ being the transition matrix see (A.25), which uses the inverse $P^{-1}$.

**Definition A.2** A linear form is a function $E \to \mathbb{R}$ (real values) which is linear. A linear form is also called a covariant vector.

The space $E^* := \mathcal{L}(E;\mathbb{R})$ is the space of linear forms on $E$ called the dual of $E$. (It $E^*$ is a vector space, sub-space of $\mathcal{F}(E;\mathbb{R})$,trivial check.)

Why a linear form is called a "covariant vector"?

Historical answer: Because of the change of basis formula $[\ell]_{|new} = [\ell]_{|old}.P$, which uses $P$, see (A.25).

**Interpretation:** A covariant vector is a linear measuring tool for vectors, because it is a linear form $\ell$ that gives reals values $\ell(\vec{x}) \in \mathbb{R}$ to vectors $\vec{x} \in E$.

## A.2 Bases

### A.2.1 Basis

• $n$ vectors $\vec{e}_1, ..., \vec{e}_n \in E$ are linearly independent iff for all $\lambda, ..., \lambda_n \in \mathbb{R}$ the equality $\sum_{i=1}^n \lambda_i \vec{e}_i = \vec{0}$ implies $\lambda_i = 0$ for all $i = 1, ..., n$.

• $n$ vectors $\vec{e}_1, ..., \vec{e}_n \in E$ span $E$ iff : $\forall \vec{x} \in E$, $\exists \lambda_1, ..., \lambda_n \in \mathbb{R}$ s.t. $\vec{x} = \sum_{i=1}^n \lambda_i \vec{e}_i$.

• A basis in $E$ is a set $\{\vec{e}_1, ..., \vec{e}_n\} \subset E$ made of $n$ linearly independent vectors which span $E$. In which case the dimension of $E$ is $n$ (all the bases in $E$ have the same number of vectors: exercise). And $\{\vec{e}_1, ..., \vec{e}_n\} =^{\text{written}} (\vec{e}_i)_{i=1,...,n} =^{\text{written}} (\vec{e}_i)$ if $n$ is implicit.

### A.2.2 Canonical basis

Consider the Cartesian product $\mathbb{R} \times ... \times \mathbb{R}$ ($n$ times) with its usual vectorial structure. Its canonical basis $(\vec{A}_i)$ is defined by

$$\vec{A}_1 = (1, 0, ..., 0), \ ..., \ \vec{A}_n = (0, ..., 0, 1), \tag{A.1}$$

with 0 the addition identity element used $n-1$ times, and 1 the multiplication identity element used once.

**Remark A.3** Consider the 3-D geometric space "we live in", and the associated vector space $\vec{\mathbb{R}^3}$ of "bi-point vectors". There is no canonical basis in $\vec{\mathbb{R}^3}$: What would the identity element 1 mean? 1 metre? 1 foot? And there is no "intrinsic" preferred direction $\vec{e}_1$. ♣

### A.2.3 Cartesian basis

(René Descartes 1596-1650.) Let $n = 1, 2, 3$, let $\mathbb{R}^n$ be the usual affine space (space of points), and let $\vec{\mathbb{R}^n} = (\vec{\mathbb{R}^n}, +, .)$ be the associated usual real vector space of bipoint vectors. Let $p \in \mathbb{R}^n$, and let $(\vec{e}_i(p))$ be a basis in $\vec{\mathbb{R}^n}$ at $p$ (e.g. the polar coordinate system see example 6.12).

Definition: $(\vec{e}_i(p))$ is a Cartesian basis in $\vec{\mathbb{R}^n}$ iff $\vec{e}_i(p)$ is independent of $p$ for all $i$ and $p$; And then $(\vec{e}_i(p)) =^{\text{written}} (\vec{e}_i)$.

Remark: A Euclidean basis described in § B.1 is a particular Cartesian basis.

## A.3 Classic and dual representation, Einstein's convention for vectors

There are to equivalent notation systems:

• the classical notation (non ambiguous), e.g. used by Arnold [3] and Germain [11], and

• the duality notation (can be ambiguous because of misuses), e.g. used by Marsden and Hughes [17].

Both classical and duality notation are equally good, but if in doubt, use the classical notations.

**Definition A.4** Let $\vec{x} \in E$ and let $(\vec{e}_i)$ be a basis in $E$. The components of $\vec{x}$ relative to $(\vec{e}_i)$ (or in $(\vec{e}_i)$) are the real numbers $x_1, ..., x_n$ (classical notation) also named $x^1, ..., x^n$ (duality notation) by

$$\vec{x} = \underbrace{x_1\vec{e}_1 + ... + x_n\vec{e}_n}_{\text{clas.}} = \underbrace{x^1\vec{e}_1 + ... + x^n\vec{e}_n}_{\text{dual}}, \quad \text{i.e.} \quad [\vec{x}]_{|\vec{e}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \tag{A.2}$$

$[\vec{x}]_{|\vec{e}}$ being the column matrix representing $\vec{x}$ relative to the basis $(\vec{e}_i)$. (Of course $x_i = x^i$ for all $i$.)

If a chosen basis is imposed to all then $[\vec{x}]_{|\vec{e}}$ is simply named $[\vec{x}]$. With the sum sign:

$$\vec{x} = \underbrace{\sum_{i=1}^n}_{\text{clas.}} x_i\vec{e}_i = \underbrace{\sum_{i=1}^n}_{\text{dual}} x^i\vec{e}_i \quad (= \sum_{J=1}^n x_J\vec{e}_J = \sum_{\alpha=1}^n x^\alpha\vec{e}_\alpha), \tag{A.3}$$

the summation index being a dummy index.

**Definition A.5** The Einstein's convention uses the duality notation. And then you can omits the sum sign $\sum$: So $\vec{x} = \sum_{j=1}^n x^j\vec{e}_j =^{\text{written}} x^j\vec{e}_j = x^i\vec{e}_i = x^J\vec{e}_J = x^\alpha\vec{e}_\alpha$. This omission was motivated by the difficulty of printing $\sum_{j=1}^n$ in the early 20th century. We won't omit the $\sum$ sign in the following, thanks to TEX-LATEX which makes the writing of $\sum_{j=1}^n$ simple.

**Example A.6** In $\vec{\mathbb{R}^2}$, let $\vec{x} = 3\vec{e}_1 + 4\vec{e}_2 = \sum_{i=1}^2 x_i\vec{e}_i = \sum_{i=1}^2 x^i\vec{e}_i$, so $x_1{=}x^1{=}3$ and $x_2{=}x^2{=}4$. And $[\vec{x}]_{|\vec{e}} = 3[\vec{e}_1]_{|\vec{e}} + 4[\vec{e}_2]_{|\vec{e}} = \sum_{i=1}^2 x_i[\vec{e}_i]_{|\vec{e}} = \sum_{i=1}^2 x^i[\vec{e}_i]_{|\vec{e}}$. So with $\delta^i_j = \delta_{ij} := \left\{\begin{matrix} = 1 \text{ if } i{=}j \\ = 0 \text{ if } i{\neq}j \end{matrix}\right\}$ (Kronecker),

$$\vec{e}_j = \underbrace{\sum_{i=1}^n \delta_{ij}\vec{e}_i}_{\text{clas.}} = \underbrace{\sum_{i=1}^n \delta^i_j\vec{e}_i}_{\text{dual}}, \quad \text{i.e.} \quad [\vec{e}_1]_{|\vec{e}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ [\vec{e}_n]_{|\vec{e}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{A.4}$$

that is, the components of $\vec{e}_j$ in $(\vec{e}_i)$ are $\delta_{ij}$ with classical notations and $\delta^i_j$ with duality notations. ∴

**Remark A.7** The basis $([\vec{e}_j]_{\vec{e}})$ is the canonical basis of the vector space $\mathcal{M}_{n1}$ of $n * 1$ column matrices. A column matrix $[\vec{x}]_{|\vec{e}}$ is also called a "column vector". So a column vector $[\vec{x}]_{|\vec{e}}$ is a matrix representation of a vector in a basis. See the change of basis formula (A.25) where the same vector is represented by two different "column vectors" (two different representation column matrices). ∴

## A.4 Dual basis

General usual notations: If $E$ and $F$ are vector spaces then $(\mathcal{F}(E;F),+,.) =^{\text{written}} \mathcal{F}(E;F)$ is the usual real vector space of functions with the internal addition $(f,g) \to f{+}g$ defined by $(f{+}g)(x) := f(x){+}g(x)$ and the external multiplication $(\lambda, f) \to \lambda.f$ defined by $(\lambda.f)(x) := \lambda(f(x))$, for all $f,g \in \mathcal{F}(E;F)$, $x \in E$, $\lambda \in \mathbb{R}$. And $\lambda.f =^{\text{written}} \lambda f$ for all $\lambda \in \mathbb{R}$ and $f \in \mathcal{F}(E;F)$.

### A.4.1 External dot notations for duality

Recall definition A.2: The dual of $E$ is the vector space $E^* := \mathcal{L}(E;\mathbb{R})$ (set of linear real valued functions), and an element $\ell \in E^*$ (a linear form) is called a covariant vector.

**Notation**: If $\ell \in E^*$ then

$$\forall \vec{u} \in E, \quad \ell(\vec{u}) \overset{\text{written}}{=} \ell.\vec{u}. \tag{A.5}$$

The dot in $\ell.\vec{u}$ in (A.5) is "the distributivity dot" since linearity $\ell(\vec{u} + \lambda\vec{v}) = \ell(\vec{u}) + \lambda\ell(\vec{v})$ follows the distributivity rule $a(x + \lambda y) = ax + \lambda ay$ : so $\ell(\vec{u} + \lambda\vec{v}) =^{\text{written}} \ell.(\vec{u} + \lambda\vec{v}) = \ell.\vec{u} + \lambda\ell.\vec{v}$. And it is an external dot for computations: $E^* \neq E$ (and $E^*$ can't be intrinsically identified with $E$), thus $\ell$ and $\vec{u}$ are in different spaces, thus $\ell.\vec{u}$ is not an inner dot product.

$\ell(\vec{u})$ is also written $\ell(\vec{u}) =^{\text{written}} \langle \ell, \vec{u} \rangle_{E^*,E}$ where $\langle .,. \rangle_{E^*,E}$ is the duality bracket (and written $\ell, \vec{u}\rangle$ for short).

**NB:** Co-variant refers to: 1- The action of a function $\ell$ on a vector $\vec{u}$ that gives the real $\ell(\vec{u})$, the calculation of $\ell(\vec{u})$ being called a co-variant calculation, and

2- The change of coordinate formula $[\ell]_{new} = [\ell]_{|old}.P$, see (A.25) (covariant formula).

**Remark A.8** More precisely, $E^*$ is the algebraic dual of $E$. If $E$ is infinite dimensional, then we may need to define a norm $||.||_E$ for which $E$ is a Banach space. E.g. $E = L^2(\Omega)$ and $||f||^2_{L^2(\Omega)} := \int_\Omega f(\vec{x})^2 \, d\Omega$. In that case $E^*$ is the name given to the set of continuous linear forms on $E$, called the topological dual of $E$: It is essential in continuum mechanics.

(If $E$ is finite dimensional then all norms are equivalent and a linear form is continuous.) ∴

**Remark A.9** $E^*$ being a vector space, an element $\ell \in E^*$ is indeed a vector. But $E^*$ has no existence if $E$ has not been specified first! And $\ell \in E^*$ can't be confused with a vector $\vec{u} \in E$ since there is no natural canonical isomorphism between $E$ and $E^*$ (no "intrinsic representation"), see § U.2. So if you want to represent a $\ell \in E^*$ by a vector then you need a tool which is observer dependent; E.g. you need some inner dot product (observer dependent) if you apply the Riesz-representation theorem, or you need to specify a basis (observer dependent) to represent $\ell$ with its matrix of components (in the dual basis). ∴

**Remark A.10** (continuing.) Misner–Thorne–Wheeler [19], box 2.1, insist: "Without it [the distinction between covariance and contravariance], one cannot know whether a vector is meant or the very different object that is a linear form." ∴

### A.4.2 Covariant dual basis (functions that give the components of a vector)

Notation: If $\vec{u}_1, ..., \vec{u}_k$ are vectors in $E$, then let $\mathrm{Vect}\{\vec{u}_1, ..., \vec{u}_k\}$ be the vector space spanned by $\vec{u}_1, ..., \vec{u}_k$.
Let $(\vec{e}_i)_{i=1,...,n}$ be a basis in $E$. Let $i \in [1, n]_{\mathbb{N}}$.

**Definition A.11** The scalar projection on $\mathrm{Vect}\{\vec{e}_i\}$ parallel to $\mathrm{Vect}\{\vec{e}_1, ..., \vec{e}_{i-1}, \vec{e}_{i+1}, ..., \vec{e}_n\}$ is the linear form named $\pi_{ei} \in E^*$ with the classical notation, named $e^i \in E^*$ with the duality notation, defined by, for all $j$,

$$\begin{cases} \text{clas. not.}: & \pi_{ei}(\vec{e}_j) = \delta_{ij}, \quad \text{i.e.} \quad \pi_{ei}.\vec{e}_j = \delta_{ij}, \\ \text{dual not.}: & e^i(\vec{e}_j) = \delta^i_j, \quad \text{i.e.} \quad e^i.\vec{e}_j = \delta^i_j. \end{cases} \tag{A.6}$$

(The dual basis $(\pi_{ei}) = (e^i)$ is intrinsic to the $(\vec{e}_i)$: The same for an English and a French observer...)

$\pi_{ei} = e^i$ being linear, if $\vec{x} = \sum_{j=1}^n x_j \vec{e}_j$ then $\pi_{ei}(\vec{x}) = \sum_{j=1}^n x_j \, \pi_{ei}(\vec{e}_j) = x_j$:

$$\pi_{ei}.\vec{x} \overset{\text{clas.}}{=} x_i \quad = \quad e^i.\vec{x} \overset{\text{dual}}{=} x^i \quad = \text{the } i\text{-th component of } \vec{x} \text{ relative to } (\vec{e}_i), \tag{A.7}$$

see figure A.1.

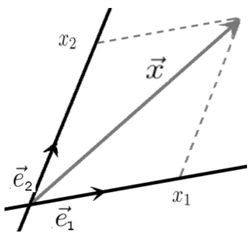


Figure A.1: Parallel projections: $\pi_{e1}(\vec{x}) = x_1$ and $\pi_{e2}(\vec{x}) = x_2$ (dual not.: $e^1(\vec{x}) = x^1$ and $e^2(\vec{x}) = x^2$).

**NB: Fundamental.** $\pi_{ei}.\vec{x}$ is not an orthogonal projection: Because orthogonality depends on the choice of an inner dot product (subjective), and $\pi_{e1}.\vec{x}$ is not an inner dot product because $\pi_{ei} = e^i \in E^*$ and $\vec{x} \in E$ do not belong to a same vector space, and (A.7) is independent of any inner dot product.

**Definition A.12** Particular case: If $(\vec{e}_i)$ is a Cartesian basis, then usually $\pi_{ei} =^{\text{written}} dx_i$ (or $dx^i$ with duality notations). So $dx_i(\vec{x}) = x_i = x^i = dx^i.\vec{x}$ with classical and/or duality notations.

**Proposition A.13 and definition .** *$(\pi_{ei})_{i=1,...,n} = (e^i)_{i=1,...,n} =^{\text{written}} (\pi_{ei}) = (e^i)$ is a basis in $E^*$, called the (covariant) dual basis of the basis $(\vec{e}_i)$. Thus $\dim E^* = \dim E = n$. And $\ell = m \in E^*$ iff $\ell(\vec{e}_i) = m(\vec{e}_i)$ for all $i$. And for all $\ell \in E^*$ the reals $\ell_i := \ell.\vec{e}_i$ are the components of $\ell$ in the dual basis:*

$$\ell \overset{\text{clas.}}{=} \sum_{i=1}^n \ell_i \pi_{ei} \overset{\text{dual}}{=} \sum_{i=1}^n \ell_i e^i \quad \textit{where} \quad \ell_i = \ell.\vec{e}_i. \tag{A.8}$$

**Proof.** If $\sum_{i=1}^n \lambda_i \pi_{ei} = 0$, then $0 = (\sum_{i=1}^n \lambda_i \pi_{ei})(\vec{e}_j) = \sum_{i=1}^n \lambda_i \pi_{ei}(\vec{e}_j) = \sum_{i=1}^n \lambda_i \delta_{ij} = \lambda_j$ for all $j$, thus $(\pi_{ei})_{i=1,...,n}$ is a family of $n$ independent vectors in $E^*$. If $\ell(\vec{e}_i) = m(\vec{e}_i)$ for all $i$ then $\ell(\vec{x}) = \sum_i x_i \ell(\vec{e}_i) = \sum_i x_i \ell(\vec{e}_i) = \sum_i x_i m(\vec{e}_i) = m(\vec{x})$ for all $\vec{x}$, thus $\ell = m$; And the converse is trivial. Then let $\ell \in E^*$ and $m = \sum_i (\ell.\vec{e}_i) \pi_{ei}$. Thus $m \in E^*$ (since $E^*$ is a vector space), and $m.\vec{e}_j = \sum_i (\ell.\vec{e}_i)(\pi_{ei}.\vec{e}_j) = \sum_i (\ell.\vec{e}_i) \delta_{ij} = \ell.\vec{e}_j$, for all $j$, thus $m = \ell$, thus $\ell = \sum_i (\ell.\vec{e}_i) \pi_{ei}$, thus $\mathrm{Vect}\{(\pi_{ei})_{i=1,...,n}\}$ span $E^*$, thus $\ell_i = \ell.\vec{e}_i$ and $(\pi_{ei})_{i=1,...,n}$ is a basis in $E^*$ and $\dim E^* = n$. (Use duality notations if you prefer.) ∎

**Example A.14** The size of a child is represented on a wall by a bipoint vector $\vec{u}$.

1- An English observer chooses the foot as unit of length and thus makes a vertical bipoint vector "one-foot long" $\vec{a}$. And then defines the linear form $\pi_a : \mathrm{Vect}\{\vec{u}\} \to \mathbb{R}$ by $\pi_a.\vec{a} = 1$. And $s_a := \pi_a.\vec{u}$ is the size of the child in foot ($\pi_a$ is a measuring instrument which gives values in foot).

2- A French observer chooses the metre as unit of length and thus makes a vertical bipoint vector "one-metre long" $\vec{b}$. And then defines the linear form $\pi_b : \mathrm{Vect}\{\vec{u}\} \to \mathbb{R}$ by $\pi_b.\vec{b} = 1$. And $s_b := \pi_b.\vec{u}$ is the size of the child in metre ($\pi_b$ is a measuring instrument which gives values in metre). ∎

**Exercice A.15** Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases and let $(\pi_{ai})$ and $(\pi_{bi})$ be the dual bases. Let $\lambda \neq 0$. Prove:

$$\text{If } \forall i \in [1, n]_{\mathbb{N}} \ \vec{b}_i = \lambda \vec{a}_i, \quad \text{then } \forall i \in [1, n]_{\mathbb{N}} \ \pi_{bi} = \frac{1}{\lambda} \pi_{ai} \quad (\text{i.e. } b^i = \tfrac{1}{\lambda} a^i). \tag{A.9}$$

**Answer.** $\pi_{bi}.\vec{b}_j = \delta_{ij} = \pi_{ai}.\vec{a}_j = \pi_{ai}.\frac{\vec{b}_j}{\lambda} = \frac{1}{\lambda} \pi_{ai}.\vec{b}_j$ for all $j$ (since $\pi_{ai}$ is linear). ∎

### A.4.3 Example: aeronautical units

Fundamental example if you fly. International aeronautical units:
- altitude = English foot (ft).
- Horizontal length = nautical mile (NM).

**Example A.16** First runway oriented South: First basis vector $\vec{e}_1$ one NM long oriented South. Second runway oriented Southwest: Second basis vector $\vec{e}_2$ one NM long oriented Southwest. $\vec{e}_3$ is the vertical vector of length 1 ft. $O$ = the position of the control tower. The referential of a traffic controller is $\mathcal{R} = (\mathcal{O}, (\vec{e}_1, \vec{e}_2, \vec{e}_3))$. The dual basis is $(\pi_{e1}, \pi_{e2}, \pi_{e3})$ (soy $\pi_{ei}(\vec{e}_j) = \delta_{ij}$ for all $i,j$). A plane $p$ is located at $t$ at $\vec{x} = \overrightarrow{Op}$. The air controller uses $\vec{x} = \sum_{i=1}^n x_i \vec{e}_i \in \vec{\mathbb{R}^n}$, hence $x_1 = \pi_{e1}(\vec{x})$ = the distance to the south in NM, $x_2 = \pi_{e2}(\vec{x})$ = the distance to the southwest in NM, $x_3 = \pi_{e3}(\vec{x})$ = the altitude in ft.

Here the basis $(\vec{e}_i)$ is not a Euclidean basis. This non Euclidean basis $(\vec{e}_i)$ is however vital if you fly: A Euclidean basis is not essential to life... Also see next remark A.17. ▪

**Remark A.17** The metre is the international unit for NASA that launched the Mars Climate Orbiter probe... But for the Mars Climate Orbiter landing procedure, NASA asked Lockheed Martin (who uses the foot) to do the computation. Result? The probe burned in the Martian atmosphere because of $\lambda \sim 3$ times too high a speed during the landing procedure: One metre is $\lambda \sim 3$ times one foot, and someone forgot it... As a matter of fact NASA and Lockheed Martin both used a Euclidean dot product... But not the same: One based on a metre, and one based on the foot. Objectivity and covariance can be useful! ▪

### A.4.4 Matrix representation of a linear form

$\ell \in E^*$, $(\vec{e}_i)$ is a basis, $(\pi_{ei})$ the dual basis, $\ell = \sum_{i=1}^n \ell_i \pi_{ei}$. The matrix of of $\ell$ relative to $(\vec{e}_i)$ is the row matrix

$$[\ell]_{|\pi_e} = (\,\ell_1 \quad \dots \quad \ell_n\,) \overset{\text{written}}{=} [\ell]_{|\vec{e}} \quad \text{(row matrix)}. \tag{A.10}$$

Thus, if $\vec{x} \in E$ and $\vec{x} = \sum_{i=1}^n x_i \vec{e}_i$, then $\ell.\vec{x} = (\sum_{i=1}^n \ell_i \pi_{ei}).(\sum_{j=1}^n x_j \vec{e}_j) = \sum_{i,j=1}^n \ell_i x_j (\pi_{ei}.\vec{e}_j) = \sum_{i,j=1}^n \ell_i x_j \delta_{ij} = \sum_{i=1}^n \ell_i x_i$, thus, with the usual matrix multiplication rule,

$$\ell.\vec{x} = [\ell]_{|\pi_e}.[\vec{x}]_{|\vec{e}} = \sum_{i=1}^n \ell_i x_i \overset{\text{written}}{=} [\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}, \tag{A.11}$$

product of a $1*n$ matrix times a $n*1$ matrix. With duality notations: $\ell.\vec{x} = \sum_{i=1}^n \ell_i x^i = [\ell]_{|\pi_e}.[\vec{x}]_{|\vec{e}}$.

In particular for the dual basis $(\pi_{ei}) = (e^i)$ (classical and duality notations),

$$[\pi_{ej}]_{|\vec{e}} = [e^j]_{|\vec{e}} = (0 \ \dots \ 0 \ \underbrace{1}_{j\text{th position}} \ 0 \ \dots \ 0) \quad (= \text{row matrix } [\vec{e}_j]_{|\vec{e}}^T), \tag{A.12}$$

and we recover $x_j = \pi_{ej}.\vec{x} = [\pi_{ej}]_{|\vec{e}}.[\vec{x}]_{|\vec{e}} = [\vec{e}_j]_{|\vec{e}}^T.[\vec{x}]_{|\vec{e}} = e^j.\vec{x} = [e^j]_{|\vec{e}}.[\vec{x}]_{|\vec{e}} = x^j$.

**Remark A.18** Relative to a basis, a vector is represented by a column matrix, cf. (A.2), and a linear form by a row matrix, cf. (A.10). This enables:

- The use of matrix calculation to compute $\ell.\vec{x} = [\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}$, cf. (A.11), not to be confused with an inner dot product calculation $\vec{x} \bullet \vec{y} := (\vec{x}, \vec{y})_g = [\vec{x}]_{|\vec{e}}^T.[g]_{\pi_e}.[\vec{y}]_{|\vec{e}}$ relative to an inner dot product $(\cdot,\cdot)_g$ in $E$.
- Not to confuse the "nature of objects": Relative to a basis, a (contravariant) vector is a mathematical object represented by a column matrix, while a linear form (covariant vector) is a mathematical object represented by a row matrix. Cf. remark A.10. ▪

### A.4.5 Example: Thermodynamic

E.g.: Cartesian space $\vec{\mathbb{R}^2} = \{\vec{X} = (T, P) \in \mathbb{R} \times \mathbb{R}\} = \{(\text{temperature,pressure})\}$. There is no meaningful inner dot product in this $\vec{\mathbb{R}^2}$: What would $||(T, P)|| = \sqrt{T^2+P^2}$ mean (Pythagoras): Can you add Kelvin degrees and pressure (kg.m$^{-1}\cdot$s$^{-2}$)? Here a (covariant) dual basis is fundamental for calculations.

Here, after a choice of temperature and pressure units, consider the basis $(\vec{E}_1{=}(1,0), \vec{E}_2{=}(0,1))$ in $\vec{\mathbb{R}^2}$, and its (covariant) dual basis $(\pi_{E1}, \pi_{E2}) =^{\text{written}} (dT, dP)$. Let $\vec{X} = T\vec{E}_1 + P\vec{E}_2 =^{\text{written}} (T, P)$.

The first principle of thermodynamics tells that the density $\alpha$ of internal energy is an exact differential form: $\exists U \in C^1(\vec{\mathbb{R}^2};\mathbb{R})$ s.t. $\alpha = dU$. Thus $\alpha(\vec{X}) = dU(\vec{X})$ (the internal energy density at $\vec{X}$) is a linear form in $(\mathbb{R}^2)^*$ with components $\alpha_1(\vec{X}) = \frac{\partial U}{\partial T}(\vec{X})$ and $\alpha_2(\vec{X}) = \frac{\partial U}{\partial P}(\vec{X})$:

$$dU(\vec{X}) = \frac{\partial U}{\partial T}(\vec{X})\,dT + \frac{\partial U}{\partial P}(\vec{X})\,dP \quad \text{and} \quad [dU(\vec{X})]_{|\vec{E}} = \begin{pmatrix} \frac{\partial U}{\partial T}(\vec{X}) & \frac{\partial U}{\partial P}(\vec{X}) \end{pmatrix} \ \text{(row matrix)}. \tag{A.13}$$

Thermodynamic notations: $dU = \frac{\partial U}{\partial T}_{|P}\,dT + \frac{\partial U}{\partial P}_{|T}\,dP$ and $[dU] = \begin{pmatrix} \frac{\partial U}{\partial T}_{|P} & \frac{\partial U}{\partial P}_{|T} \end{pmatrix}$.

With matrix computation, column matrices for vectors, row matrices for linear forms:

$$\begin{aligned} &\bullet\ [\vec{E}_1]_{|\vec{E}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [\vec{E}_2]_{|\vec{E}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [\vec{X}]_{|\vec{E}} = \begin{pmatrix} T \\ P \end{pmatrix}, \quad [\delta\vec{X}]_{|\vec{E}} = \begin{pmatrix} \delta T \\ \delta P \end{pmatrix}, \quad \text{and} \\ &\bullet\ [E^1]_{|\vec{E}} = [dT]_{|\vec{E}} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad [E^2]_{|\vec{E}} = [dP]_{|\vec{E}} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad [dU]_{|\vec{E}} = \begin{pmatrix} \frac{\partial U}{\partial T} & \frac{\partial U}{\partial P} \end{pmatrix} \end{aligned} \tag{A.14}$$

give

$$dU(\vec{X}).\delta\vec{X} = \begin{pmatrix} \frac{\partial U}{\partial T}(\vec{X}) & \frac{\partial U}{\partial P}(\vec{X}) \end{pmatrix} . \begin{pmatrix} \delta T \\ \delta P \end{pmatrix} = \frac{\partial U}{\partial T}(\vec{X})\delta T + \frac{\partial U}{\partial P}(\vec{X})\delta P. \tag{A.15}$$

Thermodynamic notations: $dU.\Delta\vec{X} = \frac{\partial U}{\partial T}_{|P}\Delta T + \frac{\partial U}{\partial P}_{|T}\Delta P$.

This is a "covariant calculation" (in particular no inner dot product has been used).

## A.5 Einstein convention

### A.5.1 Definition

When you work with components (after a choice of a basis), the goal is to visually differentiate a linear form from a vector (to visually differentiate covariance from contravariance).

Framework: a finite dimension vector space $E$, $\dim E = n$, and duality notations.

**Einstein Convention:**

1. A basis in $E$ (contravariant) is written with bottom indices: E.g., $(\vec{e}_i)$ is a basis in $E$.
2. A vector $\vec{x} \in E$ (contravariant) is quantified relative to $(\vec{e}_i)$ with its components written with up indices: $\vec{x} = \sum_{i=1}^n x^i \vec{e}_i$ and is represented by the column matrix $[\vec{x}]_{|\vec{e}} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$.
3. The (covariant) dual basis of $(\vec{e}_i)$ (in $E^* = \mathcal{L}(E;\mathbb{R})$) is $(e^i)$: Written with up indices.
4. A linear form $\ell \in E^*$ (covariant vector) is quantified relative to $(e^i)$ with its components written with bottom indices: $\ell = \sum_{i=1}^n \ell_i e^i$ and is represented by the row matrix $[\ell]_{|\vec{e}} = \begin{pmatrix} \ell_1 & \dots & \ell_n \end{pmatrix}$.
5. Optional: You can use "the repeated index convention", i.e. omit the sum sign $\sum$ when there are repeated indices at a different position. E.g. $\sum_{i=1}^n x^i \vec{e}_i =^{\text{written}} x^i \vec{e}_i$, $\sum_{i=1}^n \ell_i e^i =^{\text{written}} \ell_i e^i$, $\sum_{i=1}^n L^i{}_j \vec{e}_i =^{\text{written}} L^i_j \vec{e}_i$, $\sum_{i,j=1}^n g_{ij} x^i y^j =^{\text{written}} g_{ij} x^i y^j$, ... In fact, before computers and word processors, printing $\sum_{i=1}^n$ was not an easy task. With LaTeX it's easy: In this manuscript the sum sign $\sum$ is not omitted (and confusions are avoided).

### A.5.2 Do not mistake yourself

1. Einstein's convention is just meant not to confuse a linear function with a vector.
2. It only deals with quantification relative to a basis.
3. Classical notations are as good as duality notations, even if you are told that classical notations cannot detect obvious errors in component manipulations... But duality notations can be easily (and are often) misused in classical mechanics (cf. the paradigmatic example of the vectorial dual basis treated at § F.8), and then add confusion to the confusion.
4. The convention does not admit shortcuts; E.g. with a metric $(\cdot,\cdot)_g$: $g(\vec{x},\vec{y}) = \sum_{i,j=1}^n g_{ij} x^i y^j$ shows the observer dependence on a choice of a basis and on the chosen metric (with the $g_{ij}$); And even if $g_{ij} = \delta_{ij}$ you can***not*** write $g(\vec{x},\vec{y}) = \sum_{i,j=1}^n x^i y^j$: You must write $g(\vec{x},\vec{y}) = \sum_{i,j=1}^n \delta_{ij} x^i x^j$: Unmissable in physics because you need to see the metric and the basis in use.
5. Golden rule: Return to classical notations if in doubt. If not applied correctly, Einstein's convention can add confusions, untruths, misinterpretations, absurdities, misunderstandings...

## A.6 Matrix and transposed matrix

The definitions can be found in any elementary books, e.g., Strang [25].

• $\mathcal{M}_{mn}$ is the vector space (with the usual rules) of $m*n$ matrices.

• Product: If $M=[M_{ij}]_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in \mathcal{M}_{mn}$ and $N=[N_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,p}} \in \mathcal{M}_{np}$ then their product is the $m*p$ matrix $M.N=[(M.N)_{ij}]_{\substack{i=1,\dots,m\\ j=1,\dots,p}} \in \mathcal{M}_{mp}$ where $(M.N)_{ij}=\sum_{k=1}^{n} M_{ik}N_{kj}$.

• Transposed : If $M=[M_{ij}]_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in \mathcal{M}_{mn}$ then its transposed is the matrix $M^T=[(M^T)_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}} \in \mathcal{M}_{nm}$ defined by

$$(M^T)_{ij} := M_{ji}. \tag{A.16}$$

(Swap rows and columns).

• $M$ is symmetric iff $M^T=M$ (requires $m$=$n$).

• $(M.N)^T=N^T.M^T$ (because $\sum_k M_{jk}N_{ki}=\sum_k (N^T)_{ik}(M^T)_{kj}$).

• $M\in\mathcal{M}_{nn}$ is invertible iff $\exists N\in\mathcal{M}_{nn}$ s.t. $M.N=I$, then $N=^{\text{written}} M^{-1}$ and $N^{-1}=M$ and $N.M=I$.

**Exercice A.19** Prove: If $M$ is an $n*n$ invertible matrix then $M^T$ is invertible and $(M^T)^{-1}=(M^{-1})^T$ ( $=^{\text{written}} M^{-T}$); And if $M$ is symmetric, then $M^{-1}$ is symmetric.

**Answer**. $M.M^{-1}=I$ gives $(M^{-1})^T.M^T=I^T=I$, thus $M^T$ is invertible and $(M^T)^{-1}=(M^{-1})^T =^{\text{written}} M^{-T}$. Thus if $M=M^T$ then $M^{-1}=M^{-T}=(M^{-1})^T$. ▪

## A.7 Change of basis formulas

$E$ is a finite dimension vector space, $\dim E=n$, $(\vec{e}_{old,i})$ and $(\vec{e}_{new,i})$ are two bases in $E$, $(\pi_{old,i})$ and $(\pi_{new,i})$ are the associated dual bases in $E^*$, written $(e^i_{old})$ and $(e^i_{new})$ with duality notations.

### A.7.1 Change of basis endomorphism and transition matrix

**Definition A.20** The change of basis endomorphism $\mathcal{P}\in\mathcal{L}(E;E)$ from $(\vec{e}_{old,i})$ to $(\vec{e}_{new,i})$ is the endomorphism (= the linear map $E\to E$) defined by $\mathcal{P}.\vec{e}_{old,j}=\vec{e}_{new,j}$ for all $j\in[1,n]_{\mathbb{N}}$, so

$$\vec{e}_{new,j}=\mathcal{P}.\vec{e}_{old,j}. \tag{A.17}$$

And the transition matrix from $(\vec{e}_{old,i})$ to $(\vec{e}_{new,i})$ is $P:=[\mathcal{P}]_{|\vec{e}_{old}} =^{\text{clas.}} [P_{ij}] =^{\text{dual}} [P^i{}_j]$, thus the matrix whose $j$-th column $[\vec{e}_{new,j}]_{|\vec{e}_{old}}$ stores the components of $\vec{e}_{new,j}$ in the basis $(\vec{e}_{old,i})$:

$$\vec{e}_{new,j}=\sum_{i=1}^{n} P_{ij}\,\vec{e}_{old,i}=\sum_{i=1}^{n} P^i{}_j\,\vec{e}_{old,i},\quad\text{i.e.}\quad [\vec{e}_{new,j}]_{|\vec{e}_{old}}=\begin{pmatrix}P_{1j}\\ \vdots\\ P_{nj}\end{pmatrix}=\begin{pmatrix}P^1{}_j\\ \vdots\\ P^n{}_j\end{pmatrix}. \tag{A.18}$$

You can find other component notations: $P_{ij}=(P_j)_i=P^i{}_j=(P_j)^i$, i.e.

$$\vec{e}_{new,j}=\sum_{i=1}^{n}(P_j)_i\,\vec{e}_{old,i}=\sum_{i=1}^{n}(P_j)^i\,\vec{e}_{old,i},\quad\text{i.e.}\quad [\vec{e}_{new,j}]_{|\vec{e}_{old}}=\begin{pmatrix}(P_j)_1\\ \vdots\\ (P_j)_n\end{pmatrix}=\begin{pmatrix}(P_j)^1\\ \vdots\\ (P_j)^n\end{pmatrix}. \tag{A.19}$$

### A.7.2 Inverse of the transition matrix

The inverse endomorphism $\mathcal{Q}:=\mathcal{P}^{-1}\in\mathcal{L}(E;E)$ satisfies $\mathcal{Q}.\vec{e}_{new,j}=\vec{e}_{old,j}$ for all $j\in[1,n]_{\mathbb{N}}$, so

$$\vec{e}_{old,j}=\mathcal{Q}.\vec{e}_{new,j}=\sum_{i=1}^{n}Q_{ij}\vec{e}_{new,i}=\sum_{i=1}^{n}Q^i{}_j\vec{e}_{new,i},\quad [\vec{e}_{old,j}]_{|\vec{e}_{new}}=\begin{pmatrix}Q_{1j}\\ \vdots\\ Q_{nj}\end{pmatrix}=\begin{pmatrix}Q^1{}_j\\ \vdots\\ Q^n{}_j\end{pmatrix}, \tag{A.20}$$

i.e. $\mathcal{Q}$ is change of basis endomorphism from $(\vec{e}_{new,i})$ to $(\vec{e}_{old,i})$, and $Q:=[\mathcal{Q}]_{|\vec{e}_{new}}=[Q_{ij}]=[Q^i{}_j]$ is the transition matrix from $(\vec{e}_{new,i})$ to $(\vec{e}_{old,i})$.

**Proposition A.21**

$$Q=P^{-1}. \tag{A.21}$$

**Proof.** $\vec{e}_{new,j} = \mathcal{P}.\vec{e}_{old,j} = \sum_{i=1}^n P_{ij}\vec{e}_{old,i} = \sum_{i=1}^n P_{ij}(\sum_{k=1}^n Q_{ki}\vec{e}_{new,k}) = \sum_{k=1}^n(\sum_{i=1}^n Q_{ki}P_{ij})\vec{e}_{new,k} = \sum_{k=1}^n(Q.P)_{kj}\vec{e}_{new,k}$ for all $j$, thus $(Q.P)_{kj} = \delta_{kj}$ for all $j,k$. Hence $Q.P = I$, i.e. (A.21). ▪

**Remark A.22** $P^T \neq P^{-1}$ in general. E.g., $(\vec{e}_{old,i}) = (\vec{a}_i)$ is a foot-built Euclidean basis, $(\vec{e}_{new,i}) = (\vec{b}_i)$ is a metre-built Euclidean basis, and $\vec{b}_i = \lambda\vec{a}_i$ for all $i$ (the basis are "aligned"): Here $P = \lambda I$; Thus $P^T = \lambda I$ and $P^{-1} = \frac{1}{\lambda}I \neq P^T$, since $\lambda = \frac{1}{0.3048} \neq 1$. Thus it is essential not to confuse $P^T$ and $P^{-1}$, cf. the Mars Climate Orbiter probe crash remark A.17. ▪

**Exercice A.23** Prove $\left\{\begin{array}{l} [\mathcal{P}]_{|\vec{e}_{old}} = [\mathcal{P}]_{|\vec{e}_{new}} = P, \\ [\mathcal{Q}]_{|\vec{e}_{new}} = [\mathcal{Q}]_{|\vec{e}_{old}} = Q, \end{array}\right\}$, i.e.

$$\left\{\begin{array}{ll} \mathcal{P}.\vec{e}_{new,j} = \sum_{i,j=1}^n P_{ij}\vec{e}_{new,i} & (= \sum_{i,j=1}^n P^i{}_j\vec{e}_{new,i} = \sum_{i,j=1}^n (P_j)^i\vec{e}_{new,i}), \\ \mathcal{Q}.\vec{e}_{old,j} = \sum_{i,j=1}^n Q_{ij}\vec{e}_{old,i} & (= \sum_{i,j=1}^n Q^i{}_j\vec{e}_{old,i} = \sum_{i,j=1}^n (Q_j)^i\vec{e}_{old,i}). \end{array}\right. \tag{A.22}$$

**Answer**. $Z = [Z_{ij}] = [\mathcal{P}]_{|\vec{e}_{new}}$ means $\mathcal{P}.\vec{e}_{new,j} = \sum_i Z_{ij}\vec{e}_{new,i}$, i.e. $\vec{e}_{new,j} = \mathcal{Q}.(\sum_{i=1}^n Z_{ij}\vec{e}_{new,i}) = \sum_{i=1}^n Z_{ij}\mathcal{Q}.\vec{e}_{new,i} = \sum_{i=1}^n Z_{ij}(\sum_{k=1}^n Q_{ki}\vec{e}_{new,k}) = \sum_{k=1}^n(\sum_{i=1}^n Q_{ki}Z_{ij})\vec{e}_{new,k} = \sum_{k=1}^n(Q.Z)_{kj}\vec{e}_{new,k}$ for all $j$, thus $(Q.Z)_{kj} = \delta_{kj}$ for all $j,k$, thus $Q.Z = I$, thus $Z = P$. Idem for $Q$, thus (A.22). ▪

### A.7.3 Change of dual basis

**Proposition A.24** *$(\pi_{new,i}) = (e^i_{new})$ and $(\pi_{old,i}) = (e^i_{old})$ being the dual bases of $(\vec{e}_{new,i})$ and $(\vec{e}_{old,i})$, we have, for all $i \in [1,n]_{\mathbb{N}}$,*

$$\pi_{new,i} \overset{clas.}{=} \sum_{j=1}^n Q_{ij}\pi_{old,j} \quad = \quad e^i_{new} \overset{dual}{=} \sum_{j=1}^n Q^i{}_j e^j_{old}, \tag{A.23}$$

*i.e.*

$$[\pi_{new,i}]_{|\vec{e}_{old}} = (\,Q_{i1} \quad \ldots \quad Q_{in}\,) = [e^i_{new}]_{|\vec{e}_{old}} = (\,Q^i{}_1 \quad \ldots \quad Q^i{}_n\,) \quad (i\text{-th row of } Q). \tag{A.24}$$

**Proof.** $\pi_{new,i}(\vec{e}_{old,k}) =^{(A.20)} \pi_{new,i}(\sum_j Q_{jk}\vec{e}_{new,j}) = \sum_j Q_{jk}\,\pi_{new,i}(\vec{e}_{new,j}) = \sum_j Q_{jk}\,\delta_{ij} = Q_{ik}$, and $\sum_j Q_{ij}\pi_{old,j}(\vec{e}_{old,k}) = \sum_j Q_{ij}\delta_{jk} = Q_{ik}$, true for all $i,k$, thus $\pi_{new,i} = \sum_j Q_{ij}\pi_{old,j}$, i.e. (A.23). And the matrix of a linear form is a row matrix. ▪

### A.7.4 Change of bases formulas for vectors and linear forms

**Proposition A.25** *Let $\vec{x} \in E$ and $\ell \in E^*$. The Change of bases formulas are*

$$\begin{array}{ll} \bullet\ [\vec{x}]_{|\vec{e}_{new}} = P^{-1}.[\vec{x}]_{|\vec{e}_{old}} & \textit{(contravariance formula for vectors: between column matrices)}, \\ \bullet\ [\ell]_{|\vec{e}_{new}} = [\ell]_{|\vec{e}_{old}}.P & \textit{(covariance formula for linear forms: between row matrices)}. \end{array} \tag{A.25}$$

*And the real $\ell.\vec{x}$ is computed indifferently with one or the other basis (objective result):*

$$\ell.\vec{x} = [\ell]_{|\vec{e}_{old}}.[\vec{x}]_{|\vec{e}_{old}} = [\ell]_{|\vec{e}_{new}}.[\vec{x}]_{|\vec{e}_{new}}. \tag{A.26}$$

**Proof.** $\vec{x} = \sum_{j=1}^n x_j\vec{e}_{old,j} = \sum_{j=1}^n x_j(\sum_{i=1}^n Q_{ij}\vec{e}_{new,i}) = \sum_{i=1}^n(\sum_{j=1}^n Q_{ij}x_j)\vec{e}_{new,i}$ and $\vec{x} = \sum_i y_i\vec{e}_{new,i}$ give $y_i = \sum_j Q_{ij}x_j$ for all $i$, thus $(A.25)_1$.

$\ell = \sum_{i=1}^n \ell_i\pi_{old,i} = \sum_{ij}\ell_i P_{ij}\pi_{new,j}$ and $\ell = \sum_j m_j\pi_{new,j}$ give $m_j = \sum_i \ell_i P_{ij}$ for all $j$, thus $(A.25)_2$.

Thus $[\ell]_{|\vec{e}_{new}}.[\vec{x}]_{|\vec{e}_{new}} = ([\ell]_{|\vec{e}_{old}}.P).(P^{-1}.[\vec{x}]_{|\vec{e}_{old}}) = [\ell]_{|\vec{e}_{old}}.[\vec{x}]_{|\vec{e}_{old}}$, hence (A.26).

Use duality notations if you prefer. ▪

**Notation:** Let $\vec{x} \in E$, $\vec{x} = \sum_j x_j\vec{e}_{old,j} = \sum_i y_i\vec{e}_{new,i}$. Hence (A.25) give $y_i = \sum_{j=1}^n Q_{ij}x_j$, which tells: $y_i$ is the function defined by $y_i(x_1,...,x_n) = \sum_{j=1}^n Q_{ij}x_j$, thus $Q_{ij} = \frac{\partial y_i}{\partial x_j}(x_1,...,x_n)$; Similarly with $P_{ij}$; Which is written

$$Q_{ij} = \frac{\partial y_i}{\partial x_j}, \quad \text{and} \quad P_{ij} = \frac{\partial x_i}{\partial y_j}. \tag{A.27}$$

(Use duality notations if you prefer: $Q^i{}_j = \frac{\partial y^i}{\partial x^j}$ and $P^i{}_j = \frac{\partial x^i}{\partial y^j}$.)

**Exercice A.26** Check that (A.25) applies to $\vec{e}_{new,j}$ and $\pi_{new,i}$.

**Answer**. Let $(\vec{E}_i)$ be the canonical basis in $\mathcal{M}_{n1}$ the space of $n*1$ matrices. Thus $[\vec{e}_{new,j}]_{|\vec{e}_{new}} = \vec{E}_j$ and $P.[\vec{e}_{new,j}]_{|\vec{e}_{new}} =^{(A.25)} [\vec{e}_{new,j}]_{|\vec{e}_{old}}$ reads $P.\vec{E}_j = [\vec{e}_{new,j}]_{|\vec{e}_{old}}$ = column $j$ of $P$ : True.

$[\pi_{new,i}]_{|\vec{e}_{old}} = \vec{E}_i^T$, thus $[\pi_{new,i}]_{|\vec{e}_{new}}.Q =^{(A.25)} [\pi_{new,i}]_{|\vec{e}_{old}}$ reads $\vec{E}_i^T.Q = [\pi_{new,i}]_{|\vec{e}_{old}}$ = row $i$ of $Q$ : True. ▪

## A.8 Bidual basis (and contravariance)

**Definition A.27** The dual of $E^*$ is $E^{**} := (E^*)^* = \mathcal{L}(E^*;\mathbb{R})$ and is named the bidual of $E$. $E^{**}$ is also called the space of contravariant vectors (= the space of directional derivatives see § T.1).

If $(\vec{e}_i)$ is a basis in $E$, $(\pi_{ei})$ is its dual basis (basis in $E^*$), then the dual basis $(\partial_i)$ of $(\pi_{ei})$ is called the bidual basis of $(\vec{e}_i)$. (Duality notations: $(\partial_i)$ is the dual basis of $(e^i)$.)

Thus, for all $i$, the linear form $\partial_i \in E^{**} = \mathcal{L}(E^*;\mathbb{R})$ are characterized by, for all $j$,

$$\partial_i.\pi_{ej} = \delta_{ij} \quad (= \pi_{ej}.\vec{e}_i), \quad \text{so:} \quad \ell = \sum_{i=1}^{n} \ell_i \pi_{ei} \quad \text{iff} \quad \ell_i = \partial_i.\ell \quad (= \ell.\vec{e}_i), \tag{A.28}$$

since $\partial_i(\ell) = \partial_i(\sum_{j=1}^n \ell_j \pi_{ej}) = \sum_{j=1}^n \ell_j \partial_i(\pi_{ej}) = \sum_{j=1}^n \ell_j \delta_{ij} = \ell_i$. (Dual. not.: $\partial_i.e^j = \delta_i^j$, and $\ell_i = \partial_i.\ell$.)

**Remark A.28** $\partial_i$ refers to the derivation in the direction $\vec{e}_i$ because $\partial_i(df(\vec{x})) = df(\vec{x}).\vec{e}_i$. And thanks to the natural canonical isomorphism $\mathcal{J} : \left\{ \begin{array}{ll} E & \to E^{**} \\ \vec{u} & \to \mathcal{J}(\vec{u}) \end{array} \right\}$ given by $\mathcal{J}(\vec{u}).\ell := \ell.\vec{u}$ for all $\ell \in E^*$ (observer independent identification see (U.8)), we can identify $\vec{u}$ and $\mathcal{J}(\vec{u})$. Notation in differential geometry: $\mathcal{J}(\vec{e}_i) = \partial_i =^{\text{written}} \vec{e}_i$, and $(df(\vec{x}).\vec{e}_i =)\ \partial_i(df(\vec{x})) =^{\text{written}} \vec{e}_i(\vec{f})(\vec{x})$. ▪

## A.9 Bilinear forms

$E$ and $F$ are vector spaces.

### A.9.1 Definition

**Definition A.29** • A bilinear form is a function $\beta(\cdot,\cdot) : \left\{ \begin{array}{ll} E \times F & \to \mathbb{R} \\ (\vec{u}, \vec{w}) & \to \beta(\vec{u}, \vec{w}) \end{array} \right\}$ s.t.:

$\beta(\vec{u}_1 + \lambda\vec{u}_2, \vec{w}) = \beta(\vec{u}_1, \vec{w}) + \lambda\beta(\vec{u}_2, \vec{w})$ (linearity for the first variable) and $\beta(\vec{u}, \vec{w}_1 + \lambda\vec{w}_2) = \beta(\vec{u}, \vec{w}_1) + \lambda\beta(\vec{u}, \vec{w}_2)$ (linearity for the second variable) for all $\vec{u}, \vec{u}_1, \vec{u}_2 \in E$, $\vec{w}, \vec{w}_1, \vec{w}_2 \in F$, $\lambda \in \mathbb{R}$.

• $\mathcal{L}(E, F;\mathbb{R})$ is the set of bilinear forms $E \times F \to \mathbb{R}$.

• An elementary bilinear form in $\mathcal{L}(E, F;\mathbb{R})$ is a bilinear form $\ell \otimes m \in \mathcal{L}(E, F;\mathbb{R})$ made with a $\ell \in E^*$ and a $m \in F^*$ and defined by, for all $(\vec{u}, \vec{w}) \in E \times F$,

$$(\ell \otimes m)(\vec{u}, \vec{w}) = \ell(\vec{u})m(\vec{w}) \quad (= (\ell.\vec{u})(m.\vec{w})). \tag{A.29}$$

### A.9.2 The transposed of a bilinear form (objective)

**Definition A.30** If $\beta \in \mathcal{L}(E, F;\mathbb{R})$ then its transposed is the bilinear form $\beta^T \in \mathcal{L}(F, E;\mathbb{R})$ defined by, for all $(\vec{w}, \vec{u}) \in F \times E$,

$$\beta^T(\vec{w}, \vec{u}) = \beta(\vec{u}, \vec{w}). \tag{A.30}$$

This definition is objective = observer independent, i.e. same definition for all observers; In particular the definition of $\beta^T$ doesn't require a basis or an inner dot product.

Warning: Not to be confused with a transposed of a linear map, subjective because it depends on a choice of an inner dot product, see e.g. (A.49).

### A.9.3 Inner dot products, Cauchy–Schwarz inequality, and metrics

**Definition A.31** Here $F = E$ and $\beta \in \mathcal{L}(E, E;\mathbb{R})$.

• $\beta$ is (semi-)positive iff, for all $\vec{u} \in E$, $\beta(\vec{u}, \vec{u}) \geq 0$.

• $\beta$ is definite positive iff, for all $\vec{u} \neq \vec{0}$, $\beta(\vec{u}, \vec{u}) > 0$.

• $\beta$ is symmetric iff $\beta^T = \beta$, i.e. iff $\beta(\vec{u}, \vec{v}) = \beta(\vec{v}, \vec{u})$ for all $\vec{u}, \vec{v} \in E$.

**Definition A.32** • An "inner dot product" (or "scalar dot product", or "scalar inner dot product", or "inner scalar product", or "inner product") in a vector space $E$ is a symmetric and definite positive bilinear form $g \in \mathcal{L}(E, E;\mathbb{R})$, and

$$g \overset{\text{written}}{=} g(\cdot,\cdot) \overset{\text{written}}{=} (\cdot,\cdot)_g \overset{\text{written}}{=} \cdot \bullet_g \cdot, \quad \text{i.e.} \quad g(\vec{u}, \vec{w}) = (\vec{u}, \vec{w})_g = \vec{u} \bullet_g \vec{w}, \quad \forall \vec{u}, \vec{w} \in E. \tag{A.31}$$

• A "semi-inner dot product" is a symmetric and semi-positive bilinear form.

**Definition A.33** Let $(\cdot,\cdot)_g$ be an inner dot product in $E$.
- Two vectors $\vec{u},\vec{w}\in E$ are $(\cdot,\cdot)_g$-orthogonal iff $(\vec{u},\vec{w})_g=0$.
- The associated norm with $(\cdot,\cdot)_g$ is the function $||.||_g : E\to\mathbb{R}_+$ defined by, for all $\vec{u}\in E$,

$$||\vec{u}||_g=\sqrt{(\vec{u},\vec{u})_g}. \tag{A.32}$$

It is called a semi-norm iff $(\cdot,\cdot)_g$ is a symmetric and semi-positive bilinear form.

NB: orthogonality is subjective : It depends on a chosen $(\cdot,\cdot)_g$.

**Proposition A.34** *(Cauchy–Schwarz inequality.) If $(\cdot,\cdot)_g$ is an inner dot product in $E$ then*

$$\forall\vec{u},\vec{w}\in E,\quad |(\vec{u},\vec{w})_g|\le ||\vec{u}||_g||\vec{w}||_g, \tag{A.33}$$

*and $|(\vec{u},\vec{w})_g|=||\vec{u}||_g||\vec{w}||_g$ iff $\vec{u}$ and $\vec{w}$ are parallel. Thus $||.||_g$ in (A.32) is indeed a norm.*

**Proof.** Let $p(\lambda)=||\vec{u}{+}\lambda\vec{w}||_g^2=(\vec{u}{+}\lambda\vec{w},\vec{u}{+}\lambda\vec{w})_g$, so $p(\lambda)=a\lambda^2+b\lambda+c$ where $a=||\vec{w}||_g^2$, $b=2(\vec{u},\vec{w})_g$ and $c=||\vec{u}||_g^2$. With $p(\lambda)\ge 0$ (since$(\cdot,\cdot)_g$ is positive), we get $b^2-4ac\ge 0$, thus (A.33).

And $||u||_g=0$ iff $(\vec{u},\vec{u})_g=0$ iff $\vec{u}=0$ since $(\cdot,\cdot)_g$ is definite positive.

Then $|(\vec{u},\vec{w})_g|=||\vec{u}||_g||\vec{w}||_g$ iff $b^2-4ac=0$, i.e. iff $\exists\lambda$ s.t. $p(\lambda)=0$ i.e. $\vec{u}{+}\lambda\vec{w}=0$.

And $||u||_g\ge 0$ since $(\cdot,\cdot)_g$ is definite positive, and $||\lambda\vec{u}||_g=\sqrt{(\lambda\vec{u},\lambda\vec{u})_g}=\sqrt{\lambda^2(\vec{u},\vec{u})_g}=|\lambda|\,||\vec{u}||_g$, and $||\vec{u}+\vec{w}||_g^2=(\vec{u}+\vec{w},\vec{u}+\vec{w})_g=||\vec{u}||_g^2+2(\vec{u},\vec{w})_g+||\vec{w}||_g^2\le||\vec{u}||_g^2+2||\vec{u}||_g\,||\vec{w}||_g+||\vec{w}||_g^2=(||\vec{u}||_g+||\vec{w}||_g)^2$ thanks to Cauchy–Schwarz inequality, thus $||\vec{u}+\vec{w}||_g\le||\vec{u}||_g+||\vec{w}||_g$; Thus $||.||_g$ is a norm. ∎

**Definition A.35** (Metric.) 1- In $\mathbb{R}^n$ our usual affine geometric space, $n=1$, 2 or 3, with $\vec{\mathbb{R}^n}=$ the usual associated vector space made of bipoint vectors. Let $\Omega\subset\mathbb{R}^n$ be open in $\mathbb{R}^n$. A metric in $\Omega$ is a $C^\infty$ function $g:\begin{Bmatrix}\Omega & \to\mathcal{L}(\vec{\mathbb{R}^n},\vec{\mathbb{R}^n};\mathbb{R})\\ p & \to g(p)\overset{\text{written}}{=}g_p\end{Bmatrix}$ such that $g_p$ is an inner dot product in $\vec{\mathbb{R}^n}$ at each $p\in\Omega$. Particular Case: If the $g_p$ is independent of $p$ then a metric is simply called a inner dot product (e.g. a Euclidean metric is called a Euclidean dot product).

2- In a differentiable manifold $\Omega$, a metric is a $C^\infty$ $\binom{0}{2}$ tensor $g$ s.t. $g(p)\in\mathcal{L}(T_p\Omega,T_p\Omega;\mathbb{R})$ is an inner dot product at each $p\in\Omega$ (with $T_p\Omega$ the tangent plane at $p$). A Riemannian metric is a metric s.t. $g(p)$ is a Euclidean dot product at each $p\in\Omega$.

### A.9.4 Quantification: Matrice $[\beta_{ij}]$ and tensorial representation

$\dim E=n$, $\dim F=m$, $\beta\in\mathcal{L}(E,F;\mathbb{R})$, $(\vec{a}_i)$ is a basis in $E$ which dual basis is $(\pi_{ai})=(a^i)$, $(\vec{b}_i)$ is a basis in $F$ which dual basis is $(\pi_{bi})=(b^i)$ (classical and duality notations).

**Definition A.36** The components of $\beta\in\mathcal{L}(E,F;\mathbb{R})$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ are the $nm$ reals

$$\beta_{ij}:=\beta(\vec{a}_i,\vec{b}_j),\quad\text{and}\quad [\beta]_{|\vec{a},\vec{b}}=[\beta_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}\overset{\text{written}}{=}[\beta_{ij}] \tag{A.34}$$

is the matrix of $\beta$ relative to $(\vec{a}_i)$ and $(\vec{b}_i)$. If $F=E$ and $(\vec{b}_i)=(\vec{a}_i)$ then $[\beta]_{|\vec{a},\vec{a}}=^{\text{written}}[\beta]_{|\vec{a}}$.

**Proposition A.37** *A bilinear form $\beta\in\mathcal{L}(E,F;\mathbb{R})$ is known as soon as the $nm$ scalars $\beta_{ij}=\beta(\vec{a}_i,\vec{b}_j)$ are known: We have*

$$\begin{aligned}\beta&=\sum_{i=1}^n\sum_{j=1}^m\beta_{ij}\pi_{ai}\otimes\pi_{bj}=\sum_{i=1}^n\sum_{j=1}^m\beta_{ij}a^i\otimes b^j,\quad\text{and}\\ \beta(\vec{u},\vec{w})&=\sum_{i,j=1}^n\beta_{ij}u_iw_j=\sum_{i,j=1}^n\beta_{ij}u^iw^j=[\vec{u}]_{|\vec{a}}{}^T.[\beta]_{|\vec{a},\vec{b}}.[\vec{w}]_{|\vec{b}}\end{aligned} \tag{A.35}$$

*for all $\vec{u}=\sum_{i=1}^n u_i\vec{a}_i=\sum_{i=1}^n u^i\vec{a}_i\in E$ and $\vec{w}=\sum_{i=1}^n w_i\vec{b}_i=\sum_{i=1}^n w^i\vec{b}_i\in F$.*

*Duality notations: $\beta=\sum_{i=1}^n\sum_{j=1}^m\beta_{ij}a^i\otimes b^j$ and $\beta(\vec{u},\vec{w})=\sum_{i,j=1}^n\beta_{ij}u^iw^j$.*

*And $\dim\mathcal{L}(E,F;\mathbb{R})=nm$, a basis being given by the $nm$ functions $\pi_{ai}\otimes\pi_{bj}=a^i\otimes b^j$.*

**Proof.** $\beta$ being bilinear, $\vec{u} = \sum_{i=1}^n u_i\vec{a}_i$ and $\vec{w} = \sum_{j=1}^n w_j\vec{b}_j$ give $\beta(\vec{u},\vec{w}) = \sum_{i,j=1}^n u_iw_j\beta(\vec{a}_i,\vec{b}_j) = \sum_{i,j=1}^n u_i\beta_{ij}w_j = ([\vec{u}]_{|\vec{a}})^T.[\beta]_{|\vec{a},\vec{b}}.[\vec{w}]_{|\vec{b}}$. Thus if the $\beta_{ij}$ are known then $\beta$ is known.

And $(\pi_{ai}\otimes\pi_{bj})(\vec{a}_k,\vec{b}_\ell) =^{(A.29)} (\pi_{ai}.\vec{a}_k)(\pi_{bj}.\vec{b}_\ell) = \delta_{ik}\delta_{j\ell}$ (all the elements of the matrix $[\pi_{ai}\otimes\pi_{bj}]_{|\vec{a},\vec{b}}$ are zero except the element at the intersection of row $i$ and column $j$ which is equal to 1).

Thus $\sum_{i,j=1}^n\beta_{ij}(\pi_{ai}\otimes\pi_{bj})(\vec{u},\vec{w}) = \sum_{i,j=1}^n\beta_{ij}u_iw_j = \beta(\vec{u},\vec{v})$, for all $\vec{u},\vec{w}$, thus $\beta := \sum_{i,j=1}^n\beta_{ij}(\pi_{ai}\otimes\pi_{bj})$, thus the $\pi_{ai}\otimes\pi_{bj}$ span $\mathcal{L}(E,F;\mathbb{R})$. And $\sum_{ij}\lambda_{ij}(\pi_{ai}\otimes\pi_{bj}) = 0$ implies $0 = (\sum_{ij}\lambda_{ij}(\pi_{ai}\otimes\pi_{bj}))(\vec{a}_k,\vec{b}_\ell) = \sum_{ij}\lambda_{ij}(\pi_{ai}\otimes\pi_{bj})(\vec{a}_k,\vec{b}_\ell) = \sum_{ij}\lambda_{ij}\delta_{ik}\delta_{j\ell} = \lambda_{k\ell} = 0$ for all $k,\ell$; Thus the $\pi_{ai}\otimes\pi_{bj}$ are independent. Thus $(\pi_{ai}\otimes\pi_{bj})$ is a basis in $\mathcal{L}(E,F;\mathbb{R})$ and $\dim(\mathcal{L}(E,F;\mathbb{R})) = nm$. ▪

**Example A.38** $\dim E = \dim F = 2$. $[\beta]_{|\vec{a},\vec{b}} = \begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}$ means $\beta(\vec{a}_1,\vec{b}_1) = \beta_{11} = 1$, $\beta(\vec{a}_1,\vec{b}_2) = \beta_{12} = 2$, $\beta(\vec{a}_2,\vec{b}_1) = \beta_{21} = 0$, $\beta(\vec{a}_2,\vec{b}_2) = \beta_{22} = 3$. And $\beta_{12} = [\vec{a}_1]_{|\vec{a}}^T.[\beta]_{|\vec{a},\vec{b}}.[\vec{b}_2]_{|\vec{b}} = (1\quad 0).\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}.\begin{pmatrix}0\\ 1\end{pmatrix} = 2$. ▪

**Exercice A.39** Prove

$$\beta = \sum_{ij}\beta_{ij}\pi_{ai}\otimes\pi_{bj} \;\Rightarrow\; \beta^T = \sum_{ij}\beta_{ji}\pi_{bi}\otimes\pi_{aj}, \quad \text{i.e.}\quad [\beta^T]_{\vec{b},\vec{a}} = ([\beta]_{\vec{a},\vec{b}})^T \quad\text{written}\quad [\beta^T] = [\beta]^T. \tag{A.36}$$

I.e. $\beta = \sum_{ij}\beta_{ij}a^i\otimes b^j \;\Rightarrow\; \beta^T = \sum_{ij}\beta_{ji}b^i\otimes a^j$.

**Answer**. $\beta^T = \sum_{ij}(\beta^T)_{ij}\pi_{bi}\otimes\pi_{aj} \in \mathcal{L}(F,E;\mathbb{R})$ gives $(\beta^T)_{ij} = \beta^T(\vec{b}_i,\vec{a}_j) =^{(A.30)} \beta(\vec{a}_j,\vec{b}_i) = \beta_{ji}$. ▪

**Exercice A.40** $\beta\in\mathcal{L}(E,E;\mathbb{R})$ and $(\vec{a}_i)$ and $(\vec{b}_i)$ are two bases in $E$, and let $\lambda\in\mathbb{R}^*$. Prove:

$$\text{if, } \forall i\in[1,n]_{\mathbb{N}},\ \vec{b}_i = \lambda\vec{a}_i, \quad\text{then}\quad [\beta]_{|\vec{b}} = \lambda^2[\beta]_{|\vec{a}}. \tag{A.37}$$

(A change of unit, e.g. from foot to metre, has a big influence on the matrix of a bilinear form.)

**Answer**. $\vec{b}_i = \lambda\vec{a}_i$ give $\beta(\vec{b}_i,\vec{b}_j) = \beta(\lambda\vec{a}_i,\lambda\vec{a}_j) = \lambda^2\beta(\vec{a}_i,\vec{a}_j)$ (bilinearity), thus $[\beta]_{|\vec{b}} = \lambda^2[\beta]_{|\vec{a}}$. ▪

## A.10 Linear maps

$E$ and $F$ are vector spaces.

### A.10.1 Definition

**Definition A.41** • A function $L : E\to F$ is linear iff $L(\vec{u}_1+\lambda\vec{u}_2) = L(\vec{u}_1)+\lambda L(\vec{u}_2)$ for all $\vec{u}_1,\vec{u}_2\in E$ and all $\lambda\in\mathbb{R}$ (distributivity rule). And (distributivity notation):

$$L(\vec{u}) \overset{\text{written}}{=} L.\vec{u}, \quad\text{so}\quad L(\vec{u}_1+\lambda\vec{u}_2) = L.(\vec{u}_1+\lambda\vec{u}_2) = L.\vec{u}_1+\lambda L.\vec{u}_2. \tag{A.38}$$

NB: This dot notation $L(\vec{u}) =^{\text{written}} L.\vec{u}$ is a linearity notation (distributivity type notation);
- It is an "outer" dot product between a (linear) function and a vector;
- It is ***not*** an "inner" dot product since $L$ and $\vec{u}$ don't belong to a same space.
- It is ***not*** a matrix product (no quantification with bases has been done yet).

**Definition A.42** $\mathcal{L}(E;F)$ is the set of linear maps $E\to F$ (vector space, subspace of $(\mathcal{F}(E;F),+,.)$).

If $F=E$ then a linear map $L\in\mathcal{L}(E;E)$ is called an endomorphism in $E$.

If $F=\mathbb{R}$ then a linear map $E\to\mathbb{R}$ is called a linear form, and $E^* := \mathcal{L}(E;\mathbb{R})$ is the dual of $E$.

$L_i(E;F)$ is the space of invertible linear maps $E\to F$, i.e. $L\in L_i(E;F)$ iff $\exists M\in L_i(F;E)$ s.t. $L\circ M = I_F$ and $M\circ L = I_E$ where $I_E$ and $I_F$ are the identity maps in $E$ and $F$.

**Vocabulary:** If $E$ is a finite dimension vector space, $\dim E = n$, then, in algebra, the set $(L_i(E;E),\circ)$ of invertible endomorphisms equipped with the composition rule is called $GL_n(E)$ = "the linear group" (it is indeed a group, easy check).

Particular case: $GL_n(\mathcal{M}_n) = (L_i(\mathcal{M}_n;\mathcal{M}_n),.)$ is the set of invertible $n*n$ matrices equipped with the matrix product.

**Exercice A.43** (Math exercise.) $E=(E,||.||_E)$ and $F=(F,||.||_F)$ are Banach spaces, and $L_{ic}(E;F)$ is the space of invertible linear continuous maps $E\to F$ with its usual norm $||L||=\sup_{||\vec{x}||_E=1}||L.\vec{x}||_F$. Let $Z:\left\{\begin{array}{ll} L_{ic}(E;F) & \to L_{ic}(E;F) \\ L & \to L^{-1}\end{array}\right\}$. Prove: $dZ(L).M=-L^{-1}\circ M\circ L^{-1}$, for all $M\in L_{ic}(E;F)$.

**Answer**. Consider $\lim_{h\to 0}\frac{Z(L+hM)-Z(L)}{h}=\lim_{h\to 0}\frac{(L+hM)^{-1}-L^{-1}}{h}$, $=^{\text{written}} dZ(L).M$ if the limit exists. With $N=L^{-1}.M$ we have $L+hM=L(I+hN)$, and $(I+hN)$ is invertible as soon as $||hN||<1$, i.e. $h<\frac{1}{||N||}=\frac{1}{||L^{-1}.M||}$, its inverse being $I-hN+h^2N-...$ (Neumann series); Thus $I+hN=I-hN+o(h)$, and $(L+hM)^{-1}=(I+hN)^{-1}.L^{-1}=(I-hN+o(h)).L^{-1}=L^{-1}-hN.L^{-1}+o(h)$. Thus $\frac{(L+hM)^{-1}-L^{-1}}{h}=\frac{L^{-1}-hN.L^{-1}+o(h)-L^{-1}}{h}=-N.L^{-1}+o(1)\longrightarrow_{h\to 0}-N.L^{-1}$. ▪

### A.10.2 Quantification: Matrices $[L_{ij}]=[L^i{}_j]$

$\dim E=n$, $\dim F=m$, $L\in\mathcal{L}(E;F)$, $(\vec{a}_i)$ is a basis in $E$ which dual basis is $(\pi_{ai})=(a^i)$, $(\vec{b}_i)$ is a basis in $F$ which dual basis is $(\pi_{bi})=(b^i)$ (classical and duality notations).

**Definition A.44** The components of a linear map $L\in\mathcal{L}(E;F)$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ are the $nm$ reals named $L_{ij}$ (classical notation) $=L^i{}_j$ (duality notation), which are the components of the vectors $L.\vec{a}_j$ relative to the basis $(\vec{b}_i)$. So:

$$L.\vec{a}_j=\sum_{i=1}^m L_{ij}\vec{b}_i=\sum_{i=1}^m L^i{}_j\vec{b}_i,\quad\text{i.e.}\quad L_{ij}=\pi_{bi}.L.\vec{a}_j=L^i{}_j=b^i.L.\vec{a}_j. \tag{A.39}$$

And

$$[L]_{|\vec{a},\vec{b}}=[L_{ij}]_{\substack{i=1,...,m\\ j=1,...,n}}=[L^i{}_j]_{\substack{i=1,...,m\\ j=1,...,n}}\overset{\text{written}}{=}[L_{ij}]=[L^i{}_j] \tag{A.40}$$

is the matrix of $L$ relative to $(\vec{a}_i)$ and $(\vec{b}_i)$. So

$$[L.\vec{a}_j]_{|\vec{b}}=\begin{pmatrix}L_{1j}\\ \vdots\\ L_{mj}\end{pmatrix}=\begin{pmatrix}L^1{}_j\\ \vdots\\ L^m{}_j\end{pmatrix}=j\text{-th column of }[L]_{|\vec{a},\vec{b}}. \tag{A.41}$$

Particular case: $E=F$, so $L$ is an endomorphism in $E$, and $(\vec{b}_i)=(\vec{a}_i)$: $[L]_{|\vec{a},\vec{a}}=^{\text{written}}[L]_{|\vec{a}}$.

**Example A.45** $n=m=2$. $[L]_{|\vec{a},\vec{b}}=\begin{pmatrix}1&2\\0&3\end{pmatrix}$ means $L.\vec{a}_1=\vec{b}_1$ and $L.\vec{a}_2=2\vec{b}_1+3\vec{b}_2$ (column reading). Here $L_{11}$=1, $L_{12}$=2, $L_{21}$=0, $L_{22}$=3 (duality notations: $L^1{}_1$=1, $L^1{}_2$=2, $L^2{}_1$=0, $L^2{}_2$=3). ▪

Let $L\in\mathcal{L}(E;F)$. For all $\vec{u}\in E$, $\vec{u}=\sum_{j=1}^n u_j\vec{a}_j=\sum_{j=1}^n u^j\vec{a}_j$, we get, thanks to linearity,

$$L.\vec{u}=\sum_{i=1}^m\sum_{j=1}^n L_{ij}u_j\vec{b}_i=\sum_{i=1}^m\sum_{j=1}^n L^i{}_ju^j\vec{b}_i,\quad\text{i.e.}\quad\boxed{[L.\vec{u}]_{|\vec{b}}=[L]_{|\vec{a},\vec{b}}.[\vec{u}]_{|\vec{a}}}. \tag{A.42}$$

Shortened notation: $[L.\vec{u}]=[L].[\vec{u}]$ when the bases are implicit.

**Proposition A.46** *A linear map $L\in\mathcal{L}(E;F)$ is known as soon as the $n$ vectors $L.\vec{a}_1$, ..., $L.\vec{a}_n$ are known. And, for $i,k=1,...,n$ and $j=1,...,m$, the linear maps $\mathcal{L}_{ij}\in\mathcal{L}(E;F)$ defined by $\mathcal{L}_{ij}.\vec{a}_k=\delta_{jk}\vec{b}_i$ (all the elements of the matrix $[\mathcal{L}_{ij}]_{|\vec{a},\vec{b}}$ vanish except the element at the intersection of row $i$ and column $j$ which is equal to 1), constitute a basis $\in\mathcal{L}(E;F)$. So, $\dim(\mathcal{L}(E;F))=nm$.*

*(Duality notations: $\mathcal{L}_{ij}=^{\text{written}}\mathcal{L}_i{}^j$, and $\mathcal{L}_i{}^j.\vec{a}_k=\delta^j_k\vec{b}_i$.)*

**Proof.** $L$ is linear, $\vec{u}\in E$ and $\vec{u}=\sum_k u_j\vec{a}_j$ give $L.\vec{u}=\sum_j u_jL.\vec{a}_j$. Thus $L$ is known iff the $n$ vectors $L.\vec{a}_j$ are known, $j=1,...,n$; And $(\sum_{ij}L_{ij}\mathcal{L}_{ij}).\vec{a}_k=\sum_{ij}L_{ij}\delta_{jk}\vec{b}_i=\sum_i L_{ik}\vec{b}_i=L.\vec{a}_k$, for all $k$, thus $\sum_{ij}L_{ij}\mathcal{L}_{ij}=L$, i.e. $L=\sum_{ij}L_{ij}\mathcal{L}_{ij}$, thus the $\mathcal{L}_{ij}$ span $\mathcal{L}(E;F)$. And $\sum_{i=1}^m\sum_{j=1}^n\lambda_{ij}\mathcal{L}_{ij}=0$ implies $\sum_{i=1}^m\sum_{j=1}^n\lambda_{ij}\mathcal{L}_{ij}.\vec{a}_k=\sum_{i=1}^m\sum_{j=1}^n\lambda_{ij}\delta_{jk}\vec{b}_i=\sum_{i=1}^m\lambda_{ik}\vec{b}_i=\vec{0}$ for all $k$, thus $\lambda_{ik}=0$ for all $i,k$ (because $(\vec{b}_i)$ is a basis). Thus the $\mathcal{L}_{ij}$ are independent. Thus $(\mathcal{L}_{ij})_{\substack{i=1,...,n\\ j=1,...,m}}$ is a basis in $\mathcal{L}(E;F)$. ▪

**Exercice A.47** $L\in\mathcal{L}(E;E)$. If $\vec{b}_i=\lambda\vec{a}_i$ for all $i$ (change of unit of measurement), check $[L]_{|\vec{a}}=[L]_{|\vec{b}}$.

**Answer**. $L.\vec{b}_j=\sum_i N_{ij}\vec{b}_i$ gives $L.(\lambda\vec{a}_j)=\sum_i N_{ij}(\lambda\vec{a}_i)$, thus $L.\vec{a}_j=\sum_i N_{ij}\vec{a}_i$, thus $N=M$. ▪

## A.11 Trace of an endomorphism: Invariant

Recall: The trace of a $n*n$ matrix $[L_{ij}]$ is $\mathrm{Tr}([L_{ij}])=\sum_{i=1}^n L_{ii}=$ sum of its diagonal elements.

$E$ is a vector space, $\dim E=n$, $(\vec{a}_i)$ is a basis in $E$.

**Definition A.48** The trace of an endomorphism $L\in\mathcal{L}(E;E)$ is the real

$$\mathrm{Tr}(L)=\mathrm{Tr}([L]_{|\vec{a}})\quad\text{thus}\quad\mathrm{Tr}(L)=\sum_{i=1}^n L_{ii}=\sum_{i=1}^n L^i{}_i \tag{A.43}$$

when $L.\vec{a}_j=\sum_{i=1}^n L_{ij}\vec{a}_i=\sum_{i=1}^n L^i{}_j\vec{a}_i$ for all $j$.

And the trace operator is the linear map $\mathrm{Tr}:\left\{\begin{array}{rl}\mathcal{L}(E;E) & \to\mathbb{R}\\ L & \to\mathrm{Tr}(L)\end{array}\right\}$.

**Proposition A.49** *If $L,M\in\mathcal{L}(E;E)$ then $\mathrm{Tr}(L.M)=\mathrm{Tr}(M.L)$, which means*

$$\mathrm{Tr}(L\circ M)=\mathrm{Tr}(M\circ L)=\sum_{i,j=1}^n L_{ij}M_{ji}=\sum_{i,j=1}^n L^i{}_jM^j{}_i=\mathrm{Tr}([L]_{|\vec{a}}.[M]_{|\vec{a}}). \tag{A.44}$$

*And the real $\mathrm{Tr}(L)$ is independent of a chosen basis in $E$: If $(\vec{a}_i)$ and $(\vec{b}_i)$ are bases in $E$, then*

$$\mathrm{Tr}([L]_{|\vec{a}})=\mathrm{Tr}([L]_{|\vec{b}})=\mathrm{Tr}(L)\quad(\textit{invariant}). \tag{A.45}$$

**Proof.** $L.\vec{a}_j=\sum_i L_{ij}\vec{a}_i$ and $M.\vec{a}_j=\sum_i M_{ij}\vec{a}_i$ give $(L\circ M).\vec{a}_j=L.(M.\vec{a}_j)=\sum_k M_{kj}L.\vec{a}_k=\sum_{ik}M_{kj}L_{ik}\vec{a}_i=\sum_i(\sum_k L_{ik}M_{kj})\vec{a}_i$. Thus $\mathrm{Tr}(L\circ M)=\sum_i(\sum_k L_{ik}M_{ki})=\sum_{ij}L_{ij}M_{ji}=\mathrm{Tr}([L]_{|\vec{a}}.[M]_{|\vec{a}})=\sum_{ij}L_{ji}M_{ij}=\mathrm{Tr}(M\circ L)$.

And $[L]_{|\vec{b}}=P^{-1}.[L]_{|\vec{a}}.P$ where $P$ is the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$ (change of basis formula see (A.93)), thus $\mathrm{Tr}([L]_{|\vec{b}})=\mathrm{Tr}(P^{-1}.[L]_{|\vec{a}}.P)=\mathrm{Tr}((P^{-1}.[L]_{|\vec{a}}).P)=\mathrm{Tr}(P.(P^{-1}.[L]_{|\vec{a}}))=\mathrm{Tr}((P.P^{-1}).[L]_{|\vec{a}})=\mathrm{Tr}([L]_{|\vec{a}})$. ▪

**Exercice A.50** For $L:=\vec{w}\otimes\ell$, defined by $(\vec{w}\otimes\ell).\vec{u}=(\ell.\vec{u})\vec{w}$ for all $\vec{u}$, check:

$$\mathrm{Tr}(\vec{w}\otimes\ell)=\ell.\vec{w}. \tag{A.46}$$

**Answer.** $\vec{w}=\sum_i w_i\vec{a}_i$ and $\ell=\sum_i\ell_i\pi_{ai}$ give $[\vec{w}\otimes\ell)=[w_i\ell_j]$, thus $\mathrm{Tr}(\vec{w}\otimes\ell)=\sum_i w_i\ell_i=\sum_i\ell_iw_i=\ell.\vec{w}$. ▪

**Remark A.51** The "trace" of a bilinear form $g:E\times E\to\mathbb{R}$ (e.g. an inner dot product) defined with a basis $(\vec{a}_i)$ by $T_{\vec{a}}(g)=\sum_i g_{ii}$ is useless (not used) because it depends on the choice of the basis $(\vec{a}_i)$: E.g. if $\vec{b}_i=\lambda\vec{a}_i$ then $T_{\vec{b}}(g)=\lambda^2T_{\vec{a}}(g)\neq T_{\vec{a}}(g)$ when $\lambda\neq\pm1$. ▪

## A.12 Transposed of a linear map: depends on chosen inner dot products

Not to be confused with the transposed of a bilinear form which is objective cf. (A.30).

Not to be confused with the transposed of a matrix cf. (A.16).

Not to be confused with the adjoint of an endomorphism which is objective see § A.13.

$(E,(\cdot,\cdot)_g)$ and $(F,(\cdot,\cdot)_h)$ are Hilbert spaces, and $L\in\mathcal{L}(E;F)$ (supposed continuous when $E$ and $F$ are infinite dimensional). E.g., $E=\vec{\mathbb{R}}^n_{t_0}$, $F=\vec{\mathbb{R}}^n_t$, deformation gradient $F=d\Phi_t^{t_0}(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$, cf. (4.1), $(\cdot,\cdot)_g$ is the foot built Euclidean dot product chosen by the observer at $t_0$ (measurements at $t_0$), $(\cdot,\cdot)_h$ is the metre built Euclidean dot product chosen by the observer at $t$ (measurements at $t$).

### A.12.1 Definition (depends on inner dot products)

**Definition A.52** The transposed of $L\in\mathcal{L}(E;F)$ relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ is $L^T_{gh}\in\mathcal{L}(F;E)$ defined by, for all $(\vec{u},\vec{w})\in E\times F$,

$$(L^T_{gh}.\vec{w},\vec{u})_g=(\vec{w},L.\vec{u})_h,\quad\text{written}\quad(L^T_{gh}.\vec{w})\bullet_g\vec{u}=\vec{w}\bullet_h(L.\vec{u}). \tag{A.47}$$

where we used the dot notation $L^T_{gh}(\vec{w})\overset{\text{written}}{=}L^T_{gh}.\vec{w}$ since $L^T_{gh}$ is linear. This defines the map

$$(.)^T_{gh}:\left\{\begin{array}{rl}\mathcal{L}(E;F) & \to\mathcal{L}(F;E)\\ L & \to(.)^T_{gh}(L):=L^T_{gh}\end{array}\right. \tag{A.48}$$

(A linear map has an infinite number of transposed: depends on inner dot products, see exercise A.55.)

**Particular case of an endomorphism:** $L \in \mathcal{L}(E;E)$ and $(\cdot,\cdot)_g$ is an inner dot product in $E$. Then

$$L^T_{gg} \overset{\text{written}}{=} L^T_g = \text{the } (\cdot,\cdot)_g\text{-transposed of an endomorphism } L. \tag{A.49}$$

Isometric framework: $(\cdot,\cdot)_g$ is an imposed Euclidean dot product (English, French,...); Then $L^T_g =^{\text{written}} L^T$ and (A.47) reads

$$(L^T.\vec{w}) \bullet \vec{u} = \vec{w} \bullet (L.\vec{u}). \tag{A.50}$$

**Exercice A.53** Prove: If $E$ and $F$ are finite dimensional, if $L \in \mathcal{L}(E;F)$ is invertible then 1- $L^T_{gh}$ is invertible, and 2- $(L^T_{gh})^{-1} = (L^{-1})^T_{hg}$.

**Answer**. 1- $L$ invertible, thus $\dim E = \dim F$. Suppose $\exists \vec{w} \in E$, $\vec{w} \neq \vec{0}$, s.t. $L^T_{gh}.\vec{w} = 0$. $L$ being invertible, $\exists! \vec{u} \in E$ s.t. $L.\vec{u} = \vec{w}$, with $\vec{u} \neq \vec{0}$ because $L$ is linear and $\vec{w} \neq \vec{0}$; We get $L^T_{gh}.L.\vec{u} = 0$, thus $(L^T_{gh}.L.\vec{u},\vec{u})_g = 0 = (L.\vec{u}, L.\vec{u})_h = ||L.\vec{u}||^2_h$, thus $L.\vec{u} = 0$, thus $\vec{u} = 0$ since $L$ is linear invertible; Absurd because $\vec{u} \neq \vec{0}$. Thus $L^T_{gh}$ is one-to-one with $\dim E = \dim F$, thus invertible.

2- $(L^T_{gh}.(L^{-1})^T_{hg}.\vec{u},\vec{w})_g = ((L^{-1})^T_{hg}.\vec{u}, L.\vec{w})_h = (\vec{u}, L^{-1}.L.\vec{w})_g = (\vec{u},\vec{w})_g = (L^T_{gh}.(L^T_{gh})^{-1}.\vec{u},\vec{w})_g$ for all $\vec{u},\vec{w}$. ▪

**Exercice A.54** Prove: If $(E,(\cdot,\cdot)_g)$ is an Hilbert space and if $L \in \mathcal{L}(E;E)$ is continuous, then $L^T_g$ exists, is unique, and is continuous (apply the Riesz representation theorem F.2).

(If $E$ is finite dimensional then see next § for a direct calculation.)

**Answer**. Let $\vec{w} \in E$, let $\ell_{\vec{w}g} : \vec{u} \in E \to \ell_{\vec{w}g}(\vec{u}) := (\vec{w}, L.\vec{u})_g \in \mathbb{R}$. $\ell_{\vec{w}g}$ is linear (trivial since $L$ is linear and $(\cdot,\cdot)_g$ is bilinear) and continuous: $|\ell_{\vec{w}g}.\vec{u}| \le ||\vec{w}||_g ||L.\vec{u}||_g \le ||\vec{w}||_g ||L||\,||\vec{u}||_g$ gives $||\ell_{\vec{w}g}||_{E^*} \le ||L||\,||\vec{w}||_g < \infty$. Let $\vec{\ell}_{\vec{w}g} \in E$ be the $(\cdot,\cdot)_g$-Riesz representation of $\ell_{\vec{w}g} \in E^*$: So $\ell_{\vec{w}g}.\vec{u} = (\vec{\ell}_{\vec{w}g},\vec{u})_g$ for all $\vec{u}$ and $||\vec{\ell}_{\vec{w}g}||_g = ||\ell_{\vec{w}g}||_{E^*}$. Then define $L^T_g : \vec{w} \in E \to L^T_g(\vec{w}) := \vec{\ell}_{\vec{w}g} \in E$; So $(L^T_g(\vec{w}),\vec{u})_g = (\vec{\ell}_{\vec{w}g},\vec{u})_g = \ell_{\vec{w}g}.\vec{u} = (\vec{w}, L.\vec{u})_g$, thus $L^T_g$ is linear (since $(\cdot,\cdot)_g$ is bilinear) and continuous: $||L^T_g.\vec{w}||_g = ||\vec{\ell}_{\vec{w}g}||_g = ||\ell_{\vec{w}g}||_{E^*} \le ||L||\,||\vec{w}||_g$ gives $||L^T_g|| \le ||L||_{\mathcal{L}(E;E)} < \infty$. Uniqueness: if $M^T_g$ also satisfies $(M^T_g.\vec{w},\vec{u})_g = (\vec{w}, L.\vec{u})_g$ then $(M^T_g.\vec{w},\vec{u})_g = (L^T_g.\vec{w},\vec{u})_g$, for all $\vec{u},\vec{w}$, thus $M^T_g = L^T_g$. ▪

### A.12.2 Quantification with bases

$(\vec{a}_i)_{i=1,\dots,n}$ and $(\vec{b}_i)_{i=1,\dots,m}$ are bases in $E$ and $F$. Let

$$\begin{aligned} &\bullet\ g_{ij} = g(\vec{a}_i,\vec{a}_j),\ [g] = [g_{ij}], \quad h_{ij} = h(\vec{b}_i,\vec{b}_j),\ [h] = [h_{ij}], \quad \text{and} \\ &\bullet\ L.\vec{a}_j = \sum_{i=1}^n L_{ij}\vec{a}_i,\ [L] = [L_{ij}], \quad L^T_g.\vec{b}_j = \sum_{i=1}^n (L^T_g)_{ij},\ [L^T_{gh}] = [(L^T_{gh})_{ij}], \end{aligned} \tag{A.51}$$

i.e. $[g] := [g]_{|\vec{a}}$, $[h] := [h]_{|\vec{b}}$, $[L] := [L]_{|\vec{a},\vec{b}}$, $[L^T_{gh}] := [L^T_{gh}]_{|\vec{e}}$.

(A.47) gives $[\vec{u}]^T.[g].[L^T_{gh}.\vec{w}] = ([L.\vec{u}])^T.[h].[\vec{w}]$ for all $\vec{u},\vec{w}$, thus

$$[g].[L^T_{gh}] = [L]^T.[h], \quad \text{i.e.} \quad \sum_{k=1}^n g_{ik}(L^T_{gh})_{kj} = \sum_{k=1}^m L_{ki}\,h_{kj}, \tag{A.52}$$

i.e.

$$\boxed{[L^T_{gh}] = [g]^{-1}.[L]^T.[h]}, \quad \text{i.e.} \quad (L^T_{gh})_{ij} = \sum_{k=1}^n\sum_{\ell=1}^m ([g]^{-1})_{ik} L_{\ell k} h_{\ell j}. \tag{A.53}$$

Duality notations:

$$\sum_{k=1}^n g_{ik}(L^T_{gh})^k{}_j = \sum_{k=1}^n L^k{}_i\,h_{kj}, \quad \text{i.e.} \quad (L^T_{gh})^i{}_j = \sum_{k,\ell=1}^n g^{ik} L^\ell{}_k\,h_{\ell j} \quad \text{where} \quad g^{ij} := ([g]^{-1})_{ik}. \tag{A.54}$$

**Particular case of an endomorphism:**

$$\boxed{[L^T_g] = [g]^{-1}.[L]^T.[g]} \tag{A.55}$$

Particular case $(\vec{a}_i)$ is $(\cdot,\cdot)_g$-orthonormal: $[g] = [\delta_{ij}]$, thus $[L^T_g] = [L]^T$, i.e. $(L^T_g)_{ij} = L_{ji}$, $(L^T_g)^i{}_j = L^j{}_i$.

**Exercice A.55** Basis $(\vec a_i)$ in $\vec{\mathbb{R}^2}$, $L\in\mathcal{L}(\vec{\mathbb{R}^2};\vec{\mathbb{R}^2})$ defined by $[L]:=[L]_{|\vec a}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Find two inner dot products $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ in $\vec{\mathbb{R}^2}$ such that $L^T_g\neq L^T_h$ (a transposed endomorphism is not unique, is not intrinsic to $L$, because it depends on a choice of an inner dot product by an observer).

**Answer**. Apply (A.55):
$[g]=\begin{pmatrix}1&0\\0&1\end{pmatrix}=[I]$. Thus $[L^T_g]=[I].[L].[I]=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, i.e. $L^T_g.\vec a_1=\vec a_2$ and $L^T_g.\vec a_2=\vec a_1$.
$[h]=\begin{pmatrix}1&0\\0&2\end{pmatrix}$. Thus $[L^T_h]=[h]^{-1}.[L].[h]=\begin{pmatrix}0&2\\\frac12&0\end{pmatrix}$, i.e. $L^T_h.\vec a_1=\frac12\vec a_2$ and $L^T_h.\vec a_2=2\vec a_1$. ∴

**Exercice A.56** Prove: Two proportional inner dot products give the same transposed endomorphism: If $L\in\mathcal{L}(E;E)$ and $\exists\lambda>0$ s.t. $(\cdot,\cdot)_a=\lambda^2(\cdot,\cdot)_b$ then $L^T_a=L^T_b$.

**Answer**. $(L^T_b.\vec w,\vec u)_b=(\vec w,L.\vec u)_b=\lambda^2(\vec w,L.\vec u)_a=\lambda^2(L^T_a.\vec w,\vec u)_a=(L^T_a.\vec w,\vec u)_b$, for all $\vec u,\vec w$, so $L^T_b=L^T_a$. ∴

**Exercice A.57** Let $L\in\mathcal{L}(E;E)$. Prove: $\mathrm{Tr}(L^T_g)=\mathrm{Tr}(L)$ (independent of $g$).

**Answer**. $\mathrm{Tr}(L^T_g)=\mathrm{Tr}([L^T_g]_{|\vec e})=\mathrm{Tr}([g]^{-1}_{|\vec e}.[L]^T_{|\vec e}.[g]_{|\vec e})=\mathrm{Tr}([g]_{|\vec e}.[g]^{-1}_{|\vec e}.[L]^T_{|\vec e})=\mathrm{Tr}([L]^T_{|\vec e})=\mathrm{Tr}([L]_{|\vec e})=\mathrm{Tr}(L)$. ∴

**Remark A.58** In fact $g^{ij}$ in (A.54) is the short notation for $(g^\sharp)^{ij}$, see (F.30). Use classical notations to avoid misuses and misinterpretations. ∴

### A.12.3 Isometry

$(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are inner dot products in $E$ and $F$.

**Definition A.59** An invertible linear map $L\in\mathcal{L}_i(E;F)$ is an isometry relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ iff

$$\forall\vec u,\vec w\in E,\quad (L.\vec u,L.\vec w)_h=(\vec u,\vec w)_g,\quad\text{i.e.}\quad L^T_{gh}\circ L=I_E\ \text{(identity in } E).\tag{A.56}$$

Thus, if $L\in\mathcal{L}(E;F)$ is an isometry and $(\vec e_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis, then $(L.\vec e_i)$ is a $(\cdot,\cdot)_h$-orthonormal basis, since $(L.\vec e_i,L.\vec e_j)_h=(\vec e_i,\vec e_j)_g=\delta_{ij}$ for all $i,j$.

In particular, an endomorphism $L\in\mathcal{L}_i(E;E)$ is a $(\cdot,\cdot)_g$-isometry iff

$$\forall\vec u,\vec w\in E,\quad (L.\vec u,L.\vec w)_g=(\vec u,\vec w)_g,\quad\text{i.e.}\quad L^T_g\circ L=I.\tag{A.57}$$

**Exercice A.60** Let $\vec f:E\to F$. Prove:

$$\text{If, }\ \forall\vec u,\vec w\in E,\ \ (\vec f(\vec u),\vec f(\vec w))_h=(\vec u,\vec w)_g\ \text{ then }\ \vec f\text{ is linear (and is an isometry).}\tag{A.58}$$

**Answer**. Let $(\vec e_i)$ be a $(\cdot,\cdot)_g$-orthonormal basis; Thus $(\vec f(\vec e_i))$ is a $(\cdot,\cdot)_h$-orthonormal basis, cf. hypothesis in (A.58). And $\vec u=\sum_i u_i\vec e_i$ and $\vec w=\sum_i w_i\vec e_i$ give $\vec f(\vec u)=^{b.o.n.}\sum_i(\vec f(\vec u),\vec f(\vec e_i))_h\vec f(\vec e_i)=^{hyp.}\sum_i(\vec u,\vec e_i)_g\vec f(\vec e_i)=^{b.o.n.}\sum_i u_i\vec f(\vec e_i)$, thus $\vec f(\vec u+\lambda\vec w)=\sum_i(u_i+\lambda w_i)\vec f(\vec e_i)=\sum_i u_i\vec f(\vec e_i)+\lambda\sum_i w_i\vec f(\vec e_i)=\vec f(\vec u)+\lambda\vec f(\vec w)$, thus $\vec f$ is linear. ∴

**Exercice A.61** $\mathbb{R}^n$ is an affine space, $\vec{\mathbb{R}^n}$ is the usual associated vector space, $(\cdot,\cdot)_g$ is an inner dot product in $\vec{\mathbb{R}^n}$ and $||.||_g||$ is the associated norm. Definition: a function $f:p\in\mathbb{R}^n\to f(p)\in\mathbb{R}^n$ is $||.||_g$-distance-preserving iff

$$||\overrightarrow{f(p)f(q)}||_g=||\overrightarrow{pq}||_g,\quad\forall p,q\in\mathbb{R}^n.\tag{A.59}$$

Prove: If $f$ is a distance-preserving function, then $f$ is affine.

**Answer**. Let $O\in\mathbb{R}^n$ (an origin) and $\vec f:\vec x=\overrightarrow{Op}\in\vec{\mathbb{R}^n}\to\vec f(\vec x):=\overrightarrow{f(O)f(p)}$ (vectorial associated function). Let $\vec x=\overrightarrow{Op}$ and $\vec y=\overrightarrow{Oq}$. Then the remarkable identity $2(\vec f(\vec x),\vec f(\vec y))_g=||\vec f(\vec x)||_g^2+||\vec f(\vec y)||_g^2-||\vec f(\vec x)-\vec f(\vec y)||_g^2$ gives $2(\vec f(\vec x),\vec f(\vec y))_g=||\vec f(\vec x)||_g^2+||\vec f(\vec y)||_g^2-||\overrightarrow{f(q)f(p)}||_g^2=||\vec f(\vec x)||_g^2+||\vec f(\vec y)||_g^2-||\overrightarrow{qp}||_g^2=||\vec x||_g^2+||\vec y||_g^2-||\vec x-\vec y||_g^2=2(\vec x,\vec y)_g$, thus $\vec f$ is linear cf. (A.58), thus $f$ is affine since $f(p)=f(O)+\vec f(\overrightarrow{Op})$. ∴

### A.12.4 Symmetric endomorphism (depends on a $(\cdot,\cdot)_g$)

**Definition A.62** An inner dot product $(\cdot,\cdot)_g$ being chosen, an endomorphism $L\in\mathcal{L}(E;E)$ is $(\cdot,\cdot)_g$-symmetric iff $L^T_g=L$:

$$L\ (\cdot,\cdot)_g\text{-symmetric}\quad\Longleftrightarrow\quad L^T_g=L\quad\Longleftrightarrow\quad (L.\vec u,\vec w)_g=(\vec u,L.\vec w)_g,\quad\forall\vec u,\vec w\in E.\tag{A.60}$$

(Depends on $(\cdot,\cdot)_g$: $L$ can be $(\cdot,\cdot)_g$-symmetric and not $(\cdot,\cdot)_h$-symmetric, see exercise A.55; I.e. the symmetric character of an endomorphism is not intrinsic to the endomorphism.)

