Objectivity in classical mechanics (continuum mechanics)
Velocity-addition formula, Coriolis. Objectivity.

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In classical mechanics, there are two objectivities: 1- The covariant objectivity which concerns the general laws of physics and requires that these laws be observer independent: It deals with qualitative aspects in Mechanics. This is the main subject of this manuscript. 2- The isometric objectivity which concerns the constitutive laws of materials (frame invariance principle).

To describe the covariant objectivity, we need motions, the associated Eulerian functions, and the velocity addition formula. We also introduce the Lie derivative for vectors which might meet some needs of engineers, and which is covariant objective. (Cauchy would certainly have used it if it had existed during his lifetime; In fact, to get a stress, Cauchy had to compare two vectors, whereas one vector suffices when using the derivative of Lie.)

Thus we follow Maxwell’s needs, [13]: “Preliminary (on the measurement of quantities) [...] 2. (...) The formula at which we arrive must be such that a person of any nation, by substituting for the different symbols the numerical value of the quantities as measured by his own national units, would arrive at a true result. [...] 10. (...) The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in mathematical progress, for it reduced the methods of geometry to calculations performed on numerical quantities. The position of a point is made to depend on the length of three lines which are always drawn in determinate directions (...) But for many purposes in physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other....”

Or see the (short) historical note given in the introduction of Abraham and Marsden book “Foundations of Mechanics” [1], about qualitative versus quantitative theory: “Mechanics begins with a long tradition of qualitative