### A.12.5 Deformation gradient symmetric: Absurd

The symmetry of a linear map $L \in \mathcal{L}(E;F)$ is a nonsense if $E \neq F$.

E.g.: The gradient of deformation $F_t^{t_0}(p_{t_0}) = d\Phi_t^{t_0}(p_{t_0}) =^{\text{written}} F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ cannot be symmetric since $F^T \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0})$. Idem for the first Piola–Kirchhoff tensor $\mathbb{K}_t^{t_0}$, which motivates the introduction of the symmetric second Piola–Kirchhoff tensor $\mathbb{S}_t^{t_0}$, see Marsden–Hughes [17] or § O.2.4.

### A.12.6 Dangerous tensorial notation

To simplify the writings, we consider $F = E$.

The transposed $\beta^T \in \mathcal{L}(E,E;\mathbb{R})$ of a bilinear form $\beta \in \mathcal{L}(E,E;\mathbb{R})$ is *objective* cf. (A.30): We don't need any tool like an inner dot product to define $\beta^T$. And (quantification) with a basis $(\vec{e}_i)$ and its dual bases $(e^i)$: If $\beta = \sum_{ij} \beta_{ij} e^i \otimes e^j$ then $\beta^T = \sum_{ij} \beta_{ji} e^i \otimes e^j$, i.e. $[\beta^T]_{|\vec{e}} = [\beta]_{|\vec{e}}{}^T$.

The transposed $L_g^T$ of an endomorphism $L \in \mathcal{L}(E;E)$ is *subjective* because it depends on a choice of an inner dot products $(\cdot,\cdot)_g$. And with a basis $(\vec{e}_i)$ (quantification), $[L_g^T]_{|\vec{e}} \neq [L]_{|\vec{e}}{}^T$ in general because $[L_g^T]_{|\vec{e}} = [g]^{-1}.[L]^T.[g]$.

Hence it is *dangerous* to represent an endomorphism in a basis with its "bilinear tensorial representation" when dealing with the transposed. Details: $L \in \mathcal{L}(E;E)$ is naturally canonically represented by the bilinear form $M_L \in \mathcal{L}(E^*,E;\mathbb{R})$ (mixed tensor) defined by $M_L(\ell,\vec{u}) = \ell.L.\vec{u}$ for all $\ell \in E^*$ and $\vec{u} \in E$ (so $M_L \notin \mathcal{L}(E,E;\mathbb{R})$). Quantification:

$$L.\vec{e}_j = \sum_{i=1}^n L^i{}_j \vec{e}_i \;\simeq\; M_L = \sum_{i,j=1}^n L^i{}_j \vec{e}_i \otimes e^j \;\text{ hence }\; M_L{}^T \overset{(A.36)}{=} \sum_{i,j=1}^n L^j{}_i e^i \otimes \vec{e}_j. \tag{A.61}$$

And, $(\cdot,\cdot)_g$ being chosen, $L_g^T \in \mathcal{L}(F;E)$ is represented by the bilinear form $M_{L_g^T} \in \mathcal{L}(E^*,E;\mathbb{R})$; And

$$L_g^T.\vec{e}_j = \sum_{i=1}^n (L_g^T)^i{}_j \vec{e}_i \;\simeq\; M_{L_g^T} = \sum_{i,j=1}^n (L_g^T)^i{}_j \vec{e}_i \otimes e^j; \;\text{ Thus }\; \boxed{M_{L_g^T} \neq M_L{}^T} \tag{A.62}$$

because: 1- $\vec{e}_i \otimes e^j \neq e^i \otimes \vec{e}_j$ (!), and

2- $(L_g^T)^i{}_j = \sum_{k,\ell=1}^n ([g]^{-1})_{ik} L^\ell{}_k\, g_{\ell j} \neq L^j{}_i$ in general, while $(M_L{}^T)^i{}_j = (M_L)^j{}_i$ always: $(\beta_L)^T$ is independent of any inner dot product, while $L_g^T$ depends on a chosen inner dot product.

3- $M_{L_g^T} \in \mathcal{L}(E^*,E;\mathbb{R}) \simeq \mathcal{L}(E^*;E^*)$ is the tensorial representation of the adjoint $L^*$ of $L$, see (A.68).

So in continuum mechanics it is strongly advised ***not to use the tensorial notation*** for linear maps when dealing with transposed (you should not confuse covariance with contravariance). It can be only used for computations when a Euclidean basis and associated Euclidean dot product are imposed (isometric framework).

### A.12.7 The general flat $^\flat$ notation for an endomorphism (depends on a $(\cdot,\cdot)_g$)

The $^\flat$ notation deals with change of variance. Let $L \in \mathcal{L}(E;E)$. Choose an inner dot product $(\cdot,\cdot)_g$.

**Definition A.63** The associate bilinear form $L_g^\flat \in \mathcal{L}(E,E;\mathbb{R})$ is defined by, for all $\vec{u},\vec{w} \in E$,

$$L_g^\flat(\vec{u},\vec{w}) := (\vec{u},L.\vec{w})_g \quad \text{(it is } (\cdot,\cdot)_g \text{ dependent)}. \tag{A.63}$$

(The bilinearity of $L_g^\flat$ is trivial.) (Thus $L \in T_1^1(\Omega)$ implies $L^\flat \in T_2^0(\Omega)$.)

**Quantification:** $(\vec{e}_i)$ is a basis in $E$, $(e^i)$ is its covariant dual basis, $g = \sum_{i,j=1}^n g_{ij} e^i \otimes e^j$ i.e. $[g] = [g_{ij}]$,

$$L.\vec{e}_j = \sum_{i=1}^n L^i{}_j \vec{e}_i, \quad [L] = [L^i{}_j], \quad L_g^\flat = \sum_{i,j=1}^n (L_g^\flat)_{ij} e^i \otimes e^j, \quad [L_g^\flat] = [(L_g^\flat)_{ij}]. \tag{A.64}$$

This explains the $^\flat$ notation: The up index in $L^i{}_j$ becomes a down index in $(L_g^\flat)_{ij}$.

And $(L_g^\flat)_{ij} = L_g^\flat(\vec{e}_i,\vec{e}_j) = (\vec{e}_i, L.\vec{e}_j)_g = (\vec{e}_i, \sum_k L^k{}_j \vec{e}_k)_g = \sum_k L^k{}_j (\vec{e}_i,\vec{e}_k)_g = \sum_k L^k{}_j g_{ik} = ([g].[L])_{ij}$, thus

$$(L_g^\flat)_{ij} = \sum_k L^k{}_j g_{ik}, \quad \text{i.e.} \quad \boxed{[L_g^\flat] = [g].[L]}. \tag{A.65}$$

**Remark A.64** (A.63) defines the $(\cdot,\cdot)_g$-dependent operator

$$(.)_g^\flat = \mathcal{J}_g(.) : \begin{cases} \mathcal{L}(E;E) \simeq \mathcal{L}(E^*,E;\mathbb{R}) & \to \mathcal{L}(E,E;\mathbb{R}) \\ \qquad\qquad L & \to \mathcal{J}_g(L) := L_g^\flat, \end{cases} \tag{A.66}$$

This operator is a contravariance–covariance exchange operator. With the natural canonical isomorphism $L \in \mathcal{L}(E;E) \simeq T_L \in \mathcal{L}(E^*,E;\mathbb{R})$ given by $T_L(\ell,\vec{w}) = \ell.L.\vec{w}$ see (U.13): The "mixed tensor" $L$ has been transformed into the "twice-covariant tensor" $L_g^\flat$. With a basis:

$$L \simeq T_L = \sum_{i,j=1}^n L^i{}_j \vec{e}_i \otimes e^j \quad \text{and} \quad L_g^\flat = \sum_{i,j=1}^n (L_g^\flat)_{ij} e^i \otimes e^j, \tag{A.67}$$

with (A.65). ∴

## A.13 The adjoint of a linear map (objective)

A linear map $L \in \mathcal{L}(E;F)$ has one and only one adjoint $L^*$ (intrinsic to $L$), while it has an infinity of transposed $L^T := L^T_{gh}$ (needs chosen inner dot products). So they can't be confused.

### A.13.1 Definition

$E$ and $F$ are vector spaces, $E^* = \mathcal{L}(E;\mathbb{R})$, $F^* = \mathcal{L}(F;\mathbb{R})$.

**Definition A.65** The adjoint of a linear map $L \in \mathcal{L}(E;F)$ is the linear map $L^* \in \mathcal{L}(F^*;E^*)$ canonically defined by

$$L^* : \begin{cases} F^* & \to E^* \\ m & \to L^*(m) := m \circ L, \quad \text{written} \quad L^*.m = m.L \end{cases} \tag{A.68}$$

thanks to the linearity of $m$, $L$ and $L^*$. So, for all $(\vec{u},m) \in E \times F^*$,

$$(L^*.m).\vec{u} = m.L.\vec{u} \tag{A.69}$$

thanks to the linearity of $m$, $L$ and $L^*$.

(Remark: $||L^*.m||_{E^*} = ||m.L||_{E^*} \le ||m||_{F^*}||L||_{\mathcal{L}(E;F)}$ gives $||L^*||_{\mathcal{L}(F^*;E^*)} \le ||L||_{\mathcal{L}(E;F)}$, thus $L^*$ is continuous when $L$ is.)

### A.13.2 Quantification

$(\vec{a}_i)$ and $(\vec{b}_i)$ are bases in $E$ and $F$, $(a^i)$ and $(b^i)$ are the (covariant) dual bases, $L \in \mathcal{L}(E;F)$, so $L^* \in \mathcal{L}(F^*;E^*)$. Let

$$L.\vec{a}_j = \sum_{i=1}^m L^i{}_j \vec{b}_i, \quad [L] = [L^i{}_j], \quad L^*.b^j = \sum_{i=1}^n (L^*)_i{}^j a^i, \quad [L^*] = [(L^*)_i{}^j]. \tag{A.70}$$

(So $[L] := [L]_{|\vec{a},\vec{b}}$ and $[L^*] := [L^*]_{|b,a}$). (A.69) gives $(L^*.b^j).\vec{a}_i = b^j.L.\vec{a}_i$, thus $(\sum_{k=1}^n (L^*)_k{}^j\, a^k).\vec{a}_i = b^j.(\sum_{k=1}^m L^k{}_i \vec{b}_k)$, thus

$$(L^*)_i{}^j = L^j{}_i, \quad \text{thus} \quad [L^*] = [L]^T \quad \text{(transposed matrix).} \tag{A.71}$$

Classic notations: $L.\vec{a}_j = \sum_{i=1}^m L_{ij}\vec{b}_i$, $L^*.\pi_{bj} = \sum_{i=1}^n (L^*)_{ij}\pi_{ai}$, and $(L^*.\pi_{bj}).\vec{a}_i = \pi_{bj}.L.\vec{a}_i$, thus $(\sum_{k=1}^n (L^*)_{kj}\pi_{ak}).\vec{a}_i = \pi_{bj}.(\sum_{k=1}^m L_{ki}\vec{b}_k)$, thus $(L^*)_{ij} = L_{ji}$, thus $[L^*] = [L]^T$

### A.13.3 Relation with the transposed when inner dot products are introduced

We need the $(\cdot,\cdot)_g$-Riesz representation mapping $\vec{R}_g : \ell \in E^* \to \vec{R}_g(\ell) = \vec{\ell}_g \in E$ defined by $\ell.\vec{u} = (\vec{\ell}_g,\vec{u})_g$ for all $\vec{u} \in E$, valid when $E^*$ is the space of continuous linear forms.

Let $L \in \mathcal{L}(E;F)$ be continuous, and $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be inner dot products in $E$ and $F$: The transposed of $L$ is defined by $L^T := L^T_{gh}$ is defined by $(L^T_{gh}.\vec{w},\vec{u})_g = (\vec{w},L.\vec{u})_h$ for all $\vec{u} \in E$ and $\vec{w} \in F$.

We have

$$(L^*.m).\vec{u} \overset{(A.69)}{=} m.(L.\vec{u}), \quad \text{thus} \quad (\vec{R}_g(L^*.m),\vec{u})_g = (\vec{R}_h(m), L.\vec{u})_h, \tag{A.72}$$

thus $((\vec{R}_g \circ L^*).m,\vec{u})_g = ((L^T_{gh} \circ \vec{R}_h).m,\vec{u})_g$. Thus $\vec{R}_g \circ L^* = L^T_{gh} \circ \vec{R}_h$, i.e.

$$\boxed{L^T_{gh} = \vec{R}_g \circ L^* \circ (\vec{R}_h)^{-1}}, \quad \text{i.e.} \quad \begin{array}{ccc} E & \xleftarrow{L^T_{gh}} & F \\ \vec{R}_g \uparrow & & \uparrow \vec{R}_h \\ E^* & \xleftarrow[L^*]{} & F^* \end{array} \text{ is a commutative diagram.} \tag{A.73}$$

**Exercice A.66** From (A.73), recover (A.52), i.e. $[L^T_{gh}] = [g]^{-1}.[L]^T.[h]$.

**Answer**. $[L^T_{gh}] =^{(A.73)} [\vec{R}_g].[L^*].[\vec{R}_h]^{-1} =^{(F.7)} [g]^{-1}.[L]^T.[h]$. ▪

## A.14 Tensorial representation of a linear map (dangerous)

Consider the natural canonical isomorphism (between linear maps $E \to F$ and bilinear forms $F^* \times E \to \mathbb{R}$)

$$\widetilde{\mathcal{J}} : \left\{ \begin{array}{rl} \mathcal{L}(E;F) & \to \mathcal{L}(F^*,E;\mathbb{R}) \\ L & \to \beta_L = \widetilde{\mathcal{J}}(L) \end{array} \right\} \quad \text{where} \quad \beta_L(m,\vec{u}) := m.(L.\vec{u}), \quad \forall (m,\vec{u}) \in F^* \times E, \tag{A.74}$$

see § U.4.

**Quantification:** $(\vec{a}_i)_{i=1,...,n}$ is a basis in $E$, $(\vec{b}_i)_{i=1,...,m}$ is a basis in $F$ which dual basis is $(\pi_{bi}) = (e^i)$, $L \in \mathcal{L}(E;F)$. let

$$L.\vec{a}_j = \sum_{i=1}^m L_{ij}\vec{b}_i, \quad \beta_L = \sum_{i=1}^m\sum_{j=1}^n (\beta_L)_{ij}\vec{b}_i \otimes \pi_{aj}, \quad [L] = [L_{ij}], \quad [\beta_L] = [(\beta_L)_{ij}]. \tag{A.75}$$

(A.74) gives

$$[(\beta_L)_{ij} = \beta_L(\pi_{bi},\vec{a}_j) = \pi_{bi}.L.\vec{a}_j = L_{ij}, \quad \text{thus} \quad [\beta_L] = [L] \tag{A.76}$$

Duality notations: $L.\vec{a}_j = \sum_i L^i{}_j\vec{b}_i$ and $\beta_L = \sum_{ij}(\beta_L)^i{}_j\vec{b}_i \otimes a^j$ and $[L^i{}_j] = [(\beta_L)^i{}_j]$.

**Contraction rule.** With $\vec{u} = \sum_{i=1}^n u_i\vec{a}_i$,

$$\beta_L.\vec{u} = (\sum_{i=1}^m\sum_{j=1}^n L_{ij}\vec{b}_i \otimes \underbrace{\pi_{aj}).\vec{u}}_{\text{contraction}} := \sum_{i=1}^m\sum_{j=1}^n L_{ij}\vec{b}_i(\pi_{aj}.\vec{u}) = \sum_{j=1}^n\sum_{i=1}^m L_{ij}u_j\vec{b}_i = L.\vec{u} \tag{A.77}$$

because $L.\vec{u} = \sum_j u_jL.\vec{a}_j = \sum_{ij} u_jL_{ij}\vec{b}_i$.

Duality notations: $\beta_L.\vec{u} = (\sum_{ij} L^i{}_j\vec{b}_i \otimes \underbrace{a^j).\vec{u}}_{\text{contraction}} = \sum_{ij} L^i{}_j\vec{b}_i(a^j.\vec{u}) = \sum_j L^i{}_ju^j\vec{b}_i = L.\vec{u}$.

**Remark A.67** Warning: The bilinear form $\beta_L$ should not be confused with the linear map $L$: The domain of definition of $\beta_L$ is $F^* \times E$, and $\beta_L$ acts on the two objects $\ell$ (linear form) and $\vec{u}$ (vector) to get a ***scalar*** result; While the domain of definition of $L$ is $E$, and $L$ acts one object $\vec{u}$ to get a ***vector*** result. You can use the tensorial notation for $L$... only to calculate $L.\vec{u}$ as in (A.77) (contraction rule). ▪

## A.15 Change of basis formulas for bilinear forms and linear maps

### A.15.1 Notations

Let $A$ and $B$ be finite dimension vector spaces, $\dim A = n$, $\dim B = m$. (E.g. application to the change of basis formula for the deformation gradient $F : A{=}\vec{\mathbb{R}}^n_{t_0} \to B{=}\vec{\mathbb{R}}^n_t$.)

$(\vec{a}_{\text{old},i})$ and $(\vec{a}_{\text{new},i})$ are two bases in $A$, $(\vec{b}_{\text{old},i})$ and $(\vec{b}_{\text{new},i})$ are two bases in $B$, $(a^i_{\text{old}})$, $(a^i_{\text{new}})$, $(b^i_{\text{old}})$, $(b^i_{\text{new}})$ are the (covariant) dual bases (duality notations). Let $\mathcal{P}_A$ and $\mathcal{P}_B$ be the change of basis endomorphisms from old to new bases, and $P_A := [\mathcal{P}_A]_{|\vec{a}_{old}} = [P_{Aij}]$ and $P_B := [\mathcal{P}_B]_{|\vec{b}_{old}} = [P_{Bij}]$ be the associated transition matrices, and $Q_A = P_A^{-1}$ and $Q_B = P_B^{-1}$:

$$\begin{aligned} \vec{a}_{\text{new},j} &= \mathcal{P}_A.\vec{a}_{\text{old},i} = \sum_{i,j=1}^{n} P_A{}^i{}_j \vec{a}_{\text{old},i}, \quad a^j_{\text{new}} = \sum_{i=1}^{n} Q_A{}^i{}_j a^j_{\text{old}}, \\ \vec{b}_{\text{new},j} &= \mathcal{P}_B.\vec{b}_{\text{old},i} = \sum_{i,j=1}^{m} P_B{}^i{}_j \vec{b}_{\text{old},i}, \quad b^j_{\text{new}} = \sum_{i,j=1}^{m} Q_B{}^i{}_j b^j_{\text{old}}. \end{aligned} \tag{A.78}$$

Classical notation: $\vec{c}_{new,j} = \sum_i P_{Cij}\vec{c}_{\text{old},i}$, $\pi_{cnew,i} = \sum_j Q_{Cij}\pi_{cold,i}$.

### A.15.2 Change of coordinate system for bilinear forms $\in \mathcal{L}(A,B;\mathbb{R})$

Let $\beta \in \mathcal{L}(A,B;\mathbb{R})$, $\beta = \sum_{ij} M_{ij} a^i_{\text{old}} \otimes b^j_{\text{old}} = \sum_{ij} N_{ij} a^i_{\text{new}} \otimes b^j_{\text{new}}$, i.e., for all $(i,j) \in [1,n]_{\mathbb{N}} \times [1,m]_{\mathbb{N}}$,

$$\beta(\vec{a}_{\text{old},i}, \vec{b}_{\text{old},j}) = M_{ij}, \quad \beta(\vec{a}_{\text{new},i}, \vec{b}_{\text{new},j}) = N_{ij}, \quad \text{i.e.} \quad \begin{cases} [\beta]_{|olds} = M = [M_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \\ [\beta]_{|news} = N = [N_{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}. \end{cases} \tag{A.79}$$

**Proposition A.68** *Change of basis formula:*

$$\boxed{[\beta]_{|news} = P_A{}^T.[\beta]_{|olds}.P_B}, \quad \text{i.e.} \quad N = P_A{}^T.M.P_B. \tag{A.80}$$

*In particular, if $A = B$ and $(\vec{a}_{\text{old},i}) = (\vec{b}_{\text{old},i})$ and $(\vec{a}_{\text{new},i}) = (\vec{b}_{\text{new},i})$, then $P_A = P_B =^{\text{written}} P$, and*

$$\boxed{[\beta]_{|new} = P^T.[\beta]_{old}.P}, \quad \text{i.e.} \quad N = P^T.M.P. \tag{A.81}$$

**Proof.** $N_{ij} = \beta(\vec{a}_{\text{new},i}, \vec{b}_{\text{new},j}) = \sum_{k\ell} P_A{}^k{}_i P_B{}^\ell{}_j \beta(\vec{a}_{\text{old},k}, \vec{b}_{\text{old},\ell}) = \sum_{k\ell} P_A{}^k{}_i M_{k\ell} P_B{}^\ell{}_j = \sum_{k\ell} (P_A{}^T)^i{}_k M_{k\ell} P_B{}^\ell{}_j$. ▪

**Exercice A.69** Prove (objective result):

$$\beta(\vec{u}, \vec{w}) = [\vec{u}]^T_{|\vec{a}_{\text{new}}}.[\beta]_{|news}.[\vec{w}]_{|\vec{b}_{\text{new}}} = [\vec{u}]^T_{|\vec{a}_{\text{old}}}.[\beta]_{|olds}.[\vec{w}]_{|\vec{b}_{\text{old}}}. \tag{A.82}$$

**Answer**. $[\vec{u}]^T_{|\vec{a}_{\text{new}}}.[\beta]_{|news}.[\vec{w}]_{|\vec{b}_{\text{new}}} = (P_A{}^{-1}.[\vec{u}]_{|\vec{a}_{\text{old}}})^T.(P_A{}^T.[\beta]_{|olds}.P_B).(P_B{}^{-1}.[\vec{w}]_{|\vec{b}_{\text{old}}})$. ▪

### A.15.3 Change of coordinate system for bilinear forms $\in \mathcal{L}(A^*,B^*;\mathbb{R})$

Let $z \in \mathcal{L}(A^*,B^*;\mathbb{R})$, and, for all $(i,j) \in [1,n]_{\mathbb{N}} \times [1,m]_{\mathbb{N}}$,

$$z(a^i_{\text{old}}, b^j_{\text{old}}) = M^{ij}, \quad z(a^i_{\text{new}}, b^j_{\text{new}}) = N^{ij}, \quad \text{i.e.} \quad \begin{cases} [z]_{|olds} = M = [M^{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \\ [z]_{|news} = N = [N^{ij}]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}. \end{cases} \tag{A.83}$$

**Proposition A.70** *Change of basis formula:*

$$[z]_{|news} = P_A{}^{-T}.[z]_{|olds}.P_B{}^{-1}, \quad \text{i.e.} \quad N = P_A{}^{-T}.M.P_B{}^{-1}. \tag{A.84}$$

*In particular, if $A = B$ and $(\vec{a}_{\text{old},i}) = (\vec{b}_{\text{old},i})$ and $(\vec{a}_{\text{new},i}) = (\vec{b}_{\text{new},i})$, then $P_A = P_B =^{\text{written}} P$, and*

$$[z]_{|new} = P^{-T}.[z]_{old}.P^{-1}, \quad \text{i.e.} \quad N = P^{-T}.M.P^{-1}. \tag{A.85}$$

**Proof.** $N_{ij} = z(a^i_{\text{new}}, b^j_{\text{new}}) = \sum_{k\ell} Q_A{}^k{}_i Q_B{}^\ell_j z(a^k_{\text{old}}, b^\ell_{\text{old}}) = \sum_{k\ell} Q_A{}^k{}_i M^{k\ell} Q_B{}^\ell{}_j = \sum_{k\ell} (Q_A{}^T)^i{}_k M^{k\ell} Q_B{}^\ell{}_j$. ▪

### A.15.4 Change of coordinate system for bilinear forms $\in \mathcal{L}(B^*, A; \mathbb{R})$

(Similar to linear maps $L \in \mathcal{L}(A;B) \simeq \mathcal{L}(B^*, A; \mathbb{R})$ thanks to the natural canonical isomorphism.)

Let $T \in \mathcal{L}(B^*, A; \mathbb{R})$, and, for all $(i,j) \in [1,n]_{\mathbb{N}} \times [1,m]_{\mathbb{N}}$,

$$T(b^i_{\text{old}}, \vec{a}_{\text{old},j}) = M^i{}_j, \quad T(b^i_{\text{new}}, \vec{a}_{\text{new},j}) = N^i{}_j, \quad \text{i.e.} \quad \begin{cases} [T]_{|olds} = M = [M^i{}_j]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}, \\ [T]_{|news} = N = [N^i{}_j]_{\substack{i=1,\dots,n\\ j=1,\dots,m}}. \end{cases} \tag{A.86}$$

**Proposition A.71** *Change of basis formula:*

$$\boxed{[T]_{|news} = P_B{}^{-1}.[T]_{|olds}.P_A}, \quad \text{i.e.} \quad N = Q_A.M.P_B. \tag{A.87}$$

*In particular, if $A = B$ and $(\vec{a}_{\text{old},i}) = (\vec{b}_{\text{old},i})$ and $(\vec{a}_{\text{new},i}) = (\vec{b}_{\text{new},i})$, then $P_A = P_B =^{\text{written}} P$, and*

$$\boxed{[T]_{|new} = P^{-1}.[T]_{old}.P}, \quad \text{i.e.} \quad N = P^{-1}.M.P. \tag{A.88}$$

**Proof.** $N^i{}_j = T(b^i_{\text{new}}, \vec{a}_{\text{new},j}) = \sum_{k\ell} Q_B{}^i{}_k P_A{}^\ell{}_j T(b^i_{\text{old}}, \vec{a}_{\text{old},j}) = \sum_{k\ell} Q_B{}^i{}_k M^i{}_j P_A{}^\ell{}_j$ ∎

### A.15.5 Change of coordinate system for tri-linear forms $\in \mathcal{L}(A^*, A, A; \mathbb{R})$

(For $d^2\vec{u}(p)$: For a vector field $\vec{u} \in \Gamma(U) \simeq T^1_0(U)$, $\vec{u}(p) \in \vec{\mathbb{R}^n}$, its differential satisfies $d\vec{u}(p) \in \mathcal{L}(\vec{\mathbb{R}^n}; \vec{\mathbb{R}^n}) \simeq \mathcal{L}(\mathbb{R}^{n*}, \vec{\mathbb{R}^n}; \mathbb{R})$, and $d^2\vec{u}(p) \in \mathcal{L}(\vec{\mathbb{R}^n}; \mathcal{L}(\vec{\mathbb{R}^n}; \vec{\mathbb{R}^n})) \simeq \mathcal{L}(\mathbb{R}^{n*}, \vec{\mathbb{R}^n}, \vec{\mathbb{R}^n}; \mathbb{R})$, see § T.1.4.)

Consider a tri-linear form $T \in \mathcal{L}(A^*, A, A; \mathbb{R})$, and $[T]_{|\vec{a}_{\text{old}}} = [M^i_{jk}]$ and $[T]_{|\vec{a}_{\text{new}}} = [N^i_{jk}]$, so where

$$M^i_{jk} = T(a^i_{\text{old}}, \vec{a}_{\text{old},j}, \vec{a}_{\text{old},k}), \quad N^i_{jk} = T(a^i_{\text{new}}, \vec{a}_{\text{new},j}, \vec{a}_{\text{new},k}). \tag{A.89}$$

Then

$$N^i_{jk} = \sum_{\lambda,\mu,\nu=1}^{n} Q^i_\lambda P^\mu_j P^\nu_k \, M^\lambda_{\mu\nu}. \tag{A.90}$$

Indeed $\sum_{\lambda\mu\nu} M^\lambda_{\mu\nu} \vec{a}_{old,\lambda} \otimes a^\mu_{old} \otimes a^\nu_{old} = \sum_{\lambda\mu\nu ijk} M^\lambda_{\mu\nu} Q^i_\lambda P^\mu_j P^\nu_k \vec{a}_{\text{new},i} \otimes a^j_{\text{new}} \otimes a^k_{\text{new}}$. ∎

### A.15.6 Change of coordinate system for linear maps $\in \mathcal{L}(A;B)$

Let $L \in \mathcal{L}(A;B)$ and, for all $j = 1, ..., n$,

$$\begin{cases} L.\vec{a}_{\text{old},j} = \sum_{i=1}^m M_{ij} \vec{b}_{\text{old},i} = \sum_{i=1}^m M^i{}_j \vec{b}_{\text{old},i} \quad \text{i.e.} \quad [L]_{|olds} = M = [M_{ij}] = [M^i{}_j]_{\substack{i=1,\dots,m\\ j=1,\dots,n}}, \\ L.\vec{a}_{\text{new},j} = \sum_{i=1}^m N_{ij} \vec{b}_{\text{new},i} = \sum_{i=1}^m N^i{}_j \vec{b}_{\text{new},i} \quad \text{i.e.} \quad [L]_{|news} = N = [N_{ij}] = [N^i{}_j]_{\substack{i=1,\dots,m\\ j=1,\dots,n}}, \end{cases} \tag{A.91}$$

with classical and duality notations.

**Proposition A.72** *Change of bases formula:*

$$\boxed{[L]_{|news} = P_B{}^{-1}.[L]_{|olds}.P_A}, \quad \text{i.e.} \quad N = P_B{}^{-1}.M.P_A. \tag{A.92}$$

*Particular case $L$ endomorphism: $A = B$, $(\vec{a}_{\text{old},i}) = (\vec{b}_{\text{old},i})$, $(\vec{a}_{\text{new},i}) = (\vec{b}_{\text{new},i})$, $P_A = P_B =^{\text{written}} P$ and*

$$\boxed{[L]_{|new} = P^{-1}.[L]_{|old}.P}, \quad \text{i.e.} \quad N = P^{-1}.M.P. \tag{A.93}$$

**Proof.** $L.\vec{a}_{\text{new},j} = \sum_i N^i{}_j \vec{b}_{\text{new},i} = \sum_{ik} N^i{}_j P_B{}^k{}_i \vec{b}_{\text{old},k} = \sum_k (P_B.N)^k{}_j \vec{b}_{\text{old},k}$ and $L.\vec{a}_{\text{new},j} = L.(\sum_i P_A{}^i{}_j \vec{a}_{\text{old},i}) = \sum_i P_A{}^i{}_j \sum_k M^k{}_i \vec{b}_{\text{old},k} = \sum_k (M.P_A)^k{}_j \vec{b}_{\text{old},k}$, for all $j$, thus $P_B.N = M.P_A$. ∎

**Exercice A.73** Prove: $\ell.L.\vec{u} = [\ell]_{|\vec{b}_{\text{new}}}.[L]_{|news}.[\vec{u}]_{|\vec{a}_{\text{new}}} = [\ell]_{|\vec{b}_{\text{old}}}.[L]_{|olds}.[\vec{u}]_{|\vec{a}_{\text{old}}}$ (objective result).

**Answer.** $[\ell]_{|\vec{b}_{\text{new}}}.[L]_{|news}.[\vec{u}]_{|\vec{a}_{\text{new}}} = ([\ell]_{|\vec{b}_{\text{old}}}.P_B).(P_B{}^{-1}.[L]_{|olds}.P_A).(P_A{}^{-1}.[\vec{u}]_{|\vec{a}_{\text{old}}})$. ∎

**Remark A.74** Bilinear forms $\beta \in \mathcal{L}(A, A; \mathbb{R})$ and endomorphisms $L \in \mathcal{L}(A; A)$ behave differently: The formulas (A.81) and (A.93) should not be confused since $P^{-1} \neq P^T$ in general. E.g., if an English observer uses a Euclidean (old) basis $(\vec{a}_i) = (\vec{a}_{old,i})$ in foot, if a French observer uses a Euclidean (new) basis $(\vec{b}_i) = (\vec{a}_{new,i})$ in metre, and if (simple case) $\vec{b}_i = \lambda \vec{a}_i$ for all $i$ (change of unit), then $P = \lambda I$ and

$$[L]_{|new} = [L]_{|old}, \quad \text{while} \quad [\beta]_{|new} = \underbrace{\lambda^2}_{>10} [\beta]_{|old}. \tag{A.94}$$

Quite different results! Here $P^{-1} \neq P^T$. Cf. remark A.17 (Mars Climate Orbiter crash). ♣

# B Euclidean Frameworks

Time and space are decoupled (classical mechanics). $\mathbb{R}^n$ is the geometric affine space, and $\vec{\mathbb{R}^n}$ is the associated usual vector space made of "bi-point vectors", $n = 1, 2, 3$.

## B.1 Euclidean basis

**Manufacturing of a Euclidean basis**.

An observer chooses a unit of measurement (foot, metre, a unit of length used by Euclid, the diameter a of pipe...) and makes a "unit rod" of length 1 in this unit.

**Postulate:** The length of the rod does not depend on its direction in space.

• Space dimension $n = 1$: This rod models a vector $\vec{e}_1$ which makes a basis $(\vec{e}_1)$ called the Euclidean basis relative to the chosen unit of measure.

• Space dimension $n = 2$ and 3:

- The observers, with his unit of measurement, makes three rods of length 3, 4 and 5, to build a triangle $(A, B, C)$ (vertices $A$, $B$ and $C$) and $A$ is not on the side on length 5.
- Pythagoras: $3^2 + 4^2 = 5^2$ gives: The triangle $(A, B, C)$ is said to have a right angle at $A$.
- Two vectors $\vec{u}$ and $\vec{w}$ in $\vec{\mathbb{R}^n}$ are orthogonal iff the triangle $(A, B, C)$ can be positioned such that $\vec{AB}$ and $\vec{AC}$ are parallel to $\vec{u}$ and $\vec{w}$.
- A basis $(\vec{e}_i)_{i=1,\dots,n}$ is Euclidean relative to the chosen unit of measurement iff the $\vec{e}_i$ are two to two orthogonal and their length is 1 (relative to the chosen unit).

**Example B.1** An English observer defines a Euclidean basis $(\vec{a}_i)$ using the foot. A French observer defines a Euclidean basis $(\vec{b}_i)$ using the metre. We have (international yard and pound agreement 1959)

$$1\,\text{foot} = \mu\,\text{metre}, \quad \mu = 0.3048, \quad \text{and} \quad 1\,\text{metre} = \lambda\,\text{foot}, \quad \lambda = \frac{1}{\mu} \simeq 3.28. \tag{B.1}$$

E.g., "aligned" bases: For all $i$,

$$\vec{b}_i = \lambda \vec{a}_i, \quad \text{and} \quad P = \lambda I \tag{B.2}$$

is the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$. NB: $P^T = \lambda I = P \neq P^{-1} = \frac{1}{\lambda} I$ and $P^T.P = \lambda^2 I$. ♣

**Remark B.2** The bases used in practice are not all Euclidean. E.g., see example A.16 if you fly. ♣

## B.2 Associated Euclidean dot product

**Definition B.3** An observer has built his Euclidean basis $(\vec{e}_i)$. The associated Euclidean dot product is the bilinear form $g(\cdot, \cdot) = (\cdot, \cdot)_g \in \mathcal{L}(\vec{\mathbb{R}^n}, \vec{\mathbb{R}^n}; \mathbb{R})$ written $\cdot \bullet_g \cdot$ defined by

$$\forall i, j, \;\; g_{ij} := g(\vec{e}_i, \vec{e}_j) = \delta_{ij}, \quad \text{i.e.} \quad [g]_{|\vec{e}} = I. \tag{B.3}$$

I.e., with $(\pi_{ei}) = (e^i)$ the (covariant) dual basis (with classical and duality notations),

$$\cdot \bullet_g \cdot = (\cdot, \cdot)_g := \sum_{i=1}^n \pi_{ei} \otimes \pi_{ei} = \sum_{i=1}^n e^i \otimes e^i. \tag{B.4}$$

With Einstein's convention, $(\cdot, \cdot)_g := \sum_{ij} \delta_{ij} e^i \otimes e^j$.

Thus, for all $\vec{x},\vec{y}\in\vec{\mathbb{R}^n}$, with $\vec{x}=\sum_{i=1}^n x_i\vec{e}_i=\sum_{i=1}^n x^i\vec{e}_i$ and $\vec{y}=\sum_{i=1}^n y_i\vec{e}_i y=\sum_{i=1}^n y^i\vec{e}_i$,

$$\vec{x}\bullet_g\vec{y}=(\vec{x},\vec{y})_g=[\vec{x}]_{|\vec{e}}^T.[\vec{y}]_{|\vec{e}}=\sum_{i=1}^n x_iy_i=\sum_{i=1}^n x^iy^i. \tag{B.5}$$

With Einstein's convention: $\vec{x}\bullet_g\vec{y}=(\vec{x},\vec{y})_g:=\sum_{ij}\delta_{ij}x^iy^j$.

**Definition B.4** The associated norm is $||.||_g:=\sqrt{(\cdot,\cdot)_g}$, and the length of a vector $\vec{x}$ relative to the chosen Euclidean unit of measurement is $||\vec{x}||_g:=\sqrt{(\vec{x},\vec{x})_g}=\sqrt{\vec{x}\bullet_g\vec{x}}$.

With $\vec{x}=\sum_{i=1}^n x_i\vec{e}_i=\sum_{i=1}^n x^i\vec{e}_i$ we get $||\vec{x}||_g=\sqrt{\sum_{i=1}^n x_i^2}=\sqrt{\sum_{i=1}^n (x^i)^2}$.
Einstein convention: $||\vec{x}||_g=\sqrt{\sum_{ij}\delta_{ij}x^ix^j}$.

**Definition B.5** The angle $\theta(\vec{x},\vec{y})$ between two vectors $\vec{x},\vec{y}\in\vec{\mathbb{R}^n}-\{\vec{0}\}$ is defined by

$$\cos(\theta(\vec{x},\vec{y}))=(\frac{\vec{x}}{||\vec{x}||_g},\frac{\vec{y}}{||\vec{y}||_g})_g. \tag{B.6}$$

(With a computer, this formula gives $\theta(\vec{x},\vec{y})=\arccos((\frac{\vec{x}}{||\vec{x}||_g},\frac{\vec{y}}{||\vec{y}||_g})_g)$ in $[0,\pi]$.)

## B.3 Two Euclidean dot products are proportional

Consider two Euclidean bases in $\vec{\mathbb{R}^n}$: $(\vec{a}_i)$, e.g. built with the foot, and $(\vec{b}_i)$, e.g. built with the metre. And let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be the associated Euclidean dot products: $(\vec{a}_i,\vec{a}_j)_g=\delta_{ij}=(\vec{b}_i,\vec{b}_j)_h$.

**Proposition B.6** *If* $\lambda=||\vec{b}_1||_g$, *then* $||\vec{b}_i||_g=\lambda$ *for all* $i=1,...,n$ *and*

$$(\cdot,\cdot)_g=\lambda^2(\cdot,\cdot)_h, \quad \textit{and} \quad ||.||_g=\lambda||.||_h. \tag{B.7}$$

**Proof.** By definition of a Euclidean basis, the length of the rod that enabled to define $(\vec{b}_i)$ is independent of $i$ cf. § B.1, thus $\lambda=||\vec{b}_1||_g=||\vec{b}_i||_g$ for all $i$. Thus $||\vec{b}_i||_g^2=\lambda^2=\lambda^2||\vec{b}_i||_h^2$ for all $i$. If $i\neq j$ then $(\vec{b}_i,\vec{b}_j)_g=0=(\vec{b}_i,\vec{b}_j)_h$ since $(\vec{b}_i)$, $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are Euclidean cf. § B.1. Hence $(\vec{b}_i,\vec{b}_j)_g=\lambda^2(\vec{b}_i,\vec{b}_j)_h$ for all $i,j$, thus $(\vec{x},\vec{y})_g=\lambda^2(\vec{x},\vec{y})_h$ for all $\vec{x},\vec{y}$ (bilinearity of $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$), thus (B.7). ∎

**Example B.7** Continuation of example B.1: $(\cdot,\cdot)_a=\sum_{i=1}^n a^i\otimes a^i$ is the English Euclidean dot product (foot), and $(\cdot,\cdot)_b=\sum_{i=1}^n b^i\otimes b^i$ is the French Euclidean dot product (metre). (B.7) and (B.1) give:

$$(\cdot,\cdot)_a=\lambda^2(\cdot,\cdot)_b \quad\text{and}\quad ||.||_a=\lambda||.||_b, \quad\text{with}\quad \lambda\simeq 3.28 \quad\text{and}\quad \lambda^2\simeq 10.76. \tag{B.8}$$

E.g. if $||\vec{w}||_b=1$ (length 1 metre) then $||\vec{w}||_a=\lambda$ (length $\lambda\simeq 3.28$ foot). ∎

## B.4 Counterexample: Non existence of a Euclidean dot product

1- Thermodynamic: Consider the Cartesian vector space $\{(T,P)\}=\{(\text{temperature,pressure})\}=\mathbb{R}\times\mathbb{R}$. There is no associated Euclidean dot product: An associated norm would give $||(T,P)||=\sqrt{T^2+P^2}\in\mathbb{R}$ which is meaningless (incompatible dimensions). See § A.4.5.

2- Polar coordinate system $\vec{q}=(r,\theta)\in\mathbb{R}\times\mathbb{R}$: There is no Euclidean norm $||\vec{q}||=\sqrt{r^2+\theta^2}$ that is physically meaningful (incompatible dimensions), see example 6.12.

## B.5 Euclidean transposed of a deformation gradient

Consider a linear map $L\in\mathcal{L}(\vec{\mathbb{R}}_{t_0}^n;\vec{\mathbb{R}}_t^n)$ (e.g., $L=F_t^{t_0}(P)$ the deformation gradient).

Let $(\cdot,\cdot)_G$ be a Euclidean dot product in $\vec{\mathbb{R}}_{t_0}^n$ (used in the past by someone), and let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be Euclidean dot products in $\vec{\mathbb{R}}_t^n$ (the actual space where the results are obtained by two observers, e.g., $(\cdot,\cdot)_g$ built with a foot and $(\cdot,\cdot)_h$ built with a metre). Let $L_{Gg}^T$ and $L_{Gh}^T$ in $\mathcal{L}(\vec{\mathbb{R}}_t^n;\vec{\mathbb{R}}_{t_0}^n)$ be the transposed of $L$ relative to the dot products, cf. (A.47): For all $(\vec{X},\vec{y})\in\vec{\mathbb{R}}_{t_0}^n\times\vec{\mathbb{R}}_t^n$,

$$(L_{Gg}^T.\vec{y},\vec{X})_G=(L.\vec{X},\vec{y})_g \quad\text{and}\quad (L_{Gh}^T.\vec{y},\vec{X})_G=(L.\vec{X},\vec{y})_h. \tag{B.9}$$

**Corollary B.8**

$$\text{If}\quad (\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h \quad\text{then}\quad L^T_{Gg} = \lambda^2 L^T_{Gh}. \tag{B.10}$$

*(Do not forget $\lambda^2$, e.g. $\lambda^2 \simeq 10$ if $(\cdot,\cdot)_g$ in foot and $(\cdot,\cdot)_h$ in metre).*

**Proof.** $(L^T_{Gg}.\vec{y},\vec{X})_G \overset{(B.9)}{=} (L.\vec{X},\vec{y})_g \overset{hyp.}{=} \lambda^2(L.\vec{X},\vec{y})_h \overset{(B.9)}{=} \lambda^2(L^T_{Gh}.\vec{y},\vec{X})_G$ for all $\vec{X}\in\vec{\mathbb{R}}^n_{t_0}$ and all $\vec{y}\in\vec{\mathbb{R}}^n_t$, thus $L^T_{Gg}.\vec{y} = \lambda^2 L^T_{Gh}.\vec{y}$ for all $\vec{y}\in\vec{\mathbb{R}}^n_t$, thus $L^T_{Gg} = \lambda^2 L^T_{Gh}$. ∎

## B.6 The Euclidean transposed for endomorphisms

Consider an endomorphism $L\in\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$; E.g. $L = d\vec{v}_t(p)$ the differential of the Eulerian velocity. Let $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ be dot products in $\vec{\mathbb{R}^n}$. Let $L^T_g$ and $L^T_h$ in $\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ be the transposed of $L$ relative to $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$: For all $\vec{x},\vec{y}\in\vec{\mathbb{R}}^n_t$,

$$(L^T_g.\vec{y},\vec{x})_g = (L.\vec{x},\vec{y})_g, \quad\text{and}\quad (L^T_h.\vec{y},\vec{x})_h = (L.\vec{x},\vec{y})_h. \tag{B.11}$$

**Corollary B.9**

$$\text{If}\quad (\cdot,\cdot)_g = \lambda^2(\cdot,\cdot)_h \quad\text{then}\quad L^T_g = L^T_h \overset{\text{written}}{=} L^T \tag{B.12}$$

*(an endomorphism type relation). Hence we can speak of "the Euclidean transposed of an endomorphism".*

**Proof.** $(L^T_g.\vec{y},\vec{x})_g \overset{(B.11)}{=} (L.\vec{x},\vec{y})_g \overset{hyp.}{=} \lambda^2(L.\vec{x},\vec{y})_h \overset{(B.11)}{=} \lambda^2(L^T_h.\vec{y},\vec{x})_h \overset{hyp}{=} (L^T_h.\vec{y},\vec{x})_g$ for all $\vec{x},\vec{y}\in\vec{\mathbb{R}^n}$, thus $L^T_g.\vec{y} = L^T_h.\vec{y}$ for all $\vec{y}\in\vec{\mathbb{R}^n}$, thus $L^T_g = L^T_h$. ∎

## B.7 Unit normal vector, unit normal form

The results in this § are not objective: We need a Euclidean dot product (need a unit of length: Foot? Meter?) to get a unit normal vector. Choose a Euclidean dot product in $\vec{\mathbb{R}^n}$, and for all $\vec{u},\vec{w}\in\vec{\mathbb{R}^n}$

$$(\vec{u},\vec{w})_g \overset{\text{written}}{=} \vec{u}\bullet_g\vec{w} \tag{B.13}$$

and $=^{\text{written}}\ \vec{u}\bullet\vec{w}$ when the chosen Euclidean dot product is imposed on everyone.

$\Omega$ is a regular open bounded set in $\mathbb{R}^n$, and $\Gamma := \partial\Omega$ is its regular surface. If $p\in\Gamma$ then $T_p\Gamma$ is the tangent plane at $p$ to $\Gamma$. Let $(\vec{\beta}_1(p),...,\vec{\beta}_{n-1}(p))$ be a basis in $T_p\Gamma$ e.g. obtained thanks to a coordinate system describing $\Gamma$ (so it is not an orthonormal basis a priori).

### B.7.1 Unit normal vector

Call $\vec{n}_g(p)$ the unit outward normal vector at $p\in\Gamma$ at $T_p\Gamma$ relative to $(\cdot,\cdot)_g$:

$$\forall i=1,...,n{-}1,\quad \vec{\beta}_i\bullet_g\vec{n}_g = 0,\quad \vec{n}_g\bullet_g\vec{n}_g = 1 \quad (=||\vec{n}_g||^2_g), \tag{B.14}$$

and $\exists h_0>0$, $\forall h\in[0,h_0[$, $p-h\vec{n}(p)\in\Omega$ (drawing: outward normal).

Hence $(\vec{\beta}_1(p),...,\vec{\beta}_{n-1}(p),\vec{n}_g(p))$ is a basis at $p$ in $\vec{\mathbb{R}^n}$, written in short $(\vec{\beta}_1,...,\vec{\beta}_{n-1},\vec{n}_g)$. Drawing.

Thus, if $\vec{w}\in\vec{\mathbb{R}^n}$ is a vector at $p$, $\vec{w} = \sum_{i=1}^{n-1} w_i\vec{\beta}_i + w_n\vec{n}_g$ (classical notations) then

$$w_n = \vec{w}\bullet_g\vec{n}_g = \text{ the normal component of } \vec{w} \text{ at } p \text{ at } \Gamma. \tag{B.15}$$

($w_n$ depends on $(\cdot,\cdot)_g$.) (Duality notations: $\vec{w} = \sum_{i=1}^{n-1} w^i\vec{\beta}_i + w^n\vec{n}_g$ and $w^n = \vec{w}\bullet_g\vec{n}_g$.)

**Exercice B.10** $(\vec{a}_i)$ is a basis in $\vec{\mathbb{R}^n}$ and $[g]_{|\vec{a}} = [g_{ij}] = [g(\vec{a}_i,\vec{a}_j)]$. Call $B_{ij}$ the component of $\vec{\beta}_j$ in $(\vec{a}_i)$, i.e. $\vec{\beta}_j = \sum_{i=1}^n B_{ij}\vec{a}_i$ for $j=1,...,n{-}1$. Compute the components $n_i$ of $\vec{n}_g$ in $(\vec{a}_i)$, i.e. s.t. $\vec{n}_g = \sum_{i=1}^n n_i\vec{a}_i$. Particular case $(\vec{a}_i)$ is $(\cdot,\cdot)_g$-orthonormal?

**Answer.** (B.14) gives $[\vec{\beta}_i]^T_{|\vec{a}}.[g]_{|\vec{a}}.[\vec{n}_g]_{|\vec{a}} = 0$ for $i=1,...,n{-}1$: We get $n{-}1$ linear equations. With one more equation given by $[\vec{n}_g]^T_{|\vec{a}}.[g]_{|\vec{a}}.[\vec{n}_g]_{|\vec{a}} = 1$: We get $\vec{n}_g$ up to its sign.

If $(\vec{a}_i)$ is $(\cdot,\cdot)_g$-orthonormal then $[g]_{|\vec{a}} = I$ and $\sum_{j=1}^n B_{ij}n_j = 0$ for $i=1,...,n{-}1$, with $\sum_{i=1}^n n_i^2 = 1$. ∎

**Exercice B.11** Let $(\vec{a}_i)$ be a Euclidean basis in foot, $(\vec{b}_i)$ a Euclidean basis in metre, $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ the associated Euclidean dot products, so $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$ with $\lambda \simeq 3.28$, cf. (B.7). Let $\vec{n}_a(p)$ and $\vec{n}_b(p)$ be the corresponding unit outward normal vectors, cf. (B.14). 1- Prove (up to the sign):

$$\vec{n}_b = \lambda\vec{n}_a, \quad \text{and} \quad (\vec{w},\vec{n}_a)_a = \lambda(\vec{w},\vec{n}_b)_b \quad \forall\vec{w}\in\vec{\mathbb{R}^n} \tag{B.16}$$

2- Then let $\vec{n}_a = \sum_{i=1}^m n_{ai}\vec{a}_i$ and $\vec{n}_b = \sum_{i=1}^m n_{bi}\vec{b}_i$; Prove:

$$\text{If, } \forall i=1,...,n,\ \vec{b}_i = \lambda\vec{a}_i \quad \text{then} \quad \forall i=1,...,n,\ n_{ai} = n_{bi}. \tag{B.17}$$

So the vectors $\vec{n}_a$ and $\vec{n}_b$ are different ($\lambda > 1$), and their respective components are equal... relative to different bases! And of course $1 = ||\vec{n}_a||_a^2 = \sum_{i=1}^n (n_{ai})^2 = \sum_{i=1}^n (n_{bi})^2 = ||\vec{n}_b||_b^2 = 1$.

**Answer**. 1- $\vec{n}_a(p) \parallel \vec{n}_b(p)$, since the vectors are Euclidean orthogonal to $T_p\Gamma$. And $||\vec{n}_b||_a = \lambda||\vec{n}_b||_b = \lambda = \lambda||\vec{n}_a||_a$, thus $\vec{n}_b = \pm\lambda\vec{n}_a$. And they are outward vectors, so $\vec{n}_b = +\lambda\vec{n}_a$. Thus $(\vec{w},\vec{n}_a)_a = \lambda^2(\vec{w},\vec{n}_a)_b = \lambda^2(\vec{w},\frac{\vec{n}_b}{\lambda})_b = \lambda(\vec{w},\vec{n}_b)_b$.

2- Then $\vec{b}_i = \lambda\vec{a}_i$ gives $\sum_{i=1}^n n_b^i\vec{b}_i = \lambda\sum_{i=1}^n n_a^i\vec{a}_i = \sum_{i=1}^n n_a^i(\lambda\vec{a}_i) = \sum_{i=1}^n n_a^i\vec{b}_i$, thus $n_a^i = n_b^i$. ▪

### B.7.2 Unit normal form $n^\flat$ associated to $\vec{n}$

For mathematicians: May produce misunderstandings, bad interpretations. Don't forget: $n^\flat$ is obtained only after $\vec{n}$ has been defined.

**Definition B.12** Let $p\in\Gamma$, $(\cdot,\cdot)_g$ be an inner dot product and $\vec{n}_g(p)$ be the outward unit normal at $p$. The unit normal form $n_g^\flat(p)\in\mathbb{R}^{n*}$ is the linear form defined by $n_g^\flat(p).\vec{w} := (\vec{n}_g(p),\vec{w})_g$ for all $\vec{w}\in\vec{\mathbb{R}^n}$ vector at $p$ :

$$n_g^\flat.\vec{w} := (\vec{n}_g,\vec{w})_g. \tag{B.18}$$

( $=^{\text{written}}$ $\vec{n}\bullet\vec{w}$ if one chosen Euclidean dot product is imposed).

Quantification: Let $(\vec{e}_i)$ be a basis in $\vec{\mathbb{R}^n}$; Then (B.18) gives $[n_g^\flat]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} = [\vec{n}_g]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}$ simply written $[n^\flat].[\vec{w}] = [\vec{n}]^T.[g].[\vec{w}]$ if the basis $(\vec{e}_i)$ is imposed. Hence, with the dual basis $(e^i)$ in $\mathbb{R}^{n*}$,

$$\text{if } \vec{n} = \sum_{i=1}^n n^i\vec{e}_i \text{ and } n^\flat = \sum_{i=1}^n n_i e^i \quad \text{then} \quad n_i = \sum_{j=1}^n g_{ij}n^j, \quad \text{i.e.} \quad [n^\flat]^T = [g].[\vec{n}] \tag{B.19}$$

(recall: the matrix $[n^\flat]$ is a row matrix since $n^\flat$ is a linear form).

Particular case $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-Euclidean basis, then $n_i = n^i$ and $n^\flat = \sum_{i=1}^n n^i e^i$. Use the Einstein convention to avoid this apparent contradiction: Write $n_i = \sum_{j=1}^n \delta_{ij}n^j$ since $g_{ij} = \delta_{ij}$.

We used the duality notation to justify the $^\flat$ notation: The "top $i$" in $n^i$ becomes the "bottom $i$" in $n_i$ (change of variance). Classical notations: $\vec{n} = \sum_i n_i\vec{e}_i$, $n^\flat = \sum_i (n^\flat)_i\pi_{ei}$ and $(n^\flat)_i = \sum_j g_{ij}n_j$.

## B.8 Integration by parts (Green–Gauss–Ostrogradsky)

$\Omega$ is a regular bounded open set in $\mathbb{R}^n$, $\Gamma = \partial\Omega$, $\varphi\in C^1(\overline{\Omega};\mathbb{R})$, $(\vec{e}_i)$ is a Euclidean basis, $\frac{\partial\varphi}{\partial x_i}(p) := d\varphi(p).\vec{e}_i$, $(\cdot,\cdot)_g$ its associated Euclidean dot product, $\vec{n}_g(p) = \vec{n}(p) = \sum_{i=1}^n n_i(p)\vec{e}_i$ (classical notations) is the $(\cdot,\cdot)_g$-outward normal unit vector at $p\in\Gamma$. Then (Green), for $i=1,...,n$,

$$\int_{p\in\Omega}\frac{\partial\varphi}{\partial x_i}(p)\,d\Omega = \int_{p\in\Gamma}\varphi(p)n_i(p)\,d\Gamma, \quad \text{in short} \quad \int_\Omega\frac{\partial\varphi}{\partial x_i}\,d\Omega = \int_\Gamma\varphi n_i\,d\Gamma. \tag{B.20}$$

Thus, for any $v\in C^1(\overline{\Omega};\mathbb{R})$, with $\varphi v$ instead of $\varphi$ in (B.20), we get the integration by parts formula (Green's formula):

$$\int_\Omega\frac{\partial\varphi}{\partial x_i}v\,d\Omega = -\int_\Omega\varphi\,\frac{\partial v}{\partial x_i}\,d\Omega + \int_\Gamma\varphi v n_i\,d\Gamma. \tag{B.21}$$

Thus, for any $\vec{v}\in C^1(\overline{\Omega};\vec{\mathbb{R}^n})$ (vector field), with $\vec{v}(p) = \sum_{i=1}^n v_i(p)\vec{e}_i$ we get

$$\int_\Omega\frac{\partial\varphi}{\partial x_i}v_i\,d\Omega = -\int_\Omega\varphi\,\frac{\partial v_i}{\partial x_i}\,d\Omega + \int_\Gamma\varphi v_i n_i\,d\Gamma. \tag{B.22}$$

Thus ($\sum_i$), with $d\varphi$ the differential and $\vec{\text{grad}}\varphi = \sum_{i=1}^n\frac{\partial\varphi}{\partial x_i}\vec{e}_i$ the gradient, we get the Gauss–Ostrogradsky formula:

$$\int_\Omega d\varphi.\vec{v}\,d\Omega = \int_\Omega\vec{\text{grad}}\varphi\bullet\vec{v}\,d\Omega = -\int_\Omega\varphi\,\text{div}\vec{v}\,d\Omega + \int_\Gamma\varphi\vec{v}\bullet\vec{n}\,d\Gamma. \tag{B.23}$$

## B.9 Stokes theorem

### B.9.1 The classic Stokes theorem

$\Sigma \subset \mathbb{R}^3$ is a regular oriented 2-D surface parametrized with $\vec{r} : (u,v) \in [a,b] \times [c,d] \to \vec{x} = \vec{r}(u,v) \in \vec{\mathbb{R}^3}$. Let $\vec{n}(\vec{x}) := \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{||\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}||}(u,v)$, unit normal at $\vec{x} = \vec{r}(u,v)$. And $\Sigma$ has a boundary $\Gamma$ positively parametrized with $\vec{q} : t \in [t_1,t_2] \to \vec{q}(t) = \vec{r}(u(t),v(t)) \in \vec{\mathbb{R}^3}$ (positively means: at any $\vec{x} \in \mathrm{Im}(\vec{q}) = \Gamma$, the vector $\vec{n}(\vec{x}) \times \vec{q}\,'(\vec{x})$ points towards the surface).

**Theorem B.13** *If $\vec{f} \in C^1(\vec{\mathbb{R}^3};\vec{\mathbb{R}^3})$ then*

$$\int_\Gamma \vec{f} \bullet d\vec{\ell} = \int_\Sigma \vec{\mathrm{curl}}\vec{f} \bullet d\vec{\Sigma} \quad (= \int_\Sigma \vec{\mathrm{curl}}\vec{f} \bullet \vec{n}\, d\Sigma), \tag{B.24}$$

*i.e.* $\int_{t=t_1}^{t_2} \vec{f}(\vec{q}(t)) \bullet \vec{q}\,'(t)\, dt = \int_{u=a}^{b} \int_{v=c}^{d} \vec{\mathrm{curl}}\vec{f}(\vec{r}(u,v)) \bullet (\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v})(u,v)\, dudv.$

**Proof.** See any elementary course, e.g. https://www.isima.fr/ leborgne//Isimath1ereannee/cousur.pdf. ∴

### B.9.2 Generalized Stokes theorem

The $\vec{\mathrm{curl}}$ operator is a differential operator which acts on vectors to give vectors. From a covariant point of view, it would be nice to first define a "curl operator" curl as a (linear) function acting on vectors (objective point of view), and then eventually represented by $\vec{\mathrm{curl}}$; And because $\vec{\mathrm{curl}}$ "kill the gradient", curl should "kill the differential" i.e. $\mathrm{curl}\circ d = 0$ (in place of $\vec{\mathrm{curl}}\circ \vec{\mathrm{grad}} = 0$). To do so Cartan [5] developed the "exterior differential" $d_{ext}$ which acts on $k$-forms (skew-symmetric covariant tensors), see [5] and e.g. Marsden–Hughes [17]:

1. The set of $C^\infty(\mathbb{R}^n;\mathbb{R})$ functions is called $\Omega^0$ (the set of $\binom{0}{0}$ tensors = functions); Then define $d_{ext} := d =$ the usual differential operator on $\Omega^0$, i.e. $d_{ext}f := df$ for all $f \in \Omega^0$.
2. The set of $C^\infty(\mathbb{R}^n;\mathbb{R}^{n*})$ 1-forms is called $\Omega^1$ (the set of $\binom{0}{1}$ tensors = differential forms); In particular if $f \in \Omega^0$ then the exact differential form $d_{ext}f = df$ is in $\Omega^1$.
3. Definition: A 2-form is a bilinear skew-symmetric $\binom{0}{2}$ tensor, and the set of 2-forms is called $\Omega^2$; So $\beta \in \Omega^2$ iff $\beta$ is bilinear and $\beta(\vec{u},\vec{w}) = -\beta(\vec{w},\vec{u})$ for all $\vec{u},\vec{v} \in \mathbb{R}^n$ (a 1-form is meant to "measure a length" and a 2-form is meant to "measure a surface"). And the wedge product $\alpha \wedge \beta$ of two 1-forms $\alpha,\beta \in \Omega^1$ is the 2-form $\alpha \wedge \beta \in \Omega^2$ defined by $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ (and $\wedge$ is an exterior product defined on $\Omega^1$ to give elements in $\Omega^2$: from "lengths" you get a "surface").
4. Define the exterior differential $d_{ext} : \Omega^1 \to \Omega^2$ s.t. $d_{ext}(df) = 0$ for all $f \in \Omega^0$, and $d_{ext}(\alpha \wedge \beta) = d_{ext}\alpha \wedge \beta - \alpha \wedge d_{ext}\beta$ for any $\alpha,\beta \in \Omega^1$.
5. (Generalization.) For $k \geq 2$ define a $k$-form (also called a differential $k$-form) to be a skew-symmetric $\binom{0}{k}$ tensor (order $k$ covariant), the set of $k$-forms being called $\Omega^k$ (so $\alpha \in \Omega^k$ satisfies $\alpha(\vec{u}_{\pi(1)},...,\vec{u}_{\pi(k)}) = \mathrm{sgn}(\pi)\alpha(\vec{u}_1,...,\vec{u}_n)$ for all $\vec{u}_1,...,\vec{u}_n \in \mathbb{R}^n$ and all permutations $\pi$). On $\Omega^k \times \Omega^\ell$ define the exterior wedge product $\alpha \wedge \beta \in \Omega^{k+\ell}$ by $\alpha \wedge \beta(w_1,...,w_k,w_{k+1},...,w_{k+\ell}) := \frac{1}{k!\ell!}\sum_{\pi\in\sigma} \mathrm{sgn}(\pi)\alpha(w_{\pi_1},...,w_{\pi_k})\beta(w_{\pi_{k+1}},...,w_{\pi_{k+\ell}})$ where $\sigma$ is the set of permutations. Then define the exterior differential $d_{ext} : \Omega^k \to \Omega^{k+1}$ s.t. $d_{ext}(d_{ext}\gamma) = 0$ for all $\gamma \in \Omega^{k-1}$, and $d_{ext}(\alpha \wedge \beta) = d_{ext}\alpha \wedge \beta + (-1)^k \alpha \wedge d_{ext}\beta$ for any $\alpha \in \Omega^k$ and $\beta \in \Omega^\ell$.
6. Then $d_{ext} =^{\text{written}} d$ (this notation creates confusions for non-mathematicians).

The generalized Stokes theorem (see e.g. Abraham–Marsden [1]) is:

**Theorem B.14** *If $\Sigma$ is $n$ dimensional, if $\Gamma$ is positively oriented and if $\alpha \in \Omega^{n-1}$ then*

$$\int_\Sigma d_{ext}\alpha = \int_\Gamma \alpha, \quad \textit{written} \quad \int_\Sigma d\alpha = \int_\Gamma \alpha. \tag{B.25}$$

# C Rate of deformation tensor and spin tensor

$\widetilde{\Phi} : [t_1,t_2] \times Obj \to \mathbb{R}^n$ is a regular motion, cf. (1.4), $\bigcup_{t\in[t_1,t_2]}(\{t\} \times \Omega_t)$, and $\vec{v} : \mathcal{C} \to \vec{\mathbb{R}^n}$ is the Eulerian velocity field: $\vec{v}(t,p) = \frac{\partial \Phi}{\partial t}(t,P_{Obj})$ when $p = \widetilde{\Phi}(t,P_{Obj})$, cf. (2.5). Choose a Euclidean dot product $(\cdot,\cdot)_g$, the same at all $t$. (So loss of objectivity in what follows).

## C.1 The symmetric and antisymmetric parts of $d\vec{v}$

With the chosen Euclidean dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$, consider the transposed cf. § A.12:

$$d\vec{v}_t^T : \left\{ \begin{array}{ll} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t) \\ p & \to d\vec{v}_t^T(p) := d\vec{v}_t(p)^T \end{array} \right\}, \quad \text{where} \quad (d\vec{v}_t(p)^T.\vec{w}_1,\vec{w}_2)_g = (\vec{w}_1, d\vec{v}_t(p).\vec{w}_2)_g \tag{C.1}$$

for all $\vec{w}_1,\vec{w}_2 \in \vec{\mathbb{R}}^n_t$ vectors at $p$. Other notations (definitions): $d\vec{v}_t^T(p) = d\vec{v}(t,p)^T = d\vec{v}^T(t,p)$.

**Definition C.1** The (Eulerian) rate of deformation tensor $\mathcal{D}$, or stretching tensor, is the $(\cdot,\cdot)_g$-symmetric part of $d\vec{v}$:

$$\mathcal{D} = \frac{d\vec{v}+d\vec{v}^T}{2}, \quad \text{i.e.,} \quad \forall (t,p)\in\mathcal{C}, \ \mathcal{D}(t,p) = \frac{d\vec{v}(t,p)+d\vec{v}(t,p)^T}{2}. \tag{C.2}$$

The (Eulerian) spin tensor is the $(\cdot,\cdot)_g$-antisymmetric part of $d\vec{v}$:

$$\Omega = \frac{d\vec{v}-d\vec{v}^T}{2}, \quad \text{i.e.,} \quad \forall (t,p)\in\mathcal{C}, \ \Omega(t,p) = \frac{d\vec{v}(t,p)-d\vec{v}(t,p)^T}{2}. \tag{C.3}$$

(So $d\vec{v} = \mathcal{D}+\Omega$.)

NB: The same usual notation is used for the set of points $\Omega_t = \widetilde{\Phi}(t,Obj) \subset \mathbb{R}^n$ and for the spin tensor $\Omega_t = \frac{d\vec{v}_t - d\vec{v}_t^T}{2}$: The context removes ambiguities.

## C.2 Quantification with a basis

With a basis $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_t$, (C.1) gives $[\vec{w}_2]^T.[g].[d\vec{v}_t(p)^T.\vec{w}_1] = [d\vec{v}_t(p).\vec{w}_2]^T.[g].[\vec{w}_2]$, thus

$$[g].[d\vec{v}^T] = [d\vec{v}]^T.[g], \quad \text{thus} \quad [d\vec{v}^T] = [g]^{-1}.[d\vec{v}]^T.[g]. \tag{C.4}$$

In particular, if $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis, then $[d\vec{v}^T]_{|\vec{e}} = [d\vec{v}]^T_{|\vec{e}}$, and with $\vec{v} = \sum_i v_i\vec{e}_i$, $\mathcal{D}.\vec{e}_j = \sum_{i=1}^n \mathcal{D}_{ij}\vec{e}_i$ and $\Omega.\vec{e}_j = \sum_{i=1}^n \Omega_{ij}\vec{e}_i$ we get $\mathcal{D}_{ij} = \frac12(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})$ and $\Omega_{ij} = \frac12(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i})$:

$$[\mathcal{D}] = \frac{[d\vec{v}]+[d\vec{v}]^T}{2} \quad \text{and} \quad [\Omega] = \frac{[d\vec{v}]-[d\vec{v}]^T}{2} \quad \text{(Euclidean framework).} \tag{C.5}$$

Duality notations: $\mathcal{D}^i_j = \frac12(\frac{\partial v^i}{\partial x^j}+\frac{\partial v^j}{\partial x^i})$ (sym) and $\Omega^i{}_j = \frac12(\frac{\partial v^i}{\partial x^j}-\frac{\partial v^j}{\partial x^i})$ (antisym).

# D Interpretation of the rate of deformation tensor

We are interested in the evolution of the deformation gradient $F(t) := F^{t_0}_{p_{t_0}}(t)$ along the trajectory of a particle $P_{Obj}$ which was at $p_{t_0}$ at $t_0$. Let $\vec{A} = \vec{a}(t_0,p_{t_0})$ and $\vec{B} = \vec{b}(t_0,p_{t_0})$ be vectors at $t_0$ at $p_{t_0}\in\Omega_{t_0}$, and consider their push-forwards by the flow $\Phi^{t_0}_t$ (the transported vectors), i.e. at $t$ at $p(t) = \Phi^{t_0}_{p_{t_0}}(t)$,

$$\vec{a}(t,p(t)) := F(t).\vec{A} \quad \text{and} \quad \vec{b}(t,p(t)) := F(t).\vec{B}. \tag{D.1}$$

see (4.3) and figure 4.1. Then consider the function

$$(\vec{a},\vec{b})_g : \left\{ \begin{array}{ll} \mathcal{C} & \to \mathbb{R} \\ (t,p_t) & \to (\vec{a},\vec{b})_g(t,p_t) := (\vec{a}(t,p_t),\vec{b}(t,p_t))_g. \end{array} \right. \tag{D.2}$$

**Proposition D.1** *A unique Euclidean dot product $(\cdot,\cdot)_g$ being imposed at all $t$, the rate of deformation tensor $\mathcal{D} = \frac{d\vec{v}+d\vec{v}^T}{2}$ gives half the evolution rate between two vectors deformed by the flow:*

$$\frac{D(\vec{a},\vec{b})_g}{Dt} = 2(\mathcal{D}.\vec{a},\vec{b})_g. \tag{D.3}$$

**Proof.** Let $f(t) = (\vec{a}(t,p(t)),\vec{b}(t,p(t)))_g = (F(t).\vec{A},F(t).\vec{B})_g$. Having $(\cdot,\cdot)_g$ being independent of $t$, and $F'(t) =^{(3.33)} d\vec{v}(t,p(t)).F(t)$, we get

$$\begin{aligned} f'(t) &= \big(F'(t).\vec{A},F(t).\vec{B}\big)_g + \big(F(t).\vec{A},F'(t).\vec{B}\big)_g \\ &= \big(d\vec{v}(t,p(t)).\vec{a}(t,p(t)),\vec{b}(t,p(t))\big)_g + \big(\vec{a}(t,p(t)),d\vec{v}(t,p(t)).\vec{b}(t,p(t))\big)_g \\ &= \big((d\vec{v}(t,p(t))+d\vec{v}(t,p(t))^T).\vec{a}(t,p(t)),\vec{b}(t,p(t))\big)_g, \end{aligned} \tag{D.4}$$

thus (D.3). ∎

# E Rigid body motions and the spin tensor

Choose a Euclidean dot product $(\cdot,\cdot)_g$ the same at all times (to simply characterize a rigid body motion).

Simple definition: A rigid body motion is a motion whose Eulerian velocity satisfies $d\vec{v}+d\vec{v}^T=0$, i.e., $\mathcal{D}=0$ (Eulerian approach independent of any initial time $t_0$ chosen by some observer).

But the usual classical introduction to rigid body motion relies on some initial time $t_0$ (Lagrangian approach). So, we start with the Lagrangian approach: Consider a regular motion $\widetilde{\Phi}$, fix a $t_0\in\mathbb{R}$, the associated Lagrangian motion $\Phi^{t_0}$, and for a fixed $t$ the associated motion $\Phi^{t_0}_t$. The first order Taylor expansion of $\Phi^{t_0}_t$ in the vicinity of a $p_{t_0}\in\Omega_{t_0}$ is, with $d\Phi^{t_0}_t(p_{t_0}) =^{\text{written}} F^{t_0}_t(p_{t_0})$,

$$\Phi^{t_0}_t(q_{t_0})=\Phi^{t_0}_t(p_{t_0})+F^{t_0}_t(p_{t_0}).\overrightarrow{p_{t_0}q_{t_0}}+o(\overrightarrow{p_{t_0}q_{t_0}}). \tag{E.1}$$

Marsden–Hughes notations: $\Phi(Q)=\Phi(P)+F(P).\overrightarrow{PQ}+o(\overrightarrow{PQ})$.

## E.1 Affine motions and rigid body motions

### E.1.1 Affine motion

**Definition E.1** $\Phi^{t_0}$ is an affine motion, meaning "affine motion in space", iff $\Phi^{t_0}_t$ is an affine motion for all $t$, i.e. iff, for all $p_{t_0},q_{t_0}\in\Omega_{t_0}$ and all $t\in[t_1,t_2]$,

$$\Phi^{t_0}_t(q_{t_0})=\Phi^{t_0}_t(p_{t_0})+F^{t_0}_t(p_{t_0}).\overrightarrow{p_{t_0}q_{t_0}}. \tag{E.2}$$

Marsden–Hughes notations: $\Phi(Q)=\Phi(P)+F(P).\overrightarrow{PQ}$.

**Proposition E.2 and definition.** *If $\Phi^{t_0}$ is an affine motion then $F^{t_0}_t(p_{t_0})$ is independent of $p_{t_0}$: For all $t\in]t_1,t_2[$ and $p_{t_0},q_{t_0}\in\Omega_{t_0}$,*

$$F^{t_0}_t(p_{t_0})=F^{t_0}_t(q_{t_0})\overset{written}{=}F^{t_0}_t. \tag{E.3}$$

*Thus $dF^{t_0}_t(p_{t_0})=0$, i.e. $d^2\Phi^{t_0}_t(p_{t_0})=0$. And for all $t\in]t_1,t_2[$, $\Phi^t$ is an affine motion, i.e. for all $\tau\in]t_1,t_2[$ and all $p_t,q_t\in\Omega_t$,*

$$\Phi^t_\tau(q_t)=\Phi^t_\tau(p_t)+F^t_\tau.\overrightarrow{p_tq_t}. \tag{E.4}$$

*And $\widetilde{\Phi}$ is said to be an affine motion.*

**Proof.** $q_{t_0}=p_{t_0}+\overrightarrow{p_{t_0}q_{t_0}}$ gives $\Phi^{t_0}_t(q_{t_0})=\Phi^{t_0}_t(p_{t_0}+\overrightarrow{p_{t_0}q_{t_0}})=\Phi^{t_0}_t(p_{t_0})+d\Phi^{t_0}_t(p_{t_0}).\overrightarrow{p_{t_0}q_{t_0}}$, and, similarly, $\Phi^{t_0}_t(p_{t_0})=\Phi^{t_0}_t(q_{t_0}+\overrightarrow{q_{t_0}p_{t_0}})=\Phi^{t_0}_t(q_{t_0})+d\Phi^{t_0}_t(q_{t_0}).\overrightarrow{q_{t_0}p_{t_0}}$. Thus (addition) $\Phi^{t_0}_t(q_{t_0})+\Phi^{t_0}_t(p_{t_0})=\Phi^{t_0}_t(p_{t_0})+\Phi^{t_0}_t(q_{t_0})+(d\Phi^{t_0}_t(p_{t_0})-d\Phi^{t_0}_t(q_{t_0})).\overrightarrow{p_{t_0}q_{t_0}}$, thus $(d\Phi^{t_0}_t(p_{t_0})-d\Phi^{t_0}_t(q_{t_0})).\overrightarrow{p_{t_0}q_{t_0}}=0$, true for all $p_{t_0},q_{t_0}$, thus $d\Phi^{t_0}_t(p_{t_0})-d\Phi^{t_0}_t(q_{t_0})=0$, true for all $t,p_{t_0},q_{t_0}$, thus (E.3).

Thus $d^2\Phi^{t_0}_t(p_{t_0}).\vec{u}_{t_0}=\lim_{h\to0}\frac{d\Phi^{t_0}_t(p_{t_0}+h\vec{u}_{t_0})-d\Phi^{t_0}_t(p_{t_0})}{h}=\lim_{h\to0}\frac{d\Phi^{t_0}_t-d\Phi^{t_0}_t}{h}=0$ for all $p_{t_0}$ and all $\vec{u}_{t_0}$, thus $d^2\Phi^{t_0}_t(p_{t_0})=0$ for all $p_{t_0}$, thus $d^2\Phi^{t_0}_t=0$.

And $(\Phi^t_\tau\circ\Phi^{t_0}_t)(p_{t_0})=\Phi^{t_0}_\tau(p_{t_0})$ (composition of flows (5.16)), thus with $p_t=\Phi^{t_0}_t(p_{t_0})$ we get $d\Phi^t_\tau(p_t).d\Phi^{t_0}_t(p_{t_0})=d\Phi^{t_0}_\tau(p_{t_0})$, thus $d\Phi^t_\tau(p_t)=d\Phi^{t_0}_\tau(p_{t_0}).d\Phi^{t_0}_t(p_{t_0})^{-1}$, and (E.2) gives

$$d\Phi^t_\tau(p_t)=d\Phi^{t_0}_\tau.d\Phi^{t_0}_t{}^{-1}\overset{\text{written}}{=}d\Phi^t_\tau\quad\text{(independent of }p_t\text{)}, \tag{E.5}$$

thus (E.4). ∎

**Corollary E.3** *With $\vec{v}$ the Eulerian velocity and $\vec{V}^{t_0}$ the Lagrangian velocity: If $\widetilde{\Phi}$ is affine then, $\vec{v}_t$ is affine for all $t$, and $\vec{V}^{t_0}_t$ is affine for all $t_0,t$, i.e., $d\vec{v}_t(p_t)=d\vec{v}_t$ for all $p_t\in\Omega_t$ (independent of $p_t$), and $d\vec{V}^{t_0}_t(p_{t_0}) =^{written} d\vec{V}^{t_0}_t$ for all $p_{t_0}\in\Omega_{t_0}$ (independent of $p_{t_0}$). So, for all $p_t,q_t\in\Omega_t$ and $p_{t_0},q_{t_0}\in\Omega_{t_0}$,*

$$\begin{cases}\bullet\ \ \vec{v}_t(q_t)=\vec{v}_t(p_t)+d\vec{v}_t.\overrightarrow{p_tq_t},\\ \bullet\ \ \vec{V}^{t_0}_t(q_{t_0})=\vec{V}^{t_0}_t(p_{t_0})+d\vec{V}^{t_0}_t.\overrightarrow{p_{t_0}q_{t_0}}.\end{cases} \tag{E.6}$$

**Proof.** (E.2) gives $\Phi^{t_0}(t,q_{t_0})=\Phi^{t_0}(t,p_{t_0})+F^{t_0}(t).\overrightarrow{p_{t_0}q_{t_0}}$, and the derivation in time gives (E.6)$_2$, hence (E.6)$_1$ thanks to $d\vec{V}^{t_0}_t(p_{t_0}) =^{(3.27)} d\vec{v}_t(p_t).F^{t_0}_t$ and $\overrightarrow{p_{t_0}q_{t_0}} =^{(E.2)} (F^{t_0}_t)^{-1}.\overrightarrow{p_tq_t}$. ∎

**Example E.4** In $\mathbb{R}^2$, with a basis $(\vec{E}_1,\vec{E}_2)$ in $\vec{\mathbb{R}}^n_{t_0}$ and a basis $(\vec{e}_1,\vec{e}_2)\in\vec{\mathbb{R}}^n_t$, then $F^{t_0}_t$ given by $[F^{t_0}_t]_{|\vec{E},\vec{e}}=\begin{pmatrix}1+t&2t^2\\0&e^t\end{pmatrix}$ derives from the affine motion $[\overrightarrow{\Phi^{t_0}_t(p_{t_0})\Phi^{t_0}_t(q_{t_0})}]_{|\vec{e}}=\begin{pmatrix}1+t&2t^2\\0&e^t\end{pmatrix}.[\overrightarrow{p_{t_0}q_{t_0}}]_{|\vec{E}}$. ∎

### E.1.2 Rigid body motion

Marsden notations to lighten the notations: $\Phi := \Phi_t^{t_0}$, $F := F_t^{t_0}$, $P \in \Omega_{t_0}$, $p = \Phi(P) \in \Omega_t$, the $(\cdot,\cdot)_g$-transposed $F^T(p) \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0})$ of $F(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ is defined by

$$F^T(p) := F(P)^T : \left\{ \begin{array}{ll} \vec{\mathbb{R}}^n_t & \to \vec{\mathbb{R}}^n_{t_0} \\ \vec{w}_p & \to F^T(p).\vec{w}_p \end{array} \right\} \quad \text{where} \quad (F^T(p).\vec{w}_p, \vec{U}_P)_g = (\vec{w}_p, F(P).\vec{U}_P)_g \tag{E.7}$$

for all $\vec{U}_P \in \vec{\mathbb{R}}^n_{t_0}$. Which defines the function $F^T : \Omega_t \to \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0})$.

Particular case: For an affine motion $F$ is independent of $P$, hence $F^T$ is independent of $p$.

**Definition E.5** A rigid body motion is an affine motion $\widetilde{\Phi}$ such that angles and lengths are unchanged by $\Phi$: For all $t_0, t \in \mathbb{R}$, $P \in \Omega_{t_0}$, $\vec{U}, \vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ vectors at $P$, and with $p = \Phi(P)$,

$$(F.\vec{U}, F.\vec{W})_g = (\vec{U}, \vec{W})_g, \quad \text{i.e.} \quad (F^T.F.\vec{U}, \vec{W})_g = (\vec{U}, \vec{W})_g, \quad \text{i.e.} \quad \boxed{F^T.F = I}. \tag{E.8}$$

In other words, with the Cauchy strain tensor $C \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ defined by $C = F^T.F$, the motion is rigid iff it is affine and

$$\boxed{C = I}, \quad \text{i.e.} \quad \boxed{F^{-1} = F^T}. \tag{E.9}$$

**Proposition E.6** *If $\Phi^{t_0}$ is a rigid body motion, if $(\vec{A}_i)$ is a $(\cdot,\cdot)_g$-Euclidean basis in $\vec{\mathbb{R}}^n_{t_0}$, if $\vec{a}_{it}(p) = F_t^{t_0}(P).\vec{A}_i$ for all $i$ when $p = \Phi_t^{t_0}(P)$, then $\vec{a}_{it}(p) =^{written} \vec{a}_{it}$ is independent of $p$, and $(\vec{a}_{it})$ is a $(\cdot,\cdot)_g$-Euclidean basis with the same orientation than $(\vec{A}_i)$, for all $t$.*

**Proof.** $\Phi_t^{t_0}$ is affine, thus, for all $t, P$, $F_t^{t_0}(P) = F_t^{t_0}$ (independent of $P$), thus $\vec{a}_{i,t}(p) = F_t^{t_0}.\vec{A}_i \in \vec{\mathbb{R}}^n_t$ is independent of $p$, for all $t$. And $(\vec{a}_{it}, \vec{a}_{jt})_g = (F_t^{t_0}.\vec{A}_i, F_t^{t_0}.\vec{A}_j)_g = ({F_t^{t_0}}^T.F_t^{t_0}.\vec{A}_i, \vec{A}_j)_g =^{rigid} (I.\vec{A}_i, \vec{A}_j)_g = (\vec{A}_i, \vec{A}_j)_g = \delta_{ij}$ for all $i, j$, thus $(\vec{a}_{it})$ is $(\cdot,\cdot)_g$-orthonormal basis. And $\det(\vec{a}_{1t}, ..., \vec{a}_{nt}) = \det(F_t^{t_0}.\vec{A}_1, ..., F_t^{t_0}.\vec{A}_n) = \det(F_t^{t_0})\det(\vec{A}_1, ..., \vec{A}_n) = \det(F_t^{t_0})$ since $(\vec{A}_i)$ is a $(\cdot,\cdot)_g$- orthonormal basis. And, $\Phi^{t_0}$ being regular, $t \to \det(F_t^{t_0})$ is continuous, does not vanish, with $\det(F_{t_0}^{t_0}) = \det(I) = 1 > 0$; Thus $\det(F_t^{t_0}) > 0$ for all $t$, thus $\det(\vec{a}_1, ..., \vec{a}_n) > 0$: The bases have the same orientation. ∎

**Example E.7** In $\mathbb{R}^2$, a rigid body motion is given by $F_t^{t_0} = \begin{pmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix}$ where $\theta$ a regular function s.t. $\theta(t_0) = 0$. ∎

**Exercice E.8** Let $\widetilde{\Phi}$ be a rigid body motion. Prove

$$(F^T)' = (F')^T, \quad \text{and} \quad F^T.F' \text{ is antisymmetric: } (F')^T.F + F^T.F' = 0. \tag{E.10}$$

**Answer.** Let $F(t) := F_P^{t_0}(t)$, $p(t) = \Phi_P^{t_0}(t)$, $\vec{U}, \vec{W} \in \vec{\mathbb{R}}^n_{t_0}$ and $\vec{w}(t) = F(t).\vec{W}$. (E.7) gives $(F^T(t).\vec{w}(t), \vec{U})_g = (\vec{w}(t), F(t).\vec{U})_g$, thus $((F^T)'(t).\vec{w}(t) + F^T(t).\vec{w}'(t), \vec{U})_g = (\vec{w}'(t), F(t).\vec{U})_g + (\vec{w}(t), F'(t).\vec{U})_g$, thus $((F^T)'(t).\vec{w}(t), \vec{U})_g = (\vec{w}(t), F'(t).\vec{U})_g = ((F')^T(t).\vec{w}(t), \vec{U})_g$ where $(F')^T(t) := (F'(t))^T$, thus $(F^T)' = (F')^T$.

And (E.8) gives $F^T(t).F(t) = I_{t_0}$, thus $(F^T)'(t).F(t) + F^T(t).F'(t) = 0$, thus $(F')^T(t).F(t) + F^T(t).F'(t) = 0$, thus $((F^T)'(t).F(t) + F^T(t).F'(t) = 0$, thus $F^T(t).F'(t)$ is antisymmetric. ∎

### E.1.3 Alternative definition of a rigid body motion: $d\vec{v} + d\vec{v}^T = 0$

The stretching tensor $\mathcal{D}_t = \frac{d\vec{v}_t + d\vec{v}_t^T}{2}$ and the spin tensor $\Omega_t = \frac{d\vec{v}_t - d\vec{v}_t^T}{2}$ have been defined in (C.2)-(C.3). Here no initial time is required: Eulerian approach.

**Proposition E.9** *If $\widetilde{\Phi}$ is a rigid body motion then the endomorphism $d\vec{v}_t \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ is antisymmetric at all $t$:*

$$\mathcal{D}_t = d\vec{v}_t + d\vec{v}_t^T = 0, \quad \textit{i.e.} \quad d\vec{v}_t = \Omega_t. \tag{E.11}$$

*Converse: If, at all $t$, $d\vec{v}_t + d\vec{v}_t^T = 0$ then $\widetilde{\Phi}$ is a rigid body motion.*

*So the relation « $d\vec{v}_t + d\vec{v}_t^T = 0$ for all $t$ » gives an equivalent definition to the definition E.5.*

**Proof.** Recall: $\vec{V}^{t_0}(t,P) = \frac{\partial\Phi^{t_0}}{\partial t}(t,P)$ and $F^{t_0}(t,P) = d\Phi^{t_0}(t,P)$, thus $\frac{\partial F^{t_0}}{\partial t}(t,P) = d\vec{V}^{t_0}(t,P)$. And $p(t) = \Phi^{t_0}(t,P)$ and $\vec{V}^{t_0}(t,P) = \vec{v}(t,p(t))$ give $d\vec{V}^{t_0}(t,P) = d\vec{v}(t,p(t)).F^{t_0}(t,P)$. And $F^T.F = I$ gives $F^{-T} = F^{-1}$, thus $F.F^T = I$. Let $F(t) := F^{t_0}_P(t)$ and $V(t) := \vec{V}^{t_0}_P(t)$ and $d\vec{V}(t) = d\vec{V}^{t_0}(t,P)$.

(E.8) gives $(F.F^T)'(t) = 0 = F'(t).F^T(t) + F(t).(F^T)'(t) \overset{(E.10)}{=} F'(t).F^T(t) + (F'(t).F^T(t))^T = dV(t).F(t)^{-1} + (dV(t).F(t)^{-1})^T \overset{(3.27)}{=} d\vec{v}(t,p_t) + d\vec{v}(t,p_t)^T$. Thus (E.11).

Converse: $d\vec{v} + d\vec{v}^T = 0$ and (D.3) give $\frac{D(\vec{a},\vec{b})_g}{Dt} = 0$, thus $(\vec{a},\vec{b})_g(t,p(t)) = (\vec{a},\vec{b})_g(t_0,P)$ when $p(t) = \Phi^{t_0}_t(P)$, i.e. $(F^{t_0}_t(P).\vec{A}, F^{t_0}_t(P).\vec{B})_g = (\vec{A},\vec{B})_g$, thus $F^{t_0}_t(P)^T.F^{t_0}_t(P) = I$: $\widetilde{\Phi}$ is a rigid body motion. ∎

## E.2 Vector and pseudo-vector representations of a spin tensor $\Omega$

We are dealing here with concepts that are sometimes misunderstood or poorly known.

### E.2.1 Reminder

• The determinant $\det_{|\vec{e}}$ associated with a basis $(\vec{e}_i)$ in $\mathbb{R}^3$ is the alternating multilinear form defined by $\det_{|\vec{e}}(\vec{e}_1,\vec{e}_2,\vec{e}_3) = 1$.

• A basis $(\vec{b}_i)$ has the same orientation than the basis $(\vec{e}_i)$ iff $\det_{|\vec{e}}(\vec{b}_1,\vec{b}_2,\vec{b}_3) > 0$.

• If $(\vec{e}_i)$ is Euclidean, the algebraic volume (or signed volume) limited by three vectors $\vec{u}_1,\vec{u}_2,\vec{u}_3$ is $\det_{|\vec{e}}(\vec{u}_1,\vec{u}_2,\vec{u}_3)$; And the (positive) volume is $|\det_{|\vec{e}}(\vec{u}_1,\vec{u}_2,\vec{u}_3)|$, see § L.

• Let $A$ and $B$ be two observers (e.g. $A$=English and $B$=French), let $(\vec{a}_i)$ be a Euclidean basis chosen by $A$ (e.g. based on the foot), let $(\vec{b}_i)$ be a Euclidean basis chosen by $B$ (e.g. based on the metre). Let $\lambda = ||\vec{b}_1||_a > 0$ (change of unit of length coefficient). The relation between the determinants is:

$$\det_{|\vec{a}} = \pm\lambda^3 \det_{|\vec{b}} \quad \text{with} \quad \begin{cases} + \text{ if } \det\limits_{|\vec{a}}(\vec{b}_1,\vec{b}_2,\vec{b}_3) > 0 & \text{(the bases have the same orientation)}, \\ - \text{ if } \det\limits_{|\vec{a}}(\vec{b}_1,\vec{b}_2,\vec{b}_3) < 0 & \text{(the bases have opposite orientation)}. \end{cases} \tag{E.12}$$

In particular, if $A$ and $B$ use the same unit of length, then $\lambda = 1$ and $\det_{|\vec{a}} = \pm\det_{|\vec{b}}$.

• With an imposed Euclidean dot product $(\cdot,\cdot)_g$: An endomorphism $L$ is $(\cdot,\cdot)_g$-antisymmetric iff

$$\forall \vec{u},\vec{v}, \;\; (L.\vec{u},\vec{v})_g = -(\vec{u},L.\vec{v})_g, \quad \text{i.e.} \quad L^T := L^T_g = -L. \tag{E.13}$$

### E.2.2 Definition of the vector product (cross product)

$\mathbb{R}^n = \mathbb{R}^3$, $(\vec{e}_i)$ is a Euclidean basis, $(\cdot,\cdot)_g$ is the associated Euclidean dot product $(\cdot,\cdot)_g$ (so what follows is not objective). Let $\vec{u},\vec{v} \in \vec{\mathbb{R}^3}$, and let $\ell_{\vec{e},\vec{u},\vec{v}} \in \mathcal{L}(\vec{\mathbb{R}^3},\mathbb{R})$ be the linear form defined by

$$\ell_{\vec{e},\vec{u},\vec{v}} : \begin{cases} \vec{\mathbb{R}^3} \to \mathbb{R} \\ \quad \vec{z} \to \ell_{\vec{e},\vec{u},\vec{v}}(\vec{z}) := \det\limits_{|\vec{e}}(\vec{u},\vec{v},\vec{z}) \end{cases} \tag{E.14}$$

= the algebraic volume of the parallelepiped limited by $\vec{u},\vec{v},\vec{z}$ in the Euclidean chosen unit.

**Definition E.10** Relative to $(\vec{e}_i)$ and $(\cdot,\cdot)_g$, the vector product, or cross product, $\vec{u}\times_{eg}\vec{v}$ (written $\vec{u}\wedge_{eg}\vec{v}$ in french) of two vectors $\vec{u}$ and $\vec{v}$ is the $(\cdot,\cdot)_g$-Riesz representation vector $\vec{u}\times_{eg}\vec{v} \in \vec{\mathbb{R}^3}$ of the linear form $\ell_{\vec{e},\vec{u},\vec{v}}$: So, cf. (F.2):

$$\forall \vec{z} \in \vec{\mathbb{R}^3}, \quad \boxed{(\vec{u}\times_{eg}\vec{v},\vec{z})_g = \det\limits_{|\vec{e}}(\vec{u},\vec{v},\vec{z})}. \tag{E.15}$$

This defines the bilinear cross product operator

$$\times_{eg} : \begin{cases} \vec{\mathbb{R}^3}\times\vec{\mathbb{R}^3} \to \vec{\mathbb{R}^3} \\ \quad (\vec{u},\vec{v}) \to \times_{eg}(\vec{u},\vec{v}) := \vec{u}\times_{eg}\vec{v}. \end{cases} \tag{E.16}$$

(The bilinearity is trivial thanks to the multilinearity of the determinant, and $\vec{u}\times_{eg}\vec{v}$ depends on both $(\cdot,\cdot)_g$ and the orientation of $(\vec{e}_i)$.)

Notation: If a chosen $(\cdot,\cdot)_g$ is imposed to all, then $\vec{u} \times_{eg} \vec{v} =^{\text{written}} \vec{u} \times_e \vec{v}$. Moreover if an orthonormal basis $(\vec{e}_i)$ is imposed to all observers then $\vec{u} \times_e \vec{v} =^{\text{written}} \vec{u} \times \vec{v}$.

NB: The cross product is not an objective operator! It depends on a chosen Euclidean dot product and on a chosen Euclidean basis (its orientation).

NB: Isometric framework = imposed Euclidean basis which is positively oriented and its associated Euclidean dot product $(\cdot,\cdot)_g$: Then $(\vec{u},\vec{v})_g =^{\text{written}} \vec{u} \bullet \vec{v}$ and $\times_{eg} =^{\text{written}} \times$, and (E.15) is written

$$\forall \vec{z} \in \vec{\mathbb{R}^3}, \quad (\vec{u} \times \vec{v}) \bullet \vec{z} = \det(\vec{u},\vec{v},\vec{z}). \tag{E.17}$$

**Exercice E.11** Prove: $\vec{u} \times_{eg} \vec{v}$ is a contravariant vector.

**Answer**. $\vec{u} \times_{eg} \vec{v}$ is a Riesz representation vector, hence it is contravariant. (Or calculation: It satisfies the contravariance change of basis formula, see (F.17).) ∎

### E.2.3 Quantification

$\vec{u} = \sum_{i=1}^3 u_i \vec{e}_i$, $\vec{v} = \sum_{i=1}^3 v_i \vec{e}_i$ and (E.15) give

$$(\vec{u} \times_{eg} \vec{v}, \vec{e}_1)_g = \det_{|\vec{e}}(\vec{u},\vec{v},\vec{e}_1) = \det \begin{pmatrix} u_1 & v_1 & 1 \\ u_2 & v_2 & 0 \\ u_3 & v_3 & 0 \end{pmatrix} = \det \begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \end{pmatrix} = u_2 v_3 - u_3 v_2. \tag{E.18}$$

Similar calculation: $(\vec{u} \times_{eg} \vec{v}, \vec{e}_2)_e = u_3 v_1 - u_1 v_3$ and $(\vec{u} \times_{eg} \vec{v}, \vec{e}_3)_e = u_1 v_2 - u_2 v_1$, thus

$$[\vec{u} \times_{eg} \vec{v}]_{|\vec{e}} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}, \quad \text{i.e.} \quad \vec{u} \times_{eg} \vec{v} = \sum_{i=1}^3 (u_{i+1} v_{i+2} - u_{i+2} v_{i+1})\, \vec{e}_i \tag{E.19}$$

with the generic notation $w_4 := w_1$ and $w_5 = w_2$ (indices modulo 3): In particular $\vec{e}_i \times_{eg} \vec{e}_{i+1} = \vec{e}_{i+2}$.

**Proposition E.12** *1- $\vec{u} \times_{eg} \vec{v} = -\vec{v} \times_{eg} \vec{u}$ (the cross product $\times_{eg}$ is antisymmetric).*

*2- $\vec{u} \parallel \vec{v}$ iff $\vec{u} \times_{eg} \vec{v} = 0$.*

*3- $\vec{u} \times_{eg} \vec{v}$ is orthogonal to $\mathrm{Vect}\{\vec{u},\vec{v}\}$ the linear space generated by $\vec{u}$ and $\vec{v}$.*

*4- $\vec{u} \times_{eg} \vec{v}$ depends on the unit of measurement and on the orientation of the $(\cdot,\cdot)_g$-orthonormal basis $(\vec{e}_i)$. Precisely: Consider two Euclidean dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$, let $\lambda > 0$ s.t. $(\cdot,\cdot)_a = \lambda^2 (\cdot,\cdot)_b$, choose a $(\cdot,\cdot)_a$-orthonormal basis $(\vec{a}_i)$ and a $(\cdot,\cdot)_b$-orthonormal basis $(\vec{b}_i)$; Then*

$$\vec{u} \times_{aa} \vec{v} = \pm \lambda\, \vec{u} \times_{bb} \vec{v}, \tag{E.20}$$

*with the + sign iff $(\vec{a}_i)$ and $(\vec{b}_i)$ have the same orientation.*

**Proof.** 1- $(\vec{u} \times_{eg} \vec{v}, \vec{z})_g = \det_{|\vec{e}}(\vec{u},\vec{v},\vec{z}) = -\det_{|\vec{e}}(\vec{v},\vec{u},\vec{z}) = -(\vec{v} \times_{eg} \vec{u}, \vec{z})_g$, for all $\vec{z}$.

2- If $\vec{u} \parallel \vec{v}$ then $\det_{|\vec{e}}(\vec{u},\vec{v},\vec{z}) = 0 = (\vec{u} \times_{eg} \vec{v}, \vec{z})_e$, so $\vec{u} \times_{eg} \vec{v} \perp_g \vec{z}$, for all $\vec{z}$. Converse: If $\vec{u} \times_{eg} \vec{v} = 0$ then (E.19) gives $\vec{u} \parallel \vec{v}$.

3- If $\vec{z} \in \mathrm{Vect}\{\vec{u},\vec{v}\}$ then $\det_{|\vec{e}}(\vec{u},\vec{v},\vec{z}) = 0$, thus $(\vec{u} \times_{eg} \vec{v}, \vec{z})_g = 0$ thus $\vec{z} \perp_g \vec{u} \times_{eg} \vec{v}$.

4- $(\vec{u} \times_{aa} \vec{v}, \vec{z})_a \overset{(E.15)}{=} \det_{|\vec{a}}(\vec{u},\vec{v},\vec{z}) \overset{(E.12)}{=} \pm\lambda^3 \det_{|\vec{b}}(\vec{u},\vec{v},\vec{z}) \overset{(E.15)}{=} \pm\lambda^3 (\vec{u} \times_{bb} \vec{v}, \vec{z})_b = \pm\lambda^3 \frac{1}{\lambda^2} (\vec{u} \times_{bb} \vec{v}, \vec{z})_a$, true for all $\vec{z}$, thus (E.20). ∎

### E.2.4 Antisymmetric endomorphism represented by a vector

$(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-Euclidean basis.

**Proposition E.13 and def.** *If $\Omega \in \mathcal{L}(\vec{\mathbb{R}^3};\vec{\mathbb{R}^3})$ is $(\cdot,\cdot)_g$-antisymmetric then $\exists! \vec{\omega}_{eg} \in \vec{\mathbb{R}^3}$ s.t. $\forall \vec{y},\vec{z} \in \vec{\mathbb{R}^3}$,*

$$(\Omega.\vec{y}, \vec{z})_g = \det_{|\vec{e}}(\vec{\omega}_{eg}, \vec{y}, \vec{z}), \quad i.e. \quad \boxed{\Omega.\vec{y} = \vec{\omega}_{eg} \times_{eg} \vec{y}}. \tag{E.21}$$

*And $\vec{\omega}_{eg}$ is called the representation vector of $\Omega$ relative to $(\vec{e}_i)$ and $(\cdot,\cdot)_g$. And*

$$[\Omega]_{|\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \iff [\vec{\omega}_{eg}]_{|\vec{e}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{E.22}$$

*In particular $\Omega.\vec{\omega}_{eg} = \vec{0}$ ($= \vec{\omega}_{eg} \times_{eg} \vec{\omega}_{eg}$), i.e. $\vec{\omega}_{eg}$ is an eigenvector of $\Omega$ associated with the eigenvalue 0.*

**Proof.** $\Omega$ is antisymmetric, thus $[\Omega]_{|\vec{e}}$ is given as in (E.22). Suppose that a $\vec{\omega}_{eg}$ satisfying (E.21) exists, $\vec{\omega}_{eg} = \omega_1\vec{e}_1 + \omega_2\vec{e}_2 + \omega_3\vec{e}_3$; Hence $[\vec{\omega}_{eg} \times_{eg} \vec{e}_1]_{|\vec{e}} = \begin{pmatrix} 0 \\ \omega_3 \\ -\omega_2 \end{pmatrix}$, cf. (E.19), thus $\omega_3 = c$ and $\omega_2 = b$; Idem with $\vec{e}_2$ so that $\omega_1 = a$. Thus $\vec{\omega}_{eg}$ is unique. And $\vec{\omega}_{eg}$ given in (E.22) satisfies (E.21): It exists. ∎

**Proposition E.14** *Let $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ be two Euclidean dot products (e.g. in foot and metre), let $(\vec{a}_i)$ and $(\vec{b}_i)$ be Euclidean associated bases, let $\lambda > 0$ s.t. $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$, let $\vec{\omega}_a := \vec{\omega}_{aa}$ and $\vec{\omega}_b := \vec{\omega}_{bb}$. Then (change of representation vector for $\Omega$):*

- *If $(\vec{b}_i)$ and $(\vec{a}_i)$ have the same orientation, then* $\vec{\omega}_b = \lambda\vec{\omega}_a$,
- *If $(\vec{b}_i)$ and $(\vec{a}_i)$ have opposite orientation, then* $\vec{\omega}_b = -\lambda\vec{\omega}_a$, (E.23)

**NB:** *The formulas $\vec{\omega}_b = \pm\lambda\vec{\omega}_a$ are change of vector formulas,* ***not*** *a change of basis formula.*

**Proof.** Apply (E.20). ∎

**Interpretation of $\vec{\omega}_{eg}$:** Suppose $[\Omega]_{|\vec{e}} = \alpha \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So $\Omega$ is the rotation with angle $\frac{\pi}{2}$ in the horizontal plane composed with the dilation with ratio $\alpha$, and $[\vec{\omega}_{eg}]_{|\vec{e}} = \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. So $\vec{\omega}_{eg} = \alpha\,\vec{e}_3$ is orthogonal to the horizontal plane, hence $\vec{\omega}_{eg}\times_{eg}$ is a rotation around the $z$-axis composed with a dilation which coefficient is $\alpha$.

**Exercice E.15** Let $\Omega$ s.t. $[\Omega]_{|\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ (see (E.22)). Find a direct (relative to $(\vec{e}_i)$) orthonormal basis $(\vec{b}_i)$ s.t. $[\Omega]_{|\vec{b}} = \sqrt{a^2{+}b^2{+}c^2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

**Answer**. Let $\vec{b}_3 = \frac{\vec{\omega}_e}{||\vec{\omega}_e||_e}$, so $[\vec{b}_3]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Then let $\vec{b}_1$ be given by $[\vec{b}_1]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$, so $\vec{b}_1 \perp \vec{b}_3$. Then let $\vec{b}_2 = \vec{b}_3 \times_e \vec{b}_1$, that is, $[\vec{b}_2]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2}} \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix}$. Thus $(\vec{b}_i)$ is a direct orthonormal basis, and the transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$ is $P = \begin{pmatrix} [\vec{b}_1]_{|\vec{e}} & [\vec{b}_2]_{|\vec{e}} & [\vec{b}_3]_{|\vec{e}} \end{pmatrix}$. With $[\Omega]_{|\vec{b}} = P^{-1}.[\Omega]_{|\vec{e}}.P$ (change of basis formula), where $P^{-1} = P^T$ (change of orthonormal basis).

With $[\Omega]_{|\vec{e}}.[\vec{b}_1]_{|\vec{e}} = \frac{1}{\sqrt{b^2+c^2}} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} . \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix} = \frac{1}{\sqrt{b^2+c^2}} \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix} = \sqrt{a^2{+}b^2{+}c^2}[\vec{b}_2]_{|\vec{e}}$ (expected), $[\Omega]_{|\vec{e}}.[\vec{b}_2]_{|\vec{e}} = \frac{1}{\sqrt{b^2+c^2}} \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} . \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix} = \frac{1}{\sqrt{b^2+c^2}} \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} bc^2 + b(a^2+b^2) \\ -ac^2 - a(a^2+b^2) \\ abc - abc \end{pmatrix} = -\sqrt{a^2{+}b^2{+}c^2}[\vec{b}_1]_{|\vec{e}}$ (expected), and $[\Omega]_{|\vec{e}}.[\vec{b}_3]_{|\vec{e}} = [\vec{0}]$ (expected since $\vec{b}_3 \parallel \vec{\omega}_e$). Thus $[\Omega]_{|\vec{e}}.P = \sqrt{a^2{+}b^2{+}c^2} \begin{pmatrix} [\vec{b}_2]_{|\vec{e}} & -[\vec{b}_1]_{|\vec{e}} & [\vec{0}]_{|\vec{e}} \end{pmatrix}$. And $(P^{-1}.[\Omega]_{|\vec{e}}.P)_{ij} = (P^T.[\Omega]_{|\vec{e}}.P)_{ij} = [\vec{b}_i]^T_{|\vec{e}}.[\Omega]_{|\vec{e}}.[\vec{b}_j]_{|\vec{e}}$ gives the result. ∎

### E.2.5 Curl (rotational)

**Definition E.16** Let $(\vec{e}_i)$ be a Euclidean basis in $\vec{\mathbb{R}^3}$ and $\vec{v}$ be a $C^1$ vector field, $\vec{v} = \sum_{i=1}^3 v^i\vec{e}_i$. The curl (or rotational) of $\vec{v}$ relative to $(\vec{e}_i)$ is the $C^0$ vector field $\vec{\text{curl}}_e\vec{v}$ given by

$$\vec{\text{curl}}_e\vec{v} = \sum_{i=1}^{3}(\frac{\partial v_{i+2}}{\partial x_{i+1}} - \frac{\partial v_{i+1}}{\partial x_{i+2}})\,\vec{e}_i, \quad \text{i.e.} \quad [\vec{\text{curl}}_e\vec{v}]_{|\vec{e}} = \begin{pmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \end{pmatrix}. \tag{E.24}$$

And $\vec{\text{curl}}_e\vec{v} =^{\text{written}} \vec{\nabla} \times_e \vec{v}$ (notation due to the matrix product $\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$). This defines the curl operator $\text{curl}_e : C^1(\Omega_t;\vec{\mathbb{R}^3}) \to C^0(\Omega_t;\vec{\mathbb{R}^3})$.

**Proposition E.17** *Isometric framework = imposed Euclidean basis $(\vec{e}_i)$ which is positively oriented and its associated Euclidean dot product $(\cdot,\cdot)_g$: Then $d\vec{v}_{eg}^T =^{\text{written}} d\vec{v}^T$ and $\text{curl}_e =^{\text{written}} \text{curl}$. Let $\Omega(t,p_t) = \frac{d\vec{v}(t,p_t)-d\vec{v}(t,p_t)^T}{2}$ and let $\vec{\omega}_{eg} =^{\text{written}} \vec{\omega}$ be its associated vector relative to the Euclidean basis $(\vec{e}_i)$, cf. (E.21). Then*

$$\vec{\omega} = \frac{1}{2}\vec{\text{curl}}\vec{v}. \tag{E.25}$$

**Proof.** $[\Omega]_{|\vec{e}} = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \cdot & 0 & \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \\ \cdot & \cdot & 0 \end{pmatrix}$ is antisymmetric. Thus (E.24) gives (E.25). ∎

## E.3 Pseudo-vector, and pseudo-cross product

We enter the world of the representation of vectors with matrices. $\mathcal{M}_{mn}$ is the space of $m*n$ matrices.

### E.3.1 Definition

**Definition E.18** A column matrix is also called a pseudo-vector or a column vector. And the pseudo-cross product $\overset{\circlearrowleft}{\times} : \left\{\begin{array}{ll} \mathcal{M}_{31}\times\mathcal{M}_{31} & \to \mathcal{M}_{31} \\ (\vec{x},\vec{y}) & \to \overset{\circlearrowleft}{\times}(\vec{x},\vec{y}) \overset{\text{written}}{=} \vec{x}\overset{\circlearrowleft}{\times}\vec{y} \end{array}\right\}$ is defined by

$$\vec{x}\overset{\circlearrowleft}{\times}\vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \overset{\circlearrowleft}{\times} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}, \quad \text{when} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{E.26}$$

### E.3.2 Antisymmetric matrix represented by a pseudo-vector

**Definition E.19** The pseudo-vecteur $\overset{\circlearrowleft}{\omega} \in \mathcal{M}_{31}$ associated to the antisymmetric matrix $A = [A_{ij}] = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \mathcal{M}_{33}$ is the matrix $\overset{\circlearrowleft}{\omega} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathcal{M}_{31}$: So $\overset{\circlearrowleft}{\omega}$ satisfies

$$\boxed{A.\vec{y} = \overset{\circlearrowleft}{\omega}\overset{\circlearrowleft}{\times}\vec{y}}, \quad \forall \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \tag{E.27}$$

### E.3.3 Pseudo-vector representations of an antisymmetric endomorphism

$(\cdot,\cdot)_g$ is a chosen Euclidean dot product and $\Omega \in \mathcal{L}(\vec{\mathbb{R}^3};\vec{\mathbb{R}^3})$ is a $(\cdot,\cdot)_g$-antisymmetric endomorphism i.e. s.t. $\Omega^T = -\Omega$ (i.e. $(\Omega^T.\vec{u},\vec{w})_g = -(\vec{u},\Omega.\vec{w})_g$ for all $\vec{u},\vec{w} \in \vec{\mathbb{R}^3}$).

Then choose a positively oriented $(\cdot,\cdot)_g$-Euclidean basis. Thus $[\Omega]_{|\vec{e}}$ is an antisymmetric matrix and call $\overset{\circlearrowleft}{\omega}$ the associated pseudo-vector: $\forall \vec{v} \in \vec{\mathbb{R}^3}$, $[\Omega]_{|\vec{e}}.[\vec{v}]_{|\vec{e}} = \overset{\circlearrowleft}{\omega}\overset{\circlearrowleft}{\times}[\vec{v}]_{|\vec{e}}$, written

$$[\Omega].[\vec{v}] = \overset{\circlearrowleft}{\omega}\overset{\circlearrowleft}{\times}[\vec{v}]. \tag{E.28}$$

This formula is widely used in mechanics, and unfortunately sometimes written $\Omega.\vec{v} = \vec{\omega}\times\vec{v}$:

**Be careful:** (E.28) is ***not*** a vectorial formula; This is just a formula for matrix calculations which can give **false results** if a change of basis is considered; E.g., consider the basis $(\vec{b}_1,\vec{b}_2,\vec{b}_3) = (-\vec{e}_1,\vec{e}_2,\vec{e}_3)$: $(\vec{b}_i)$ is also a $(\cdot,\cdot)_g$-Euclidean basis but with a different orientation.

1- Vector approach: The transition matrix $P$ from $(\vec{e}_i)$ to $(\vec{b}_i)$ is $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $\Omega$ being an endomorphism, we have $[\Omega]_{|\vec{b}} = P^{-1}.[\Omega]_{|\vec{e}}.P$ (change of basis formula). Thus $[\Omega]_{|\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$

gives

$$[\Omega]_{|\vec{b}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} . \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & c & -b \\ -c & 0 & -a \\ b & a & 0 \end{pmatrix} . \tag{E.29}$$

Thus the representation vectors $\vec{\omega}_e$ and $\vec{\omega}_b$ are, cf. (E.22):

$$[\vec{\omega}_e]_{|\vec{e}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad [\vec{\omega}_b]_{|\vec{b}} = \begin{pmatrix} a \\ -b \\ -c \end{pmatrix}, \quad \text{i.e.} \quad \left\{ \begin{aligned} \vec{\omega}_e &= a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3, \\ \vec{\omega}_b &= a\vec{b}_1 - b\vec{b}_2 - c\vec{b}_3, \end{aligned} \right\} \quad \text{thus} \quad \boxed{\vec{\omega}_b = -\vec{\omega}_e}. \tag{E.30}$$

(Or simply apply (E.23).)

2- Matrix approach and pseudo-vectors: (E.27) gives $[\Omega]_{|\vec{e}}.[\vec{v}]_{|\vec{e}} = \overset{\circlearrowleft}{\omega}_e \overset{\circlearrowleft}{\times} [\vec{v}]_{|\vec{e}}$ and $[\Omega]_{|\vec{b}}.[\vec{v}]_{|\vec{b}} = \overset{\circlearrowleft}{\omega}_b \overset{\circlearrowleft}{\times} [\vec{v}]_{|\vec{b}}$, with

$$\overset{\circlearrowleft}{\omega}_e = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \overset{\circlearrowleft}{\omega}_b = \begin{pmatrix} a \\ -b \\ -c \end{pmatrix}, \quad \text{so} \quad \boxed{\overset{\circlearrowleft}{\omega}_b \neq -\overset{\circlearrowleft}{\omega}_e} : \text{can't be written} = \overset{\circlearrowleft}{\omega} \tag{E.31}$$

because a unique notation $\overset{\circlearrowleft}{\omega}$ of both $\overset{\circlearrowleft}{\omega}_e$ and $\overset{\circlearrowleft}{\omega}_b$ is absurd. Moreover such a $\overset{\circlearrowleft}{\omega}$ can't represent a single vector because it does not satisfy the vector change of basis formula $\overset{\circlearrowleft}{\omega}_b \neq P^{-1}.\overset{\circlearrowleft}{\omega}_e$ (in fact $[\Omega]_{|\vec{b}} = P^{-1}.[\Omega]_{|\vec{e}}.P$). Thus the matrix notation $\overset{\circlearrowleft}{\omega}$ can only be used if no change of basis (even Euclidean) will ever be used...

# F Riesz representation theorem

Framework: $E := (E, (\cdot,\cdot)_g)$ is a Hilbert space, i.e. a vector space $E$ with an inner dot product $(\cdot,\cdot)_g$ s.t. $(E.||.||_g)$ is complete (with $||.||_g := \sqrt{(\cdot,\cdot)_g}$ the associated norm).

And $E^* = \mathcal{L}(E;\mathbb{R})$ is the space of continuous linear forms on $E$ = the space of linear "measuring tools" on $E$.

(A linear function $\ell : E \to \mathbb{R}$ is continuous iff $\exists c > 0$ s.t. $\forall \vec{x} \in E$, $|\ell(\vec{x})| \leq c\,||\vec{x}||_E$. And then $||\ell||_{E^*} := \sup_{||\vec{x}||_E=1} |\ell.\vec{x}|$ defines a norm in $E^*$, and $(E^*, ||.||_{E^*})$ is a complete space, easy to prove.)

## F.1 The Riesz representation theorem

The Riesz representation theorem establishes the converse of the easy statement:

**Proposition F.1** *If $(E, (\cdot,\cdot)_g)$ is a Hilbert space then*

$$\forall \vec{v} \in E \text{ (vector)}, \ \exists! v_g \in E^* \text{ (linear continuous form) s.t. } v_g.\vec{x} = (\vec{v}, \vec{x})_g, \ \ \forall \vec{x} \in E, \tag{F.1}$$

*and moreover $||v_g||_{E^*} = ||\vec{v}||_g$.*

**Proof.** Define $v_g : E \to \mathbb{R}$ by $v_g(\vec{x}) = (\vec{v}, \vec{x})_g$ for all $\vec{x} \in E$. We have $v_g$ linear on $E$ thanks to the bilinearity of an inner dot product. And the Cauchy–Schwarz inequality gives $|v_g(\vec{x})| = |(\vec{v}, \vec{x})_g| \leq ||\vec{v}||_g\,||\vec{x}||_g$ for all $\vec{x} \in E$, thus $||v_g||_{E^*} \leq ||\vec{v}||_g < \infty$, thus $v_g$ is continuous. And $|v_g(\vec{v})| = |(\vec{v}, \vec{v})_g| = ||\vec{v}||_g\,||\vec{v}||_g$, thus $||v_g||_{E^*} \geq ||\vec{v}||_g$, thus $||v_g||_{E^*} = ||\vec{v}||_g$. Uniqueness: Another $w_g$ satisfying $w_g.\vec{x} = (\vec{v}, \vec{x})_g$ gives $(w_g - v_g).\vec{x} = 0$ for all $\vec{x} \in E$, thus $w_g - v_g = 0$. ∎

**Theorem F.2 (Riesz representation theorem, and definition)** *$(E, (\cdot,\cdot)_g)$ being a Hilbert space, any "measuring tool" $\ell \in E^*$ can be represented by a vector $\vec{\ell}_g \in E$:*

$$\forall \ell \in E^* \text{ (linear continuous form)}, \ \exists! \vec{\ell}_g \in E \text{ (vector) s.t. } \ell.\vec{x} = (\vec{\ell}_g, \vec{x})_g, \ \ \forall \vec{x} \in E, \tag{F.2}$$

*and moreover $||\vec{\ell}_g||_g = ||\ell||_{E^*}$. And $\vec{\ell}_g$ is called the $(\cdot,\cdot)_g$-Riesz representation vector of $\ell$.*

**Proof.** Easy in finite dimension: $(\vec{e}_i)$ being a basis in $E$, if $[\ell]_{|\vec{e}} = (\, \ell_1 \ \ \dots \ \ \ell_n \,)$ (row matrix since $\ell$ is a linear form) then (F.2) gives $[\ell]_{|\vec{e}}.[\vec{x}]_{|\vec{e}} = [\vec{\ell}_g]_{\vec{e}}^T.[g]_{|\vec{e}}.[\vec{x}]_{|\vec{e}}$, thus $[\vec{\ell}_g]_{\vec{e}} = [g]_{|\vec{e}}^{-1}.[\ell]_{|\vec{e}}^T$ (column matrix), thus $\vec{\ell}_g$. Then $|\ell.\vec{x}| = |(\vec{\ell}_g, \vec{x})_g| \leq ||\vec{\ell}_g||_g||\vec{x}||_g$, with $|\ell.\vec{\ell}_g| = |(\vec{\ell}_g, \vec{\ell}_g)_g| = ||\vec{\ell}_g||_g||\vec{\ell}_g||_g$, thus $||\ell||_{E^*} = ||\vec{\ell}_g||_g$.

General case (infinite dimension e.g. $E = L^2(\Omega)$). $\ell \in E^*$ being linear and continuous, its kernel $\mathrm{Ker}\ell = \ell^{-1}(\{0\})$ is a closed sub-vector space in $E$. If $\ell = 0$ then $\vec\ell_g = \vec 0$ (trivial). Suppose $\ell \neq 0$, thus $\mathrm{Ker}\ell \subsetneq E$, thus $\exists \vec z \in E$ s.t. $\vec z \notin \mathrm{Ker}\ell$ and call $\vec z_0$ its $(\cdot,\cdot)_g$-orthogonal projection on $\mathrm{Ker}\ell$ (which exists and is unique because $\mathrm{Ker}\ell$ is closed): We have, $\forall \vec y_0 \in \mathrm{Ker}\ell$, $(\vec z - \vec z_0, \vec y_0)_g = 0$. Thus $\vec n := \frac{\vec z - \vec z_0}{||\vec z - \vec z_0||_g}$ is a unit vector in $(\mathrm{Ker}\ell)^\perp$. And $1 = \dim \mathbb{R} = \dim(\text{codomain of } \ell) = \dim(\mathrm{Ker}\ell)^\perp$ gives $(\mathrm{Ker}\ell)^\perp = \mathrm{Vect}\{\vec n\}$ (see exercise F.3). With $E = \mathrm{Ker}\ell \oplus (\mathrm{Ker}\ell)^\perp$ since both vector spaces are closed (an orthogonal is always closed in a Hilbert space), any $\vec x \in E$ satisfies $\vec x = \vec x_0 + (\vec x - \vec x_0) = \vec x_0 + \lambda \vec n \in \mathrm{Ker}\ell \oplus (\mathrm{Ker}\ell)^\perp$. We get $(\vec x, \vec n)_g = 0 + \lambda ||\vec n||_g^2 = \lambda$ and $\ell(\vec x) = 0 + \lambda \ell(\vec n)$, thus $\ell(\vec x) = (\vec x, \vec n)_g \ell(\vec n) = (\vec x, \ell(\vec n)\vec n)_g$ (bilinearity of $(\cdot,\cdot)_g$). Thus $\vec\ell_g := \ell(\vec n)\vec n$ satisfies (F.2). And if $\vec\ell_{g1}$ and $\vec\ell_{g2}$ satisfy (F.2) then $0 = (\ell - \ell).\vec x = (\vec\ell_{g1} - \vec\ell_{g2}, \vec x)_g$ for all $\vec x \in E$, thus $\vec\ell_{g1} - \vec\ell_{g2} = 0$. Thus $\vec\ell_g$ is unique. And $||\ell||_{E^*} := \sup_{||\vec x||_g=1} |\ell(\vec x)| = \sup_{||\vec x||_g=1} |(\vec\ell_g, \vec x)_g| =_{Schwarz}^{Cauchy} ||\vec\ell_g||_g$. ▪

**Exercice F.3** Prove: If $\ell \in E^*$ and $\ell \neq 0$ then $\dim(\mathrm{Ker}\ell)^\perp = 1$ $(= \dim(\mathrm{Im}(\ell)) = \dim \mathbb{R})$.

**Answer**. Consider the restriction $\ell_{|\mathrm{Ker}\ell^\perp} : \left\{ \begin{array}{ll} (\mathrm{Ker}\ell)^\perp & \to \mathbb{R} \\ \vec x & \to \ell_{|\mathrm{Ker}\ell^\perp}.\vec x := \ell.\vec x \end{array} \right\}$. It is linear (since $\ell$ is), it is onto since $\ell$ is linear and $\ell \neq 0$. And it is one to one since $\ell_{|\mathrm{Ker}\ell^\perp}(\vec x) = 0 = \ell(\vec x)$ gives $\vec x \in (\mathrm{Ker}\ell)^\perp \bigcap \mathrm{Ker}\ell = \{\vec 0\}$ thus $\vec x = 0$; Thus $\ell_{|\mathrm{Ker}\ell^\perp}$ is (linear) bijective, thus $\dim(\mathrm{Ker}\ell)^\perp = \dim(\mathbb{R}) = 1$. ▪

## F.2 The $(\cdot,\cdot)_g$-Riesz representation operator

(F.2) defines the $(\cdot,\cdot)_g$-Riesz representation operator (linear)

$$\vec R_g : \left\{ \begin{array}{ll} E^* & \to E \\ \ell & \to \vec R_g(\ell) := \vec\ell_g \overset{\text{written}}{=} \vec R_g.\ell \end{array} \right\} \quad \text{where} \quad (\vec R_g(\ell), \vec v)_g = \ell.\vec v, \ \ \forall \vec v \in E. \tag{F.3}$$

$\vec R_g$ is a change of variance operator: Transforms the covariant $\ell$ into the contravariant $\vec\ell_g$ thanks to the tool $(\cdot,\cdot)_g$. With components see (F.6): $i$ down in $\ell_i$, and $i$ up in $(\vec\ell_g)^i$.

**NB (fundamental):** $\vec R_g$ is ***not*** objective since it requires a man made tool (an inner dot product e.g. English or French) to be defined. In fact, an isomorphism $E \leftrightarrow E^*$ cannot be objective, see § U.2.

With $\mathcal{G}$ the set of inner dot products in $E$, we have thus defined the Riesz representation mapping

$$\vec R : \left\{ \begin{array}{ll} \mathcal{G} \times E^* & \to E \\ (g, \ell) & \to \vec R(g, \ell) := \vec R_g(\ell) = \vec\ell_g = \vec\ell(g). \end{array} \right. \tag{F.4}$$

So $\vec R$ has two inputs: A choice $(\cdot,\cdot)_g$ by an observer for the first slot, a linear form for the second slot.

**Proposition F.4** *$\vec R_g$ is an isomorphism between Banach spaces.*

**Proof.** Linearity: $(\vec R_g(\ell + \lambda m), \vec x)_g = (\ell + \lambda m).\vec x = \ell.\vec x + \lambda m.\vec x = (\vec R_g(\ell), \vec x)_g + \lambda(\vec R_g(m), \vec x)_g = (\vec R_g(\ell) + \lambda \vec R_g(m), \vec x)_g$, for all $\vec x$, gives $\vec R_g(\ell + \lambda m) = \vec R_g(\ell) + \lambda \vec R_g(m)$. Bijectivity thanks to (F.1) and (F.2), and $||\vec\ell_g||_g = ||\ell||_{E^*}$ thanks to the Riesz representation theorem. ▪

## F.3 Quantification

$\dim E = n$, $\ell \in E^*$ (a linear form), $(\vec e_i)$ is a basis, $(e^i)$ is the dual basis, notations:

$$g_{ij} = g(\vec e_i, \vec e_j), \quad \ell = \sum_{j=1}^n \ell_j e^j, \quad \vec\ell_g = \sum_{i=1}^n (\vec\ell_g)^i \vec e_i, \quad \vec R_g.e^j = \sum_{i=1}^n R_g^{ij} \vec e_i, \quad [g^{ij}] := [g_{ij}]^{-1}, \tag{F.5}$$

so $[g]_{|\vec e} = [g_{ij}] \in \mathcal{M}_{nn}$, $[\ell]_{|e} = [\ell_j] \in \mathcal{M}_{1n}$, $[\vec\ell_g]_{|\vec e} = [(\vec\ell_g)^i] \in \mathcal{M}_{n1}$, $[\vec R]_{|e,\vec e} = [R^{ij}] \in \mathcal{M}_{nn}$ are the matrices representing $g(\cdot,\cdot)$, $\ell$, $\vec\ell_g$ and $\vec R$ in the bases $(\vec e_i)$ and $(e^i)$. Then (F.2) gives $\ell_i = \ell.\vec e_i = (\vec\ell_g, \vec e_i)_g =$

$\sum_j (\vec{\ell}_g)^j g(\vec{e}_j, \vec{e}_i) = \sum_j g_{ij} (\vec{\ell}_g)^j$ thus $[\ell]^T = [g].[\vec{\ell}_g]$, thus

$$\boxed{[\vec{\ell}_g] = [g]^{-1}.[\ell]^T}, \quad \text{i.e.} \quad (\vec{\ell}_g)^i = \sum_{j=1}^n g^{ij} \ell_j, \ \forall i. \tag{F.6}$$

And $\vec{\ell}_g =^{(F.3)} \vec{R}_g.\ell = \sum_{ij} R_g^{kj} \ell_j \vec{e}_i = \sum_{ij} R_g^{ij} \ell_j \vec{e}_i$ gives $(\vec{\ell}_g)^i = e^i.\vec{\ell}_g = \sum_j R_g^{ij} \ell_j$, thus $[\vec{\ell}_g] = [\vec{R}_g].[\ell]^T$, thus

$$\boxed{[\vec{R}_g] = [g]^{-1}}, \quad \text{i.e.} \quad [R_g^{ij}] = [g^{ij}], \quad \text{and} \quad (\vec{\ell}_g)^i = \sum_{j=1}^n R^{ij} \ell_j, \ \forall i. \tag{F.7}$$

**Remark F.5** Isometric framework: A chosen Euclidean dot product $(\cdot,\cdot)_g$ is imposed. If the duality notations are used, then $(\vec{\ell}_g)^i =^{\text{written}} \ell^i$ and $\vec{\ell}_g =^{\text{written}} \ell^\sharp$ because the bottom index $i$ in $\ell_i$ has been raised by $\vec{R}_g$ to give $\ell^i$. So $\ell^\sharp = \sum_i \ell^i \vec{e}_i$ and (F.6) gives

$$\ell.\vec{x} = \ell^\sharp \bullet \vec{x} \quad \text{and} \quad \begin{pmatrix} \ell^1 \\ \vdots \\ \ell^n \end{pmatrix} = [g]^{-1}. \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_n \end{pmatrix} = [R^{ij}]. \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_n \end{pmatrix} \quad \text{(isometric framework).} \tag{F.8}$$

We won't use this $\ell^\sharp$ notation (we deal with objectivity: No isometric framework imposed). ∴

## F.4 Change of Riesz representation vector, and Euclidean case

$(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are two inner dot products, $\ell \in E^*$, $\vec{\ell}_g := \vec{R}_g(\ell)$ and $\vec{\ell}_h := \vec{R}_h(\ell)$. Thus, $\forall \vec{x} \in E$,

$$(\vec{\ell}_g, \vec{x})_g = \ell.\vec{x} = (\vec{\ell}_h, \vec{x})_h. \tag{F.9}$$

**Proposition F.6** *For one given basis $(\vec{e}_i)$ in $E$, we have the change of Riesz representation vector formula:*

$$[h].[\vec{\ell}_h] = [g].[\vec{\ell}_g], \quad \textit{i.e.} \quad \boxed{[\vec{\ell}_h] = [h]^{-1}.[g].[\vec{\ell}_g]}, \tag{F.10}$$

*short notation for $[h]_{|\vec{e}}.[\vec{\ell}_h]_{|\vec{e}} = [g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}}$, i.e. $[\vec{\ell}_h]_{|\vec{e}} = [h]_{|\vec{e}}^{-1}.[g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}}$. In particular*

$$\textit{If} \ \ (\cdot,\cdot)_g = \lambda^2 (\cdot,\cdot)_h \quad \textit{then} \quad \vec{\ell}_h = \lambda^2 \vec{\ell}_g. \tag{F.11}$$

*So, a linear form $\ell$ **can't** be identified with a Riesz representation vector (which one: $\vec{\ell}_g$? $\vec{\ell}_h$?).*

*Conversely, if $\vec{\ell}_h = \lambda^2 \vec{\ell}_g$ for all linear forms $\ell \in E^*$, then $(\cdot,\cdot)_g = \lambda^2 (\cdot,\cdot)_h$.*

*NB: (F.10) is a "change of vector" formula (from $\vec{\ell}_g$ to $\vec{\ell}_h$ due to the change of inner dot product to represent $\ell$); Not a "change of basis" formula (one vector expressed with two bases).*

**Proof.** (F.9) gives $[\vec{x}]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}} = [\vec{x}]_{|\vec{e}}^T.[h]_{|\vec{e}}.[\vec{\ell}_h]_{|\vec{e}}$ for all $\vec{x}$, hence $[g]_{|\vec{e}}.[\vec{\ell}_g]_{|\vec{e}} = [h]_{|\vec{e}}.[\vec{\ell}_h]_{|\vec{e}}$, i.e. (F.10).

In particular $\lambda^2 (\cdot,\cdot)_h = (\cdot,\cdot)_g$ give $\lambda^2 (\vec{\ell}_g, \vec{x})_h = (\vec{\ell}_g, \vec{x})_g =^{(F.9)} (\vec{\ell}_h, \vec{x})_h$ for all $\vec{x}$, hence $\lambda^2 \vec{\ell}_g = \vec{\ell}_h$.

Converse: For all $\ell \in E^*$, $\lambda^2 \vec{\ell}_g = \vec{\ell}_h$ gives $\lambda^2 (\vec{\ell}_g, \vec{x})_h = (\vec{\ell}_h, \vec{x})_h \overset{(F.9)}{=} (\vec{\ell}_g, \vec{x})_g$, for all $\vec{x}$ and for all $\vec{\ell}_g$ because $\vec{R}_g$ is an isomorphism cf. prop. (F.4), thus $\lambda^2 (\cdot,\cdot)_h = (\cdot,\cdot)_g$. ∴

**Example F.7** If $(\cdot,\cdot)_g$ and $(\cdot,\cdot)_h$ are the Euclidean dot products made with the foot and the metre then (F.11) gives

$$\vec{\ell}_h = \lambda^2 \vec{\ell}_g \quad \text{with} \quad \lambda^2 > 10 \ : \tag{F.12}$$

$\vec{\ell}_g$ and $\vec{\ell}_h$ are quite different! So a Riesz representation vector is (very) subjective, and certainly not "canonical" (a word that you may find in books where... nothing is defined... nor justified...).

Thus, aviation: If you do want to use a Riesz representation vector to represent a $\ell \in \mathbb{R}^{n*}$, it is vital to know which Euclidean dot product is in use, cf. the Mars Climate Orbiter probe crash (remark A.17). Recall: The foot is the international unit of altitude for aviation. ∴

## F.5 Riesz representation vectors and gradients

$f \in C^1(\mathbb{R}^n;\mathbb{R})$, $p \in \mathbb{R}^n$. The differential of $f$ at $p$ is the linear form $df(p) \in \mathbb{R}^{n*}$ defined by

$$df(p).\vec{w} := \lim_{h\to 0} \frac{f(p+h\vec{w})-f(p)}{h}, \quad \forall \vec{w} \in \vec{\mathbb{R}^n} \tag{F.13}$$

(definition independent of any inner dot product or basis).

If you choose an inner dot product $(\cdot,\cdot)_g$ then you can define the gradient $\vec{\text{grad}}_g f(p)$: It is the $(\cdot,\cdot)_g$-Riesz representation vector of $df(p)$:

$$\vec{\text{grad}}_g f(p) := \vec{R}_g(df(p)), \quad \text{i.e.} \quad \boxed{df(p).\vec{w} = (\vec{\text{grad}}_g f(p), \vec{w})_g}, \quad \forall \vec{w} \in \vec{\mathbb{R}^n}. \tag{F.14}$$

E.g. (F.12) gives

$$\vec{\text{grad}}_h f(p) = \lambda^2 \vec{\text{grad}}_g f(p) \quad \text{with} \quad \lambda^2 > 10 \;\; \text{(English vs French)} : \tag{F.15}$$

The gradient is very dependent on the observer (a gradient is subjective, the differential is objective).

**Remark F.8** Without inner dot products but with a basis, we also have an observer dependence. E.g., in the 1-D case with $f : x \in \mathbb{R} \to f(x) \in \mathbb{R}$, question: What does $f'(x)$ mean? Answer:

11- For one observer, it means $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$... where in the departure space the observer has chosen a basis vector $\vec{a}$ of length 1 for him (e.g. 1 foot) which he calls $\vec{a} = 1$; So, with explicit notations, his derivative $f'(x)$ is in fact $f'_a(x) := df(x).\vec{a} = \lim_{h\to 0} \frac{f(x+h\vec{a})-f(x)}{h}$.

12- For another observer, it means $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$... where in the departure space the observer has chosen a basis vector $\vec{b}$ of length 1 for him (e.g. 1 metre), and he write $\vec{b} = 1$; So, with explicit notations, his derivative $f'(x)$ is in fact $f'_b(x) := df(x).\vec{b} = \lim_{h\to 0} \frac{f(x+h\vec{b})-f(x)}{h}$.

13- If $\vec{b} = \lambda\vec{a}$, then

$$\lim_{h\to 0} \frac{f(x+h\vec{b})-f(x)}{h} = \lim_{h\to 0} \frac{f(x+h\lambda\vec{a})-f(x)}{h} = \lambda \lim_{h\to 0} \frac{f(x+h\lambda\vec{a})-f(x)}{h\lambda} = \lambda \lim_{k\to 0} \frac{f(x+k\vec{a})-f(x)}{k}.$$

Thus, e.g. with foot and metre,

$$f'_b(x) = \lambda f'_a(x), \quad \text{with} \quad \lambda \simeq 3.28, \quad \text{so} \quad f'_b(x) \neq f'_a(x). \tag{F.16}$$

In other words, $f'(x) = \frac{\text{opposite side}}{\text{adjacent side}}$ depends on the length unit of the adjacent side: foot? metre? ▟

**Exercice F.9** We have $f'_b(x) =^{(F.16)} \lambda f'_a(x)$ and $\vec{\text{grad}}_b f(x) =^{(F.15)} \lambda^2 \vec{\text{grad}}_a f(x)$. Why?

**Answer**. Because (F.16) does not use the Riesz representation theorem. Details: $(\vec{a})$ and $(\vec{b})$ are two bases in $\vec{\mathbb{R}}$, associated inner dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$, and $\vec{b} = \lambda\vec{a}$; thus $(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b$. And $f'_b(x) = \lambda f'_a(x)$ gives $(\vec{\text{grad}} f_b(x), \vec{b})_b =^{(F.14)} df(x).\vec{b} = f'_b(x) = \lambda f'_a(x) = \lambda df(x).\vec{a} =^{(F.14)} \lambda(\vec{\text{grad}} f_a(x), \vec{a})_a = (\vec{\text{grad}} f_b(x), \lambda\vec{a})_b = \lambda^2(\vec{\text{grad}} f_a(x), \vec{a})_b$, so $\vec{\text{grad}} f_b(x) = \lambda^2 \vec{\text{grad}} f_a(x)$ as expected. ▟

**Exercice F.10** With $||.||_g = \lambda||.||_h$ we have $||\vec{\ell}_h||_g = \lambda||\vec{\ell}_h||_h$. Does it contradict the Riesz representation theorem which gives $||\ell|| = ||\vec{\ell}_g||$?

**Answer**. No, because $||\ell|| := \sup_{\vec{x}} \frac{|\ell.\vec{x}|}{||\vec{x}||_{\mathbb{R}^n}}$ depends on the norm $||.||_{\mathbb{R}^n}$ chosen; Here $||.||_{\mathbb{R}^n}$ is either $||.||_g$ or $||.||_h$. And if $||\ell||_g := \sup_{\vec{x}} \frac{|\ell.\vec{x}|}{||\vec{x}||_g}$ (you have chosen the $||.||_{\mathbb{R}^n} := ||.||_g$), then $||\ell||_h = \sup_{\vec{v}\in\vec{\mathbb{R}^n}} \frac{|\ell.\vec{v}|}{||\vec{v}||_h} = \sup_{\vec{v}\in\vec{\mathbb{R}^n}} \frac{|\ell.\vec{v}|}{\frac{1}{\lambda}||\vec{v}||_g} = \lambda \sup_{\vec{v}\in\vec{\mathbb{R}^n}} \frac{|\ell.\vec{v}|}{||\vec{v}||_g} = \lambda||\ell||_g$. Don't forget: $||\ell|| = \sup(...)$ depends on the choice of a norm: $||.||_g$? $||.||_h$? ▟

## F.6 A Riesz representation vector is contravariant

$\vec{\ell}_g$ is a vector in $E$ so it is contravariant. To be convinced:

**Exercice F.11** Check:

$$[\vec{\ell}_g]_{|new} = P^{-1}.[\vec{\ell}_g]_{|old} \quad \text{(contravariance formula)}. \tag{F.17}$$

**Answer**. Consider two bases $(\vec{a}_i)$ and $(\vec{b}_i)$ in $E$ and the transition matrix $P$ from $(\vec{a}_i)$ to $(\vec{b}_i)$. Thus $[\vec{x}]_{|\vec{b}} = P^{-1}.[\vec{x}]_{|\vec{a}}$ and $[g]_{|\vec{b}} = P^T.[g]_{|\vec{a}}.P$, and $\ell.\vec{x} = (\vec{\ell}_g, \vec{x})_g$ for all $\vec{x}$ gives

$$[\vec{x}]^T_{|\vec{a}}.[g]_{|\vec{a}}.[\vec{\ell}_g]_{|\vec{a}} = \ell.\vec{x} = [\vec{x}]^T_{|\vec{b}}.[g]_{|\vec{b}}.[\vec{\ell}_g]_{|\vec{b}} = ([\vec{x}]^T_{|\vec{a}}.P^{-T}).(P^T.[g]_{|\vec{a}}.P).[\vec{\ell}_g]_{|\vec{b}} = [\vec{x}]^T_{|\vec{a}}.[g]_{|\vec{a}}.(P.[\vec{\ell}_g]_{|\vec{b}}), \tag{F.18}$$

thus $[\vec{\ell}_g]_{|\vec{a}} = P.[\vec{\ell}_g]_{|\vec{b}}$ since $[g]$ is invertible (an inner dot product is positive definite). ▟

**Remark F.12** • Dont forget: A representation vector $\vec{\ell}_g$ is not intrinsic to the linear form $\ell$ because it depends on a $(\cdot,\cdot)_g$ (depends on a observer: foot? metre?).
• It is impossible to identify a linear form with a vector (which one?).
• $\vec{\ell}_g$ is ***not*** compatible with the use of push-forwards, cf. § 7.2.
• $\vec{\ell}_g$ is ***not*** compatible with the use of Lie derivatives, cf. (9.64).

## F.7 What is a vector versus a $(\cdot,\cdot)_g$-vector?

1. Originally, a vector is a bipoint vector $\vec{x} = \overrightarrow{AB}$ in $\mathbb{R}^3$ used to represent a "material object". E.g. the height of a child is represented on a wall by a vertical bipoint vector $\vec{x}$ starting from $A$ the ground up to $B$ a pencil line. The vector $\vec{x}$ is objective: The same vector for all observers; Then to get the height of the child an observer uses "its own unit" (foot, metre...) to give a value (subjective).
2. Then (mid 19th century), the concept of vector space was introduced: It is a quadruplet $(E,+,K,.)$ where $+$ is an inner law, $(E,+)$ is a group, $K$ is a field, . is a external law on $E$ (called a scalar multiplication) compatible with $+$ (see any math book).
3. Then a scalar inner dot product $(\cdot,\cdot)_g$ in a vector space $E$ was introduced.
4. We can then get non "material" vectors ("subjectively built vectors"). E.g., usual vector space $\vec{\mathbb{R}^3}$ of bi-point vectors, its dual $\mathbb{R}^{3*} := \mathcal{L}(\vec{\mathbb{R}^3};\mathbb{R})$, $\ell \in \mathbb{R}^{3*}$ (a measuring device), foot built Euclidean dot product $(\cdot,\cdot)_g$, metre built Euclidean dot product $(\cdot,\cdot)_h$. We get the artificial (man made) Riesz representation vectors $\vec{\ell}_g = \vec{R}_g(\ell)$ and $\vec{\ell}_h = \vec{R}_h(\ell)$, cf (F.12), and $\vec{\ell}_g \neq \vec{\ell}_h$.
5. Remark: with differential geometry, a vector $\vec{v}$ is redefined: It is a "tangent vector", which means that there exists a $C^1$ curve $c : s \in [a,b] \to c(s) \in E$ such that $\vec{v}$ is defined at a $p = c(s) \in \mathrm{Im}(c)$ by $\vec{v}(p) := \vec{c}\,'(s)$. Advantage: This definition of a tangent vector is applicable to "tangent vectors to a surface" (and to a manifold), see e.g. § 9.1.1,2-. Then it is shown that $\vec{v}$ is equivalent to $\frac{\partial}{\partial \vec{v}}$ = the directional derivative in the direction $\vec{v}$ (natural canonical isomorphism $E \simeq E^{**}$ see § U.3).

   For other equivalent definitions of vectors, see e.g. Abraham–Marsden [1].

## F.8 The "$(\cdot,\cdot)_g$-dual vectorial basis" of a basis (and warnings)

### F.8.1 A basis and its many associated "dual vectorial basis"

$E$ vector space, $\dim E = n$, inner dot product $(\cdot,\cdot)_g$ (e.g. Euclidean foot-built).

**Definition F.13** The $(\cdot,\cdot)_g$-dual vectorial basis $(\vec{e}_{ig})$ (or $(\cdot,\cdot)_g$-vectorial dual basis, or $(\cdot,\cdot)_g$-dual basis) of a basis $(\vec{e}_i)$ in $E$ is the (contravariant) basis in $E$ defined by

$$\forall j = 1,...,n, \quad (\vec{e}_{ig},\vec{e}_j)_g = \delta_{ij}, \quad \text{i.e.} \quad \vec{e}_{ig} \bullet_g \vec{e}_j = \delta_{ij}. \tag{F.19}$$

NB: A vectorial dual basis is not unique: It depends on the chosen inner dot product, see e.g. (F.21).
NB: $\vec{e}_{ig}$ is contravariant: $\vec{e}_{ig} \in E$. So with Einstein's convention the index $i$ in $\vec{e}_{ig}$ is a down index.

**Exercice F.14** Prove that the vectors $\vec{e}_{ig}$ satisfy the contravariant change of basis formula

$$[\vec{e}_{ig}]_{|new} = P^{-1}.[\vec{e}_{ig}]_{|old} \quad \text{(the } \vec{e}_{jg} \text{ are "contravariant vectors")}. \tag{F.20}$$

**Answer.** • First answer: $\vec{e}_{ig}$ is a vector in $E$, thus it is contravariant.
• Second answer = direct computation: Consider two bases $(\vec{a}_i)$ and $(\vec{b}_i)$ and the transition matrix $P$ from $(\vec{a}_i)$ to $(\vec{b}_i)$. (F.19) and the change of basis formulas give $[\vec{e}_j]^T_{|\vec{a}}.[g]_{|\vec{a}}.[\vec{e}_{ig}]_{|\vec{a}} = (\vec{e}_{ig},\vec{e}_j)_g = [\vec{e}_j]^T_{|\vec{b}}.[g]_{|\vec{b}}.[\vec{e}_{ig}]_{|\vec{b}} = (P^{-1}.[\vec{e}_j]_{|\vec{a}})^T.(P^T.[g]_{|\vec{a}}.P).[\vec{e}_{ig}]_{|\vec{b}} = [\vec{e}_j]^T_{|\vec{a}}.[g]_{|\vec{a}}.P.[\vec{e}_{ig}]_{|\vec{b}}$, for all $i,j$, thus $[\vec{e}_{ig}]_{|\vec{a}} = P.[\vec{e}_{ig}]_{|\vec{b}}$, for all $i$, i.e. (F.20).
• Third answer: Apply (F.17) since $\vec{e}_{ig}$ is the Riesz-representation vector of $e^i$, see (F.22).

**Exercice F.15** One basis $(\vec{e}_i)$ in $E$, two inner dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ (e.g., foot and metre built). Call $(\vec{e}_{ia})$ and $(\vec{e}_{ib})$ the $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$-dual vectorial bases of the basis $(\vec{e}_i)$. Prove:

$$(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b \implies \vec{e}_{ib} = \lambda^2\vec{e}_{ia}, \ \forall i. \tag{F.21}$$

E.g., $\lambda^2 > 10$ with foot and metre built Euclidean bases: $\vec{e}_{ib}$ is much bigger than $\vec{e}_{ia}$ : A vectorial dual basis is ***not*** intrinsic to $(\vec{e}_i)$ (***not*** objective).

**Answer.** (F.19) gives $(\vec{e}_{ib},\vec{e}_j)_b = \delta_{ij} = (\vec{e}_{ia},\vec{e}_j)_a = \lambda^2(\vec{e}_{ia},\vec{e}_j)_b$, thus $(\vec{e}_{ib} - \lambda^2\vec{e}_{ia},\vec{e}_j)_b = \delta_{ij}$, for all $i,j$.

**Remark F.16** If $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis we trivially get $\vec{e}_{ig} = \vec{e}_i$ for all $i$, i.e., $(\vec{e}_{ig}) = (\vec{e}_i)$.This particular case is not compatible with joint work by an English (foot) and a French (metre) observers. ∴

**Definition F.17 (Equivalent definition.)** Let $(e^i)$ in $E^*$ be the (covariant) dual basis of the basis $(\vec{e}_i)$ (the linear forms defined by $e^i.\vec{e}_j = \delta_{ij}$ for all $j$, cf. (A.6)). The $(\cdot,\cdot)_g$-dual vectorial basis of the basis $(\vec{e}_i)$ is the basis $(\vec{e}_{ig})$ in $E$ made of the $(\cdot,\cdot)_g$-Riesz representative vectors of the $e^i$:

$$\vec{e}_{ig} := \vec{R}_g(e^i), \quad \text{i.e. defined by} \quad (\vec{e}_{ig}, \vec{v})_g = e^i.\vec{v}, \ \ \forall \vec{v} \in E. \tag{F.22}$$

where $\vec{R}_g$ is the $(\cdot,\cdot)_g$-Riesz operator (change of variance operator cf. (F.3)).

### F.8.2 Components of $\vec{e}_{jg}$ in the basis $(\vec{e}_i)$

Basis $(\vec{e}_i)$, inner dot product $(\cdot,\cdot)_g$, $[g] = [g_{ij}] := [(\vec{e}_i, \vec{e}_j)_g]$, $[g]^{-1} =^{\text{written}} [g^{ij}]$.

**Proposition F.18** *The transition matrix from $(\vec{e}_i)$ to $(\vec{e}_{ig})$ is $P = [g]^{-1}$, i.e. the components of $\vec{e}_{jg}$ are ${P^i}_j = g^{ij}$: for any $j \in [1,n]_{\mathbb{N}}$,*

$$\vec{e}_{jg} = \sum_{i=1}^n g^{ij} \vec{e}_i, \quad \text{i.e.} \quad \vec{e}_{jg} = \sum_{i=1}^n {P^i}_j \vec{e}_i \ \text{ where } \ {P^i}_j = g^{ij}, \quad \text{i.e.} \quad [\vec{e}_{jg}]_{|\vec{e}} = [g]_{|\vec{e}}^{-1}.[\vec{e}_j]_{|\vec{e}}. \tag{F.23}$$

*(Einstein's convention is not satisfied because $\vec{R}_g$ is a change of variance operator.)*

*Use classic notations if you prefer: $\vec{e}_{jg} = \sum_i P_{ij} \vec{e}_i$ where $[P_{ij}] = [g_{ij}]^{-1}$.*

*And the matrix of $g(\cdot,\cdot)$ in the basis $(\vec{e}_{ig})$ is the inverse of the matrix of $g(\cdot,\cdot)$ in the basis $(\vec{e}_i)$:*

$$[g(\vec{e}_{ig}, \vec{e}_{jg})] = \quad [g]_{|\vec{e}_{ig}} = {[g]_{|\vec{e}_i}}^{-1} \quad = ([g(\vec{e}_i, \vec{e}_j)])^{-1}. \tag{F.24}$$

**Proof.** (F.19) gives

$$\forall i,j, \ \ [\vec{e}_j]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{e}_{ig}]_{|\vec{e}} = \delta_{ij} = [\vec{e}_j]_{|\vec{e}}^T.[\vec{e}_i]_{|\vec{e}}, \quad \text{thus} \quad [g]_{|\vec{e}}.[\vec{e}_{ig}]_{|\vec{e}} = [\vec{e}_i]_{|\vec{e}}, \ \ \forall i, \tag{F.25}$$

thus (F.23). (Or apply (F.7) = generic Riesz representation result.)

Then, $[g]_{|\vec{e}}$ being symmetric, $g(\vec{e}_{ig}, \vec{e}_{jg}) = [\vec{e}_{ig}]_{|\vec{e}}^T.[g]_{|\vec{e}}.[\vec{e}_{jg}]_{|\vec{e}} = [\vec{e}_i]_{|\vec{e}}^T.{[g]_{|\vec{e}}}^{-1}.[g]_{|\vec{e}}.{[g]_{|\vec{e}}}^{-1}.[\vec{e}_j]_{|\vec{e}} = [\vec{e}_i]_{|\vec{e}}^T.{[g]_{|\vec{e}}}^{-1}.[\vec{e}_j]_{|\vec{e}} = ({[g]_{|\vec{e}}}^{-1})_{ij}$, thus (F.24). ∴

**Example F.19** $\vec{\mathbb{R}^2}$, $[g]_{|\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, thus $[g]_{|\vec{e}}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. Thus $\vec{e}_{1g} = \vec{e}_1$ and $\vec{e}_{2g} = \frac{1}{2}\vec{e}_2$. ∴

**Remark F.20** $M = [g]_{|\vec{e}} = [M_{ij}]$ is a matrix, and its inverse is the matrix $M^{-1} = [M_{ij}]^{-1} = |N_{ij}]$: A matrix is just a collection of scalars, it is not tensorial (has nothing to do with the Einstein convention), and its inverse is also a collection of scalars, and you don't change this fact by calling $M^{-1} = [M^{ij}]$.

And because ${P^i}_j$ equals $([g]_{|\vec{e}}^{-1})_{ij} =^{\text{written}} g^{ij}$, some people rename $\vec{e}_{jg}$ as $\vec{e}^j$... to get $\vec{e}^j = \sum_{i=1}^n g^{ij} \vec{e}_i$... to have the illusion to satisfy Einstein's convention, which is false: They confuse covariance and contravariance... and add confusion to the confusion... ∴

### F.8.3 Multiple admissible notations for the components of $\vec{e}_{jg}$

Let $\mathcal{P} \in \mathcal{L}(E;E)$ be the change of basis endomorphism from $(\vec{e}_i)$ to $(\vec{e}_{ig})$,i.e. $\mathcal{P}.\vec{e}_j = \vec{e}_{jg}$ for all $j$. And let $P = [\mathcal{P}]_{|\vec{e}} =$ the transition matrix from $(\vec{e}_i)$ to $(\vec{e}_{ig})$. We have multiple admissible notations

$$\vec{e}_{jg} = \mathcal{P}.\vec{e}_j = \sum_{j=1}^n P_{ij} \vec{e}_i = \sum_{j=1}^n (P_j)_i \vec{e}_i = \sum_{j=1}^n (P_j)^i \vec{e}_i = \sum_{j=1}^n {P^i}_j \vec{e}_i, \tag{F.26}$$

i.e. the $i$-th component of the vector $\vec{e}_{jg}$ has the names $P_{ij} = (P_j)_i = (P_j)^i = {P^i}_j$ or $P^i_j$, i.e. $P = [\mathcal{P}]_{|\vec{e}} = [P_{ij}] = [(P_j)_i] = [(P_j)^i] = [{P^i}_j]$ (four different notations for the same matrix), i.e.

$$\forall j, \quad [\vec{e}_{jg}]_{|\vec{e}} = P.[\vec{e}_j]_{|\vec{e}} = \begin{pmatrix} P_{1j} \\ \vdots \\ P_{nj} \end{pmatrix} = \begin{pmatrix} (P_j)_1 \\ \vdots \\ (P_j)_n \end{pmatrix} = \begin{pmatrix} {P^1}_j \\ \vdots \\ {P^n}_j \end{pmatrix} = \begin{pmatrix} (P_j)^1 \\ \vdots \\ (P_j)^n \end{pmatrix} \tag{F.27}$$

$=$ the $j$-th column of $P$. You can choose any notation, depending on your current need or mood...

### F.8.4 (Huge) differences between "the (covariant) dual basis" and "a dual vectorial basis"

1. A basis $(\vec{e}_i)$ has an infinite number of vectorial dual bases $(\vec{e}_{ig})$, as many as the number of inner dot products $(\cdot,\cdot)_g$ (observer dependents), see (F.23).
2. While a basis $(\vec{e}_i)$ has a unique intrinsic (covariant) dual basis $(\pi_{ei}) = (e^i)$, cf. (A.6): Two observers who consider the same basis $(\vec{e}_i)$ have the same (covariant) dual basis.
3. If you fly, it is vital to use the dual basis $(\pi_{ei}) = (e^i)$: It is possibly fatal if you confuse foot and metre at takeoff and at landing (if you survived takeoff...).
4. Einstein's convention can help... only if it is properly applied.

### F.8.5 About the notation $g^{ij}$ = shorthand notation for $(g^\sharp)^{ij}$

**Definition F.21** $g(\cdot,\cdot) = (\cdot,\cdot)_g$ being an inner dot product in $E$, the Riesz associated inner dot product $g^\sharp(\cdot,\cdot) = (\cdot,\cdot)_{g^\sharp}$ in $E^*$ is the bilinear form in $\mathcal{L}(E^*,E^*;\mathbb{R})$ defined by, for all $\ell, m \in E^*$,

$$(\ell,m)_{g^\sharp} := (\vec{\ell}_g, \vec{m}_g)_g \quad \text{when} \quad \vec{\ell}_g = \vec{R}_g(\ell) \text{ and } \vec{m}_g = \vec{R}_g(m). \tag{F.28}$$

($g^\sharp(\cdot,\cdot)$ is indeed an inner dot product in $E^*$: easy check.)

So the $\binom{2}{0}$ tensor $g^\sharp$ is created from the $\binom{0}{2}$ tensor $g$ using twice the $(\cdot,\cdot)_g$-Riesz representation theorem.

**Quantification:** Basis $(\vec{e}_i)$ in $E$, covariant dual basis $(e^i)$ in $E^*$ (duality notations). (F.28) gives:

$$(g^\sharp)^{ij} := g^\sharp(e^i,e^j) \overset{(F.28)}{=} g(\vec{e}_{ig},\vec{e}_{jg}), \quad \text{thus} \quad [g^\sharp]_{|e} \overset{(F.23)}{=} = [g]_{|\vec{e}}^{-1}, \quad \text{i.e.} \quad \boxed{[(g^\sharp)^{ij}] = [g_{ij}]^{-1}}, \tag{F.29}$$

$$\text{shorthand notation: } \boxed{[(g^\sharp)^{ij}] \overset{\text{written}}{=} [g^{ij}]}. \tag{F.30}$$

Classical notations: $[g^\sharp]_{|e} = [(g^\sharp)_{ij}] = [g^\sharp(\pi_{ei},\pi_{ej})] = [g(\vec{e}_{ig},\vec{e}_{jg})] = [g_{ij}]^{-1} = ([g]_{|\vec{e}})^{-1}$.

**Exercice F.22** How do we compute $g^\sharp(\ell,m)$ with matrix computations?

**Answer**. $\ell = \sum_{i=1}^n \ell_i e^i$ and $m = \sum_{j=1}^n m_j e^j$ give $g^\sharp(\ell,m) = \sum_{i,j=1}^n \ell_i m_j g^\sharp(e^i,e^j) = \sum_{i,j=1}^n \ell_i (g^\sharp)^{ij} m_j = [\ell]_{|\vec{e}}.[g^\sharp]_{|\vec{e}}.[m]_{|\vec{e}}^T = [\ell]_{|\vec{e}}.[g]_{|\vec{e}}^{-1}.[m]_{|\vec{e}}^T$ (a linear form is represented by a row matrix,). ∎

**Exercice F.23** Purpose: Prove $I \simeq g^\natural$ and $(g^\natural)^\flat = g$ and $(g^\natural)^\sharp = g^\sharp$.

1- Start with the $\binom{0}{2}$ tensor $g$, use the $(\cdot,\cdot)_g$-Riesz representation theorem just once: Prove that you get the $\binom{1}{1}$ tensor $g^\natural \in \mathcal{L}(E^*,E;\mathbb{R}) \simeq \mathcal{L}(E;E)$ which is the identity endomorphism:

$$g^\natural \simeq I. \tag{F.31}$$

2- Show that if you start with the $\binom{1}{1}$ tensor $g^\natural$ and you apply the $(\cdot,\cdot)_g$-Riesz representation theorem once then you get the $\binom{2}{0}$ tensor $g^\sharp$.

**Answer**. 1- $g^\natural \in \mathcal{L}(E^*,E;\mathbb{R})$ is defined by $g^\natural(\ell,\vec{w}) = (\vec{\ell}_g,\vec{w})_g$ for all $(\ell,\vec{w}) \in E^* \times E$, where $\vec{\ell}_g$ is the $(\cdot,\cdot)_g$-Riesz representation vector of $\ell$. Thus $g^\natural(\ell,\vec{w}) = \ell.\vec{w} = \ell.I.\vec{w}$, for all $(\ell,\vec{w}) \in E^* \times E$, hence $g^\natural \in \mathcal{L}(E^*,E;\mathbb{R})$ is naturally canonically associated with the identity $I \in \mathcal{L}(E;E)$.

2- $g^\natural(\ell,\vec{w}) = \ell.\vec{w} = (\vec{\ell}_g,\vec{w})_g = (\ell,w_g)_{g^\sharp}$ where $\vec{w}_g = (\vec{R}_g)^{-1}.\vec{w}$. ∎

# G Cauchy–Green deformation tensor $C = F^T.F$

$\widetilde{\Phi} : \left\{ \begin{array}{ll} [t_0,T] \times Obj & \to \mathbb{R}^n \\ (t,P_{Obj}) & \to \widetilde{\Phi}(t,P_{Obj}) \end{array} \right\}$ is a motion of $Obj$, $\Omega_t = \widetilde{\Phi}(,P_{Obj})$ is the configuration of $Obj$ at $t$, $t_0$ is fixed, $\Phi^{t_0}(t,p_{t_0}) := \widetilde{\Phi}(t,P_{Obj})$ when $p_{t_0} = \widetilde{\Phi}(t_0,p_{t_0})$, $\Phi_t^{t_0}(p_{t_0}) := \Phi^{t_0}(t,p_{t_0})$. When $t$ is fixed, $\Phi := \Phi_t^{t_0}$ : $\left\{ \begin{array}{ll} \Omega_{t_0} & \to \Omega_t \\ p_{t_0} & \to p_t = \Phi(p_{t_0}) \end{array} \right\}$, and $F(p_{t_0}) := d\Phi(p_{t_0})$ : $\left\{ \begin{array}{ll} \vec{\mathbb{R}}^n_{t_0} & \to \vec{\mathbb{R}}^n_t \\ \vec{W} & \to \vec{w} = F(p_{t_0}).\vec{W} := \lim\limits_{h\to 0} \dfrac{\Phi(p_{t_0}+h\vec{W}) - \Phi(p_{t_0})}{h} \end{array} \right\}$ (deformation gradient at $p_{t_0}$ between $t_0$ and $t$).

## G.0 Summary

**Construction of $C$ (summary of Cauchy's approach):** $t$ and $t_0$ are fixed and

1- At $t_0$, consider two vectors $\vec{W}_1$ and $\vec{W}_2$ at a point $P \in \Omega_{t_0}$.

2- At $t$, they have been distorted by the motion to become the vectors $F.\vec{W}_1$ and $F.\vec{W}_2$ at $p = \Phi(P)$.

3- Then choose a Euclidean dot product $(\cdot,\cdot)_g =^{\text{written}} \cdot \bullet \cdot$, the same at all $t$.

4- Then, by definition of the transposed, $(F.\vec{W}_1) \bullet (F.\vec{W}_2) = (F^T.F.\vec{W}_1) \bullet \vec{W}_2$: You have got the Cauchy strain tensor $C := F^T.F$; We have $(F.\vec{W}_1) \bullet (F.\vec{W}_2) = (C.\vec{W}_1) \bullet \vec{W}_2$.

5- Then $(F.\vec{W}_1) \bullet (F.\vec{W}_2) - \vec{W}_1 \bullet \vec{W}_2 = ((C-I).\vec{W}_1) \bullet \vec{W}_2$ gives a measure of the deformation relative to $\vec{W}_1$ and $\vec{W}_2$, value used to build a first order constitutive law for Cauchy's stress.

## G.1 Transposed $F^T$: Inner dot products required

### G.1.1 Definition of the function $F^T$

$t_0$ and $t$ are fixed, $t_0 < t$, $\Phi_t^{t_0} =^{\text{written}} \Phi$, $d\Phi_t^{t_0} = F_t^{t_0} = d\Phi =^{\text{written}} F : \left\{ \begin{array}{ll} \Omega_{t_0} & \to \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \\ P & \to F(P) \end{array} \right\}$. At $t_0$, a past observer chose an inner dot product $(\cdot,\cdot)_G$ in $\vec{\mathbb{R}}^n_{t_0}$, and at $t$ a present observer chooses an inner dot product $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$. With $P \in \Omega_{t_0}$ and $p = \Phi(P) \in \Omega_t$, the transposed of the linear map $F(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ relative to $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ is the linear map $F(P)^T_{Gg} \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0})$ defined by, for all $\vec{U}_P \in \vec{\mathbb{R}}^n_{t_0}$ vector at $P$ and $\vec{w}_p \in \vec{\mathbb{R}}^n_t$ vector at $p$:

$$(F(P)^T_{Gg}.\vec{w}_p, \vec{U}_P)_G = (F(P).\vec{U}_P, \vec{w}_p)_g, \tag{G.1}$$

when $t_0$, $t$, $P$ are implicit. Full notation: $(F_t^{t_0}(P)^T_{Gg}.\vec{w}_p, \vec{U}_P)_G = (F_t^{t_0}(P).\vec{U}_P, \vec{w}_p)_g$. This defines

$$(F_t^{t_0})^T_{Gg} \overset{\text{written}}{=} F^T_{Gg} : \left\{ \begin{array}{ll} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0}) \\ p & \to \boxed{F^T_{Gg}(p) := F(P)^T_{Gg}} \quad \text{where} \quad P = \Phi^{-1}(p), \end{array} \right. \tag{G.2}$$

so

$$(F^T_{Gg}(p).\vec{w}_p, \vec{U}_P)_G = (F(P).\vec{U}_P, \vec{w}_p)_g, \quad \text{written in short} \quad \boxed{(F^T.\vec{w}) \bullet_G \vec{U} = \vec{w} \bullet_g (F.\vec{U})} \tag{G.3}$$

**Exercice G.1** 1. With the ambiguous notation $F^T.\vec{z}.\vec{W} = \vec{z}.F.\vec{W} = F.\vec{W}.\vec{z} = \vec{W}.F^T.\vec{z}$, which dots are inner dot products?

2. With ambiguous notations, what does $F.\vec{W}_1.F.\vec{W}_2 = \vec{W}_1.F^T.F.\vec{W}_2$ mean?

**Answer.** 1. No choice: $(\vec{W}, \vec{z}) \in \vec{\mathbb{R}}^n_{t_0} \times \vec{\mathbb{R}}^n_t$ and meaning $(F^T.\vec{z}) \bullet_G \vec{W} = \vec{z} \bullet_g (F.\vec{W}) = (F.\vec{W}) \bullet_g \vec{z} = \vec{W} \bullet_G (F^T.\vec{z})$.

2. No choice: $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$ and meaning $(F.\vec{W}_1) \bullet_g (F.\vec{W}_2) = \vec{W}_1 \bullet_G (F^T.F.\vec{W}_2)$. ∎

**Remark G.2** On a surface $\Omega$ (a manifold), (G.1) is defined for all $(\vec{U}_P, \vec{w}_p) \in T_P\Omega_{t_0} \times T_p\Omega_t$. ∎

### G.1.2 Quantification with bases (matrix representation)

Classical notations: $(\vec{a}_i)$ is a basis in $\vec{\mathbb{R}}^n_{t_0}$, and $(\vec{b}_i)$ is a basis in $\vec{\mathbb{R}}^n_t$. Marsden–Hughes notations: $(\vec{E}_I)$ is a basis in $\vec{\mathbb{R}}^n_{t_0}$ and $(\vec{e}_i)$ is a basis in $\vec{\mathbb{R}}^n_t$. Let (lighten notations)

$$G_{ij} = (\vec{a}_i, \vec{a}_j)_G, \; g_{ij} = (\vec{b}_i, \vec{b}_j)_g, \; F.\vec{a}_j = \sum_{i=1}^n F_{ij}\vec{b}_i, \; F^T.\vec{b}_j = \sum_{i=1}^n (F^T)_{ij}\vec{a}_j, \tag{G.4}$$

so $[G] := [G]_{|\vec{a}} = [G_{ij}]$, $[g] := [g]_{|\vec{b}} = [g_{ij}]$, $[F] := [F]_{|\vec{a},\vec{b}} = [F_{ij}]$, $[F^T] := [F^T]_{|\vec{b},\vec{a}} = [(F^T)_{ij}]$.

(G.1) gives $[\vec{U}]^T.[G].[F^T.\vec{w}] = [F.\vec{U}]^T.[g].[\vec{w}]$ for all $\vec{U}, \vec{w}$, thus

$$[G].[F^T] = [F]^T.[g], \quad \text{i.e.} \quad \boxed{[F^T] = [G]^{-1}.[F]^T.[g]}, \tag{G.5}$$

i.e.

$$\sum_{k=1}^n G_{ik}(F^T)_{kj} = \sum_{k=1}^n F_{ki}g_{kj}, \quad \text{i.e.} \quad (F^T)_{ij} = \sum_{k,\ell=1}^n ([G]^{-1})_{ik}F_{\ell k}g_{\ell j} \tag{G.6}$$

Duality notations: $G_{IJ} = G(\vec{E}_I, \vec{E}_j)$, $g_{ij} = g(\vec{e}_i, \vec{e}_j)$, $F.\vec{E}_J = \sum_{i=1}^n F^i{}_J\vec{e}_i$, $F^T.\vec{e}_j = \sum_{I=1}^n (F^T)^I{}_j\vec{E}_I$ and

$$\sum_{K=1}^n G_{IK}(F^T)^K{}_j = \sum_{k=1}^n F^k{}_I g_{kj}, \quad \text{i.e.} \quad (F^T)^I{}_j = \sum_{K,k=1}^n G^{IK} F^k{}_K g_{kj} \quad \text{where} \quad [G^{IJ}] := [G_{IJ}]^{-1}.$$

**Remark G.3** If $(\vec{a}_i)$ and $(\vec{b}_i)$ are $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$-orthonormal bases, then $[G] = I = [g]$, thus $[C] = [F]^T.[F]$. But recall: If you work with coordinate systems then the coordinate system bases are ***not*** orthonormal in general, i.e. $[G]^{-1} \neq I$ and/or $[g]^{-1} \neq I$ in general. ▪

**Exercice G.4** Detail the obtaining of (G.6).

**Answer**. $(F^T.\vec{b}_j, \vec{a}_i)_G = (\vec{b}_j, F.\vec{a}_i)_g$ gives $(\sum_{k=1}^n (F^T)_{kj}\vec{a}_k, \vec{a}_i)_G = (\vec{b}_j, \sum_{k=1}^n F_{ki}\vec{b}_k)_g$, thus $\sum_{k=1}^n (F^T)_{kj}(\vec{a}_k, \vec{a}_i)_G = \sum_{k=1}^n F_{ki}(\vec{b}_j, \vec{b}_k)_g$,thus $\sum_{k=1}^n (F^T)_{kj} G_{ki} = \sum_{k=1}^n F_{ki} g_{jk}$, thus (G.6). ▪

### G.1.3 Remark: Usual classical mechanics isometric framework

We can choose a unique Euclidean basis $(\vec{a}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$ at all time, so $(\vec{b}_i) = (\vec{a}_i) \in \vec{\mathbb{R}}^n_t$, and $(\cdot,\cdot)_G = (\cdot,\cdot)_g$ is the associated Euclidean dot product; Thus $[G]_{|\vec{a}} = I = [g]_{|\vec{a}}$, and $F^T_{Gg} \stackrel{\text{written}}{=} F^T$ and thus $[F^T]_{|\vec{a}} = ([F]_{|\vec{a}})^T$, written

$$[F^T] = [F]^T : \text{ usual classical mechanics isometric framework.} \tag{G.7}$$

### G.1.4 Remark: $F^*$

For mathematicians (no "magic tricks"):

**Definition G.5** The adjoint of the linear map $F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ (acting on vectors) is the linear map $F^* \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t; \vec{\mathbb{R}}^{n*}_{t_0})$ (acting on functions) canonically defined by,

$$\forall m \in \vec{\mathbb{R}}^{n*}_t, \quad F^*(m) := m \circ F, \quad \text{written} \quad F^*.m = m.F \ \ (\in \vec{\mathbb{R}}^{n*}_{t_0}) \tag{G.8}$$

because $F^*$ is linear. I.e. $F^*$ is characterized by, for all $(m, \vec{W}) \in \vec{\mathbb{R}}^{n*}_t \times \vec{\mathbb{R}}^n_{t_0}$,

$$(F^*.m).\vec{W} = m.F.\vec{W} \ \ (\in \mathbb{R}). \tag{G.9}$$

NB: There is no inner dot product, no basis here: This is an objective definition.

**Quantification** = matrix representation. With Marsden notations: $(E^i)$ and $(e^i)$ are the (covariant) dual bases of $(\vec{E}_i)$ and $(\vec{e}_i)$, and $F^i{}_J$ and $(F^*)_I{}^j$ are the components of $F$ and $F^*$ relative to the chosen bases: So

$$\begin{aligned} &F.\vec{E}_J = \sum_{i=1}^n F^i{}_J \vec{e}_i, \quad \text{i.e.} \quad [F] := [F]_{\vec{E},\vec{e}} = [F^i{}_J]_{\substack{i=1,\dots,n \\ J=1,\dots,n}} \stackrel{\text{written}}{=} [F^i{}_J], \text{ and} \\ &F^*.e^j = \sum_{I=1}^n (F^*)_I{}^j E^I, \quad \text{i.e.} \quad [F^*] := [F^*]_{|e,E} = [(F^*)_I{}^j]_{\substack{I=1,\dots,n \\ j=1,\dots,n}} \stackrel{\text{written}}{=} [(F^*)_I{}^j]. \end{aligned} \tag{G.10}$$

And (G.9) gives $(F^*.e^j).\vec{E}_I = e^j.F.\vec{E}_I$, thus

$$\forall i, j, \ (F^*)_I{}^j = F^j{}_I, \quad \text{i.e.} \quad [F^*] = [F]^T. \tag{G.11}$$

Classic notations: $F.\vec{a}_j = \sum_i F_{ij}\vec{b}_i$, $[F] = [F_{ij}$, $F^*.\pi_{bj} = \sum_{I=1}^n (F^*)_{ij}\pi_{ai}$, $[F^*] = [(F^*)_{ij}]$, and

$$(F^*.\pi_{bj}).\vec{a}_i = \pi_{aj}.F.\vec{a}_i \quad \text{gives} \quad (F^*)_{ij} = F_{ji}. \tag{G.12}$$

NB: There is no inner dot product.

**Interpretation of $F^*$ in classical mechanics:** We introduce Euclidean dot products, $(\cdot,\cdot)_G$ in $\vec{\mathbb{R}}^n_{t_0}$ and $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$. Then we use the $(\cdot,\cdot)_G$-Riesz representation vector $\vec{R}_G(F^*.m) \in \vec{\mathbb{R}}^n_{t_0}$ of $F^*.m \in \vec{\mathbb{R}}^{n*}_{t_0}$, and the $(\cdot,\cdot)_g$-Riesz representation vector $\vec{R}_g(m) \in \vec{\mathbb{R}}^n_t$ of $m \in \vec{\mathbb{R}}^{n*}_t$. Thus (G.9) and (F.3) give $(\vec{R}_G(F^*.m), \vec{W})_G = (\vec{R}_g(m), F.\vec{W})_g = (F^T.\vec{R}_g(m), \vec{W})_G$, thus $\vec{R}_G(F^*.m) = F^T.\vec{R}_g(m)$, written

$$\vec{R}_G.F^* = F^T.\vec{R}_g, \quad \text{i.e.} \quad F^* = \vec{R}_G{}^{-1}.F^T.\vec{R}_g. \tag{G.13}$$

NB: The definition of $F^*$ is intrinsic to $F$ (objective), while the definition of $F^T$ is ***not*** intrinsic to $F$ (***not*** objective) because its definition requires inner dot products (observers choices).

## G.2 Cauchy–Green deformation tensor $C$

### G.2.1 Definition of $C$

$t_0$ and $t$ are fixed, $\Phi := \Phi_t^{t_0}$, $F := F_t^{t_0}$, $P \in \Omega_{t_0}$ and $p = \Phi(P) \in \Omega_t$, $i = 1, 2$. Consider the $\vec{W}_i(P) \in \vec{\mathbb{R}}^n_{t_0}$ vectors at $P$ and their push forwards at $p$:

$$\vec{w}_i(p) = F(P).\vec{W}_i(P) \in \vec{\mathbb{R}}^n_t, \quad \text{written} \quad \vec{w}_i = F.\vec{W}_i. \tag{G.14}$$

Choose inner dot products $(\cdot,\cdot)_G$ in $\vec{\mathbb{R}}^n_{t_0}$ and $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_t$. Thus

$$(\vec{w}_1, \vec{w}_2)_g = (F.\vec{W}_1, F.\vec{W}_2)_g = (\underbrace{F^T.F}_{C}.\vec{W}_1, \vec{W}_2)_G. \tag{G.15}$$

More precisely: $(\vec{w}_{1p}, \vec{w}_{2p})_g = (F(P).\vec{W}_{1P}, F(P).\vec{W}_{2P})_g = (F^T_{Gg}(p).F(P).\vec{W}_{1P}, \vec{W}_{2P})_G$.

**Definition G.6** The (right) Cauchy–Green deformation tensor between $t_0$ and $t$ at $P \in \Omega_{t_0}$ relative to $(\cdot,\cdot)_G$, $(\cdot,\cdot)_g$ is the endomorphism $C^{t_0}_{t,Gg}(P) =^{\text{written}} C_{Gg}(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ defined by

$$C_{Gg}(P) := F^T_{Gg}(p) \circ F(P), \quad \text{written} \quad \boxed{C = F^T \circ F = F^T.F}, \tag{G.16}$$

the last notation because $F^T$ is linear.

So

$$C = F^T \circ F = F^T.F : \begin{cases} \vec{\mathbb{R}}^n_{t_0} \to \quad \vec{\mathbb{R}}^n_t \quad \to \vec{\mathbb{R}}^n_{t_0} \\ \vec{W} \to F(\vec{W}) \to F^T(F(\vec{W})) = C(\vec{W}), \end{cases} \tag{G.17}$$

and (G.15) tells that, for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$,

$$\vec{w}_1 \bullet_g \vec{w}_2 = (C.\vec{W}_1) \bullet_G \vec{W}_2 = (F.\vec{W}_1) \bullet_g (F.\vec{W}_2). \tag{G.18}$$

Moreover $C$ is a $(\cdot,\cdot)_G$-symmetric endomorphism in $\vec{\mathbb{R}}^n_{t_0}$, i.e., for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$,

$$(C.\vec{W}_1, \vec{W}_2)_G = (\vec{W}_1, C.\vec{W}_2)_G, \quad \text{i.e.} \quad (C.\vec{W}_1) \bullet_G \vec{W}_2 = \vec{W}_1 \bullet_G (C.\vec{W}_2), \tag{G.19}$$

since $(F^T.F.\vec{W}_1, \vec{W}_2)_G = (F.\vec{W}_1, F.\vec{W}_2)_g = (\vec{W}_1, F^T.F.\vec{W}_2)_G$ and $(\cdot,\cdot)_G$ is symmetric.

### G.2.2 Quantification

With $(\vec{a}_i)$ and $(\vec{b}_i)$ bases in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$, $[C] =^{(G.16)} [F^T].[F]$, with $[F^T] =^{(G.5)} [G]^{-1}.[F]^T.[g]$, thus

$$\boxed{[C] = [G]^{-1}.[F]^T.[g].[F]} \quad (= [F^T].[F]), \tag{G.20}$$

short notation for $[C_{Gg}]_{|\vec{a}} = [G]^{-1}_{|\vec{a}}.([F]_{|\vec{a},\vec{b}})^T.[g]_{|\vec{b}}.[F]_{|\vec{a},\vec{b}}$.

**Exercice G.7** Use classical notation, then duality notations, to express (G.20) with components.

**Answer**. Classical notations: $F.\vec{a}_j = \sum_{i=1}^n F_{ij}\vec{b}_i$, $C.\vec{a}_j = \sum_{i=1}^n C_{ij}\vec{a}_i$, $[F]_{|\vec{a},\vec{b}} = [F_{ij}]$, and $[C]_{|\vec{a}} = [C_{ij}]$ give $\sum_k C_{kj}(\vec{a}_i, \vec{a}_k)_G = (\vec{a}_i, \sum_k C_{kj}\vec{a}_k)_G = (\vec{a}_i, C.\vec{a}_j)_G = (F.\vec{a}_i, F.\vec{a}_j)_g = (\sum_k F_{ki}\vec{b}_k, \sum_\ell F_{\ell j}\vec{b}_\ell)_g = \sum_{k\ell} F_{ki}(\vec{b}_k, \vec{b}_\ell)_g F_{\ell j}$, thus

$$\sum_{k=1}^n G_{ik}C_{kj} = \sum_{k,\ell=1}^n F_{ki}\, g_{k\ell} F_{\ell j} = \sum_{k,\ell=1}^n ([F]^T)_{ik}\, g_{k\ell} F_{\ell j}, \quad \text{thus} \quad C_{ij} = \sum_{k,\ell,m=1}^n ([G]^{-1})_{im} F_{km}\, g_{k\ell} F_{\ell j}. \tag{G.21}$$

Marsden duality notations: $F.\vec{E}_J = \sum_{i=1}^n F^i{}_J \vec{e}_i$, $C.\vec{E}_J = \sum_{I=1}^n C^I{}_J \vec{E}_I$, $[F]_{|\vec{E},\vec{e}} = [F^i{}_J]$, $[C]_{|\vec{E}} = [C^I{}_J]$ and

$$\sum_{K=1}^n G_{IK}C^K{}_J = \sum_{k,\ell=1}^n F^k{}_I\, g_{k\ell} F^\ell{}_J, \quad \text{and} \quad C^I{}_J = \sum_{k,\ell,M=1}^n G^{IM} F^k{}_M\, g_{k\ell} F^\ell{}_J \quad \text{when} \quad [G^{IJ}] := [G_{IJ}]^{-1}. \tag{G.22}$$

Matrix equalities: $[G].[C] = [F]^T.[g].[F]$ and $[C] = [G]^{-1}.[F]^T.[g].[F]$. ∎

**Exercice G.8** $(\cdot,\cdot)_G$ is a Euclidean dot product in foot, $(\cdot,\cdot)_g$ is a Euclidean dot product in metre, so $(\cdot,\cdot)_g = \mu^2(\cdot,\cdot)_G$ with $\mu = 0.3048$; And $(\vec{a}_i)$ is a $(\cdot,\cdot)_G$-orthonormal basis, and $(\vec{b}_i) := (\vec{a}_i)$. Prove:

$$[C] = \mu^2 [F]^T.[F]. \tag{G.23}$$

**Answer**. $[C]_{|\vec{a}} =^{(G.20)} [G]^{-1}_{|\vec{a}}.[F]^T_{|\vec{a},\vec{a}}.[g]_{|\vec{a}}.[F]_{|\vec{a},\vec{a}}$ gives $[C]_{|\vec{a}} = I.[F]^T_{|\vec{a},\vec{a}}.\mu^2 I.[F]_{|\vec{a},\vec{a}}$. Shorten notation = (G.23). ∎

## G.3 Time Taylor expansion of $C$

A time Taylor expansion implicitly imposes "along a trajectory of a fixed particle".

So let $P$ be fixed, $F^{t_0}(t,P) := F^{t_0}_P(t) =^{\text{written}} F(t)$, and $C(t) = F^T(t).F(t)$.

And we use a unique Euclidean basis $(\vec{a}_i)$ and the associate Euclidean dot product $(\cdot,\cdot)_G = (\cdot,\cdot)_g$ at all time. So $[C(t)] = [F(t)]^T.[F(t)] = [C](t) = [F]^T(t).[F](t)$.

And $\vec{V}^{t_0}(t,P) =^{\text{written}} \vec{V}(t)$ and $\vec{A}^{t_0}(t,P) =^{\text{written}} \vec{A}(t)$ (Lagrangian velocities and accelerations).

And $\Phi(t{+}h,P) = \Phi(t,P) + h\,\vec{V}(t,P) + \frac{h^2}{2}\,\vec{A}(t,P) + o(h^2)$ gives $F(t{+}h,P) = F(t,P) + h\,d\vec{V}(t,P) + \frac{h^2}{2}\,d\vec{A}(t,P) + o(h^2)$, written $F(t{+}h) = F(t) + h\,d\vec{V}(t) + \frac{h^2}{2}\,d\vec{A}(t) + o(h^2)$, thus

$$\begin{aligned}
[C(t{+}h)] &= [F(t{+}h)]^T.[F(t{+}h)] = [F]^T(t{+}h).[F(t{+}h)] \\
&= \Big([F]^T + h\,d[\vec{V}]^T + \frac{h^2}{2}\,d[\vec{A}]^T + o(h^2)]\Big)\Big([F + h\,d[\vec{V}] + \frac{h^2}{2}\,d[\vec{A}] + o(h^2)]\Big)(t) \\
&= \Big([C] + h\,([F^T].[d\vec{V}]{+}[d\vec{V}]^T.[F]) + \frac{h^2}{2}\,([F]^T.[d\vec{A}]{+}2[d\vec{V}]^T.[d\vec{V}]{+}[d\vec{A}]^T.[F]) + o(h^2)\Big)(t).
\end{aligned}$$

Together with $[C(t{+}h)] = [C(t)] + h\,[C'(t)] + \frac{h^2}{2}\,[C''(t)] + o(h^2)$ we get

$$[C'] = [F^T].[d\vec{V}]{+}[d\vec{V}]^T.[F] \quad\text{and}\quad [C''] = [F]^T.[d\vec{A}]{+}2[d\vec{V}]^T.[d\vec{V}]{+}[d\vec{A}]^T.[F]. \tag{G.24}$$

In particular $[C'(t_0)] = [d\vec{V}(t_0)]{+}[d\vec{V}(t_0)]^T$ and $[C''(t_0)] = [d\vec{A}(t_0)]{+}2[d\vec{V}(t_0)]^T.[d\vec{V}(t_0)]{+}[d\vec{A}(t_0)]^T$, thus

$$[C(t_0{+}h)] = I + h\,([d\vec{V}]{+}[d\vec{V}]^T)(t_0) + \frac{h^2}{2}\,([d\vec{A}]{+}2[d\vec{V}]^T.[d\vec{V}]{+}[d\vec{A}]^T)(t_0) + o(h^2). \tag{G.25}$$

Abusively written $C(t_0{+}h) = I + (d\vec{V} + d\vec{V}^T)(t_0) + \frac{h^2}{2}\,(d\vec{A} + 2d\vec{V}^T.d\vec{V} + d\vec{A}^T)(t_0) + o(h^2)$, but don't forget it is a matrix meaning.

With Eulerian variables and $\vec{v}(t,p)$ and $\vec{\gamma}(t,p)$ the Eulerian velocities and accelerations at $t$ at $p = \Phi^{t_0}_t(t,P)$ : We have $d\vec{V}^{t_0}(t,P) = d\vec{v}(t,p(t)).F(t)$ and $d\vec{A}^{t_0}(t,P) = d\vec{\gamma}(t,p(t)).F(t)$, thus

$$\begin{aligned}
C^{t_0}_P(t{+}h) = C^{t_0}_P(t) &+ h\,(F^T(t).(d\vec{v} + d\vec{v}^T)(t,p(t)).F(t)) \\
&+ \frac{h^2}{2}\,(F^T(t).(d\vec{\gamma} + 2d\vec{v}^T.d\vec{v} + d\vec{\gamma}^T)(t,p(t)).F(t)) + o(h^2).
\end{aligned} \tag{G.26}$$

abusive notation of $[C^{t_0}_P(t{+}h)] = \ldots$ (matrices).

**Remark G.9** $F'' = d\vec{A}$ is easy to interpret, but $C'' = F^T.d\vec{A}{+}2d\vec{V}^T.d\vec{V}{+}d\vec{A}^T.F = (F^T.d\vec{A}{+}d\vec{V}^T.d\vec{V}){+}(F^T.d\vec{A} + d\vec{V}^T.d\vec{V})^T$ is not that easy to interpret (and in not linear in $\vec{V}$).

We already had a problem with the composition of flows: The (deterministic) formula $F^{t_0}_{t_2} = F^{t_1}_{t_2}.F^{t_0}_{t_1}$ is straightforward, but the formula $C^{t_0}_{t_2} = (F^{t_0}_{t_2})^T.F^{t_0}_{t_2} = (F^{t_0}_{t_1})^T.(F^{t_1}_{t_2})^T.F^{t_1}_{t_2}.F^{t_0}_{t_1} = (F^{t_0}_{t_1})^T.C^{t_1}_{t_2}.F^{t_0}_{t_1}$ is "not that simple" ($\neq C^{t_1}_{t_2}.C^{t_0}_{t_1}$).

Since $C'(t_0) = d\vec{V}(t_0) + d\vec{V}(t_0)^T$ this may have little consequences for linear approximation near $t_0$, but ultimately not small consequences for second-order approximations (and large deformations) if $C''$ is used to make constitutive laws. The consideration of Lie derivatives may be an interesting alternative. ▪

## G.4 Remark: $C^\flat$

For mathematicians. For the general $^\flat$ notation see § A.12.7.

### G.4.1 Definition of $C^\flat$

**Definition G.10** At $P \in \Omega_{t_0}$, the bilinear form $C^\flat_{Gg}(P) =^{\text{written}} C^\flat \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}, \vec{\mathbb{R}}^n_{t_0}; \mathbb{R})$ associated with the linear map $C_{Gg}(P) =^{\text{written}} C(P) =^{\text{written}} C \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ is defined by, for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$ vectors at $P$,

$$C^\flat(\vec{W}_1, \vec{W}_2) := (\vec{W}_1, C.\vec{W}_2)_G \quad (= (F.\vec{W}_1, F.\vec{W}_2)_g). \tag{G.27}$$

NB: From the $\binom{1}{1}$ tensor $C$ we have built the $\binom{0}{2}$ tensor $C^\flat$, thanks to inner dot products: Change in variance.

$C^\flat$ is a bilinear symmetric form (trivial) and is a metric in $\vec{\mathbb{R}}^n_{t_0}$ since $F$ si a diffeomorphism; But $C^\flat$ is not a Euclidean metric (unless $C = I$ i.e. for rigid body motions i.e. no deformations).

**Quantification:** (G.27) gives $[\vec{W}_2]^T.[C^\flat].[\vec{W}_1] = [\vec{W}_2]^T.[G].[C].[\vec{W}_1]$ for all $\vec{W}_1, \vec{W}_2$ since $C^\flat$ and $(\cdot,\cdot)_G$ are symmetric, thus

$$[C^\flat] = [G].[C] \quad (= [F]^T.[g].[F]). \tag{G.28}$$

More precisely: $[C^\flat]_{|\vec{E}} = [G]_{|\vec{E}}.[C]_{|\vec{E}} = ([F]_{|\vec{E},\vec{e}})^T.[g]_{|\vec{e}}.[F]_{|\vec{E},\vec{e}}$.

Duality notations: $C^\flat = \sum_{IJ} C_{IJ} E^I \otimes E^J$ and $C.\vec{E}_j = \sum_I C^I_J \vec{E}_i$ and $G = \sum_{IJ} G_{IJ} E^I \otimes E^J$ give

$$C_{IJ} = \sum_K C^K{}_J G_{KI} \quad (= \sum_{k\ell} F^k{}_I g_{k\ell} F^\ell{}_J). \tag{G.29}$$

Explains the flat notation: $I$ is up in $[C] = [C^I{}_J]$ and $I$ is down in $[C^\flat] = [C_{IJ}]$ (change of variance).

Classical notations: $C^\flat = \sum_{ij} C_{ij} \pi_{ai} \otimes \pi_{aj}$ and $C.\vec{a}_j = \sum_i C_{ij} \vec{a}_i$ and $G = \sum_{ij} G_{ij} \pi_{ai} \otimes \pi_{aj}$ give

$$(C^\flat)_{ij} = \sum_k G_{ik} C_{kj} \quad (= \sum_{k\ell} F_{ki} g_{k\ell} F_{\ell j}). \tag{G.30}$$

### G.4.2 Remarks, and Jaumann

$C^\flat$ can also be defined only with $(\cdot,\cdot)_g$ by, for all $\vec{W}_1, \vec{W}_2 \in \vec{\mathbb{R}}^n_{t_0}$,

$$C^\flat_g(\vec{W}_1, \vec{W}_2) := (F.\vec{W}_1, F.\vec{W}_2)_g, \tag{G.31}$$

i.e., $C^\flat := C^\flat_g := g^*$ the pull-back of the metric $(\cdot,\cdot)_g$ by $\Phi$, see (8.9).

• $C^\flat$ is mainly useful to characterize a deformation: To compare the value $C^\flat(\vec{W}_1, \vec{W}_2)$ with $(\vec{W}_1, \vec{W}_2)_G$, i.e. if a Euclidean dot product $(\cdot,\cdot)_G$ was introduced in $\vec{\mathbb{R}}^n_{t_0}$: This is why $C^\flat$ is classically defined from $C$, cf. (G.27).

• There is no objective "trace" for a $\binom{0}{2}$ tensor like $C^\flat$, while $\mathrm{Tr}(C)$ is objective (endomorphism).

• The Lie derivatives of a second order tensor depends on the type of the tensor, and the Lie derivative of the $\binom{1}{1}$ tensor like $C$ gives the Jaumann derivative, which is usually preferred to the Lie derivative of the $\binom{0}{2}$ tensor like $C^\flat$ which is the lower convected Lie derivative, see next remark G.11.

• So the introduction and use of $C^\flat$ in mechanics mostly complicate things unnecessarily, and interferes with basic understandings like the distinction between covariance and contravariance.

**Remark G.11** Interpretation issue with Jaumann (and the use of $C^\flat$ should be avoided in mechanics).

With $\frac{D(d\vec{v})}{Dt} \stackrel{(2.30)}{=} d(\frac{D\vec{v}}{Dt}) - d\vec{v}.d\vec{v} = d\vec{\gamma} - d\vec{v}.d\vec{v}$ and with orthonormal bases, $2\mathcal{D} = \frac{D(d\vec{v})}{Dt} + \frac{D(d\vec{v})^T}{Dt} = d\vec{\gamma} + d\vec{\gamma}^T - d\vec{v}.d\vec{v} - d\vec{v}^T.d\vec{v}^T$ (matrix meaning), thus, with (G.26) (matrix meaning),

$$\begin{aligned} C''(t) &= F(t)^T.(2\frac{D\mathcal{D}}{Dt} + d\vec{v}.d\vec{v} + d\vec{v}^T.d\vec{v}^T + 2d\vec{v}^T.d\vec{v})(t, p(t)).F(t) \\ &= 2F(t)^T.(\frac{D\mathcal{D}}{Dt} + \mathcal{D}.d\vec{v} + d\vec{v}^T.\mathcal{D})(t, p(t)).F(t). \end{aligned} \tag{G.32}$$

The $\frac{D\mathcal{D}}{Dt} + \mathcal{D}.d\vec{v} + d\vec{v}^T.\mathcal{D}$ term looks like a lower-convected Lie derivative, but with $d\vec{v}^T$ instead of $d\vec{v}^*$, cf. (9.71); So you may find (G.32) written $C'' = 2F^T.\mathcal{L}_{\vec{v}}\mathcal{D}.F$ (isometric framework); But should not be written $(C^\flat)'' = 2F^T.\mathcal{L}_{\vec{v}}\mathcal{D}^\flat_g.F$ where $\mathcal{D}^\flat_g := \frac{d\vec{v}^\flat_g + (d\vec{v}^\flat_g)^T}{2}$ because then $\mathcal{L}_{\vec{v}}\mathcal{D}^\flat_g$ a priori means the lower convected Lie derivative (disappointing results). ▪

## G.5 Stretch ratio and deformed angle

Here $(\cdot,\cdot)_g = (\cdot,\cdot)_G$, i.e. at $t_0$ and $t$ we use the same Euclidean dot product, to be able to compare the lengths relative to the same unit of measurement. (If $(\cdot,\cdot)_g \neq (\cdot,\cdot)_G$ then use $(\cdot,\cdot)_g = \mu^2(\cdot,\cdot)_G$.)

### G.5.1 Stretch ratio

The stretch ratio at $P \in \vec{\mathbb{R}}^n_{t_0}$ between $t_0$ and $t$ for a $\vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$ is defined by

$$\lambda(\vec{W}_P) := \frac{||\vec{w}_p||_G}{||\vec{W}_P||_G} = \frac{||F_P.\vec{W}_P||_G}{||\vec{W}_P||_G} \quad (= ||F_P.(\frac{\vec{W}_P}{||\vec{W}_P||_G})||_G) \tag{G.33}$$

where $\vec{w}_p = F_P.\vec{W}_P$ is the deformed vector by the motion at $p = \Phi(P)$. I.e., in short

$$\forall \vec{W} \in \vec{\mathbb{R}}^n_{t_0} \text{ s.t. } ||\vec{W}|| = 1, \quad \lambda(\vec{W}) := ||F.\vec{W}||. \tag{G.34}$$

(You may find: $\lambda(d\vec{X}) = ||F.d\vec{X}||$ with $d\vec{X}$ a unit vector(!); This notation should be avoided, see § 4.3.)

### G.5.2 Deformed angle

Recall: The angle $\theta_{t_0} = (\widehat{\vec{W}_1, \vec{W}_2})$ between two vectors $\vec{W}_1$ and $\vec{W}_2$ in $\vec{\mathbb{R}}^n_{t_0} - \{\vec{0}\}$ at $P \in \Omega_{t_0}$ is defined by

$$\cos(\theta_{t_0}) = \frac{\vec{W}_1}{||\vec{W}_1||_G} \bullet_G \frac{\vec{W}_2}{||\vec{W}_2||_G}. \tag{G.35}$$

And the deformed angle $\theta_t$ between the deformed vectors $\vec{w}_i = F.\vec{W}_i$ at $p = \Phi^{t_0}_t(P)$, with $(\cdot,\cdot)_g = (\cdot,\cdot)_G$,

$$\cos(\theta_t) := (\widehat{\vec{w}_1, \vec{w}_2}) = \frac{\vec{w}_1}{||\vec{w}_1||_G} \bullet_G \frac{\vec{w}_2}{||\vec{w}_2||_G} = \frac{(C.\vec{W}_1) \bullet_G \vec{W}_2}{||\vec{w}_1||_G \, ||\vec{w}_2||_G}. \tag{G.36}$$

## G.6 Decompositions of $C$

### G.6.1 Spherical and deviatoric tensors

**Definition G.12** The deformation spheric tensor is

$$C_{sph} = \frac{1}{n} \mathrm{Tr}(C)\, I, \tag{G.37}$$

with $\mathrm{Tr}(C)$ = the trace of the endomorphism $C$ (there is no "trace" for the $\binom{0}{2}$ tensor $C^\flat$).

**Definition G.13** The deviatoric tensor is

$$C_{dev} = C - C_{sph}. \tag{G.38}$$

So $\mathrm{Tr}(C_{dev}) = 0$ and $C = C_{sph} + C_{dev}$.

### G.6.2 Rigid motion

The deformation is rigid iff, for all $t_0, t$,

$$(F^{t_0}_t)^T.F^{t_0}_t = I, \quad \text{i.e.} \quad C^{t_0}_t = I, \quad \text{written} \quad C = I = F^T.F. \tag{G.39}$$

After a rigid body motion, lengths and angles are left unchanged.

### G.6.3 Diagonalization of $C$

**Proposition G.14** *$C = F^T.F$ being symmetric positive, $C$ is diagonalizable, its eigenvalues are positive, and $\vec{\mathbb{R}}^n_{t_0}$ has an orthonormal basis made of eigenvectors of $C$.*

**Proof.** $(C(P).\vec{W}_1, \vec{W}_2)_G = (F(P).\vec{W}_1, F(P).\vec{W}_2)_g = (\vec{W}_1, C(P).\vec{W}_2)_G$, thus $C$ is $(\cdot,\cdot)_G$-symmetric.

$(C.\vec{W}_1, \vec{W}_1)_G = (F.\vec{W}_1, F.\vec{W}_1)_g = ||F.\vec{W}_1||^2_g > 0$ when $\vec{W}_1 \neq \vec{0}$, since $F$ invertible ($\Phi^{t_0}_t$ is supposed to be a diffeomorphism). Thus $C$ est $(\cdot,\cdot)_G$-symmetric definite positive real endomorphism. ∎

**Definition G.15** Let $\lambda_i$ be the eigenvalues of $C$. Then the $\sqrt{\lambda_i}$ are called the principal stretches. And the associated eigenvectors give the principal directions.

### G.6.4 Mohr circle

This § deals with general properties of $3*3$ symmetric positive endomorphism, like $C_t^{t_0}(P)$.

Consider $\vec{\mathbb{R}^3}$ with a Euclidean dot product $(\cdot,\cdot)_{\mathbb{R}^3}$ and a $(\cdot,\cdot)_{\mathbb{R}^3}$-orthonormal basis $(\vec{a}_i)$.

Let $\mathcal{M} : \vec{\mathbb{R}^3} \to \vec{\mathbb{R}^3}$ be a symmetric positive endomorphism. Thus $\mathcal{M}$ is diagonalizable in a $(\cdot,\cdot)_{\mathbb{R}^3}$-orthonormal basis $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$, that is, $\exists\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}$, $\exists\vec{e}_1,\vec{e}_2,\vec{e}_3 \in \vec{\mathbb{R}^3}$ s.t.

$$\mathcal{M}.\vec{e}_i = \lambda_i\vec{e}_i \quad \text{and} \quad (\vec{e}_i,\vec{e}_j)_{\mathbb{R}^3} = \delta_{ij}, \quad \text{so} \quad [\mathcal{M}]_{|\vec{e}} = \text{diag}(\lambda_1,\lambda_2,\lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{G.40}$$

And the orthonormal basis $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ is ordered s.t. $\lambda_1 \geq \lambda_2 \geq \lambda_3$ $(> 0)$.

Let $S$ be the unit sphere in $\mathbb{R}^3$, that is the set $\{(x,y,z) : x^2+y^2+z^2 = 1\}$. Its image $\mathcal{M}(S)$ by $\mathcal{M}$ is the ellipsoid $\{(x,y,z) : \frac{x^2}{\lambda_1^2} + \frac{y^2}{\lambda_2^2} + \frac{z^2}{\lambda_3^2} = 1\}$. Then consider $\vec{n} = \sum_i n_i\vec{e}_i$ s.t. $||\vec{n}||_{\mathbb{R}^3} = 1$:

$$[\vec{n}]_{|\vec{e}} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \quad \text{with} \quad n_1^2+n_2^2+n_3^2 = 1. \tag{G.41}$$

Thus its image $\vec{A} = \mathcal{M}.\vec{n} \in \mathcal{M}(S)$ satisfies

$$\vec{A} = \mathcal{M}.\vec{n}, \quad [\vec{A}]_{|\vec{e}} = \begin{pmatrix} \lambda_1 n_1 \\ \lambda_2 n_2 \\ \lambda_3 n_3 \end{pmatrix}. \tag{G.42}$$

Then define

$$A_n = (\vec{A},\vec{n})_{\mathbb{R}^3}, \quad \vec{A}_\perp = \vec{A} - A_n\vec{n}, \quad A_\perp := ||\vec{A}_\perp||. \tag{G.43}$$

So $\vec{A} = A_n\vec{n} + \vec{A}_\perp \in \text{Vect}\{\vec{n}\} \otimes \text{Vect}\{\vec{n}\}^\perp$. (Remark: $\vec{A}_\perp$ is not orthonormal to the ellipsoid $\mathcal{M}(S)$, but is orthonormal to the initial sphere $S$.)

**Mohr Circle purpose:** To find a relation:

$$A_\perp = f(A_n), \tag{G.44}$$

relation between "the normal force $A_n$" (to the initial sphere) and the "tangent force$A_\perp$" (to the initial sphere).

(G.41), (G.42) and $A_n = (\mathcal{M}.\vec{n},\vec{n})_{\mathbb{R}^3}$ give

$$\begin{cases} n_1^2+n_2^2+n_3^2 = 1, \\ \lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2 = A_n \\ \lambda_1^2 n_1^2 + \lambda_2^2 n_2^2 + \lambda_3^2 n_3^2 = ||\vec{A}||^2 = A_n^2 + A_\perp^2. \end{cases} \tag{G.45}$$

This is linear system with the unknowns $n_1^2, n_2^2, n_3^2$. The solution is

$$\begin{cases} n_1^2 = \dfrac{A_\perp^2 + (A_n-\lambda_2)(A_n-\lambda_3)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \\ n_2^2 = \dfrac{A_\perp^2 + (A_n-\lambda_3)(A_n-\lambda_1)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}, \\ n_3^2 = \dfrac{A_\perp^2 + (A_n-\lambda_1)(A_n-\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}. \end{cases} \tag{G.46}$$

The $n_i^2$ being non negative, and with $\lambda_1 > \lambda_2 > \lambda_3 \geq 0$, we get

$$\begin{cases} A_\perp^2 + (A_n-\lambda_2)(A_n-\lambda_3) \geq 0, \\ A_\perp^2 + (A_n-\lambda_3)(A_n-\lambda_1) \leq 0, \\ A_\perp^2 + (A_n-\lambda_1)(A_n-\lambda_2) \geq 0. \end{cases} \tag{G.47}$$

Then let $x = A_n$ and $y = A_\perp$, and consider, for some $a,b \in \mathbb{R}$, the equation

$$y^2 + (x-a)(x-b) = 0, \quad \text{so} \quad (x - \frac{a+b}{2})^2 + y^2 = \frac{(a-b)^2}{4}.$$

This is the equation of a circle centered at $(\frac{a+b}{2}, 0)$ with radius $\frac{|a-b|}{2}$.

Thus $(G.47)_2$ tells that $A_n$ and $A_\perp$ are inside the circle centered at $(\frac{\lambda_1+\lambda_3}{2},0)$ with radius $\frac{\lambda_1-\lambda_3}{2}$, and $(G.47)_{1,3}$ tell that $A_n$ and $A_\perp$ are outside the other circles (adjacent and included in the first, drawing).

**Exercice G.16** What happens if $\lambda_1=\lambda_2=\lambda_3>0$?

**Answer**. Then $\left\{\begin{aligned} & n_1^2+n_2^2+n_3^2=1,\\ & n_1^2+n_2^2+n_3^2=\frac{A_n}{\lambda_1},\\ & n_1^2+n_2^2+n_3^2=\frac{A_n^2+A_\perp^2}{\lambda_1^2}. \end{aligned}\right\}$ Thus $A_n=\lambda_1$ and $A_n^2+A_\perp^2=\lambda_1^2$, thus $A_\perp=0$. Here $C=\lambda_1 I$, and we deal with a dilation: $A_\perp=0$. ▪

**Exercice G.17** What happens if $\lambda_1=\lambda_2>\lambda_3>0$?

**Answer**. Then $\left\{\begin{aligned} & n_1^2+n_2^2+n_3^2=1,\\ & \lambda_1(1-n_3^2)+\lambda_3 n_3^2=A_n,\\ & \lambda_1^2(1-n_3^2)+\lambda_3^2 n_3^2=A_n^2+A_\perp^2. \end{aligned}\right\}$ Thus $A_n=\lambda_1-(\lambda_1-\lambda_3)n_3^2\in[\lambda_3,\lambda_1]$, and $A_\perp=\pm(\lambda_1^2-(\lambda_1^2-\lambda_3^2)n_3^2-A_n^2)^{\frac12}$, with $A_n^2+A_\perp^2$ a point on the circle with radius $\lambda_1^2(1-n_3^2)+\lambda_3^2 n_3^2$. ▪

## G.7 Green–Lagrange deformation tensor $E$

(G.15) gives $(\vec w_1,\vec w_2)_g=(F.\vec W_1,F.\vec W_2)_g=(F^T.F.\vec W_1,\vec W_2)_G=(C.\vec W,\vec W)_G$ at $p=\Phi(P)$, thus

$$(\vec w_1,\vec w_2)_g-(\vec W_1,\vec W_2)_G=((C-I).\vec W_1,\vec W_2)_G. \tag{G.48}$$

**Definition G.18** The Green–Lagrange tensor (or Green–Saint Venant tensor) at $P$ relative to $t_0$ and $t$ is the endomorphism $E_t^{t_0}(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ defined by

$$E_t^{t_0}(P):=\frac{C_t^{t_0}(P)-I_{t_0}}{2},\quad\text{in short}\quad \boxed{E=\frac{C-I}{2}}\quad(=\frac{F^T.F-I}{2}). \tag{G.49}$$

(In particular $E=0$ for rigid body motions.) And $E_t^{t_0}:\Omega_{t_0}\to\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ is the Green–Lagrange tensor relative to $t_0$ and $t$.

The $\frac12$ is introduced because $\frac12(C.,.)=\frac12(F.,F.)$ corresponds to the "motion squared", see the following linearization.

And we get the time Taylor expansion of $E_P^{t_0}(t)=\frac12(C_P^{t_0}(t)-I_{t_0})$ with $p(t)=\Phi_P^{t_0}(t)$ and (G.26):

$$\begin{aligned} E_P^{t_0}(t+h)&=F_P^{t_0}(t)^T.\Big(h\,\frac{d\vec v+d\vec v^T}{2}+\frac{h^2}{2}\,(\frac{d\vec\gamma+d\vec\gamma^T}{2}+d\vec v^T.d\vec v)\Big)(t,p(t)).F_P^{t_0}(t)+o(h^2)\\ &=F_P^{t_0}(t)^T.\Big(h\,\mathcal{D}+h^2\,(\frac{D\mathcal{D}}{Dt}+\mathcal{D}.d\vec v+d\vec v^T.\mathcal{D})\Big)(t,p(t)).F_P^{t_0}(t)+o(h^2). \end{aligned} \tag{G.50}$$

## G.8 Small deformations (linearization): The infinitesimal strain tensor $\underline{\underline{\varepsilon}}$

### G.8.1 Landau notations big-$O$ and little-$o$

Reminder. Let $f,g:\mathbb{R}\to\mathbb{R}$ and $x_0\in\mathbb{R}$.

- $$f=O(g)\text{ near }x_0\quad\Longleftrightarrow\quad\exists C>0,\ \exists\eta>0,\ \forall x\ s.t.\ |x-x_0|<\eta,\ |f(x)|<C|g(x)|. \tag{G.51}$$

and $f$ is said to be "comparable with $g$" near $x_0$. If $g\neq0$ near $x_0$ then it reads $\frac{|f(x)|}{|g(x)|}<C$.

And $\frac{|f(x)|}{|x^n|}<C$ near $x$=0 means $f=O(x^n)$ near $x_0$=0.

- $$f=o(g)\text{ near }x_0\quad\Longleftrightarrow\quad\forall\varepsilon>0,\ \exists\eta>0,\ \forall x\ s.t.\ |x-x_0|<\eta,\ |f(x)|<\varepsilon|g(x)|. \tag{G.52}$$

and $f$ is said to be "negligible compared with $g$ near $x_0$". If $g\neq0$ near $x_0$ then it reads $\frac{|f(x)|}{|g(x)|}\longrightarrow_{x\to x_0}0$.

And $\frac{|f(x)|}{|x^n|}\longrightarrow_{x\to0}0$ means $f=o(x^n)$ near $x_0$=0.

### G.8.2 Definition of the infinitesimal strain tensor $\underline{\underline{\varepsilon}}$

The motion is $C^2$. Along a trajectory, having $F_P^{t_0}(t_0) = I$ = identity in $\vec{\mathbb{R}}^n_{t_0}$, we have near $t_0$:

$$F_P^{t_0}(t_0{+}h) = I + O(h), \tag{G.53}$$

thus $F_P^{t_0}(t_0{+}h).\vec{W} = \vec{W} + O(h)$ for all $\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, i.e., near $t_0$, with $(\cdot,\cdot)_g = (\cdot,\cdot)_G$,

$$||\vec{w} - \vec{W}|| = O(h) \quad \text{when} \quad \vec{w} = F_P^{t_0}(t_0{+}h).\vec{W}. \tag{G.54}$$

Full notation: $||F_P^{t_0}(t).\vec{W}_P - \vec{W}_P||_g = O(t{-}t_0)$ near $t_0$. (More precisely $||F_P^{t_0}(t).\vec{W}_P - S_t^{t_0}.\vec{W}_P||_g = O(t{-}t_0)$ with Marsden shifter $S_t^{t_0}$, to avoid using any ubiquity gift.)

**Definition G.19** Isometric framework: The same Euclidean dot product $(\cdot,\cdot)_g$ used at all time and $(\vec{e}_i)$ a $(\cdot,\cdot)_g$-orthonormal basis. The infinitesimal strain tensor at $P$ is the matrix defined by

$$[\underline{\underline{\varepsilon}}(P)]_{|\vec{e}} = \frac{[F(P)]_{|\vec{e}} + [F(P)]^T_{|\vec{e}}}{2} - [I], \tag{G.55}$$

written

$$\underline{\underline{\varepsilon}} := \frac{F + F^T}{2} - I \quad \text{(matrix meaning)}. \tag{G.56}$$

More precisely, at $P \in \Omega_{t_0}$ and between $t_0$ and $t$, $[\underline{\underline{\varepsilon}}_t^{t_0}(P)]_{|\vec{e}} = \frac{[F_t^{t_0}(P)]_{|\vec{e}} + [F_t^{t_0}(P)]^T_{|\vec{e}}}{2} - [I]$.

So $\underline{\underline{\varepsilon}}.\vec{W} = \frac{F.\vec{W} + F^T.\vec{W}}{2} - \vec{W}$ means $[\underline{\underline{\varepsilon}}]_{|\vec{e}}.[\vec{W}]_{|\vec{e}} = \frac{[F]_{|\vec{e}}.[\vec{W}]_{|\vec{e}} + [F]^T_{|\vec{e}}.[\vec{W}]_{|\vec{e}}}{2} - [\vec{W}]_{|\vec{e}}$.

**Remark G.20** $\underline{\underline{\varepsilon}}$ in (G.56) can***not*** be a tensor (cannot be a function) since $F_t^{t_0}(P) : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_t$ and $F_t^{t_0}(P)^T : \vec{\mathbb{R}}^n_t \to \vec{\mathbb{R}}^n_{t_0}$ and $I_{t_0} : \vec{\mathbb{R}}^n_{t_0} \to \vec{\mathbb{R}}^n_{t_0}$ don't have the same definition domain. In particular $F^T(p).\vec{W}(P)$ is ill-defined.

So $\underline{\underline{\varepsilon}}$ is not a function, is not a tensor: It is a matrix... But is called "the infinitesimal strain tensor"... ∎

**Proposition G.21** *The Green–Lagrange tensor* $E = \frac{F^T.F - I}{2} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ *satisfies near* $t_0$ *(linearization and matrix meaning):*

$$E = \underline{\underline{\varepsilon}} + o(t{-}t_0) \quad \Big(= \frac{F + F^T}{2} - I + o(t{-}t_0)\Big), \tag{G.57}$$

*which means* $[E] = [\underline{\underline{\varepsilon}}] + o(t{-}t_0) = \frac{[F]+[F^T]}{2} - [I] + o(t{-}t_0))$.

*Interpretation: (G.57) is a linearization of the "quadratic"* $E = \frac{1}{2}(F^T.F - I)$ *which satisfies* $(E.\vec{U},\vec{U})_g = \frac{1}{2}\big(||F.\vec{U}||_g^2 - ||\vec{U}||_g^2\big)$ *for all* $\vec{U} \in \vec{\mathbb{R}}^n_{t_0}$ *("motion squared" cf. the* $(F\cdot,F\cdot)_g = ||F.||_g^2$ *term).*

**Proof.** (Isometric framework.) $[F^T] =^{(G.5)} [F]^T$, thus $[C] = [F]^T.[F]$, thus

$$2[E] = [C] - [I] = [F]^T.[F] - [I] = ([F]^T - [I]).([F] - [I]) + [F]^T + [F] - 2[I]. \tag{G.58}$$

Then, near $t_0$ and with $h = t{-}t_0$, (G.53) gives $([F]^T - [I]).([F] - I]) = O(h)O(h) = O(h^2)$, thus $2[E] = [F]^T + [F] - 2[I] + O(h)$, thus (G.57). ∎

# H Finger tensor $F.F^T$ (left Cauchy–Green tensor)

Finger's approach is consistent with the foundations of relativity (Galileo classical relativity or Einstein general relativity): We can only do measurements at the current time $t$, and we can refer to the past.

There are maby misunderstandings, as was the case for the Cauchy–Green deformation tensor $C$, due to the lack of precise definitions: Definition domain? Value domain? Points at stake ($p$ or $P$)? Euclidean dot product (English? French?)? Covariance? Contravariance?...

## H.1 Definition

$\widetilde{\Phi}$ is a motion, $t_0 \in \mathbb{R}$, $\Phi^{t_0}$ is the associated motion, $P \in \Omega_{t_0}$, $p_t = \Phi_t^{t_0}(P)$, $t \in \mathbb{R}$, $F_t^{t_0}(P) := d\Phi_t^{t_0}(P) \in \mathcal{L}(\vec{\mathbb{R}}_{t_0}^n; \vec{\mathbb{R}}_t^n)$, $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ are Euclidean dot products in $\vec{\mathbb{R}}_{t_0}^n$ and $\vec{\mathbb{R}}_t^n$.

**Definition H.1** The Finger tensor or left Cauchy–Green deformation tensor $\underline{\underline{b}}_t^{t_0}(p_t)$ at $t$ at $p_t = \Phi_t^{t_0}(P)$ relative to $t_0$ is the endomorphism $\in \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_t^n)$ defined by

$$\underline{\underline{b}}_t^{t_0}(p_t) := F_t^{t_0}(P).(F_t^{t_0})_{Gg}^T(p_t) \quad \text{written in short} \quad \boxed{b = F.F^T}, \tag{H.1}$$

i.e. is defined by $(\underline{\underline{b}}_t^{t_0}(p_t).\vec{w}_1, \vec{w}_2)_g = (F_t^{t_0}(P)^T.\vec{w}_1, F_t^{t_0}(P)^T.\vec{w}_2)_G = ((F_t^{t_0})^T(p_t).\vec{w}_1, (F_t^{t_0})^T(p_t).\vec{w}_2)_G$, for all $\vec{w}_1, \vec{w}_2$ vectors at $p_t \in \Omega_t$, written in short

$$(\underline{\underline{b}}.\vec{w}_1, \vec{w}_2)_g = (F^T.\vec{w}_1, F^T.\vec{w}_2)_G. \tag{H.2}$$

(To compare with $C = F^T.F$ and $(C.\vec{W}_1, \vec{W}_2)_G = (F.\vec{W}_1, F.\vec{W}_2)_g$.) The Finger tensor relative to $t_0$ is

$$\underline{\underline{b}}^{t_0} : \begin{cases} \mathcal{C} = \bigcup_t (\{t\} \times \Omega_t) & \to \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_t^n) \\ (t, p_t) & \to \underline{\underline{b}}^{t_0}(t, p_t) := \underline{\underline{b}}_t^{t_0}(p_t). \end{cases} \tag{H.3}$$

NB: $\underline{\underline{b}}^{t_0}$ looks like a Eulerian function, but isn't, because it depends on a $t_0$.

Other definition found:

$$B_t^{t_0} := \underline{\underline{b}}_t^{t_0} \circ (\Phi_t^{t_0})^{-1}, \quad \text{i.e.} \quad B_t^{t_0}(P) := \underline{\underline{b}}_t^{t_0}(p_t) = F_t^{t_0}(P).F_t^{t_0}(P)^T, \quad \text{written} \quad B = F.F^T. \tag{H.4}$$

Pay attention: $B_t^{t_0}(P) \in \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_t^n)$ is an endomorphism at $t$ at $p_t$, not at $t_0$ at $P$: E.g., $B_t^{t_0}(P).\vec{w}_t(p_t) = \underline{\underline{b}}_t^{t_0}(p_t).\vec{w}_t(p_t)$ is meaningful, while $B_t^{t_0}(P).\vec{W}_{t_0}(P)$ is absurd.

**Remark H.2** For mathematicians. The push-forward by $\Phi := \Phi_t^{t_0}$ of the Cauchy–Green deformation tensor $C = F^T.F$ is $\Phi_*(C) = F.C.F^{-1} = F.F^T = \underline{\underline{b}}$, cf. (8.16): It is the Finger tensor. So the endomorphism $C$ in $\vec{\mathbb{R}}_{t_0}^n$ is the pull-back of the endomorphism $\underline{\underline{b}}$ in $\vec{\mathbb{R}}_t^n$. (However a push-forward and a pull-back don't depend on any inner dot product while the transposed $F^T$ does...). ▪

## H.2 $\underline{\underline{b}}^{-1}$

With pull-backs (towards the virtual power principle at $t$). With $p_t = \Phi_t^{t_0}(P)$ and $\vec{W}_i(P) = (F_t^{t_0}(P))^{-1}.\vec{w}_i(p_t)$:

$$(\vec{W}_1, \vec{W}_2)_G = (F^{-1}.\vec{w}_1, F^{-1}.\vec{w}_2)_G = (F^{-T}.F^{-1}.\vec{w}_1, \vec{w}_2)_g = (\underline{\underline{b}}^{-1}.\vec{w}_1, \vec{w}_2)_g. \tag{H.5}$$

So $\underline{\underline{b}}^{-1} := (\underline{\underline{b}}_t^{t_0})^{-1}$ is useful:

$$(\underline{\underline{b}}_t^{t_0})^{-1} : \begin{cases} \Omega_t & \to \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_t^n) \\ p_t & \to (\underline{\underline{b}}_t^{t_0})^{-1}(p_t) = F_t^{t_0}(P)^{-T}.F_t^{t_0}(P)^{-1} = H_t^{t_0}(p_t)^T.H_t^{t_0}(p_t) \end{cases} \tag{H.6}$$

with $p_t = \Phi_t^{t_0}(P)$ and $H_t^{t_0}(p_t) = (F_t^{t_0}(P))^{-1}$ cf. (4.44). Thus we can define

$$(\underline{\underline{b}}^{t_0})^{-1} : \begin{cases} \bigcup_t (\{t\} \times \Omega_t) & \to \mathcal{L}(\vec{\mathbb{R}}_t^n; \vec{\mathbb{R}}_t^n) \\ (t, p_t) & \to (\underline{\underline{b}}^{t_0})^{-1}(t, p_t) := (\underline{\underline{b}}_t^{t_0})^{-1}(p_t). \end{cases} \tag{H.7}$$

Remark: $(\underline{\underline{b}}^{t_0})^{-1}$ looks like a Eulerian function, but isn't, because it depends on $t_0$.

In short:

$$\underline{\underline{b}}^{-1} = H^T.H, \quad \text{to compare with} \quad C = F^T.F, \tag{H.8}$$

and with $\vec{w} = F.\vec{W}$,

$$\underline{\underline{b}}^{-1}.\vec{w} = H^T.\vec{W}, \quad \text{to compare with} \quad C.\vec{W} = F^T.\vec{w}, \tag{H.9}$$

and with $\vec{W}_i = F^{-1}.\vec{w}_i$, i.e. $\vec{w}_i = F.\vec{W}_i$,

$$(\underline{\underline{b}}^{-1}.\vec{w}_1, \vec{w}_2)_g = (\vec{W}_1, \vec{W}_2)_G, \quad \text{to compare with} \quad (C.\vec{W}_1, \vec{W}_2)_G = (\vec{w}_1, \vec{w}_2)_g. \tag{H.10}$$

**Remark H.3** For mathematicians. $p_t = \Phi_t^{t_0}(P)$, $b(p_t) = F(P).F(P)^T$ and $C(P) = F(P)^T.F(P)$ give

$$\underline{\underline{b}}(p_t).F(P) = F(P).C(P), \tag{H.11}$$

written $\underline{\underline{b}} = F.C.F^{-1}$. Thus $\underline{\underline{b}}^{-1} = F.C^{-1}.F^{-1}$, so

$$\Phi_t^{t_0 *}\underline{\underline{b}}^{-1} = F^{-1}.\underline{\underline{b}}^{-1}.F = F^{-1}.F^{-T} = (F^T.F)^{-1} = C^{-1}, \tag{H.12}$$

i.e. the pull-back of $\underline{\underline{b}}^{-1}$ is $C^{-1}$, i.e. $\underline{\underline{b}}^{-1}$ is the push-forward of $C^{-1}$. ∎

## H.3 Time derivatives of $\underline{\underline{b}}^{-1}$

With (H.7) let $(\underline{\underline{b}}^{t_0})^{-1} =^{\text{written}} \underline{\underline{b}}^{-1} = H^T.H$. Thus, along a trajectory, and with (4.48), we get

$$\begin{aligned} \frac{D\underline{\underline{b}}^{-1}}{Dt} &= \frac{DH^T}{Dt}.H + H^T.\frac{DH}{Dt} = -d\vec{v}^T.H^T.H - H^T.H.d\vec{v} \\ &= -\underline{\underline{b}}^{-1}.d\vec{v} - d\vec{v}^T.\underline{\underline{b}}^{-1}. \end{aligned} \tag{H.13}$$

**Exercice H.4** Prove (H.13) with (H.10).

**Answer**. (H.10) gives $\frac{D}{Dt}(\underline{\underline{b}}^{-1}.\vec{w}_1, \vec{w}_2)_g = 0 = (\frac{D\underline{\underline{b}}^{-1}}{Dt}.\vec{w}_1, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.\frac{D\vec{w}_1}{Dt}, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.\vec{w}_1, \frac{D\vec{w}_2}{Dt})_g$, and $\vec{w}_i(t,p(t)) = F^{t_0}(t,P).\vec{W}t_0(P)$ gives $\frac{D\vec{w}_i}{Dt} = d\vec{v}.\vec{w}_i$, thus $(\frac{D\underline{\underline{b}}^{-1}}{Dt}.\vec{w}_1, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.d\vec{v}.\vec{w}_1, \vec{w}_2)_g + (\underline{\underline{b}}^{-1}.\vec{w}_1, d\vec{v}.\vec{w}_2)_g = 0$, thus (H.13). ∎

**Exercice H.5** Prove (H.13) with $F^T.\underline{\underline{b}}^{-1}.F = I_{t_0}$.

**Answer**. $\underline{\underline{b}}^{-1} = (F.F^T)^{-1} = F^{-T}.F^{-1}$ gives $F^T.\underline{\underline{b}}^{-1}.F = I_{t_0}$, thus $(F^T)'.\underline{\underline{b}}^{-1}.F + F^T.\frac{D\underline{\underline{b}}^{-1}}{Dt}.F + F^T.\underline{\underline{b}}^{-1}.F' = 0$, thus $F^T.d\vec{v}^T.\underline{\underline{b}}^{-1}.F + F^T.\frac{D\underline{\underline{b}}^{-1}}{Dt}.F + F^T.\underline{\underline{b}}^{-1}.d\vec{v}.F = 0$, thus (H.13). ∎

## H.4 Euler–Almansi tensor $\underline{\underline{a}}$

Euler–Almansi approach is consistent with the foundations of relativity (Galileo relativity or Einstein general relativity): We can only do measurements at the current time $t$, and we can refer to the past.

At $t$ in $\Omega_t$, consider the Finger tensor $\underline{\underline{b}} = F.F^T$ and its inverse $\underline{\underline{b}}^{-1} = F^{-T}.F^T = H^T.H$ cf. (H.8).

**Definition H.6** Euler–Almansi tenor at $p_t \in \Omega_t$ is the endomorphism $\underline{\underline{a}}_t^{t_0}(p_t) \in \mathcal{L}(\vec{\mathbb{R}_t^n}; \vec{\mathbb{R}_t^n})$ defined by

$$\underline{\underline{a}}_t^{t_0}(p_t) = \frac{1}{2}(I_t - \underline{\underline{b}}_t^{t_0}(p_t)^{-1}) = \frac{1}{2}(I_t - H(p_t)^T.H(p_t)), \tag{H.14}$$

written

$$\underline{\underline{a}} = \frac{1}{2}(I - \underline{\underline{b}}^{-1}) = \frac{1}{2}(I - H^T.H), \tag{H.15}$$

to compare with the Green–Lagrange tensor $E = \frac{1}{2}(C - I) = \frac{1}{2}(F^T.F - I) \in \mathcal{L}(\vec{\mathbb{R}_{t_0}^n}; \vec{\mathbb{R}_{t_0}^n})$.

Remark: $\underline{\underline{a}}^{t_0}$ looks like a Eulerian function, but isn't, because it depends on $t_0$.

(H.10) gives $(\vec{w}_i = F.\vec{W}_i)$

$$(\vec{w}_1, \vec{w}_2)_g - (\vec{W}_1, \vec{W}_2)_G = 2(\underline{\underline{a}}.\vec{w}_1, \vec{w}_2)_g, \tag{H.16}$$

to compare with $(\vec{w}_1, \vec{w}_2)_g - (\vec{W}_1, \vec{W}_2)_G = 2(E.\vec{W}_1, \vec{W}_2)_G$. (This also gives $(\underline{\underline{a}}.\vec{w}_1, \vec{w}_2)_g = (E.\vec{W}_1, \vec{W}_2)_G$.) And (H.15) gives

$$F^T.\underline{\underline{a}}.F = E, \quad \text{i.e.} \quad \underline{\underline{a}} = F^{-T}.E.F^{-1}, \tag{H.17}$$

standing for $F_t^{t_0}(P)^T.\underline{\underline{a}}_t^{t_0}(p).F_t^{t_0}(P) = E_t^{t_0}(P)$ when $p = \Phi_t^{t_0}(P)$.

**Remark H.7** $\underline{\underline{a}}_t^{t_0}$ is not the push-forward of $E_t^{t_0}$ by $\Phi_t^{t_0}$ (the push-forward is $F.E.F^{-1}$). ∎

## H.5 Time Taylor expansion for $\underline{\underline{a}}$

(H.13) gives

$$\frac{D\underline{\underline{a}}}{Dt} = \frac{\underline{\underline{b}}^{-1}.d\vec{v} + d\vec{v}^T.\underline{\underline{b}}^{-1}}{2}. \tag{H.18}$$

## H.6 Almansi modified Infinitesimal strain tensor $\underline{\underline{\widetilde{\varepsilon}}}$

Same Euclidean framework as in § G.8.2, and matrix meaning again.

We have $I - \underline{\underline{b}}^{-1} = I - H^T.H = -(I-H^T).(I-H) + 2I - H^T - H$ where $H$ stands for $H_t^{t_0}(p_t)$. Thus, for small displacement we get $I - \underline{\underline{b}}^{-1} = 2I - H^T - H + O(h)$, so

$$\underline{\underline{a}}(t,p(t)) = \underline{\underline{\widetilde{\varepsilon}}}(t,p(t)) + O(h) \quad \text{where} \quad \underline{\underline{\widetilde{\varepsilon}}} := I - \frac{H+H^T}{2}. \tag{H.19}$$

And, with $t = t_0 + h$ we have $F^{t_0}(t,P) = I + (t{-}t_0)\, d\vec{v}(t,P) + o(t{-}t_0)$, cf. (4.38), thus we have $H^{t_0}(t,p(t)) = F^{t_0}(t,P)^{-1} = I - (t{-}t_0)\, d\vec{v}(t,P) + o(t{-}t_0)$ when $p(t) = \Phi^{t_0}(t,P)$. Thus

$$F^{t_0}(t,P) - I = I - H^{t_0}(t,p(t)) + O(t{-}t_0). \tag{H.20}$$

Therefore, for small displacements ($|t-t_0| << 1$):

$$\underline{\underline{a}}(t,p(t)) \simeq \underline{\underline{\widetilde{\varepsilon}}}(t,p(t)) \simeq \underline{\underline{\varepsilon}}^{t_0}(t,P) \quad \text{(matrix meaning)}. \tag{H.21}$$

# I Polar decompositions of $F$ (“isometric objectivity”)

Regular motion $\widetilde{\Phi} : (t, P_{Obj}) \in [t_0,T] \times Obj \to p_t = \widetilde{\Phi}(t,P_{Obj}) \in \mathbb{R}^n$, $\Omega_t = \widetilde{\Phi}(t,Obj)$, associated Lagrangian motion $\Phi_t^{t_0} : (t,p_{t_0}) \in [t_0,T] \times \Omega_{t_0} \to p_t = \Phi^{t_0}(t,p_{t_0}) := \widetilde{\Phi}(t,P_{Obj}) \in \mathbb{R}^n$ when $p_{t_0} = \widetilde{\Phi}(t_0,P_{Obj})$, deformation gradient $F_t^{t_0}(p_{t_0}) := d\Phi_t^{t_0}(p_{t_0}) =^{\text{written}} F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$.

The covariant objectivity is abandoned here, due to the need for inner dot products $(\cdot,\cdot)_G$ and $(\cdot,\cdot)_g$ in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$ to define $F^T \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_{t_0})$ and build $C = F^T.F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$.

$(\vec{E}_i)$ is a $(\cdot,\cdot)_G$-orthonormal basis in $\vec{\mathbb{R}}^n_{t_0}$, and $(\vec{e}_i)$ a $(\cdot,\cdot)_g$-orthonormal basis in $\vec{\mathbb{R}}^n_t$.

Recall: $(F_t^{t_0})^T_{Gg}(p_t) =^{\text{written}} F^T$ is defined by $(F^T.\vec{w},\vec{U})_G = (F.\vec{U},\vec{w})_g$ for all $(\vec{U},\vec{w}) \in \vec{\mathbb{R}}^n_{t_0} \times \vec{\mathbb{R}}^n_t$, and $C^{t_0}_{t,Gg}(p_{t_0}) := (F_t^{t_0})^T_{Gg}(p_t) \circ F_t^{t_0}(p_{t_0}) =^{\text{written}} C = F^T.F$ is a $(\cdot,\cdot)_G$-symmetric endomorphism in $\vec{\mathbb{R}}^n_{t_0}$ since $(C.\vec{X},\vec{Y})_G = (F^T.F.\vec{X},\vec{Y})_G = (F.\vec{X},F.\vec{Y})_g = (\vec{X},F^T.F.\vec{Y})_G = (\vec{X},C.\vec{Y})_G$ for all $\vec{X},\vec{Y} \in \vec{\mathbb{R}}^n_{t_0}$.

## I.1 $F = R.U$ (right polar decomposition)

$C$ being $(\cdot,\cdot)_G$-symmetric, $\exists \alpha_1,...,\alpha_n \in \mathbb{R}$ (the eigenvalues), $\exists \vec{c}_1,...,\vec{c}_n \in \vec{\mathbb{R}}^n_{t_0}$ (associated eigenvectors), s.t.

$$\forall i \in [1,n]_{\mathbb{N}},\ \ C.\vec{c}_i = \alpha_i \vec{c}_i, \quad \text{and} \quad (\vec{c}_i) \text{ is a } (\cdot,\cdot)_G\text{-orthonormal basis in } \vec{\mathbb{R}}^n_{t_0}, \tag{I.1}$$

i.e. $(\vec{c}_i,\vec{c}_j)_G = \delta_{ij}$ for all $i,j \in [1,n]_{\mathbb{N}}$. With $[C]_{\vec{c}} = D := \text{diag}(\alpha_1,...,\alpha_n)$ the diagonal matrix of eigenvalues and $P = [P_{ij}]$ the transition matrix from $(\vec{E}_i)$ to $(\vec{c}_i)$, i.e. $\vec{c}_j = \sum_i P_{ij}\vec{E}_i$ for all $j$, (I.1) reads

$$[C]_{\vec{E}}.P = P.D, \quad \text{and} \quad P^T.P = I. \tag{I.2}$$

And $F$ being regular, $0 < ||F.\vec{c}_i||_g^2 = (F.\vec{c}_i,F.\vec{c}_i)_g = (C.\vec{c}_i,\vec{c}_i)_G = \alpha_i ||\vec{c}_i||_G^2$, thus $\alpha_i > 0$, for all $i$.

**Definition I.1** The right stretch tensor $U =^{\text{written}} \sqrt{C} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ is the endomorphism defined by

$$\forall i \in [1,n]_{\mathbb{N}},\ \ U.\vec{c}_i = \sqrt{\alpha_i}\,\vec{c}_i, \quad \text{i.e.} \quad [U]_{\vec{c}} = \text{diag}(\sqrt{\alpha_1},...,\sqrt{\alpha_n}), \tag{I.3}$$

the $\sqrt{\alpha_i}$ being called the principal stretches. (Full notation: $U := U^{t_0}_{t,Gg}(p_{t_0})$.) Which reads

$$[U]_{|\vec{E}}.P = P.\sqrt{D} \quad (\text{so } [U]_{|\vec{E}} = P.\sqrt{D}.P^{-1} = P.\sqrt{D}.P^T). \tag{I.4}$$

**Proposition I.2** *$U$ is $(\cdot,\cdot)_G$-symmetric positive definite and* $C = U \circ U =^{\text{written}} U.U =^{\text{written}} U^2$.

**Proof.** $(U^T.\vec{c}_i,\vec{c}_j)_G = (\vec{c}_i,U.\vec{c}_j)_G = (\vec{c}_i,\sqrt{\alpha_j}\vec{c}_j)_G = \sqrt{\alpha_j}\delta_{ij} = \sqrt{\alpha_i}\delta_{ij} = (\sqrt{\alpha_i}\vec{c}_i,\vec{c}_j)_G = (U.\vec{c}_i,\vec{c}_j)_G$ for all $i,j$. And $(U\circ U).\vec{c}_j = U(U.\vec{c}_j) = U(\sqrt{\alpha_j}\vec{c}_j) = \sqrt{\alpha_j}U(\vec{c}_j) = \sqrt{\alpha_j}\sqrt{\alpha_j}\vec{c}_i = \alpha_j\vec{c}_j = C.\vec{c}_j$ for all $j$. ∎

**Definition I.3** The orthogonal transformation $R \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ is the linear map defined by

$$R := F\circ U^{-1} \overset{\text{written}}{=} F.U^{-1}. \tag{I.5}$$

(Full notation: $R^{t_0}_{t,Gg}(p_{t_0}) = F^{t_0}_t(p_{t_0})\circ(U^{t_0}_t(p_{t_0})_{Gg})^{-1}$.) And

$$\boxed{F = R\circ U} \overset{\text{written}}{=} R.U \quad \text{is called the right polar decomposition of } F. \tag{I.6}$$

**Proposition I.4** *1-*

$$R^T\circ R = I, \quad i.e. \quad R^{-1} = R^T, \tag{I.7}$$

*written $R^T.R = I$, i.e. $R$ sends a $(\cdot,\cdot)_G$-orthonormal basis in $\vec{\mathbb{R}}^n_{t_0}$ to a $(\cdot,\cdot)_g$-orthonormal basis in $\vec{\mathbb{R}}^n_t$.*

*2- The right polar decomposition $F = R\circ U$ is unique: If $F = R_2\circ U_2$ with $U_2 \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ symmetric definite positive and $R_2 \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ s.t. $R_2^{-1} = R_2^T$, then $U_2 = U$ and $R_2 = R$.*

**Proof.** 1- $R^T\circ R =^{(I.5)} U^{-T}\circ F^T\circ F\circ U^{-1} = U^{-1}\circ C\circ U^{-1} = U^{-1}\circ(U\circ U)\circ U^{-1} = I$ identity in $\vec{\mathbb{R}}^n_{t_0}$. Thus $(R.\vec{E}_i,R.\vec{E}_j)_g = (R^T.R.\vec{E}_i,\vec{E}_j)_G = (\vec{E}_i,\vec{E}_j)_G = \delta_{ij}$ for all $i,j$: $(R.\vec{E}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis.

2- $U_2$ being symmetric definite positive, call $\sqrt{\beta_i}$ its eigenvalues (all positive) and $(\vec{d}_i)$ a $(\cdot,\cdot)_G$-orthonormal basis made of associated eigenvectors. We have $C = (U_2^T.R_2^T).(R_2.U_2) = U_2.(R_2^T.R_2).U_2 = U_2.I.U_2 = U_2^2$, thus $C.\vec{d}_j = U_2^2.\vec{d}_i = \beta_i\vec{d}_i$, thus the $\beta_i$ are eigenvalues of $C$ and the $\vec{d}_i$ are associated eigenvectors. Thus, even if it means reordering $(\beta_i)$, $\beta_i = \alpha_i$ and $\vec{d}_i \in \mathrm{Ker}(C-\alpha_i I)$, for all $i$, and $U.\vec{d}_i =^{(I.3)} \sqrt{\alpha_i}\vec{d}_i = U_2.\vec{d}_i$ for all $i$, thus $U_2 = U$. Thus $R_2 = F.U_2^{-1} = F.U^{-1} = R$. ∎

**Exercice I.5** Express $R$ in terms of vectors and matrices.

**Answer**. Call $\vec{r}_j = R.\vec{E}_j = \sum_i R_{ij}\vec{e}_i$. Then $[R]_{|\vec{E},\vec{e}} = [R_{ij}] = \big([\vec{r}_1]_{|\vec{e}} \ \ \dots \ \ [\vec{r}_n]_{|\vec{e}}\big)$ and $(\vec{r}_i,\vec{r}_j)_g = \delta_{ij}$, i.e. $[\vec{r}_i]^T_{|\vec{e}}.[\vec{r}_i]_{|\vec{e}} = \delta_{ij}$ for all $i,j$, i.e. $[R]^T_{|\vec{E},\vec{e}}.[R]_{|\vec{E},\vec{e}} = I$. ∎

## I.2 $F = S.R_0.U$ (shifted right polar decomposition)

We need to be more precise if the gift of ubiquity is prohibited: Because we work with the affine space $\mathbb{R}^n$, we can consider Marsden's shifter, with $p_t = \Phi^{t_0}_t(p_{t_0})$,

$$S := S^{t_0}_t(p_{t_0}) : \begin{cases} T_{p_{t_0}}(\Omega_{t_0}) & \to T_{p_t}(\Omega_t) \\ (p_{t_0},\vec{w}_{t_0,p_{t_0}}) & \to S(p_{t_0},\vec{w}_{t_0,p_{t_0}}) := (p_t,\vec{w}_{t,p_t}) \quad \text{where} \quad \vec{w}_{t,p_t} := \vec{w}_{t_0,p_{t_0}}. \end{cases} \tag{I.8}$$

Shorten notation:

$$S := S^{t_0}_t(P) : \begin{cases} \vec{\mathbb{R}}^n_{t_0} & \to \vec{\mathbb{R}}^n_t \\ \vec{W} & \to \vec{w} = S.\vec{W} := \vec{W} \quad (\text{vector at } p = \Phi(P)). \end{cases} \tag{I.9}$$

NB: 1- $S$ is not "the identity" unless you have time and space ubiquity gift, since $\vec{W}$ is defined at $t_0$ at $p_{t_0}$ while $\vec{w} = S.\vec{W}$ is defined at $t$ at $p_t$, and $t \neq t_0$ and $p_t \neq p_{t_0}$ in general;

2- $S$ is not a topological identity since it changes the norms in general: You consider $||\vec{W}||_G$ in $\vec{\mathbb{R}}^n_{t_0}$ and $||S.\vec{W}||_g = ||\vec{w}||_g$ in $\vec{\mathbb{R}}^n_t$.

With $R = F.U^{-1}$ cf. (I.5), let $R_0 \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ be the endomorphism defined by

$$R_0 := S^{-1}\circ R \overset{\text{written}}{=} S^{-1}.R, \quad \text{so} \quad R = S.R_0 \quad (= S\circ R_0). \tag{I.10}$$

(Full notations: $(R_0)^{t_0}_{t,Gg}(p_{t_0}) := (S^{t_0}_t(p_{t_0}))^{-1}(R^{t_0}_{t,Gg}(p_{t_0})) \in \mathcal{L}(T_{p_{t_0}}(\Omega_{t_0});T_{p_{t_0}}(\Omega_{t_0}))$.) We have

$$F = S\circ R_0\circ U \quad \text{written} \quad \boxed{F = S.R_0.U}. \tag{I.11}$$

**Proposition I.6** *If $(\cdot,\cdot)_g = (\cdot,\cdot)_G$ (same inner dot product in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$) then*

$$S^T.S = I, \quad \text{i.e.} \quad S^{-1} = S^T. \tag{I.12}$$

*And the endomorphism $R_0 = S^{-1}.R \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ is a change of $(\cdot,\cdot)_G$-orthonormal basis:*

$$R_0^T.R_0 = I, \quad \text{i.e.} \quad R_0^{-1} = R_0^T. \tag{I.13}$$

**Proof.** $(S^T.S.\vec{U},\vec{W})_G = (S.\vec{U},S.\vec{W})_g \overset{(I.8)}{=} (\vec{U},\vec{W})_g = (\vec{U},\vec{W})_G$ (here $(\cdot,\cdot)_g = (\cdot,\cdot)_G$), for all $\vec{U},\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, thus $S^T.S = I$, thus $S^{-1} = S^T$.

Thus $I = S.S^T$ and $R_0 = S^T.R$, thus $(R_0^T.R_0.\vec{U},\vec{W})_G = (R_0.\vec{U},R_0.\vec{W})_G = (S^T.R.\vec{U},S^T.R.\vec{W})_G = (S.S^T.R.\vec{U},R.\vec{W})_g = (R.\vec{U},R.\vec{W})_g =^{(I.7)} (\vec{U},\vec{W})_G$, for all $\vec{U},\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, thus $R_0^T.R_0 = I$. ∴

**Interpretation of (I.11):** $F$ is composed of: The pure deformation $U$ (endomorphism in $\vec{\mathbb{R}}^n_{t_0}$), the change of orthonormal basis with $R_0$ (endomorphism in $\vec{\mathbb{R}}^n_{t_0}$), and the shift operator $S : T_{p_{t_0}}(\Omega_{t_0}) \to T_{p_t}(\Omega_t)$ from past to present (time and position).

## I.3 $F = V.R$ (left polar decomposition)

Same steps. Let $\underline{\underline{b}}^{t_0}_t(p_t) := F^{t_0}_t(p_{t_0}) \circ (F^{t_0}_t)^T(p_t) \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ (the Finger tensor), written $\underline{\underline{b}} = F.F^T$. The endomorphism $\underline{\underline{b}}$ is symmetric definite positiven thus $\exists \beta_1,...,\beta_n \in \mathbb{R}^*_+$ (the eigenvalues) and $\exists \vec{z}_1,...,\vec{z}_n \in \vec{\mathbb{R}}^n_t$ (associated eigenvectors) s.t.

$$\forall i \in [1,n]_{\mathbb{N}}, \quad \underline{\underline{b}}.\vec{z}_i = \beta_i \vec{z}_i, \quad \text{and} \quad (\vec{z}_i) \text{ is a } (\cdot,\cdot)_g\text{-orthonormal basis in } \vec{\mathbb{R}}^n_t. \tag{I.14}$$

The left stretch tensor $V \in \mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ is the endomorphism defined by,

$$\forall i \in [1,n]_{\mathbb{N}}, \quad V.\vec{z}_i = \sqrt{\beta_i}\vec{z}_i, \quad \text{and} \quad V \overset{\text{written}}{=} \sqrt{\underline{\underline{b}}}. \tag{I.15}$$

(Full notation: $V^{t_0}_{t,Gg}(p_t) = \sqrt{\underline{\underline{b}}^{t_0}_t(p_t)_{Gg}}$.) Then define the linear map $R_\ell \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ by

$$R_\ell := V^{-1}.F, \tag{I.16}$$

so that

$$\boxed{F = V.R_\ell}, \quad \text{called the left polar decomposition of } F. \tag{I.17}$$

**Proposition I.7** *1- $\underline{\underline{b}} = V.V =^{written} V^2$, $V$ is symmetric definite positive, $R_\ell^{-1} = R_\ell^T$. And the left polar decomposition $F = V.R_\ell$ is unique.*

*2- $R_\ell = R$ and $V = R.U.R^{-1}$ (so $U$ and $V$ are similar), thus $U$ and $V$ have the same eigenvalues (square root of those of $C$): $\alpha_i = \beta_i$ and, with (I.1), $\vec{z}_i = R.\vec{c}_i$ is an associated eigenvector of $\underline{\underline{b}}$, for all $i$.*

**Proof.** 1- "Copy" the proof of prop. I.4 with $F^{-1}$ and $\underline{\underline{b}}^{-1} = (F^{-1})^T.(F^{-1})$ instead of $F$ and $C = F^T.F$.

2- $F = V.R_\ell = R_\ell.(R_\ell^{-1}.V.R_\ell)$ with $R_\ell^{-1}.V.R_\ell$ symmetric (since $(R_\ell^{-1}.V.R_\ell)^T = R_\ell^T.V^T.R_\ell^{-T} = R_\ell^{-1}.V.R_\ell$) and definite positive (since $(R_\ell^{-1}.V.R_\ell.\vec{y}_i,\vec{y}_j)_g = (R_\ell^{-1}.V.R_\ell.\vec{y}_i,R_\ell^{-T}.\vec{y}_j)_g = (V.R_\ell.\vec{y}_i,R.\vec{y}_j)_g = (V.\vec{z}_i,\vec{z}_j)_g = \beta_i$ where the $\vec{y}_i := R_\ell^{-1}\vec{z}_i$ make a basis). Thus $F = R.U = R_\ell.(R_\ell^{-1}.V.R_\ell)$ gives $R = R_\ell$ (uniqueness of the right polar decomposition). Hence $R.U = V.R$ (so $V$ and $U$ are similar), hence $V$ and $U$ have the same eigenvalues and if $\vec{c}_i$ is an eigenvector of $U$ then $R.\vec{c}_i$ is an eigenvector of $V$: Indeed $V.(R.\vec{c}_i) = R.U.\vec{c}_i = R.(\alpha_i\vec{c}_i) = \alpha_i(R.\vec{c}_i)$ for all $i$. ∴

# J Linear elasticity: A Classical "tensorial" approach

## J.1 Hookean linear elasticity

### J.1.1 Young's modulus and Poisson's ratio

Elastic cylindrical bar. At rest: Length $L_0$, transverse area $S_0 = \pi R_0^2$. After a longitudinal traction: Length $L$, transverse area $S = \pi R^2$. Let

$$\varepsilon_L = \frac{L-L_0}{L_0} = \frac{\delta L}{L_0} \quad \text{and} \quad \varepsilon_R = \frac{R-R_0}{R_0} = \frac{\delta R}{R_0}, \tag{J.1}$$

the length deformation per unit length and the radial deformation per unit length. Longitudinal force $f$ along the bar, $\sigma = \frac{f}{S}$ = force per unit area (pressure). Hooke's law:

$$\sigma = E\varepsilon_L \; (= E\frac{\delta L}{L_0}), \quad \text{with } E = \text{Young's modulus} \in \mathbb{R}_+^*. \tag{J.2}$$

Thus $f = \sigma S = E\varepsilon_L S = \frac{ES}{L_0}\delta L$, so $f = k\,\delta L$ (elastic spring) where $k = \frac{ES}{L_0}$. And

$$\nu := -\frac{\varepsilon_R}{\varepsilon_L} = -\frac{\frac{\delta R}{R_0}}{\frac{\delta L}{L_0}} = \text{ Poisson's ratio } \in ]-1, \frac{1}{2}[. \tag{J.3}$$

**Exercice J.1** Detail the motion, give $F = R.U$, $E$, $\nu$ and the change of volume $\frac{\delta V}{V_0} := \frac{V-V_0}{V_0}$.

**Answer**. $\Omega_{t_0} = \{(X,Y,Z) : X \in [0,L_0],\, Y^2+Z^2 \le R_0^2\}$, volume $V_0 = L_0\pi R_0^2$. At $t$, length $L = \alpha L_0$, radius $R = \beta R_0$ (with $\alpha,\beta > 0$), $\Omega_t = \{(x,y,z) : x \in [0,L],\, y^2+z^2 \le R^2\}$, volume $V = L\pi R^2 = \alpha\beta^2 V_0$, so $\frac{V-V_0}{V_0} = \alpha\beta^2 - 1$, motion $\Phi_t^{t_0}(X,Y,Z) = \begin{pmatrix} x = \alpha X \\ y = \beta Y = r\beta\cos\theta \\ z = \beta Z = r\beta\sin\theta \end{pmatrix}$ defined in $\Omega_{t_0}$, $\delta L = (\alpha-1)L_0$, $\varepsilon_L = \alpha-1$, $\delta R = (\beta-1)R_0$, $\varepsilon_R = \beta-1$, $\frac{f}{\pi R^2} = \sigma = E(\alpha-1)$, and

$$E = \frac{\sigma}{\alpha-1} \quad \text{and} \quad \nu = -\frac{\alpha-1}{\beta-1}. \tag{J.4}$$

And $k\delta L = f = E\pi R^2(\alpha-1)$, so $k = \frac{\pi R^2}{L_0}E$. And $F := d\Phi_t^{t_0}(X,Y,Z) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{pmatrix}$, thus $\alpha,\beta$ are the (positive) eigenvalues, hence $U = F$ (diagonal matrix) and $R = I$. And $F^T = F$, $\frac{F+F^T}{2} = F$, $\underline{\underline{\varepsilon}} = F - I = \begin{pmatrix} \alpha-1 & 0 & 0 \\ 0 & \beta-1 & 0 \\ 0 & 0 & \beta-1 \end{pmatrix}$. Elasticity: With Lamé coefficients, see § J.1.3, $\underline{\underline{\sigma}} = \lambda \mathrm{Tr}(F-I)I + 2\mu(F-I)$, so

$$\begin{aligned} &\underline{\underline{\sigma}} = \lambda\Big(\alpha+2\beta-3\Big)I + 2\mu\begin{pmatrix} \alpha-1 & 0 & 0 \\ 0 & \beta-1 & 0 \\ 0 & 0 & \beta-1 \end{pmatrix}, \\ &\sigma = ||\underline{\underline{\sigma}}.\vec{e}_1|| = \lambda(\alpha+2\beta-3) + 2\mu(\alpha-1) = (\lambda+2\mu)(\alpha-1) + 2\lambda(\beta-1) = E(\alpha-1). \end{aligned} \tag{J.5}$$

Remark:

$$\begin{aligned} \underline{\underline{\sigma}}_t^{t_0} &= \Big(\lambda \mathrm{Tr}(F_t^{t_0})I + 2\mu F_t^{t_0}\Big) - (3\lambda+2\mu)F_{t_0}^{t_0} \\ &= \Big(\lambda(\alpha+2\beta)I + 2\mu F_t^{t_0}\Big) - (3\lambda+2\mu)F_{t_0}^{t_0} \\ &= \Big(\mathrm{pres_t I} + 2\mu \mathrm{F_t^{t_0}}\Big) - \Big(\mathrm{pres_{t_0} I} + 2\mu \mathrm{F_{t_0}^{t_0}}\Big). \end{aligned} \tag{J.6}$$

$F = F_D + F_S$ sum of the deviatoric part $F_D$ (s.t. $\mathrm{Tr}(F_D) = 0$) and the spherical part:

$$F_S = \frac{\mathrm{Tr}(F)}{3}I = \frac{\alpha+2\beta}{3}I, \quad F_D = F - F_S = \begin{pmatrix} \frac{2(\alpha-\beta)}{3} & 0 & 0 \\ 0 & \frac{\beta-\alpha}{3} & 0 \\ 0 & 0 & \frac{\beta-\alpha}{3} \end{pmatrix}. \tag{J.7}$$

($F_D = 0$ iff $\beta = \alpha$ and then $F_S = \alpha I$. Incompressibility iff $V = V_0$, i.e. $\beta = \frac{1}{\sqrt{\alpha}}$.) ∎

### J.1.2 Shear modulus

Euclidean basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, point $O$, cubic bloc $(O, L\vec{e}_1, L\vec{e}_2, L\vec{e}_3)$ glued on a horizontal table. Area of the upper face: $S = L^2$. Horizontal force (traction) $f$ along $\vec{e}_1$ on the upper horizontal face, horizontal force per unit area $\tau = \frac{f}{S}$, horizontal displacement of the upper face $\delta x$, horizontal displacement per unit area of the upper face $\frac{\delta x}{L} = \tan\gamma$. Hooke's law:

$$\tau = G\frac{\delta x}{L} = G\tan\gamma, \quad \text{with} \quad G = \text{ shear modulus}. \tag{J.8}$$

(If $\gamma \ll 1$ then $\tan\gamma \simeq \gamma$.) And

$$G = \frac{E}{2(1+\nu)}. \tag{J.9}$$

**Exercice J.2** Detail the motion, give $F = R.U$.

**Answer**. $\Omega_{t_0} = \{(X,Y,Z) \in [0,L]^3\}$, $\alpha > 0$, at $t$: $\Phi(X,Y,Z) = \begin{pmatrix} x = X+\alpha Y \\ y = Y \\ z = Z \end{pmatrix}$, upper face $S = \{(x,y,z) : x = X+\alpha L,\ y = Y,\ z = Z\}$, $\delta x = \alpha L$, $\tan\gamma = \alpha$, $\frac{f}{S} = \tau = G\alpha$. And $F := F(X,Y,Z) = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $C = F^T.F = \begin{pmatrix} 1 & \alpha & 0 \\ \alpha & 1{+}\alpha^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; Eigenvalues called $\beta$: with $\theta = 1{-}\beta$, $\det(C-\beta I) = \det \begin{pmatrix} \theta & \alpha & 0 \\ \alpha & \alpha^2{+}\theta & 0 \\ 0 & 0 & \theta \end{pmatrix} = \theta(\theta^2+\alpha^2\theta-\alpha^2)$, and with $\Delta = \alpha^4+4\alpha^2 = \alpha^2(\alpha^2+4)$, we get the three roots $\theta_1 = \frac{-\alpha^2-\alpha\sqrt{\alpha^2+4}}{2}$ and $\theta_2 = \frac{-\alpha^2+\alpha\sqrt{\alpha^2+4}}{2}$, $\theta_3 = 0$, thus the 3 eigenvalues $\beta_1 = 1+\alpha\frac{\alpha+\sqrt{\alpha^2+4}}{2}$, $\beta_2 = 1+\alpha\frac{\alpha-\sqrt{\alpha^2+4}}{2}$, $\beta_3 = 1$.

Associated eigenvectors: $\vec v_1 = \begin{pmatrix} 2 \\ \alpha+\sqrt{\alpha^2{+}4} \\ 0 \end{pmatrix}$, $\vec v_2 = \begin{pmatrix} 2 \\ \alpha-\sqrt{\alpha^2{+}4} \\ 0 \end{pmatrix}$, $\vec v_3 = \vec e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

(Remark: $g(\alpha) = \sqrt{\alpha^2{+}4}$ gives $g(0) = 2$, thus, near $\alpha{=}0$, $g(\alpha) = 2+o(1)$, thus $\beta_1 = 1+\alpha\frac{\alpha+2+o(1)}{2} = 1+\alpha+o(\alpha)$, and $\beta_2 = 1+\alpha\frac{\alpha-2+o(1)}{2} = 1-\alpha+o(\alpha)$; And $\vec v_1 = \begin{pmatrix} 2 \\ 2+o(1) \\ 0 \end{pmatrix}$, $\vec v_2 = \begin{pmatrix} 2 \\ -2+o(1) \\ 0 \end{pmatrix}$.)

And $P = \big( [\frac{\vec v_1}{||\vec v_1||}]_{|\vec e} \ \ [\frac{\vec v_2}{||\vec v_2||}]_{|\vec e} \ \ [\vec e_3]_{|\vec e} \big)$ = transition matrix between orthonormal basis, $P^{-1} = P^T$, $U = \sqrt{C} = P.\mathrm{diag}(\sqrt{\beta_i}).P^T$, $U^{-1} = P.\mathrm{diag}(\frac{1}{\sqrt{\beta_i}}).P^T$, then $R = F.U^{-1}$.

$F-I = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $(F-I).\vec e_2 = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \alpha\vec e_1 = 2(\frac{F-I+(F-I)^T}{2}).\vec e_2 = 2\underline{\underline{\varepsilon}}.\vec e_2$ (since $(F-I)^T.\vec e_2 = \vec 0$) is twice the Cauchy stress vector on the upper face. ∎

### J.1.3 Lamé coefficients

The Lamé coefficients (Lamé parameters) are the reals $\lambda, \mu$ given by (for a Hookean solid)

$$\underline{\underline{\sigma}} = \lambda \mathrm{Tr}(\underline{\underline{\varepsilon}})I + 2\mu\underline{\underline{\varepsilon}}, \quad \text{where} \quad \underline{\underline{\varepsilon}} = \frac{F+F^T}{2} - I. \tag{J.10}$$

And

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} = G, \quad E = \mu\frac{3\lambda+2\mu}{\lambda+\mu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}. \tag{J.11}$$

## J.2 Classical definition of elasticity

Motion $\widetilde\Phi : [t_1,t_2]\times Obj \to \mathbb{R}^n$, $\Omega_t := \widetilde\Phi(t, Obj) \subset \mathbb{R}^n$ for all $t \in [t_1,t_2]$, $t_0 \in [t_1,t_2]$, associated motion $\Phi^{t_0} : (t,P) \in [t_1,t_2]\times\Omega_{t_0} \to \Phi^{t_0}(t,P) \in \mathbb{R}^n$, $\Phi^{t_0}_t(P) := \Phi^{t_0}(t,P)$, $F^{t_0}_t := d\Phi^{t_0}_t$ (deformation gradient), imposed Euclidean basis $(\vec e_i)$ and associated Euclidean dot product $(\cdot,\cdot)_g =^{\text{written}} . \bullet .$, first Piola–Kirchhoff stress tensor $I\!\!K^{t_0}_g =^{\text{written}} I\!\!K$, see (O.17). Then, see e.g. Ciarlet [8], Marsden–Hughes [17]:

**Definition J.3** A material is elastic iff, at any $t \in [t_1,t_2]$ and $P \in \Omega_{t_0}$, there exists a mapping $\widehat{I\!\!K^{t_0}_g} =^{\text{written}} \widehat{I\!\!K} : \Omega_{t_0}\times\mathcal{M}_{nn} \to \vec{\mathbb{R}^n}$ (constitutive equation) s.t., at any $t \in [t_1,t_2]$ and $P \in \Omega_{t_0}$,

$$I\!\!K(t,P) = \widehat{I\!\!K}(P, F(t,P)), \tag{J.12}$$

short notation for $I\!\!K^{t_0}_g(t,P) = \widehat{I\!\!K^{t_0}_g}(P, [F^{t_0}(t,P)]_{|\vec e})$ (the first Piola–Kirchhoff stress tensor value $I\!\!K(t,P)$ only depends on $P$ and on the deformation gradient $F^{t_0}_t(P)$ expressed in a Euclidean basis).

## J.3 Classical approach ("isometric objectivity"): An issue

The infinitesimal strain "tensor" is the matrix defined relative to a $(\cdot,\cdot)_g$-Euclidean basis $(\vec e_i)$ (the same at all time) by

$$[\underline{\underline{\varepsilon}}]_{|\vec e} = \frac{[F]_{|\vec e} + [F]^T_{|\vec e}}{2} - I, \quad \text{written} \quad \underline{\underline{\varepsilon}} = \frac{F+F^T}{2} - I. \tag{J.13}$$

The homogeneous isotropic elasticity constitutive law is given by, with $\lambda,\mu$ the Lamé coefficients and $\underline{\underline{\sigma}}$ the Cauchy stress "tensor":

$$\underline{\underline{\sigma}} = \lambda\mathrm{Tr}(\underline{\underline{\varepsilon}})I + 2\mu\underline{\underline{\varepsilon}} = \big(\lambda\mathrm{Tr}(F)-(\lambda+2\mu)\big)I + \mu(F+F^T) \quad \text{(matrix equation)}. \tag{J.14}$$

(Recall: $F$ is not an endomorphism, so $\mathrm{Tr}(F)$ is meaningless: It is $\mathrm{Tr}([F]_{|\vec{e}})$ which is meant in (J.14)).

**Issue**: Adding $F$ and $F^T$ (and $I$) to make $2\underline{\underline{\varepsilon}}$ (in (J.13)) is a mathematical nonsense since they don't have the same domain or codomain: $F:\vec{\mathbb{R}}^n_{t_0}\to\vec{\mathbb{R}}^n_t$ while $F^T:\vec{\mathbb{R}}^n_t\to\vec{\mathbb{R}}^n_{t_0}$ (and $I$ is some identity operator which codomain = domain). Thus $\underline{\underline{\varepsilon}}$ can't be a function: It is the matrix in (J.14) (obtained with some Euclidean basis).

And $F$ is not an endomorphism, so $\mathrm{Tr}(F)$ is meaningless: It is $\mathrm{Tr}([F]_{|\vec{e}})$ which is meant in (J.14), and $\mathrm{Tr}(\underline{\underline{\varepsilon}}) := \mathrm{Tr}([\underline{\underline{\varepsilon}}]_{|\vec{e}}) = \frac{\mathrm{Tr}([F]_{|\vec{e}})+\mathrm{Tr}([F^T]_{|\vec{e}})}{2} - n = \mathrm{Tr}([F]_{|\vec{e}}) - n$ (trace of a matrix).

Idem

$$\underline{\underline{\sigma}}.\vec{n} = \lambda\mathrm{Tr}(\underline{\underline{\varepsilon}})\vec{n} + 2\mu\underline{\underline{\varepsilon}}.\vec{n} \quad \text{means} \quad \underline{\underline{\sigma}}.[\vec{n}]_{|\vec{e}} = \lambda\mathrm{Tr}([\underline{\underline{\varepsilon}}]_{|\vec{e}})[\vec{n}]_{|\vec{e}} + 2\mu[\underline{\underline{\varepsilon}}]_{|\vec{e}}.[\vec{n}]_{|\vec{e}} \tag{J.15}$$

with $\vec{n}$ the $(\cdot,\cdot)_g$-normal unit out of $\Omega_t$ (not out of $\Omega_{t_0}$). So, despite the eventual claims, neither $\underline{\underline{\varepsilon}}$ nor $\underline{\underline{\sigma}}$ are tensors (they don't have any functional meaning): They are matrices.

**Remark J.4** You may read: "For small displacements the Eulerian variable $p=p_t$ and the Lagrangian variable $P=p_{t_0}$ can be confused": $p_t \simeq p_{t_0}$ (so $\Omega_{t_0}$ and $\Omega_t$ are "almost equal"). Which means that you use the zero-th order Taylor expansions $p_t = \Phi^{t_0}_{p_{t_0}}(t) = p_{t_0} + o(1)$. But then you can***not*** use the first order Taylor expansion (in time) in following calculations (you cannot use velocities)... ▪

## J.4 A functional formulation ("isometric objectivity")

Can the constitutive law (J.14) be modified into a functional expression? Yes:

1. Consider the "right polar decomposition" $F = R.U$ where $U \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$, cf. (I.5). Thus $C = F^T.F = U^T.R^T.R.U = U^2$ because $U = U^T$ and $R^T.R = I$, thus the Green Lagrange tensor $E = \frac{C-I}{2} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ (endomorphism) reads

$$E = \frac{U^2-I_{t_0}}{2} = \frac{(U-I_{t_0})^2 + 2(U-I_{t_0})}{2}. \tag{J.16}$$

Then, with $U - I_{t_0} = O(h)$ (small deformation approximation), we get the modified infinitesimal strain tensor at $t_0$ at $p_{t_0}\in\Omega_{t_0}$

$$\boxed{\underline{\underline{\widetilde{\varepsilon}}} = U - I_{t_0}} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0}), \tag{J.17}$$

endomorphism in $\vec{\mathbb{R}}^n_{t_0}$. (Full notation $\underline{\underline{\widetilde{\varepsilon}}}^{t_0}_{t,Gg}(p_{t_0}) = U^{t_0}_{t,Gg}(p_{t_0}) - I_{t_0}(p_{t_0})$.) I.e., for all $\vec{W}\in\vec{\mathbb{R}}^n_{t_0}$,

$$\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} = U.\vec{W} - \vec{W} = R^{-1}.\vec{w} - \vec{W} \in \vec{\mathbb{R}}^n_{t_0}, \quad \text{when} \quad \vec{w} = F.\vec{W} \text{ (push-forward)}. \tag{J.18}$$

Interpretation: From $\vec{w} = F.\vec{W} = R.U.\vec{W} \in \vec{\mathbb{R}}^n_t$ (the deformed by the motion), first remove the "shifted rotation" to get $R^{-1}.\vec{w} = U.\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$, then remove the initial $\vec{W}$ to obtain $R^{-1}.\vec{w} - \vec{W} = \underline{\underline{\widetilde{\varepsilon}}}.\vec{W} \in \vec{\mathbb{R}}^n_{t_0}$. In particular $||\underline{\underline{\widetilde{\varepsilon}}}.\vec{W}||_G = ||(U-I_{t_0}).\vec{W}||_G$ measures the relative elongation undergone by $\vec{W}$.

2. Then you get a constitutive law with the stress "tensor" $\widetilde{\Sigma}(\Phi) =^{\text{written}} \widetilde{\Sigma} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ functionally well defined:

$$\boxed{\widetilde{\Sigma} = \lambda\mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})I_{t_0} + 2\mu\underline{\underline{\widetilde{\varepsilon}}}} = \lambda\mathrm{Tr}(U-I_{t_0})I_{t_0} + 2\mu(U-I_{t_0}). \tag{J.19}$$

(The trace $\mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})$ is well defined since $\underline{\underline{\widetilde{\varepsilon}}}$ is an endomorphism.) And, at $p_{t_0}\in\Omega_{t_0}$, for any $\vec{W}\in\vec{\mathbb{R}}^n_{t_0}$,

$$\widetilde{\Sigma}.\vec{W} = \lambda\mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})\vec{W} + 2\mu\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} = \lambda\mathrm{Tr}(U-I_{t_0})\vec{W} + 2\mu(U.\vec{W}-\vec{W}) \in \vec{\mathbb{R}}^n_{t_0}. \tag{J.20}$$

3. Then "rotate and shift" with $R$ to get the two point "tensor" (functionally well defined), in $\vec{\mathbb{R}}^n_t$ at $p_t$,

$$R.\widetilde{\Sigma} = \lambda\mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})R + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t). \tag{J.21}$$

I.e., for all $\vec{W}\in\vec{\mathbb{R}}^n_{t_0}$,

$$\begin{aligned} R.\widetilde{\Sigma}.\vec{W} &= \lambda\mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})R.\vec{W} + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} = \lambda\mathrm{Tr}(U-I_{t_0})R.\vec{W} + 2\mu R.(U-I_{t_0}).\vec{W} \\ &= \lambda\mathrm{Tr}(U-I_{t_0})R.\vec{W} + 2\mu(F-R).\vec{W}, \\ &= \lambda\mathrm{Tr}(U-I_{t_0})R.\vec{W} + 2\mu(\vec{w}-R.\vec{W}), \quad \text{where} \quad \vec{w} = F.\vec{W} = R.U.\vec{W}. \end{aligned} \tag{J.22}$$

4. You get the constitutive law for the stress "tensor" (endomorphism) in $\vec{\mathbb{R}}^n_t$:

$$(\underline{\underline{\widetilde{\sigma}}}(\Phi) =) \quad \boxed{\underline{\underline{\widetilde{\sigma}}} = R \circ \widetilde{\Sigma} \circ R^{-1}} \overset{\text{written}}{=} R.\widetilde{\Sigma}.R^{-1} \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t). \tag{J.23}$$

I.e., for any $\vec{w} \in \Omega_t$,

$$\underline{\underline{\widetilde{\sigma}}}.\vec{w} = R.\widetilde{\Sigma}.R^{-1}.\vec{w} \quad \in \vec{\mathbb{R}}^n_t. \tag{J.24}$$

**Interpretation** of (J.23)-(J.24): Shift and rigid rotate backward by applying $R^{-1}$, apply the elastic stress law with $\widetilde{\Sigma}$ which corresponds to a rotation free motion, then shift and rigid rotate forward by applying $R$.

Detailed expression for (J.23)-(J.24): With $\text{Tr}(R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1}) = \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}})$ (see exercise J.6), we get, at $(t, p_t)$,

$$\begin{aligned} \underline{\underline{\widetilde{\sigma}}} = \lambda \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}})\, I_t + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1} &= \lambda \text{Tr}(U - I_{t_0})\, I_t + 2\mu R.(U - I_{t_0}).R^{-1} \\ &= \lambda \text{Tr}(U - I_{t_0})\, I_t + 2\mu(F.R^{-1} - I_t). \end{aligned} \tag{J.25}$$

And for any $\vec{w} \in \vec{\mathbb{R}}^n_t$, and with $\vec{w} = R.\vec{W}$, you get

$$\begin{aligned} \underline{\underline{\widetilde{\sigma}}}.\vec{w} = \lambda \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}})\, \vec{w} + 2\mu R.\underline{\underline{\widetilde{\varepsilon}}}.\vec{W} &= \lambda \text{Tr}(U - I_{t_0})\, \vec{w} + 2\mu R.(U - I_{t_0}).\vec{W} \\ &= \lambda \text{Tr}(U - I_{t_0})\, \vec{w} + 2\mu(R.U.R^{-1}.\vec{w} - \vec{w}). \end{aligned} \tag{J.26}$$

To compare with the classical "functionally meaningless" (J.15).

**Remark J.5** Doing so, you avoid the use of the Piola–Kirchhoff tensors. ♣

**Exercice J.6** Prove: $\text{Tr}(R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1}) = \text{Tr}(\underline{\underline{\widetilde{\varepsilon}}}) = \sum_i(\alpha_i - 1)$. (NB: $\underline{\underline{\widetilde{\varepsilon}}}$ is an endomorphism in $\vec{\mathbb{R}}^n_{t_0}$ while $R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1}$ is an endomorphism in $\vec{\mathbb{R}}^n_t$.)

**Answer**. $\det_{|\vec{e}}(R.\underline{\underline{\widetilde{\varepsilon}}}.R^{-1} - \lambda I_t) = \det_{|\vec{e}}(R.(\underline{\underline{\widetilde{\varepsilon}}} - \lambda I_{t_0}).R^{-1}) = \det_{|\vec{E},\vec{e}}(R).\det_{|\vec{E}}(\underline{\underline{\widetilde{\varepsilon}}} - \lambda I).\det_{|\vec{e},\vec{E}}(R^{-1}) = \det_{|\vec{E}}(\underline{\underline{\widetilde{\varepsilon}}} - \lambda I)$ for all Euclidean bases $(\vec{E}_i)$ and $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$. (With $L = \underline{\underline{\widetilde{\varepsilon}}}$ and components, $\text{Tr}(R.L.R^{-1}) = \sum_i (R.L.R^{-1})^i_i = \sum_{ijk} R^i_j L^j_k (R^{-1})^k_i = \sum_{jk} (R^{-1}.R)^k_j L^j_k = \sum_{jk} \delta^k_j L^j_k = \sum_j L^j_j = \text{Tr}(L)$.) ♣

**Exercice J.7** Elongation in $\mathbb{R}^2$ along the first axis : origin $O$, same Euclidean basis $(\vec{E}_1, \vec{E}_2)$ and Euclidean dot product at all time, $\xi > 0$, $t \geq t_0$, $L, H > 0$, $P \in [0, L] \times [0, H]$, $[\overrightarrow{OP}]_{|\vec{E}} = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}$, and $[\overrightarrow{O\Phi^{t_0}_t(P)}]_{|\vec{E}} = \begin{pmatrix} X_0 + \xi(t - t_0)X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} X_0(\kappa+1) \\ Y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = [\overrightarrow{Op}]_{|\vec{E}}$, where $\kappa = \xi(t - t_0) > 0$ for $t > t_0$.

1- Give $F$, $C$, $U = \sqrt{C}$ and $R = F.U^{-1}$. Relation with the classical expression ?

2- Spring $\overrightarrow{OP} = \overrightarrow{Oc_{t_0}}(s) = X_0\vec{E}_1 + Y_0\vec{E}_2 + s\vec{W}$, i.e. $[\overrightarrow{OP}]_{|\vec{E}} = [\overrightarrow{Oc_{t_0}}]_{|\vec{E}} = \begin{pmatrix} X_0 + sW_1 \\ Y_0 + sW_2 \end{pmatrix}_{|\vec{E}}$ with $s \in [0, L]$ and $\vec{W} = W_1\vec{E}_1 + W_2\vec{E}_2$. Give the deformed spring, i.e. give $p = c_t(s) = \Phi^{t_0}_t(c_{t_0}(s))$, and $\vec{c}_t{}'$, and the stretch ratio.

**Answer**. 1- $[F] = [d\Phi] = \begin{pmatrix} \kappa+1 & 0 \\ 0 & 1 \end{pmatrix}$, same Euclidean dot product and basis at all time, thus $[F^T] = [F]^T = [F]$, then $[C] = [F^T].[F] = [F]^2 = \begin{pmatrix} (\kappa+1)^2 & 0 \\ 0 & 1 \end{pmatrix}$, thus $[U] = [F] = \begin{pmatrix} \kappa+1 & 0 \\ 0 & 1 \end{pmatrix}$, thus $[R] = [I]$. All the matrices are given relative to the basis $(\vec{E}_i)$, thus $F, C, U, R$ (e.g., $C.\vec{E}_1 = (\kappa+1)^2\vec{E}_1$ and $C.\vec{E}_2 = \vec{E}_2$).

Since $R = I$ and $[\underline{\underline{\varepsilon}}] = [\underline{\underline{\widetilde{\varepsilon}}}]$, (J.25) gives the usual result $[\underline{\underline{\sigma}}] = \lambda \text{Tr}([\underline{\underline{\varepsilon}}])I + 2\mu[\underline{\underline{\varepsilon}}]$, cf (J.14) (matrix meaning).

2- $\overrightarrow{Oc_t(s)} = \overrightarrow{O\Phi^{t_0}_t(c_{t_0}(s))} = \begin{pmatrix} (X_0 + sW_1)(\kappa+1) \\ Y_0 + sW_2 \end{pmatrix}_{|\vec{E}}$, thus $\vec{c}_t{}'(s) = \begin{pmatrix} W_1(\kappa+1) \\ W_2 \end{pmatrix}_{|\vec{E}}$, stretch ration $\frac{W_1^2(\kappa+1)^2 + W_2^2}{W_1^2 + W_2^2}$ at $(t, p_t)$. ♣

**Exercice J.8** Simple shear in $\mathbb{R}^2$ : $[\overrightarrow{O\Phi^{t_0}_t(P)}]_{|\vec{E}} = \begin{pmatrix} X + \xi(t - t_0)Y \\ Y \end{pmatrix} =^{\text{written}} \begin{pmatrix} X + \kappa Y \\ Y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = [\overrightarrow{Op}]_{|\vec{E}}$. Same questions, and moreover give the eigenvalues of $C$.

**Answer**. 1- $[F] = \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix}$, $[C] = \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix}.\begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \kappa \\ \kappa & \kappa^2+1 \end{pmatrix}$. Eigenvalues: $\det(C - \lambda I) = \lambda^2 - (2+\kappa^2)\lambda + 1$. Discriminant $\Delta = (2+\kappa^2)^2 - 4 = \kappa^2(\kappa^2+4)$. Eigenvalues $\alpha_\pm = \frac{1}{2}(2+\kappa^2 \pm \kappa\sqrt{\kappa^2+4})$. (We check that $\alpha_\pm > 0$.) Eigenvectors $\vec{v}_\pm$(main directions of deformations) given by $(1 - \alpha_\pm)x + \kappa y = 0$, i.e., $y = \frac{1}{2}(\kappa \pm \sqrt{\kappa^2+4})x$, thus, e.g., $\vec{v}_\pm = \begin{pmatrix} 2 \\ \kappa \pm \sqrt{\kappa^2+4} \end{pmatrix}$. (We check that $\vec{v}_+ \perp \vec{v}_-$.) With $P$ the transition matrix from $(\vec{E}_1, \vec{E}_2)$ to $(\frac{\vec{v}_+}{||\vec{v}_+||}, \frac{\vec{v}_-}{||\vec{v}_-||})$ and $D = \mathrm{diag}(\alpha_+, \alpha_-)$ we get $C = P.D.P^{-1}$ (with $P^{-1} = P^T$ since here $(\frac{\vec{v}_+}{||\vec{v}_+||}, \frac{\vec{v}_-}{||\vec{v}_-||})$ is an orthonormal basis), thus $U = P.\sqrt{D}.P^{-1}$ (we check that $U^T = U$ and $U^2 = C$). And $R = F.U^{-1}$.

2- $\overrightarrow{Oc_t(s)} = \overrightarrow{O\Phi_t^{t_0}(c_{t_0}(s))} = \begin{pmatrix} (X_0{+}sW_1) + \kappa(Y_0{+}sW_2) \\ Y_0{+}sW_2 \end{pmatrix}$, thus $[\vec{c_t}'(s)] = \begin{pmatrix} W_1 + \kappa W_2 \\ W_2 \end{pmatrix}$. Stretch ratio $\frac{(W_1+\kappa W_2)^2+W_2^2}{W_1^2+W_2^2}$ at $(t, p_t)$.

## J.5 Second functional formulation: With the Finger tensor

The above approach uses the push-forward, i.e. uses $F$ (you arrive with your memory). You may prefer to use the pull-back, i.e. use $F^{-1}$ (you remember the past which is Cauchy's point of view): Then you use $F^{-1} = R^{-1}.V^{-1}$ the right polar decomposition of $F^{-1}$, and you consider the "tensor"

$$\underline{\underline{\tilde{\tilde{\varepsilon}}}}_t = V^{-1} - I_t \ \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t), \tag{J.27}$$

and

$$\underline{\underline{\sigma}}_t = \lambda \mathrm{Tr}(\underline{\underline{\tilde{\tilde{\varepsilon}}}}_t) I_t + 2\mu \underline{\underline{\tilde{\tilde{\varepsilon}}}}_t, \quad \text{and} \quad \underline{\underline{\sigma}}_t.\vec{n}_t = \lambda \mathrm{Tr}(\underline{\underline{\tilde{\tilde{\varepsilon}}}}_t)\, \vec{n}_t + 2\mu \underline{\underline{\tilde{\tilde{\varepsilon}}}}_t.\vec{n}_t. \tag{J.28}$$

(Quantities functionally well defined).

# K Displacement

## K.1 The displacement vector $\vec{\mathcal{U}}$

In $\mathbb{R}^n$, let $p_t = \Phi_t^{t_0}(p_{t_0})$. Then the bi-point vector

$$\vec{\mathcal{U}}_t^{t_0}(p_{t_0}) = \Phi_t^{t_0}(p_{t_0}) - I_{t_0}(p_{t_0}) = p_t - p_{t_0} = \overrightarrow{p_{t_0}p_t} \tag{K.1}$$

is called the displacement vector at $p_{t_0}$ relative to $t_0$ and $t$. This defines the map

$$\vec{\mathcal{U}}_t^{t_0} : \begin{cases} \Omega_{t_0} & \to \vec{\mathbb{R}^n} \\ p_{t_0} & \to \vec{\mathcal{U}}_t^{t_0}(p_{t_0}) := p_t - p_{t_0} = \overrightarrow{p_{t_0}p_t} \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}). \end{cases} \tag{K.2}$$

Thus we have defined

$$\vec{\mathcal{U}}^{t_0} : \begin{cases} [t_0, T] \times \Omega_{t_0} & \to \vec{\mathbb{R}^n} \\ (t, p_{t_0}) & \to \vec{\mathcal{U}}^{t_0}(t, p_{t_0}) := \vec{\mathcal{U}}_t^{t_0}(p_{t_0}), \end{cases} \quad \text{and} \quad \vec{\mathcal{U}}^{t_0}_{p_{t_0}} : \begin{cases} [t_0, T] & \to \vec{\mathbb{R}^n} \\ t & \to \vec{\mathcal{U}}^{t_0}_{p_{t_0}}(t) := \vec{\mathcal{U}}_t^{t_0}(p_{t_0}). \end{cases} \tag{K.3}$$

**Remark K.1** $\vec{\mathcal{U}}_t^{t_0}(p_{t_0})$ doesn't define a vector field (it is not tensorial), because $\vec{\mathcal{U}}_t^{t_0}(p_{t_0}) = p_t - p_{t_0} = \overrightarrow{p_{t_0}p_t}$ is a bi-point vector which is neither in $\vec{\mathbb{R}}^n_{t_0}$ nor in $\vec{\mathbb{R}}^n_t$ since $p_{t_0} \in \Omega_{t_0}$ and $p_t \in \Omega_t$ (it requires time and space ubiquity gift). In particular, it makes no sense on a non-plane surface (manifold). More at § K.5.

**Remark K.2** For elastic solids in $\mathbb{R}^n$, the function $\vec{\mathcal{U}}^{t_0}$ is often considered to be the unknown; But the "real" unknown is the motion $\Phi^{t_0}$. And it is sometimes confused with the extension of a 1-D spring. But see figure 4.1: $||\vec{w}_{t_0}(p_{t_0})||$ represents the initial length and $||\vec{w}_{t_0*}(t, p_t)||$ represents the current length of the spring, and the difference $||\vec{w}_{t_0*}(t, p_t)|| - ||\vec{w}_{t_0}(p_{t_0})||$ can be very small ($\ll 1$) while the length of the displacement vector $||\vec{\mathcal{U}}_t^{t_0}|| = p_t - p_{t_0}$ can be very long ($\gg 1$).

**Exercice K.3** Notation in mechanics courses. At $t_0$ let $V_0$ be a small squared volume at $P$ at $t_0$, and at $t$ close to $t_0$ let $V = \Phi(V_0)$ = the deformed volume by $\Phi$. Prove, for $t$ close to $t_0$:

$$\frac{V - V_0}{V_0} = \det(F) - 1 \simeq \mathrm{div}(\vec{\mathcal{U}}) = \mathrm{Tr}(F - I) = \mathrm{Tr}(\underline{\underline{\varepsilon}}) \ \text{ (relative rate of change in volume).} \tag{K.4}$$

**Answer**. 2-D case (idem in 3-D). Implicit hypotheses: A unique chosen Euclidean basis $(\vec{e}_i)$ in $\vec{\mathbb{R}^2}$ at all time, $F := [d\Phi(P)]_{|\vec{e}}$, and the vector notation means the representation matrix in the basis $(\vec{e}_i)$. We have Tr linear and $\mathrm{Tr}(F) = \mathrm{Tr}(F^T)$ and $d\vec{\mathcal{U}} =^{(K.1)} F - I$, thus $\mathrm{div}\vec{\mathcal{U}} = \mathrm{Tr}(F - I) = \mathrm{Tr}(\frac{F+F^T}{2} - I) = \mathrm{Tr}(\underline{\underline{\varepsilon}})$. At $t_0$ $V_0$ is a small squared volume which sides are $\overrightarrow{PP_1}$ and $\overrightarrow{PP_2}$ where $P_1 = P + h\vec{e}_1$ and $P_2 = P + h\vec{e}_2$ (by choice of an adequate Euclidean basis). Thus $V_0 = \det(h\vec{e}_1, h\vec{e}_2) = h^2 \det(\vec{e}_1, \vec{e}_2) = h^2$. Let $p = \Phi(P)$, $p_1 = \Phi(P_1)$, $p_2 = \Phi(P_2)$; Thus $p_j - p = \Phi(P_j) - \Phi(P) = d\Phi(P).(P_j - P) + o(P_j - P)$, which, with the linear hypothesis, is reduced to $p_j - p = d\Phi(P).(P_j - P) = h\, F.\vec{e}_j$. Thus the deformed volume $V$ has its sides given by $\overrightarrow{pp_1}$ and $\overrightarrow{pp_2}$ and $V = \det(\overrightarrow{pp_1}, \overrightarrow{pp_2}) = h^2 \det(F.\vec{e}_1, F.\vec{e}_2) = h^2 \det(F) \det(\vec{e}_1, \vec{e}_2) = h^2 \det(F)$. Thus $\frac{V-V_0}{V_0} = \det(F) - 1 = \det(d\vec{\mathcal{U}} + I) - 1 =^{(L.20)} \Big(1 + \mathrm{Tr}(d\vec{\mathcal{U}}) + o(d\vec{\mathcal{U}})\Big) - 1 = \mathrm{Tr}(d\vec{\mathcal{U}}) + o(d\vec{\mathcal{U}}) = \mathrm{div}\vec{\mathcal{U}} + o(d\vec{\mathcal{U}})$.

## K.2 The differential of the displacement vector

The differential of $\vec{\mathcal{U}}_t^{t_0}$ at $p_{t_0}$ is (matrix meaning)

$$d\vec{\mathcal{U}}_t^{t_0}(p_{t_0}) = d\Phi_t^{t_0}(p_{t_0}) - I_{t_0} = F_t^{t_0}(p_{t_0}) - I_{t_0}, \quad \text{written} \quad d\vec{\mathcal{U}} = F - I, \tag{K.5}$$

which means $[d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})] = [d\Phi_t^{t_0}(p_{t_0})] - [I_{t_0}]$ relative to some basis. It doesn't defined a function, because $F_t^{t_0}(p_{t_0}) : \vec{\mathbb{R}}_{t_0}^n \to \vec{\mathbb{R}}_t^n$ while $I_{t_0} : \vec{\mathbb{R}}_{t_0}^n \to \vec{\mathbb{R}}_{t_0}^n$. Idem, with $\vec{W} \in \vec{\mathbb{R}}_{t_0}^n$, matrix meaning

$$d\vec{\mathcal{U}}.\vec{W} = F.\vec{W} - \vec{W} : \quad \text{means} \quad [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})].[\vec{W}] = [F_t^{t_0}(p_{t_0})].[\vec{W}] - [\vec{W}]. \tag{K.6}$$

## K.3 Deformation "tensor" $\underline{\underline{\varepsilon}}$ (matrix), bis

(K.5) gives (matrix meaning)

$$F_t^{t_0}(p_{t_0}) = I_{t_0} + d\vec{\mathcal{U}}_t^{t_0}(p_{t_0}), \quad \text{written} \quad F = I + d\vec{\mathcal{U}}. \tag{K.7}$$

Therefore, Cauchy–Green deformation tensor $C = F^T.F$ reads, in short, (matrix meaning)

$$C = I + d\vec{\mathcal{U}} + d\vec{\mathcal{U}}^T + d\vec{\mathcal{U}}^T.d\vec{\mathcal{U}} \quad \text{(matrix meaning)}, \tag{K.8}$$

i.e. $[C_t^{t_0}(p_{t_0})] = [I_{t_0}] + [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})] + [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})]^T + [d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})]^T.[d\vec{\mathcal{U}}_t^{t_0}(p_{t_0})]$.

Thus the Green–Lagrange deformation tensor $E = \frac{C-I}{2}$, cf. (G.49), reads, in short, (matrix meaning)

$$E = \frac{d\vec{\mathcal{U}} + d\vec{\mathcal{U}}^T}{2} + \frac{1}{2} d\vec{\mathcal{U}}^T.d\vec{\mathcal{U}} \quad \text{(matrix meaning)}. \tag{K.9}$$

Thus the deformation tensor $\underline{\underline{\varepsilon}}$, cf. (G.56), reads (matrix meaning)

$$\underline{\underline{\varepsilon}} = E - \frac{1}{2}(d\vec{\mathcal{U}})^T.d\vec{\mathcal{U}}, \tag{K.10}$$

with $\underline{\underline{\varepsilon}}$ the "linear part" of $E$ (small displacements: we only used the first order derivative $d\Phi_t^{t_0}$).

## K.4 Small displacement hypothesis, bis

(Usual introduction.) Let $p_t = \Phi_t^{t_0}(p_{t_0})$, $i = 1, 2$, $\vec{W}_i \in \vec{\mathbb{R}}_{t_0}^n$, $\vec{w}_i(p_t) = F_t^{t_0}(p_{t_0}).\vec{W}_i(p_{t_0}) \in \vec{\mathbb{R}}_t^n$ (the push-forwards), written $\vec{w}_i = F.\vec{W}_i$. Then define (matrix meaning)

$$\vec{\Delta}_i := \vec{w}_i - \vec{W}_i = d\mathcal{U}.\vec{W}_i, \quad \text{and} \quad ||\vec{\Delta}||_\infty = \max(||\vec{\Delta}_1||_{\mathbb{R}^n}, ||\vec{\Delta}_2||_{\mathbb{R}^n}). \tag{K.11}$$

Then the small displacement hypothesis reads (matrix meaning):

$$||\vec{\Delta}||_\infty = o(||\vec{W}||_\infty). \tag{K.12}$$

Thus $\vec{w}_i = \vec{W}_i + \vec{\Delta}_i$ (with $\vec{\Delta}_i$ "small") and the hypothesis $(\cdot,\cdot)_g = (\cdot,\cdot)_G$ (same inner dot product at $t_0$ and $t$) give

$$(\vec{w}_1, \vec{w}_2)_G - (\vec{W}_1, \vec{W}_2)_G = (\vec{\Delta}_1, \vec{W}_2)_G + (\vec{\Delta}_2, \vec{W}_1)_G + (\vec{\Delta}_1, \vec{\Delta}_2)_G.$$

So (K.10) gives $2(E.\vec{W}_1, \vec{W}_2)_G = 2(\underline{\underline{\varepsilon}}.\vec{W}_1, \vec{W}_2)_G + (d\vec{\mathcal{U}}^T.d\vec{\mathcal{U}}.\vec{W}_1, \vec{W}_2)_G$, And (K.12) gives

$$(E.\vec{W}_1, \vec{W}_2)_G = (\underline{\underline{\varepsilon}}.\vec{W}_1, \vec{W}_2)_G + O(||\vec{\Delta}||_\infty^2), \tag{K.13}$$

so $E_t^{t_0}$ is approximated by $\underline{\underline{\varepsilon}}_t^{t_0}$, that is, $E_t^{t_0} \simeq \underline{\underline{\varepsilon}}_t^{t_0} = \frac{F+F^T}{2} - I = \frac{d\vec{\mathcal{U}}+d\vec{\mathcal{U}}^T}{2}$ (matrix meaning).

## K.5 Displacement vector with differential geometry

### K.5.1 The shifter

We give the steps, see Marsden–Hughes [17].

• **Affine case** $\mathbb{R}^n$ (continuum mechanics). Recall (I.8): With $p = \Phi_t^{t_0}(P)$, the shifter is:

$$\widetilde{S_t^{t_0}} : \begin{cases} \Omega_{t_0} \times \vec{\mathbb{R}}^n_{t_0} & \to \Omega_t \times \vec{\mathbb{R}}^n_t \\ (P, \vec{Z}_P) & \to \widetilde{S_t^{t_0}}(P, \vec{Z}_P) = (p, S_t^{t_0}(\vec{Z}_P)) \quad \text{with} \quad S_t^{t_0}(\vec{Z}_P) = \vec{Z}_P. \end{cases} \tag{K.14}$$

(The vector is unchanged but the time and the application point have changed: A real observer has no ubiquity gift). So:

$$S_t^{t_0} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \quad \text{and} \quad [S_t^{t_0}]_{|\vec{e}} = I \text{ identity matrix,} \tag{K.15}$$

the matrix equality being possible after the choice of a unique basis at $t_0$ and at $t$. And (simplified notation) $\widetilde{S_t^{t_0}}(P, \vec{Z}_P) =^{\text{written}} S_t^{t_0}(\vec{Z}_P)$. Then the deformation tensor $\underline{\underline{\varepsilon}}$ at $P$ can be defined by

$$\underline{\underline{\varepsilon}}_t^{t_0}(P).\vec{Z}(P) = \frac{(S_t^{t_0})^{-1}(F_t^{t_0}(P).\vec{Z}(P)) + F_t^{t_0}(P)^T.(S_t^{t_0}(P).\vec{Z}(P))}{2} - \vec{Z}(P), \tag{K.16}$$

in short: $\underline{\underline{\varepsilon}}.\vec{Z} = \frac{(S_t^{t_0})^{-1}(F.\vec{Z}) + F^T.(S_t^{t_0}.\vec{Z})}{2} - \vec{Z})$.

• **In a manifold:** $\Omega$ is a manifold (like a surface in $\mathbb{R}^3$ from which we cannot take off). Let $T_P\Omega_{t_0}$ be the tangent space à $P$ (the fiber at $P$), and $T_p\Omega_t$ be the tangent space à $p$ (the fiber at $p$). In general $T_P\Omega_{t_0} \neq T_p\Omega_t$ (e.g. on the sphere "the Earth"). The bundle (the union of fibers) at $t_0$ is $T\Omega_{t_0} = \bigcup_{P\in\Omega_{t_0}}(\{P\} \times T_P\Omega_{t_0})$, and the bundle at $t$ is $T\Omega_t = \bigcup_{p\in\Omega_t}(\{p\} \times T_p\Omega_t)$. Then the shifter

$$\widetilde{S_t^{t_0}} : \begin{cases} T\Omega_{t_0} & \to T\Omega_t \\ (P, \vec{Z}_P) & \to \widetilde{S_t^{t_0}}(P, \vec{Z}_P) = (p, S_t^{t_0}(\vec{Z}_P)), \end{cases} \tag{K.17}$$

where $S_t^{t_0}(\vec{Z}_P)$ is defined such that it distorts $\vec{Z}_P$ "as little as possible" along geodesics.

E.g., on a sphere along a path which is a geodesic, if $\theta_{t_0}$ is the angle between $\vec{Z}_P$ and the tangent vector to the geodesic at $P$, then $\theta_{t_0}$ is also the angle between $S_t^{t_0}(\vec{Z}_P)$ and the tangent vector to the geodesic at $p$, and $S_t^{t_0}(\vec{Z}_P)$ has the same length than $\vec{Z}_P$ (at constant speed in a car you think the geodesic is a straight line, although $S_t^{t_0}(\vec{Z}_P) \neq \vec{Z}_P$: the Earth is not flat).

### K.5.2 The displacement vector

(Affine space framework, $\Omega_{t_0}$ open set in $\mathbb{R}^n$.) Let $P \in \Omega_{t_0}$, $\vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$, $p = \Phi_t^{t_0}(P) \in \Omega_t$, and $d\Phi_t^{t_0} = F_t^{t_0} \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$. Define

$$\widetilde{\delta\vec{\mathcal{U}}_t^{t_0}} : \begin{cases} \Omega_{t_0} \times \vec{\mathbb{R}}^n_{t_0} & \to \Omega_t \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) \\ (P, \vec{Z}_P) & \to \widetilde{\delta\vec{\mathcal{U}}_t^{t_0}}(P, \vec{Z}_P) = (p, \delta\vec{\mathcal{U}}_t^{t_0}(\vec{Z}_P)) \quad \text{with} \quad \delta\vec{\mathcal{U}}_t^{t_0}(\vec{Z}_P) = (F_t^{t_0} - S_t^{t_0}).\vec{Z}_P. \end{cases} \tag{K.18}$$

Then $\widetilde{\delta\vec{\mathcal{U}}_t^{t_0}} = F_t^{t_0} - S_t^{t_0} : P \in \Omega_{t_0} \to \widetilde{\delta\vec{\mathcal{U}}_t^{t_0}}(P) = F_t^{t_0}(P) - S_t^{t_0}(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ is a two-point tensor. And

$$\begin{aligned} C_t^{t_0} &= (F_t^{t_0})^T.F_t^{t_0} = (\delta\mathcal{U}_t^{t_0} + S_t^{t_0})^T.(\delta\mathcal{U}_t^{t_0} + S_t^{t_0}) \\ &= I + (S_t^{t_0})^T.\delta\mathcal{U}_t^{t_0} + (\delta\mathcal{U}_t^{t_0})^T.S_t^{t_0} + (\delta\mathcal{U}_t^{t_0})^T.\delta\mathcal{U}_t^{t_0}, \end{aligned} \tag{K.19}$$

since $(S_t^{t_0})^T.S_t^{t_0} = I$ identity in $T\Omega_{t_0}$: Indeed, $((S_t^{t_0})^T.S_t^{t_0}.\vec{A}, \vec{B})_{\mathbb{R}^n} = (S_t^{t_0}.\vec{A}, S_t^{t_0}.\vec{B})_{\mathbb{R}^n} = (\vec{A}, \vec{B})_{\mathbb{R}^n}$, cf. (K.14), for all $\vec{A}, \vec{B}$. Then the Green–Lagrange tensor is defined on $\Omega_{t_0}$ by

$$E_t^{t_0} = \frac{1}{2}(C_t^{t_0} - I_{t_0}) = \frac{(S_t^{t_0})^T.\delta\mathcal{U}_t^{t_0} + S_t^{t_0}.(\delta\mathcal{U}_t^{t_0})^T}{2} + \frac{1}{2}(\delta\mathcal{U}_t^{t_0})^T.\delta\mathcal{U}_t^{t_0}, \tag{K.20}$$

to compare with (G.49).

# L Determinants

## L.1 Alternating multilinear form

Let $E$ be a vector space, and let $\mathcal{L}(E, ..., E; \mathbb{R}) =^{\text{written}} \mathcal{L}(E^n; \mathbb{R})$ be the set of $n$-multilinear forms, i.e. $m \in \mathcal{L}(E^n; \mathbb{R})$ iff

$$m(..., \vec{x} + \lambda\vec{y}, ...) = m(..., \vec{x}, ...) + \lambda m(..., \vec{y}, ...) \tag{L.1}$$

for all $\vec{x}, \vec{y} \in E$, all $\lambda \in \mathbb{R}$, and any "slot" the other slots being unchanged.

E.g., $m(\lambda_1\vec{x}_1,...,\lambda_n\vec{x}_n) = (\prod_{i=1,...,n}\lambda_i)\, m(\vec{x}_1,...,\vec{x}_n)$, for all $\lambda_1,...,\lambda_n \in \mathbb{R}$ and all $\vec{x}_1,...,\vec{x}_n \in E$.

In particular a 1-multilinear form is a linear form, also called a 1-form. And the set of 1-forms is $\Omega^1(E) = E^*$. Suppose $n \geq 2$.

**Definition L.1** The multilinear form $\mathcal{A}\!\ell : \left\{ \begin{array}{rl} E^n & \to \mathbb{R} \\ (\vec{v}_1,...,\vec{v}_n) & \to \mathcal{A}\!\ell(\vec{v}_1,...,\vec{v}_n) \end{array} \right\} \in \mathcal{L}(E^n;\mathbb{R})$ is a $n$-alternating iff, for all $\vec{u},\vec{v} \in E$,

$$\mathcal{A}\!\ell(...,\vec{u},...,\vec{v},...) = -\mathcal{A}\!\ell(...,\vec{v},...,\vec{u},...), \tag{L.2}$$

the other elements being unchanged. The set of $n$-alternating multilinear forms is

$$\Omega^n(E) = \{\mathcal{A}\!\ell \in \mathcal{L}(E^n;\mathbb{R}) : \mathcal{A}\!\ell \text{ is alternating}\}. \tag{L.3}$$

If $\mathcal{A}\!\ell, \mathcal{B}\!\ell \in \Omega^n(E)$ and $\lambda \in \mathbb{R}$ then $\mathcal{A}\!\ell + \lambda\mathcal{B}\!\ell \in \Omega^n(E)$ thanks to the linearity for each variable. Thus $\Omega^n(E)$ is a vector space, sub-space of $\mathcal{L}(E^n;\mathbb{R})$.

## L.2 Leibniz formula

Formula made for the particular case $\dim E{=}n$ and $\mathcal{A}\!\ell \in \Omega^n(E)$ (a $n$-alternating multilinear form).

Recall (see e.g. Cartan [5]):

1. A permutation $\sigma : [1,n]_\mathbb{N} \to [1,n]_\mathbb{N}$ is a bijective map, i.e. one-to-one (injective) and onto (surjective); Let $S_n$ be the set of permutations of $[1,n]_\mathbb{N}$.
2. A transposition $\tau : [1,n]_\mathbb{N} \to [1,n]_\mathbb{N}$ is a permutation that exchanges two elements, that is, $\exists i \neq j$ s.t. $\tau(...,i,...,j,...) = \tau(...,j,...,i,...)$, the other elements being unchanged.
3. A permutation is a composition of transpositions And a permutation is even iff the number of transpositions is even, and a permutation is odd iff the number of transpositions is odd. The parity (even or odd character) of a permutation is an invariant. (See Cartan.)
4. The signature $\varepsilon(\sigma) = \pm 1$ of a permutation $\sigma$ is $+1$ if $\sigma$ is even, and is $-1$ if $\sigma$ is odd.

**Proposition L.2 (Leibniz formula)** *Setting:* $\dim E = n$, $\mathcal{A}\!\ell \in \Omega^n(E)$, $(\vec{e}_i)$ *is a basis in* $E$. *For all vectors* $\vec{v}_1,...,\vec{v}_n \in E$ *with* $\vec{v}_j = \sum_{i=1}^n v_j^i \vec{e}_i$ *for all* $j$ *we have*

$$\mathcal{A}\!\ell(\vec{v}_1,...,\vec{v}_n) = c \sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{j=1}^n v_j^{\sigma(j)} = c \sum_{\tau\in S_n} \varepsilon(\tau) \prod_{i=1}^n v_{\tau(i)}^i \quad \textit{with} \quad c = \mathcal{A}\!\ell(\vec{e}_1,...,\vec{e}_n). \tag{L.4}$$

*Thus if* $c = \mathcal{A}\!\ell(\vec{e}_1,...,\vec{e}_n)$ *is known, then* $\mathcal{A}\!\ell$ *is known, and*

$$\dim(\Omega^n(E)) = 1. \tag{L.5}$$

*(Classic not.:* $\vec{v}_j = \sum_{i=1}^n v_{ij}\vec{e}_i$, $\mathcal{A}\!\ell(\vec{v}_1,...,\vec{v}_n) = c\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{j=1}^n v_{\sigma(j),j} = c\sum_{\tau\in S_n}\varepsilon(\tau)\prod_{i=1}^n v_{i,\tau(i)}$.*)*

**Proof.** Trivial if $\mathcal{A}\!\ell = 0$. Suppose $\mathcal{A}\!\ell \neq 0$. Let $F$ be the set of functions $i : \left\{ \begin{array}{rl} [1,n]_\mathbb{N} & \to [1,n]_\mathbb{N} \\ k & \to i_k = i(k) \end{array} \right\}$. $\mathcal{A}\!\ell$ being multilinear, $\vec{v}_1 = \sum_{i_1=1}^n v_1^{i_1}\vec{e}_i$ gives $\mathcal{A}\!\ell(\vec{v}_1,...,\vec{v}_n) = \sum_{i_1=1}^n v_1^{i_1}\mathcal{A}\!\ell(\vec{e}_{i_1},\vec{v}_2,...,\vec{v}_n)$ ("first column development"). By recurrence we get $\mathcal{A}\!\ell(\vec{v}_1,...,\vec{v}_n) = \sum_{i_1,i_2=1}^n v_1^{i_1}v_2^{i_2}\mathcal{A}\!\ell(\vec{e}_{i_1},\vec{e}_{i_2},\vec{v}_3,...,\vec{v}_n) = ... = \sum_{i_1,...,i_n=1}^n v_1^{i_1}...v_n^{i_n}\mathcal{A}\!\ell(\vec{e}_{i_1},...,\vec{e}_{i_n}) = \sum_{i\in F}\prod_{j=1}^n v_j^{i(j)}\mathcal{A}\!\ell(\vec{e}_{i(1)},...,\vec{e}_{i(n)})$. And $\mathcal{A}\!\ell$ is alternate, thus $\mathcal{A}\!\ell(\vec{e}_{i(1)},...,\vec{e}_{i(n)}) \neq 0$ iff $i \in F$ is one-to-one thus bijective, i.e. iff $i \in S_n$. Thus $\mathcal{A}\!\ell(\vec{v}_1,...,\vec{v}_n) = \sum_{\sigma\in S_n}\prod_{j=1}^n v_j^{\sigma(j)}\mathcal{A}\!\ell(\vec{e}_{\sigma(1)},...,\vec{e}_{\sigma(n)}) = \sum_{\sigma\in S_n}\prod_{j=1}^n v_j^{\sigma(j)}\varepsilon(\sigma)\mathcal{A}\!\ell(\vec{e}_1,...,\vec{e}_n)$ which is the first equality in (L.4) with $c = \mathcal{A}\!\ell(\vec{e}_1,...,\vec{e}_n)$. Then $\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{j=1}^n v_j^{\sigma(j)} = \sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{i=1}^n v_{\sigma^{-1}(i)}^{\sigma(\sigma^{-1}(i))}$ since $\sigma$ is bijectif; Thus the second equality in (L.4). Thus if $\mathcal{B}\!\ell \in \Omega^n(E)$ then $\mathcal{B}\!\ell(\vec{v}_1,...,\vec{v}_n) =^{(L.4)} \mathcal{B}\!\ell(\vec{e}_1,...,\vec{e}_n)\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{j=1}^n v_j^{\sigma(j)} = \beta\mathcal{A}\!\ell(\vec{v}_1,...,\vec{v}_n)$ where $\beta = \frac{\mathcal{B}\!\ell(\vec{e}_1,...,\vec{e}_n)}{\mathcal{A}\!\ell(\vec{e}_1,...,\vec{e}_n)}$, thus (L.5). ∎

**Exercice L.3** Let $\sigma \in S_n$. Prove: if $\sigma \neq I$ (identity) then $\sigma(i) \neq i$ at least for two $i \in [1,n]_\mathbb{N}$.

**Answer**. If $\sigma(i) \neq i$ and $\sigma(j) = j$ for all $j \neq i$ then $\sigma$ is not one-to-one, thus not bijective, thus $\sigma \notin S_n$. ∎

## L.3 Determinant of vectors

**Definition L.4** $(\vec{e}_i)_{i=1,...,n}$ being a basis in $E$, the determinant relative to $(\vec{e}_i)$ is the alternating multilinear form $\det_{|\vec{e}} \in \Omega^n(E)$ defined thanks to (L.5) by

$$\det_{|\vec{e}}(\vec{e}_1, ..., \vec{e}_n) = 1, \tag{L.6}$$

i.e. the determinant of $n$ vectors $\vec{v}_i$ relative to $(\vec{e}_i)$ is, with $\vec{v}_j = \sum_{i=1}^n v_j^i \vec{e}_i$ for all $j$,

$$\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = \sum_{\sigma\in S_n} \varepsilon(\sigma) \prod_{j=1}^n v_j^{\sigma(j)} = \sum_{\tau\in S_n} \varepsilon(\tau) \prod_{i=1}^n v_{\tau(i)}^i. \tag{L.7}$$

Thus

$$\Omega^n(E) = \text{Vect}\{\det_{|\vec{e}}\} \quad \text{the 1-D vector space spanned by } \det_{|\vec{e}}, \tag{L.8}$$

and any $\mathcal{A} \in \Omega^n(E)$ reads

$$\mathcal{A} = \mathcal{A}(\vec{e}_1, ..., \vec{e}_n) \det_{|\vec{e}}. \tag{L.9}$$

Hence: If $(\vec{b}_i)$ is another basis then $\det_{|\vec{b}} \in \Omega^n(E)$ gives

$$\det_{|\vec{b}} = c \det_{|\vec{e}} \quad \text{where} \quad c = \det_{|\vec{b}}(\vec{e}_1, ..., \vec{e}_n). \tag{L.10}$$

**Exercice L.5** Change of measuring unit: If $(\vec{a}_i)$ is a basis and $\vec{b}_j = \lambda\vec{a}_j$ for all $j$, prove

$$\forall j = 1, ..., n, \quad \vec{b}_j = \lambda\vec{a}_j \quad \Longrightarrow \quad \det_{|\vec{a}} = \lambda^n \det_{|\vec{b}} \tag{L.11}$$

(gives the relation between volumes relative to a change of measuring unit in the Euclidean case).

**Answer**. $\det_{|\vec{a}}(\vec{b}_1, ..., \vec{b}_n) = \det_{|\vec{a}}(\lambda\vec{a}_1, ..., \lambda\vec{a}_n) \overset{multi}{\underset{linear}{=}} \lambda^n \det_{|\vec{a}}(\vec{a}_1, ..., \vec{a}_n) \overset{(L.6)}{=} \lambda^n \overset{(L.6)}{=} \lambda^n \det_{|\vec{b}}(\vec{b}_1, ..., \vec{b}_n)$. ▪

**Definition L.6** Two bases $(\vec{e}_i)$ and $(\vec{b}_i)$ have the same orientation iff $\det_{|\vec{e}}(\vec{b}_1, ..., \vec{b}_n) > 0$, i.e. iff $\det_{|\vec{b}}(\vec{e}_1, ..., \vec{e}_n) > 0$ (we use (L.10) to justify this definition).

**Proposition L.7** $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0$ *iff* $(\vec{v}_1, ..., \vec{v}_n)$ *is a basis; Or equivalently,* $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0$ *iff* $\vec{v}_1, ..., \vec{v}_n$ *are linearly dependent.*

**Proof.** If $\sum_{i=1}^n c_i\vec{v}_i = 0$ and one of the $c_i \neq 0$ then $\vec{v}_i$ is a linear combination of the others thus $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0$ (since $\det_{|\vec{e}}$ is alternate); Thus $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0 \Rightarrow$ the $\vec{v}_i$ are independent, and hence make a basis ($n$ vectors in $E$ s.t. $\dim E = n$). And if $(\vec{v}_1, ..., \vec{v}_n)$ is a basis then $\det_{|\vec{v}}(\vec{v}_1, ..., \vec{v}_n) = 1 \neq 0$, with $\det_{|\vec{v}} = c \det_{|\vec{e}}$, thus $\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0$. ▪

**Exercice L.8** In $\mathbb{R}^2$. Let $\vec{v}_1 = \sum_{i=1}^2 v_1^i\vec{e}_i$ and $\vec{v}_2 = \sum_{j=1}^2 v_2^j\vec{e}_j$ (duality notations). Prove:

$$\det_{|\vec{e}}(\vec{v}_1, \vec{v}_2) = v_1^1v_2^2 - v_1^2v_2^1. \tag{L.12}$$

**Answer**. Development relative to the first column: $\det_{|\vec{e}}(\vec{v}_1, \vec{v}_2) = \det_{|\vec{e}}(v_1^1\vec{e}_1 + v_1^2\vec{e}_2, \vec{v}_2) = v_1^1 \det_{|\vec{e}}(\vec{e}_1, \vec{v}_2) + v_1^2 \det_{|\vec{e}}(\vec{e}_2, \vec{v}_2)$. Then $\det_{|\vec{e}}(\vec{v}_1, \vec{v}_2) = 0 + v_1^1v_2^2 \det(\vec{e}_1, \vec{e}_2) + v_1^2v_2^1 \det(\vec{e}_2, \vec{e}_1) + 0 = v_1^1v_2^2 - v_1^2v_2^1$. ▪

**Exercice L.9** In $\mathbb{R}^3$, with $\vec{v}_j = \sum_{i=1}^3 v_j^i\vec{e}_i$, prove:

$$\det(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \sum_{i,j,k=1}^3 \varepsilon_{ijk}v_1^iv_2^jv_3^k, \tag{L.13}$$

where $\varepsilon_{ijk} = \frac{1}{2}(j-i)(k-j)(k-i)$, i.e. $\varepsilon_{ijk} = 1$ if $(i,j,k) = (1,2,3)$, $(3,1,2)$ or $(2,3,1)$ (even signature), $\varepsilon_{ijk} = -1$ if $(i,j,k) = (3,2,1)$, $(1,3,2)$ and $(2,1,3)$ (odd signature), and $\varepsilon_{ijk} = 0$ otherwise.

**Answer**. Development relative to the first column then the second column give $= v_1^1v_2^2v_3^3 + v_2^1v_3^2v_1^3 + v_3^1v_1^2v_2^3 - v_1^3v_2^2v_3^1 - v_2^3v_3^2v_1^1 - v_3^3v_1^2v_2^1$. ▪

## L.4 Volume

**Definition L.10** *Let $(\vec{e}_i)$ be a Euclidean basis. Consider a parallelepiped in $\mathbb{R}^n$ which sides are given by the vectors $\vec{v}_1, ..., \vec{v}_n$; Its algebraic volume and its volume relative to $(\vec{e}_i)$ are*

$$\textit{algebraic volume} = \det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n), \quad \textit{and} \quad \textit{volume} = \big|\det_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n)\big|. \tag{L.14}$$

*If $n = 2$ then volume is also called an area. If $n = 1$ then volume is also called a length.*

For mathematicians, notations: Let $(\vec{e}_i)$ be a Cartesian basis and $(e^i) = (dx^i)$ be the dual basis. Then, cf. Cartan [6],

$$\det_{|\vec{e}} \overset{\text{written}}{=} e^1 \times ... \times e^n = dx^1 \times ... \times dx^n. \tag{L.15}$$

And, for integration, the volume element (non negative) uses a Euclidean basis $(\vec{e}_i)$ and is

$$d\Omega = |\det_{|\vec{e}}| = |dx^1 \times ... \times dx^n| \underset{\text{integration}}{\overset{\text{written for}}{=}} dx^1 ... dx^n : \tag{L.16}$$

$$0 \le \text{volume } = |\Omega| = \int_\Omega d\Omega = \int_{\vec{x} \in \Omega} dx^1 ... dx^n. \tag{L.17}$$

(cf. Riemann approach: any regular volume $\Omega$ can be approximated with cubes as small as wished.)

**Exercice L.11** Let $\Psi : \left\{ \begin{array}{ll} [a_1, b_1] \times ... \times [a_n, b_n] & \to \Omega \\ \vec{q} = (q_1, ..., q_n) & \to \vec{x} = (x_1 = \Psi_1(\vec{q}), ..., x_n = \Psi_n(\vec{q})) \end{array} \right\}$ be a parametric description $\Omega$. Prove

$$d\Omega(\vec{x}) = |J_\Psi(\vec{q})|\, dq^1 ... dq^n, \quad \text{and} \quad |\Omega| = \int_{\vec{q}} |J_\Psi(\vec{q})|\, dq^1 ... dq^n, \tag{L.18}$$

where $J_\Psi(\vec{q}) = \det_{|\vec{e}}[d\Psi(\vec{q})]_{|\vec{e}} = \det[\frac{\partial \Psi}{\partial q_i}(\vec{q})] = \det_{|\vec{e}}(\vec{p}_1, ... \vec{p}_n) =$ the Jacobian matrix of $\Psi$ at $\vec{q} =$ the volume at $\vec{x} = \Psi(\vec{q})$ limited by the tangent vectors $\vec{p}_i(\vec{x}) = \frac{\partial \Psi}{\partial q_i}(\vec{q})$.

**Answer**. Polar coordinates for illustration purpose (immediate generalization): Consider the disk $\Omega$ parametrized with the polar coordinate system $\Psi : \left\{ \begin{array}{ll} ]0, R] \times [0, 2\pi] & \to \mathbb{R}^2 \\ \vec{q} = (\rho, \theta) & \to \vec{x} = (x = \rho \cos\theta, y = \rho \sin\theta) \end{array} \right\}$ where a Euclidean basis $(\vec{e}_1, \vec{e}_2)$ is used in $\mathbb{R}^2$ (so $\vec{x} = \rho \cos\theta \vec{e}_1 + \rho \sin\theta \vec{e}_2$). The associated polar basis at $\vec{x} = \Psi(\vec{q})$ is $(\vec{p}_1(\vec{x}) = \frac{\partial \Psi}{\partial \rho}(\rho, \theta), \vec{p}_2(\vec{x}) = \frac{\partial \Psi}{\partial \theta}(\rho, \theta))$, so $[\vec{p}_1(\vec{x})]_{|\vec{e}} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $[\vec{p}_2(\vec{x})]_{|\vec{e}} = \begin{pmatrix} -\rho \sin\theta \\ \rho \cos\theta \end{pmatrix}$. Thus $\det_{|\vec{e}}(\vec{p}_1(\vec{x}), \vec{p}_2(\vec{x})) = \rho$ ($> 0$ here), thus $d\Omega = |\rho|\, d\rho d\theta = \rho\, d\rho d\theta$. Thus the volume is $|\Omega| = \int_{\vec{x} \in \Omega} d\Omega = \int_{\rho=0}^R \int_{\theta=0}^{2\pi} \rho\, d\rho d\theta = \pi R^2$. ∎

**Exercice L.12** What is the "volume element" on a regular surface $\Sigma$ in $\mathbb{R}^3$, called the "surface element"?

**Answer**. Let $\Psi : \left\{ \begin{array}{ll} [a_1, b_2] \times [a_2, b_2] & \to \mathbb{R}^3 \\ (u, v) & \to \vec{x} = \Psi(u, v) = x_1(u, v)\vec{e}_1 + ... + x_3(u, v)\vec{e}_3 \end{array} \right\}$ be a regular parametrization of the geometric surface $\Sigma = \text{Im}(\Psi)$, where $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is a Euclidean basis in $\mathbb{R}^3$. Thus $\vec{t}_1(\vec{x}) = \frac{\partial \Psi}{\partial u}(u, v)$ and $\vec{t}_2(\vec{x}) = \frac{\partial \Psi}{\partial v}(u, v)$ are the tangent vectors at $\Sigma$ at $\vec{x} = \Psi(u, v)$. Hence a normal unit vector is $\vec{n}(\vec{x}) = \frac{\vec{t}_1(\vec{x}) \times \vec{t}_2(\vec{x})}{||\vec{t}_1(\vec{x}) \times \vec{t}_2(\vec{x})||}$, thus $\det_{|\vec{e}}(\vec{t}_1, \vec{t}_2, \vec{n}) = ||\vec{t}_1(\vec{x}) \times \vec{t}_2(\vec{x})||$ is the area of the parallelogram which sides are given by $\vec{t}_1$ and $\vec{t}_2$ (volume with height 1). Thus the surface element at $\vec{x} = \Psi(u, v)$ is $d\Sigma(\vec{x}) = ||\vec{t}_1(\vec{x}) \times \vec{t}_2(\vec{x})||\, dudv = ||\frac{\partial \Psi}{\partial u}(u, v) \times \frac{\partial \Psi}{\partial v}(u, v)||\, dudv$. Thus $|\Sigma| = \int_{\vec{x} \in \Sigma} d\Sigma(\vec{x}) = \int_{u=a_1}^{b_1} \int_{v=a_2}^{b_2} ||\frac{\partial \Psi}{\partial u}(u, v) \times \frac{\partial \Psi}{\partial v}(u, v)||\, dudv$. ∎

## L.5 Determinant of a matrix

$\mathcal{M}_{mn}$ is the vector space of $m * n$ matrices, $(\vec{E}_i)$ is the canonical basis in $\mathcal{M}_{n1}$ (column matrices). Let $\vec{m}_j \in \mathcal{M}_{n1}$, $\vec{m}_j = \sum_{i=1}^n M_{ij} \vec{E}_i$, $M = [M_{ij}]$. So $[\vec{m}_j]_{|\vec{E}} = M.[\vec{E}_j]_{|\vec{E}}$. And $[\vec{m}_j]_{|\vec{E}} =^{\text{written}} \vec{m}_j$ because the canonical basis will be systematically used in $\mathcal{M}_{n1}$. So $M = (\,\vec{m}_1, ..., \vec{m}_n\,) = \big(\, M.\vec{E}_1, ..., M.\vec{E}_n \,\big) = [M_{ij}]$.

**Definition L.13** The determinant of the matrix $M = (\,\vec{m}_1, ..., \vec{m}_n\,) = [M_{ij}]$ is

$$\det(M) = \det([M_{ij}]) := \det_{|\vec{E}}(\vec{m}_1, ..., \vec{m}_n) \quad (= \det_{|\vec{E}} \big(\, M.\vec{E}_1, ..., M.\vec{E}_n \,\big)). \tag{L.19}$$

This defines $\det : \mathcal{M}_{nn} \to \mathbb{R}$.

**Proposition L.14** • $\det(I) = 1$.
• *$M \in \mathcal{M}_{nn}$ is invertible iff* $\det(M) \neq 0$.
• *If $M, N \in \mathcal{M}_{nn}$ then* $\det(M.N) = \det(M)\det(N)$, *and if $M$ is invertible then* $\det(M^{-1}) = \frac{1}{\det(M)}$.
• *If $M \in \mathcal{M}_{nn}$ then* $\det(M^T) = \det(M)$.

**Proof.** • $\det(I) := \det_{|\vec{E}}(\vec{E}_1, ..., \vec{E}_n) = 1$.
• Apply prop. L.7.
• If $M$ is not invertible then $M.N$ is not invertible, thus $\det(M) = 0$ and $\det(M.N) = 0$, and $0 = 0$.
If $M$ is invertible, define $a : {\mathcal{M}_{n1}}^n \to \mathbb{R}$ by $a(\vec{v}_1, ..., \vec{v}_n) := \det_{|\vec{E}}(M.\vec{v}_1, ..., M.\vec{v}_n)$. We have $a \in \Omega^1(\mathcal{M}_{n1})$ (multilinear alternate) because the matrix products are linear and $\det_{|\vec{E}} \in \Omega^1(\mathcal{M}_{n1})$. Idem $b : {\mathcal{M}_{n1}}^n \to \mathbb{R}$ defined by $b(\vec{v}_1, ..., \vec{v}_n) := \det_{|\vec{E}}(N.M.\vec{v}_1, ..., N.M.\vec{v}_n)$ is in $\Omega^1(\mathcal{M}_{n1})$. And $\dim(\Omega^n(\mathcal{M}_{n1})) = 1$ gives $\exists \lambda > 0$ s.t. $b = \lambda a$. Thus $\det(N.M.\vec{w}_1, ..., N.M.\vec{w}_n) = \lambda \det_{|\vec{E}}(M.\vec{w}_1, ..., M.\vec{w}_n)$, thus with $\vec{w}_j = \vec{E}_j$ for all $j$ we get $\det(N.M) = \lambda \det(M)$, and with $\vec{w}_j = M^{-1}.\vec{E}_j$ for all $j$ we get $\det(N) = \lambda$.
• $\det[M_{ij}] = \det\limits_{|\vec{E}}(\vec{v}_1, ..., \vec{v}_n) \overset{(L.7)}{=} \sum\limits_{\sigma \in S_n} \varepsilon(\sigma) \prod\limits_{i=1}^{n} v_i^{\sigma(i)} = \sum\limits_{\tau \in S_n} \varepsilon(\tau) \prod\limits_{i=1}^{n} v_{\tau(i)}^i = \det[M_{ji}]$. ∎

**Exercice L.15** Let $g(\cdot,\cdot)$ be an inner dot product, $(\vec{e}_i)$ be a basis, $g_{ij} = g(\vec{e}_i, \vec{e}_j)$. Prove $\det([g_{ij}]) > 0$.

**Answer**. $[g]_{|\vec{e}}$ is symmetric definite $> 0$, $[g]_{|\vec{e}} = P^T.D.P$, $\det([g]_{|\vec{e}}) = \det(P)^2 \prod_{i=1}^n (\lambda_i) > 0$. ∎

**Exercice L.16** If $M$ is a matrix s.t. $M$ is small, prove

$$\det(I + M) = I + \mathrm{Tr}(M) + o(M). \tag{L.20}$$

**Answer**. Let $M = hN \in \mathcal{M}_{nn}$ with $h \ll 1$. Let $\vec{v}_j$ be the $j$-th column of $N$. We have $\det(I + hN) =^{(L.7)} \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^n (\delta_i^{\sigma(i)} + hv_i^{\sigma(i)})$. And if $\sigma \neq$ identity then $\exists j \neq k$ $\sigma(j) = j$ and $\sigma(k) \neq k$, cf. exercise L.3, hence $\prod_{i=1}^n (\delta_i^{\sigma(i)} + hv_i^{\sigma(i)}) = (\prod_{i \neq j, i \neq k})(0 + hv_j^{\sigma(j)})(0 + hv_k^{\sigma(k)}) = O(h^2) = o(h)$. Thus only the identity produces $O(h)$ terms, thus $\det(I + hN) = \prod_{i=1}^n (1 + hv_i^i) + o(h) = 1 + h\sum_i v_i^i + o(h) + o(h) = 1 + h\mathrm{Tr}(N) + o(h) = 1 + \mathrm{Tr}(M) + o(M)$. ∎

## L.6 Determinant of an endomorphism

### L.6.1 Definition and basic properties

**Definition L.17** The determinant of an endomorphism $L \in \mathcal{L}(E; E)$ relative to a basis $(\vec{e}_i)$ in $E$ is

$$\widetilde{\det\limits_{|\vec{e}}}(L) := \det\limits_{|\vec{e}}(L.\vec{e}_1, ..., L.\vec{e}_n). \tag{L.21}$$

This defines $\widetilde{\det}_{|\vec{e}} : \mathcal{L}(E; E) \to \mathbb{R}$.

**Proposition L.18** *Let $L \in \mathcal{L}(E; E)$.*
*1- If $L = I$ the identity, then $\widetilde{\det}_{|\vec{e}}(I) = 1$, for all basis $(\vec{e}_i)$.*
*2- For all $\vec{v}_1, ..., \vec{v}_n \in E$,*

$$\det\limits_{|\vec{e}}(L.\vec{v}_1, ..., L.\vec{v}_n) = \widetilde{\det\limits_{|\vec{e}}}(L) \det\limits_{|\vec{e}}(\vec{v}_1, ..., \vec{v}_n). \tag{L.22}$$

*3- If $L.\vec{e}_j = \sum_{i=1}^n L_{ij}\vec{e}_i$ for all $j$, i.e. if $[L]_{|\vec{e}} = [L_{ij}]$, then*

$$\widetilde{\det\limits_{|\vec{e}}}(L) = \det([L]_{|\vec{e}}) \quad (= \det([L_{ij}])). \tag{L.23}$$

*4- For all $M \in \mathcal{L}(E; E)$, and with $M \circ L =^{written} M.L$ (thanks to linearity),*

$$\widetilde{\det\limits_{|\vec{e}}}(M.L) = \widetilde{\det\limits_{|\vec{e}}}(M) \widetilde{\det\limits_{|\vec{e}}}(L) = \widetilde{\det\limits_{|\vec{e}}}(L.M). \tag{L.24}$$

*5- $L$ is invertible iff $\widetilde{\det}_{|\vec{e}}(L) \neq 0$, and then*

$$\widetilde{\det\limits_{|\vec{e}}}(L^{-1}) = \frac{1}{\widetilde{\det}_{|\vec{e}}(L)}. \tag{L.25}$$

*6- If $(\cdot,\cdot)_g$ is an inner dot product in $E$ and $L_g^T$ is the $(\cdot,\cdot)_g$ transposed of $L$ (i.e., $(L_g^T\vec{w}, \vec{u})_g = (\vec{w}, L.\vec{u})_g$*

*for all $\vec{u},\vec{w}\in E$) then*

$$\widetilde{\det_{|\vec{e}}}(L_g^T)=\widetilde{\det_{|\vec{e}}}(L). \tag{L.26}$$

*7- If $(\vec{e}_i)$ and $(\vec{b}_i)$ are two $(\cdot,\cdot)_g$-orthonormal bases in $\vec{\mathbb{R}}^n_t$, then $\det_{|\vec{b}}=\pm\det_{|\vec{e}}$ with $+$ iff $(\vec{b}_i)$ and $(\vec{e}_i)$ have the same orientation.*

**Proof.** 1- $\widetilde{\det}_{|\vec{e}}(I)=\det_{|\vec{e}}(I.\vec{e}_1,...,I.\vec{e}_n)=\det_{|\vec{e}}(\vec{e}_1,...,\vec{e}_n)=1$, true for all basis.

2- Let $m:(\vec{v}_1,...,\vec{v}_n)\to m(\vec{v}_1,...,\vec{v}_n):=\det_{|\vec{e}}(L.\vec{v}_1,...,L.\vec{v}_n)$: It is a multilinear form since $L$ is linear, and alternated since $\det_{|\vec{e}}$ is; Thus $m=^{(L.9)}m(\vec{e}_1,...,\vec{e}_n)\det_{|\vec{e}}=\widetilde{\det}_{|\vec{e}}(L)$, thus (L.22).

3- Apply (L.19) with $M=[L]_{|\vec{e}}$ to get (L.23).

4- $\det_{|\vec{e}}((M.L).\vec{e}_1,...,(M.L).\vec{e}_n)=\det_{|\vec{e}}(M.(L.\vec{e}_1),...,M.(L.\vec{e}_n))=^{(L.22)}\widetilde{\det}_{|\vec{e}}(M)\det_{|\vec{e}}(L.\vec{e}_1,...,L.\vec{e}_n)$.

5- If $L$ is invertible, then $1=\widetilde{\det}_{|\vec{e}}(I)=\widetilde{\det}_{|\vec{e}}(L.L^{-1})=\widetilde{\det}_{|\vec{e}}(L)\widetilde{\det}_{|\vec{e}}(L^{-1})$, thus $\widetilde{\det}_{|\vec{e}}(L)\neq 0$.

If $\widetilde{\det}_{|\vec{e}}(L)\neq 0$ then $\det_{|\vec{e}}(L.\vec{e}_1,...,L.\vec{e}_n)\neq 0$, thus $(L.\vec{e}_1,...,L.\vec{e}_n)$ is a basis, thus $L$ is invertible. Then (L.24) gives $1=\widetilde{\det}_{|\vec{e}}(I)=\widetilde{\det}_{|\vec{e}}(L^{-1}.L)=\widetilde{\det}_{|\vec{e}}(L).\widetilde{\det}_{|\vec{e}}(L^{-1})$, thus (L.25).

6- $[g]_{|\vec{e}}.[L_g^T]_{|\vec{e}}=([L]_{|\vec{e}})^T.[g]_{|\vec{e}}$ gives $\det([g]_{|\vec{e}})\det([L_g^T]_{|\vec{e}})=\det(([L]_{|\vec{e}})^T)\det([g]_{|\vec{e}})$,

7- Let $\mathcal{P}$ be the change of basis endomorphism from $(\vec{e}_i)$ to $(\vec{b}_i)$, and $P=[\mathcal{P}]_{|\vec{e}}$ (the transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$). We have $\widetilde{\det}_{|\vec{e}}(\mathcal{P})=\det_{|\vec{e}}(\vec{b}_1,...,\vec{b}_n)=\det(P)$, and both basis being $(\cdot,\cdot)_g$-orthonormal, $P^T.P=I$, thus $\det(P)^2=1$, thus $\det_{|\vec{e}}(\vec{b}_1,...,\vec{b}_n)=\det(P)=\pm 1=\pm\det_{|\vec{b}}(\vec{b}_1,...,\vec{b}_n)$, thus $\det_{|\vec{e}}=\pm\det_{|\vec{b}}$. And apply the definition L.6. ∎

**Exercice L.19** Prove $\widetilde{\det}_{|\vec{e}}(\lambda L)=\lambda^n\widetilde{\det}_{|\vec{e}}(L)$.

**Answer**. $\widetilde{\det_{|\vec{e}}}(\lambda L)=\det_{|\vec{e}}(\lambda L.\vec{e}_1,...,\lambda L.\vec{e}_n)=\lambda^n\det_{|\vec{e}}(L.\vec{e}_1,...,L.\vec{e}_n)=\lambda^n\widetilde{\det_{|\vec{e}}}(L)$. ∎

### L.6.2 The determinant of an endomorphism is objective

**Proposition L.20** *Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases in $E$. The determinant of an endomorphism $L\in\mathcal{L}(E;E)$ is objective (observer independent, here basis independent):*

$$(\det([L]_{|\vec{a}})=)\quad \widetilde{\det_{|\vec{a}}}(L)=\widetilde{\det_{|\vec{b}}}(L)\quad (=\det([L]_{|\vec{b}})). \tag{L.27}$$

**NB:** *But the determinant of $n$ vectors is* **not** *objective, cf. (L.10) (compare the change of basis formula for vectors $[\vec{w}]_{|\vec{b}}=P^{-1}.[\vec{w}]_{|\vec{a}}$ with the change of basis formula for endomorphisms $[L]_{|\vec{b}}=P^{-1}.[L]_{|\vec{a}}.P$).*

**Proof.** Let $P$ be the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$. Hence $[L]_{|\vec{b}}=P^{-1}.[L]_{|\vec{a}}.P$, thus $\widetilde{\det}_{|\vec{b}}(L)=\det([L]_{|\vec{b}})=\det(P^{-1})\det([L]_{|\vec{a}})\det(P)=\det([L]_{|\vec{a}})=\widetilde{\det}_{|\vec{a}}(L)$. ∎

**Exercice L.21** Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases in $E$, and define $\mathcal{P}\in\mathcal{L}(E;E)$ by $\mathcal{P}.\vec{a}_j=\vec{b}_j$ for all $j$ (the change of basis endomorphism). Prove

$$\det_{|\vec{a}}(\vec{b}_1,...,\vec{b}_n)=\widetilde{\det_{|\vec{a}}}(\mathcal{P}),\quad\text{thus}\quad \det_{|\vec{a}}=\widetilde{\det_{|\vec{a}}}(\mathcal{P})\det_{|\vec{b}}. \tag{L.28}$$

**Answer**. $\det_{|\vec{a}}(\vec{b}_1,...,\vec{b}_n)=\det_{|\vec{a}}(\mathcal{P}.\vec{a}_1,...,\mathcal{P}.\vec{a}_n)\overset{(L.22)}{=}\widetilde{\det_{|\vec{a}}}(\mathcal{P})\det_{|\vec{a}}(\vec{a}_1,...,\vec{a}_n)=\widetilde{\det_{|\vec{a}}}(\mathcal{P})=\widetilde{\det_{|\vec{a}}}(\mathcal{P})\det_{|\vec{b}}(\vec{b}_1,...,\vec{b}_n)$, thus $\det_{|\vec{a}}=\widetilde{\det}_{|\vec{a}}(\mathcal{P})\det_{|\vec{b}}$. ∎

## L.7 Determinant of a linear map

(Needed for the deformation gradient $F_t^{t_0}(P)=d\Phi_t^{t_0}(P):\vec{\mathbb{R}}^n_{t_0}\to\vec{\mathbb{R}}^n_t$.)

Let $A$ and $B$ be vector spaces, $\dim A=\dim B=n$, and $(\vec{a}_i)$ and $(\vec{b}_i)$ be bases in $A$ and $B$.

### L.7.1 Definition and first properties

**Definition L.22** The determinant of a linear map $L \in \mathcal{L}(A;B)$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ is

$$\widetilde{\det}_{|\vec{a},\vec{b}}(L) := \det_{|\vec{b}}(L.\vec{a}_1, ..., L.\vec{a}_n). \tag{L.29}$$

If $(\vec{b}_i) = (\vec{a}_i)$ then $\widetilde{\det}_{|\vec{a},\vec{b}}(L) =^{\text{written}} \widetilde{\det}_{|\vec{a}}(L)$. If $(\vec{b}_i) = (\vec{a}_i)$ is implicit then $\widetilde{\det}_{|\vec{a},\vec{b}}(L) =^{\text{written}} \det(L)$.

Thus, if $L.\vec{a}_j = \sum_{i=1}^n L_{ij}\vec{b}_i$, i.e. $[L]_{|\vec{a},\vec{b}} = [L_{ij}]$, then with (L.19):

$$\widetilde{\det}_{|\vec{a},\vec{b}}(L) = \det([L_{ij}]). \tag{L.30}$$

**Proposition L.23** *Let $\vec{u}_1, ..., \vec{u}_n \in A$. Then*

$$\det_{|\vec{b}}(L.\vec{u}_1, ..., L.\vec{u}_n) = \widetilde{\det}_{|\vec{a},\vec{b}}(L) \det_{|\vec{a}}(\vec{u}_1, ..., \vec{u}_n). \tag{L.31}$$

**Proof.** $m : (\vec{u}_1, ..., \vec{u}_n) \in A^n \to m(\vec{u}_1, ..., \vec{u}_n) := \det_{|\vec{b}}(L.\vec{u}_1, ..., L.\vec{u}_n) \in \mathbb{R}$ is a multilinear alternated form since $L$ is linear; And $m(\vec{a}_1, ..., \vec{a}_n) = \det_{|\vec{b}}(L.\vec{a}_1, ..., L.\vec{a}_n) =^{(L.29)} \widetilde{\det}_{|\vec{a},\vec{b}}(L) = \widetilde{\det}_{|\vec{a},\vec{b}}(L) \det_{|\vec{a}}(\vec{a}_1, ..., \vec{a}_n)$. Thus $m = \widetilde{\det}_{|\vec{a},\vec{b}}(L) \det_{|\vec{a}}$, cf. (L.10), thus (L.31). ∎

**Corollary L.24** *Let $A, B, C$ be vector spaces such that $\dim A = \dim B = \dim C = n$ and $(\vec{a}_i)$, $(\vec{b}_i)$, $(\vec{c}_i)$ be bases in $A, B, C$. If $L : A \to B$ and $M : B \to C$ are linear then, with $M \circ L =^{\text{written}} M.L$ (thanks to linearity),*

$$\widetilde{\det}_{|\vec{a},\vec{c}}(M.L) = \widetilde{\det}_{|\vec{a},\vec{b}}(L) \, \widetilde{\det}_{|\vec{b},\vec{c}}(M). \tag{L.32}$$

**Proof.** $\widetilde{\det}_{|\vec{a},\vec{c}}(M.L) = \det_{|\vec{c}}(M.L.\vec{a}_1), ..., M.L.\vec{a}_n)) = \widetilde{\det}_{|\vec{b},\vec{c}}(M) \det_{|\vec{b}}(L.\vec{a}_1, ..., L.\vec{a}_n) = \widetilde{\det}_{|\vec{b},\vec{c}}(M) \, \widetilde{\det}_{|\vec{a},\vec{b}}(L)$. ∎

### L.7.2 Jacobian of a motion, and dilatation

$F := F_t^{t_0}(p_{t_0}) := d\Phi_t^{t_0}(p_{t_0}) : \vec{\mathbb{R}}_{t_0}^n \to \vec{\mathbb{R}}_t^n$ is the deformation gradient at $p_{t_0} \in \Omega_{t_0}$ relative to $t_0$ and $t$, cf. (4.1). Let $(\vec{E}_i)$ be a Euclidean basis in $\vec{\mathbb{R}}_{t_0}^n$ and $(\vec{e}_i)$ be a Euclidean basis in $\vec{\mathbb{R}}_t^n$ for all $t \geq t_0$. Let $F_{ij}$ be the components of $F$ relative to these bases, so $F.\vec{E}_j = \sum_{i=1}^n F_{ij}\vec{e}_i$ for all $j$ and $[F]_{|\vec{E},\vec{e}} = [F_{ij}]$.

**Definition L.25** The "volume dilatation rate" at $p_{t_0}$ relative to the Euclidean bases $(\vec{E}_i)$ and $(\vec{e}_i)$ is

$$J_{|\vec{E},\vec{e}}(\Phi_t^{t_0})(p_{t_0}) := \widetilde{\det}_{|\vec{E},\vec{e}}(F) \quad (= \det_{|\vec{e}}(F.\vec{E}_1, ..., F.\vec{E}_n) = \det([F_{ij}])), \tag{L.33}$$

often written $J_{|\vec{E},\vec{e}} := \det([F]_{|\vec{E},\vec{e}})$ (or simply $J = \det(F)$ if $(\vec{e}_i) = (\vec{E}_i)$ is implicit).

So, at $t_0$ at $p_{t_0}$, $(\vec{E}_1, ..., \vec{E}_n)$ is a unit parallelepiped which volume is 1 (relative to the unit of measurement chosen in $\vec{\mathbb{R}}_{t_0}^n$), and, at $t$ at $p_t = \Phi_t^{t_0}(p_{t_0})$, $J_{|\vec{E},\vec{e}}(\Phi_t^{t_0})(p_{t_0}) = \det_{|\vec{e}}(F.\vec{E}_1, ..., F.\vec{E}_n)$ is the volume of the parallelepiped $(p_t, F.\vec{E}_1, ..., F.\vec{E}_n)$ at $p_t = \Phi_t^{t_0}(p_{t_0})$ (relative to the unit of measurement chosen in $\vec{\mathbb{R}}_t^n$).

Interpretation: With $t_2 > t_1 \geq t_0$, and $(\vec{e}_i)$ is the basis at $t_1$ and $t_2$:
- Dilatation if $J_{|\vec{E},\vec{e}}(\Phi_{t_2}^{t_0})(p_{t_0}) > J_{|\vec{E},\vec{e}}(\Phi_{t_1}^{t_0})(p_{t_0})$ (volume increase),
- contraction if $J_{|\vec{E},\vec{e}}(\Phi_{t_2}^{t_0})(p_{t_0}) < J_{|\vec{E},\vec{e}}(\Phi_{t_1}^{t_0})(p_{t_0})$ (volume decrease), and
- incompressibility if $J_{|\vec{E},\vec{e}}(\Phi_{t_2}^{t_0})(p_{t_0}) = J_{|\vec{E},\vec{e}}(\Phi_{t_1}^{t_0})(p_{t_0})$ for all $t$ (volume conservation).

In particular, if $(\vec{e}_i) = (\vec{E}_i)$ then $J_{|\vec{e},\vec{e}}(\Phi_{t_0}^{t_0})(p_{t_0}) = 1$, and if $t > t_0$, then
- Dilatation if $J_{|\vec{e},\vec{e}}(\Phi_t^{t_0})(p_{t_0}) > 1$ (volume increase),
- contraction if $J_{|\vec{e},\vec{e}}(\Phi_t^{t_0})(p_{t_0}) < 1$ (volume decrease), and
- incompressibility if $J_{|\vec{e},\vec{e}}(\Phi_t^{t_0})(p_{t_0}) = 1$ for all $t$ (volume conservation).

**Exercice L.26** Let $(\vec{E}_i)$ be a Euclidean basis in $\mathbb{R}^n_{t_0}$, and let $(\vec{a}_i)$ and $(\vec{b}_i)$ be two Euclidean bases in $\mathbb{R}^n_t$ for the same Euclidean dot product $(\cdot,\cdot)_g$. Prove:

$$J_{|\vec{E},\vec{a}}(\Phi^{t_0}_t(P)) = \pm J_{|\vec{E},\vec{b}}(\Phi^{t_0}_t(P)). \tag{L.34}$$

**Answer**. $P$ being the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, $\det(P)=\pm1$ here. And (4.28) gives $[F]_{|\vec{E},\vec{a}} = P.[F]_{|\vec{E},\vec{b}}$, thus $\det([F]_{|\vec{E},\vec{a}}) = \pm\det([F]_{|\vec{E},\vec{b}})$, thus $\det_{|\vec{a}}(F.\vec{E}_1,...,F.\vec{E}_n) = \pm\det_{|\vec{b}}(F.\vec{E}_1,...,F.\vec{E}_n)$. ▪

### L.7.3 Determinant of the transposed

Let $(A,(\cdot,\cdot)_g)$ and $(B,(\cdot,\cdot)_h)$ be finite dimensional Hilbert spaces. Let $L\in\mathcal{L}(A;B)$ (a linear map). Recall: The transposed $L^T_{gh}\in\mathcal{L}(B;A)$ is defined by, for all $\vec{u}\in A$ and all $\vec{w}\in B$, cf. (A.47)

$$(L^T_{gh}.\vec{w},\vec{u})_g := (\vec{w},L.\vec{u})_h. \tag{L.35}$$

Thus if $(\vec{a}_i)$ is a basis in $A$ and $(\vec{b}_i)$ is a basis in $B$ then

$$\widetilde{\det}([L^T_{gh}]_{|\vec{b},\vec{a}}) = \det([L]_{|\vec{a},\vec{b}})\frac{\det([(\cdot,\cdot)_g]_{|\vec{a}})}{\det([(\cdot,\cdot)_h]_{|\vec{b}})}. \tag{L.36}$$

Indeed, (L.35) gives $[(\cdot,\cdot)_g]_{|\vec{a}}.[L^T_{gh}]_{|\vec{b},\vec{a}} = ([L]_{|\vec{a},\vec{b}})^T.[(\cdot,\cdot)_h]_{|\vec{b}}$.

## L.8 Dilatation rate

A unique Euclidean basis $(\vec{e}_i)$ at all time is chosen, and $(\cdot,\cdot)_g$ is the associated inner dot product. $C^2$ motion. The Eulerian velocity is $\vec{v}(t,p_t) = \frac{\partial\widetilde{\Phi}}{\partial t}(t,P_{Obj})$ at $p_t=\widetilde{\Phi}(t,P_{Obj})$. $t_0$ is fixed, the Lagrangian velocity is $\vec{V}^{t_0}(t,p_{t_0}) = \frac{\partial\Phi^{t_0}}{\partial t}(t,p_{t_0}) = \vec{v}(t,p_t)$ at $p_t=\widetilde{\Phi}(t,P_{Obj}) = \Phi^{t_0}(t,p_{t_0})$, the deformation gradient is $F^{t_0}(t,p_{t_0}) := d\Phi^{t_0}_t(t,p_{t_0})$, the Jacobian is

$$J^{t_0}(t,p_{t_0}) = \det_{|\vec{e}}(F^{t_0}(t,p_{t_0})), \quad \text{and} \quad J^{t_0} \overset{\text{written}}{=} \det(F^{t_0}). \tag{L.37}$$

$\mathcal{O}$ is a origin in $\mathbb{R}^n$, $\overrightarrow{\mathcal{O}\Phi^{t_0}} = \sum_{i=1}^n\Phi^i\vec{e}_i$, $\vec{V}^{t_0} = \sum_{i=1}^n V^i\vec{e}_i$, $\vec{v} = \sum_{i=1}^n v^i\vec{e}_i$, $V^i(t,p_{t_0}) = v^i(t,p_t) = v^i(t,\Phi^{t_0}(t,p_{t_0}))$, $d\Phi^{t_0}(t,p_{t_0}).(.) = \sum_{i=1}^n(d\Phi^i(t,p_{t_0}).(.))\vec{e}_i$, $F^{t_0}.\vec{e}_j = \sum_{i=1}^n F_{ij}\vec{e}_i$, $F_{ij} = d\Phi^i.\vec{e}_j = \frac{\partial\Phi^i}{\partial X^j}$, $[d\Phi^i]_{|\vec{e}} = \begin{pmatrix}\frac{\partial\Phi^i}{\partial X^1} & \dots & \frac{\partial\Phi^i}{\partial X^n}\end{pmatrix} =^{\text{written}} d\Phi^i$ (row matrix)

### L.8.1 $\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0}) = J^{t_0}(t,p_{t_0})\,\mathrm{div}\vec{v}(t,p_t)$

**Lemma L.27** *$\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0})$ satisfies, with $p_t=\Phi^{t_0}_t(p_{t_0})$,*

$$\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0}) = J^{t_0}(t,p_{t_0})\,\mathrm{div}\vec{v}(t,p_t) \tag{L.38}$$

*(value to be considered at $t$ at $p_t$). In particular, $\widetilde{\Phi}$ is incompressible iff $\mathrm{div}\vec{v}(t,p_t)=0$.*

**Proof.** A determinant is multilinear thus

$$J^{t_0} = \det F^{t_0} = \det\begin{pmatrix} d\Phi^1\\ \vdots\\ d\Phi^n\end{pmatrix}, \quad \frac{\partial J^{t_0}}{\partial t} = \det\begin{pmatrix}\frac{\partial(d\Phi^1)}{\partial t}\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} + \ldots + \det\begin{pmatrix} d\Phi^1\\ \vdots\\ d\Phi^{n-1}\\ \frac{\partial(d\Phi^n)}{\partial t}\end{pmatrix}.$$

With $\Phi^{t_0}$ $C^2$, $\frac{\partial(d\Phi^i)}{\partial t}(t,p_{t_0}) = d(\frac{\partial\Phi^i}{\partial t})(t,p_{t_0}) = dV^i(t,p_{t_0}) = dv^i(t,p_t).F^{t_0}(t,p_{t_0})$. Thus

$$\det\begin{pmatrix}\frac{\partial(d\Phi^1)}{\partial t}\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} = \det\begin{pmatrix}\sum_{i=1}^n\frac{\partial v^1}{\partial x^i}d\Phi^i\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} \underset{\text{alternating}}{\overset{\text{det is}}{=}} \det\begin{pmatrix}\frac{\partial v^1}{\partial x^1}d\Phi^1\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} = \frac{\partial v^1}{\partial x^1}\det\begin{pmatrix}d\Phi^1\\ d\Phi^2\\ \vdots\\ d\Phi^n\end{pmatrix} = \frac{\partial v^1}{\partial x^1}J^{t_0}$$

Idem for the other terms, thus

$$\frac{\partial J^{t_0}}{\partial t}(t,p_{t_0}) = \frac{\partial v^1}{\partial x^1}(t,p_t)\;J(t,p_{t_0}) + \ldots + \frac{\partial v^n}{\partial x^n}(t,p_t)\;J(t,p_{t_0}) = \mathrm{div}\vec{v}(t,p_t)\;J^{t_0}(t,p_{t_0}),$$

i.e. (L.38). ▪

**Definition L.28** $\mathrm{div}\vec{v}(t,p_t)$ is the dilatation rate.

### L.8.2 Leibniz formula

**Proposition L.29 (Leibniz formula)** *Under regularity assumptions (e.g. hypotheses of the Lebesgue theorem to be able to differentiate under $\int$) we have*

$$\begin{aligned}\frac{d}{dt}\int_{p\in\Omega_t} f\,d\Omega &= \int_{p\in\Omega_t} \frac{Df}{Dt} + f\,\mathrm{div}\vec{v}\,d\Omega \\ &= \int_{p\in\Omega_t} \frac{\partial f}{\partial t} + df.\vec{v} + f\,\mathrm{div}(\vec{v})\,d\Omega = \int_{p\in\Omega_t} \frac{\partial f}{\partial t} + \mathrm{div}(f\vec{v})\,d\Omega.\end{aligned} \tag{L.39}$$

**Proof.** With $p_t = \Phi^{t_0}(t,P) \in \Omega_t = \Phi^{t_0}(t,\Omega_{t_0})$, the Jacobian being positive (regular motion),

$$Z(t) = \int_{p_t\in\Omega_t} f(t,p_t)\,d\Omega = \int_{P\in\Omega_{t_0}} f(t,\Phi^{t_0}(t,P))\,J^{t_0}(t,P)\,d\Omega_{t_0}.$$

Thus

$$\begin{aligned}\frac{dZ}{dt}(t) &= \int_{P\in\Omega_{t_0}} \frac{Df}{Dt}(t,\Phi^{t_0}(t,P))\,J^{t_0}(t,P)\,d\Omega_{t_0} + \int_{P\in\Omega_{t_0}} f(t,\Phi^{t_0}(t,P)) \underbrace{\frac{\partial J^{t_0}}{\partial t}(t,P)}_{J^{t_0}(t,P)\,\mathrm{div}\vec{v}(t,p(t))} d\Omega_{t_0} \\ &= \int_{p_t\in\Omega_t} \frac{Df}{Dt}(t,p_t)\,d\Omega_t + \int_{p_t\in\Omega_t} f(t,p_t)\mathrm{div}\vec{v}(t,p_t)\,d\Omega_t.\end{aligned}$$

And $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + df.\vec{v}$ and $\mathrm{div}(f\vec{v}) = df.\vec{v} + f\,\mathrm{div}\vec{v}$. ∎

**Corollary L.30** *With $(\vec{u},\vec{w})_g =^{\text{written}} \vec{u}\bullet\vec{w}$ (in the given Euclidean framework),*

$$\frac{d}{dt}\int_{\Omega_t} f(t,p_t)\,d\Omega_t = \int_{\Omega_t} \frac{\partial f}{\partial t}(t,p_t)\,d\Omega_t + \int_{\partial\Omega_t} (f\vec{v}\bullet\vec{n})(t,p_t)\,d\Gamma_t, \tag{L.40}$$

*sum of the temporal variation within $\Omega_t$ and the flux through the surface $\partial\Omega_t$.*

**Proof.** Apply the last equality of (L.39). ∎

## L.9 $\partial J/\partial F = J\,F^{-T}$

### L.9.1 Meaning of $\frac{\partial \det}{\partial M_{ij}}$?

$\mathcal{M}_{nn} = \{M = [M_{ij}] \in \mathbb{R}^{n^2}\}$ is the set of $n*n$ matrices. Consider the function

$$Z := \det : \left\{ \begin{array}{rl} \mathcal{M}_{nn} & \to \mathbb{R} \\ M = [M_{ij}] & \to Z(M) := \det(M) = \det([M_{ij}]). \end{array}\right. \tag{L.41}$$

Question: What does $\frac{\partial Z}{\partial M_{ij}}(M)$ mean?

Answer: It is the “standard meaning” of a directional derivative $\frac{\partial f}{\partial x_i}(\vec{x}) = df(\vec{x}).\vec{e}_i$... where here $f = Z$, thus $\vec{x} =^{\text{written}} M$ is a matrix (a vector in $\mathcal{M}_{nn}$), and the canonical basis $(\vec{e}_i)$ is the canonical basis $(m_{ij})$ in $\mathcal{M}_{nn}$ (all the elements of the matrix $m_{ij}$ vanish but the element at intersection of line $i$ and column $j$ which equals 1). So:

$$\frac{\partial Z}{\partial M_{ij}}(M) := dZ(M).m_{ij} = \lim_{h\to 0} \frac{Z(M+hm_{ij}) - Z(M)}{h} \quad (\in \mathbb{R}). \tag{L.42}$$

### L.9.2 Calculation of $\frac{\partial \det}{\partial M_{ij}}$

**Proposition L.31**

$$\forall i,j,\quad \frac{\partial Z}{\partial M_{ij}}(M) = Z(M)\,(M^{-T})_{ij}, \quad \textit{written} \quad \frac{\partial Z}{\partial M} = Z.M^{-T}. \tag{L.43}$$

**Proof.** $\frac{\partial Z}{\partial M_{ij}}(M) := \lim_{h\to 0}\frac{\det(M+hm_{ij})-\det(M)}{h}$; The development of the determinant $\det(M+hm_{ij})$ relative to the column $j$ gives

$$\det(M+h[m_{ij}]) = \det(M) + h\,c_{ij} \tag{L.44}$$

where $c_{ij}$ is the $(i,j)$-th cofactor of $M$; Thus $\frac{\partial Z}{\partial M_{ij}}(M) = \lim_{h\to 0}\frac{Z(M+hm_{ij})-Z(M)}{h} = c_{ij}$; And since $M^{-1} = \frac{1}{\det(M)}[c_{ij}]^T$, i.e. $[c_{ij}] = \det(M)M^{-T}$, we get $\frac{\partial Z}{\partial M_{ij}}(M) = \det(M)(M^{-T})_{ij}$, i.e. (L.43). ∎

### L.9.3 $\partial J/\partial F = J\,F^{-T}$

Setting of § L.8 with $t_0$ implicit: $F := d\Phi(p_{t_0})$ with $F.\vec{E}_j = \sum_{i=1}^n F_{ij}\vec{e}_i$ where $F_{ij} = \frac{\partial \Phi_i}{\partial X_j}(p_{t_0})$, and

$$\widehat{J}_{|\vec{E},\vec{e}} \overset{\text{written}}{=} \widehat{J} : \begin{cases} \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t) & \to \mathbb{R} \\ \qquad F & \to \widehat{J}(F) := \det([F_{ij}]) \quad (= \det([\frac{\partial \Phi_i}{\partial X_j}(p_{t_0})] = \widetilde{\underset{|\vec{E},\vec{e}}{\det}}(d\Phi(p_{t_0}))), \end{cases} \tag{L.45}$$

so, $\widehat{J}(F) = J(\Phi)$ is the Jacobian of $\Phi$ at $p_{t_0}$ relative to $(\vec{E}_i)$ and $(\vec{e}_i)$. Thus (L.43) gives:

**Corollary L.32**

$$\forall i,j,\quad \frac{\partial \widehat{J}}{\partial F_{ij}}(F) = \widehat{J}(F)\,([F]^{-T})_{ij}, \quad \textit{written} \quad \frac{\partial J}{\partial F} = J\,F^{-T}. \tag{L.46}$$

### L.9.4 Interpretation of $\frac{\partial J}{\partial F_{ij}}$?

The first derivations into play are along the directions $\vec{E}_j$ at $t_0$ because $F_{ij} = \frac{\partial \Phi_i}{\partial X_j} := d\Phi_i.\vec{E}_j$, when $\Phi = \sum_i \Phi_i\vec{e}_i$, so $F.\vec{E}_j = F_{ij}\vec{e}_i$.

Question: What does $\frac{\partial J}{\partial F_{ij}}$ mean ? That is, derivative of $J$ in which direction(s) ?

Answer: 1- "Identify" $F \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ with the tensor $\widetilde{F} \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t,\vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ given by $\widetilde{F}(\ell,\vec{U}) = \ell.(F.\vec{U})$; So $F.\vec{E}_j = \sum_{i=1}^n F_{ij}\vec{e}_i$ iff $\widetilde{F} = \sum_{i,j=1}^n F_{ij}\vec{e}_i \otimes \pi_{Ej}$, where $\vec{e}_i$ is a basis in $\vec{\mathbb{R}}^n_t$ and $(\pi_{Ei})$ is the covariant dual basis of $(\vec{E}_i)$ basis in $\vec{\mathbb{R}}^n_{t_0}$.

2- Define the function Jac : $\left\{\begin{array}{ll} \mathcal{L}(\vec{\mathbb{R}}^{n*}_t,\vec{\mathbb{R}}^n_{t_0};\mathbb{R}) & \to \mathbb{R} \\ \qquad \widetilde{F} & \to \text{Jac}(\widetilde{F}) := J(F) = \underset{\vec{E},\vec{e}}{\det}(F) \end{array}\right\}$.

3- Then it is meaningful to differentiate Jac along the direction $\vec{e}_i \otimes \pi_{Ej} \in \mathcal{L}(\vec{\mathbb{R}}^{n*}_t,\vec{\mathbb{R}}^n_{t_0};\mathbb{R})$ to get

$$\frac{\partial \text{Jac}}{\partial F_{ij}}(\widetilde{F}) = \lim_{h\to 0}\frac{\text{Jac}(\widetilde{F}+h\vec{e}_i\otimes\pi_{Ej}) - \text{Jac}(\widetilde{F})}{h} \overset{\text{written}}{=} \frac{\partial J}{\partial F_{ij}}(F). \tag{L.47}$$

(Duality notation: $\frac{\partial \text{Jac}}{\partial F^i_j}(\widetilde{F}) = \lim_{h\to 0}\frac{\text{Jac}(\widetilde{F}+h\vec{e}_i\otimes E^J)-\text{Jac}(\widetilde{F})}{h}$.)

4- So $\frac{\partial \text{Jac}}{\partial F_{ij}}$ is a derivation in both directions $\vec{e}_i$ in $\vec{\mathbb{R}}^n_t$ (present at $p_t$) and $\pi_{Ej}$ in $\vec{\mathbb{R}}^n_{t_0}$ (past at $p_{t_0}$ and dual basis vector).

So, what does this derivative mean? (The author does not know.)

# M Transport of volumes and areas

Here $\mathbb{R}^n = \mathbb{R}^3$ the usual affine space, $t_0, t \in \mathbb{R}$, $\Phi := \Phi^{t_0}_t : \mathbb{R}\times\Omega_{t_0} \to \Omega_t$ is a regular motion, and $F_P = d\Phi(P)$. We need a $(\cdot,\cdot)_g$ be a Euclidean dot product in $\vec{\mathbb{R}}^n$, the same at all time. And $(\vec{E}_i)$ and $(\vec{e}_i)$ are $(\cdot,\cdot)_g$-Euclidean bases in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$.

## M.1 Transport of volumes

### M.1.1 Transformed parallelepiped

The Jacobian of $\Phi$ at $P$ relative to the chosen Euclidean bases is

$$J_P = J(P) := \det_{|\vec{E},\vec{e}}(F_t^{t_0}(P)) \quad (= \det_{|\vec{e}}(F_t^{t_0}(P).\vec{E}_1, ..., F_t^{t_0}(P).\vec{E}_n))), \tag{M.1}$$

cf. (L.33); The motion being regular, $J_P > 0$. And if $(\vec{U}_{1P}, ..., \vec{U}_{nP})$ is a parallelepiped at $t_0$ at $P$, if $\vec{u}_{ip} = F_P.\vec{U}_{iP}$, then $(\vec{u}_{1p}, ..., \vec{u}_{np})$ is a parallelepiped at $t$ at $p = \Phi(P)$ which algebraic volume is

$$\det_{|\vec{e}}(\vec{u}_{1p}, ..., \vec{u}_{np}) = J_P \det_{|\vec{E}}(\vec{U}_{1P}, ..., \vec{U}_{nP}). \tag{M.2}$$

### M.1.2 Transformed volumes

Riemann integrals and (M.2) give the change of variable formula: For any regular function $f : \Omega_t \to \mathbb{R}$,

$$\int_{p\in\Omega_t} f(p)\, d\Omega_t = \int_{P\in\Omega_{t_0}} f(\Phi(P))\, |J(P)|\, d\Omega_{t_0}. \tag{M.3}$$

Here $J_P > 0$ (regular motion), hence

$$\int_{p\in\Omega_t} f(p)\, d\Omega_t = \int_{P\in\Omega_{t_0}} f(\Phi(P))\, J(P)\, d\Omega_{t_0}. \tag{M.4}$$

In particular, $|\Omega_t| = \int_{p\in\Omega_t} d\Omega_t = \int_{P\in\Omega_{t_0}} J(P)\, d\Omega_{t_0}$.

## M.2 Transformed surface

### M.2.1 Transformed parallelogram and its area

Consider two independent vectors $\vec{U}_{1P}, \vec{U}_{2P}$ in $\vec{\mathbb{R}}^n_{t_0}$ at $t_0$ at $P$, and, $\Phi$ being a diffeomorphism, the two independent vectors $\vec{u}_{1p} = F_P.\vec{U}_{1P}$ and $\vec{u}_{2p} = F_P.\vec{U}_{2P}$ in $\vec{\mathbb{R}}^n_t$ at $t$ at $p = \Phi(P)$. The areas of the associated quadrilaterals are $||\vec{U}_{1P} \times \vec{U}_{2P}||_g$ $||\vec{u}_{1p} \times \vec{u}_{2p})||_g$, and the unit normal vectors to the quadrilaterals are (up to the sign)

$$\vec{N}_P = \frac{\vec{U}_{1P} \times \vec{U}_{2P}}{||\vec{U}_{1P} \times \vec{U}_{2P}||_g}, \quad \text{and} \quad \vec{n}_p = \frac{\vec{u}_{1p} \times \vec{u}_{2p}}{||\vec{u}_{1p} \times \vec{u}_{2p}||_g}. \tag{M.5}$$

**Proposition M.1**

$$\vec{u}_{1p} \times \vec{u}_{2p} = J_P\, F_P^{-T}.\big(\vec{U}_{1P} \times \vec{U}_{2P}\big), \quad \textit{in short} \quad \vec{u}_1 \times \vec{u}_2 = J\, F^{-T}.\big(\vec{U}_1 \times \vec{U}_2\big), \tag{M.6}$$

*and*

$$\vec{n}_p = \frac{F_P^{-T}.\vec{N}_P}{||F_P^{-T}.\vec{N}_P||_g} \quad (\neq F_P.\vec{N}_P \textit{ in general}), \quad \textit{in short} \quad \vec{n} = \frac{F^{-T}.\vec{N}}{||F^{-T}.\vec{N}||_g}. \tag{M.7}$$

*Thus with the (unique) polar decomposition* $F_P =^{(I.17)} R_P.U_p$, *where* $U_P^T = U_P > 0$ *in* $\mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_{t_0})$ *and* $R_P^{-1} = R_P^T \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t)$ *(rigid body motion), we have* $\vec{n}_p = \frac{R_P.(U_P^{-1}.\vec{N}_P)}{||R_P.(U_P^{-1}.\vec{N}_P)||_g} = R_P.\frac{U_P^{-1}.\vec{N}_P}{||U_P^{-1}.\vec{N}_P||_g}$.

**Proof.** Let $\vec{W}_P \in \vec{\mathbb{R}}^n_{t_0}$, and $\vec{w}_p = F_P.\vec{W}_P$. The volume of the parallelepiped $(\vec{u}_{1p}, \vec{u}_{2p}, \vec{w}_p)$ is

$$\begin{aligned}(\vec{u}_{1p} \times \vec{u}_{2p}, \vec{w}_p)_g &= \det_{|\vec{e}}(\vec{u}_{1p}, \vec{u}_{2p}, \vec{w}_p) = J_P \det_{|\vec{E}}(\vec{U}_{1P}, \vec{U}_{2P}, \vec{W}_P) = J_P\, (\vec{U}_{1P} \times \vec{U}_{2P}, \vec{W}_P)_g \\ &= J_P\, (\vec{U}_{1P} \times \vec{U}_{2P}, F_P^{-1}.\vec{w}_p)_g = J_P\, (F_P^{-T}.(\vec{U}_{1P} \times \vec{U}_{2P}), \vec{w}_p)_g,\end{aligned}$$

for all $\vec{w}_p$, thus (M.6), thus $\frac{\vec{u}_{1p}\times\vec{u}_{2p}}{||\vec{u}_{1p}\times\vec{u}_{2p}||_g} = \frac{J_P\, F_P^{-T}.(\vec{U}_{1P}\times\vec{U}_{2P})}{J_P||F_P^{-T}.(\vec{U}_{1P}\times\vec{U}_{2P})||_g}$ (here $J_P > 0$), thus (M.7). ∎

### M.2.2 Deformation of a surface

A parametrized surface $\Psi_{t_0}$ in $\Omega_{t_0}$ and the associated geometric surface $S_{t_0}$ are defined by

$$\Psi_{t_0} : \left\{ \begin{array}{rl} [a,b]\times[c,d] & \to \Omega_{t_0} \\ (u,v) & \to P = \Psi_{t_0}(u,v) \end{array} \right\} \quad \text{and} \quad S_{t_0} = \mathrm{Im}(\Psi_{t_0}) \subset \Omega_{t_0}. \tag{M.8}$$

Consider the basis $(\vec{E}_1 = (1,0), \vec{E}_2 = (0,1))$ in the space $\mathbb{R}\times\mathbb{R} \supset [a,b]\times[c,d] = \{(u,v)\}$ of parameters, and suppose that $\Psi_{t_0}$ is regular. Thus the tangent vectors at $P = \Psi_{t_0}(u,v) \in S_{t_0}$ given by

$$\left\{ \begin{aligned} \vec{T}_{1P} &:= d\Psi_{t_0}(u,v).\vec{E}_1 \overset{\text{written}}{=} \frac{\partial \Psi_{t_0}}{\partial u}(u,v), \\ \vec{T}_{2P} &:= d\Psi_{t_0}(u,v).\vec{E}_2 \overset{\text{written}}{=} \frac{\partial \Psi_{t_0}}{\partial v}(u,v), \end{aligned} \right. \tag{M.9}$$

are independent: $\vec{T}_{1P} \times \vec{T}_{2P} \neq \vec{0}$.

Call $\Psi_t := \Phi_t^{t_0} \circ \Psi_{t_0} = \Phi \circ \Psi_{t_0}$ and $S_t$ the transformed parametric and geometric surfaces:

$$\Psi_t := \Phi \circ \Psi_{t_0} : \left\{ \begin{array}{rl} [a,b]\times[c,d] & \to \Omega_{t_0} \\ (u,v) & \to p = \Psi_t(u,v) = \Phi(\Psi_{t_0}(u,v)) \ \ (= \Phi(P)) \end{array} \right\} \quad \text{and} \quad S_t = \Phi(S_{t_0}). \tag{M.10}$$

The tangent vectors at $S_t$ at $p = \Phi_t^{t_0}(P)$ at $t$:

$$\left\{ \begin{aligned} \vec{t}_{1p} &:= d\Psi_t(u,v).\vec{E}_1 = \frac{\partial \Psi_t}{\partial u}(u,v) = d\Phi_t^{t_0}(P).\frac{\partial \Psi_{t_0}}{\partial u}(u,v), \quad \text{i.e.} \quad \vec{t}_{1p} = F_P.\vec{T}_{1P}, \\ \vec{t}_{2p} &:= d\Psi_t(u,v).\vec{E}_2 = \frac{\partial \Psi_t}{\partial v}(u,v) = d\Phi_t^{t_0}(P).\frac{\partial \Psi_{t_0}}{\partial v}(u,v), \quad \text{i.e.} \quad \vec{t}_{2p} = F_P.\vec{T}_{2P}, \end{aligned} \right. \tag{M.11}$$

are independent since $\Phi_t^{t_0}$ is a diffeomorphism and $\Psi_{t_0}$ is regular.

### M.2.3 Euclidean dot product and unit normal vectors

Relative to $(\cdot,\cdot)_g$, the scalar area elements $d\Sigma_P$ at $P$ at $S_{t_0}$ relative to $\Psi_{t_0}$, and $d\sigma_p$ at $p$ at $S_t$ relative to $\Psi_t$, are

$$\left\{ \begin{aligned} d\Sigma_P &:= ||\frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v)||_g \, du\, dv \quad (= ||\vec{T}_{1P} \times \vec{T}_{2P}||_g \, du\, dv), \\ d\sigma_p &:= ||\frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v)||_g \, du\, dv \quad (= ||\vec{t}_{1p} \times \vec{t}_{2p}||_g \, du\, dv). \end{aligned} \right. \tag{M.12}$$

And the areas of $S_{t_0}$ and $S_t$ are

$$\left\{ \begin{aligned} |S_{t_0}| &= \int_{P\in S_{t_0}} d\Sigma_P = \int_{u=a}^{b} \int_{v=c}^{d} ||\frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v)||_g \, du\, dv, \\ |S_t| &= \int_{p\in S_t} d\sigma_p = \int_{u=a}^{b} \int_{v=c}^{d} ||\frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v)||_g \, du\, dv. \end{aligned} \right. \tag{M.13}$$

And the unit normal vectors $\vec{N}_P$ at $S_{t_0}$ at $P$ at $t_0$ and $\vec{n}_p$ at $S_t$ at $p$ at $t$ are (up to the sign)

$$\left\{ \begin{aligned} \vec{N}_P &= \frac{\frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v)}{||\frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v)||_g} \quad (= \frac{\vec{T}_{1P} \times \vec{T}_{2P}}{||\vec{T}_{1P} \times \vec{T}_{2P}||_g}) \\ \vec{n}_p &= \frac{\frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v)}{||\frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v)||_g} \quad (= \frac{\vec{t}_{1p} \times \vec{t}_{2p}}{||\vec{t}_{1p} \times \vec{t}_{2p}||_g}). \end{aligned} \right. \tag{M.14}$$

And the vectorial area elements $d\vec{\Sigma}_P$ at $P$ at $S_{t_0}$ and $d\vec{\sigma}_p$ at $p$ at $S_t$ are

$$\left\{ \begin{aligned} d\vec{\Sigma}_P &:= \vec{N}_P \, d\Sigma_P = \frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v) \, du\, dv \quad (= \vec{T}_{1P} \times \vec{T}_{2P} \, du\, dv) \\ d\vec{\sigma}_p &:= \vec{n}_p \, d\sigma_p = \frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v) \, du\, dv \quad (= \vec{t}_{1p} \times \vec{t}_{2p} \, du\, dv). \end{aligned} \right. \tag{M.15}$$

(And the flux through a surface is $\int_\Gamma \vec{f} \bullet \vec{n} \, d\sigma =^{\text{written}} \int_\Gamma \vec{f} \bullet d\vec{\sigma}$.)

### M.2.4 Relations between area elements

$\vec{t}_{1p} \times \vec{t}_{2p} = J_P\, F_P^{-T}.(\vec{T}_{1P} \times \vec{T}_{2P})$, cf. (M.6), gives

$$\frac{\partial \Psi_t}{\partial u}(u,v) \times \frac{\partial \Psi_t}{\partial v}(u,v) = J_P\, F_P^{-T}.(\frac{\partial \Psi_{t_0}}{\partial u}(u,v) \times \frac{\partial \Psi_{t_0}}{\partial v}(u,v)). \tag{M.16}$$

And

$$\vec{n}\, d\sigma_p = \boxed{d\vec{\sigma}_p = J_P\, F_P^{-T}.d\vec{\Sigma}_P} = J_P\, F_P^{-T}.\vec{N}_P\, d\Sigma_P, \quad \text{and} \quad d\sigma_p = J_P\, ||F_P^{-T}.\vec{N}_P||_g\, d\Sigma_P. \tag{M.17}$$

(Check with (M.7).)

### M.2.5 Piola identity...

The divergence (in continuum mechanics) of a $3 * 3$ matrix function $M = [M^i_j]$ is: $\text{div}M := \begin{pmatrix} \sum_{j=1}^n \frac{\partial M^1_j}{\partial X^j} \\ \sum_{j=1}^n \frac{\partial M^2_j}{\partial X^j} \\ \sum_{j=1}^n \frac{\partial M^3_j}{\partial X^j} \end{pmatrix} \quad (= \begin{pmatrix} \frac{\partial M^1_1}{\partial X^1} + \frac{\partial M^1_2}{\partial X^2} + \frac{\partial M^1_3}{\partial X^3} \\ \frac{\partial M^2_1}{\partial X^1} + \frac{\partial M^2_2}{\partial X^2} + \frac{\partial M^2_3}{\partial X^3} \\ \frac{\partial M^3_1}{\partial X^1} + \frac{\partial M^3_2}{\partial X^2} + \frac{\partial M^3_3}{\partial X^3} \end{pmatrix})$, cf. (T.88). Its matrix of cofactors $\text{Cof}(M)$ is given by $\text{Cof}(M)^i_j = M^{i+1}_{j+1} M^{i+2}_{j+2} - M^{i+1}_{j+2} M^{i+2}_{j+1}$, and $(\det M) M^{-1} = \text{Cof}(M)^T$.

Application: $\det([F(P)]_{|\vec{E},\vec{e}})\, ([F(P)]_{|\vec{E},\vec{e}})^{-T} = \text{Cof}([F(P)]_{|\vec{E},\vec{e}})$; Written in short $\det(F(P))\, F(P)^{-T} = \text{Cof}(F(P))$ (matrix meaning); So, in $\Omega_{t_0}$,

$$JF^{-T} = \text{Cof}(F) \quad \text{(matrix meaning)}. \tag{M.18}$$

**Proposition M.2 (Piola identity)**

$$\text{div}(JF^{-T}) = 0, \quad \textit{i.e.} \quad \forall i,\ \forall P,\ \sum_{j=1}^n \frac{\partial \text{Cof}(F)_{ij}}{\partial X_j}(P) = 0 \ \textit{ or } \sum_{J=1}^n \frac{\partial \text{Cof}(F)^i_J}{\partial X^J}(P) = 0. \tag{M.19}$$

*Also sometimes ambiguously written* $\sum_{j=1}^n \frac{\partial}{\partial X_j}(J \frac{\partial X_i}{\partial x_j}) = 0$ *or* $\sum_{J=1}^n \frac{\partial}{\partial X^J}(\text{Jac}(\frac{\partial X^i}{\partial x^J})) = 0...$

**Proof.** $\text{Cof}(F)^i_j = F^{i+1}_{j+1} F^{i+2}_{j+2} - F^{i+1}_{j+2} F^{i+2}_{j+1} = \frac{\partial \varphi^{i+1}}{\partial X^{j+1}} \frac{\partial \varphi^{i+2}}{\partial X^{j+2}} - \frac{\partial \varphi^{i+1}}{\partial X^{j+2}} \frac{\partial \varphi^{i+2}}{\partial X^{j+1}}$. Thus

$$\frac{\partial \text{Cof}(F)^i_j}{\partial X^j} = \frac{\partial^2 \varphi^{i+1}}{\partial X^j \partial X^{j+1}} \frac{\partial \varphi^{i+2}}{\partial X^{j+2}} + \frac{\partial \varphi^{i+1}}{\partial X^{j+1}} \frac{\partial^2 \varphi^{i+2}}{\partial X^j \partial X^{j+2}} - \frac{\partial^2 \varphi^{i+1}}{\partial X^j \partial X^{j+2}} \frac{\partial \varphi^{i+2}}{\partial X^{j+1}} - \frac{\partial \varphi^{i+1}}{\partial X^{j+2}} \frac{\partial^2 \varphi^{i+2}}{\partial X^j \partial X^{j+1}}.$$

And summation: The terms cancel out two by two. ∎

### M.2.6 ... and Piola transformation

Goal: for a $\vec{u} : \Omega_t \to \vec{\mathbb{R}}^n_t$, find $\vec{U}_{\text{Piola}} : \Omega_{t_0} \to \vec{\mathbb{R}}^n_{t_0}$ s.t., for all $\omega_t = \Phi^{t_0}_t(\omega_{t_0})$ (with $\omega_{t_0}$ open subset in $\Omega_{t_0}$),

$$\int_{\partial\omega_{t_0}} \vec{U}_{\text{Piola}} \bullet \vec{N}\, d\Sigma = \int_{\partial\omega_t} \vec{u} \bullet \vec{n}\, d\sigma, \tag{M.20}$$

i.e.

$$\int_{\omega_{t_0}} \text{div}\vec{U}_{\text{Piola}}\, d\Omega_{t_0} = \int_{\omega_t} \text{div}\vec{u}\, d\Omega_t, \tag{M.21}$$

i.e., with (M.4), for all $P \in \Omega_{t_0}$,

$$\text{div}\vec{U}_{\text{Piola}}(P) = J(P)\, \text{div}\vec{u}(\Phi(P)). \tag{M.22}$$

**Proposition M.3** *With* $p = \Phi(P)$,

$$\vec{U}_{\text{Piola}}(P) = J(P) F(P)^{-1}.\vec{u}(p), \tag{M.23}$$

*i.e.,* $\vec{U}_{\text{Piola}} := J\, \Phi^*(\vec{u})$, *i.e.* $= J$ *times the pull-back of* $\vec{u}$ *by* $\Phi$. *Hence*

$$\int_{p\in\partial\omega_t} \vec{u}(p) \bullet \vec{n}(p)\, d\sigma = \int_{P\in\partial\omega_{t_0}} \big(J(P)F(P)^{-1}.\vec{u}(\Phi(P))\big) \bullet \vec{N}(P)\, d\Sigma. \tag{M.24}$$

**Proof.** $d(\vec{u}\circ\Phi)(P) = d\vec{u}(p).F(P)$, thus

$$\begin{aligned}\mathrm{div}((JF^{-1}).(\vec{u}\circ\Phi_t^{t_0}))(P) &\overset{(T.81)}{=} \widetilde{\mathrm{div}}_P(JF^{-1})(P).\vec{u}(\Phi_t^{t_0}(P)) + J(P)F(P)^{-1} \odot (d\vec{u}(p).F(P))\\ &= \mathrm{div}(JF^{-T})(P)\bullet\vec{u}(p) + J(P)(F(P).F(P)^{-1}) \odot d\vec{u}(p)\\ &\overset{(M.19)}{=} 0 + J(P)I_t \odot d\vec{u}(p) = J(P)\mathrm{div}\vec{u}(p),\end{aligned}$$

thus $\vec{U}_{\mathrm{Piola}}(P) := J(P)F(P)^{-1}.\vec{u}(p)$ satisfies (M.22). (Check with components if you prefer.) ∎

**Definition M.4** The Piola transform is the map (between vector fields in $\Omega_t$ and $\Omega_{t_0}$)

$$\begin{cases} T\Omega_t \to T\Omega_{t_0} \\ \vec{u} \to \vec{U}_{\mathrm{Piola}}, \quad \vec{U}_{\mathrm{Piola}}(P) := J(P)F(P)^{-1}.\vec{u}(p) \quad \text{when} \quad p = \Phi_t^{t_0}(P). \end{cases} \tag{M.25}$$

# N Conservation of mass

Motion $\widetilde{\Phi} : [t_1,t_2]\times Obj \to \mathbb{R}^n$, $\Omega_t = \widetilde{\Phi}(t, Obj)$, $\vec{v}(t,p_t) = \frac{\partial\widetilde{\Phi}}{\partial t}(t, P_{Obj})$ associated Eulerian velocity at $t$ at $p_t = \widetilde{\Phi}(t, P_{Obj})$, $t_0$ fixed, $\Phi^{t_0} : [t_1,t_2]\times\Omega_{t_0} \to \mathbb{R}^n$ the associated motion supposed regular, so with $\Phi^{t_0}(t,p_{t_0}) = \widetilde{\Phi}(t,P_{Obj}) = p_t$, $F^{t_0}(t,p_{t_0}) = d\Phi^{t_0}(t,p_{t_0})$ the deformation gradient and $J^{t_0}(t,p_{t_0}) = \det(F^{t_0}(t,p_{t_0}))$ the Jacobian at $t$ at $p_t = \widetilde{\Phi}(t,P_{Obj})$.

Let $\rho(t,p) = \rho_t(p)$ be the (Eulerian) positive mass density at $t$ at $p\in\Omega_t$, The mass $m(\omega_t)$ of a subset $\omega_t\subset\Omega_t$ is

$$m(\omega_t) = \int_{p\in\omega_t} \rho_t(p)\,d\omega_t. \tag{N.1}$$

**Conservation of mass principle**: For all $\omega_{t_0}\subset\Omega_{t_0}$, all $t$, and with $\omega_t = \Phi^{t_0}(t,\omega_{t_0})$,

$$m(\omega_t) = m(\omega_{t_0}), \quad \text{i.e.} \quad \int_{p\in\omega_t}\rho_t(p)\,d\omega_t = \int_{P\in\omega_{t_0}}\rho_{t_0}(P)\,d\omega_{t_0}. \tag{N.2}$$

**Proposition N.1** *If (N.2) then, for all $t$,*

$$\rho_t(p) = \frac{\rho_{t_0}(P)}{J_t^{t_0}(P)}, \tag{N.3}$$

*and*

$$\frac{D\rho}{Dt} + \rho\,\mathrm{div}\vec{v} = 0, \quad \text{i.e.} \quad \frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\,\vec{v}) = 0. \tag{N.4}$$

*Thus, for all $\omega_t\subset\Omega_t$,*

$$\int_{\omega_t}\frac{\partial\rho}{\partial t}\,d\omega_t = -\int_{\partial\omega_t}\rho\vec{v}.\vec{n}\,d\sigma_t. \tag{N.5}$$

**Proof.** The change of variable formula gives

$$\int_{p\in\omega_t}\rho_t(p)\,d\omega_t = \int_{P\in\omega_{t_0}}\rho_t(\Phi_t^{t_0}(P))\,J_t^{t_0}(P)\,d\omega_{t_0},$$

thus (N.2) gives $\rho_t(\Phi_t^{t_0}(P))J_t^{t_0}(P) = \rho_{t_0}(P)$, thus (N.3). And (N.2) gives $\frac{d}{dt}(\int_{p(t)\in\omega_t}\rho(t,p(t))\,d\omega_t) = 0$, with $\frac{d}{dt}(\int_{p(t)\in\omega_t}\rho(t,p(t))\,d\omega_t) =^{(L.39)} \int_{p_t\in\omega_t}\big(\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\,\vec{v})\big)(t,p_t)\,d\omega_t$ (Leibniz formula), true for all $\omega_t$, thus (N.4). Hence the Green formula $\int_{\Omega_t}\mathrm{div}(\rho\,\vec{v})\,d\Omega_t = \int_{\partial\Omega_t}\rho\vec{v}.\vec{n}\,d\sigma_t$ gives (N.5). ∎

# O Work and power

## O.1 Definitions

### O.1.1 Work along a trajectory

Let $\alpha$ be a differential form in $[t_0,T]\times\Omega$ (unmissable in thermodynamics, e.g. $\alpha = dU$ the internal energy density, $\alpha = \delta W$ the elementary work, $\alpha = \delta Q$ the elementary heat...).

And consider a regular curve $c : t\in[t_0,T]\to c(t)\in\Omega$ and let $\vec{v}(t,c(t)) := \vec{c}\,'(t)$.

**Definition O.1** The work of the differential form $\alpha$ along the curve $c$ is

$$\begin{aligned} W_T^{t_0}(\alpha,c) = \int_c \alpha \ &:= \int_{t=t_0}^T \alpha(t,c(t)).\vec{c}\,'(t)\,dt \overset{\text{written}}{=} \int_{t=t_0}^T \alpha.d\vec{c} \\ &= \int_{t=t_0}^T \alpha(t,c(t)).\vec{v}(t,c(t))\,dt \overset{\text{written}}{=} \int_{t=t_0}^T \alpha.\vec{v}\,dt. \end{aligned} \tag{O.1}$$

(This definition is objective: No inner dot product and no basis needed.)

**Definition O.2** If we have a Euclidean dot product $(\cdot,\cdot)_g =^{\text{written}} . \bullet_g .$ then the work of a vector (force) field $\vec{f}$ along the curve $c$ is

$$\widetilde{W_T^{t_0}}(\vec{f},c) = \int_{t=t_0}^T \vec{f} \bullet_g d\vec{c} = \int_{t=t_0}^T \vec{f} \bullet_g \vec{v}\,dt = \int_{t=t_0}^T \vec{f}(t,c(t)) \bullet \vec{c}\,'(t)\,dt = \int_{t=t_0}^T \vec{f}(t,c(t)) \bullet \vec{v}(t,c(t))\,dt. \tag{O.2}$$

**Link** between the two definitions if you use a Euclidean dot product $(\cdot,\cdot)_g$: $\vec{f}$ is the $(\cdot,\cdot)_g$-Riesz representation vector of $\alpha$, i.e. $(\vec{f},\vec{w})_g = \alpha.\vec{w}$ for all $\vec{w}$.

**Remark O.3** If the differential form $\alpha$ is exact, i.e. iff $\exists U \in C^1(\Omega;\mathbb{R})$ s.t. $\alpha(t,p) = dU(p)$ for all $t,p$, then $W_T^{t_0}(dU,c) = \int_c dU = \int_{t=t_0}^T dU(c(t)).\vec{c}\,'(t)\,dt = \int_{t=t_0}^T \frac{d(U\circ c)}{dt}(t)\,dt = [U\circ c]_{t_0}^T$, thus

$$\int_c dU = U(c(T)) - U(c(t_0)) \overset{\text{written}}{=} \Delta U : \tag{O.3}$$

The work of an exact differential form is independent of the trajectory joining two points in $\Omega$. ▪

Then consider an object $Obj$, a motion $\widetilde{\Phi} : (t, P_{Obj}) \in [t_0,T] \times Obj \to p(t) = \widetilde{\Phi}(t,P_{Obj}) = \widetilde{\Phi}_{P_{Obj}}(t) \in \mathbb{R}^n$, the trajectories $c_{P_{Obj}} = \widetilde{\Phi}_{P_{Obj}} : t \in [t_0,T] \to p(t) = \widetilde{\Phi}_{P_{Obj}}(t) \in \mathbb{R}^n$, $\Omega_t = \widetilde{\Phi}(t,Obj)$, $\vec{v}(t,p(t)) = c_{P_{Obj}}{}'(t)$ (Eulerian velocities), and $p_{t_0} = \widetilde{\Phi}(t_0,P_{Obj})$, $p_t = \widetilde{\Phi}(t,P_{Obj}) = \widetilde{\Phi}_{P_{Obj}}(t) = \Phi_{p_{t_0}}^{t_0}(t)$.

**Definition O.4** The work of $\alpha$ along $\widetilde{\Phi}$ is the sum of the works of $\alpha$ along all the trajectories, i.e. $W_T^{t_0}(\widetilde{\Phi}) = \int_{p_{t_0}\in\Omega_{t_0}} (W_T^{t_0}(\alpha,\widetilde{\Phi}_{P_{Obj}}))\,d\Omega_{t_0} = \int_{p_{t_0}\in\Omega_{t_0}} (\int_{\widetilde{\Phi}_{P_{Obj}}} \alpha)\,d\Omega_{t_0}$:

$$W_T^{t_0}(\widetilde{\Phi}) = \int_{p_{t_0}\in\Omega_{t_0}} (\int_{t=t_0}^T \alpha.\vec{v}\,dt)\,d\Omega_{t_0} := \int_{p_{t_0}\in\Omega_{t_0}} \Big(\int_{t=t_0}^T \alpha(t,\Phi_{p_{t_0}}^{t_0}(t)).\vec{v}(t,\Phi_{p_{t_0}}^{t_0}(t))\,dt\Big)\,d\Omega_{t_0}, \tag{O.4}$$

written with $\sum_{P_{Obj}\in Obj}(.)$ instead of $\int_{p_{t_0}\in\Omega_{t_0}}(.)d\Omega_{t_0}$ for a finite number of particles.

### O.1.2 The associated power density

**Definition O.5** Definition: The power density of a differential form $\alpha$ relative to a Eulerian velocity field $\vec{v}$ is the Eulerian function

$$\psi := \alpha.\vec{v} : \begin{cases} \mathcal{C} = \bigcup_{t\in[t_0,T]} (\{t\}\times\Omega_t) \ \to \mathbb{R} \\ \qquad\qquad (t,p) \ \to \psi(t,p) = \alpha(t,p).\vec{v}(t,p). \end{cases} \tag{O.5}$$

And the power at $t$ is, with $\psi_t(p) := \psi(t,p)$,

$$\mathcal{P}_t(\vec{v}_t) := \int_{p\in\Omega_t} \psi_t(p)\,d\Omega = \int_{p\in\Omega_t} \alpha_t(p).\vec{v}_t(p)\,d\Omega \overset{\text{written}}{=} \int_{\Omega_t} \alpha.\vec{v}\,d\Omega. \tag{O.6}$$

E.g. with a differential form $\mathcal{L}_{\vec{w}}\alpha$ (a Lie derivative of a differential form): $\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_t} \mathcal{L}_{\vec{w}}\alpha.\vec{v}\,d\Omega = \int_{\Omega_t} (\frac{\partial\alpha}{\partial t} + d\alpha.\vec{w} + \alpha.d\vec{w}).\vec{v}\,d\Omega = \int_{\Omega_t} (\frac{\partial\alpha}{\partial t}(t,p) + d\alpha_t(p).\vec{w}_t(p) + \alpha_t(p).d\vec{w}_t(p)).\vec{v}_t(p)\,d\Omega$.

**Particular case:** If $\alpha_t$ is an exact differential form, i.e. $\exists U_t$ s.t. $\alpha_t = dU_t$, then, with a Euclidean dot product $(\cdot,\cdot)_g$,

$$\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_t} dU_t.\vec{v}_t\,d\Omega = -\int_{\Omega_t} U_t\,\mathrm{div}\vec{v}_t\,d\Omega + \int_\Gamma U_t\,\vec{v}_t \bullet \vec{n}_t\,d\Omega. \tag{O.7}$$

And with the $(\cdot,\cdot)_g$-Riesz representation vector $\vec{f}_t$ of $\alpha_t = dU_t$ we get

$$\mathcal{P}_t(\vec{v}_t) = \int_{p\in\Omega_t} \vec{f}_t(p) \bullet \vec{v}_t(p)\,d\Omega, \tag{O.8}$$

and if $\vec{f}_t = \vec{\mathrm{grad}}U_t$ then $\mathcal{P}_t(\vec{v}_t) = -\int_{\Omega_t} U_t\,\mathrm{div}\vec{v}_t\,d\Omega + \int_\Gamma U_t\,\vec{v}_t \bullet \vec{n}_t\,d\Omega$.

**Remark O.6** To measure something we need time: To get a power we must first measure work. ▪

## O.2 Piola–Kirchhoff tensors

### O.2.1 Internal power: Classical presentation (subjective)

**First order classical hypothesis for the internal stress**: At all time, a Euclidean basis $(\vec{e}_i)$ and associated Euclidean dot product $.\bullet.$ are imposed. The power density is of the type (subjective quantity)

$$\psi = \underline{\underline{\sigma}} : d\vec{v} = \sum_{i,j=1}^{n} \sigma_{ij} \frac{\partial v_i}{\partial x_j}, \tag{O.9}$$

where $[\underline{\underline{\sigma}}]_{|\vec{e}} = [\sigma_{ij}]$, $\vec{v} = \sum_{i=1}^{n} v_i \vec{e}_i$ and $d\vec{v}.\vec{e}_j = \sum_i \frac{\partial v_i}{\partial x_j} \vec{e}_i$ . This is a subjective formulation, $\underline{\underline{\sigma}} : d\vec{v}$ being a matrix product (term to term product $\sum_{ij} \sigma_{ij} v_{i,j}$ meaningful once a basis has been chosen). And the power at $t$ is

$$\mathcal{P}_t(\vec{v}_t) = \int_{p\in\Omega_t} \psi(t,p)\, d\Omega_t = \int_{p\in\Omega_t} \underline{\underline{\sigma}}_t(p) : d\vec{v}_t(p)\, d\Omega_t. \tag{O.10}$$

### O.2.2 Internal power: Objective presentation

**First order objective hypothesis for the internal stress**: At any $t$, there exists a $C^1$ (field of) endomorphism $\underline{\underline{\tau}}_t \in \mathcal{F}(\Omega_t; \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t))$ (so $\underline{\underline{\tau}}_t(p) \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_t)$ for all $p \in \Omega_t$), s.t. the power density is

$$\psi_t = \underline{\underline{\tau}}_t \odot d\vec{v}_t \quad \text{(objective)}, \tag{O.11}$$

where $\underline{\underline{\tau}}_t \odot d\vec{v}_t = \mathrm{Tr}(\underline{\underline{\tau}}_t.d\vec{v}_t)$ is the objective double contraction (trace of the compound endomorphism). And the density is

$$\mathcal{P}_t(\vec{v}_t) = \int_{p\in\Omega_t} \psi(t,p)\, d\Omega_t = \int_{p\in\Omega_t} \underline{\underline{\tau}}_t(p) \odot d\vec{v}_t(p)\, d\Omega_t \tag{O.12}$$

**Quantification**. Basis $(\vec{e}_i)$ at $t$ at $p$, $\underline{\underline{\tau}}.\vec{e}_j = \sum_{i=1}^{n} \tau_{ij} \vec{e}_i$, $\vec{v} = \sum_{i=1}^{n} v_i \vec{e}_i$, $d\vec{v}.\vec{e}_j = \sum_{i=1}^{n} v_{i|j} \vec{e}_i$, so $[\underline{\underline{\tau}}]_{|\vec{e}} = [\tau_{ij}]$ and $[d\vec{v}]_{|\vec{e}} = [v_{i|j}]$ and

$$\psi = \sum_{i,j=1}^{n} \tau_{ij} v_{j|i} \quad \text{and} \quad \mathcal{P}_t(\vec{v}_t) = \sum_{i,j=1}^{n} \int_{p\in\Omega_t} \tau_{ij}(p) v_{j|i}(p)\, d\Omega_t. \tag{O.13}$$

(Duality notations : $[\underline{\underline{\tau}}]_{|\vec{e}} = [\tau^i_{|j}]$, $[d\vec{v}]_{|\vec{e}} = [v^i_{|j}]$, and $\psi = \sum_{i,j=1}^{n} \tau^i_j v^j_{|i}$.)

(Cartesian basis: $v_{i|j} = \frac{\partial v_i}{\partial x_j} = v^i_{|j} = \frac{\partial v^i}{\partial x^j}$.)

(Euclidean basis and $\underline{\underline{\sigma}} := [\underline{\underline{\tau}}]^T_{|\vec{e}}$: We recover (O.9).)

### O.2.3 The first Piola–Kirchhoff tensor

The Piola–Kirchhoff approach consists in transforming Eulerian quantities into Lagrangian quantities to refer to an initial configuration. $t_0$ and $t$ are fixed, and we write $\Phi = \Phi^{t_0}_t$, $F = F^{t_0}_t = d\Phi$, $J = \det(F)$, $V = \vec{V}^{t_0}_t$ (Lagrangian velocity). Recall: $\vec{V}(P) = \vec{v}_t(\Phi(P))$ gives $d\vec{V}(P) = d\vec{v}_t(p).F(P)$ when $p = \Phi(t,P)$. (O.11) gives (objective quantity), the Jacobian $J(P)$ being positive (regular motion),

$$\begin{aligned} \mathcal{P}_t(\vec{v}_t) &= \int_{P\in\Omega_{t_0}} \underline{\underline{\tau}}_t(\Phi(P)) \odot \Big(d\vec{V}(P).F(P)^{-1}\Big)\, J(P)\, d\Omega_{t_0} \\ &= \int_{P\in\Omega_{t_0}} \Big(J(P)\, F(P)^{-1}.\underline{\underline{\tau}}_t(\Phi(P))\Big) \odot d\vec{V}(P)\, d\Omega_{t_0} \overset{\text{written}}{=} \int_{\Omega_{t_0}} (JF^{-1}.\underline{\underline{\tau}}) \odot d\vec{V}\, d\Omega_{t_0}. \end{aligned} \tag{O.14}$$

**Definition O.7** (From Marsden–Hughes definitions page 135.) The first Piola–Kirchhoff tensor at $P \in \Omega_{t_0}$ relative to $t_0$ and $t$ is the linear map $\widetilde{\mathbb{K}}^{t_0}_t(P) =^{\text{written}} \widetilde{\mathbb{K}}(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_t; \vec{\mathbb{R}}^n_{t_0})$ defined with $p = \Phi^{t_0}_t(P)$ by

$$\widetilde{\mathbb{K}}(P) := J(P)\, F(P)^{-1}.\underline{\underline{\tau}}_t(p), \quad \text{i.e.} \quad \widetilde{\mathbb{K}} := J\, F^{-1}.(\underline{\underline{\tau}} \circ \Phi). \tag{O.15}$$

(So $\widetilde{\mathbb{K}^{t_0}_t}(P).\vec{w}_p = J^{t_0}_t(P)\, F^{t_0}_t(P)^{-1}.\underline{\underline{\tau}}_t(p).\vec{w}_p$ for all $\vec{w}_p \in \vec{\mathbb{R}}^n_t$.)

So

$$\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_{t_0}} \widetilde{\mathbb{K}} \odot d\vec{V}_t\, d\Omega_{t_0} \quad \Big(= \int_{P\in\Omega_{t_0}} \widetilde{\mathbb{K}}(P) \odot d\vec{V}_t(P)\, d\Omega_{t_0}\Big). \tag{O.16}$$

**Definition O.8** Usual classic definition which requires a Euclidean basis $(\vec{e}_i)$ and the associated Euclidean dot product $(\cdot,\cdot)_g$ (subjective quantification): The first Piola–Kirchhoff tensor at $P\in\Omega_{t_0}$ relative to $t_0,t$ and $(\cdot,\cdot)_g$ is the linear map $\widetilde{I\!\!K}_t^{t_0}(P) := \widetilde{I\!\!K_t^{t_0}}(P)_g^T \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$, written $I\!\!K := \widetilde{I\!\!K}^T$, defined by

$$\boxed{I\!\!K := J\,\underline{\underline{\sigma}}.F^{-T}},\quad \text{where}\quad \underline{\underline{\sigma}}_t := \underline{\underline{\tau}}_t^T, \tag{O.17}$$

which means, with $p=\Phi_t^{t_0}(P)$ and $J_t^{t_0}(P)=\det([F_t^{t_0}(P)]_{|\vec{e}})$ and $[.]:=[.]_{|\vec{e}}$,

$$(I\!\!K_{t,g}^{t_0}(P)=)\quad I\!\!K(P) = J(P)\,[\underline{\underline{\tau}}_t(p)]^T.[F(P)]^{-T}, \tag{O.18}$$

Thus

$$\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_{t_0}} I\!\!K : d\vec{V}_t\,d\Omega_{t_0}\quad (=\int_{P\in\Omega_{t_0}} I\!\!K(P) : d\vec{V}_t(P)\,d\Omega_{t_0}). \tag{O.19}$$

### O.2.4 The second Piola–Kirchhoff tensor

$I\!\!K(P)$ is not symmetric: It can't be since $I\!\!K(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ is not an endomorphism.

**Definition O.9** The second Piola–Kirchhoff tensor at $P\in\Omega_{t_0}$, relative to $t_0,t$ and $(\cdot,\cdot)_g$, is the endomorphism $S\!\!K_{t,g}^{t_0}(P) =^{\text{written}} S\!\!K(P)\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0})$ defined by, in short (matrix meaning),

$$S\!\!K = F^{-1}.I\!\!K = JF^{-1}.\underline{\underline{\sigma}}.F^{-T} \tag{O.20}$$

Thus: If $\underline{\underline{\sigma}}_t(p)\in\mathcal{L}(\vec{\mathbb{R}}^n_t;\vec{\mathbb{R}}^n_t)$ is $(\cdot,\cdot)_g$-symmetric, then $S\!\!K$ is (trivially) symmetric, and (O.19) gives

$$\mathcal{P}_t(\vec{v}_t) = \int_{\Omega_{t_0}} (F.S\!\!K) : d\vec{V}_t\,d\Omega_{t_0} = \int_{\Omega_{t_0}} S\!\!K : (d\vec{V}_t.F^T)\,d\Omega_{t_0} = \int_{\Omega_{t_0}} S\!\!K : \frac{d\vec{V}_t.F^T + F.d\vec{V}_t^T}{2}\,d\Omega_{t_0}. \tag{O.21}$$

**Example O.10** Saint-Venant–Kirchhoff model: With $E=\frac12(F^T.F-I)$ the Green–Lagrange tensor, $S=\lambda\mathrm{Tr}(E)I+2\mu E$ is the second Piola–Kirchhoff "tensor" for classical elasticity. And $W=\frac{\lambda}{2}(\mathrm{Tr}(E))^2+\mu\mathrm{Tr}(E^2)$ is the associated hyperelastic potential that gives $S=\frac{\partial W}{\partial E}$. ▟

**Remark O.11** It is a "chosen time derivative" of $S\!\!K(t)=J(t)F(t)^{-1}.\underline{\underline{\sigma}}(t).F(t)^{-T}$ that leads to some kind of Lie derivative as explain in books in continuum mechanics, see footnote page 26. ▟

## O.3 Classical hyper-elasticity and the notation $\partial W/\partial F$

### O.3.1 Notation $\partial W/\partial F$

$A$ and $B$ are finite dimensional spaces, $\dim A=n$, $\dim B=m$, and $\widehat{W}\in C^1(\mathcal{L}(A;B);\mathbb{R})$, so

$$\widehat{W}:\left\{\begin{array}{rl}\mathcal{L}(A;B) &\to \mathbb{R}\\ L &\to \widehat{W}(L)\end{array}\right\},\quad\text{and}\quad d\widehat{W}:\left\{\begin{array}{rl}\mathcal{L}(A;B) &\to \mathcal{L}(\mathcal{L}(A;B);\mathbb{R})\\ L &\to d\widehat{W}(L)\end{array}\right\} \tag{O.22}$$

is given by $d\widehat{W}(L)(M) \underset{\text{notation}}{\overset{\text{linearity}}{=}} d\widehat{W}(L).M = \lim_{h\to0}\dfrac{\widehat{W}(L+hM)-\widehat{W}(L)}{h}$ for all $M\in\mathcal{L}(A;B)$. Notation when $L$ is the name of the variable:

$$d\widehat{W}(L) \overset{\text{written}}{=} \frac{\partial\widehat{W}}{\partial L}(L),\quad\text{so}\quad d\widehat{W}(L).M \overset{\text{written}}{=} \frac{\partial\widehat{W}}{\partial L}(L).M. \tag{O.23}$$

**Example O.12** $A=B=\vec{\mathbb{R}}^n$, and $\widehat{W}(L):=\mathrm{Tr}(L)$ (the trace of an endomorphism $L\in\mathcal{L}(\vec{\mathbb{R}}^n;\vec{\mathbb{R}}^n)$). Here $d\mathrm{Tr}(L)(M)=\lim_{h\to0}\frac{\mathrm{Tr}(L+hM)-\mathrm{Tr}(L)}{h}=\mathrm{Tr}(M)$ since the trace is linear: $\frac{\partial\mathrm{Tr}}{\partial L}(L):=d\mathrm{Tr}(L)=\mathrm{Tr}$. ▟

**Example O.13** $A=\vec{\mathbb{R}}^n_{t_0}$, $B=\vec{\mathbb{R}}^n_t$, $L=F=d\Phi_t^{t_0}(p_{t_0})$. Then $d\widehat{W}(F).M =^{\text{written}} \frac{\partial\widehat{W}}{\partial F}(F).M\in\mathbb{R}$ is the derivative of $\widehat{W}$ at $F\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$ in a direction $M\in\mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$. ▟

**Example O.14** $A=B=\mathcal{M}_{nn}$ (space of $n*n$ matrices), $L=[F]=[d\Phi_t^{t_0}(p_{t_0})]$, $\frac{\partial W}{\partial L}$ cf. (O.23). ▟

### O.3.2 Expression with bases (quantification) and the notation $\partial W/\partial L_{ij}$

Quantification: $(\vec{a}_i) \in A^n$ and $(\vec{b}_i) \in B^m$ are bases in $A$ and $B$, $(\pi_{ai}) \in (A^*)^n$ is the dual basis of $(\vec{a}_i)$. Consider the basis $(\mathcal{L}_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} =^{\text{written}} (\vec{b}_i \otimes \pi_{aj})$ in $\mathcal{L}(A;B)$ (defined by $\mathcal{L}_{ij}.\vec{a}_\ell = \delta_{j\ell}\vec{b}_i$ for all $i,j,\ell$).

The derivation of $\widehat{W}$ at a $L \in \mathcal{L}(A;B)$ in the direction of a basis vector $\mathcal{L}_{ij}$ is, cf. (T.14),

$$\partial_{\mathcal{L}_{ij}}(L) = \frac{\partial\widehat{W}}{\partial\mathcal{L}_{ij}}(L) = d\widehat{W}(L).\mathcal{L}_{ij} \overset{\text{written}}{=} \frac{\partial\widehat{W}}{\partial L_{ij}}(L) \quad (= \lim_{h\to 0}\frac{\widehat{W}(L+h\mathcal{L}_{ij}) - \widehat{W}(L)}{h}) \tag{O.24}$$

notation used when the components of $L$ are called $L_{ij}$, i.e. when $[L]_{|\vec{a},\vec{b}} = [L_{ij}]$, i.e. $L.\vec{a}_j = \sum_{i=1}^m L_{ij}\vec{b}_i$ for all $j$, written $L = \sum_{i=1}^m\sum_{j=1}^n L_{ij}\mathcal{L}_{ij}$. So, the Jacobian matrix of $\widehat{W}$ at $L$ relative to $(\mathcal{L}_{ij})$ is

$$[d\widehat{W}(L)]_{|\mathcal{L}_{ij}} = [\frac{\partial\widehat{W}}{\partial L_{ij}}(L)]_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \overset{\text{written}}{=} [d\widehat{W}(L)]_{|\vec{a},\vec{b}} = [d\widehat{W}(L)_{ij}]. \tag{O.25}$$

So if $M = \sum_{ij} M_{ij}\mathcal{L}_{ij}$, i.e. if $[M]_{|\vec{a},\vec{b}} = [M_{ij}]$, then (linearity of $d\widehat{W}(L)$)

$$d\widehat{W}(L).M = \sum_{ij} M_{ij}\, d\widehat{W}(L).\mathcal{L}_{ij} = \sum_{ij} M_{ij}\frac{\partial\widehat{W}}{\partial L_{ij}}(L) = [M]_{|\vec{a},\vec{b}} : [d\widehat{W}(L)]_{|\vec{a},\vec{b}} \tag{O.26}$$

($= [d\widehat{W}(L)]_{|\vec{a},\vec{b}} : [M]_{|\vec{a},\vec{b}}$) with the double matrix contraction.

Duality notations: $a^i := \pi_{ai}$, $\mathcal{L}_i{}^j = \vec{b}_i \otimes a^j$, $[L]_{|\vec{a},\vec{b}} = [L^i{}_j]$ i.e. $L.\vec{a}_i = \sum_{ij} L^i{}_j\vec{b}_i$, $[M]_{|\vec{a},\vec{b}} = [M^i{}_j]$ i.e. $M.\vec{a}_j = \sum_j M^i{}_j\vec{b}_i$,

$$d\widehat{W}(L).M = \sum_{ij}\frac{\partial\widehat{W}}{\partial L^i{}_j}(L)M^i{}_j. \tag{O.27}$$

NB: $d\widehat{W}(L) \in \mathcal{L}(\mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^m});\mathbb{R})$ and $M = \mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^m})$ are different kinds of mathematical objects, hence $[M]_{|\vec{a},\vec{b}} : [d\widehat{W}(L)]_{|\vec{a},\vec{b}}$ is nothing but a "term to term product" called "double matrix contraction".

**Example O.15** Continuing example O.12: $\widehat{W}(L) = \mathrm{Tr}(L)$ gives $d\widehat{W}(L).M = \mathrm{Tr}(M) = \sum_i M_{ii}$, thus $\frac{\partial\widehat{W}}{\partial L_{ij}}(L) = \delta_{ij}$ for all $i,j$, thus $[d\widehat{W}(L)]_{|\vec{e}} = [I] = [\frac{\partial\mathrm{Tr}}{\partial L_{ij}}(L)]$ (identity matrix), and we recover $d\mathrm{Tr}(L)(M) = [\frac{\partial\mathrm{Tr}}{\partial L_{ij}}(L)] : [M] = [I] : [M] = \sum_{i=1}^n M_{ii} = \mathrm{Tr}(M)$. ∴

**Remark O.16** Continuing example O.13: $\frac{\partial\widehat{W}}{\partial F_{ij}}(F) = d\widehat{W}(F).\mathcal{L}_{ij} = d\widehat{W}(F).(\vec{e}_i \otimes \pi_{Ej}) = \frac{\partial\widehat{W}}{\partial F^i_J}(F) = d\widehat{W}(F).(\vec{e}_i \otimes E^J)$ is a derivation in the directions $\vec{e}_i$ at $(t,p)$ and $\pi_{Ej}$ at $(t_0,P)$. ∴

### O.3.3 Motions and $\omega$-lemma

Generalization of (O.22): With $U_{\mathcal{A}}$ open subset in a affine space which associated vector space is $A$, let

$$\widehat{W} : \left\{\begin{array}{rl} U_{\mathcal{A}} \times \mathcal{L}(A;B) & \to \mathbb{R} \\ (P,L) & \to \widehat{W}(P,L) \end{array}\right\}, \quad\text{and}\quad \widehat{W}_P(L) := \widehat{W}(P,L) \ \ \text{(at any fixed } P \in U_{\mathcal{A}}). \tag{O.28}$$

And let (usual notation) $d\widehat{W}_P(L) =^{\text{written}} \frac{\partial\widehat{W}}{\partial L}(P,L)$, so, for all $M \in \mathcal{L}(A;B)$,

$$\frac{\partial\widehat{W}}{\partial L}(P,L).M = \lim_{h\to 0}\frac{\widehat{W}(P,L+hM) - \widehat{W}(P,L)}{h} \quad (= d(\widehat{W}_P)(L).M = \partial_2\widehat{W}(P,L).M). \tag{O.29}$$

Then consider a motion $\Phi := \Phi_t^{t_0} : \Omega_{t_0} \to \Omega_t$, and $F := d\Phi : P \in \Omega_{t_0} \to d\Phi(P) \in \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_t)$; And define

$$f : \left\{\begin{array}{rl} C^1(\overline{\Omega_{t_0}};\overline{\Omega_t}) & \to C^0(\overline{\Omega_{t_0}};\mathbb{R}) \\ \Phi & \to \boxed{f(\Phi) := \widehat{W}(.,d\Phi(.))} \end{array}\right\}, \quad\text{so}\quad f(\Phi)(P) = \widehat{W}(P,d\Phi(P)) = \widehat{W}_P(d\Phi(P)). \tag{O.30}$$

(Toward the classical power: Only the first order derivative $F = d\Phi$ of $\Phi$ is taken into account.)

Hence $df : \left\{\begin{array}{rl} C^1(\overline{\Omega_{t_0}};\overline{\Omega_t}) & \to \mathcal{L}(C^1(\overline{\Omega_{t_0}};\overline{\Omega_t});C^0(\overline{\Omega_{t_0}};\mathbb{R}) \\ \Phi & \to df(\Phi) \end{array}\right\}$, $df(\Phi) : \left\{\begin{array}{rl} C^1(\overline{\Omega_{t_0}};\overline{\Omega_t}) & \to C^0(\overline{\Omega_{t_0}};\mathbb{R}) \\ \Psi & \to df(\Phi).\Psi \end{array}\right\}$, and

$$(df(\Phi).\Psi)(P) = df(\Phi(P)).\Psi(P). \tag{O.31}$$

**Lemma O.17 ($\omega$-lemma)** *If $f$ and $\widehat{W}$ are $C^1$ then, for all $\Phi, \Psi \in C^1(\overline{\Omega_{t_0}}; \overline{\Omega_t})$, and with $F = d\Phi$,*

$$\boxed{df(\Phi).\Psi = \frac{\partial \widehat{W}}{\partial F}(.,d\Phi).d\Psi}, \quad i.e. \quad (df(\Phi).\Psi)(P) = \frac{\partial \widehat{W}}{\partial F}(P, d\Phi(P)).d\Psi(P). \tag{O.32}$$

**Proof.** $df(\Phi)(\Psi) = \lim_{h\to 0} \frac{f(\Phi+h\Psi)-f(\Phi)}{h} \in C^0(\overline{\Omega_{t_0}}; \overline{\Omega_t})$, i.e. $P \in \overline{\Omega_{t_0}}$ being fixed, $df(\Phi)(\Psi)(P) = \lim_{h\to 0} \frac{f(\Phi+h\Psi)(P)-f(\Phi)(P)}{h} = \lim_{h\to 0} \frac{\widehat{W}_P(d\Phi(P)+h\,d\Psi(P))-\widehat{W}_P(d\Phi(P))}{h} = d\vec{W}_P(d\Phi(P)).d\Psi(P)$, i.e. (O.32). ∎

Quantification: With bases $(\vec{E}_i)$ and $(\vec{e}_i)$ in $\vec{\mathbb{R}}^n_{t_0}$ and $\vec{\mathbb{R}}^n_t$ and $d\Psi.\vec{E}_j = \sum_{i=1}^n \frac{\partial \Psi_i}{\partial X_j}\vec{e}_i$, we get

$$df(\Phi).\Psi = \sum_{i,j=1}^n \frac{\partial \widehat{W}}{\partial F_{ij}}(.,d\Phi)\frac{\partial \Psi_i}{\partial X_j}(.) = [\frac{\partial \widehat{W}}{\partial F_{ij}}(.,d\Phi)] : [\frac{\partial \Psi_i}{\partial X_j}(.)] = [\frac{\partial \widehat{W}}{\partial F}] : [d\Psi], \tag{O.33}$$

Marsden duality notations: $df(\Phi).\Psi = \sum_{i,J=1}^n \frac{\partial \widehat{W}}{\partial F^i_J}\frac{\partial \Psi^i}{\partial X^J} = [\frac{\partial \widehat{W}}{\partial F^i_J}] : [\frac{\partial \Psi^i}{\partial X^J}] = [\frac{\partial \widehat{W}}{\partial F}] : [d\Psi]$.

### O.3.4 Application to classical hyper-elasticity: $I\!\!K = \partial W/\partial F$

$(\vec{e}_i) = (\vec{E}_I)$ is a Euclidean basis, $(\cdot,\cdot)_g$ is its associated Euclidean dot product, the same at all times, $\underline{\underline{\sigma}}_t(p)$ is the Cauchy stress tensor at $t$ at $p = \Phi(P)$. Let $I\!\!K = I\!\!K(\Phi)$ be the first Piola–Kirchhoff tensor:

$$I\!\!K(\Phi)(P) \overset{(O.17)}{=} \det(d\Phi(P))\,\underline{\underline{\sigma}}_t(\Phi(P)).d\Phi(P)^{-T}. \tag{O.34}$$

**Definition O.18** If there exists a function $\widehat{I\!\!K} : (C^1(\Omega_{t_0}; \Omega_t) \to C^0(\Omega_{t_0}; \Omega_t)$ s.t., with $F = d\Phi$,

$$I\!\!K(\Phi) = \widehat{I\!\!K}(.,F), \quad \text{i.e.} \quad I\!\!K(\Phi)(P) = \widehat{I\!\!K}(P, F(P)) \quad \text{(first order hypothesis)}, \tag{O.35}$$

then $\widehat{I\!\!K}$ is called a constitutive function. So $I\!\!K(\Phi) = \widehat{I\!\!K}(.,d\Phi)$.

**Definition O.19** The material is hyper-elastic iff $\exists \widehat{W} : \left\{ \begin{array}{ll} \Omega_{t_0} \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0}; \vec{\mathbb{R}}^n_t) & \to \mathbb{R} \\ (P, L) & \to \widehat{W}(P, L) \end{array} \right\}$ s.t.

$$\widehat{I\!\!K} = \frac{\partial W}{\partial F}, \quad \text{i.e.} \quad I\!\!K(\Phi)(P) = \frac{\partial \widehat{W}}{\partial F}(P, F(P)), \tag{O.36}$$

with $F = d\Phi$ and for all $P \in \Omega_{t_0}$.

Quantification (Marsden notations): $(E^I)$ dual basis of $(\vec{E}_I)$, $F.\vec{E}_J = \sum_{i=1}^n F^i_J \vec{e}_i$, $I\!\!K.\vec{E}_J = \sum_{i=1}^n I\!\!K^i{}_J \vec{e}_i$, and $[I\!\!K(.,\Phi)] = [I\!\!K^i{}_J(.,\Phi)] = [\frac{\partial \widehat{W}}{\partial F^i_J}(.,F)]$. For any (virtual) motion $\Psi : \Omega_{t_0} \to \Omega_t$,

$$\widehat{I\!\!K}(.,d\Phi).d\Psi = \sum_{iJ} \frac{\partial \widehat{W}}{\partial F^i_J}(.,F)\frac{\partial \Psi^i}{\partial X^J} = [\widehat{I\!\!K}(.,F)] : [d\Psi], \tag{O.37}$$

i.e. $\widehat{I\!\!K}(P, d\Phi(P))(d\Psi(P)) = \sum_{iJ} \frac{\partial \widehat{W}}{\partial F^i_J}(P, F(P))\frac{\partial \Psi^i}{\partial X^J}(P)$ for all $P \in \Omega_{t_0}$.

**Exercice O.20** With $C = F^T.F = C(F)$, and with $F = \sum_{iK} F^i_K \vec{e}_i \otimes E^K$, prove

$$\frac{\partial C}{\partial F^i_J}(F) = \sum_K F^i_K(\vec{E}_J \otimes E^K + \vec{E}_K \otimes E^J) \quad (= dC(F).(\vec{e}_i \otimes E^J)), \tag{O.38}$$

$$\text{and} \quad \frac{\partial \sqrt{C}}{\partial F} = \frac{1}{2}\big(\sqrt{C}\big)^{-1}.\frac{\partial C}{\partial F}, \quad \text{i.e.} \quad 2\sqrt{C}.d(\sqrt{C}) = dC. \tag{O.39}$$

**Answer**. Euclidean basis, thus $(\vec{e}_i \otimes E^J)^T = \vec{E}_J \otimes e^j$, and $F^T = \sum_{Ik} F^k_I \vec{E}_I \otimes e^k$. Thus

$$\begin{aligned} C(F + h\vec{e}_i \otimes E^J) &= (F + h\vec{e}_i \otimes E^J)^T.(F + h\vec{e}_i \otimes E^J) = (F^T + h\vec{E}_J \otimes e^i).(F + h\vec{e}_i \otimes E^J) \\ &= F^T.F + h\,(\vec{E}_J \otimes e^i).F + h\,F^T.(\vec{e}_i \otimes E^J) + h^2\,(\vec{E}_J \otimes e^i).(\vec{e}_i \otimes E^J) \\ &= C(F) + h\,(\sum_K F^i_K \vec{E}_J \otimes E^K + \sum_K F^i_K \vec{E}_K \otimes E^J) + h^2\,\vec{E}_J \otimes E^J. \end{aligned} \tag{O.40}$$

Thus (O.38). And $dC(F)$ is linear, hence $dC(F).L = \sum_{iJ} L^i_J\, dC(F).\vec{e}_i \otimes E^J$.

With $\sqrt{f} : \vec{x} \to \sqrt{f}(\vec{x}) := \sqrt{f(\vec{x})}$ we have $\frac{f(\vec{x}+h\vec{z}_k)-f(\vec{x})}{h} = (\sqrt{f}(\vec{x}+h\vec{z}_k) + \sqrt{f}(\vec{x})).\frac{\sqrt{f}(\vec{x}+h\vec{z}_k)-\sqrt{f}(\vec{x})}{h}$, thus $h \to 0$ gives $df(\vec{x}).\vec{z}_k = 2\sqrt{f}(\vec{x}).d\sqrt{f}(\vec{x}).\vec{z}_k$, thus $df(\vec{x}) = 2\sqrt{f}(\vec{x}).d\sqrt{f}(\vec{x})$.

In particular, $f = C$ and $\vec{x} = F$ give $dC(F) = 2\sqrt{C}(F).d\sqrt{C}(F)$, thus (O.39). ∎

### O.3.5 Corollary (hyper-elasticity): $\mathcal{SK} = \partial W/\partial C$

For the second Piola–Kirchhoff tensor $\mathcal{SK} = F^{-1}.\mathcal{PK}$: We get the existence of a function $\widetilde{W}$ : $\left\{\begin{array}{ll} \Omega_{t_0} \times \mathcal{L}(\vec{\mathbb{R}}^n_{t_0};\vec{\mathbb{R}}^n_{t_0}) & \to \mathbb{R} \\ (P,L) & \to \widetilde{W}(P,L) \end{array}\right\}$ s.t. (constitutive function), with $C = F^T.F$,

$$\widehat{\mathcal{SK}^{t_0}_t}(.,C) = \frac{\partial \widetilde{W}}{\partial C}(.,C). \tag{O.41}$$

See Marsden and Hughes [17] for details and the thermodynamical hypotheses required.

**Remark O.21** Hyper-elasticity is also often proposed in terms of $\frac{\partial \mathcal{W}}{\partial C}$, although the results are not convincing when elasticity is proposed in terms of $E = C - I$ without any linearization of $E$. ∴

# P Balance of momentum

Formerly expansion-contraction normal forces and bending forces were considered. Cauchy proposed reducing these forces to a single force deduced from tensions exerted on three orthogonal planes; So, with a Euclidean basis $(\vec{e}_i)$ and at a point $p$, the force $\vec{T}_p(\vec{n})$ in a direction $\vec{n} = \sum_i n_i\vec{e}_i$ is linear in $\vec{n}$, i.e. $\vec{T}_p(\vec{n}) = \vec{T}_p(\vec{e}_1)n_1 + \vec{T}_p(\vec{e}_2)n_2 + \vec{T}_p(\vec{e}_3)n_3 = \underline{\underline{\sigma}}(p).\vec{n}$ where $\underline{\underline{\sigma}}(p) = \big( [\vec{T}_p(\vec{e}_1)] \quad [\vec{T}_p(\vec{e}_2)] \quad [\vec{T}_p(\vec{e}_3)] \big)$. Cf. the introduction of [7], $\vec{n}$ being a direction of measurement.

Later Cauchy's hypothesis was transformed into the master balance law (to satisfy newton's principle $\sum \vec{f} = m\vec{\gamma}$) and its consequence called Cauchy's theorem (which is in fact Cauchy's original hypothesis).

$\Omega$ is an open bounded regular set in $\mathbb{R}^n$ and $\Gamma = \partial\Omega$ (boundary).

## P.1 Trace theorem 1: $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ is onto

The trace operator $\gamma_0$ : $\left\{\begin{array}{ll} H^1(\Omega) & \to L^2(\Gamma) \\ w & \to \gamma_0(w) = w_{|\Gamma} \end{array}\right\}$ is (well-defined, linear, continuous) onto its image $H^{\frac12}(\Gamma) := \gamma_0(H^1(\Omega))$, see e.g. Brezis [4].

(The onto character is given by: If $\psi \in H^{\frac12}(\Gamma)$ then $u_d$ solution of $-\Delta u + u = 0$ with $u_{|\Gamma} = \psi$ gives $u_d \in H^1(\Omega)$; Moreover the inner dot product $(\psi,\widetilde{\psi})_{H^{\frac12}(\Gamma)} := (u_d,\widetilde{u}_d)_{H^1(\Omega)}$ makes $H^{\frac12}(\Gamma)$ a Hilbert space.)

And the topological dual of $H^{\frac12}(\Gamma)$ is called $H^{-\frac12}(\Gamma)$. And $L^2(\Gamma) \subset H^{-\frac12}(\Gamma)$.

## P.2 Isometric framework and trace theorem 2: $\gamma_{\vec{n}}$ is onto

Let (for minimal regularity hypothesis)

$$H^{\rm div}(\Omega) := \{\vec{v} \in L^2(\Omega)^n,\ {\rm div}\vec{v} \in L^2(\Omega)\}, \quad ||\vec{v}||_{H^{\rm div}} = (||\vec{v}||^2_{L^2(\Omega)} + ||{\rm div}\vec{v}||^2_{L^2(\Omega)})^{\frac12}. \tag{P.1}$$

(Thus $H^{\rm div}(\Omega)$ with the induced dot product $(\vec{v},\vec{w})_{H^{\rm div}} = (\vec{v},\vec{w})_{L^2} + ({\rm div}\vec{v},{\rm div}\vec{w})_{L^2}$ is a Hilbert.)

Then we need Euclidean orthogonality. Choose a Euclidean basis $(\vec{e}_i)$ and its associated Euclidean dot product $(\cdot,\cdot)_g$. Call $\vec{n} \in \vec{\mathbb{R}}^n$ the unit outward normal vector field on $\Gamma$.

### P.2.1 Scalar valued functions: $\gamma_{\vec{n}}$ is onto

Here $(\vec{u},\vec{w})_g =^{\rm written} \vec{u} \bullet \vec{w}$.

**Theorem P.1** *The trace operator* $\gamma_{\vec{n}}$ : $\left\{\begin{array}{ll} H^{\rm div}(\Omega) & \to H^{-\frac12}(\Gamma) \\ \vec{u} & \to \gamma_{\vec{n}}(\vec{u}) = \vec{u} \bullet \vec{n} \end{array}\right\}$ *is (linear continuous and) onto:*

$$\forall \psi \in H^{-\frac12}(\Gamma),\ \exists \vec{u} \in H^{\rm div}(\Omega)\ \ s.t.\ \ \psi = \vec{u} \bullet \vec{n}\ \ (= \gamma_{\vec{n}}\vec{u})\ \ on\ \Gamma, \tag{P.2}$$

*i.e.,* $\forall \zeta \in H^1(\Omega)$ *(so* $\zeta_{|\Gamma} \in H^{\frac12}(\Gamma)$*), with* $\langle \psi,\zeta\rangle_{H^{-\frac12}(\Gamma),H^{\frac12}(\Gamma)} =^{\rm written} \int_\Gamma \zeta\,\psi\,d\Gamma$*: (Gauss–Green formula)*

$$\int_\Gamma \psi\,\zeta\,d\Gamma = \int_\Gamma \zeta\,\vec{u}.\vec{n}\,d\Gamma = \int_\Omega \zeta\,{\rm div}\vec{u} + \vec{u} \bullet \vec{\rm grad}\zeta\,d\Omega \quad (= \int_\Omega {\rm div}(\zeta\,\vec{u})\,d\Omega). \tag{P.3}$$

*In particular with* $\zeta = 1$ *(knowing* $\Omega$ *is bounded) and with* $\psi \in L^2(\Gamma)$, $\int_\Gamma \psi\,d\Gamma = \int_\Gamma \vec{u}.\vec{n}\,d\Gamma = \int_\Omega {\rm div}\vec{u}\,d\Omega$.

**Proof.** See `https://perso.ensta-paris.fr/~ciarlet/MS03/MS03_Notes-de-cours_2024.pdf`. ▪

**Exercice P.2** Give the classic proof of the classic form of (P.2) (cf. Cauchy [7]) which is:

If $\varphi \in C^0(\overline{\Omega};\mathbb{R})$, if $\psi \in C^0(\overline{\Omega},\mathbb{R}^3;\mathbb{R})$, if $\forall\omega$ open $\subset \Omega$, $\int_{p\in\omega}\varphi(p)\,d\Omega = \int_{p\in\partial\omega}\psi(p,\vec{n}(p))\,d\Gamma$, then $\exists!\vec{k}\in C^1(\overline{\Omega};\mathbb{R}^3)$ s.t. $\psi = (\vec{k},\vec{n})_g$.

**Answer**. Let $p\in\Omega\subset\mathbb{R}^3$, $h_1,h_2,h_3>0$ and $\partial\omega$ be the tetrahedral defined by its vertices $p$, $p+h_1\vec{e}_1$, $p+h_2\vec{e}_2$, $p+h_3\vec{e}_3$. Call $\partial\omega_1$ (square triangle) the face of $\partial\omega$ which outer unit normal is $\vec{n}_1=-\vec{E}_1$: Its area is $a_1=\frac{1}{2}h_2h_3$. Idem for $\partial\omega_2$ and $\partial\omega_3$. Let $\partial\omega_4$ be the fourth face: Its area is $a_4=\frac{1}{2}\sqrt{h_2^2h_3^2+h_3^2h_1^2+h_1^2h_2^2}$ and its outer unit normal is $\vec{n}_4=\frac{1}{2a_4}(h_2h_3,h_3h_1,h_1h_2)=(\nu_1,\nu_2,\nu_3)$ where $\nu_i=\frac{a_i}{a_4}$ for $i=1,2,3$ (see exercise P.3). The volume of the tetrahedral is $v^3=\frac{1}{6}h_1h_2h_3$. Let $M:=\sup_{p\in\overline{\Omega}}|\varphi(p)|$; We have $M<\infty$, since $\varphi$ is continuous in $\overline{\Omega}$. Hence

$$Mv^3 \geq |\int_{p\in\omega}\varphi(p)\,d\Omega| = |\int_{\partial\omega}\psi(p,\vec{n}(p))\,d\Gamma|. \tag{P.4}$$

And, $\psi$ being continuous, the mean value theorem applied on $\partial\omega_i$ gives: For $i=1,...,4$ there exists $p_i\in\partial\omega_i$ s.t.

$$\int_{\partial\omega_i}\psi(p,\vec{n}(p))\,d\Gamma = a_i\psi(p_i,\vec{n}_i). \tag{P.5}$$

Thus $\int_{\partial\omega}=\sum_i\int_{\partial\omega_i}$ gives

$$|\int_{\partial\omega}\psi(p,\vec{n}(p))\,d\Gamma| = |a_1\psi(p_1,-\vec{E}_1)+a_2\psi(t,p_2,-\vec{E}_2)+a_3\psi(p_3,-\vec{E}_3)+a_4\psi(p_4,\vec{n}_4)| \leq Mv^3. \tag{P.6}$$

We flatten the tetrahedron on the $yz$ face by choosing $h_2=h_3=^{\text{written}} h$ and $h_1=h^2$, with $h\ll 1$; Thus $a_1=\frac{1}{2}h^2$, $a_2=a_3=\frac{1}{2}h^3=o(h^2)$, $a_4=\frac{1}{2}\sqrt{h^4+2h^6}=\frac{h^2}{2}\sqrt{1+2h^2}=\frac{1}{2}h^2+o(h^2)\simeq a_1$, $v^3=\frac{1}{6}h^4=o(h^2)$, $\vec{n}_4=\frac{1}{h^2+o(h^2)}(h^2,h^3,h^3)=(1-o(1),h-o(h),h-o(h))\sim\vec{E}_1=-\vec{n}_1$ and $p_i\sim p$; Thus (P.6) gives $|\psi(p,-\vec{E}_1)+\psi(p,+\vec{E}_1)|=o(1)$, thus

$$\psi(p,-\vec{E}_1)+\psi(p,+\vec{E}_1)=0. \tag{P.7}$$

Idem with $xz$ and $xy$. Then for the initial tetrahedron $\partial\omega$ consider the smaller tetrahedron with $\varepsilon h_1,\varepsilon h_2,\varepsilon h_3$. And (P.6) with (P.7) give, as $\varepsilon\to 0$,

$$\psi(p,\vec{n}_4) = -\frac{a_1}{a}\psi(p,-\vec{E}_1)-\frac{a_2}{a}\psi(p,-\vec{E}_2)-\frac{a_3}{a}\psi(p,-\vec{E}_3)=\sum_{i=1}^3 n_i\psi(p,\vec{E}_i).$$

Thus $\psi_p$ is a linear map in $\vec{n}_p$, that is, there exists a linear form $\alpha_p$ s.t. $\psi_p(\vec{n}_p)=\alpha_p.\vec{n}_p$ for any $p\in\partial\omega$. And the Riesz representation theorem gives: $\exists\vec{k}_p$ s.t. $\alpha_p.\vec{n}_p=(\vec{k}_p,\vec{n}_p)_g=^{\text{written}}\vec{k}_p\bullet\vec{n}_p$. ▪

**Exercice P.3** Consider a triangle $T$ in $\mathbb{R}^3$ which vertices are $A=(h_1,0,0)$, $B=(0,h_2,0)$, $C=(0,0,h_3)$. Prove that $\vec{n}=(h_2h_3,h_3h_1,h_1h_2)$ is orthogonal to $T$ and $\text{area}(T)=\frac{1}{2}\sqrt{h_2^2h_3^2+h_3^2h_1^2+h_1^2h_2^2}$.

**Answer**. Consider the parallelogram (including the triangle) given by $\vec{r}(t,u)=A+t\vec{AB}+u\vec{AC}$ for $t,u\in[0,1]$. Thus $\pm\vec{n}=\frac{\partial\vec{r}}{\partial t}\times\frac{\partial\vec{r}}{\partial u}=\vec{AB}\times\vec{AC}=\begin{pmatrix}-h_1\\h_2\\0\end{pmatrix}\times\begin{pmatrix}-h_1\\0\\h_3\end{pmatrix}=\begin{pmatrix}h_2h_3\\h_3h_1\\h_1h_2\end{pmatrix}$. And $d\sigma=||\frac{\partial\vec{r}}{\partial t}\times\frac{\partial\vec{r}}{\partial u}||dudt=\sqrt{h_2^2h_3^2+h_3^2h_1^2+h_1^2h_2^2}dudt$ gives twice $\text{area}(T)=\text{area}(\vec{r})=\int_{t=0}^1\int_{u=0}^1 d\sigma=\sqrt{h_2^2h_3^2+h_3^2h_1^2+h_1^2h_2^2}$. ▪

### P.2.2 Vector valued functions: $\gamma_{\vec{n}}$ is onto

For the usual notations in classic mechanics, we need to use matrices: A basis being imposed,

1- A vector $\vec{w}=\sum_i w_i\vec{e}_i$ will abusively be written $\vec{w}:=[\vec{w}]_{|\vec{e}}=\begin{pmatrix}w_1\\\vdots\\w_n\end{pmatrix}\in\mathcal{M}_{n1}$ (column matrix).

2- The row matrices $\vec{\sigma}_i=(\,\sigma_{i1}\;\;...\;\;\sigma_{in}\,)\in\mathcal{M}_{1n}$ make the $n*n$ matrix $\underline{\underline{\sigma}}=[\sigma_{ij}]=\begin{pmatrix}\vec{\sigma}_1\\\vdots\\\vec{\sigma}_n\end{pmatrix}\in\mathcal{M}_{nn}$,

$$\text{div}\vec{\sigma}_i:=\sum_{j=1}^n\frac{\partial\sigma_{ij}}{\partial x_j},\quad \text{div}(\underline{\underline{\sigma}}):=\begin{pmatrix}\text{div}\vec{\sigma}_1\\\vdots\\\text{div}\vec{\sigma}_n\end{pmatrix}\in\mathcal{M}_{n1}\text{ (mechanical divergence)}, \tag{P.8}$$

$$H^{\text{div}}(\Omega)^n := \{\underline{\underline{\sigma}}\in L^2(\Omega)^{n^2} : \forall i\in[1,n]_{\mathbb{N}},\ \text{div}(\vec{\sigma}_i)\in L^2(\Omega)\}. \tag{P.9}$$

3- The canonical inner dot product in $\mathcal{M}_{n1}$ is given by $\vec{u}\bullet\vec{w}=\sum_i u_iw_i$. So $\vec{\sigma}_i.\vec{w}$ (matrix product) $=\vec{\sigma}_i^T\bullet\vec{w}$ (inner dot product) $=\sum_j\sigma_{ij}w_j$.

**Corollary P.4** *The trace operator* $\gamma_{\vec{n}} : \left\{ \begin{array}{rl} H^{\mathrm{div}}(\Omega)^n & \to H^{-\frac{1}{2}}(\Gamma)^n \\ \underline{\underline{\sigma}} & \to \gamma_{\vec{n}}(\underline{\underline{\sigma}}) = \underline{\underline{\sigma}}.\vec{n} \end{array} \right\}$ *is (linear continuous and) onto:*

$$\forall \vec{\psi} \in L^2(\Gamma)^n,\ \exists \underline{\underline{\sigma}} \in H^{\mathrm{div}}(\Omega)^n \ \ s.t.\ \ \vec{\psi} = \underline{\underline{\sigma}}.\vec{n} \ \text{ on } \Gamma. \tag{P.10}$$

*Thus,* $\forall \vec{\psi} \in L^2(\Gamma)^n$,

$$\int_\Gamma \vec{\varphi} \bullet \vec{\psi}\, d\Gamma = \int_\Gamma \vec{\varphi} \bullet (\underline{\underline{\sigma}}.\vec{n})\, d\Gamma = \int_\Omega \vec{\varphi} \bullet \mathrm{div}\underline{\underline{\sigma}} + \underline{\underline{\sigma}} : d\vec{\varphi}\, d\Omega, \tag{P.11}$$

*i.e.* $\sum_i \int_\Gamma \varphi_i \psi_i\, d\Gamma = \sum_i \int_\Omega \varphi_i\, \mathrm{div}\vec{\sigma}_i + \vec{\sigma}_i.\vec{\mathrm{grad}}\varphi_i\, d\Omega = \sum_i \int_\Gamma \varphi_i \vec{\sigma}_i.\vec{n}\, d\Gamma$.

**Proof.** (P.3) component wise gives $\int_\Gamma \varphi_i \psi_i\, d\Gamma = \int_\Gamma \varphi_i \vec{\sigma}_i.\vec{n}\, d\Gamma = \int_\Omega \varphi_i\, \mathrm{div}\vec{\sigma}_i + \vec{\sigma}_i.\vec{\mathrm{grad}}\varphi_i\, d\Omega$, and $\vec{\sigma}_i.\vec{\mathrm{grad}}\varphi_i = \sum_j \sigma_{ij} \frac{\partial \varphi_i}{\partial x_j}$. ∎

## P.3 Master balance law

Let $\rho : \left\{ \begin{array}{rl} \bigcup_{t\in[t_0,T]} (\{t\} \times \Omega_t) & \to \mathbb{R} \\ (t, p_t) & \to \rho(t, p_t) \end{array} \right\}$ (a mass density), let $\vec{f} : \left\{ \begin{array}{rl} \bigcup_{t\in[t_0,T]} (\{t\} \times \Omega_t) & \to \vec{\mathbb{R}^n} \\ (t, p_t) & \to \vec{f}(t, p_t) \end{array} \right\}$ (a body force density), and let $\vec{T} : \left\{ \begin{array}{rl} \bigcup_{t\in[t_0,T]} (\{t\} \times \partial\omega_t \times \vec{\mathbb{R}^n_t}) & \to \vec{\mathbb{R}^n} \\ (t, p_t, \vec{n}(p_t)) & \to \vec{T}(t, p_t, \vec{n}(p_t)) \end{array} \right\}$ (a surface force density) defined for any regular subset $\omega_t \subset \Omega_t$.

**Definition P.5** The balance of momentum is satisfied by $\rho$, $\vec{f}$ and $\vec{T}$ regular enough iff for all $t$ and all regular open subset $\omega_t \subset \Omega_t$

$$\frac{d}{dt}\Big(\int_{\omega_t} \rho\vec{v}\, d\Omega\Big) = \int_{\omega_t} \vec{f}\, d\Omega + \int_{\partial\omega_t} \vec{T}\, d\Gamma \quad \text{(master balance law)}, \tag{P.12}$$

i.e., with (L.39),

$$\int_{\omega_t} \frac{D(\rho\vec{v})}{Dt} + \rho\vec{v}\,\mathrm{div}\vec{v}\, d\Omega = \int_{\omega_t} \vec{f}\, d\Omega + \int_{\partial\omega_t} \vec{T}\, d\Gamma. \tag{P.13}$$

Thus with $\vec{\gamma} = \frac{D\vec{v}}{Dt}$ and the conservation of mass hypothesis $\frac{D\rho}{Dt} + \rho\,\mathrm{div}\vec{v} = 0$ cf. (N.4):

$$\int_{\omega_t} \rho\vec{\gamma}\, d\Omega = \int_{\omega_t} \vec{f}\, d\Omega + \int_{\partial\omega_t} \vec{T}\, d\Gamma. \tag{P.14}$$

## P.4 Cauchy theorem $\vec{T} = \underline{\underline{\sigma}}.\vec{n}$ (Cauchy stress "tensor" $\underline{\underline{\sigma}}$)

Purpose: Transform $\int_{\Gamma_t} \vec{T}\, d\Gamma$ into $\int_{\Omega_t} \mathrm{div}\underline{\underline{\sigma}}\, d\Omega$.

**Theorem P.6 (Cauchy first law: Cauchy stress tensor)** *Let* $\vec{T} \in L^2(\Gamma)^n$ *and* $\underline{\underline{\sigma}}$ *s.t.* $\vec{T} =^{(P.10)} \underline{\underline{\sigma}}.\vec{n}$. *If the motion is* $C^2$, *if* $\vec{f} \in L^2(\Omega_t)$, *and if the conservation of mass and the master balance law are satisfied then*

$$\vec{f} + \mathrm{div}\underline{\underline{\sigma}} = \rho\vec{\gamma}. \tag{P.15}$$

**Proof.** Apply (P.14) with $\int_{\partial\Omega_t} \vec{T}\, d\Gamma = \int_{\Omega_t} \mathrm{div}\underline{\underline{\sigma}}\, d\Gamma$. ∎

Thus

$$\int_{\Omega_t} (\rho\vec{\gamma} - \vec{f}) \bullet \vec{w}\, d\Omega = -\int_{\Omega_t} \underline{\underline{\sigma}} : d\vec{w}\, d\Omega + \int_{\Gamma_t} \vec{T} \bullet \vec{w}\, d\Gamma \quad \Big(= \int_{\Omega_t} \mathrm{div}\underline{\underline{\sigma}} \bullet \vec{w}\, d\Omega\Big). \tag{P.16}$$

## P.5 Alternative approach: div : $H^{\text{div}}(\Omega) \to L^2(\Omega)$ is onto

Cauchy starts with $\vec{T}$ defined on $\Gamma_t$, and then extends $\vec{T}$ inside $\Omega_t$ (computations on inside tetrahedrons), makes computations with $\underline{\underline{\sigma}}$ define in $\Omega_t$. Doing so he implicitly uses:

**Theorem P.7** *The divergence operator* div : $H^{\text{div}}(\Omega) \to L^2(\Omega)$ *is (linear continuous and) onto*[4]*:*

$$\forall \varphi \in L^2(\Omega), \quad \exists \vec{w} \in H^{\text{div}}(\Omega) \quad s.t. \quad \varphi = \text{div}\vec{w}. \tag{P.17}$$

*($\vec{w}$ is not unique since you can add any "incompressible" vector field $\vec{u}$, i.e. s.t. $\text{div}\vec{u} = 0$.)*

*Hence the divergence operator* div : $H^{\text{div}}(\Omega)^n \to L^2(\Omega)^n$ *is (linear continuous and) onto:* $\forall \vec{T} \in L^2(\Omega)^n$, $\exists \underline{\underline{\sigma}} \in H^{\text{div}}(\Omega)^n$ *s.t.*

$$\vec{T} = \text{div}\underline{\underline{\sigma}} \quad in\ \Omega. \tag{P.18}$$

**Proof.** Application of the Petree–Tartar compactness theorem, see e.g. [15] for references. In finite element courses, it is "the inf-sup condition for the divergence" (vocabulary introduced by Brezzi because surjectivity between Banach spaces can be expressed thanks to an inf-sup equality).

Then component by component for $\vec{T}$. ∎

**Definition:** The balance of momentum is satisfied by $\rho$, $\vec{f}$ and $\vec{T}$ iff for all $t$ and all $\omega_t \subset \Omega_t$

$$\int_{\omega_t} \rho\vec{\gamma}\, d\Omega = \int_{\omega_t} \vec{f}\, d\Omega + \int_{\omega_t} \vec{T}\, d\Omega, \quad \text{i.e.} \quad \rho\vec{\gamma} = \vec{f} + \text{div}\underline{\underline{\sigma}}, \tag{P.19}$$

Hence, with a Euclidean dot product, $\int_{\Omega_t} \rho\vec{\gamma}\, d\Omega = \int_{\Omega_t} \vec{f}\, d\Omega + \int_{\Gamma_t} \underline{\underline{\sigma}}.\vec{n}\, d\Gamma$.

# Q Balance of moment of momentum

**Definition Q.1** The "master balance of moment of momentum law", or simply the "balance of moment of momentum", is satisfied by $\rho$, $\vec{f}$ and $\vec{T}$ iff for all regular sub-open set $\omega \subset \Omega$

$$\frac{d}{dt}\int_\omega \rho\,\overrightarrow{OM} \times \vec{v}\, d\Omega = \int_\omega \rho\,\overrightarrow{OM} \times \vec{f}\, d\Omega + \int_{\partial\omega_t} \overrightarrow{OM} \times \vec{T}\, d\Gamma_t, \tag{Q.1}$$

(This excludes e.g. Cosserat continua materials.)

**Theorem Q.2** *(Cauchy second law.) If the master balance law (so $\vec{T} = \underline{\underline{\sigma}}.\vec{n}$) and the master balance of moment of momentum law are satisfied then $\underline{\underline{\sigma}}$ is symmetric.*

**Proof.** Let $\vec{x} = \overrightarrow{OM} = \sum_i x_i\vec{E}_i$, and $\vec{T} = \sum_i T_i\vec{E}_i = \underline{\underline{\sigma}}.\vec{n} = \sum_{ij}\sigma_{ij}n_j\vec{E}_i$. Then (first component) $(\vec{x}\times\vec{T})_1 = x_2T_3 - x_3T_2 = x_2(\sigma_{31}n_1+\sigma_{32}n_2+\sigma_{33}n_3) - x_3(\sigma_{21}n_1+\sigma_{22}n_2+\sigma_{23}n_3) = (x_2\sigma_{31}-x_3\sigma_{21})n_1 + (x_2\sigma_{32}-x_3\sigma_{22})n_2 + (x_2\sigma_{33}-x_3\sigma_{23})n_3$. Thus $\int_{\partial\omega_t}(\vec{x}\times\vec{T})_1\, d\Gamma_t = \int_{\omega_t} \frac{\partial(x_2\sigma_{31}-x_3\sigma_{21})}{\partial x_1} + \frac{\partial(x_2\sigma_{32}-x_3\sigma_{22})}{\partial x_2} + \frac{\partial(x_2\sigma_{33}-x_3\sigma_{23})}{\partial x_3}\, d\Omega = \int_{\omega_t} x_2(\text{div}\underline{\underline{\sigma}})_3 + x_3(\text{div}\underline{\underline{\sigma}})_2 + \sigma_{32} - \sigma_{23}\, d\omega$.

(P.15) gives $\rho\vec{\gamma} - \vec{f} = \text{div}\underline{\underline{\sigma}}$, thus $\vec{x}\times(\rho\vec{\gamma}-\vec{f}) = \vec{x}\times\text{div}\underline{\underline{\sigma}}$, thus the first component of $\vec{x}\times(\rho\vec{\gamma}-\vec{f})$ is $x_2(\text{div}\underline{\underline{\sigma}})_3 - x_3(\text{div}\underline{\underline{\sigma}})_2$. Thus (Q.1) gives $\int_{\omega_t}\sigma_{32}-\sigma_{23}\, d\omega = 0$. True for all $\omega\subset\Omega$, thus $\sigma_{32}-\sigma_{23} = 0$. Idem for the other components: $\underline{\underline{\sigma}}$ is symmetric. ∎

# R Uniform tensors in $\mathcal{L}^r_s(E)$

Uniform tensors enable to define without ambiguity the "objective contraction rules". Uniform tensors are scalar valued multilinear functions acting on both vectors and linear forms.

NB: In classical mechanics courses, what is called a "tensor" generally not a tensor but a matrix. E.g. you may encounter the expression "Euclidean tensor" which means: The matrix representation of "something" with respect to a Euclidean basis (based on the foot, metre,...) chosen by some observer.

[4] (P.17) tells that one regular function $\varphi$ can be deduced from a vector $\vec{v} \in \mathcal{M}_{n1}$ ($n$ functions) s.t. $\text{div}\vec{v} = \varphi$. Not an impressive result compared with the rare result: "A vector field ($n$ components) derives from one potential (one component)".

## R.1 Tensorial product and multilinear forms

Let $A_i$, $i \in [1,n]_{\mathbb{N}}$, be finite dimension vector spaces, and $A_i^* = \mathcal{L}(A_i;\mathbb{R})$ their duals (set of linear forms).

### R.1.1 Tensorial product of functions

The tensorial product of functions $f_i : A_i \to \mathbb{R}$, $i \in [1,n]_{\mathbb{N}}$, is the "separate variable" function $f_1 \otimes ... \otimes f_n : A_1 \times ... \times A_n \to \mathbb{R}$ defined by

$$(f_1 \otimes ... \otimes f_n)(\vec{x}_1, ..., \vec{x}_n) = f_1(\vec{x}_1)...f_n(\vec{x}_n). \tag{R.1}$$

E.g., $n = 2$, $A_1 = A_2 = \mathbb{R}$ and $(\cos \otimes \sin)(x, y) = \cos(x)\sin(y)$.

### R.1.2 Tensorial product of linear forms: multilinear forms

Let $\mathcal{L}(A_1, ..., A_n; \mathbb{R})$ be the set of $\mathbb{R}$-multilinear forms on the Cartesian product $A_1 \times ... \times A_n$, i.e. the set of the functions $M : A_1 \times ... \times A_n \to \mathbb{R}$ s.t., for all $i = 1, ..., n$, all $\vec{x}_i, \vec{y}_i \in A_i$ and all $\lambda \in \mathbb{R}$,

$$M(..., \vec{x}_i + \lambda\vec{y}_i, ...) = M(..., \vec{x}_i, ...) + \lambda\, M(..., \vec{y}_i, ...), \tag{R.2}$$

the other variables being unchanged.

Definition: An elementary tensor is multilinear form $M = \ell_1 \otimes .... \otimes \ell_n$, with $\ell_i \in A_i^*$ for all $i$; So

$$\forall (\vec{x}_i)_{i \in [1,n]_{\mathbb{N}}} \in \prod_{i=1}^n A_i, \quad (\ell_1 \otimes ... \otimes \ell_n)(\vec{x}_1, ..., \vec{x}_n) = (\ell_1.\vec{x}_1)...(\ell_n.\vec{x}_n) \ \in \mathbb{R}. \tag{R.3}$$

(The dot in $\ell_i.\vec{x}_i$ is **not** an inner dot product: It is the duality outer product $\ell_i.\vec{x}_i := \ell_i(\vec{x}_i) = \langle \ell, \vec{x} \rangle_{A_i, A_i^*}$.)

## R.2 Uniform tensors in $\mathcal{L}_s^0(E)$

Let $E$ be a real vector space, with $\dim(E) = n \in \mathbb{N}^*$. In this section we consider the first overlay on $E$ made of multilinear forms $M$ on $E$, called the uniform tensors of type 0 $s$ or of type $\binom{0}{s}$.

E.g., $M \in \mathcal{L}_1^0(E)$ a linear form, $M \in \mathcal{L}_2^0(E)$ an inner dot product, $M \in \mathcal{L}_n^0(E)$ a determinant...

Notations for quantification purposes: $(\vec{e}_i)$ is a basis in $E$, $(\pi_{ei})$ is its (covariant) dual basis (basis in $E^* = \mathcal{L}(E;\mathbb{R})$), $(\partial_i)$ is its bidual basis (basis in $E^{**} = \mathcal{L}(E^*;\mathbb{R})$).

### R.2.1 Definition of type $\binom{0}{s}$ uniform tensors

$\mathcal{L}_0^0(E) := \mathbb{R}$, and if $s \in \mathbb{N}^*$ then

$$\mathcal{L}_s^0(E) := \mathcal{L}(\underbrace{E \times ... \times E}_{s \text{ times}}; \mathbb{R}) \text{ is called the set of uniform tensors of type } \binom{0}{s} \text{ on } E. \tag{R.4}$$

### R.2.2 Example: Type $\binom{0}{1}$ uniform tensor = linear forms

A type $\binom{0}{1}$ uniform tensor is an element of $\mathcal{L}_1^0(E) = \mathcal{L}(E;\mathbb{R}) = E^*$: It is a linear form $\ell \in \mathcal{L}_1^0(E) = E^*$.

**Quantification:** With $\ell_i := \ell(\vec{e}_i)$ we have, cf. (A.10),

$$\ell = \sum_{i=1}^n \ell_i \pi_{ei}, \quad \text{and} \quad [\ell]_{|\pi_e} = (\, \ell_1 \quad ... \quad \ell_n \,) \overset{\text{written}}{=} [\ell]_{|\vec{e}} \tag{R.5}$$

(row matrix for a linear form). Duality notations: $(e^i)$ is the covariant dual basis and $\ell = \sum_{i=1}^n \ell_i e^i$.

Thus, if $\vec{v} \in E$, $\vec{v} = \sum_{i=1}^n v_i \vec{e}_i$, then $\vec{v}$ is represented by $[\vec{v}]_{|\vec{e}} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ (column matrix for a vector), and the matrix calculation rules give

$$\ell(\vec{v}) = [\ell]_{|\vec{e}}.[\vec{v}]_{|\vec{e}} = (\, \ell_1 \quad ... \quad \ell_n \,) . \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^n \ell_i v_i \overset{\text{written}}{=} \ell.\vec{v}. \tag{R.6}$$

Duality notations: $\vec{v} = \sum_{i=1}^n v^i \vec{e}_i$ and $\ell(\vec{v}) = \sum_{i=1}^n \ell_i v^i$, and Einstein's convention is satisfied.

### R.2.3 Example: Type $\binom{0}{2}$ uniform tensor

A type $\binom{0}{2}$ uniform tensor is an element of $\mathcal{L}^0_2(E) = \mathcal{L}(E,E;\mathbb{R})$: It is a bilinear form $T \in \mathcal{L}(E,E;\mathbb{R})$ (e.g., an inner dot product). In particular an elementary uniform tensor in $\mathcal{L}^0_2(E)$ is a tensor $T = \ell \otimes m$ where $\ell, m \in E^*$.

**Quantification:** Let $[T]_{|\vec{e}} = [T_{ij}] := [T(\vec{e}_i, \vec{e}_j)]$. Then, with $\vec{v} = \sum_{i=1}^n v_i \vec{e}_i$ and $\vec{w} = \sum_{i=1}^n w_i \vec{e}_i$,

$$T(\vec{v},\vec{w}) = \sum_{i,j=1}^n T_{ij} v_i w_j = [\vec{v}]^T_{|\vec{e}}.[T]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}, \quad \text{i.e.} \quad T = \sum_{i,j=1}^n T_{ij} \pi_{ei} \otimes \pi_{ej}. \tag{R.7}$$

Duality notations: $T = \sum_{ij} T_{ij} e^i \otimes e^j$ and $T(\vec{v},\vec{w}) = \sum_{ij} T_{ij} v^i w^j$.

### R.2.4 Example: Determinant

The determinant is a alternating $\binom{0}{n}$ uniform tensor, cf. (L.2).

## R.3 Uniform tensors in $\mathcal{L}^r_s(E)$

They are multilinear forms acting on both vectors ($\in E$) and functions $\in E^*$ (linear forms).

### R.3.1 Definition of type $\binom{r}{s}$ uniform tensors

Let $r, s \in \mathbb{N}$ s.t. $r + s \geq 1$. The set of multilinear forms

$$\mathcal{L}^r_s(E) := \mathcal{L}(\underbrace{E^* \times ... \times E^*}_{r \text{ times}}, \underbrace{E \times ... \times E}_{s \text{ times}}; \mathbb{R}) \tag{R.8}$$

is called the set of uniform tensors of type $\binom{r}{s}$ on $E$. The case $r = 0$ has been considered at § R.2.

### R.3.2 Example: Type $\binom{1}{0}$ uniform tensor: Identified with a vector

A uniform $\binom{1}{0}$ tensor is a element $w \in \mathcal{L}^1_0(E) = \mathcal{L}(E^*;\mathbb{R}) = E^{**}$. And we have the natural canonical isomorphism

$$\mathcal{J} : \begin{cases} E & \to E^{**} = \mathcal{L}^1_0(E) \\ \vec{w} & \to \mathcal{J}(\vec{w}) = w \quad \text{defined by} \quad w(\ell) := \ell(\vec{w}) \quad \forall \ell \in E^*. \end{cases} \tag{R.9}$$

And linearity gives $w(\ell) =^{\text{written}} w.\ell$ and $\ell(\vec{w}) =^{\text{written}} \ell.\vec{w}$. Hence a $\binom{1}{0}$ uniform tensor $w$ can be identified with the vector $\vec{w} = \mathcal{J}^{-1}(w)$:

$$w.\ell \overset{\text{written}}{=} \vec{w}.\ell \quad (= \ell.\vec{w}). \tag{R.10}$$

**Interpretation:** $E^{**}$ is the set of directional derivatives. Indeed, if $p \in \mathcal{E}$, and if $f$ is a differentiable function at $p$, then $w.df(p) =^{(R.9)} df(p).\vec{w}$ is the directional derivative along $\vec{w}$.

Remark: In differential geometry, $w.df$ is written $\vec{w}(f)$, so $\vec{w}(f)(p) := df(p).\vec{w}$, the definition of a vector being a directional derivative.

**Quantification:** Basis $(\vec{e}_i)$, dual basis $(\pi_{ei})$, bidual basis $(\partial_i)$. For all $i, j$,

$$\partial_i.\pi_{ej} = \delta_{ij} = \pi_{ej}.\vec{e}_i, \quad \text{thus} \quad \partial_i = \mathcal{J}(\vec{e}_i) \overset{\text{written}}{=} \vec{e}_i. \tag{R.11}$$

And $w = \sum_i w_i \partial_i =^{\text{written}} \sum_i w_i \vec{e}_i = \vec{w}$ and $\ell = \sum_i \ell_i \pi_{ei}$ give $w.\ell = \sum_i w_i \ell_i$.

Duality notations: $\partial_i.e^j = \delta_i^j = e^j.\vec{e}_i$, $w = \sum_i w^i \partial_i \simeq \sum_i w^i \vec{e}_i$, $\ell = \sum_i \ell_i \pi_{ei}$, $w.\ell = \sum_i w^i \ell_i$.

### R.3.3 Example: Type $\binom{1}{1}$ uniform tensor

A uniform $\binom{1}{1}$ tensor is a element $T \in \mathcal{L}^1_1(E) = \mathcal{L}(E^*, E;\mathbb{R})$, thus $T$ is bilinear and $T(\ell, \vec{w}) \in \mathbb{R}$ for all $\ell \in E^*$ and $\vec{w} \in E$. In particular an elementary uniform $\binom{1}{1}$ tensor is a tensor $T = u \otimes \beta$ where $u \in E^{**}$ and $\beta \in E^*$; Also written $T = \vec{u} \otimes \beta$ thanks to the identification $J(u) =^{(R.9)} \vec{u} =^{\text{written}} u$. Then thanks to linearity $(u \otimes \beta)(\ell, \vec{w}) = u(\ell)\beta(\vec{w}) =^{\text{written}} (u.\ell)(\beta.\vec{w}) = (\ell.\vec{u})(\beta.\vec{w})$.

**Quantification:** Basis $(\vec{e}_i)$, dual basis $(\pi_{ei})$, bidual basis $(\partial_i)$. Let $T_{ij} := T(\pi_{ei}, \vec{e}_j)$, $[T]_{|\vec{e}} = [T_{ij}] = [T]$, $\vec{w} = \sum_i w_i \vec{e}_i$, $[\vec{w}]_{\vec{e}} = [w_i] = [\vec{w}]$ (column matrix), $\ell = \sum_i \ell_i \pi_{ei}$, $([\ell]_{|\vec{e}}[= [\ell_j] = [\ell]$ (row matrix),

$$T = \sum_{i,j=1}^n T_{ij}\ \vec{e}_i \otimes \pi_{ej}, \quad \text{and} \quad T(\ell, \vec{w}) = \sum_{i,j=1}^n T_{ij} \ell_i w_j = [\ell].[T.[\vec{w}]. \tag{R.12}$$

Duality notations: $T^i{}_j := T(e^i, \vec{e}_j)$, $\vec{w} = \sum_i w^i \vec{e}_i$, $\ell = \sum_i \ell_i e^i$, $T(\ell, \vec{w}) = \sum_{i,j=1}^n T^i{}_j \ell_i w^j$.

### R.3.4 Example: Type $\binom{1}{2}$ uniform tensor

Same steps: $T \in \mathcal{L}^1_2(E) = L(E^*, E, E; \mathbb{R})$, duality notations $T^i{}_{jk} := T(e^i, \vec{e}_j, \vec{e}_k)$,

$$T = \sum_{i,j,k=1}^n T^i{}_{jk} \vec{e}_i \otimes e^j \otimes e^k, \quad \text{and} \quad T(\ell, \vec{u}, \vec{w}) = \sum_{i,j,k=1}^n T^i{}_{jk} \ell_i u^j w^k. \tag{R.13}$$

## R.4 Exterior tensorial products

The tensorial product of $T_1 \in \mathcal{L}^{r_1}_{s_1}(E)$ and $T_2 \in \mathcal{L}^{r_2}_{s_2}(E)$ is the tensor $T_1 \otimes T_2 \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}(E)$ defined by

$$(T_1 \otimes T_2)(\ell_{1,1}, ..., \ell_{2,1}, ..., \vec{u}_{1,1}, ..., \vec{u}_{2,1}, ...) := T_1(\ell_{1,1}, ..., \vec{u}_{1,1}, ...) T_2(\ell_{2,1}, ..., \vec{u}_{2,1}, ...). \tag{R.14}$$

Particular case: with $\lambda \in \mathcal{L}^0_0(E) = \mathbb{R}$ and $T \in \mathcal{L}^r_s(E)$,

$$\lambda \otimes T = T \otimes \lambda := \lambda T \in \mathcal{L}^r_s(E). \tag{R.15}$$

**Example R.1** $T_1, T_2 \in \mathcal{L}^1_1(E)$ gives $T_1 \otimes T_2 \in \mathcal{L}^2_2(E)$. Quantification: $T_1 = \sum_{i,j=1}^n (T_1)^i_j \vec{e}_i \otimes e^j$ and $T_2 = \sum_{k,m=1}^n (T_2)^k_m \vec{e}_k \otimes e^m$ give $T_1 \otimes T_2 = \sum_{i,j,k,m=1}^n (T_1)^i_k (T_2)^j_m \vec{e}_i \otimes \vec{e}_j \otimes e^k \otimes e^m$. ▲

**Remark R.2** Another definition: $T_1 \widetilde{\otimes} T_2 := \sum_{i,j,k,m=1}^n (T_1)^i_j (T_2)^k_m \vec{e}_i \otimes e^j \otimes \vec{e}_k \otimes e^m \in \mathcal{L}(E^*, E, E^*, E; \mathbb{R})$. And we get back to the previous definition thanks to the natural canonical isomorphism $\widetilde{J} : \widetilde{T} \in \mathcal{L}(E^*, E, E^*, E; \mathbb{R}) \to T \in \mathcal{L}(E^*, E^*, E, E; \mathbb{R}) = \mathcal{L}^2_2(E)$ defined by $T(\ell, m, \vec{v}, \vec{w}) = \widetilde{T}(\ell, \vec{v}, m, \vec{w})$. ▲

## R.5 Objective contractions

### R.5.1 Contraction of a linear form with a vector

Let $\ell \in \mathcal{L}^0_1(E) = E^*$ and $\vec{w} \in E$. Their contraction is the value

$$\ell(\vec{w}) \overset{\text{linearity}}{=} \ell.\vec{w} \overset{\text{written}}{=} \vec{w}.\ell. \tag{R.16}$$

**Quantification:** Basis $(\vec{e}_i)$, dual basis $(\pi_{ei})$, $\ell = \sum_{i=1}^n \ell_i \pi_{ei}$ and $\vec{w} = \sum_{i=1}^n w_i \vec{e}_i$ give

$$\ell.\vec{w} = \sum_{i=1}^n \ell_i w_i = [\ell]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} = \sum_{i=1}^n w_i \ell_i = \vec{w}.\ell = \mathrm{Tr}(\vec{w} \otimes \ell), \tag{R.17}$$

where Tr is the objective trace operator $\mathrm{Tr} : \mathcal{L}(E; E) \simeq \mathcal{L}^1_1(E) \to \mathbb{R}$.

Duality notations: Dual basis $(e^i)$, $\ell = \sum_i \ell_i e^i$, $\vec{w} = \sum_i w^i \vec{e}_i$, $\ell.\vec{w} = \sum_i \ell_i w^i$.

### R.5.2 Contraction of a $\binom{1}{1}$ tensor and a vector

Let $\ell \in E^*$ and $\vec{w}, \vec{u} \in E$. The contraction of the elementary tensor $\vec{w} \otimes \ell \in \mathcal{L}^1_1(E)$ with $\vec{u}$ is defined by:

$$(\vec{w} \otimes \underbrace{\ell).\vec{u}}_{\text{contraction}} = (\ell.\vec{u})\vec{w}. \tag{R.18}$$

The contraction of a tensor $T \in \mathcal{L}^1_1(E)$ with $\vec{u} \in E$ is the linear operator

$$\left\{ \begin{array}{rl} \mathcal{L}^1_1(E) \times E & \to E \\ (T, \vec{u}) & \to T.\vec{u} \end{array} \right\} \quad \text{s.t.} \quad \text{(R.18) is satisfied for all } T = \vec{w} \otimes \ell. \tag{R.19}$$

**Quantification:** Basis $(\vec{e}_i)$, dual basis $(e^i)$, $T = \sum_{ij} T^i{}_j \vec{e}_i \otimes e^j \in \mathcal{L}^1_1(E)$, $\vec{u} = \sum_j u^j \vec{e}_j \in E$,

$$T.\vec{u} = \sum_{i,j=1}^n T^i{}_j u^j \vec{e}_i \tag{R.20}$$

because $(\sum_{ij} T^i{}_j \vec{e}_i \otimes e^j).\vec{u} = \sum_{ij} T^i{}_j (\vec{e}_i \otimes e^j).\vec{u} = \sum_{ij} T^i{}_j (e^j.\vec{u})\vec{e}_i$.

Classical notations: $T.\vec{u} = \sum_{ij} T_{ij} u_j \vec{e}_i$.

NB: With the natural canonical isomorphism $(\mathcal{L}^1_1(E) =)\ \mathcal{L}(E, E^*; \mathbb{R}) \simeq \mathcal{L}(E; E)$, see (U.10), any endomorphism $L \in \mathcal{L}(E; E)$ defined by $L.\vec{e}_j = \sum_{i=1}^n L^i{}_j \vec{e}_i$ can, for calculation purposes, be written

$$\widetilde{L} = \sum_{i,j=1}^n L^i{}_j \vec{e}_i \otimes e^j \overset{\text{written}}{=} L, \quad \text{which means} \quad L.\vec{u} \overset{(R.18)}{=} \sum_{i=1}^n L^i{}_j u^j \vec{e}_i \tag{R.21}$$

when $\vec{u} = \sum_i u^j \vec{e}_j$.

Generalization: If $\ell \in E^*$, $\vec{u} \in E$ and $\vec{w} \in F$ then the objective contraction $(\vec{w} \otimes \ell).\vec{u} \in F$ is defined by

$$(\vec{w} \otimes \underbrace{\ell).\vec{u}}_{\text{contraction}} := (\ell.\vec{u})\vec{w}. \tag{R.22}$$

### R.5.3 Contractions of uniform tensors

More generally: Let $T_1 \in \mathcal{L}^{r_1}_{s_1}(E)$, $T_2 \in \mathcal{L}^{r_2}_{s_2}(E)$, $\ell \in E^*$ and $\vec{u} \in E$.

**Definition R.3** The objective contraction of $T_1 \otimes \ell \in \mathcal{L}^{r_2}_{s_2+1}(E)$ and $\vec{u} \otimes T_2 \in \mathcal{L}^{r_2+1}_{s_2}(E)$ is the tensor $(T_1 \otimes \ell).(\vec{u} \otimes T_2) \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}$ given by

$$(T_1 \otimes \underbrace{\ell).(\vec{u}}_{\text{contraction}} \otimes T_2) := (\ell.\vec{u})\, T_1 \otimes T_2. \tag{R.23}$$

And the objective contraction of $T_1 \otimes \vec{u} \in \mathcal{L}^{r_2+1}_{s_2}(E)$ and $\ell \otimes T_2 \in \mathcal{L}^{r_2}_{s_2+1}(E)$ is the tensor $(T_1 \otimes \vec{u}).(\ell \otimes T_2) \in \mathcal{L}^{r_1+r_2}_{s_1+s_2}$ given by

$$(T_1 \otimes \vec{u}).(\ell \otimes T_2) = (\vec{u}.\ell)\, T_1 \otimes T_2 \quad (= (\ell.\vec{u})\, T_1 \otimes T_2). \tag{R.24}$$

Quantification with a basis $(\vec{e}_i)$, examples to avoid cumbersome notations:

**Example R.4** $T \in \mathcal{L}^1_1(E)$, $T = \sum_{i,j=1}^n T^i{}_j \vec{e}_i \otimes e^j$, $\vec{w} \in \mathcal{L}^1_0(E) = E^{**} \sim E$, $\vec{w} = \sum_{j=1}^n w^j \vec{e}_j$. (R.23) gives $T.\vec{w} \in \mathcal{L}^1_0(E) = E^{**} \simeq E$ and

$$T.\vec{w} = \sum_{i,j=1}^n T^i{}_j w^j \vec{e}_i, \quad \text{i.e.} \quad [T.\vec{w}]_{|\vec{e}} = [T]_{|\vec{e}}.[\vec{w}]_{|\vec{e}} \quad \text{(column matrix)}. \tag{R.25}$$

Indeed, $T.\vec{w} = \sum_{ijk} T^i{}_j w^k (\vec{e}_i \otimes e^j).\vec{e}_k = \sum_{ijk} T^i{}_j w^k \vec{e}_i (e^j.\vec{e}_k) = \sum_{ijk} T^i{}_j w^k \vec{e}_i (\delta^j_k) = \sum_{ij} T^i{}_j w^j \vec{e}_i$. ▪

**Example R.5** $\ell = \sum_{i=1}^n \ell_i e^i \in E^* = \mathcal{L}^0_1(E)$, $T = \sum_{j,k=1}^n T^j{}_k \vec{e}_j \otimes e^k \in \mathcal{L}^1_1(E)$, $\ell.T \in \mathcal{L}^0_1(E) = E^*$ and

$$\ell.T = \sum_{i,j=1}^n \ell_i T^i{}_j e^j, \quad \text{i.e.} \quad [\ell.T]_{|\vec{e}} = [\ell]_{|\vec{e}}.[T]_{|\vec{e}} \quad \text{(row matrix)}. \tag{R.26}$$

Indeed $\ell.T = \sum_{ijk} \ell_i T^j{}_k (e^i.\vec{e}_j) e^k = \sum_{ijk} \ell_i T^j{}_k \delta^i_j e^k = \sum_{ik} \ell_i T^i{}_k e^k$. ▪

**Example R.6** $S, T \in \mathcal{L}^1_1(E)$, $S = \sum_{i,k=1}^n S^i_k \vec{e}_i \otimes e^k$ and $T = \sum_{j,k=1}^n T^k_j \vec{e}_k \otimes e^j$. Then

$$S.T = \sum_{i,j,k=1}^n S^i_k T^k_j \vec{e}_i \otimes e^j, \quad \text{i.e.} \quad [S.T]_{|\vec{e}} = [S]_{|\vec{e}}.[T]_{|\vec{e}} \tag{R.27}$$

Indeed $S.T = (\sum_{ik} S^i_k \vec{e}_i \otimes e^k).(\sum_{j,m} T^m_j \vec{e}_m \otimes e^j) = \sum_{ijkm} S^i_k T^m_j \vec{e}_i (e^k.\vec{e}_m) \otimes e^j = \sum_{ijk} S^i_k T^k_j \vec{e}_i \otimes e^j$. ▪

**Example R.7** $T \in \mathcal{L}^1_2(E)$, $\vec{u}, \vec{w} \in \mathcal{L}^1_0(E) \simeq E$, $T = \sum_{ijk} T^i_{jk} \vec{e}_i \otimes e^j \otimes e^k$, $\vec{w} = \sum_i w^i \vec{e}_i$, $\vec{u} = \sum_{i=1}^n u^i \vec{e}_i$,

$$T.\vec{w} = \sum_{i,j,k=1}^n T^i_{jk} w^k \vec{e}_i \otimes e^j \in \mathcal{L}^1_1(E), \quad (T.\vec{w}).\vec{u} = \sum_{i,j,k=1}^n T^i_{jk} w^k u^j \vec{e}_i \overset{\text{written}}{=} T(\vec{u}, \vec{w}) \in \mathcal{L}^1_0(E) \simeq E. \tag{R.28}$$

And $\ell = \sum_i \ell_i e^i \in E^*$ gives

$$((T.\vec{w}).\vec{u}).\ell = \ell.(T.\vec{w}).\vec{u} = \sum_{i,j,k=1}^n \ell_i T^i_{jk} w^k u^j = \sum_{i,j,k=1}^n \ell_i u^j T^i_{jk} w^k = T(\ell, \vec{u}, \vec{w}) = \ell.T(\vec{u}, \vec{w}). \tag{R.29}$$

▪

### R.5.4 Objective double contractions of uniform tensors

**Definition R.8** If $S,T \in \mathcal{L}^1_1(E)$ then the double objective contraction $S \mathbin{(\!:\!)} T$ of $S$ and $T$ is defined by

$$S \mathbin{(\!:\!)} T = \mathrm{Tr}(S.T). \tag{R.30}$$

**Quantification** Basis $(\vec{e}_i)$, dual basis $(e^i)$, bidual basis $(\partial_i)$, $S^i{}_j := S(e^i,\vec{e}_j)$, $T^i{}_j := T(e^i,\vec{e}_j)$, thus $S = \sum_{ij} S^i{}_j \vec{e}_i \otimes e^j$, $T = \sum_{jk} T^j{}_k \vec{e}_j \otimes e^k$, $S.T = \sum_{ijk} S^i{}_j T^j{}_k \vec{e}_i \otimes e^k$,

$$S \mathbin{(\!:\!)} T = \sum_{i,j=1}^n S^i{}_j T^j{}_i \quad (= T \mathbin{(\!:\!)} S). \tag{R.31}$$

**Proposition R.9** *$S \mathbin{(\!:\!)} T$ is an invariant: It is the trace $\mathrm{Tr}(L_S \circ L_T)$ of the endomorphisms $L_S, L_T \in \mathcal{L}(E;E)$ naturally canonically associated to $S$ and $T$ (defined by $\ell.L_S.\vec{u} := S(\ell,\vec{u})$ and $\ell.L_T.\vec{u} := T(\ell,\vec{u})$ for all $(\vec{u},\ell) \in E \times E^*$).*

*So the real $S \mathbin{(\!:\!)} T = \sum_{i,j=1}^n S^i{}_j T^j{}_i$ has the same value regardless of the chosen basis $(\vec{e}_i)$ used to compute it. (Which is not the case of the "term to term" matrix multiplication $S : T = \sum_{i,j=1}^n S^i_j T^i_j$, see next § R.5.5 and example R.13.)*

**Proof.** $L_S.\vec{e}_j = \sum_i S^i{}_j \vec{e}_i$ and $L_T.\vec{e}_j = \sum_k T^k{}_j \vec{e}_k$ (immediate check), thus $(L_S \circ L_T).\vec{e}_j = L_S(T.\vec{e}_j) = \sum_k T^k{}_j (L.\vec{e}_k) = \sum_i (\sum_k T^k{}_j S^i{}_k)\vec{e}_i$, thus $[L_S \circ L_T]_{\vec{e}} = [\sum_k T^k{}_j S^i{}_k]_{\substack{i\\j}}$, thus $\mathrm{Tr}(L_S \circ L_T) = \sum_k T^k{}_i S^i{}_k = S \mathbin{(\!:\!)} T$. And the trace of an endomorphism, here $L_S \circ L_T$, is objective. ∎

**Definition R.10** More generally, the objective double contractions $S \mathbin{(\!:\!)} T$ of uniform tensors $S \in \mathcal{L}^{r_1}_{s_1}(E)$ and $T \in \mathcal{L}^{r_2}_{s_2}(E)$ is obtained by applying the objective simple contraction twice consecutively, when applicable.

E.g., $T_1 \otimes \ell_{1,1} \otimes \ell_{1,2}$ and $\vec{u}_{2,1} \otimes \vec{u}_{2,2} \otimes T_2$ give

$$\begin{aligned}(T_1 \otimes \ell_{1,1} \otimes \underbrace{\ell_{1,2}).(\vec{u}_{2,1}}_{\text{first}} \otimes \vec{u}_{2,2} \otimes T_2) &= (\ell_{1,2}.\vec{u}_{2,1})(T_1 \otimes \underbrace{\ell_{1,1}) \otimes (\vec{u}_{2,2}}_{\text{second}} \otimes T_2) \\ &= (\ell_{1,2}.\vec{u}_{2,1})(\ell_{1,1}.\vec{u}_{2,2})\, T_1 \otimes T_2.\end{aligned} \tag{R.32}$$

**Example R.11** $S = \sum_{ijk} S^i_{jk} \vec{e}_i \otimes e^j \otimes e^j \in \mathcal{L}^1_2(E)$ and $T = \sum_{\alpha,\beta,\gamma} T^{\alpha\beta}{}_\gamma \vec{e}_\alpha \otimes \vec{e}_\beta \otimes e^\gamma \in \mathcal{L}^2_1(E)$ give

$$S.T = \sum_{i,j,k,\beta,\gamma=1}^n S^i{}_{jk} T^{k\beta}{}_\gamma \vec{e}_i \otimes e^j \otimes \vec{e}_\beta \otimes e^\gamma, \quad \text{and} \quad S \mathbin{(\!:\!)} T = \sum_{i,j,k,\gamma=1}^n S^i{}_{jk} T^{kj}{}_\gamma \vec{e}_i \otimes e^\gamma. \tag{R.33}$$

Similarly we define the triple objective contraction (apply the simple contraction three times consecutively). E.g., with (R.33) we get

$$S \mathbin{(\!\vdots\!)} T = \sum_{i,j,k=1}^n S^i{}_{jk} T^{kj}{}_i. \tag{R.34}$$

∎

**Exercice R.12** More generally. If $S \in \mathcal{L}(E,F;\mathbb{R})$, $T \in \mathcal{L}(F,E;\mathbb{R})$ then prove

$$S \mathbin{(\!:\!)} T = T \mathbin{(\!:\!)} S. \tag{R.35}$$

2- If $S \in \mathcal{L}(E,F;\mathbb{R})$, $T \in \mathcal{L}(F,G;\mathbb{R})$ and $U \in \mathcal{L}(G,E;\mathbb{R})$ then prove

$$S \mathbin{(\!:\!)} (T.U) = (S.T) \mathbin{(\!:\!)} U = U \mathbin{(\!:\!)} (S.T) = (U.S) \mathbin{(\!:\!)} T \quad \text{(circular permutation)}. \tag{R.36}$$

**Answer**. 1- $S = \sum_{ij} S^i_j \vec{a}_i \otimes b^j$ and $T = \sum_{ij} T^i_j \vec{b}_i \otimes a^j$ give then $S.T = \sum_{ijk\ell} S^i_j T^k_\ell (\vec{a}_i \otimes b^j).(\vec{b}_k \otimes a^\ell) = \sum_{ij\ell} S^i_j T^j_\ell (\vec{a}_i \otimes a^\ell)$, thus $S \mathbin{(\!:\!)} T = \sum_{ij} S^i_j T^j_i$. And $T.S = \sum_{ijk\ell} T^i_j S^k_\ell (\vec{b}_i \otimes a^j).(\vec{a}_k \otimes b^\ell) = \sum_{ij\ell} T^i_j S^j_\ell (\vec{b}_i \otimes b^\ell)$ thus $T \mathbin{(\!:\!)} S = \sum_{ij} T^i_j S^j_i = S \mathbin{(\!:\!)} T$.

2- $S = \sum_{ij} S^i_j \vec{a}_i \otimes b^j$, $T = \sum_{ij} T^i_j \vec{b}_i \otimes a^j$ and $U = \sum U^i_j \vec{c}_i \otimes a^j$ give $T.U = \sum T^i_k U^k_j \vec{b}_i \otimes a^j$, thus $S \mathbin{(\!:\!)} (T.U) = \sum S^i_m T^m_k U^k_i$, and $S.T = \sum S^i_k T^k_j \vec{a}_i \otimes c^j$, so $(S.T) \mathbin{(\!:\!)} U = \sum S^i_k T^k_m U^m_i$. And the second equality thanks to the symmetry of $\mathbin{(\!:\!)}$, i.e. $(S.T) \mathbin{(\!:\!)} U = U \mathbin{(\!:\!)} (S.T) = (U.S) \mathbin{(\!:\!)} T$ with the previous calculation. ∎

### R.5.5 Non objective double contraction: Double matrix contraction

The double matrix contraction of second order tensors is the "term to term multiplication" of the matrix representations: $(\vec{e}_i)$ is a basis, $S,T \in L(E;E) \simeq L(E^*,E;\mathbb{R})$, $S.\vec{e}_j = \sum_i S_{ij}\vec{e}_i$ i.e. $[S]_{|\vec{e}} = [S_{ij}]$, and $T.\vec{e}_j = \sum_i T_{ij}\vec{e}_i$ i.e. $[T]_{|\vec{e}} = [T_{ij}]$ give

$$[S]_{|\vec{e}} : [T]_{|\vec{e}} := \sum_{i,j=1}^{n} S_{ij}T_{ij} \underset{written}{\overset{unfortunately}{=}} S:T \tag{R.37}$$

Or $[S]_{|\vec{e}} : [T]_{|\vec{e}} := \sum_{i,j=1}^{n} S^i{}_j T^i{}_j$ with duality notations: Einstein's convention is ***not*** satisfied.

Unfortunate notation: Because the result is basis dependent (observer dependent, not objective, not invariant, not intrinsic...):

**Example R.13** $(\vec{e}_i)$ is a basis, $S \in \mathcal{L}(E;E)$ given by $[S]_{\vec{e}} = \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$ (so $S.\vec{e}_1 = 2\vec{e}_2$ and $S.\vec{e}_2 = 4\vec{e}_1$):

$$S:S = [S]_{\vec{e}} : [S]_{\vec{e}} = 4*4+2*2 = 20. \tag{R.38}$$

Basis $(\vec{b}_1 = \vec{e}_1, \vec{b}_2 = 2\vec{e}_2)$: The transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$ is $P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Thus $[S]_{\vec{b}} = P^{-1}.[S]_{\vec{e}}.P = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.\begin{pmatrix} 0 & 8 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 8 \\ 1 & 0 \end{pmatrix}$. Thus

$$S:S = [S]_{\vec{b}} : [S]_{\vec{b}} = 8*8+1*1 = 65 \neq 20 \quad \text{thus} \quad S:S \neq S:S \quad !!! \tag{R.39}$$

To be compared with the double objective contraction: $[S]_{\vec{e}} \odot [S]_{\vec{e}} = 4*2+2*4 = 16 = 8*1+1*8 = [S]_{\vec{b}} \odot [S]_{\vec{b}} = S \odot S$ (observer independent result = objective result).

Recall that the foot is the international vertical unit in aviation, and thus the use of the double **objective** contraction $\odot$ is vital, while the use of the double matrix contraction can be fatal (really). Also see the Mars climate orbiter probe crash.

**Exercice R.14** $(\vec{a}_i)$ is a Euclidean basis in foot, $(\vec{b}_i) = (\lambda\vec{a}_i)$ is the euclidean basis in metre (change of unit with $\lambda = \frac{1}{0.3048}$). Let $S \in \mathcal{L}^0_2(E)$. Compare $[S]_{|\vec{a}} : [S]_{|\vec{a}}$ with $[S]_{|\vec{b}} : [S]_{|\vec{b}}$. (The simple and double objective contractions are impossible here because $S$ and $T$ are not compatible.)

**Answer**. Let $S = \sum_{i,j=1}^{n} S_{a,ij} a^i \otimes a^j = \sum_{i,j=1}^{n} S_{b,ij} b^i \otimes b^j$. Since $(\vec{b}_i) = (\lambda\vec{a}_i)$ we have $b^i = \frac{1}{\lambda}a^i$. Thus $\sum_{i,j=1}^{n} S_{a,ij} a^i \otimes a^j = \sum_{i,j=1}^{n} S_{a,ij}\lambda^2 b^i \otimes b^j$, thus $\lambda^2 S_{a,ij} = S_{b,ij}$. Thus

$$[S]_{|\vec{b}} : [S]_{|\vec{b}} = \sum_{i,j=1}^{n} (S_{b,ij})^2 = \lambda^4 \sum_{i,j=1}^{n} (S_{a,ij})^2 = \lambda^4 [S]_{|\vec{a}} : [S]_{|\vec{a}}, \tag{R.40}$$

with $\lambda^4 \geq \frac{1}{100}$: Quite a difference isn't it?

## R.6 Kronecker (contraction) tensor, trace

**Definition R.15** The Kronecker tensor is the $\binom{1}{1}$ uniform tensor $\underline{\underline{\delta}} \in \mathcal{L}^1_1(E)$ defined by

$$\forall(\ell,\vec{u}) \in E^* \times E, \quad \underline{\underline{\delta}}(\ell,\vec{u}) := \ell.\vec{u}. \tag{R.41}$$

Quantification: Basis $(\vec{e}_i)$, dual basis $(e^i)$, components $\delta^i_j$ = the Kronecker symbols:

$$\delta = \sum_{i=1}^{n} \vec{e}_i \otimes e^i = \sum_{i,j=1}^{n} \delta^i_j \vec{e}_i \otimes e^j, \quad \text{i.e.} \quad [\delta]_{|\vec{e}} = I = [\delta^i_j], \quad \text{where} \quad \delta^i_j = \begin{cases} 1 \text{ if } i=j, \\ 0 \text{ if } i \neq j, \end{cases} \tag{R.42}$$

(identity matrix whatever the basis). Classic notations: $\delta = \sum_{i=1}^{n} \vec{e}_i \otimes \pi_{ei} = \sum_{i,j=1}^{n} \delta_{ij} \vec{e}_i \otimes \pi_{ej}$.

**Definition R.16** The trace of a $\binom{1}{1}$ uniform tensor $T \in \mathcal{L}^1_1(E)$ is

$$\widetilde{\mathrm{Tr}}(T) = \underline{\underline{\delta}} \odot T \tag{R.43}$$

($= \mathrm{Tr}(L_T)$ with the natural canonical isomorphism $T \in \mathcal{L}^1_1(E) \simeq L_T \in \mathcal{L}(E;E)$ given by $T(\ell,\vec{v}) := \ell.L_T.\vec{v}$).

Thus $\widetilde{\mathrm{Tr}}(T) = \sum_{i=1}^{n} T^i{}_i$ whatever the basis.

In particular $\widetilde{\mathrm{Tr}}(\underline{\underline{\delta}}) = n$, and $\widetilde{\mathrm{Tr}}(\vec{v} \otimes \ell) = \sum_i v^i \ell_i = \ell.\vec{v}$ when $\vec{v} = \sum_i v^i \vec{e}_i$ and $\ell = \sum_j \ell_j e^j$.

# S Tensors in $T^r_s(U)$

## S.1 Fundamental counter-example (derivation), and modules

Recall: Let $A$ and $B$ be any sets and $\mathcal{F}(A;B)$ be the set of functions $A \to B$. The "plus" inner operation and the "dot" outer operation are defined by, for all $f,g \in \mathcal{F}(A;B)$, all $\lambda \in \mathbb{R}$ and all $p \in A$,

$$\begin{cases} (f+g)(p) := f(p) + g(p), \quad \text{and} \\ (\lambda.f)(p) := \lambda\, f(p), \quad \lambda.f \overset{\text{written}}{=} \lambda f. \end{cases} \tag{S.1}$$

$(\mathcal{F}(A;B),+,.,\mathbb{R})$ is thus a vector space on the field $\mathbb{R}$ (see any elementary course) called $\mathcal{F}(A;B)$.

But the field $\mathbb{R}$ is "too small" to define tensors which are "tools that satisfy the change of coordinate system rules":

**Example S.1 Fundamental counter-example: Derivation.** $U$ being an open set in $\mathbb{R}^n$, the derivation $d : \vec{w} \in C^1(U;\vec{\mathbb{R}^n}) \to d\vec{w} \in C^0(U;\mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^n}))$ is $\mathbb{R}$-linear: In particular $d(\lambda\vec{w}) = \lambda(d\vec{w})$ for all $\lambda \in \mathbb{R}$...

...but $d$ doesn't satisfy the change of coordinate system rules, see (T.56).

So a derivation it ***not*** a tensor (it is a "spray", see Abraham–Marsden [1]).

In fact, one requirement for $T$ to be a tensor is, e.g. with $T \in T^0_1(U)$: For all $\varphi \in C^\infty(U;\mathbb{R})$, and all $\vec{w} \in \Gamma(U)$ ($C^\infty$-vector field),

$$T(\varphi\vec{w}) = \varphi\, T(\vec{w}). \tag{S.2}$$

While

$$d(\varphi\vec{w}) \neq \varphi\, d(\vec{w}), \quad \text{because} \quad d(\varphi\vec{w}) = \varphi\, d(\vec{w}) + d(\varphi).\vec{w}. \tag{S.3}$$

Thus the elementary $\mathbb{R}$-linearity requirement "$T.(\lambda\vec{w}) = \lambda(T.\vec{w})$ for all $\lambda \in \mathbb{R}$ is not sufficient to characterize a tensor: The $\mathbb{R}$-linearity has to be replaced by the $C^\infty(U;\mathbb{R})$-linearity, cf. (S.2).

Thus we will have to replace a real vector space $(V,+,.,\mathbb{R})$ over the field $\mathbb{R}$ with the "module" $(V,+,.,C^\infty(U;\mathbb{R}))$ over the ring $C^\infty(U;\mathbb{R})$, which mainly amounts to consider (S.1) for all $\lambda = \varphi \in C^\infty(U;\mathbb{R})$. Remark: The use of a module is very similar to the use of a vector space, but for the use of the inverse: all real $\lambda \neq 0$ has a multiplicative inverse in $\mathbb{R}$ (namely $\frac{1}{\lambda}$), but a function $f \in C^\infty(U;\mathbb{R})$ "that vanishes somewhere" doesn't have a multiplicative inverse in $C^\infty(U;\mathbb{R})$. ▟

## S.2 Field of functions and vector fields

$U$ is an open set in an affine space $\mathcal{E}$ which associated space is $E$ . The definition of tensors is done at a fixed time $t$ (concerns the space variables in classical mechanics). The approach is first qualitative, then quantitative with, at any $p \in \mathcal{E}$, a basis $(\vec{e}_i(p))$ and its dual basis $(\pi_{ei}(p)) = (e^i(p))$.

### S.2.1 Framework of classical mechanics

$\mathcal{E}$ is the affine space $\mathbb{R}$, $\mathbb{R}^2$ or $\mathbb{R}^3$ made of points $p$, and $E = \vec{\mathbb{R}^n}$ is the usual associated vector space $\vec{\mathbb{R}}$, $\vec{\mathbb{R}^2}$ or $\vec{\mathbb{R}^3}$ made of bipoint vectors $\vec{w} = \vec{pq} =^{\text{written}} q - p$, and we then write $q = p + \vec{w}$, which means: If $O \in \mathcal{E}$ (an origin) then $\overrightarrow{Oq} = \overrightarrow{Op} + \vec{w}$ (which is Chasles' relation $\vec{pq} = \overrightarrow{pO} + \overrightarrow{Oq}$), relation independent of the choice of $O$; And hence the vectors $\vec{w}$ in $E$ are called "free vectors": congruence relation: $\vec{u}\mathcal{R}\vec{w}$ iff $\vec{u} = \vec{w}$, i.e. $\overrightarrow{p_1q_1}\mathcal{R}\overrightarrow{p_2q_2}$ iff $\overrightarrow{p_1q_1} = \overrightarrow{p_2q_2}$.

### S.2.2 Vector fields

Let $\vec{w} : \left\{ \begin{array}{l} U \to E \\ p \to \vec{w}(p) \end{array} \right\}$ be a vector valued function. The associated vector field is

$$\widetilde{\vec{w}} : \begin{cases} U \to U \times E \\ p \to \widetilde{\vec{w}}(p) = (p, \vec{w}(p)) \quad \text{called a vector at } p. \end{cases} \tag{S.4}$$

So the range $\mathrm{Im}\widetilde{\vec{w}} = \{(p,\vec{w}(p)) : p \in U\}$ is the graph of $\vec{w}$, and the use of $\widetilde{\vec{w}}$ tells that the vector $\vec{w}(p)$ has to be drawn at "the so called base point" $p$ (first component of $\widetilde{\vec{w}}(p)$); And $\widetilde{\vec{w}}(p)$ is called a vector at $p$.

+ and . (usual) rules for vector fields: $\widetilde{\vec{u}}+\widetilde{\vec{w}}$ and $\lambda\widetilde{\vec{u}}$ are defined by

$$(\widetilde{\vec{u}}+\widetilde{\vec{w}})(p)=(p,\vec{u}(p)+\vec{w}(p)),\quad\text{and}\quad(\lambda\widetilde{\vec{u}})(p):=(p,\lambda\vec{u}(p))\tag{S.5}$$

(usual rules for "vectors at $p$"). To lighten the notations, $\widetilde{\vec{w}}(p)=^{\text{written}}\vec{w}(p)$, but then don't forget it is a pointed vector.

Notation:

$$\Gamma(U)=T_0^1(U):=\text{ the set of vector fields on } U=\text{the set of } \binom{1}{0} \text{ tensors on } U.\tag{S.6}$$

**More precisely,** we will use the definition of vector fields (see e.g. Abraham–Marsden [1]): A vector field is built from tangent vectors to curves. It makes sense on non planar surfaces, and more generally on differential manifolds.

**Example S.2** Discrete case: $n$ "force vectors" $\vec{f}_i(p_i)$ applied at $n$ points $p_i\in\mathbb{R}^3$ give the discrete vector field $\widetilde{\vec{f}}:p_i\in\{p_1,...,p_n\}\subset\mathbb{R}^3\to\widetilde{\vec{f}}(p_i)=(p_i,\vec{f}_i(p_i))\in\mathbb{R}^3\times\mathbb{R}^3$ where $p_i$ is "the point of application" of $\vec{f}_i(p_i)$, and $\widetilde{\vec{f}}(p_i)=(p_i,\vec{f}_i)$ is a pointed vector. Essential in mechanics, e.g. see screws (torsors). ▟

### S.2.3 Field of functions

Let $f:\left\{\begin{matrix}U&\to\mathbb{R}\\ p&\to f(p)\end{matrix}\right\}$ be a $C^\infty$ scalar valued function. The associated field of functions is

$$\widetilde{f}:\begin{cases}U&\to U\times\mathbb{R}\\ p&\to\widetilde{f}(p):=(p,f(p)),\end{cases}\tag{S.7}$$

and the first component $p$ of the couple $\widetilde{f}(p)=(p;f(p))$ is called the base point. So Im$\widetilde{f}=\{(p,f(p)):p\in U\}$ is the graph of $f$.

Notation:

$$T_0^0(U):=\{\text{field of functions}\}=\text{the set of }\binom{0}{0}\text{ tensors on }U,\tag{S.8}$$

or the set of tensors of order 0 on $U$. Abusive short notations (to lighten the writings):

$$\widetilde{f}(p)\overset{\text{written}}{=}f(p),\quad\text{and}\quad T_0^0(U)\overset{\text{written}}{=}C^\infty(U;\mathbb{R}),\tag{S.9}$$

but then keep the base point in mind (no ubiquity gift).

+ and . (usual) rules:

$$(\widetilde{f}+\widetilde{g})(p)=(p,\widetilde{f}(p)+\widetilde{g}(p)),\quad\text{and}\quad(\lambda\widetilde{f})(p):=(p,\lambda\widetilde{f}(p))\tag{S.10}$$

## S.3 Differential forms

The basic concept is that of vector fields. Then a first over-layer is made of differential forms (which "measure vector fields"):

**Definition S.3** Let $\alpha:\left\{\begin{matrix}U&\to E^*\\ p&\to\alpha(p)\end{matrix}\right\}$ (so $\alpha(p)$ is a linear form at $p$). The associated differential form (also called a 1-form) is "the field of linear forms" defined by

$$\widetilde{\alpha}:\begin{cases}U&\to U\times E^*\\ p&\to\widetilde{\alpha}(p)=(p,\alpha(p))\quad\text{called a linear form at }p.\end{cases}\tag{S.11}$$

And $p$ is called the base point, and Im$\widetilde{\alpha}=\{(p,\alpha(p)):p\in U\}$ is the graph of $\alpha$.

Notation:

$$\Omega^1(U) = T^0_1(U) := \{\text{differential forms}\} = \text{the set of } \binom{0}{1} \text{ tensors on } U. \tag{S.12}$$

Thus, if $\widetilde{\alpha} \in \Omega^1(U)$ (differential form) and $\widetilde{\vec{w}} \in \Gamma(U)$ (vector field), then $\widetilde{\alpha}.\widetilde{\vec{w}} \in T^0_0(U)$ (field of scalar valued functions) satisfies

$$\widetilde{\alpha}.\widetilde{\vec{w}} : \begin{cases} U & \to U \times \mathbb{R} \\ p & \to (\widetilde{\alpha}.\widetilde{\vec{w}})(p) = (p, (\alpha.\vec{w})(p)) = (p, \alpha(p).\vec{w}(p)) \ \in U \times \mathbb{R}. \end{cases} \tag{S.13}$$

Abusive short notations (to lighten the writings):

$$\widetilde{\alpha}(p) \overset{\text{written}}{=} \alpha(p), \quad \text{instead of } \widetilde{\alpha}(p) = (p, \alpha(p)), \tag{S.14}$$

but then keep the base point in mind.

## S.4 Tensors

### S.4.1 Definition of tensors, and $T^r_s(U)$

A second over-layer is introduced with the tensors with are "functions defined on vector fields and on differential forms" (which "measure vector fields and differential forms").

Let $r, s \in \mathbb{N}$, $r+s \geq 1$, and let $T : \left\{ \begin{array}{ll} U & \to \mathcal{L}^r_s(E) \\ p & \to T(p) \end{array} \right\}$ (so $T(p)$ is a uniform $\binom{r}{s}$ cf. (R.3.1)). And consider the associated function

$$\widetilde{T} : \begin{cases} U & \to U \times \mathcal{L}^r_s(E) \\ p & \to \widetilde{T}(p) = (p; T(p)) \end{cases} \tag{S.15}$$

Abusive short notation:

$$\widetilde{T}(p) \overset{\text{written}}{=} T(p) \quad \text{instead of } \widetilde{T}(p) = (p; T(p)), \tag{S.16}$$

but then keep the base point in mind.

**Definition S.4** (Abraham–Marsden [1].) $\widetilde{T}$ is a tensor of type $\binom{r}{s}$ iff $T$ is $C^\infty(U;\mathbb{R})$-multilinear (not only $\mathbb{R}$-multilinear), i.e., for all $f \in C^\infty(U;\mathbb{R})$, all $z_1, z_2$ vector field or differentiable form where applicable, and all $p \in U$,

$$\begin{cases} T(p)(..., z_1(p) + z_2(p), ...) = T(p)(..., z_1(p), ...) + T(p)(..., z_2(p), ...), \quad \text{and} \\ T(p)(..., f(p)z_1(p), ...) = f(p)\, T(p)(..., z_1(p), ...), \end{cases} \tag{S.17}$$

written in short

$$\begin{cases} T(..., z_1 + z_2, ...) = T(..., z_1, ...) + T(..., z_2, ...), \quad \text{and} \\ T(..., f z_1, ...) = f\, T(..., z_1, ...). \end{cases} \tag{S.18}$$

And

$$T^r_s(U) := \text{ the set of } \binom{r}{s} \text{ type tensors on } U. \tag{S.19}$$

**Remark S.5** Differential geometry vocabulary: A tensor is a section of a bundle over a manifold. For classical mechanics, definition S.4 gives an equivalent definition (Abraham–Marsden [1]). ∴

### S.4.2 Type $\binom{0}{1}$ tensor = differential forms

If $T \in T^0_1(U)$ then $T(p) \in E^*$, so $T = \alpha \in \Omega^1(U)$ is a differential form: $T^0_1(U) \subset \Omega^1(U)$.

Converse: Does a differential form $\alpha \in \Omega^1(U)$ defines a $\binom{0}{1}$ type tensor on $U$? Yes: $\alpha(p).(\vec{w}_1(p)+\vec{w}_2(p)) = \alpha(p).\vec{w}_1(p)+\alpha(p).\vec{w}_2(p)$ and $\alpha(p).(f\vec{w})(p) = \alpha(p).(f(p)\vec{w}(p)) = f(p)\alpha(p).\vec{w}(p)$, so (S.17) is checked. So $\Omega^1(U) \subset T^0_1(U)$. Thus

$$T^0_1(U) = \Omega^1(U). \tag{S.20}$$

### S.4.3 Type $\binom{1}{0}$ tensor (identified to a vector field)

Let $T \in T_1^0(U)$, so $T(p) \in \mathcal{L}_0^1(E) = \mathcal{L}(E^*;\mathbb{R}) = E^{**}$ for all $p \in U$. Thus, thanks to the natural canonical isomorphism $E^{**} \simeq E$, $T(p)$ can be identified to a vector, thus $T_1^0(U) \subset \Gamma(U)$.

Converse: Does a vector field $\vec{w} \in \Gamma(U)$ defines a $\binom{1}{0}$ type tensor on $U$? Yes: $\vec{w}(p).(\alpha_1(p)+\alpha_2(p)) = (\alpha_1(p)+\alpha_2(p)).\vec{w}(p) = \alpha_1(p).\vec{w}(p)+\alpha_2(p).\vec{w}(p) = \vec{w}(p).\alpha_1(p))+\vec{w}(p).\alpha_2(p)$ and $\vec{w}(p).(f(p)\alpha(p)) = (f(p)\alpha(p)).\vec{w}(p) = f(p)\alpha(p).\vec{w}(p) = f(p)\vec{w}(p).\alpha(p)$, so (S.17) is checked. So $\Gamma(U) \subset T_0^1(U)$. Thus

$$T_0^1(U) \simeq \Gamma(U). \tag{S.21}$$

### S.4.4 A metric is a $\binom{0}{2}$ tensor

Let $T \in T_2^0(U)$, so $T(p) \in \mathcal{L}_2^0(E)$ for all $p \in U$, and $T(\vec{u},\vec{w}) \in T_0^0(U)$ for all $\vec{u},\vec{w} \in \Gamma(U)$.

**Definition S.6** A metric $g$ on $U$ is a $\binom{0}{2}$ type tensor on $U$ such that, for all $p \in E$, $g(p) =^{\text{written}} g_p$ is an inner dot product on $E$.

### S.4.5 $\binom{1}{1}$ tensor, identification with fields of endomorphisms

Let $T \in T_1^1(U)$, so $T(p) \in \mathcal{L}_1^1(E)$ for all $p \in U$, and $T(\alpha,\vec{w}) \in T_0^0(U)$ for all $\alpha \in \Omega^1(U)$ and $\vec{w} \in \Gamma(U)$ (so $T(p)(\alpha(p),\vec{w}(p)) \in \mathbb{R}$ for all $p$).

The associated field of endomorphisms on $U$ is $\widetilde{L}_T : \left\{ \begin{array}{ll} U & \to U \times \mathcal{L}(E;E) \\ p & \to \widetilde{L}_T(p) = (p, L_T(p)) \end{array} \right\}$ where $L_T(p)$ is identified with $T(p)$ thanks to the natural canonical isomorphism $\mathcal{L}(E;E) \simeq \mathcal{L}(E^*,E;\mathbb{R}) = \mathcal{L}_1^1(E)$ given by

$$\forall \ell \in E^*,\ \forall \vec{w} \in E, \quad \ell.(L_T(p).\vec{w}) = T(p)(\ell,\vec{w}). \tag{S.22}$$

## S.5 Unstationary tensor

Let $t \in [t_1,t_2] \subset \mathbb{R}$. Let $(T_t)_{t\in[t_1,t_2]}$ be a family of $\binom{r}{s}$ tensors, cf. (S.15). Then $T : t \to T(t) := T_t$ is called an unstationary tensor. And the set of unstationary tensors is also noted $T_s^r(U)$. E.g., a Eulerian velocity field is a $\binom{1}{0}$ unstationary vector field.

# T Differential, possible associated gradients, divergence

Classical framework. $\mathcal{E}$ and $\mathcal{F}$ are affine spaces associated with vector spaces $E$ and $F$, and $||.||_E$ and $||.||_F$ are norms in $E$ and $F$ s.t. $(E,||.||_E)$ and $(F,||.||_F)$ are complete (we need limits "that stay in the space as $h \to 0$"). If applicable, $\mathcal{E}$ and/or $\mathcal{F}$ can be replaced by $E$ and/or $F$.

$U$ is an open set in $\mathcal{E}$, and $\Phi : \left\{ \begin{array}{ll} U & \to \mathcal{F} \\ p & \to p_{\mathcal{F}} = \Phi(p) \end{array} \right\}$ is a function. Reminder:

$\Phi$ is continuous at $p \in U$ iff $\Phi(q) \longrightarrow_{q\to p} \Phi(p)$ relative to $||.||_E$ and $||.||_F$, i.e. iff

$$||\Phi(q) - \Phi(p)||_F \underset{||q-p||_E\to 0}{\longrightarrow} 0, \quad \text{written} \quad \Phi(q) = \Phi(p) + o(1) \ \text{ near } p \quad \text{(Landau Notation)} \tag{T.1}$$

and called "the zero-th order Taylor expansion of $\Phi$ near $p$". It means:

$$\forall \varepsilon > 0,\ \exists \eta > 0 \text{ s.t. } \forall q \in \mathcal{E} \text{ satisfying } ||q-p||_E < \eta \text{ we have } ||\Phi(q) - \Phi(p)||_F < \varepsilon. \tag{T.2}$$

And $C^0(U;\mathcal{F})$ is the set of functions that are continuous at all $p \in U$.

## T.1 Differential

The definition of "derivative" or "differential" is observer independent: All observers (English with foot, French with metre...) have the same definition. So it is an objective (qualitative) definition: Does not require any man made tool like a basis or an inner dot product.

Warning: The usual (historical) notations are sometimes confusing and can make understanding difficult. So pay attention to the notations.

### T.1.1 Directional derivative (Gâteaux)

$p \in U$, $\vec{u} \in E$. Define $f : \mathbb{R} \to \mathcal{F}$ by

$$f(h) := \Phi(p + h\vec{u}). \tag{T.3}$$

(In a manifold: $f(h) := \Phi(c(h))$ where $c$ is a $C^1$ curve in $U$ s.t. $c(0) = p$ and $c'(0) = \vec{u}$.)

**Definition T.1** The function $\Phi$ is differentiable at $p$ in the direction $\vec{u}$ iff $f$ is derivable at 0, i.e. iff the limit $f'(0) := \lim_{h\to 0} \frac{\Phi(p+h\vec{u})-\Phi(p)}{h} =^{\text{written}} d\Phi(p)(\vec{u})$ exists in $F$, i.e. iff, near $p$,

$$\Phi(p + h\vec{u}) = \Phi(p) + h\, d\Phi(p)(\vec{u}) + o(h) \tag{T.4}$$

(first order Taylor expansion of $\Phi$ near $p$ in the direction $\vec{u}$); $d\Phi(p)(\vec{u})$ is called the directional derivative of $\Phi$ at $p$ in the direction $\vec{u}$.

And if, for all $\vec{u} \in E$, $d\Phi(p)(\vec{u})$ exists (in $F$) then $\Phi$ is called Gâteaux differentiable at $p$.

**Exercice T.2** Prove: If $\Phi$ is differentiable at $p$ in the direction $\vec{u}$ then $d\Phi(p)(\vec{0}) = 0$ and for all $f \in \mathcal{F}(U; \mathbb{R})$

$$d\Phi(p)(f(p)\vec{u}) = f(p)\, d\Phi(p)(\vec{u}). \tag{T.5}$$

In particular $d\Phi(p)$ is homogeneous in the direction $\vec{u}$: $d\Phi(p)(\lambda\vec{u}) = \lambda\, d\Phi(p)(\vec{u})$ for $\lambda \in \mathbb{R}$.

**Answer**. $d\Phi(p)(\vec{0}) = \lim_{h\to 0} \frac{\Phi(p+\vec{0})-\Phi(p)}{h} = 0$. Then suppose $f(p) \neq 0$ (trivial if $f(p) = 0$).
$d\Phi(p)(f(p)\vec{u}) = \lim_{h\to 0} \frac{\Phi(p+hf(p)\vec{u})-\Phi(p)}{h} = f(p) \lim_{h\to 0} \frac{\Phi(p+hf(p)\vec{u})-\Phi(p)}{hf(p)} = f(p) \lim_{k\to 0} \frac{\Phi(p+k\vec{u})-\Phi(p)}{k}$. ▪

### T.1.2 Differential (Fréchet)

**Definition T.3** If $\Phi$ is Gateaux differentiable at $p$ in all directions $\vec{u} \in E$, if $d\Phi(p)$ is linear and if $d\Phi$ is continuous at $p$, then $\Phi$ is said to be differentiable at $p$ (or Fréchet differentiable at $p$).

Hence, If $\Phi$ is differentiable at $p$ then (T.4) gives near $p$, for all $\vec{u} \in E$ and $h$ near 0,

$$\Phi(p + h\vec{u}) = \Phi(p) + h\, d\Phi(p).\vec{u} + o(h), \tag{T.6}$$

where $d\Phi(p).\vec{u} := d\Phi(p)(\vec{u})$ (linearity notation).

**Definition T.4** The affine function $\mathit{aff}_p : q \in U \to \mathit{aff}_p(q) := \Phi(p) + d\Phi(p).\vec{pq} \in \mathcal{F}$ is the affine approximation of $\Phi$ at $p$ (the graph of $\mathit{aff}_p$ is the tangent plane to the graph of $\Phi$ at $p$).

**Definition T.5** $\Phi : U \to \mathcal{F}$ is differentiable in $U$ iff $\Phi$ is differentiable at all $p \in U$, and then its differential is the map

$$d\Phi : \begin{cases} U & \to \mathcal{L}(E; F) \\ p & \to d\Phi(p). \end{cases} \tag{T.7}$$

And $C^1(U; \mathcal{F})$ is the set of differentiable functions $\Phi$ such that $d\Phi \in C^0(U; \mathcal{L}(E; F))$.

And $C^2(U; \mathcal{F})$ is the set of differentiable functions $\Phi$ such that $d\Phi \in C^1(U; \mathcal{L}(E; F))$.

... And $C^k(U; \mathcal{F})$ is the set of differentiable functions $\Phi$ such that $d\Phi \in C^{k-1}(U; \mathcal{L}(E; F))$....

And $C^\infty(U; \mathcal{F}) = \bigcap_{k\in\mathbb{N}} C^k(U; \mathcal{F})$.

The differentiation (or derivation) operator is $d : \left\{ \begin{matrix} C^1(U; \mathcal{F}) & \to C^0(U; \mathcal{L}(E; F)) \\ \Phi & \to d\Phi \end{matrix} \right\}$.

If $\Phi \in C^1(U; \mathcal{F})$ and $\vec{u} \in U$ then the directional differential (or derivative) operator along $\vec{u}$ is

$$\partial_{\vec{u}} : \begin{cases} C^1(U; \mathcal{F}) & \to C^0(U; F) \\ \qquad \Phi & \to \partial_{\vec{u}}(\Phi) := d\Phi.\vec{u}, \quad \text{so defined by} \quad \partial_{\vec{u}}(\Phi)(p) := d\Phi(p).\vec{u}(p). \end{cases} \tag{T.8}$$

(And $\partial_{\vec{u}}(\Phi) =^{\text{written}} \vec{u}(\Phi)$ in differential geometry thanks to $E \simeq E^{**}$ which gives $\partial_{\vec{u}} =^{\text{written}} \vec{u}$.)

**Proposition T.6** *"A derivation is linear", i.e. $d$ is $\mathbb{R}$-linear: $d(\Phi + \lambda\Psi) = d\Phi + \lambda d\Psi$.*

**Proof.** $d(\Phi + \lambda\Psi)(p).\vec{u} = \lim_{h\to 0} \frac{(\Phi+\lambda\Psi)(p+h\vec{u})-(\Phi+\lambda\Psi)(p)}{h} = \lim_{h\to 0} \frac{\Phi(p+h\vec{u})-\Phi(p)+\lambda\Psi(p+h\vec{u})-\lambda\Psi(p)}{h} = \lim_{h\to 0} \frac{\Phi(p+h\vec{u})-\Phi(p)}{h} + \lambda \lim_{h\to 0} \frac{\Psi(p+h\vec{u})-\Psi(p)}{h} = d\Phi(p).\vec{u} + \lambda d\Psi(p).\vec{u} = (d\Phi(p) + \lambda d\Psi(p)).\vec{u}$ for all $\Phi, \Psi \in C^1(U; \mathcal{F})$, $p \in U$, $\vec{u} \in \Gamma(U)$, $\lambda \in \mathbb{R}$. ▪

**Exercice T.7** (Differentiation of a product) $f \in C^1(U;\mathbb{R})$ (scalar valued), $\Phi \in C^1(U;\mathcal{F})$, $\vec{u} \in E$. Prove

$$d(f\Phi).\vec{u} = (df.\vec{u})\Phi + f(d\Phi.\vec{u}), \quad \text{written} \quad d(f\Phi) = \Phi \otimes df + f\, d\Phi \tag{T.9}$$

(tensor notation for computations with contraction rules).

**Answer**.

$$\begin{aligned} d(f\Phi)(p).\vec{u} =& \lim_{h\to 0} \frac{f(p+h\vec{u})\Phi(p+h\vec{u}) - f(p)\Phi(p)}{h} \\ =& \lim_{h\to 0} \frac{f(p+h\vec{u})\Phi(p+h\vec{u}) - f(p)\Phi(p+h\vec{u})}{h} + \frac{f(p)\Phi(p+h\vec{u}) - f(p)\Phi(p)}{h} \\ =& \lim_{h\to 0} \frac{f(p+h\vec{u}) - f(p)}{h}(\Phi(p) + o(1)) + \lim_{h\to 0} f(p)\frac{\Phi(p+h\vec{u}) - \Phi(p)}{h} \end{aligned} \tag{T.10}$$

$= (df(p).\vec{u})\Phi(p) + f(p)(d\Phi(p).\vec{u})$. And $(\Phi \otimes df).\vec{u} + (f\, d\Phi).\vec{u} = \Phi(df.\vec{u}) + f\, d\Phi.\vec{u}$.

**Exercice T.8** (Differentiation of a compound) $f \in C^1(\mathcal{E};\mathcal{F})$, $g \in C^1(\mathcal{F};\mathcal{G})$, $p \in E$, $\vec{u} \in E$. Prove

$$d(g \circ f)(p) = dg(f(p)).df(p), \quad \forall p \in U, \quad \text{written} \quad d(g\circ f) = dg(f).df. \tag{T.11}$$

**Answer**. $g(q) - g(q_0) = dg(q_0).(q-q_0) + o(q-q_0)$ and $f(p) - f(p_0) = df(p_0).(p-p_0) + o(p-p_0)$. Thus $q = f(p)$ and $q_0 = f(p_0)$ give

$$\begin{aligned} g(f(p)) - g(f(p_0)) &= dg(f(p_0)).(df(p_0).(p-p_0) + o(p-p_0)) + o(df(p_0).(p-p_0) + o(p-p_0)) \\ &= (dg(f(p_0)).df(p_0)).(p-p_0) + o(p-p_0) + o(p-p_0). \end{aligned} \tag{T.12}$$

Indeed $dg$ and $df$ $C^0$ give $||dg(f(p_0))|| < \infty$ and $||df(p_0)|| < \infty$ (bounded) near $p_0$ thus $||dg(f(p_0)).df(p_0)|| \le ||dg(f(p_0))||\,||df(p_0)|| < \infty$ (bounded) near $p_0$.

**Definition T.9** Differential geometry: The tangent map is

$$T\Phi : \begin{cases} U \times E & \to \mathcal{F} \times F \\ (p,\vec{u}) & \to T\Phi(p,\vec{u}) = (\Phi(p), d\Phi(p).\vec{u}). \end{cases} \tag{T.13}$$

The two points $p$ (input) and $\Phi(p)$ (output) are the base points, and the two vectors $\vec{u}$ (input) and $d\Phi(p).\vec{u}$ (output) are the initial vector and its push-forward by $\Phi$.

### T.1.3 Quantification: A basis and the $j$-th partial derivative (subjective)

Quantification: In $E$, choose a basis $(\vec{e}_i(p))$ defined at $p \in U$.

**Definition T.10** When $\Phi \in C^1(U;\mathcal{F})$, the $j$-th partial derivative of $\Phi$ at $p$ is

$$\partial_{\vec{e}_j}\Phi(p) \overset{\text{written}}{=} \boxed{\Phi_{|j}(p) := d\Phi(p).\vec{e}_j(p)} = \lim_{h\to 0} \frac{\Phi(p+h\vec{e}_j(p)) - \Phi(p)}{h} \overset{\text{written}}{=} \partial_j\Phi(p) \overset{\text{written}}{=} \frac{\partial\Phi}{\partial\vec{e}_j}(p). \tag{T.14}$$

And the $j$-th directional derivative operator is

$$\partial_{\vec{e}_j} \overset{\text{written}}{=} \partial_j \overset{\text{written}}{=} \frac{\partial}{\partial\vec{e}_j} : \begin{cases} C^1(U;\mathcal{F}) & \to C^0(U;F) \\ \Phi & \to \partial_j\Phi = \boxed{\Phi_{|j} := d\Phi.\vec{e}_j} = \partial_{\vec{e}_j}\Phi = \dfrac{\partial\Phi}{\partial\vec{e}_j}. \end{cases} \tag{T.15}$$

**Cartesian case:** $(\vec{e}_i)$ is a Cartesian basis in $E$, $O$ is an origin in $\mathcal{E}$, $\vec{x} = \overrightarrow{Op} = \sum_{i=1}^n x_i\vec{e}_i \in E$, then

$$\partial_{\vec{e}_j}\Phi(p) = \Phi_{|j}(p) \overset{\text{written}}{=} \frac{\partial\Phi}{\partial x_j}(p) \overset{\text{written}}{=} \Phi_{,j}(p), \quad \text{i.e.} \quad \partial_{\vec{e}_j}\Phi = \Phi_{|j} \overset{\text{written}}{=} \frac{\partial\Phi}{\partial x_j} \overset{\text{written}}{=} \Phi_{,j}. \tag{T.16}$$

(Duality notations: $\partial_{\vec{e}_j}\Phi =^{\text{written}} \frac{\partial\Phi}{\partial x^j}$.) And if $\vec{x} = \overrightarrow{Op} = \sum_{i=1}^n y_i\vec{e}_i \in E$ then $\partial_{\vec{e}_j}\Phi(p) =^{\text{written}} \frac{\partial\Phi}{\partial y_j}$... So: Warning: The notation $\frac{\partial}{\partial x_j}$ is ambiguous since it depends on the names of the components.

### T.1.4 Notation for the second order Differential

Let $\Phi \in C^2(U;\mathcal{F})$; Thus $d\Phi \in C^1(U;\mathcal{L}(E;F))$ and $d(d\Phi) \in C^0(U;\mathcal{L}(E;\mathcal{L}(E;F)))$: For $p \in U$ and $\vec{u},\vec{v} \in E$, we have $d(d\Phi)(p).\vec{u} \in \mathcal{L}(E;F)$ and $(d(d\Phi)(p).\vec{u}).\vec{v} \in F$.

**Definition T.11** With the natural canonical isomorphism $L \in \mathcal{L}(E;\mathcal{L}(E;F)) \leftrightarrow T_L \in \mathcal{L}(E,E;F)$ given by $T_L(\vec{u}_1,\vec{u}_2) := (L.\vec{u}_1).\vec{u}_2$ for all $\vec{u}_1,\vec{u}_2 \in E$, we get the bilinear map $d^2\Phi(p)$ defined by

$$d^2\Phi(p)(\vec{u},\vec{v}) := (d(d\Phi)(p).\vec{u}).\vec{v}, \quad \text{written} \quad d^2\Phi(\vec{u},\vec{v}) := (d(d\Phi).\vec{u}).\vec{v}. \tag{T.17}$$

We then get the usual second order Taylor expansion of $\Phi$ near $p$ in the direction $\vec{u}$:

$$\Phi(p+h\vec{u}) = \Phi(p) + h\,d\Phi(p).\vec{u} + \frac{h^2}{2}\,d^2\Phi(p)(\vec{u},\vec{u}) + o(h^2), \tag{T.18}$$

which is the second order Taylor expansion of $f : h \to f(h) = \Phi(p+h\vec{u})$ near $h=0$.

**Theorem T.12** *(Schwarz's theorem) If $\Phi$ is $C^2$ then $d^2\Phi(p)$ is symmetric: $d^2\Phi(\vec{u},\vec{v}) = d^2\Phi(\vec{v},\vec{u})$.*

**Proof.** See any elementary course (the proof is quite simple but long to write). ∎

## T.2 Scalar valued functions

### T.2.1 Differential of a scalar valued function

$\Phi = f : \left\{\begin{matrix} \Omega & \to \mathbb{R} \\ p & \to f(p) \end{matrix}\right\} \in C^1(\Omega;\mathbb{R})$ (scalar valued), so $df \in C^0(\Omega;E^*)$ (differential form), with $df(p).\vec{u} = \lim_{h\to 0} \frac{f(p+h\vec{u})-f(p)}{h} \in \mathbb{R}$ for all $\vec{u} \in E$. Hence the first order Taylor expansion near any $p$:

$$f(p+h\vec{u}) = f(p) + h\,df(p).\vec{u} + o(h), \quad \forall \vec{u} \in E. \tag{T.19}$$

### T.2.2 Quantification

$(\vec{e}_i(p))$ is a basis at $p$ and $(e^j(p))$ is its dual basis. Call $f_{|j}(p)$ the components of $df(p)$, i.e. $df(p) = \sum_j f_{|j}(p)e^j(p)$ and $f_{|j}(p) = df(p).\vec{e}_j(p)$, i.e. $[df(p)]_{|\vec{e}} = (\,f_{|1}(p) \quad \dots \quad f_{|n}(p)\,)$:

$$df = \sum_{j=1}^n f_{|j}e^j, \quad \text{i.e.} \quad f_{|j} = df.\vec{e}_j, \quad \text{i.e.} \quad [df]_{|\vec{e}} = (\,f_{|1} \quad \dots \quad f_{|n}\,) \quad \text{(row matrix).} \tag{T.20}$$

So $df(p).\vec{u} = [df(p)]_{|\vec{e}}.[\vec{u}]_{|\vec{e}} = \sum_{j=1}^n f_{|j}(p)u^j$ for $\vec{u} = \sum_j u^j\vec{e}_j$ vector at $p$:

$$df.\vec{u} = [df]_{|\vec{e}}.[\vec{u}]_{|\vec{e}} = \sum_{j=1}^n f_{|j}u^j \quad \text{where} \quad f_{|j} = df.\vec{e}_i. \tag{T.21}$$

Classic notations: $df = \sum_j f_{|j}\pi_{ej}$, $\vec{u} = \sum_j u_j\vec{e}_j$, $df.\vec{u} = [df]_{|\vec{e}}.[\vec{u}]_{|\vec{e}} = \sum_{j=1}^n f_{|j}u_j$.

E.g.:

$$\text{Cartesian basis:} \quad f_{|j} \overset{\text{written}}{=} \frac{\partial f}{\partial x^j} \overset{\text{written}}{=} f_{,j}, \quad \text{and} \quad df = \sum_{j=1}^n \frac{\partial f}{\partial x^j}\,dx^j \tag{T.22}$$

Classic notation (Cartesian basis): $f_{,j} = \frac{\partial f}{\partial x_j}$ and $df = \sum_j \frac{\partial f}{\partial x_j}dx_j$.

**Exercice T.13** Prove

$$(fg)_{|i} = f_{|i}\,g + fg_{|i}. \tag{T.23}$$

**Answer**. $d(fg).\vec{e}_i = (df.\vec{e}_i)g + f(dg.\vec{e}_i)$, cf. (T.9). ∎

**Exercice T.14** Cartesian basis, $\vec{r} \in C^1(\mathbb{R};\mathbb{R}^n)$, $\vec{r}(t) = (x_1(t),\dots,x_n(t))$, $g \in C^1(\mathbb{R}^2;\mathbb{R})$, $f := g\circ\vec{r}$, i.e. $f(t) = g(\vec{r}(t)) = g(x_1(t),\dots,x_n(t))$. Prove (derivative of a compound function)

$$(f'(t) =) \quad (g\circ\vec{r})'(t) = dg(\vec{r}(t)).\vec{r}\,'(t) = \frac{\partial g}{\partial x_1}(\vec{x}(t))x_1'(t) + \dots + \frac{\partial g}{\partial x_n}(\vec{x}(t))x_n'(t). \tag{T.24}$$

**Answer**. See (T.11). ∎

### T.2.3 A partial differential $\frac{\partial f}{\partial x_i}$ is subjective

An English observer chooses a Euclidean basis $(\vec{a}_i)$ made with the foot, writes $\vec{x} = \sum_i x_i\vec{a}_i$ and uses $\frac{\partial f}{\partial x_i}$. A French observer chooses a Euclidean basis $(\vec{b}_i)$ made with the metre, writes $\vec{x} = \sum_i x_i\vec{b}_i$ and uses $\frac{\partial f}{\partial x_i}$. But the English $\frac{\partial f}{\partial x_i}$ is not equal to the French $\frac{\partial f}{\partial x_i}$...!

Indeed, if $\vec{x} = \sum_i x_{a,i}\vec{a}_i = \sum_i x_{b,i}\vec{b}_i$, then $\frac{\partial f}{\partial x_{a,i}}(p) = df(p).\vec{a}_i$ while $\frac{\partial f}{\partial x_{b,i}}(p) = df(\vec{x}).\vec{b}_i$, and e.g.

$$\text{if } \vec{b}_i = \lambda\vec{a}_i, \ \forall i, \quad \text{then} \quad \boxed{\frac{\partial f}{\partial x_{b,i}} = \lambda\frac{\partial f}{\partial x_{a,i}}} \quad \text{(change of unit formula)}, \tag{T.25}$$

because $df(p).\vec{b}_j = df(p).(\lambda\vec{a}_j) = \lambda\, df(p).\vec{a}_j$ (linearity of $df(p)$). (Duality notations: $\frac{\partial f}{\partial x_b^j} = \lambda\frac{\partial f}{\partial x_a^j}$.)

Remark: $\vec{b}_i \simeq \frac{\partial}{\partial x_{b,i}}$ and $\vec{a}_i \simeq \frac{\partial}{\partial x_{a,i}}$ (differential geometry), so $\vec{b}_i = \lambda\vec{a}_i$ gives $\vec{b}_i(f) = \lambda\vec{a}_i(f)$, i.e. (T.25).

More generally, with $P(p)$ the transition matrix from $(\vec{a}_i(p))$ to $(\vec{b}_i(p))$, we have (change of basis formula for linear forms):

$$[df]_{|\vec{b}} = [df]_{|\vec{a}}.P, \quad \text{i.e.} \quad \frac{\partial f}{\partial x_{b,j}} = \sum_{i=1}^n \frac{\partial f}{\partial x_{a,i}}P_{ij} \quad \text{written} \quad \frac{\partial f}{\partial x_{b,j}} = \sum_{i=1}^n \frac{\partial f}{\partial x_{a,i}}\frac{\partial x_{a,i}}{\partial x_{b,j}}. \tag{T.26}$$

(Duality notations: $\frac{\partial f}{\partial x_b^j} = \sum_{i=1}^n \frac{\partial f}{\partial x_a^i}P^i_j$ written $\frac{\partial f}{\partial x_b^j} = \sum_{i=1}^n \frac{\partial f}{\partial x_a^i}\frac{\partial x_a^i}{\partial x_b^j}$.)

**Remark T.15** Why this last notation $P_{ij} \overset{\text{written}}{=} \frac{\partial x_{a,i}}{\partial x_{b,j}}$?
Answer : $[\vec{x}]_{|\vec{a}} = P.[\vec{x}]_{|\vec{b}}$, tells that $[\vec{x}]_{|\vec{a}}$ is a function of $[\vec{x}]_{|\vec{b}}$: Full notation: $[\vec{x}]_{|\vec{a}}([\vec{x}]_{|\vec{b}}) = P.[\vec{x}]_{|\vec{b}}$, i.e.

$$x_a^i(x_b^1,...,x_b^n) = \sum_{j=1}^n P^i_j x_b^j \quad \text{thus} \quad \frac{\partial x_a^i}{\partial x_b^j}(x_b^1,...,x_b^n) = P^i_j, \ \forall i,j. \tag{T.27}$$

More details: With an origin $O \in \mathcal{E}$ and $\vec{x} = \overrightarrow{Op}$, define $f_a, f_b \in C^1(\mathcal{M}_{n1};\mathbb{R})$ by $f_a([\vec{x}]_{|\vec{a}}) := f(p)$ and $f_b([\vec{x}]_{|\vec{b}}) := f(p)$. Thus $f_b([\vec{x}]_{|\vec{b}}) = f_a([\vec{x}]_{|\vec{a}}) = (f_a \circ [\vec{x}]_{|\vec{a}})([\vec{x}]_{|\vec{b}})$, hence (T.26) should be written (with no abusive notations):

$$\frac{\partial f_b}{\partial x_b^i}([\vec{x}]_{|\vec{b}}) = \sum_{j=1}^n \frac{\partial f_a}{\partial x_a^j}([\vec{x}]_{|\vec{a}})\frac{\partial x_a^j}{\partial x_b^i}([\vec{x}]_{|\vec{b}}). \tag{T.28}$$

Question: Why did we need to introduce $f_a$ and $f_b$ (and not just keep $f$)? Answer: Because $\vec{x} \in \mathbb{R}^n$ while $[\vec{x}]_{|\vec{a}}, [\vec{x}]_{|\vec{b}} \in \mathcal{M}_{n1}$ and $[\vec{x}]_{|\vec{a}} \neq [\vec{x}]_{|\vec{b}}$: A vector $\vec{x}$ can't be reduced to a matrix of components (which one?).

### T.2.4 Gradient (subjective: requires an inner dot product)

Let $f \in C^1(\Omega;\mathbb{R})$ (scalar valued), let $p \in \Omega$. Choose an inner dot product $(\cdot,\cdot)_g$ in $E$.

**Definition T.16** The $(\cdot,\cdot)_g$-conjugate gradient $\vec{\text{grad}}_g f(p) \overset{\text{written}}{=} \vec{\nabla}_g f(p)$ of $f$ relative to $(\cdot,\cdot)_g$ is the $(\cdot,\cdot)_g$-Riesz representation vector $\vec{R}_g(df(p))$ in $E$ of the linear form $df(p) \in E^*$, i.e. the vector defined by

$$\forall \vec{u} \in E, \quad \boxed{df(p).\vec{u} = (\vec{\text{grad}}_g f(p), \vec{u})_g} = \vec{\text{grad}}_g f(p) \bullet_g \vec{u} = \vec{\nabla}_g f(p) \bullet_g \vec{u}. \tag{T.29}$$

If a chosen inner dot product $(\cdot,\cdot)_g$ is imposed to all, e.g. in an isometric Euclidean framework (foot? metre?), then $\vec{\text{grad}}_g f \overset{\text{written}}{=} \vec{\text{grad}} f = \vec{\nabla} f$ and (T.29) is written $df.\vec{u} = \vec{\text{grad}} f \bullet \vec{u} = \vec{\nabla} f \bullet \vec{u}$, and $\vec{\text{grad}} f = \vec{\nabla} f$ is called "the" gradient of $f$, although "the" is misleading (which one: foot? metre?).

**Fundamental:** An English observer with his foot, his Euclidean basis $(\vec{a}_i)$ and associated Euclidean dot product $(\cdot,\cdot)_a$, and a French observer with his metre, his Euclidean basis $(\vec{b}_i)$ and associated Euclidean dot product $(\cdot,\cdot)_b$: They have the same differential $df(p)$ (objective), but they do ***not*** have the same gradient (subjective). E.g. if $(\vec{b}_i) = (\lambda\vec{a}_i)$ then

$$\vec{\text{grad}}_b f \overset{(F.12)}{=} \lambda^2 \vec{\text{grad}}_a f \quad \text{with} \quad \lambda^2 > 10. \tag{T.30}$$

So $\vec{\text{grad}}_b f$ is quite different from $\vec{\text{grad}}_a f$ isn't it? And to forget this fact leads to accidents like the crash of the Mars Climate Orbiter probe, cf. remark A.17.

**Subjective first order Taylor expansion:** With a chosen inner dot product $(\cdot,\cdot)_g$, the first order Taylor expansion (T.4) gives

$$f(p+h\vec{u}) = f(p) + h\,(\vec{\text{grad}}_g f(p),\vec{u})_g + o(h) \qquad (= f(p) + h\,\vec{\text{grad}}_g f(p) \bullet_g \vec{u} + o(h)). \tag{T.31}$$

**Fundamental once again** (we insist), for $f$ $C^1$:

• An inner dot product does not always exist (as a meaningful tool), see § B.4 (thermodynamics), hence a gradient does not always exists (contrary to a differential).

• $df(p)$ is a linear form (covariant) while $\vec{\text{grad}}_g f(p)$ is a vector (contravariant). In particular the change of basis formulas differ, cf. (A.25):

$$[df]_{|new} = [df]_{|old}.P, \quad \text{while} \quad [\vec{\text{grad}}_g f]_{|new} = P^{-1}.[\vec{\text{grad}}_g f]_{|old}. \tag{T.32}$$

• $df$ cannot be identified with $\vec{\text{grad}} f$ (with one?) (Recall: there is no natural canonical isomorphims between $E$ and $E^*$.)

• Vocabulary: The differential $df$ is also called the "covariant gradient of $f$", while the vector $\vec{\text{grad}}_g f$ is called the "contravariant gradient of $f$ relative to $(\cdot,\cdot)_g$".

**Exercice T.17** $f$ $C^1$, Cartesian basis $(\vec{e}_i)$ and $(\cdot,\cdot)_g$ given by $[g]_{|\vec{e}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Give $[\vec{\text{grad}}_g f]_{|\vec{e}}$.

**Answer**. $[df]_{|\vec{e}} = (\ \frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2}\ )$ (row matrix), thus (T.29) gives $[\vec{\text{grad}}_g f]_{|\vec{e}} = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{1}{2}\frac{\partial f}{\partial x_2} \end{pmatrix}$ (column matrix $\neq [df]^T$). ▪

## T.3 Coordinate system, associated basis, Christoffel symbols

### T.3.1 Coordinate system

Purpose = quantification along coordinate lines relative to the coordinate basis. We use duality notations for readability, usage and mnemonic device (for the Christoffel symbols $\gamma^i_{jk}$).

Classical framework to simplify: $n \in [1,3]_{\mathbb{N}}$, $\mathbb{R}^n$ is the geometric affine space we live in, $\vec{\mathbb{R}^n}$ is the vector space of bi-point vectors, $(\vec{a}_i)$ is a (chosen) Euclidean basis in $\vec{\mathbb{R}^n}$, $O$ is a point (an origin) in $\mathbb{R}^n$, $\vec{x} = \overrightarrow{Op} = \sum_{i=1}^n x^i \vec{a}_i \in \vec{\mathbb{R}^n}$.

Let $m \le n$. $\vec{\mathbb{R}}^m_{par} = \mathbb{R} \times ... \times \mathbb{R}$ ($m$-times) is the theoretical canonical Cartesian vector space called the space of parameters $\vec{q}$, $(\vec{A}_i)$ is its canonical basis, $\vec{q} = \sum_{i=1}^m q^i \vec{A}_i$, and $U^m_{par} = ]a_1,b_1[\times ... \times ]a_m,b_m[$ is non empty. E.g., polar representation: $m=n=2$ and $U^2_{par} = \mathbb{R}_+ \times ]-\pi,\pi[ = \{\vec{q} = (r,\theta) \in \mathbb{R}_+ \times ]-\pi,\pi[\}$.

Consider a $C^2$ map $\Psi$ and its associated map $\vec{\Psi}$:

$$\Psi : \left\{ \begin{array}{rl} U^m_{par} \subset \vec{\mathbb{R}}^m_{par} & \to \Omega_m \subset \mathbb{R}^n \\ \vec{q} & \to p = \Psi(\vec{q}) \end{array} \right\}, \quad \vec{\Psi} = \overrightarrow{O\Psi} : \left\{ \begin{array}{rl} U^m_{par} \subset \vec{\mathbb{R}}^m_{par} & \to \vec{\Omega}_m \subset \vec{\mathbb{R}^n} \\ \vec{q} & \to \vec{x} = \vec{\Psi}(\vec{q}) := \overrightarrow{O\Psi(\vec{q})} = \overrightarrow{Op} \end{array} \right\}, \tag{T.33}$$

where $\Omega_m := \Psi(U^m_{par})$ and $\vec{\Omega}_m := \vec{\Psi}(U^m_{par})$. (In particular $d\Psi(p) = d\vec{\Psi}(\vec{x}) \in \mathcal{L}(\vec{\mathbb{R}}^m_{par};\vec{\mathbb{R}^n})$ when $\vec{x} = \overrightarrow{Op}$.)

**Definition T.18** $\Psi$ (and $\vec{\Psi}$) is a coordinate system iff $\Psi$ is a diffeomorphism onto its image.

In that case $\Omega_m = \Psi(U^m_{par})$ is a dimension $m$ "surface" (a dimension $m$ manifold) in $\mathbb{R}^n$ (a curve when $m=1$, a surface when $m=2$, a volume when $m=3$).

**Quantification:** $\vec{q} = (q^1,...,q^m) = \sum_{i=1}^m q^i \vec{A}_i \in \vec{\mathbb{R}}^m_{par}$ and $\overrightarrow{Op} = \vec{x} = \vec{\Psi}(\vec{q}) \in \vec{\mathbb{R}^n}$,

$$\vec{x} = \sum_{i=1}^n x^i \vec{a}_i = \vec{\Psi}(\vec{q}) = \sum_{i=1}^n \Psi^i(\vec{q})\vec{a}_i, \quad \text{i.e.} \quad [\vec{x}]_{\vec{a}} = [\vec{\Psi}(\vec{q})]_{\vec{a}} = \begin{pmatrix} x^1 = \Psi^1(\vec{q}) \\ \vdots \\ x^n = \Psi^n(\vec{q}) \end{pmatrix}. \tag{T.34}$$

And $d\Psi(\vec{q}).\vec{A}_j = \sum_{i=1}^n (d\Psi^i(\vec{q}).\vec{A}_j)\vec{a}_i = \sum_{i=1}^n \frac{\partial \Psi^i}{\partial q^j}(\vec{q})\vec{a}_i \in \vec{\mathbb{R}^n}$, i.e., with $(A^i)$ the dual basis of $(\vec{A}_i)$,

$$d\Psi = \sum_{i=1}^n \sum_{j=1}^m \frac{\partial \Psi^i}{\partial q^j} \vec{a}_i \otimes A^j, \quad \text{i.e.} \quad [d\Psi]_{|\vec{A},\vec{a}} = [\frac{\partial \Psi^i}{\partial q^j}] \in \mathcal{M}_{nm} \ \text{(Jacobian matrix)}. \tag{T.35}$$

**Example T.19** Polar coordinate $\vec{\Psi}$: $m=n=2$. Cartesian vector space of parameters: $\mathbb{R}^2_{par}=\mathbb{R}\times\mathbb{R}=\{\vec{q}=(q_1,q_2)=(r,\theta)=(\text{length,angle})\}$, $U_{par}=\mathbb{R}^*_+\times]-\pi,\pi[$, $\Omega=\mathbb{R}^2-\{(x,0),x\le 0\}$ (the affine plane without the negative part of the $x$ axis). The polar coordinate system is the diffeomorphism $\Psi:\vec{q}=(r,\theta)\in U_{par}\to p\in\Omega$ given by $\vec{x}=\overrightarrow{Op}=\vec{\Psi}(r,\theta)=r\cos\theta\vec{a}_1+r\sin\theta\vec{a}_2$, thus

$$[\vec{x}]_{|\vec{a}}=[\vec{\Psi}(r,\theta)]_{|\vec{a}}=\begin{pmatrix}x=r\cos\theta\\ y=r\sin\theta\end{pmatrix},\quad [d\Psi]_{|\vec{A},\vec{a}}=\begin{pmatrix}[d\Psi^1]_{|\vec{a}}\\ [d\Psi^2]_{|\vec{a}}\end{pmatrix}=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}\in\mathcal{M}_{22}. \tag{T.36}$$

And $[\vec{A}_1]_{|\vec{A}}=\begin{pmatrix}1\\0\end{pmatrix}$ and $[\vec{A}_2]_{|\vec{A}}=\begin{pmatrix}0\\2\end{pmatrix}$ give $d\Psi(r,\theta).\vec{A}_1=\begin{pmatrix}\cos\theta\\ \cos\theta\end{pmatrix}$ and $d\Psi(r,\theta).\vec{A}_2=\begin{pmatrix}-R\sin\theta\\ R\cos\theta\end{pmatrix}$. ∴

**Example T.20** Polar coordinate $\vec{\Psi}_R$ on the circle: $m=1$, $n=2$: It is the restriction of (T.36) when $r=R>0$ is fixed. So, Cartesian parameter vector space (angles) $\vec{\mathbb{R}}^1_{par}=\{\vec{q}=(\theta)=\theta\vec{B},\ \theta\in]-\pi,\pi[\}$ where $\vec{B}=\vec{1}=(1)$ is the canonical basis in $\vec{\mathbb{R}}^1_{par}$, $\Psi_R(\theta)=\Psi(R,\theta)$, $p_R=\Psi_R(\theta)$, $\vec{x}=\overrightarrow{Op_R}$ and

$$[\vec{x}]_{|\vec{a}}=[\vec{\Psi}_R(\theta)]_{|\vec{a}}=\begin{pmatrix}x=R\cos\theta\\ y=R\sin\theta\end{pmatrix},\quad [d\Psi_R]_{|\vec{A},\vec{a}}=\begin{pmatrix}[d\Psi^1_R]_{|\vec{a}}\\ [d\Psi^2_R]_{|\vec{a}}\end{pmatrix}=\begin{pmatrix}-R\sin\theta\\ R\cos\theta\end{pmatrix}\in\mathcal{M}_{21}. \tag{T.37}$$

And $[\vec{B}]_{|\vec{B}}=(\,1\,)$ gives $d\Psi_R(\theta).\vec{B}=\lim_{h\to 0}\frac{\Psi_R(\theta+h)-\Psi_R(\theta)}{h}\in\mathbb{R}^2$ and $[d\Psi_R(\theta).\vec{B}]_{|\vec{a}}=\begin{pmatrix}-R\sin\theta\\ R\cos\theta\end{pmatrix}$. ∴

### T.3.2 The $j$-th coordinate line through a point

Let $\vec{q}_0=(q_{01},...,q_{0m})\in U_{par}$. The $j$-th Cartesian parametric straight line in $\vec{\mathbb{R}}^m$ through $\vec{q}_0$ is

$$\vec{z}_{\vec{q}_0 j}:\begin{cases}]a_j-q_{0j},b_j-q_{0j}[ & \to\vec{\mathbb{R}}^m_{par}\\ \qquad s & \to\vec{z}_{\vec{q}_0 j}(s)=\vec{q}_0+s\vec{A}_j.\end{cases} \tag{T.38}$$

The $j$-th coordinate line in $\vec{\mathbb{R}}^n$ through $\vec{x}_0=\vec{\Psi}(\vec{q}_0)$ is the curve:

$$\vec{c}_{\vec{x}_0 j}\begin{cases}]a_j-q_{0j},b_j-q_{0j}[ & \to\vec{\mathbb{R}}^n\\ \qquad s & \to\vec{c}_{\vec{x}_0 j}(s)=\vec{\Psi}(\vec{z}_{0j}(s)).\end{cases} \tag{T.39}$$

**Example T.21** Polar coordinate basis: $\vec{q}_0=(r_0,\theta_0)$ gives $\vec{z}_{\vec{q}_0 1}(s)=\vec{q}_0+s\vec{A}_1=(r_0{+}s,\theta_0)$, and $\vec{z}_{\vec{q}_0 2}(s)=\vec{q}_0+s\vec{A}_2=(r_0,\theta_0{+}s)$; Let $\vec{x}_0=\vec{\Psi}(\vec{q}_0)$; Thus $\vec{c}_{\vec{x}_0 1}(s)=\vec{\Psi}(r_0{+}s,\theta_0)$ (a radial straight line through $\vec{x}_0$ at angle $\theta_0$), i.e. $[\vec{c}_{\vec{x}_0 1}(s)]_{|\vec{a}}=\begin{pmatrix}(r_0{+}s)\cos\theta_0\\ (r_0{+}s)\sin\theta_0\end{pmatrix}=[\vec{x}_0]_{|\vec{a}}+s\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix}$; And $\vec{c}_{\vec{x}_0 2}(s)=\vec{\Psi}(r_0,\theta_0{+}s)$ (circle of radius $r_0$ through $\vec{x}_0$), i.e. $[\vec{c}_{\vec{x}_0 2}(s)]_{|\vec{a}}=\begin{pmatrix}r_0\cos(\theta_0{+}s)\\ r_0\sin(\theta_0{+}s)\end{pmatrix}$. ∴

**Example T.22** Restricted polar coordinate to the radius $R$ circle with $\vec{x}_0=\vec{\Psi}(R,\theta_0)$ : There is only one coordinate line: $\vec{c}_{\vec{x}_0}(s)=\vec{\Psi}_R(\theta_0{+}s)=\vec{\Psi}(R,\theta_0{+}s)=\vec{c}_{\vec{x}_0 2}(s)$ (the circle of radius $R$ itself). ∴

### T.3.3 Coordinate basis

**Definition T.23** The $\Psi$-coordinate system basis $(\vec{e}_i(p))_{i=1,...,m}$ in the tangent plane of Im$\Psi$ at $p=\Psi(\vec{q})\in\Omega$ is the basis $(\vec{e}_i(p))$ at $p$ (along the coordinate lines) in $\vec{\mathbb{R}}^n$ defined by, for $j=1,...,m$,

$$\boxed{\vec{e}_j(p):=d\Psi(\vec{q}).\vec{A}_j}\overset{\text{written}}{=}\frac{\partial\Psi}{\partial q^j}(\vec{q}). \tag{T.40}$$

Its dual basis at $p$ is made of the $m$ linear forms named $e^i(p)=^{\text{written}}dq^i(p)$ defined by, for all $j=1,...,m$,

$$e^i(p).\vec{e}_j(p)=\delta^i_j,\quad\text{written}\quad dq^i(p).\vec{e}_j(p)=\delta^i_j. \tag{T.41}$$

Hence with a chosen Euclidean basis $(\vec{a}_i)$ in $\mathbb{R}^n$ and $\vec{\Psi}(\vec{q})=\sum_i\Psi^i(\vec{q})\vec{a}_i\in\vec{\mathbb{R}}^n$, for all $j=1,...,m$:

$$\vec{e}_j(p)=\sum_{i=1}^n\frac{\partial\Psi^i}{\partial q^j}(\vec{q})\vec{a}_i,\quad\text{i.e.}\quad[\vec{e}_j(p)]_{|\vec{a}}=\begin{pmatrix}\frac{\partial\Psi^1}{\partial q^j}(\vec{q})\\ \vdots\\ \frac{\partial\Psi^n}{\partial q^j}(\vec{q})\end{pmatrix} \tag{T.42}$$

is the $j$-th column of $n*m$ Jacobian matrix $[d\Psi(\vec{q})]_{|\vec{A}_i,\vec{a}_i}=[\frac{\partial\Psi^i}{\partial q^j}(\vec{q})]$.

**Example T.24** Polar coordinate basis at $p=\Psi(\vec q)$: $[d\Psi(\vec q)]_{|\vec A,\vec a} = [\frac{\partial\Psi^i}{\partial q^j}] = \begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix}$, gives

$$[\vec e_1(p)]_{\vec a} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix},\quad [\vec e_2(p)]_{\vec a} = \begin{pmatrix}-r\sin\theta\\ r\cos\theta\end{pmatrix},\quad \text{i.e.}\quad \begin{cases}\vec e_1(p) = \cos\theta\,\vec a_1 + \sin\theta\,\vec a_2,\\ \vec e_2(p) = -r\sin\theta\,\vec a_1 + r\cos\theta\,\vec a_2.\end{cases} \tag{T.43}$$

So $P = \begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix} = [d\Psi(\vec q)]_{|\vec A,\vec a}$ is the transition matrix from $(\vec a_i)$ to $(\vec e_i(p))$. Call $(a^i) =^{\text{written}} (dx^i) =^{\text{written}} (dx,dy)$ the dual basis of the Euclidean basis $(\vec a_i)$. $P^{-1} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\frac1r\sin\theta & \frac1r\cos\theta\end{pmatrix}$ gives thus the dual basis $(e^1(p),e^2(p)) = (dq^1(p),dq^2(p)) =^{\text{written}} (dr(p),d\theta(p))$ of $(\vec e_1(p),\vec e_2(p))$ at $p=\Psi(\vec q)$:

$$\begin{cases}[dr(p)]_{|\vec a} = (\,\cos\theta \quad \sin\theta\,),\\ [d\theta(p)]_{|\vec a} = (\,-\frac1r\sin\theta \quad \frac1r\cos\theta\,),\end{cases}\quad \text{i.e.}\quad \begin{cases}dr(p) = \cos\theta\,dx + \sin\theta\,dy,\\ d\theta(p) = -\dfrac1r\sin\theta\,dx + \dfrac1r\cos\theta\,dy.\end{cases} \tag{T.44}$$

(The matrices of the linear forms $dq^i(p)$ are a row matrices.) ∴

**Example T.25** Restricted polar basis on the radius $R$ circle at $p=\vec\Psi_R(\theta)=\vec\Psi(R,\theta)$: Basis $(\vec e_R(p)) = (\vec e_2(p))$, dual basis $(d\theta_R(p)) = (d\theta(p))$. ∴

**Remark T.26** Pay attention to the notations that could contradict themselves: In $U_{par}$ the dual basis $(A^i)$ of the canonical basis $(\vec A_i)$ is uniform (independent of $\vec q$), and, in the context of coordinate systems, $(A^i)$ is not written $(dq^i)$ (unless expressly indicated): In the context of coordinate systems the notation $dq^i$ means $dq^i(p)$ at $p$ cf. (T.41): The $dq^i$ are defined in $\vec\Omega$ (not in $U_{par}$). E.g. polar coordinates: $dq^1(p)=dr(p)$ and $dq^2(p)=d\theta(p)$ are of $p$ (in the geometric space), not functions of $\vec q$ (in the parametric space). ∴

### T.3.4 Parametric expression of a differential: notation $\frac{\partial f}{\partial q_i}(p)$

With a coordinate system $\Psi$, a $C^1$ function $f : p\in\Omega\to f(p)\in\mathbb R$ can be studied thanks to

$$g := f\circ\Psi : \begin{cases}U_{par} & \to \mathbb R\\ \quad\vec q & \to g(\vec q) := f(\Psi(\vec q)) = f(p) \text{ when } p=\Psi(\vec q).\end{cases} \tag{T.45}$$

(Polar example: $g(r,\theta) := f(x,y) = f(r\cos\theta, r\sin\theta)$.) Thus with $p=\Psi(\vec q)$ and $\vec e_i(p) = d\Psi(p).\vec A_i$:

$$dg(\vec q) = df(p).d\Psi(\vec q) \quad\text{and}\quad \frac{\partial g}{\partial q^j}(\vec q) = dg(\vec q).\vec A_j = \boxed{df(p).\vec e_j(p) \overset{\text{written}}{=} \frac{\partial f}{\partial q^j}(p)}. \tag{T.46}$$

**Warning (notations):** $f$ is a function of $p$, not of $\vec q$ (!), so the notation $\frac{\partial f}{\partial q^j}(p)$ is absurd... unless it is defined to be $\frac{\partial f}{\partial q^j}(p) := \frac{\partial g}{\partial q^j}(\vec q) = \frac{\partial(f\circ\Psi)}{\partial q^j}(\vec q) = df(p).\vec e_j(p)$ when $p=\Psi(\vec q)$.

Thus, at $p$, with the dual basis $(e^j(p)) =^{\text{written}} (dq^j(p))$ of the coordinate basis $(\vec e_i(p))$,

$$df(p) = \sum_{j=1}^n f_{|j}(p)\,dq^j(p) = \sum_{j=1}^n \frac{\partial f}{\partial q^j}(p)\,dq^j(p), \quad \text{written} \quad df = \sum_{j=1}^n f_{|j}\,dq^j = \sum_{j=1}^n \frac{\partial f}{\partial q^j}\,dq^j, \tag{T.47}$$

unfortunate notation because $f$ is a function of $p$, not of $\vec q$. Check: $(\sum_j \frac{\partial f}{\partial q^j}(p)\,dq^j(p)).\vec e_i(p) = \sum_j \frac{\partial f}{\partial q^j}(p)\,(dq^j(p).\vec e_i(p)) = \sum_j \frac{\partial f}{\partial q^j}(p)\,\delta_{ij} = \frac{\partial f}{\partial q^j}(p) =^{(T.46)} df(p).\vec e_j(p)$.

Classic notations: $df(p) = \sum_j \frac{\partial f}{\partial q_j}(p)\,dq_j(p)$.

## T.4 Christoffel symbols

Purpose: Express $d\vec e_i.\vec e_j$, and then express $d\vec v$, in the coordinate system basis $(\vec e_i)$.

First: We have a problem; Example of the polar coordinate system restricted to the circle: $\vec\Psi_R(\theta) := \Psi(R,\theta) = R\cos\theta\vec a_1 + R\sin\theta\vec a_2$ gives $\vec e_2(\vec\Psi_R(\theta)) = \frac{d\vec\Psi_R}{d\theta}(\theta) = -R\sin\theta\vec a_1 + R\cos\theta\vec a_2$, thus $d\vec e_2(p).\vec e_2(p) = d\vec e_2(p).d\vec\Psi_R(\theta).\vec A_2 = \frac{\partial^2\vec\Psi}{\partial\theta^2}(\vec q) = -r\cos\theta\vec a_1 - r\sin\theta\vec a_2 = -r\vec e_1(p)$ perpendicular (thus not parallel) to $\vec e_2(p)$;

So $d\vec{e}_2(p).\vec{e}_2(p)$ is not tangent to $\vec{e}_2(p)$: The derivation of a tangent vector in the direction of a tangent vector can give a non tangent vector. Thus, to make things simple, we will "enlarge" $\Psi$ ($m$-D manifold) into $\Psi_+$ ($n$-D manifold), to have straight forward computations. General setting:

The dimension $m$ surface (manifold) $\Omega_m = \Psi(U_{par}^m)$ is considered as a part of a set $\Omega_n = \Psi_+(U_{par}^n)$ of dimension $n$ so that $\Psi$ is the restriction of a $C^2$ map $\Psi_+$: With $U_{par}^n = ]a_1, b_1[\times...\times]a_n, b_n[$,

$$\vec{\Psi}_+ : \begin{cases} U_{par+} & \to \vec{\Omega}_+ \subset \vec{\mathbb{R}^n} \\ \vec{q} = (q_1, ..., q_n) & \to \vec{x} = \vec{\Psi}_+(\vec{q}) \end{cases} \tag{T.48}$$

and when $q_{m+1}, ..., q_n$ are fixed then

$$\vec{\Psi}(q_1, ..., q_m) := \vec{\Psi}_{+q_{m+1},...,q_n}(q_1, ..., q_m) := \vec{\Psi}_+(q_1, ..., q_m, q_{m+1}, ..., q_n). \tag{T.49}$$

E.g. 1: Polar coordinate system: $m = n$ thus $\vec{\Psi}_+ = \vec{\Psi}$ and (T.48) is useless.
E.g. 2: $R$ fixed, restricted coordinate system $\Psi_R : \theta \to \Psi_R(\theta) = \Psi_+(R, \theta)$ where $\Psi_+ = \Psi$.

Then we consider the Coordinate system $\Psi_+$: At $p = \Psi_+(\vec{q})$, coordinate basis $(\vec{e}_j(p))_{j=1,...,n}$ and dual basis $(e^i(p))_{i=1,...,n}$ where

$$\forall j = 1, ..., n, \quad \vec{e}_j(p) = d\Psi_+(\vec{q}).\vec{A}_j = \frac{\partial \Psi_+}{\partial q^j}(\vec{q}) \in \vec{\mathbb{R}^n}. \tag{T.50}$$

**Definition T.27** Relative to $\vec{\Psi}_+$: The $n^3$ Christoffel symbol $\gamma_{ji}^k(p) \in \mathbb{R}$ are the components of $d\vec{e}_i(p).\vec{e}_j(p)$ in the basis $(\vec{e}_k(p))$, i.e. $d\vec{e}_i(p).\vec{e}_j(p) = \sum_k \gamma_{ji}^k(p)\vec{e}_k(p)$, i.e. $\gamma_{ji}^k(p) := e^k(p).d\vec{e}_i(p).\vec{e}_j(p)$, i.e., for all $i, j, k \in [1, n]_{\mathbb{N}}$,

$$\boxed{d\vec{e}_i.\vec{e}_j = \sum_{k=1}^n \gamma_{ji}^k \vec{e}_k} \quad \text{i.e.} \quad [d\vec{e}_i.\vec{e}_j]_{|\vec{e}} = \begin{pmatrix} \gamma_{ji}^1 \\ \vdots \\ \gamma_{ji}^n \end{pmatrix}, \quad \text{i.e.} \quad \gamma_{ji}^k := e^k.d\vec{e}_i.\vec{e}_j, \tag{T.51}$$

$$\text{i.e.} \quad d\vec{e}_i = \sum_{j,k=1}^n \gamma_{ji}^k \vec{e}_k \otimes e^j, \quad \text{or} \quad d\vec{e}_k = \sum_{i,j=1}^n \gamma_{jk}^i \vec{e}_i \otimes e^j, \quad \text{i.e.} \quad [d\vec{e}_k]_{|\vec{e}} = [\gamma_{jk}^i] \in \mathcal{M}_{nn}. \tag{T.52}$$

Notation:

$$\frac{\partial \vec{e}_i}{\partial q^j}(p) := d\vec{e}_i(p).\vec{e}_j(p). \tag{T.53}$$

Warning: The $\vec{e}_i(p)$ depend on $p$, not on $\vec{q}$, thus $\frac{\partial \vec{e}_i}{\partial q^j}(p)$ is nothing but the notation for $d\vec{e}_i(p).\vec{e}_j(p)$, derivation of $\vec{e}_i$ along the $j$-th coordinate line at $p$.

**Remark T.28** Cartesian basis: The $\vec{e}_i$ are independent of $p$, hence $\gamma_{jk}^i = 0$ for all $i, j, k$. ∴

**Remark T.29** Relative to $\vec{\Psi}$: A Euclidean dot product being chosen, and $\text{Proj}_{T_p(\Omega_m)}$ being the orthogonal projection on the tangent space at $p$, the $m^3$ Christoffel symbol $\gamma_{ji}^k(p) \in \mathbb{R}$ for $i, j, k \in [1, m]_{\mathbb{N}}$ are the components of $\nabla_{\vec{e}_j}\vec{e}_i = \text{Proj}_{T(\Omega_m)}(d\vec{e}_i.\vec{e}_j)$, i.e. given by $\nabla_{\vec{e}_j}\vec{e}_i = \sum_{i=1}^m \gamma_{ji}^k \vec{e}_k$ for all $i, j \in [1, m]_{\mathbb{N}}$. ∴

**Proposition T.30** *A coordinate system $\Psi_+$ being $C^2$, at $p = \Psi_+(\vec{q})$ we have*

$$d\vec{e}_i(p).\vec{e}_j(p) = \frac{\partial^2 \Psi_+}{\partial q^i \partial q^j}(\vec{q}) = d\vec{e}_j(p).\vec{e}_i(p), \quad \text{and} \quad \gamma_{ij}^k = \gamma_{ji}^k, \tag{T.54}$$

*for all $i, j, k$: Symmetry of the lower indices of the Christoffel symbols.*

**Proof.** $\vec{e}_i(\Psi(\vec{q})) =^{(T.42)} d\Psi(\vec{q}).\vec{A}_i$ gives $d\vec{e}_i(p).d\Psi(\vec{q}) = d^2\Psi(\vec{q}).\vec{A}_i$, thus $d\vec{e}_i(p).\vec{e}_j(p) = d\vec{e}_i(p).d\Psi(\vec{q}).\vec{A}_j = (d^2\Psi(\vec{q}).\vec{A}_i).\vec{A}_j = \frac{\partial^2 \Psi}{\partial q^j \partial q^i}(\vec{q})$. And $\Psi$ is $C^2$ gives $\frac{\partial^2 \Psi}{\partial q^i \partial q^j} = \frac{\partial^2 \Psi}{\partial q^j \partial q^i}$ (Schwarz theorem). ∴

**Corollary T.31** *Divergence of the vector field $\vec{e}_k$: (T.52) gives:*

$$\text{div}\vec{e}_k = \sum_{i=1}^n \gamma_{ik}^i \quad (= \text{Tr}([d\vec{e}_k]_{|\vec{e}})). \tag{T.55}$$

**Exercice T.32** Polar coordinate system, prove: $d\vec{e}_1.\vec{e}_1 = \vec{0}$, $d\vec{e}_1.\vec{e}_2 = \frac{1}{r}\vec{e}_2$, $d\vec{e}_2.\vec{e}_2 = -r\vec{e}_1$, that is $\gamma_{12}^2 = \frac{1}{r} = \gamma_{21}^2$, $\gamma_{22}^1 = -r$ and the other $\gamma_{jk}^i$ vanish.

**Answer**. With (T.43), $d\vec{e}_1.\vec{e}_1 = \frac{\partial\vec{e}_1}{\partial r} = \vec{0}$, thus $\gamma_{11}^1 = \gamma_{11}^2 = 0$.

$d\vec{e}_1.\vec{e}_2 = \frac{\partial\vec{e}_1}{\partial\theta} = -\sin\theta\vec{a}_1 + \cos\theta\vec{a}_2 = -\sin\theta(\cos\theta\vec{e}_1 - \frac{1}{r}\sin\theta\,\vec{e}_2) + \cos\theta(\sin\theta\,\vec{e}_1 + \frac{1}{r}\cos\theta\,\vec{e}_2) = \frac{1}{r}\vec{e}_2$, thus $\gamma_{12}^1 = 0 = \gamma_{21}^1$ and $\gamma_{12}^2 = \frac{1}{r} = \gamma_{21}^2$.

$d\vec{e}_2.\vec{e}_2 = \frac{\partial\vec{e}_2}{\partial\theta} = -r\cos\theta\vec{a}_1 - r\sin\theta\vec{a}_2 = -r\cos\theta(\cos\theta\vec{e}_1 - \frac{1}{r}\sin\theta\,\vec{e}_2) - r\sin\theta(\sin\theta\,\vec{e}_1 + \frac{1}{r}\cos\theta\,\vec{e}_2) = -r\vec{e}_1$, thus $\gamma_{22}^1 = -r$ and $\gamma_{22}^2 = 0$. ▪

**Exercice T.33** Consider two coordinate system bases $(\vec{a}_i(p))$ and $(\vec{b}_i(p))$ at $p$, and $P(p) = [P_j^i(p)]$ the transition matrix from $(\vec{a}_i(p))$ to $(\vec{b}_i(p))$. Let $Q = P^{-1}$. Using the generic notation $d\vec{e}_k.\vec{e}_j = \sum_{i=1}^n \gamma_{jk,e}^i \vec{e}_i$, prove the change of basis formula for the Christoffel symbols:

$$\gamma_{jk,b}^i = \sum_{\lambda,\mu,\nu=1}^n Q_\lambda^i P_j^\mu P_k^\nu \gamma_{\mu\nu,a}^\lambda + \sum_{\lambda,\mu=1}^n Q_\lambda^i P_j^\mu (dP_k^\lambda.\vec{a}_\mu) \quad (= \sum_{\lambda,\mu,\nu=1}^n Q_\lambda^i P_j^\mu P_k^\nu \gamma_{\mu\nu,a}^\lambda + \sum_{\lambda=1}^n Q_\lambda^i (dP_k^\lambda.\vec{b}_j)). \tag{T.56}$$

So a derivation is not a tensor (although it is linear) because of the term $\sum_{\mu\nu} Q_\lambda^i P_j^\mu (dP_k^\lambda.\vec{a}_\mu)$.

**Answer**. $\vec{b}_k(p) = \sum_\nu P_k^\nu(p)\vec{a}_\nu(p)$ gives $d\vec{b}_k.\vec{b}_j = \sum_\nu (dP_k^\nu.\vec{b}_j)\vec{a}_\nu + \sum_\nu P_k^\nu (d\vec{a}_\nu.\vec{b}_j) = \sum_{\mu\nu} P_j^\mu (dP_k^\nu.\vec{a}_\mu)\vec{a}_\nu + \sum_{\mu\nu} P_k^\nu P_j^\mu (d\vec{a}_\nu.\vec{a}_\mu)$; And $b^i = \sum_\lambda Q_\lambda^i a^\lambda$, thus

$$\gamma_{jk,b}^i = b^i.d\vec{b}_k.\vec{b}_j = \sum_{\lambda\mu\nu} Q_\lambda^i P_j^\mu (dP_k^\nu.\vec{a}_\mu) a^\lambda.\vec{a}_\nu + \sum_{\lambda\mu\nu} Q_\lambda^i P_j^\mu P_k^\nu a^\lambda.(d\vec{a}_\nu.\vec{a}_\mu) = \sum_{\lambda\mu} Q_\lambda^i P_j^\mu (dP_k^\lambda.\vec{a}_\mu) + \sum_{\lambda\mu\nu} Q_\lambda^i P_j^\mu P_k^\nu \gamma_{\mu\nu,a}^\lambda,$$

thus (T.56). ▪

**Remark T.34** Differential geometry notations: $d\vec{e}_i.\vec{e}_j =^{\text{written}} \nabla_{\vec{e}_j}\vec{e}_i = \sum_{k=1}^n \gamma_{ji}^k \vec{e}_k$, so $\gamma_{ji}^k = e^k.\nabla_{\vec{e}_j}\vec{e}_i$, or $\gamma_{jk}^i = e^i.\nabla_{\vec{e}_j}\vec{e}_k$; And the $\gamma_{jk}^i$ are the components of the connection $\nabla$; And the usual connection in a surface in $\mathbb{R}^n$ is the Riemannian connection given by: $\nabla_{\vec{e}_j}\vec{e}_i$ is, relative to a Euclidean dot product, the orthogonal projection of $d\vec{e}_i.\vec{e}_j$ on the tangent plane. And $\nabla\vec{e}_j = \sum_i \vec{e}_i \otimes \omega_j^i$, where the $\omega_j^i := \sum_k \gamma_{kj}^i e^k$ are called the connections one forms. ▪

## T.5 Warning: Notations!

Issue: $\frac{\partial f}{\partial q^i}(p) := \frac{\partial f_U}{\partial q^i}(\vec{q}) = df(p).\vec{e}_i(p)$, but $\frac{\partial^2 f_U}{\partial q^j \partial q^i}(\vec{q}) \neq d^2 f(p)(\vec{e}_i(p), \vec{e}_j(p))$, and $d^2 f(p)(\vec{e}_i(p), \vec{e}_j(p))$ is not written $\frac{\partial^2 f}{\partial q^i \partial q^j}(p)$ in general...! (Ambiguous notations.)

In fact: Let $f \in C^2(\Omega;\mathbb{R})$, hence $f_U := f \circ \Psi \in C^2(U_{par};\mathbb{R})$. By definition $\frac{\partial f}{\partial q^i}(p) := \frac{\partial f_U}{\partial q^i}(\vec{q})$ when $p = \Psi(\vec{q})$, where $\frac{\partial f_U}{\partial q^i}(\vec{q}) := df_U(\vec{q}).\vec{A}_i$, thus $f_U(\vec{q}) = f(\Psi(\vec{q}))$ gives $\frac{\partial f}{\partial q^i}(p) = df(p).d\Psi(\vec{q}).\vec{A}_i = df(p).\vec{e}_i(p) = df(\Psi(\vec{q})).\vec{e}_i(\Psi(\vec{q}))$.

Thus $\frac{\partial \frac{\partial f_U}{\partial q^i}}{\partial q^j}(\vec{q}) = d(\frac{\partial f_U}{\partial q^i}).\vec{A}_j = (d(df)(\Psi(\vec{q})).d\Psi(\vec{q}).\vec{A}_j).\vec{e}_i(\Psi(\vec{q})) + df(\Psi(\vec{q})).(d\vec{e}_i(\Psi(\vec{q})).d\Psi(\vec{q}).\vec{A}_j)$, so, with the usual notation $d^2 f(\vec{u}, \vec{w}) := d(df.\vec{u}).\vec{w})$ and $d\vec{e}_i.\vec{e}_j = \sum_k \gamma_{ji}^k \vec{e}_k$,

$$\frac{\partial^2 f_U}{\partial q^j \partial q^i}(\vec{q}) = d^2 f(p)(\vec{e}_i(p), \vec{e}_j(p)) + \sum_{k=1}^n \gamma_{ij}^k(p) \frac{\partial f}{\partial q^k}(p) : \quad \text{First order terms remain} \tag{T.57}$$

(in a non Cartesian setting). I.e. $d^2 f(p)(\vec{e}_i(p), \vec{e}_j(p)) = \frac{\partial^2 f_U}{\partial q^j \partial q^i}(\vec{q}) - \sum_{k=1}^n \gamma_{ij}^k(p) \frac{\partial f}{\partial q^k}(p)$. Thus, when a coordinate system is used, the notation $\frac{\partial^2 f}{\partial q^i \partial q^j}(p)$ should be avoided because it is not reduced to a second order derivative term of $f$, contrary to what $\frac{\partial^2 f}{\partial q^i \partial q^j}(p)$ could mean.

Remark: (T.57) gives $d^2 f(\vec{e}_i, \vec{e}_j) = d^2 f(\vec{e}_j, \vec{e}_i)$ in a coordinate system.

## T.6 Differential of a vector field

$F = E = \vec{\mathbb{R}^n}$, $\Phi = \vec{w} \in \Gamma(\Omega)$ and $\vec{u} \in \Gamma(\Omega)$ are vector fields. Thus $d\vec{w}(p) \in \mathcal{L}(E;E)$ and $d\vec{w}.\vec{u}$ is a vector field in $E$ given by $(d\vec{w}.\vec{u})(p) = d\vec{w}(p).\vec{u}(p) = \lim_{h\to 0} \frac{\vec{w}(p+h\vec{u}(p)) - \vec{w}(p)}{h} \in E$.

**Proposition T.35** *(derivation of a product.) If $\vec{w} \in \Gamma(\Omega) \cap C^1(\Omega)$ and $f \in C^1(\Omega)$ then*

$$d(f\vec{w}) = \vec{w} \otimes df + f\,dw. \tag{T.58}$$

**Proof.** $d(f\vec{w})(p).\vec{u}(p) = \lim_{h\to0}\frac{f(p+h\vec{u}(p))\vec{w}(p+h\vec{u}(p))-f(p)\vec{w}(p)}{h} = \lim_{h\to0}\frac{f(p+h\vec{u}(p))-f(p)}{h}\vec{w}(p+h\vec{u}(p)) + \lim_{h\to0} f(p)\frac{\vec{w}(p+h\vec{u}(p))-\vec{w}(p)}{h} = (df(p).\vec{u}(p))\vec{w}(p) + f(p)\,d\vec{w}(p).\vec{u}(p)$, since $\vec{w}(p+h\vec{u}(p)) = \vec{w}(p)+o(1)$. ∎

**Quantification:** $(\vec{e}_i(p))$ is a basis at $p$ and $(e^i(p))$ is its dual basis, $\vec{w} = \sum_{i=1}^n w^i\vec{e}_i \in \Gamma(\Omega)$, the $w^i_{|j}$ are the components of $d\vec{w}$:

$$d\vec{w}.\vec{e}_j = \sum_{i=1}^n w^i_{|j}\vec{e}_i, \quad \text{i.e.} \quad d\vec{w} = \sum_{i,j=1}^n w^i_{|j}\vec{e}_i\otimes e^j, \quad \text{i.e.} \quad [d\vec{w}]_{|\vec{e}} = [w^i_{|j}],$$
$$\text{where} \quad \boxed{w^i_{|j} = \frac{\partial w^i}{\partial q^j} + \sum_{k=1}^n \gamma^i_{jk}w^k} = e^i.d\vec{w}.\vec{e}_j. \tag{T.59}$$

Indeed $\vec{w} = \sum_i w^i\vec{e}_i$ gives $d\vec{w}.\vec{e}_j = \sum_i(dw^i.\vec{e}_j)\vec{e}_i + \sum_i w^i(d\vec{e}_i.\vec{e}_j) = \sum_i \frac{\partial w^i}{\partial q^j}\vec{e}_i + \sum_{ik} w^i\gamma^k_{ji}\vec{e}_k = \sum_i \frac{\partial w^i}{\partial q^j}\vec{e}_i + \sum_{ik} w^k\gamma^i_{jk}\vec{e}_i$. Cartesian basis: $w^i_{|j} = \dfrac{\partial w^i}{\partial x^j} = w^i_{,j}$.

**Example T.36** $\vec{w} = \vec{e}_i = \sum_\ell \delta^\ell_i\vec{e}_\ell$ gives $d\vec{e}_i.\vec{e}_j = \sum_\ell(\frac{\partial\delta^\ell_i}{\partial q^j} + \sum_k \gamma^\ell_{jk}\delta^k_i)\vec{e}_\ell = \sum_\ell \gamma^\ell_{ji}\vec{e}_\ell$: OK, cf. (T.51). ∎

**Exercice T.37** $d\vec{w}(p)$ being an endomorphism, with $\vec{w} = \sum_i u^i\vec{a}_i = \sum_i v^i\vec{b}_i$ and $Q = P^{-1}$, check with exercise T.33:

$$[d\vec{w}]_{|\vec{b}} = P^{-1}.[d\vec{w}]_{|\vec{a}}.P, \quad \text{i.e.} \quad v^i_{|j} = \sum_{k,\ell=1}^n Q^i_k u^k_{|\ell}P^\ell_j. \tag{T.60}$$

**Answer**. $[\vec{w}]_{|\vec{b}} = Q.[\vec{w}]_{|\vec{a}}$, i.e. $v^i = \sum_k Q^i_k u^k$ for all $i$, thus $dv^i.\vec{b}_j = \sum_\lambda(dQ^i_\lambda.\vec{b}_j)u^\lambda + \sum_\lambda Q^i_\lambda(du^\lambda.\vec{b}_j)$, thus

$$v^i_{|j} \overset{(T.51)}{=} dv^i.\vec{b}_j + \sum_k v^k\gamma^i_{jk,b}$$
$$\overset{(T.56)}{=} \sum_{\lambda\mu} u^\lambda P^\mu_j(dQ^i_\lambda.\vec{a}_\mu) + \sum_{\lambda\mu} Q^i_\lambda P^\mu_j(du^\lambda.\vec{a}_\mu) + \sum_{k\omega\lambda\mu\nu}(Q^k_\omega u^\omega)Q^i_\lambda P^\mu_j P^\nu_k\gamma^\lambda_{\mu\nu,a} + \sum_{k\lambda\mu\nu}(Q^k_\lambda u^\lambda)Q^i_\nu P^\mu_j(dP^\nu_k.\vec{a}_\mu),$$

and $Q^k_\omega P^\lambda_k = \delta^\lambda_\omega$ gives $(dQ^k_\omega.\vec{a}_\mu)P^\lambda_k + Q^k_\omega(dP^\lambda_k.\vec{a}_\mu) = 0$, thus the fourth term reads

$$\sum_{k\lambda\mu\nu} u^\lambda Q^i_\nu P^\mu_j Q^k_\lambda(dP^\nu_k.\vec{a}_\mu) = -\sum_{k\lambda\mu\nu} u^\lambda Q^i_\nu P^\mu_j P^\nu_k(dQ^k_\lambda.\vec{a}_\mu) = -\sum_{\lambda\mu} u^\lambda P^\mu_j(dQ^i_\lambda.\vec{a}_\mu),$$

which cancels the first term: Thus $v^i_{|j} = \sum_{\lambda\mu} Q^i_\lambda P^\mu_j(du^\lambda.\vec{a}_\mu) + \sum_{\lambda\mu\nu} u^\nu Q^i_\lambda P^\mu_j\gamma^\lambda_{\mu\nu} = \sum_{\lambda\mu} Q^i_\lambda u^i_{|j}P^\mu_j$, i.e. (T.60). ∎

## T.7 Christoffel symbols and dual basis

$e^i.\vec{e}_j = \delta^i_j$, thus $d(e^i.\vec{e}_j) = 0 = de^i.\vec{e}_j + e^i.d\vec{e}_j =^{(T.52)} de^i.\vec{e}_j + e^i.(\sum_{k\ell}\gamma^\ell_{kj}\vec{e}_\ell\otimes e^k)$,

$$\boxed{de^i.\vec{e}_j = -\sum_{k=1}^n \gamma^i_{kj}e^k}, \quad \text{i.e.} \quad (de^i.\vec{e}_j).\vec{e}_k = -\gamma^i_{kj}. \tag{T.61}$$

(I.e. $de^i(p).\vec{e}_j(p) = -\sum_{k=1}^n \gamma^i_{kj}(p)e^k(p)$.)

## T.8 Differential of a differential form

$F = \mathbb{R}$, $\Phi = \ell \in \Omega^1(\Omega)$ (differential form) in $C^1(\Omega;E^*)$, $p\in\Omega$; So $\ell(p)\in E^*$, $d\ell(p)\in\mathcal{L}(E;E^*)$, and with $\vec{u},\vec{v}\in E$,

$$d\ell(p).\vec{u} = \lim_{h\to0}\frac{\ell(p+h\vec{u})-\ell(p)}{h} \in E^*, \quad \text{and} \quad (d\ell(p).\vec{u}).\vec{v} = \lim_{h\to0}\frac{\ell(p+h\vec{u}).\vec{v}-\ell(p).\vec{v}}{h} \in \mathbb{R}. \tag{T.62}$$

**Proposition T.38** *(derivation of a product.) If $\ell\in\Omega^1(\Omega)\cap C^1(\Omega)$ and $f\in C^1(\Omega)$ then*

$$d(f\ell) = \ell\otimes df + f\,d\ell, \tag{T.63}$$

*i.e. $d(f\ell).\vec{u} = (df.\vec{u})\ell + f\,d\ell.\vec{u}$ for all vector field $\vec{u}$ in $\Omega$.*

**Proof.** $d(f\ell)(p).\vec{u}(p) = \lim_{h\to0}\frac{f(p+h\vec{u}(p))\ell(p+h\vec{u}(p))-f(p)\ell(p)}{h} = \lim_{h\to0}\frac{f(p+h\vec{u}(p))-f(p)}{h}\ell(p+h\vec{u}(p)) + \lim_{h\to0} f(p)\frac{\ell(p+h\vec{u}(p))-\ell(p)}{h} = (df(p).\vec{u}(p))\ell(p) + f(p)\,d\ell(p).\vec{u}(p)$, since $\ell(p+h\vec{u}(p)) = \ell(p)+o(1)$;
And $(\ell\otimes df + f\,d\ell)(p).\vec{u} = (\ell(p)\otimes df(p) + f(p)\,d\ell(p)).\vec{u} = \ell(p)(df(p).\vec{u}) + f(p)\,(d\ell(p).\vec{u})$. ∎

**Quantification:** $(\vec{e}_i(p))$ coordinate system basis at $p$, $(e^i(p))$ its dual basis. When $\ell = \sum_{i=1}^n \ell_i e^i$ let $\ell_{i|j}$ be the components of $d\ell$:

$$d\ell.\vec{e}_j = \sum_{i=1}^n \ell_{i|j}e^i, \quad \text{i.e.} \quad \ell_{i|j} = (d\ell.\vec{e}_j).\vec{e}_i, \quad \text{i.e.} \quad d\ell = \sum_{i,j=1}^n \ell_{i|j}e^i\otimes e^j \quad \text{and} \quad [d\ell]_{|\vec{e}} = [\ell_{i|j}],$$
$$\text{where} \quad \boxed{\ell_{i|j} = \frac{\partial \ell_i}{\partial q^j} - \sum_{k=1}^n \ell_k\gamma_{ij}^k}. \tag{T.64}$$

Indeed $\ell = \sum_i \ell_i e^i$ gives $d\ell.\vec{e}_j = \sum_i (d\ell_i.\vec{e}_j)e^i + \sum_i \ell_i(de^i.\vec{e}_j) = \sum_i \frac{\partial \ell_i}{\partial q^j}e^i - \sum_{ik}\ell_i\gamma_{kj}^i e^k = \sum_i \frac{\partial \ell_i}{\partial q^j}e^i - \sum_{ik}\ell_k\gamma_{ij}^k e^i$.
(Cartesian basis: $(e^i)$ is uniform, thus $\ell_{i|j} = \frac{\partial \ell_i}{\partial x^j} =^{\text{written}} \ell_{i,j}$.)

**Exercice T.39** With $\ell = \sum_i \ell_i e^i$ and $\vec{w} = \sum_j w^j\vec{e}_j$ a vector field and $f = \ell.\vec{w} = \sum_i \ell_i w^i$, check $d(\ell.\vec{w}) = df = \sum_j \frac{\partial f}{\partial q^j}e^j = \sum_{ij}\frac{\partial \ell_i}{\partial q^j}w^i + \ell_i\frac{\partial w^i}{\partial q^j}$, and the Christoffel symbols vanish.

**Answer**. $f = \sum_i \ell_i w^i$ gives $df.\vec{e}_j = \sum_i (d\ell_i.\vec{e}_j)w^i + \ell_i(dw^i.\vec{e}_j) = \sum_i \frac{\partial \ell_i}{\partial q^j}w^i + \ell_i\frac{\partial w^i}{\partial q^j}$ since $\ell_i$ and $w^i$ are scalar valued function. For the vanishing of the Christoffel symbols: $d(\ell.\vec{w}).\vec{e}_j = (d\ell.\vec{e}_j).\vec{w} + \ell.(d\vec{w}.\vec{e}_j) = (\sum_i \ell_{i|j}e^i).\vec{w} + \ell.(\sum_i w^i_{|j}\vec{e}_i) = \sum_i(\frac{\partial \ell_i}{\partial q^j} - \sum_k \ell_k\gamma_{ji}^k)w^i + \sum_i \ell_i(\frac{\partial w^i}{\partial q^j} + \sum_k w^k\gamma_{jk}^i)$, and $-\sum_{ik}\ell_k\gamma_{ji}^k w^i + \sum_{ik}\ell_i w^k\gamma_{jk}^i = -\sum_{ik}\ell_k\gamma_{ji}^k w^i + \sum_{ik}\ell_k w^i\gamma_{ji}^k = 0$. ∎

## T.9 Differential of a 1 1 tensor

Consider a $C^1$ $\binom{1}{1}$ tensor $\underline{\underline{\tau}} : \left\{\begin{array}{ll}\Omega & \to \mathcal{L}(E^*,E;\mathbb{R})\\ p & \to \underline{\underline{\tau}}(p)\end{array}\right\} \in T_1^1(\Omega)$. Its differential

$$d\underline{\underline{\tau}} : \left\{\begin{array}{ll}\Omega & \to \mathcal{L}(E;\mathcal{L}(E^*,E;\mathbb{R})) \simeq \mathcal{L}(E^*,E,E;\mathbb{R})\\ p & \to d\underline{\underline{\tau}}(p)\end{array}\right\} \in T_2^1(\Omega) \tag{T.65}$$

is defined by $d\underline{\underline{\tau}}(p).\vec{u} = \lim_{h\to0}\frac{\underline{\underline{\tau}}(p+h\vec{u})-\underline{\underline{\tau}}(p)}{h}$, i.e. $(d\underline{\underline{\tau}}(p).\vec{u})(\ell,\vec{v}) = \lim_{h\to0}\frac{\underline{\underline{\tau}}(p+h\vec{u})(\ell,\vec{v})-\underline{\underline{\tau}}(p)(\ell,\vec{v})}{h}$ $(\in\mathbb{R})$, for all $\vec{u},\vec{v}\in E$ and $\ell\in E^*$ defined at $p$.

**Quantification**: Coordinate system basis $(\vec{e}_i(p))$ in $E$ at $p$, dual basis $(e^i(p))$:

$$d(\vec{e}_i\otimes e^j).\vec{e}_k = (d\vec{e}_i.\vec{e}_k)\otimes e^j + \vec{e}_i\otimes(de^j.\vec{e}_k) = \sum_{\ell=1}^n \gamma_{ki}^\ell\vec{e}_\ell\otimes e^j - \sum_{\ell=1}^n \gamma_{k\ell}^j\vec{e}_i\otimes e^\ell, \tag{T.66}$$

thus

$$d(\vec{e}_i\otimes e^j) = \sum_{k\ell}\gamma_{ki}^\ell\vec{e}_\ell\otimes e^j\otimes e^k - \sum_{k\ell}\gamma_{k\ell}^j\vec{e}_i\otimes e^\ell\otimes e^k. \tag{T.67}$$

Thus for $\underline{\underline{\tau}} = \sum_{ij}\tau_{ij}\vec{e}_i\otimes e^j$ we get $d\underline{\underline{\tau}}.\vec{e}_k = \sum_{ij}(d\tau_{ij}.\vec{e}_k)\,\vec{e}_i\otimes e^j + \sum_{ij}\tau_{ij}d(\vec{e}_i\otimes e^j).\vec{e}_k = \sum_{ij}\frac{\partial \tau_{ij}}{\partial q^k}\,\vec{e}_i\otimes e^j + \sum_{ij\ell}\tau_j^i\gamma_{ki}^\ell\vec{e}_\ell\otimes e^j - \sum_{ij\ell}\tau_j^i\gamma_{\ell k}^j\vec{e}_i\otimes e^\ell = \sum_{ij}\frac{\partial \tau_{ij}}{\partial q^k}\,\vec{e}_i\otimes e^j + \sum_{ij\ell}\tau_j^\ell\gamma_{k\ell}^i\vec{e}_i\otimes e^j - \sum_{ij\ell}\tau_\ell^i\gamma_{jk}^\ell\vec{e}_i\otimes e^j$, thus

$$d\underline{\underline{\tau}}.\vec{e}_k = \sum_{i,j=1}^n \tau^i_{j|k}\vec{e}_i\otimes e^j \quad \text{i.e.} \quad \tau^i_{j|k} = e^i.(d\underline{\underline{\tau}}.\vec{e}_k).\vec{e}_j \quad \text{i.e.} \quad d\underline{\underline{\tau}} = \sum_{i,j,k=1}^n \tau^i_{j|k}\vec{e}_i\otimes e^j\otimes e^k,$$
$$\text{where} \quad \tau^i_{j|k} = \frac{\partial \tau^i_j}{\partial q^k} + \sum_{\ell=1}^n(\gamma_{k\ell}^i\tau_j^\ell - \tau_\ell^i\gamma_{jk}^\ell) \quad \text{and} \quad \frac{\partial \tau^i_j}{\partial q^k} = d\tau^i_j.\vec{e}_k. \tag{T.68}$$

(The $+$ sign from vector fields, cf. (T.59), and the $-$ sign from differential forms, cf. (T.64).) (Cartesian basis: $\tau^i_{j|k} = \frac{\partial \tau^i_j}{\partial x^k} = \tau^i_{j,k}$.)

Application: $\vec{v} = \sum_i v^i\vec{e}_i$, $d\vec{v} = \sum_{ij}v^i_{|j}\vec{e}_i\otimes e^j$ with $v^i_{|j} = (dv^i.\vec{e}_j) + \sum_m \gamma_{jm}^i v^m$, and $d(d\vec{v}).\vec{e}_k = \sum_{ijk}(v^i_{|j})_{|k}\vec{e}_i\otimes e^j$ with $(v^i_{|j})_{|k} = dv^i_{|j}.\vec{e}_k + \sum_m(\gamma_{km}^i v^m_{|j} - \gamma_{jk}^m v^i_{|m})$.

And $dv^i_{|j}.\vec{e}_k = d^2v^i(\vec{e}_k,\vec{e}_j) + dv^i.d\vec{e}_j.\vec{e}_k + \sum_m (d\gamma^i_{jm}.\vec{e}_k)v^m + \sum_m \gamma^i_{jm}\,dv^m.\vec{e}_k = d^2v^i(\vec{e}_k,\vec{e}_j) + \sum_m \gamma^m_{kj}(dv^i.\vec{e}_m) + \sum_m (d\gamma^i_{jm}.\vec{e}_k)v^m + \sum_m \gamma^i_{jm}\,dv^m.\vec{e}_k$.

And $\gamma^i_{km}v^m_{|j} - \gamma^m_{jk}v^i_{|m} = \gamma^i_{km}(\frac{\partial v^m}{\partial q^j} + \sum_\ell \gamma^m_{j\ell}v^\ell) - \gamma^m_{jk}(\frac{\partial v^i}{\partial q^m} + \sum_\ell \gamma^i_{m\ell}v^\ell)$. Thus

$$\begin{aligned}(v^i_{|j})_{|k} &= d^2v^i(\vec{e}_k,\vec{e}_j) + \sum_m \frac{\partial v^i}{\partial q^m}\gamma^m_{kj} + \sum_m \frac{\partial \gamma^i_{jm}}{\partial q^k}v^m + \sum_m \gamma^i_{jm}\frac{\partial v^m}{\partial q^k}\\ &\quad + \sum_m \gamma^i_{km}\frac{\partial v^m}{\partial q^j} + \sum_{\ell m}\gamma^i_{km}\gamma^m_{j\ell}v^\ell - \sum_m \gamma^m_{jk}\frac{\partial v^i}{\partial q^m} - \sum_{\ell m}\gamma^m_{jk}\gamma^i_{m\ell}v^\ell\\ &= d^2v^i(\vec{e}_k,\vec{e}_j) + \sum_m(\gamma^i_{jm}\frac{\partial v^m}{\partial q^k} + \gamma^i_{km}\frac{\partial v^m}{\partial q^j}) + \sum_m \frac{\partial\gamma^i_{jm}}{\partial q^k}v^m + \sum_{\ell m}(\gamma^i_{km}\gamma^m_{j\ell} - \gamma^m_{jk}\gamma^i_{m\ell})v^\ell\end{aligned} \tag{T.69}$$

Cartesian case: $(v^i_{|j})_{|k} = \frac{\partial^2 v^i}{\partial x^j\partial x^k}$ and $d^2\vec{v} = \sum_{ij}\frac{\partial^2 v^i}{\partial x^j\partial x^k}dx_i\otimes dx^j\otimes dx^k$.

## T.10 Divergence of a vector field: Objective

$C^1(\Omega;\vec{\mathbb{R}^n})$ is the set vector fields in $\Omega$, and $\mathrm{Tr} : \mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^n}) \to \mathbb{R}$ is the trace operator.

**Definition T.40** The divergence operator is

$$\mathrm{div} := \mathrm{Tr}\circ d : \begin{cases} C^1(\Omega;\vec{\mathbb{R}^n}) & \to C^0(\Omega;\mathbb{R})\\ \qquad\vec{w} & \to \mathrm{div}\vec{w} := \mathrm{Tr}(d\vec{w}),\end{cases} \tag{T.70}$$

i.e. $\mathrm{div}\vec{w}(p) = \mathrm{Tr}(d\vec{w}(p))$ is the trace of the endomorphism $d\vec{w}(p) \in \mathcal{L}(\vec{\mathbb{R}^n};\vec{\mathbb{R}^n})$.

Tr and $d$ are linear, hence $\mathrm{div} = \mathrm{Tr}\circ d$ is linear.

**Quantification:** $(\vec{e}_i(p))$ is a basis at $p$, $\vec{w} = \sum_i w^i\vec{e}_i$ and $d\vec{w} = \sum_{ij} w^i_{|j}\vec{e}_i\otimes e^j$ give (contraction of $i$ and $j$)

$$(\mathrm{Tr}(d\vec{w}) =)\quad \mathrm{div}\vec{w} = \sum_{i=1}^n w^i_{|i} \quad (\overset{(T.59)}{=} \sum_{i=1}^n \frac{\partial w^i}{\partial q^i} + \sum_{i,k=1}^n \gamma^i_{ik}w^k). \tag{T.71}$$

(Cartesian basis: $\mathrm{div}\vec{w} = \sum_i \frac{\partial w^i}{\partial x^i}$). In particular $d\vec{e}_i =^{(T.52)} \sum_{jk}\gamma^k_{ji}\vec{e}_k\otimes e^j$ gives

$$\mathrm{div}\vec{e}_i = \sum_{j=1}^n \gamma^j_{ji}. \tag{T.72}$$

(If you prefer $\vec{e}_\ell = \sum_i \delta^i_\ell\vec{e}_i$, thus (T.71) gives $\mathrm{div}\vec{e}_\ell = \sum_i(\delta^i_\ell)_{|i} = 0 + \sum_{ik}\gamma^i_{ik}\delta^k_\ell = \sum_i \gamma^i_{i\ell}$.)

**Proposition T.41** *The divergence* $\mathrm{div}\vec{w}$ *of a* $C^1$ *vector field* $\vec{w}$ *is objective (is an invariant): Same value for all observers (intrinsic to* $\vec{w}$*).*

**Proof.** The differential and the trace are objective. (Computation: $\vec{w} = \sum_i u^i\vec{a}_i = \sum_i v^i\vec{b}_i$ gives $v^i_{|j} = \sum_{k\ell} Q^i_k u^k_{|\ell}P^\ell_j$, see (T.60), thus $\sum_i v^i_{|i} = \sum_{ik\ell} P^\ell_i Q^i_k u^k_{|\ell} = \sum_{k\ell}\delta^\ell_k u^k_{|\ell} = \sum_k u^k_{|k}$.) ∎

**Exercice T.42** Prove:

$$\mathrm{div}(f\vec{w}) = df.\vec{w} + f\,\mathrm{div}\vec{w}. \tag{T.73}$$

**Answer**. $d(f\vec{w}) = \vec{w}\otimes df + f\,d\vec{w}$ gives $\mathrm{Tr}(d(f\vec{w})) = \mathrm{Tr}(\vec{w}\otimes df) + \mathrm{Tr}(f\,d\vec{w}) = df.\vec{w} + f\,\mathrm{Tr}(d\vec{w})$. Use a coordinate system if you prefer. ∎

**Remark T.43** If $\alpha = \sum_{i=1}^n \alpha_i e^i$ is a differential form then $d\alpha = \sum_{i=1}^n \alpha_{i|j}e^i\otimes e^j$ where $\alpha_{i|j} := \vec{e}_i.d\alpha.\vec{e}_j$. Here it is impossible to define an objective trace of $d\alpha$ like $\sum_{i=1}^n \alpha_{i|i}$: The result depends on the choice of the basis (the Einstein convention is not satisfied, and e.g. with a Euclidean basis the result depends on the choice of unit of length: Foot? Metre?). Thus the objective (or intrinsic) divergence of a differential form is a nonsense. Similarly the trace of an inner dot product $(\cdot,\cdot)_g$ is a nonsense. ∎

## T.11 Objective divergence for 1 1 tensors

Analysis: Consider a second order $\binom{1}{1}$ tensor $\underline{\underline{\tau}} = \sum_{ij} \tau^i_j \vec{e}_i \otimes e^j \in T^1_1(U)$. Hence $\underline{\underline{\tau}} = \sum_{ij} \tau^i_j \vec{e}_i \otimes e^j \in T^1_1(U)$, and to create an objective divergence ("a trace of something") we have not choice an admissible index with the "differential index $k$": Only $i$ can be contacted with $k$, and we get the differential form

$$\begin{cases} \widetilde{\mathrm{div}}\underline{\underline{\tau}} := \displaystyle\sum_{i,j=1}^n \tau^i_{j|i} e^j, \quad \text{i.e.} \quad [\widetilde{\mathrm{div}}\underline{\underline{\tau}}]_{|\vec{e}} = \begin{pmatrix} \displaystyle\sum_{i=1}^n \tau^i_{1|i} & \dots & \displaystyle\sum_{i=1}^n \tau^i_{n|i} \end{pmatrix} \text{ (row matrix)}, \\ \quad \text{where} \quad \tau^i_{j|i} = \dfrac{\partial \tau^i_j}{\partial q^i} + \displaystyle\sum_{\ell=1}^n \gamma^i_{i\ell} \tau^\ell_j - \sum_{\ell=1}^n \tau^i_\ell \gamma^\ell_{ji}, \end{cases} \tag{T.74}$$

cf. (T.68). To get a qualitative (objective result) and justify (T.74):

**Definition T.44** (qualitative.) Let $\vec{u} \in \Gamma(\Omega)$ and $\ell \in \Omega^1(\Omega)$ be $C^1$. The objective divergence of the elementary $\binom{1}{1}$ tensor $\vec{u} \otimes \ell \in T^1_1(U)$ is the differential form $\widetilde{\mathrm{div}}(\vec{u} \otimes \ell) \in \Omega^1(\Omega)$ defined by

$$\widetilde{\mathrm{div}}(\vec{u} \otimes \ell) := (\mathrm{div}\vec{u})\ell + d\ell.\vec{u}, \quad \text{i.e.} \quad \widetilde{\mathrm{div}}(\vec{u} \otimes \ell).\vec{w} := (\mathrm{div}\vec{u})(\ell.\vec{w}) + (d\ell.\vec{u}).\vec{w} \tag{T.75}$$

for all $\vec{w} \in E$. (No basis and no inner dot product needed.) And the objective divergence operator is the linear operator

$$\widetilde{\mathrm{div}} : \left\{ \begin{aligned} T^1_1(U) &\to \Omega^1(U) \\ \underline{\underline{\tau}} &\to \widetilde{\mathrm{div}}\underline{\underline{\tau}} \end{aligned} \right\} \tag{T.76}$$

defined by (T.75) on elementary tensors.

Hence (T.75) gives, with $f \in C^1(\Omega;\mathbb{R})$, $\widetilde{\mathrm{div}}(f\vec{u} \otimes \ell) = \widetilde{\mathrm{div}}(f\vec{u})\ell + f\, d\ell.\vec{u}$, with $\widetilde{\mathrm{div}}(f\vec{u}) = df.\vec{u} + f\,\mathrm{div}\vec{u}$, thus

$$\widetilde{\mathrm{div}}(f\vec{u} \otimes \ell) = (df.\vec{u})\ell + f\,(\mathrm{div}\vec{u})\,\ell + f\,d\ell.\vec{u} = df.(\vec{u} \otimes \ell) + f\,\widetilde{\mathrm{div}}(\vec{u} \otimes \ell). \tag{T.77}$$

Thus, a tensor being a finite sum of elementary tensors (in $\vec{\mathbb{R}^n}$) and $\widetilde{\mathrm{div}}$ being linear, we get

$$\widetilde{\mathrm{div}}(f\underline{\underline{\tau}}) = df.\underline{\underline{\tau}} + f\,\widetilde{\mathrm{div}}(\underline{\underline{\tau}}). \tag{T.78}$$

**Quantification:** Coordinate system basis $(\vec{e}_i(p))$ at $p$, dual basis $(e^i(p))$. (T.77) gives $\widetilde{\mathrm{div}}(f\vec{e}_i \otimes e^j) = (df.\vec{e}_i)e^j + f\,(\mathrm{div}\vec{e}_i)\,e^j + f\,de^j.\vec{e}_i$, thus with (T.72) and (T.61),

$$\widetilde{\mathrm{div}}(\vec{e}_i \otimes e^j) = \sum_k \gamma^k_{ki} e^j - \sum_k \gamma^j_{ki} e^k, \quad \text{and} \quad \widetilde{\mathrm{div}}(f\vec{e}_i \otimes e^j) = \frac{\partial f}{\partial q^i} e^j + f \sum_k \gamma^k_{ki} e^j - f \sum_k \gamma^j_{ki} e^k. \tag{T.79}$$

Thus, div being linear, $\underline{\underline{\tau}} = \sum_{ij} \tau^i_j \vec{e}_i \otimes e^j$ gives $\widetilde{\mathrm{div}}\underline{\underline{\tau}} = \sum_{ij} \widetilde{\mathrm{div}}(\tau^i_j \vec{e}_i \otimes e^j) = \sum_{ij} \frac{\partial \tau^i_j}{\partial q^i} e^j + \sum_{ijk} \tau^i_j \gamma^k_{ki} e^j - \sum_{ijk} \tau^i_j \gamma^j_{ki} e^k = \sum_{i,j=1}^n \Big( \frac{\partial \tau^i_j}{\partial q^i} + \sum_{k=1}^n \gamma^k_{ki} \tau^i_j - \sum_{k=1}^n \tau^i_k \gamma^k_{ji} \Big) e^j = \sum_{i,j=1}^n \tau^i_{j|i} e^j$, i.e. (T.74) as announced, i.e. contraction of the indices $i$ and $k$ in $d\underline{\underline{\tau}} = \sum_{ijk} \tau^i_{j|k} \vec{e}_i \otimes e^j \otimes e^k$.

In particular

$$\widetilde{\mathrm{div}}(I) = 0, \tag{T.80}$$

since $I = \sum_i \vec{e}_i \otimes e^i$ and thus $\widetilde{\mathrm{div}}(I) = \widetilde{\mathrm{div}}(\sum_i \vec{e}_i \otimes e^i) = \sum_{ik} \gamma^k_{ki} e^i - \sum_{ik} \gamma^i_{ki} e^k = \sum_{ik} \gamma^k_{ki} e^i - \sum_{ik} \gamma^k_{ik} e^i = 0$ thanks to $\gamma^k_{ij} = \gamma^k_{ji}$ for all $i,j,k$, $(\vec{e}_i)$ being a basis of a coordinate system.

And $\vec{w} = \sum_i w^i \vec{e}_i$ gives $\widetilde{\mathrm{div}}(\underline{\underline{\tau}}).\vec{w} = \sum_{ij} \tau^i_{j|i} w^j = [\widetilde{\mathrm{div}}(\underline{\underline{\tau}})]_{|\vec{e}}.[\vec{w}]_{|\vec{e}}$, product of the row matrix $[\mathrm{div}(\underline{\underline{\tau}})]_{|\vec{e}} \in \mathcal{M}_{1n}$ (matrix of a differential form) and of the column matrix $[\vec{w}]_{|\vec{e}} \in \mathcal{M}_{n1}$.

**Exercice T.45** Prove: If $\underline{\underline{\tau}} \in T^1_1(U)$ and $\vec{w} \in \Gamma(\Omega)$ then

$$\boxed{\mathrm{div}(\underline{\underline{\tau}}.\vec{w}) = \widetilde{\mathrm{div}}(\underline{\underline{\tau}}).\vec{w} + \underline{\underline{\tau}} \odot d\vec{w}}. \tag{T.81}$$

**Answer**. $\underline{\underline{\tau}} = \sum_{ij} \tau^i_j \vec{e}_i \otimes e^j$ and $\vec{w} = \sum_i w^i \vec{e}_i$ give $\underline{\underline{\tau}}.\vec{w} = \sum_{ij} \tau^i_j w^j \vec{e}_i$, thus $\mathrm{div}(\underline{\underline{\tau}}.\vec{w}) = \sum_{ij} \tau^i_{j|i} w^j + \tau^i_j w^j_{|i}$. ∎

**Exercice T.46** Prove if $f$ is $C^1$ and $\vec{v}$ is $C^2$ then:

$$\widetilde{\operatorname{div}}(fI) = df, \quad \text{e.g.} \quad \widetilde{\operatorname{div}}((\operatorname{div}\vec{v})I) = d(\operatorname{div}\vec{v}). \tag{T.82}$$

**Answer**. (T.78) gives $\widetilde{\operatorname{div}}(fI) = df.I + f\widetilde{\operatorname{div}}(I) = df + 0$. Cartesian framework: $\vec{v} = \sum_i v^i \vec{e}_i$, $d\vec{v} = \sum_{ij} \frac{\partial v^i}{\partial x^j} \vec{e}_i \otimes e^j$, $\operatorname{div}\vec{v} = \sum_i \frac{\partial v^i}{\partial x^i}$, $d(\operatorname{div}\vec{v}) = \sum_{ij} \frac{\partial^2 v^i}{\partial x^i \partial x^j} e^j$; And $d^2\vec{v} = \sum_{ijk} \frac{\partial^2 v^i}{\partial x^j \partial x^k} \vec{e}_i \otimes e^j \otimes e^k$ and $\widetilde{\operatorname{div}}(d\vec{v}) = \sum_{ij} \frac{\partial^2 v^i}{\partial x^i \partial x^j} e^j$. Thus $d(\operatorname{div}\vec{v}) = \widetilde{\operatorname{div}}(d\vec{v})$. ▪

**Exercice T.47** For $\underline{\underline{\tau}} \in T_1^1(U)$, check with component calculations that $\widetilde{\operatorname{div}}(\underline{\underline{\tau}}) \in T_1^0(U)$ is objective:

$$[\widetilde{\operatorname{div}}(\underline{\underline{\tau}})]_{|b} = [\widetilde{\operatorname{div}}(\underline{\underline{\tau}})]_{|a}.P \quad \text{(covariance formula)}, \tag{T.83}$$

where $P$ is the transition matrix from a basis $(\vec{a}_i)$ to a basis $(\vec{b}_i)$ (and call $Q = P^{-1}$).

**Answer**. $\underline{\underline{\tau}} = \sum_{ij} \tau^i_j \vec{b}_i \otimes b^j = \sum_{\lambda\mu} \sigma^\lambda_\mu \vec{a}_\lambda \otimes a^\mu = \sum_{ij\lambda\mu} \sigma^\lambda_\mu (Q^i_\lambda \vec{b}_i) \otimes (P^\mu_j b^j)$, so $\tau^i_j = \sum_{\lambda\mu} Q^i_\lambda \sigma^\lambda_\mu P^\mu_j$.

1- Cartesian bases: $\sum_i \tau^i_{j|i} = \sum_i d\tau^i_j.\vec{b}_i = \sum_i d(\sum_{\lambda\mu} Q^i_\lambda \sigma^\lambda_\mu P^\mu_j).(\sum_\nu P^\nu_i.\vec{a}_\nu) = \sum_{i\lambda\mu\nu} Q^i_\lambda P^\mu_j P^\nu_i (d\sigma^\lambda_\mu.\vec{a}_\nu) = \sum_{\lambda\mu\nu} \delta^\nu_\lambda P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\nu) = \sum_{\lambda\mu} P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\lambda) = \sum_\mu (\sum_\lambda \sigma^\lambda_{\mu|\lambda}) P^\mu_j$, i.e. (T.83).

2- Coordinate system bases: $\sum_i \tau^i_{j|i} = \sum_i d\tau^i_j.\vec{e}_i + \sum_{i\ell} \tau^\ell_j \gamma^i_{i\ell,b} - \sum_{i\ell} \tau^i_\ell \gamma^\ell_{ij,b}$ (with $j$ fixed); With

$$\begin{aligned}
\sum_i (d\tau^i_j.\vec{b}_i) &= \sum_{i\lambda\mu} Q^i_\lambda\,(d\sigma^\lambda_\mu.\vec{b}_i)\,P^\mu_j + \sum_{i\lambda\mu} (dQ^i_\lambda.\vec{b}_i)\,\sigma^\lambda_\mu\,P^\mu_j + \sum_{i\lambda\mu} Q^i_\lambda\,\sigma^\lambda_\mu\,(dP^\mu_j.\vec{b}_i)\\
&= \sum_{i\lambda\mu\nu} Q^i_\lambda P^\mu_j P^\nu_i (d\sigma^\lambda_\mu.\vec{a}_\nu) + \sum_{i\lambda\mu\nu} \sigma^\lambda_\mu\,P^\mu_j P^\nu_i (dQ^i_\lambda.\vec{a}_\nu) + \sum_{i\lambda\mu\nu} \sigma^\lambda_\mu\,Q^i_\lambda P^\nu_i (dP^\mu_j.\vec{a}_\nu)\\
&= \sum_{\lambda\mu} P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\lambda) - \sum_{i\lambda\mu\nu} \sigma^\lambda_\mu\,P^\mu_j Q^i_\lambda (dP^\nu_i.\vec{a}_\nu) + \sum_{\lambda\mu} \sigma^\lambda_\mu\,(dP^\mu_j.\vec{a}_\lambda)
\end{aligned}$$

since $P^\nu_i Q^i_\lambda = \delta^\nu_\lambda$ gives $P^\nu_i (dQ^i_\lambda.\vec{a}_\nu) - Q^i_\lambda (dP^\nu_i.\vec{a}_\nu)$. And, with (T.56),

$$\begin{aligned}
\sum_{i\ell} \tau^\ell_j \gamma^i_{i\ell,b} &= \sum_{i\ell} (\sum_{\lambda\mu} Q^\ell_\lambda \sigma^\lambda_\mu P^\mu_j)(\sum_{\alpha\beta\omega} Q^i_\alpha P^\beta_i P^\omega_\ell \gamma^\alpha_{\beta\omega,a} + \sum_{\alpha\beta} Q^i_\alpha P^\beta_i (dP^\alpha_\ell.\vec{a}_\beta))\\
&= \sum_{\lambda\mu\alpha} \sigma^\lambda_\mu P^\mu_j \gamma^\alpha_{\alpha\lambda,a} + \sum_{\ell\lambda\mu\alpha} \sigma^\lambda_\mu Q^\ell_\lambda P^\mu_j (dP^\alpha_\ell.\vec{a}_\alpha),
\end{aligned} \tag{T.84}$$

and

$$\begin{aligned}
-\sum_{i\ell} \tau^i_\ell \gamma^\ell_{ij,b} &= -\sum_{i\ell} (\sum_{\lambda\mu} Q^i_\lambda \sigma^\lambda_\mu P^\mu_\ell)(\sum_{\alpha\beta\omega} P^\alpha_i P^\beta_j Q^\ell_\omega \gamma^\omega_{\alpha\beta,a} + \sum_{\alpha\omega} P^\alpha_i Q^\ell_\omega (dP^\omega_j.\vec{a}_\alpha))\\
&= -\sum_{\lambda\mu\beta} \sigma^\lambda_\mu P^\beta_j \gamma^\mu_{\lambda\beta,a} - \sum_{\lambda\mu} \sigma^\lambda_\mu (dP^\mu_j.\vec{a}_\lambda).
\end{aligned} \tag{T.85}$$

Thus $\sum_i \tau^i_{j|i} = \sum_{\lambda\mu} P^\mu_j (d\sigma^\lambda_\mu.\vec{a}_\lambda) + \sum_{\lambda\mu\alpha} \sigma^\lambda_\mu P^\mu_j \gamma^\alpha_{\alpha\lambda,a} - \sum_{\lambda\mu\beta} \sigma^\lambda_\mu P^\beta_j \gamma^\mu_{\lambda\beta,a} = \sum_{\lambda\mu} P^\mu_j \sigma^\lambda_{\mu|\lambda}$, i.e. (T.83). ▪

**Remark T.48** Divergence of a 2 0 tensor. Let $\underline{\underline{\tau}} \in T^2_0(U)$ and $\underline{\underline{\tau}} = \sum_{i,j=1}^n \tau^{ij} \vec{e}_i \otimes \vec{e}_j$, thus $d\underline{\underline{\tau}} = \sum_{i,j,k=1}^n \tau^{ij}_{\ |k} \vec{e}_i \otimes \vec{e}_j \otimes e^k$; Then two objective divergences may be defined: by contracting $k$ with $i$, or $k$ with $j$. (The Einstein convention is then satisfied.) ▪

**Remark T.49** Divergence of a 0 2 tensor. Let $\underline{\underline{\tau}} = \sum_{i,j=1}^n \tau_{ij} e^i \otimes e^j \in T^0_2(U)$. Thus $d\underline{\underline{\tau}} = \sum_{i,j,k=1}^n \tau_{ij|k} e^i \otimes e^j \otimes e^k$, and there are no indices to contract to satisfy Einstein convention: There is no objective divergence of 0 2 tensors (e.g. metric). ▪

**Remark T.50** Divergence of a 1 2 tensor. Let $\underline{\underline{\underline{T}}} = \sum_{i,j,k=1}^n T^i_{jk} \vec{e}_i \otimes e^j \otimes e^j \in T^1_2(U)$. Thus $d\underline{\underline{\underline{T}}} = \sum_{i,j,k,\ell=1}^n T^i_{jk|\ell} \vec{e}_i \otimes e^j \otimes e^k \otimes e^\ell$, thus, to get an objective quantity, the only admissible index to be contracted with the "differential index $\ell$" is $i$. So, no choice, objective quantity

$$\operatorname{div}(\underline{\underline{\underline{T}}}) = \sum_{i,j,k=1}^n T^i_{jk|i} e^j \otimes e^k. \tag{T.86}$$

(Objective definition given by $\operatorname{div}(\vec{u} \otimes \ell \otimes m) = (\operatorname{div}\vec{u})\ell \otimes m + (d\ell.\vec{u}) \otimes m + \ell \otimes (dm.\vec{u})$ for all $\vec{u} \in \Gamma(U)$, $\ell, m \in \Omega^1(U)$, and linearity of div.) ▪

## T.12 Divergence of a column or row matrix

**Definition T.51** For classical mechanics calculations,

$$\text{If } [\vec{v}] = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \quad \text{or} \quad [\vec{v}] = (\, v_1 \quad \dots \quad v_n \,) \quad \text{then} \quad \operatorname{div}([\vec{v}]) := \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}. \tag{T.87}$$

(Unfortunately sometimes called "divergences of a tensor" in classical mechanics.)

## T.13 Euclidean framework and "usual divergence" of a "tensor" (subjective)

**Definition T.52** The divergence of a matrix fonction $M = [M_{ij}] \in \mathcal{M}_{nn}$ is the column matrix $\operatorname{div}(M) \in \mathcal{M}_{n1}$ given by

$$\operatorname{div}(M) := \begin{pmatrix} \operatorname{div}(\vec{M}_1) \\ \vdots \\ \operatorname{div}(\vec{M}_n) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} \frac{\partial M_{1j}}{\partial x^j} \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial M_{nj}}{\partial x^j} \end{pmatrix} \quad \text{where} \quad \vec{M}_i := (\, M_{i1} \quad \dots \quad M_{in} \,) \text{ (row matrix).} \tag{T.88}$$

So: Take the divergences of the "row vectors" $\vec{M}_i$ of $M$ to make the "column vector" $\operatorname{div}(M)$.

**Definition T.53** In continuum mechanics, the usual (classical) divergence $\operatorname{div}\underline{\underline{\sigma}}$ of a so-called "tensor" $\underline{\underline{\sigma}} = [\sigma_{ij}]$ is the column matrix

$$\operatorname{div}\underline{\underline{\sigma}} : \stackrel{(T.88)}{=} \begin{pmatrix} \operatorname{div}(\vec{\sigma}_1) \\ \vdots \\ \operatorname{div}(\vec{\sigma}_n) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} \frac{\partial \sigma_{1j}}{\partial x_j} \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial \sigma_{nj}}{\partial x_j} \end{pmatrix} \quad \text{where} \quad \vec{\sigma}_i := (\, \sigma_{i1} \quad \dots \quad \sigma_{in} \,) \text{ (row matrix),} \tag{T.89}$$

where (T.87) is used. More precisely, if $\underline{\underline{\sigma}}$ is a order 2 tensor of any kind (covariant, contravariant, mixed), then relative to a basis $(\vec{e}_i)$ the notation $\operatorname{div}\underline{\underline{\sigma}}$ means $\operatorname{div}([\underline{\underline{\sigma}}]_{|\vec{e}})$:

$$\operatorname{div}\underline{\underline{\sigma}} := \operatorname{div}([\underline{\underline{\sigma}}]_{|\vec{e}}) = \begin{pmatrix} \sum_{j=1}^{n} \frac{\partial \sigma_{1j}}{\partial x^j} \\ \vdots \\ \sum_{j=1}^{n} \frac{\partial \sigma_{nj}}{\partial x^j} \end{pmatrix} = \begin{pmatrix} \operatorname{div}(\vec{\sigma}_1) \\ \vdots \\ \operatorname{div}(\vec{\sigma}_n) \end{pmatrix} \in \mathcal{M}_{n1} \quad \text{where} \quad \vec{\sigma}_i := (\, \sigma_{i1} \quad \dots \quad \sigma_{in} \,) \in \mathcal{M}_{1n}. \tag{T.90}$$

And $\operatorname{div}\underline{\underline{\sigma}}$ is never a vector (neither contravariant nor covariant), see next exercise: It is just a matrix.

Remark, particular case $\underline{\underline{\sigma}}$ is $\binom{1}{1}$ tensor: we have $\operatorname{div}\underline{\underline{\sigma}} = [\widetilde{\operatorname{div}\underline{\underline{\sigma}}}]_{\vec{e}}^T$.

**Exercice T.54** Prove: For any order two tensor $\underline{\underline{\sigma}}$, with $[\underline{\underline{\sigma}}]_{|\vec{e}} = [\sigma_{ij}]$,

$$\operatorname{div}\underline{\underline{\sigma}} = \sum_{i,j=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_j} \vec{e}_i \text{ is not a vector of any kind.} \tag{T.91}$$

**Answer**. We have to prove that: If $(\vec{a}_i)$ and $(\vec{b}_i)$ are bases, if $P$ is the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, then

$$\text{neither} \quad [\operatorname{div}\underline{\underline{\sigma}}]_{|\vec{b}} \neq P^{-1}.[\operatorname{div}\underline{\underline{\sigma}}]_{|\vec{a}} \quad \text{nor} \quad [\operatorname{div}\underline{\underline{\sigma}}]_{|\vec{b}}^T = [\operatorname{div}\underline{\underline{\sigma}}]_{|\vec{a}}^T.P, \tag{T.92}$$

i.e. the divergence as defined in (T.91) is neither contravariant nor covariant (does not satisfy any change of basis formula). (Compare with (T.83))

Consider the simple case $\vec{b}_i = \lambda\vec{a}_i$, for all $i$, $\lambda > 1$: Transition matrix $P = \lambda I$, and $P^{-1} = \frac{1}{\lambda} I$.

For a $\binom{1}{1}$ tensor: $\underline{\underline{\sigma}} = \sum_{ij} (\sigma_b)^i_j \vec{b}_i \otimes b^j = \sum_{ij} (\sigma_a)^i_j \vec{a}_i \otimes a^j$, $[\underline{\underline{\sigma}}]_{|\vec{b}} = P^{-1}.[\underline{\underline{\sigma}}]_{|\vec{a}}.P = \frac{1}{\lambda}.[\underline{\underline{\sigma}}]_{|\vec{a}}.\lambda = [\underline{\underline{\sigma}}]_{|\vec{a}}$, i.e. $(\sigma_a)^i_j = (\sigma_b)^i_j$ for all $i,j$. Thus (T.91) gives $\operatorname{div}_b\underline{\underline{\sigma}} = \sum_{ij} (d(\sigma_b)^i_j.\vec{b}_j)\vec{b}_i = \sum_{ij} (d(\sigma_a)^i_j.(\lambda\vec{a}_j))(\lambda\vec{a}_i) = \lambda^2 \operatorname{div}_a\underline{\underline{\sigma}}$. Thus $[\operatorname{div}_b\underline{\underline{\sigma}}]_{|\vec{b}} \neq P^{-1}.[\operatorname{div}_b\underline{\underline{\sigma}}]_{|\vec{a}}$ and $[\operatorname{div}_b\underline{\underline{\sigma}}]_{|\vec{b}}^T \neq [\operatorname{div}_a\underline{\underline{\sigma}}]_{|\vec{a}}^T.P$.

For a $\binom{0}{2}$ tensor: $\underline{\underline{\sigma}} = \sum_{ij} \sigma_{b,ij} b^i \otimes b^j = \sum_{ij} \sigma_{a,ij} a^i \otimes a^j$, and $[\underline{\underline{\sigma}}]_{|\vec{b}} = P^T.[\underline{\underline{\sigma}}]_{|\vec{a}}.P = \lambda^2 [\underline{\underline{\sigma}}]_{|\vec{a}}$, i.e. $\sigma_{b,ij} = \lambda^2 \sigma_{a,ij}$ for all $i,j$. Thus (T.91) gives $\operatorname{div}_b\underline{\underline{\sigma}} = \sum_{ij} (d\sigma_{b,ij}.\vec{b}_j)\vec{b}_i = \lambda^2 \sum_{ij} (d\sigma_{a,ij}.(\lambda\vec{a}_j))(\lambda\vec{a}_i) = \lambda^4 \operatorname{div}_a\underline{\underline{\sigma}}$. Thus $[\operatorname{div}_b\underline{\underline{\sigma}}]_{|\vec{b}} \neq P^{-1}.[\operatorname{div}_b\underline{\underline{\sigma}}]_{|\vec{a}}$ and $[\operatorname{div}_b\underline{\underline{\sigma}}]_{|\vec{b}}^T \neq [\operatorname{div}_a\underline{\underline{\sigma}}]_{|\vec{a}}^T.P$.

For a $\binom{2}{0}$ tensor: $\underline{\underline{\sigma}} = \sum_{ij} \sigma_b^{ij} \vec{b}_i \otimes \vec{b}_j = \sum_{ij} \sigma_a^{ij} \vec{a}_i \otimes \vec{a}_j$, and $[\underline{\underline{\sigma}}]_{|\vec{b}} = P^{-T}.[\underline{\underline{\sigma}}]_{|\vec{a}}.P^{-1} = \frac{1}{\lambda^2}[\underline{\underline{\sigma}}]_{|\vec{a}}$, i.e. $\sigma_b^{ij} = \frac{1}{\lambda^2}\sigma_a^{ij}$ for all $i,j$. Thus (T.91) gives $\operatorname{div}_b\underline{\underline{\sigma}} = \sum_{ij} (d\sigma_b^{ij}.\vec{b}_j)\vec{b}_i = \frac{1}{\lambda^2} \sum_{ij} (d\sigma_a^{ij}.(\lambda\vec{a}_j))(\lambda\vec{a}_i) = \operatorname{div}_a\underline{\underline{\sigma}}$. Thus $[\operatorname{div}_b\underline{\underline{\sigma}}]_{|\vec{b}} \neq P^{-1}.[\operatorname{div}_b\underline{\underline{\sigma}}]_{|\vec{a}}$ and $[\operatorname{div}_b\underline{\underline{\sigma}}]_{|\vec{b}}^T \neq [\operatorname{div}_a\underline{\underline{\sigma}}]_{|\vec{a}}^T.P$. ∎

**Exercice T.55** Prove, with $[\vec{\text{grad}}p]_{|\vec{e}}$ abusively written $\vec{\text{grad}}p$,

$$\text{div}(pI) = \vec{\text{grad}}p, \quad \text{and} \quad \text{div}(\vec{\text{grad}}p) = \Delta p. \tag{T.93}$$

**Answer**. $\text{div}(pI) =^{(T.91)} \sum_{ij} \frac{\partial(p\delta_{ij})}{\partial x_j}\vec{e}_i = \sum_i \frac{\partial p}{\partial x_i}\vec{e}_i =^{\text{written}} \vec{\text{grad}}p$, which means $\text{div}(pI) = [dp]^T_{|\vec{e}} = [\vec{\text{grad}}p]_{|\vec{e}}$ (last equality thanks to the Euclidean framework). Thus $\text{div}(\vec{\text{grad}}p) =^{(T.87)} \sum_i \frac{\partial \frac{\partial p}{\partial x_i}}{\partial x_i} = \sum_i \frac{\partial^2 p}{\partial x_i^2} = \Delta p$. ▪

**Exercice T.56** Prove, when $\vec{v}$ is $C^2$, and $(\vec{E}_i)$ is the canonical basis in $\mathcal{M}_{n1}$,

$$\begin{cases} \text{div}(d\vec{v}) = \begin{pmatrix} \Delta v_1 \\ \vdots \\ \Delta v_n \end{pmatrix} = \sum_{i=1}^n (\Delta v_i)\vec{E}_i = \sum_{i,j=1}^n v^i_{|jj}\vec{E}_i \overset{\text{written}}{=} \Delta\vec{v} \overset{\text{written}}{=} \text{div}(\text{grad}\vec{v}) = \text{div}(\nabla\vec{v}), \\ \text{div}((\text{div}\vec{v})I) = \vec{\text{grad}}(\text{div}\vec{v}) = \begin{pmatrix} \frac{\partial(\text{div}\vec{v})}{\partial x_1} \\ \vdots \\ \frac{\partial(\text{div}\vec{v})}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \sum_j \frac{\partial^2 v_j}{\partial x_1 \partial x_j} \\ \vdots \\ \sum_j \frac{\partial^2 v_j}{\partial x_n \partial x_j} \end{pmatrix} = \sum_{i,j=1}^n \frac{\partial^2 v_j}{\partial x_i \partial x_j}\vec{E}_i. \end{cases} \tag{T.94}$$

**Answer**. Let $\overrightarrow{(d\vec{v})}_i = \left(\frac{\partial v^i}{\partial x^1} \ \ldots \ \frac{\partial v^i}{\partial x^n}\right) \in \mathcal{M}_{1n}$: Thus $\text{div}(\overrightarrow{(d\vec{v})}_i) = \sum_{ij} \frac{\partial \frac{\partial v^i}{\partial x^j}}{\partial x_j}\vec{e}_i = \sum_{ij} \frac{\partial^2 v^i}{\partial x_j^2}\vec{e}_i = \sum_i \Delta v^i \vec{e}_i =^{\text{written}} \Delta\vec{v}$. And $\text{div}((\text{div}\vec{v})I) =^{(T.93)} \vec{\text{grad}}(\text{div}\vec{v})$. ▪

**Exercice T.57** Prove: If $\vec{u} \in \Gamma(U)$ (vector field) and $\underline{\underline{\sigma}} \in T^1_1(U)$ then

$$[\text{div}(\underline{\underline{\sigma}}.\vec{u})]_{|\vec{e}} = \text{div}\underline{\underline{\sigma}}^T.[\vec{u}]_{|\vec{e}} + [\underline{\underline{\sigma}}]^T_{|\vec{e}} : [d\vec{u}]_{|\vec{e}}. \tag{T.95}$$

**Answer**. $\underline{\underline{\sigma}}.\vec{u}$ is a vector field, thus (T.81) gives $\text{div}(\underline{\underline{\sigma}}.\vec{u}) = \widetilde{\text{div}}(\underline{\underline{\sigma}}).\vec{u} + \underline{\underline{\sigma}} \odot d\vec{u}$, thus $[\text{div}(\underline{\underline{\sigma}}.\vec{u})]_{|\vec{e}} = [\widetilde{\text{div}}(\underline{\underline{\sigma}})]_{|\vec{e}}.[\vec{u}]_{|\vec{e}} + [\underline{\underline{\sigma}}]_{|\vec{e}} \odot [d\vec{u}]_{|\vec{e}} = [\text{div}(\underline{\underline{\sigma}})]^T.[\vec{u}]_{|\vec{e}} + [\underline{\underline{\sigma}}]^T_{|\vec{e}} : [d\vec{u}]_{|\vec{e}}$. Or direct calculation. ▪

# U Natural canonical isomorphisms

$E$ and $F$ are finite dimensional vector spaces, $E^* = \mathcal{L}(E;\mathbb{R})$ and $F^* = \mathcal{L}(F;\mathbb{R})$ are their dual spaces, $\mathcal{L}(E;F)$ is the space of linear maps, $\mathcal{L}_i(E;F)$ is its sub-space of invertible linear maps.

## U.1 The adjoint of a linear map

**Definition U.1** The adjoint of a linear map $\mathcal{P} \in \mathcal{L}(E;F)$ is the linear map $\mathcal{P}^* \in \mathcal{L}(F^*;E^*)$ canonically defined by

$$\mathcal{P}^* : \begin{cases} F^* & \to E^* \\ \ell & \to \mathcal{P}^*(\ell) := \ell \circ \mathcal{P}, \quad \text{written} \quad \mathcal{P}^*.\ell = \ell.\mathcal{P} \end{cases} \tag{U.1}$$

(dot notations since $\ell$, $\mathcal{P}$ and $\mathcal{P}^*$ are linear), i.e., for all $(\ell, \vec{u}) \in F^* \times E$,

$$(\mathcal{P}^*.\ell).\vec{u} = \ell.\mathcal{P}.\vec{u} \quad (\in \mathbb{R}). \tag{U.2}$$

Interpretation: If $\mathcal{P}$ is a push-forward of vector fields, then $\mathcal{P}^*$ is the pull-back of differential forms, see remark 7.5. And $\mathcal{P} \in \mathcal{L}_i(E;F)$ (linear and invertible) is a change of observer.

**Application:** $\mathcal{P}^{**} := (\mathcal{P}^*)^* \in \mathcal{L}(E^{**};F^{**})$ is given by, for all $u \in E^{**}$ and $\ell \in E^*$,

$$\mathcal{P}^{**}.u = u.\mathcal{P}^*, \quad \text{i.e.} \quad (\mathcal{P}^{**}.u).\ell = u.(\ell.\mathcal{P}) \quad (= u.\mathcal{P}^*.\ell). \tag{U.3}$$

## U.2 An isomorphism $E \simeq E^*$ is never natural (never objective)

Two observers A and B consider a linear map $L \in \mathcal{L}(E;E^*)$ (a "change of variance operator"), and they want to check their results: With $\mathcal{P} \in \mathcal{L}_i(E;E)$ the change of observer endomorphism from A to B, they ("naturally") consider the diagram

$$\begin{array}{ccccl} & E & \xrightarrow{L} & E^* & \leftarrow \text{computation by observer } A \\ \mathcal{P}\downarrow & & & \uparrow \mathcal{P}^* & \\ & E & \underset{L}{\longrightarrow} & E^* & \leftarrow \text{computation by observer } B \end{array} \tag{U.4}$$

**Definition U.2** (Spivak [23].) $L \in \mathcal{L}(E;E^*)$ is natural iff (U.4) commutes for all $\mathcal{P} \in \mathcal{L}(E;E)$:

$$L \in \mathcal{L}(E;E^*) \text{ is natural} \iff \forall \mathcal{P} \in \mathcal{L}(E;E),\ \ \mathcal{P}^* \circ L \circ \mathcal{P} = L. \tag{U.5}$$

In that case, if $A$ computes $L.\vec{u}$ (top line) and if $B$ computes $L.(\mathcal{P}.\vec{u})$ (bottom line) then they can easily check their results: They must have $L.\vec{u} = \mathcal{P}^*.(L.(\mathcal{P}.\vec{u}))$.

**Theorem U.3** *A non-zero linear map $L \in \mathcal{L}(E;E^*)$ is not natural (a change of variance operator is subjective):*

$$\exists \mathcal{P} \in \mathcal{L}(E;E) \quad s.t. \quad L \neq \mathcal{P}^* \circ L \circ \mathcal{P}. \tag{U.6}$$

*(So the observers can't check their results with a diagram like (U.4).)*

**Proof.** (Spivak [23].) Take $P = 2I \in \mathcal{L}(E;E)$. Thus $\mathcal{P}^*(\ell) =^{(U.1)} \ell \circ \mathcal{P} = 2\ell$ for all $\ell \in E^*$, thus $\mathcal{P}^* = 2I$, thus $\mathcal{P}^*.(L \circ \mathcal{P}) = 2I.(2L) = 4L \neq L$ when $L \neq 0$, thus (U.6). ▪

**Remark U.4** The definition U.2 was made to generalize: 1- A representation with matrices is not objective (see exercise U.5), 2- A Riesz-representation vector is not objective (see exercise U.6). ▪

**Exercice U.5** $\dim E = 1$, bases $(\vec{a})$ and $(\vec{b})$, $\vec{b} = \beta\vec{a}$, $\beta \neq \pm 1$ (change of unit of measurement), dual bases $(\pi_a)$ and $(\pi_b)$.

1- Consider the $L \in \mathcal{L}(E;E^*)$ that sends the basis $(\vec{a})$ onto its dual basis $(\pi_a)$, i.e. defined by $L.\vec{a} := \pi_a$. Prove: $L.\vec{b} \neq \pi_b$ (i.e. $L$ does not send $(\vec{b})$ onto its dual basis $(\pi_b)$), then $L$ is not natural.

2- Let $L_A$ and $L_B$ be defined by $L_A.\vec{a} = \pi_a$ and $L_B.\vec{b} = \pi_b$. Prove: $L_A \neq L_B$.

**Answer**. 1- $\vec{b} = \beta\vec{a}$ gives $\pi_{b1} = \frac{1}{\beta}\pi_a$, thus $L.\vec{b} = \beta L.\vec{a} = \beta\pi_a = \beta^2\pi_{b1} \neq \pi_{b1}$ since $\beta^2 \neq 1$. Here $P = \beta I$, $P^* = \beta I$, thus $\mathcal{P}^* \circ L \circ \mathcal{P} = \beta^2 L \neq L$ because $L \neq 0$ and $\beta^2 \neq 1$.

2- $\vec{b} = \beta\vec{a}$ gives $\pi_{b1} = \frac{1}{\beta}\pi_a$, thus $L_A.\vec{b}_j = \beta L_A.\vec{a}_j = \beta\pi_{aj} = \beta^2\pi_{bj} = \beta^2 L_B.\vec{b}_j \neq L_B.\vec{b}_j$ since $\beta^2 \neq 1$. ▪

**Exercice U.6** $E = \vec{\mathbb{R}^n}$, $(\cdot,\cdot)_g$ Euclidean dot product in $\vec{\mathbb{R}^n}$, and $\vec{R}_g \in \mathcal{L}(E^*;E)$ is the Riesz representation map $\vec{R}_g \in \mathcal{L}(E^*;E)$ i.e. defined by $\vec{R}_g(\ell) := \vec{\ell}_g$ where $(\vec{\ell}_g, \vec{v})_g = \ell.\vec{v}$ for all $\vec{v} \in \vec{\mathbb{R}^n}$, cf (F.3).

1- Prove: If $(\cdot,\cdot)_h = \lambda^2(\cdot,\cdot)_g$ is another inner dot product, $\lambda^2 \neq 1$, then $\vec{R}_h \neq \vec{R}_g$.

2- $\vec{R}_g^{-1} \in \mathcal{L}(E;E^*)$ is not natural ($\vec{R}_g$ and $\vec{R}_g^{-1}$ change the variance), and $\vec{R}_g \in \mathcal{L}(E^*;E)$ is not natural.

**Answer**. 1- $\vec{R}_g.\ell = \vec{\ell}_g$ and $\vec{R}_h.\ell = \vec{\ell}_h$ with $(\vec{\ell}_g, \vec{v})_g = \ell.\vec{v} = (\vec{\ell}_h, \vec{v})_h$ for all $\vec{v} \in \vec{\mathbb{R}^n}$. And $(\vec{\ell}_h, \vec{v})_h = \lambda^2(\vec{\ell}_h, \vec{v})_g = (\lambda^2\vec{\ell}_h, \vec{v})_g$ for all $\vec{v} \in \vec{\mathbb{R}^n}$, thus $\vec{\ell}_g = \lambda^2\vec{\ell}_h$. Thus $\vec{R}_g = \lambda^2\vec{R}_h \neq \vec{R}_h$ because $\lambda^2 \neq 1$.

2- Choose $\mathcal{P} = 2I$, thus $\mathcal{P}^* = 2I$. Thus $\mathcal{P}.\vec{R}_g^{-1}.\mathcal{P}^*.m = 4\vec{R}_g^{-1}.m \neq \vec{R}_g^{-1}.m$, thus$\vec{R}_g$ is not natural. To prove that $\vec{R}_g$ is not natural, we use the natural canonical isomorphism $E^{**} \simeq E$, see next § U.3: we get $\vec{R}_g \in \mathcal{L}(E^*, E) \simeq \mathcal{L}(E^*, (E^*)^*)$ and same steps as with $\vec{R}_g^{-1}$. ▪

## U.3 Natural canonical isomorphism $E \simeq E^{**}$

Let $E^{**} := (E^*)^*$ ($= \mathcal{L}(E^*;\mathbb{R})$). Two observers $A$ and $B$ consider a linear map $L \in \mathcal{L}(E;E^{**})$, and they want to check their results: With $\mathcal{P} \in \mathcal{L}_i(E;E)$ the change of observer endomorphism from A to B, they ("naturally") consider the diagram

$$\begin{array}{ccccl} & E & \xrightarrow{L} & E^{**} & \leftarrow \text{computation by observer } A, \\ \mathcal{P}\downarrow & & & \downarrow \mathcal{P}^{**} & \\ & E & \xrightarrow[L]{} & E^{**} & \leftarrow \text{computation by observer } B. \end{array} \tag{U.7}$$

In particular they consider the linear map $L = \mathcal{J}_E \in \mathcal{L}(E;E^{**})$ (the derivation operator in the direction $\vec{u}$) canonically defined by

$$\mathcal{J}_E : \begin{cases} E & \to E^{**} \\ \vec{u} & \to u = \mathcal{J}_E(\vec{u}), \quad u(\ell) := \ell.\vec{u}, \ \ \forall \ell \in E^*. \end{cases} \tag{U.8}$$

**Proposition U.7** *$\mathcal{J}_E$ is a natural canonical isomorphism, i.e., for all $\mathcal{P} \in \mathcal{L}(E;E)$,*

$$\text{the diagram}\quad \begin{array}{ccc} E & \stackrel{\mathcal{J}_E}{\longrightarrow} & E^{**} \\ \mathcal{P}\downarrow & & \downarrow \mathcal{P}^{**} \\ E & \underset{\mathcal{J}_E}{\longrightarrow} & E^{**} \end{array} \quad \text{commutes,}\quad \text{i.e.}\quad \mathcal{J}_E \circ \mathcal{P} = \mathcal{P}^{**} \circ \mathcal{J}_E, \tag{U.9}$$

*and we write $E \simeq E^{**}$. Thus we can use the unambiguous notation (observer independent)*

$$\mathcal{J}_E(\vec{u}) \stackrel{\text{written}}{=} \vec{u}, \quad \text{and} \quad \vec{u}.\ell = \ell.\vec{u}. \tag{U.10}$$

*More generally, if $F$ is a vector space s.t. $\dim F = \dim E$ then, for all $\mathcal{P} \in \mathcal{L}(E;F)$,*

$$\text{the diagram}\quad \begin{array}{ccc} E & \stackrel{\mathcal{J}_E}{\longrightarrow} & E^{**} \\ \mathcal{P}\downarrow & & \downarrow \mathcal{P}^{**} \\ F & \underset{\mathcal{J}_F}{\longrightarrow} & F^{**} \end{array} \quad \text{commutes,}\quad \text{i.e.}\quad \mathcal{J}_F \circ \mathcal{P} = \mathcal{P}^{**} \circ \mathcal{J}_E. \tag{U.11}$$

**Proof.** (Spivak [23].) $\mathcal{J}_E$ and $\mathcal{J}_F$ are linear and bijective (trivial): They are isomorphisms.

$(\mathcal{P}^{**}\circ\mathcal{J}_E(\vec{u}))(\ell) \stackrel{(U.3)}{=} \mathcal{J}_E(\vec{u})(\ell.\mathcal{P}) \stackrel{(U.8)}{=} (\ell.\mathcal{P})(\vec{u}) = \ell(\mathcal{P}(\vec{u})) \stackrel{(U.8)}{=} \mathcal{J}_F(\mathcal{P}(\vec{u}))(\ell)$, for all $\ell \in F^*$ and all $\vec{u} \in E$, thus $\mathcal{P}^{**} \circ \mathcal{J}_E(\vec{u}) = \mathcal{J}_F(\mathcal{P}(\vec{u}))$, for all $\vec{u} \in E$, thus $\mathcal{P}^{**} \circ \mathcal{J}_E = \mathcal{J}_F \circ \mathcal{P}$. ∎

**Notation:** $\mathcal{J}_E =^{\text{written}} \mathcal{J}$ and $\mathcal{J}_F =^{\text{written}} \mathcal{J}$ (implicit notations), thus, for all $\mathcal{P} \in \mathcal{L}(E;F)$,

$$\begin{array}{ccc} E & \stackrel{\mathcal{J}}{\longrightarrow} & E^{**} \\ \mathcal{P}\downarrow & & \downarrow \mathcal{P}^{**} \\ F & \underset{\mathcal{J}}{\longrightarrow} & F^{**} \end{array} \quad \text{commutes,}\quad \text{i.e.}\quad \mathcal{P}^{**} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{P}. \tag{U.12}$$

**Proposition U.8** *(A characterization of $\mathcal{J}$.) $\mathcal{J}$ sends any basis $(\vec{a}_i)$ onto its bidual basis.*

**Proof.** Basis $(\vec{a}_i)$, $(\pi_{ai})$ its dual basis. $\mathcal{J}(\vec{a}_j).\pi_{ai} =^{(U.8)} \pi_{ai}.\vec{a}_j = \delta_{ij}$ for all $i,j$, thus $(\mathcal{J}(\vec{a}_j))$ is the dual basis of $(\pi_{ai})$, thus the bidual basis of $(\vec{a}_i)$. True for all basis $(\vec{a}_i)$. ∎

## U.4 Natural canonical isomorphisms $\mathcal{L}(E;F) \simeq \mathcal{L}(F^*,E;\mathbb{R}) \simeq \mathcal{L}(E^*;F^*)$

Consider the canonical isomorphism

$$\mathcal{J}_{EF} : \begin{cases} \mathcal{L}(E;F) & \to \mathcal{L}(F^*,E;\mathbb{R}) \\ \quad L & \to \widetilde{L} = \mathcal{J}_{EF}(L), \quad \widetilde{L}(m,\vec{u}) := m.L.\vec{u}, \ , \ \forall (m,\vec{u}) \in F^* \times E. \end{cases} \tag{U.13}$$

**Exercice U.9** With $(\vec{a}_i)$ and $(\vec{b}_i)$ bases in $E$ and $F$, with $(a^i)$ and $(b^i)$ their (covariant) dual bases, prove:

$$L.\vec{a}_j = \sum_{i=1}^{n} L^i{}_j \vec{b}_i \quad \Longrightarrow \quad \widetilde{L} = \sum_{i,j=1}^{n} L^i{}_j \vec{b}_i \otimes a^j. \tag{U.14}$$

**Answer**. $\widetilde{L}(b^k,\vec{a}_\ell) =^{(U.13)} b^k.L.\vec{a}_\ell = L^k{}_\ell$. And $(\sum_{ij} L^i{}_j \vec{b}_i \otimes a^j)(b^k,\vec{a}_\ell) = \sum_{ij} L^i{}_j (\vec{b}_i.b^k)(a^j.\vec{a}_\ell) = \sum_{ij} L^i{}_j \delta_i^k \delta_\ell^j = L^k{}_\ell$. True for all $k,\ell$, thus $\widetilde{L} = \sum_{ij} L^i{}_j \vec{b}_i \otimes a^j$. ∎

$A, B$ are also finite dimensional vector spaces. Let $\mathcal{P}_1 \in \mathcal{L}_i(E;A)$ and $\mathcal{P}_2 \in \mathcal{L}(F;B)$, and consider the diagram

$$\begin{array}{ccc} \mathcal{L}(E;F) & \stackrel{\mathcal{J}_{EF}}{\longrightarrow} & \mathcal{L}(F^*,E;\mathbb{R}) \\ \mathcal{I}_{\mathcal{P}} \downarrow & & \downarrow \widetilde{\mathcal{I}_{\mathcal{P}}} \\ \mathcal{L}(A;B) & \underset{\mathcal{J}_{AB}}{\longrightarrow} & \mathcal{L}(B^*,A;\mathbb{R}) \end{array} \tag{U.15}$$

where $\mathcal{I}_{\mathcal{P}}$ and $\widetilde{\mathcal{I}_{\mathcal{P}}}$ are the push-forwards of the linear map $L \in \mathcal{L}(E;F)$ and of the bilinear form $\widetilde{L} \in \mathcal{L}(F^*,E;\mathbb{R})$ (we need $\mathcal{P}_1$ invertible), i.e.

$$\mathcal{I}_{\mathcal{P}}(L) := \mathcal{P}_2.L.\mathcal{P}_1^{-1} \quad \text{and} \quad \widetilde{\mathcal{I}_{\mathcal{P}}}(\widetilde{L})(b,\vec{a}) := \widetilde{L}(b.\mathcal{P}_2, \mathcal{P}_1^{-1}.\vec{a}) \quad \forall (b,\vec{a}) \in B^* \times A. \tag{U.16}$$

**Proposition U.10** *The canonical isomorphism $\mathcal{J}_{EF}$ is natural, that is, the diagram (U.15) commutes for all $\mathcal{P}_1 \in \mathcal{L}_i(E,A)$ and all $\mathcal{P}_2 \in \mathcal{L}(F,B)$:*

$$\widetilde{\mathcal{I}_{\mathcal{P}}} \circ \mathcal{J}_{EF} = \mathcal{J}_{AB} \circ \mathcal{I}_{\mathcal{P}}, \quad \text{and} \quad \mathcal{L}(E;F) \simeq \mathcal{L}(F^*,E;\mathbb{R}) \simeq \mathcal{L}(E;F). \tag{U.17}$$

**Proof.** $\mathcal{J}_{AB}(\mathcal{I}_{\mathcal{P}}(L))(b,\vec{a}) \overset{(U.13)}{=} b.\mathcal{I}_{\mathcal{P}}(L).\vec{a} \overset{(U.16)}{=} b.(\mathcal{P}_2.L.\mathcal{P}_1^{-1}).\vec{a} = (b.\mathcal{P}_2).L.(\mathcal{P}_1^{-1}.\vec{a}) \overset{(U.13)}{=} \mathcal{J}_{EF}(L)(b.\mathcal{P}_2,\mathcal{P}_1^{-1}.\vec{a}) \overset{(U.16)}{=} \widetilde{\mathcal{I}_{\mathcal{P}}}(\mathcal{J}_{EF}(L))(b,\vec{a})$, thus $\mathcal{L}(E;F) \simeq \mathcal{L}(F^*,E;\mathbb{R})$.

Thus $\mathcal{L}(E^*;F^*) \simeq \mathcal{L}((F^*)^*,E^*;\mathbb{R}) \simeq \mathcal{L}(F,E^*;\mathbb{R}) \simeq \mathcal{L}(E^{**};F) \simeq \mathcal{L}(E;F)$. ∎

Then consider the canonical isomorphism (transposed of a bilinear map)

$$\mathcal{K}_{EF} : \left\{ \begin{array}{rl} \mathcal{L}(E,F;\mathbb{R}) & \to \mathcal{L}(F,E;\mathbb{R}) \\ T & \to \mathcal{K}_{EF}(T) \end{array} \right\}, \quad \mathcal{K}_{EF}(T)(\vec{u},\vec{v}) := T(\vec{v},\vec{u}), \quad \forall(\vec{u},\vec{v}) \in E \times F, \tag{U.18}$$

and

$$Z_{AB} \left\{ \begin{array}{rl} \mathcal{L}(E,F;\mathbb{R}) & \to \mathcal{L}(A,B;\mathbb{R}) \\ T & \to Z_{AB}(T) \end{array} \right\}, \quad Z_{AB}(T)(\vec{a},\vec{b}) := T(\mathcal{P}_1^{-1}.\vec{a},\mathcal{P}_2^{-1}.\vec{b}), \quad \forall(\vec{a},\vec{b}) \in A \times B. \tag{U.19}$$

**Proposition U.11** *The canonical isomorphism $\mathcal{K}_{EF}$ is natural: For all $(\mathcal{P}_1,\mathcal{P}_2) \in \mathcal{L}_i(E;A) \times \mathcal{L}_i(F;B)$,*

$$\text{the diagram} \quad \begin{array}{ccc} \mathcal{L}(E,F;\mathbb{R}) & \overset{\mathcal{K}_{EF}}{\longrightarrow} & \mathcal{L}(F,E;\mathbb{R}) \\ Z_{AB}\downarrow & & \downarrow Z_{BA} \\ \mathcal{L}(A,B;\mathbb{R}) & \underset{\mathcal{K}_{AB}}{\longrightarrow} & \mathcal{L}(B,A;\mathbb{R}) \end{array} \quad \text{commutes:} \quad \mathcal{L}(E,F;\mathbb{R}) \simeq \mathcal{L}(F,E;\mathbb{R}). \tag{U.20}$$

**Proof.** $\mathcal{K}_{EF}(Z_{AB}(T))(\vec{b},\vec{a}) = Z_{AB}(T)(\vec{a},\vec{b}) = T(\mathcal{P}_2^{-1}.\vec{b},\mathcal{P}_1^{-1}.\vec{a})$ and $Z_{BA}(\mathcal{K}_{EF}(T))(\vec{a},\vec{b}) = \mathcal{K}_{EF}(T)(\mathcal{P}_1^{-1}.\vec{a},\mathcal{P}_2^{-1}.\vec{b}) = T(\mathcal{P}_2^{-1}.\vec{b},\mathcal{P}_1^{-1}.\vec{a})$, thus $\mathcal{K}_{AB} \circ Z_{AB} = Z_{BA} \circ \mathcal{K}_{EF}$. ∎

## U.5 Natural canonical isomorphisms $\mathcal{L}(E;\mathcal{L}(E;F)) \simeq \mathcal{L}(E,E;F) \simeq \mathcal{L}(F^*,E,E;\mathbb{R})$

E.g. for application to the second order derivative $d(d\vec{u}) \simeq d^2\vec{u}$. Let $\vec{u} \in T_0^1(U)$, and use the notations $d\vec{u} \in T_1^1(U)$ (since $d\vec{u}(p) \in \mathcal{L}(E;E) \simeq \mathcal{L}(E^*,E;\mathbb{R})$), then $d^2\vec{u} \in T_2^1(U)$, ..., $d^k\vec{u} \in T_k^1(U)$, ...

Consider the canonical isomorphisms

$$\mathcal{J}_{12E} : \left\{ \begin{array}{rl} \mathcal{L}(E;\mathcal{L}(E;F)) & \to \mathcal{L}(E,E;F) \\ T_1 & \to T_2 = \mathcal{J}_{12E}(T_1), \quad T_2(\vec{u}_1,\vec{u}_2) := T_1(\vec{u}_1).\vec{u}_2 \in F, \ \forall \vec{u}_1,\vec{u}_2 \in E, \end{array} \right. \tag{U.21}$$

and

$$\mathcal{J}_{23E} : \left\{ \begin{array}{rl} \mathcal{L}(E,E;F) & \to \mathcal{L}(F^*,E,E;\mathbb{R}) \\ T_2 & \to \mathcal{J}_{23E}(T_2) = T_3, \quad T_3(\ell,\vec{u},\vec{v}) := \ell.T_2(\vec{u}_1,\vec{u}_2), \ \forall \vec{u}_1,\vec{u}_2 \in E, \ \forall \ell \in F^*. \end{array} \right. \tag{U.22}$$

**Proposition U.12** *$\mathcal{J}_{12}$ and $\mathcal{J}_{23}$ are natural. Thus $\mathcal{J}_{23} \circ \mathcal{J}_{12}$ is natural.*

**Proof.** 1- We have to prove that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(E;\mathcal{L}(E;F)) & \overset{\mathcal{J}_{12E}}{\longrightarrow} & \mathcal{L}(E,E;F) \\ Z_{AB}\downarrow & & \downarrow Y_{AB} \\ \mathcal{L}(A;\mathcal{L}(A;B)) & \underset{\mathcal{J}_{12A}}{\longrightarrow} & \mathcal{L}(A,A;B) \end{array} \quad \text{where} \quad \begin{array}{l} Z_{AB}(T_1)(\vec{a}_1).\vec{a}_2 := T_1(\mathcal{P}_1^{-1}.\vec{a}_1).(\mathcal{P}_1^{-1}.\vec{a}_2), \\ Y_{AB}(T_2)(\vec{a}_1,\vec{a}_2) = T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2), \end{array} \tag{U.23}$$

(the "push-forwards) for all $\vec{a}_1,\vec{a}_2 \in A$ and $L_{AB} \in \mathcal{L}(A;B)$.

Let $T_1 \in \mathcal{L}(E;\mathcal{L}(E;F))$. We have
$\mathcal{J}_{12A}(Z_{AB}(T_1))(\vec{a}_1).\vec{a}_2 = Z_{AB}(T_1)(\vec{a}_1).\vec{a}_2 = T_1(\mathcal{P}_1^{-1}.\vec{a}_1).(\mathcal{P}_1^{-1}.\vec{a}_2)$, and
$Y_{AB}(\mathcal{J}_{12E}(T_1))(\vec{a}_1,\vec{a}_2) = \mathcal{J}_{12E}(T_1)(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2) = T_1(\mathcal{P}_1^{-1}.\vec{a}_1).(\mathcal{P}_1^{-1}.\vec{a}_2)$,
thus $\mathcal{J}_{12A} \circ Z_{AB} = Y_{AB} \circ \mathcal{J}_{12E}$, thus $\mathcal{J}_{12}$ is natural.

2- We have to prove that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(E,E;F) & \overset{\mathcal{J}_{23E}}{\longrightarrow} & \mathcal{L}(F^*,E,E;\mathbb{R}) \\ Z_{AB}\downarrow & & \downarrow Y_{AB} \\ \mathcal{L}(A,A;B) & \underset{\mathcal{J}_{23A}}{\longrightarrow} & \mathcal{L}(B^*,A,A;\mathbb{R}) \end{array} \quad \text{where} \quad \begin{array}{l} \ell_B.Z_{AB}(T_2)(\vec{a}_1,\vec{a}_2) := (\ell_B.\mathcal{P}_2).T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2), \\ Y_{AB}(T_3)(\ell_B,\vec{a}_1,\vec{a}_2) = T_3(\ell_B.\mathcal{P}_2,\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2), \end{array} \tag{U.24}$$

(the "push-forwards) for all $\vec{a}_1,\vec{a}_2\in A$ and $\ell_B\in B^*$.

Let $T_2\in\mathcal{L}(E,E;F)$. We have
$\mathcal{J}_{23A}(\ell_B,Z_{AB}(T_2)(\vec{a}_1,\vec{a}_2)) = \ell_B.Z_{AB}(T_2)(\vec{a}_1,\vec{a}_2) = (\ell_B.\mathcal{P}_2).T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2)$, and
$Y_{AB}(\mathcal{J}_{23A}(T_2))(\ell_B,\vec{a}_1,\vec{a}_2) = \mathcal{J}_{23A}(T_2)(\ell_B.\mathcal{P}_2,\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2) = \ell_B.\mathcal{P}_2.T_2(\mathcal{P}_1^{-1}.\vec{a}_1,\mathcal{P}_1^{-1}.\vec{a}_2)$
thus $\mathcal{J}_{23A}\circ Z_{AB} = Y_{AB}\circ\mathcal{J}_{23E}$, thus $\mathcal{J}_{23}$ is natural. ∎

# V Distribution in brief: A covariant concept

For a full description, see the books by Laurent Schwartz.

## V.1 Definitions

Usual notations with $\Omega$ an open set in $\vec{\mathbb{R}^n}$: $p\in[1,\infty[$ (e.g. $p=2$ for finite energy functions), and $L^p(\Omega)$ is the space of functions $f$ such that $|f|^p$ is Lebesgue integrable, and $||.||_p$ is its usual norm:

$$L^p(\Omega) := \{f:\Omega\to\mathbb{R} : \int_\Omega |f(\vec{x})|^p\,d\Omega<\infty\}, \quad \text{and} \quad ||f||_p = (\int_\Omega |f(\vec{x})|^p\,d\Omega)^{\frac{1}{p}}. \tag{V.1}$$

And $L^\infty(\Omega)$ is the space of bounded functions, and $||.||_\infty$ is its usual norm

$$L^\infty(\Omega) := \{f:\Omega\to\mathbb{R} : \sup_{\vec{x}\in\Omega}(|f(\vec{x})|)<\infty\}, \quad \text{and} \quad ||f||_\infty = \sup_{\vec{x}\in\Omega}(|f(\vec{x})|). \tag{V.2}$$

We have: $(L^p(\Omega),||.||_p)$ is a Banach space (a complete normed space) for all $p\in[1,\infty]$, see e.g. Brezis [4].

**Definition V.1** If $f\in\mathcal{F}(\Omega;\mathbb{R})$, then its support is the set

$$\text{supp}(f) := \overline{\{\vec{x}\in\Omega : f(\vec{x})\neq 0\}} = \text{the closure of } \{\vec{x}\in\Omega : f(\vec{x})\neq 0\}. \tag{V.3}$$

The closure in the definition of supp$(f)$ is required: E.g., $n=1$, $\Omega=]0,2\pi[$ and $f(x)=\sin x$: $\{f\neq 0\} := \{x\in\Omega : f(x)\neq 0\} = ]0,\pi[\cup]\pi,2\pi[$. Here $\pi\notin\{f\neq 0\}$, but $\pi$ is a point of interest since sin varies in its vicinity: $f'(\pi)=-1\neq 0$. So the set $\{f\neq 0\}$ is "too small", and it is its closure supp$(f) := \overline{\{f\neq 0\}} = [0,2\pi]$ that is needed: supp$(f)$ is closed.

**Schwartz notation:**

$$\mathcal{D}(\Omega) := C_c^\infty(\Omega;\mathbb{R}) = \{\varphi\in C^\infty(\Omega;\mathbb{R}) \text{ s.t. supp}(\varphi) \text{ is compact in } \Omega\}. \tag{V.4}$$

E.g., $\Omega=\mathbb{R}$, $\varphi(x) := e^{-\frac{1}{1-x^2}}$ if $x\in]-1,1[$ and $\varphi(x) := 0$ elsewhere: $\varphi\in\mathcal{D}(\mathbb{R})$ with supp$(\varphi)=[-1,1]$.

**Result** (Schwartz): $\mathcal{D}(\Omega)$ is a vector space which is dense in $(L^p(\Omega),||.||_{L^p})$ for any $p\in[1,\infty[$.

**Definition V.2** A distribution in $\Omega$ is a linear $\mathcal{D}(\Omega)$-continuous[5] function

$$T : \begin{cases} \mathcal{D}(\Omega) & \to\mathbb{R} \\ \varphi & \to T(\varphi) \overset{\text{written}}{=} \langle T,\varphi\rangle \end{cases} \tag{V.5}$$

The space of distribution in $\Omega$ is named $\mathcal{D}'(\Omega)$ (the dual of $\mathcal{D}(\Omega)$).

The notation $\langle T,\varphi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle T,\varphi\rangle$ is the "duality bracket" = the "covariance–contravariance bracket" between a continuous linear form $T\in\mathcal{D}'(\Omega)$ and a vector $\varphi\in\mathcal{D}(\Omega)$.

[5]The $\mathcal{D}(\Omega)$-continuity of $T$ is defined by: 1- A sequence $(\varphi_n)_{\mathbb{N}^*}$ in $\mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ towards a function $\varphi\in\mathcal{D}(\Omega)$ iff there exists a compact $K\subset\Omega$ s.t. supp$(\varphi_n)\subset K$ for all $n$, and $||\frac{\partial^k\varphi}{\partial x_{i_1}\ldots\partial x_{i_k}} - \frac{\partial^k\varphi_n}{\partial x_{i_1}\ldots\partial x_{i_k}}||_\infty \longrightarrow_{n\to\infty} 0$ for all $k,i_j$; 2- $T$ is continuous at $\varphi\in\mathcal{D}(\Omega)$ iff $T(\varphi_n)\underset{n\to\infty}{\longrightarrow}T(\varphi)$ for any sequence $(\varphi_n)_{\mathbb{N}}\in\mathcal{D}(\Omega)^{\mathbb{N}}\underset{n\to\infty}{\longrightarrow}\varphi$ in $\mathcal{D}(\Omega)$.

**Definition V.3** Let $f \in L^p(\Omega)$. The regular distribution $T_f \in \mathcal{D}'(\Omega)$ associated to $f$ is defined by

$$\forall \varphi \in \mathcal{D}(\Omega), \quad T_f(\varphi) := \int_\Omega f(\vec{x})\varphi(\vec{x})\,d\Omega \quad = \int_\Omega \varphi(\vec{x})\,(f(\vec{x})\,d\Omega), \tag{V.6}$$

i.e. $T_f$ is a measuring instrument for the density $dm_f(\vec{x}) = f(\vec{x})\,d\vec{x}$ at $\vec{x}$: $T_f(\varphi) := \int_\Omega \varphi(\vec{x})\,dm_f(\vec{x})$.

**Definition V.4** Let $\vec{x}_0 \in \mathbb{R}^n$. The Dirac measure at $\vec{x}_0$ is the distribution $T \overset{\text{written}}{=} \delta_{\vec{x}_0} \in \mathcal{D}'(\mathbb{R})$ defined by, for all $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\delta_{\vec{x}_0}(\varphi) = \varphi(\vec{x}_0), \quad \text{i.e.} \quad \langle \delta_{\vec{x}_0}, \varphi \rangle = \varphi(\vec{x}_0). \tag{V.7}$$

$\delta_{x_0}$ is indeed a distribution (easy check). But $\delta_{x_0}$ is not a regular distribution ($\delta_{x_0}$ is not a density measure): There is no integrable function $f$ such that $T_f = \delta_{x_0}$. Interpretation: $\delta_{x_0}$ is an ideal measuring device: The precision is perfect at $x_0$ (gives the exact value $\varphi(x_0)$ at $x_0$). In real life $\delta_{x_0}$ is the ideal approximation of $T_{f_n}$ where $f_n$ is e.g. given by $f_n(x) = n1_{[x_0, x_0+\frac{1}{n}]}$ with $n$ "very large" (drawing): We have $T_{f_n}(\varphi) \longrightarrow_{n\to\infty} \varphi(x_0) = \delta_{x_0}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$ (easy check).

**Generalization of the definition:** In (V.5) $\mathcal{D}(\Omega) = C_c^\infty(\Omega; \mathbb{R})$ is replaced by $C_c^\infty(\Omega; \vec{\mathbb{R}^n})$. So if you consider a basis $(\vec{e}_i)$ then $\vec{\varphi} \in C_c^\infty(\Omega; \vec{\mathbb{R}^n})$ reads $\vec{\varphi} = \sum_{i=1}^n \varphi^i \vec{e}_i$ with $\varphi^i \in \mathcal{D}(\Omega)$ for all $i$.

**Example V.5** Power: Let $\alpha : \Omega \to T_1^0(\Omega)$ be a differential form. Then the distribution $P_\alpha$ defined by $P_\alpha(\vec{v}) = \int_\Omega \alpha.\vec{v}\,d\Omega$ gives the virtual power associated to $\alpha$ for any vector field $\vec{v} \in C_c^\infty(\Omega; \mathbb{R})$. ∎

## V.2 Derivation of a distribution

$(\vec{e}_i)$ is a basis in $\vec{\mathbb{R}^n}$. Generic notation: $\vec{x} = \sum_{i=1}^n x_i \vec{e}_i$.

**Definition V.6** The derivative $\frac{\partial T}{\partial x_i}$ of a distribution $T \in \mathcal{D}'(\Omega)$ is the distribution in $\mathcal{D}'(\Omega)$ defined by, for all $\varphi \in \mathcal{D}(\Omega)$,

$$\frac{\partial T}{\partial x_i}(\varphi) := -T(\frac{\partial \varphi}{\partial x_i}), \quad \text{i.e.} \quad \langle \frac{\partial T}{\partial x_i}, \varphi \rangle := -\langle T, \frac{\partial \varphi}{\partial x_i} \rangle. \tag{V.8}$$

($\frac{\partial T}{\partial x_i}$ is indeed a distribution: Easy check.)

**Example V.7** If $T = T_f$ is a regular distribution with $f \in C^1(\Omega)$, then $\frac{\partial (T_f)}{\partial x_i} = T_{(\frac{\partial f}{\partial x_i})}$. Indeed, for all $\varphi \in \mathcal{D}(\Omega)$, $\frac{\partial (T_f)}{\partial x_i}(\varphi) = -T_f(\frac{\partial \varphi}{\partial x_i}) = -\int_\Omega f(\vec{x})\frac{\partial \varphi}{\partial x_i}\,d\Omega = +\int_\Omega \frac{\partial f}{\partial x_i}\varphi(\vec{x})\,d\Omega + \int_\Gamma 0\,d\Gamma$, since $\varphi$ vanishes on $\Gamma = \partial\Omega$ (the support of $\varphi$ is compact in $\Omega$), thus $\frac{\partial (T_f)}{\partial x_i}(\varphi) = T_{(\frac{\partial f}{\partial x_i})}(\varphi)$ for all $\varphi \in \mathcal{D}(\Omega)$. ∎

**Example V.8** $n = 1$. Consider the Heaviside function (the unit step function) $H_0 := 1_{\mathbb{R}_+}$ and the associated distribution $T = T_{H_0}$. Then $\langle (T_{H_0})', \varphi \rangle := -\langle T_{H_0}, \varphi' \rangle = -\int_\Omega H_0(x)\varphi'(x)\,dx = -\int_0^\infty \varphi'(x)\,dx = \varphi(0) = \langle \delta_0, \varphi \rangle$ for any $\varphi \in \mathcal{D}(\mathbb{R})$, thus $(T_{H_0})' = \delta_0$. Written $H_0{}' = \delta_0$ in $\mathcal{D}'(\Omega)$, which is not in a equality between functions, because $H_0$ is not derivable at 0 as a function, and $\delta_0$ is not a function; It is equality between distributions: The notation $H_0{}'$ can only be used to compute $\langle H_0{}', \varphi \rangle := -\langle H_0, \varphi' \rangle$. ∎

## V.3 Hilbert space $H^1(\Omega)$

### V.3.1 Motivation

$n = 1$. Consider the hat function $\Lambda(x) \left\{ \begin{array}{l} = x+1 \text{ if } x \in [-1,0], \\ = 1-x \text{ if } x \in [0,1], \\ = 0 \text{ otherwise} \end{array} \right\}$ (drawing). When applying the finite element method, it is well-known that, if you use integrals (use the virtual power principle which makes you compute average values) then you can consider the derivative of the hat function $\Lambda$ as if it was the usual derivative, i.e. at the points where the usual computation of $\Lambda'$ is meaningful, that is,

$$\Lambda'(x) \left\{ \begin{array}{l} = 1 \text{ if } x \in ]-1,0[, \\ = -1 \text{ if } x \in ]0,1[, \\ = 0 \text{ if } x \in \mathbb{R} - ([-1,0] \cup [0,1]) \end{array} \right. \tag{V.9}$$

(drawing).

**Problem:** $\Lambda'$ is not defined at $-1,0,1$ (the function $\Lambda$ is not derivable at $-1,0,1$);
**Question:** So does (V.9) and the "usual" computation $I=\int_{\mathbb{R}}\Lambda'(x)\varphi(x)\,dx$ gives the good result?

This is not a trivial question: E.g., with $H_0=1_{\mathbb{R}_+}$ instead of $\Lambda$, we would get the absurd result $H_0'=0$, absurd because $H_0'=\delta_0$.

**Answer:** Yes in the distribution meaning, i.e.:

1- Consider $T_\Lambda$ the regular distribution associated to $\Lambda$, cf. (V.6);

2- Then consider $(T_\Lambda)'$, cf. (V.8): We get $\langle (T_\Lambda)',\varphi\rangle \overset{(V.8)}{=} -\langle T_\Lambda,\varphi'\rangle = -\int_{\mathbb{R}}\Lambda(x)\varphi'(x)\,dx = -\int_{-1}^0 \Lambda(x)\varphi'(x)\,dx - \int_0^1 \Lambda(x)\varphi'(x)\,dx = +\int_{-1}^0 1_{]-1,0[}(x)\varphi(x)\,dx + \int_0^1 1_{]0,1[}\varphi(x)\,dx$, for any $\varphi\in\mathcal{D}(\mathbb{R})$;

3- Thus $(T_\Lambda)'=T_f$ where $f=1_{]-1,0[}+1_{]0,1[}$, that is $(T_\Lambda)'$ is the regular distribution $T_f$.

4- Then $T_f=(T_\Lambda)' =^{\text{written}} \Lambda'$ when used within the distribution framework, i.e. when used with the Lebesgue integral $\int_\Omega \Lambda'(x)\varphi(x)\,dx := -\int_\Omega \Lambda(x)\varphi'(x)\,dx$: Ok for finite element methods.

### V.3.2 Definition of $L^2(\Omega)$ and its dual

$n=1$: The space of finite energy functions $L^2(\Omega):=\{f:\Omega\to\mathbb{R} : \int_\Omega |f(x)|^2\,dx<\infty\}$ is equipped with its usual inner dot product and associated norm defined by

$$(u,v)_{L^2}=\int_\Omega u(x)v(x)\,dx \quad\text{and}\quad ||v||_{L^2}=\sqrt{(v,v)_{L^2}}=(\int_\Omega v(x)^2\,dx)^{\frac12}. \tag{V.10}$$

Result: $(L^2(\Omega),(\cdot,\cdot)_{L^2})$ is a Hilbert space (Riesz-Fisher theorem).

The dual space of $L^2(\Omega)$ is the space

$$L^2(\Omega)'=\mathcal{L}(L^2(\Omega);\mathbb{R}):=\{\ell : L^2(\Omega)\to\mathbb{R} \text{ linear and continuous}\}, \tag{V.11}$$

i.e. the space of linear forms $\ell : L^2(\Omega)\to\mathbb{R}$ s.t.: $\exists C>0$, $\forall v\in H^1(\Omega)$, $|\ell(v)|\le C||v||_{L^2}$.

$L^2(\Omega)'$ equipped with tbe norm $||\ell||_{L^2(\Omega)'} := \sup_{||v||_{L^2(\Omega)}=1} |\ell(v)|$ is a Banach space.

Duality bracket:

$$\text{If}\quad \ell\in L^2(\Omega)' \quad\text{then}\quad \ell(v) \overset{\text{written}}{=} \langle \ell,v\rangle_{L^{2\prime},L^2},\ \ \forall v\in L^2(\Omega). \tag{V.12}$$

And thanks to the $(\cdot,\cdot)_{L^2}$-Riesz representation theorem, a $\ell\in L^2(\Omega)'$ being linear and continuous, $\ell\in L^2(\Omega)'$ can be represented by function $f\in L^2(\Omega)$:

$$\text{if}\quad \ell\in L^2(\Omega)' \quad\text{then}\quad \exists f\in L^2(\Omega),\ \forall v\in L^2(\Omega),\ \langle \ell,v\rangle=(f,v)_{L^2}\quad (=\int_\Omega f(x)v(x)\,dx). \tag{V.13}$$

NB: $L^2(\Omega)$ is called the "pivot space" (the central space).

$n\ge 2$: Idem with $\ell\in L^2(\Omega)^{n\prime}$, after an inner dot product $.\bullet.$ in $\vec{\mathbb{R}^n}$ has been chosen:

$$\text{if}\quad \ell\in L^2(\Omega)' \quad\text{then}\quad \exists f\in L^2(\Omega),\ \forall v\in L^2(\Omega),\ \langle \ell,v\rangle=(f,v)_{L^2}\quad (=\int_\Omega f(\vec{x})v(\vec{x})\,d\Omega). \tag{V.14}$$

### V.3.3 Definition of $H^1(\Omega)$ and its dual

The space $C^1(\Omega;\mathbb{R})$ is too small in many applications (e.g., for the $\Lambda$ function above). We need a larger space where the functions are "derivable is a weaker sense": The distribution sense.

$(\vec{e}_i)$ is a Cartesian basis in $\vec{\mathbb{R}^n}$

**Definition V.9** The Sobolev space $H^1(\Omega)$ is the subspace of $L^2(\Omega)$ restricted to functions whose generalized derivatives are in $L^2(\Omega)$:

$$H^1(\Omega):=\{v\in L^2(\Omega) : \frac{\partial v}{\partial x_i}\in L^2(\Omega),\ \forall i=1,...,n\} \overset{\text{written}}{=} \{v\in L^2(\Omega) : \vec{\text{grad}}v\in L^2(\Omega)^n\}. \tag{V.15}$$

**Remark V.10** So to check that $v\in H^1(\Omega)$, even if $\frac{\partial v}{\partial x_i}$ does not exists in the classic way (see the above hat function $\Lambda$), you have to: 1- Consider its associated regular distribution $T_v$, 2- Compute $\frac{\partial T_v}{\partial x_i}$ in $\mathcal{D}'(\Omega)$, 3- and if, for all $i$, there exists $f_i\in L^2(\Omega)$ s.t. $\frac{\partial T_v}{\partial x_i}=T_{f_i}$, then $v\in H^1(\Omega)$. 4- Then $T_{f_i}=\frac{\partial T_v}{\partial x_i}$ is noted $\frac{\partial v}{\partial x_i}$ when used with $\varphi\in\mathcal{D}(\Omega)$ and the Lebesgue integral: $\int_\Omega \frac{\partial v}{\partial x_i}(x)\varphi(x)\,dx := \int_\Omega v(x)\frac{\partial \varphi}{\partial x_i}(x)\,dx$. ▟

With the inner dot product $.\bullet.$ defined by $\vec{e}_i \bullet \vec{e}_j = \delta_{ij}$ for all $i,j$, define

$$(\vec{\text{grad}}u, \vec{\text{grad}}v)_{L^2} := \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i})_{L^2} \overset{\text{written}}{=} \int_\Omega \vec{\text{grad}}u(\vec{x}) \bullet \vec{\text{grad}}v(\vec{x})\, d\Omega. \tag{V.16}$$

**Definition V.11** The usual inner dot product and associated norm in $H^1(\Omega)$ are

$$(u,v)_{H^1} = (u,v)_{L^2} + (\vec{\text{grad}}u, \vec{\text{grad}}v)_{L^2}, \quad \text{and} \quad ||v||_{H^1} = (v,v)_{H^1}^{\frac{1}{2}}, \tag{V.17}$$

Thus $(H^1(\Omega), (\cdot,\cdot)_{H^1})$ is a Hilbert space (Riesz–Fisher).

The dual space of $H^1(\Omega)$ is

$$H^1(\Omega)' := \mathcal{L}(H^1(\Omega);\mathbb{R}) := \{\ell : H^1(\Omega) \to \mathbb{R} \text{ linear and continuous}\} \tag{V.18}$$

i.e. the space of linear forms $\ell : H^1(\Omega) \to \mathbb{R}$ s.t. $\ell$ is linear and $\exists C > 0$, $\forall v \in H^1(\Omega)$, $|\ell(v)| \le C||v||_{H^1}$.

And (duality bracket) if $\ell \in H^1(\Omega)'$ then $\ell(v) =^{\text{written}} \langle \ell, v\rangle_{H^{1\prime},H^1} =^{\text{written}} \langle \ell, v\rangle$ for all $v \in H^1(\Omega)$.

**Theorem V.12** *$\ell \in H^1(\Omega)'$ iff: $\exists (f,\vec{u}) \in L^2(\Omega)\times L^2(\Omega)^n$, $\forall \psi \in H^1(\Omega)$,*

$$\ell(\psi) = (f,\psi)_{L^2} + (\vec{u}, \vec{\text{grad}}\psi)_{L^2}. \tag{V.19}$$

**Proof.** From Brézis [4] (application of the Riesz representation theorem). The space $Z = L^2(\Omega)\times L^2(\Omega)^n$ with its inner dot product $((f,\vec{u}),(g,\vec{v}))_Z := (f,g)_{L^2} + (\vec{u},\vec{v})_{L^2}$ is a Hilbert space. Let $T : H^1(\Omega) \to Z$ be defined by $T(\psi) = (\psi, \vec{\text{grad}}\psi)$; $T$ is linear and $||T(\psi)||_Z = ||\psi||_{H^1}$, thus $T(\psi) = 0$ imply $\psi = 0$, so $T$ is one-to-one, thus $T^{-1} : \text{Im}T \to H^1(\Omega)$ is well defined. And $T^{-1}$ continuous since $T^{-1}(\psi, \vec{\text{grad}}\psi) = \psi$. (Remark: $\text{Im}T$ is not closed in $Z$.) Let $\ell \in H^1(\Omega)'$, then define $L : \text{Im}(T) \to \mathbb{R}$ by $\langle L, (\psi, \vec{\text{grad}}\psi)\rangle_{Z',Z} = \langle \ell, T^{-1}(\psi, \vec{\text{grad}}\psi)\rangle_{H^{1\prime},H^1}$: so $L = \ell \circ T^{-1}$ is linear continuous since $\ell$ and $T^{-1}$ are, and $\langle L, (\psi, \vec{\text{grad}}\psi)\rangle_{Z',Z} = \langle \ell, \psi\rangle_{H^{1\prime},H^1}$; With Hahn–Banach theorem, extend $L : \text{Im}(T) \to \mathbb{R}$ to $L_Z : Z \to \mathbb{R}$ linear continuous. Apply Riesz representation theorem: $\exists (f,\vec{u}) \in Z$ s.t. $\langle L_Z, (\psi,\vec{w})\rangle_{Z',Z} = ((f,\vec{u}),(\psi,\vec{w}))_Z = (f,\psi)_{L^2} + (\vec{u},\vec{w})_{L^2}$ for all $(\psi,\vec{w}) \in Z$, in particular for all $(\psi,\vec{w}) \in \text{Im}T$, thus $\langle \ell,\psi\rangle_{H^{1\prime},H^1} = (f,\psi)_{L^2} + (\vec{u}, \vec{\text{grad}}\psi)_{L^2}$ for all $\psi \in H^1(\Omega)$. ∎

### V.3.4 Subspace $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$

Definition:

$$H_0^1(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1} \quad \text{the closure of } \mathcal{D}(\Omega) \text{ in } H^1(\Omega). \tag{V.20}$$

So $H_0^1(\Omega)$ is closed in $H^1(\Omega)$, hence $(H_0^1(\Omega), (\cdot,\cdot)_{H^1})$ is a Hilbert space. If the boundary $\Gamma = \partial\Omega$ of $\Omega$ is bounded and regular then

$$\boxed{H_0^1(\Omega) = \{v \in H^1(\Omega) : v_{|\Gamma} = 0\}}. \tag{V.21}$$

(See Brézis [4].) The dual space of $(H_0^1(\Omega), ||.||_{H^1})$ is the space

$$(H_0^1(\Omega))' := \mathcal{L}(H_0^1(\Omega);\mathbb{R}) := \{\ell : H_0^1(\Omega) \to \mathbb{R} \text{ linear and continuous}\} \overset{\text{written}}{=} H^{-1}(\Omega), \tag{V.22}$$

i.e. space of linear forms $\ell : H_0^1(\Omega) \to \mathbb{R}$ s.t. $\exists C > 0$, $\forall \psi \in H_0^1(\Omega)$, $|\ell(\psi)| \le C||\psi||_{H^1}$. And then $\ell(\psi) =^{\text{written}} \langle \ell,\psi\rangle_{H^{-1},H_0^1}$ (duality bracket).

**Theorem V.13** *$\ell \in H^{-1}(\Omega) = (H_0^1(\Omega))'$ iff $\exists (f,\vec{g}) \in L^2(\Omega) \times L^2(\Omega)^n$ s.t.*

$$\ell = f - \text{div}\vec{g} \quad (\in \mathcal{D}'(\Omega)), \tag{V.23}$$

*i.e., for all $\psi \in H_0^1(\Omega)$,*

$$\langle \ell,\psi\rangle_{H^{-1},H_0^1} = \int_\Omega f\psi\, d\Omega + \int_\Omega d\psi.\vec{g}\, d\Omega. \tag{V.24}$$

*And if $\Omega$ is bounded then we can choose $f = 0$, and moreover if $\vec{g} \in H_{\text{div}}(\Omega)$ then*

$$\langle \ell,\psi\rangle_{H^{-1},H_0^1} = -\int_\Omega \text{div}\vec{g}(x)\psi(x)\, dx. \tag{V.25}$$

**Proof.** Apply (V.19) here with $\psi \in \mathcal{D}(\Omega)$ (or $\psi \in H_0^1(\Omega)$) and with $\psi_{|\Gamma} = 0$ for the integration by parts. ∎

# W Basics of thermodynamics

See https://perso.isima.fr/leborgne/IsimathMeca/Thermo.pdf

# References


[1] Abraham R., Marsden J.E.: *Foundation of mechanics, 2nd edition.* Addison-Wesley, 1985.

[2] Arnold V.I.: *Equations différentielles ordinaires.* Editions Mir, 1974.

[3] Arnold V.I.: *Mathematical Methods of Classical Mechanics.* Second Edition, Springer 1989.

[4] Brézis H.: *Functional Analysis, Sobolev spaces and Partial differential equations.* Springer, 2011.

[5] Cartan H.: *Formes différentielles.* Hermann 1967.

[6] Cartan H.: *Cours de calcul différentiel.* Hermann 1990.

[7] Cauchy A.L.: *Exercices de mathématiques, seconde année.* De Bures Frères, 1827.

[8] Ciarlet Ph.: *Mathematical Elasticity, vol.1: Three-dimensional elasticity.* North-Holland 1993.

[9] Debard Y.: Elasticité. https://iut.univ-lemans.fr/ydlogi/cours/elasticite.pdf, 2011.

[10] Forest et al: *Mécanique de milieux continus.* Ecole des Mines de Paris. 2020. http://mms2.ensmp.fr/mmc_paris/poly/MMC2020web.pdf

[11] Germain P.: *Cours de Mécanique des Milieux Continus. Tome 1: théorie générale.* Masson. Paris. 1973.

[12] Germain P.: The Method of Virtual Power in Continuum Mechanics. Part 2: Microstructure. SIAM Journal on Applied Mathematics, Vol. 25, No. 3 (Nov., 1973), pp. 556-575.

[13] Hughes T.J.R., Marsden J.E.: *A short Course in Fluid Mechanics.* Publish or Perish, 1976.

[14] Eva Kanso, Marino Arroyo, Yiying Tong, Arash Yavari, Jerrold e. Marsden, Mathieu Deshbrun: *On the geometric character of stress in continuum mechanics.* Z. angew. Math. Phys. 58 (2007) 1-14.

[15] Leborgne G.: Inf-sup condition and locking: Understanding and circumventing. Stokes, Laplacian, bi-Laplacian, Kirchhoff–Love and Mindlin–Reissner locking type, boundary conditions. https://arxiv.org/abs/2301.04373

[16] Le Dret Hervé: *Méthodes mathématiques en élasticité, notes de cours de DEA 2003–2004.* Université Pierre et Marie Curie.

[17] Marsden J.E., Hughes T.J.R.: *Mathematical Foundations of Elasticity.* Dover publications, 1993.

[18] Maxwell: A treatise on electricity and magnetism, Clarendon press series, 1873.

[19] Misner C.W., Thorne K.S., Wheeler J.A.: *Gravitation.* W.H. Freeman and Cie, 1973

[20] Petropoulos M.: Relativité Générale. La gravitation en une leçon et demie. https://gargantua.polytechnique.fr/siatel-web/linkto/mICYYYTEsNY

[21] Romano G.: *Geometry & Continuum Mechanics.* Short course in Innsbruck, 24-25 november 2014.

[22] Sadd Martin H.: *Elasticity. Theory, Applications, and Numerics.* Elsevier 2005.

[23] Spivak M.: *A comprehensive introduction to differential geometry, Volume 1.* Publish or Perish, Inc. 1979.

[24] Stefanou I.: Definition-and-Uses-of-the-Principle-of-Virtual-Power.pdf

[25] Strang G.: *Introduction to linear algebra.* 5th edition. Wellesley - Cambridge Press 2016.

[26] Truesdell C., Noll W.: *The Non-Linear Field Theories of Mechanics.* Springer, 2004.

[27] https://planet-terre.ens-lyon.fr/article/force-de-coriolis.xml, 2018.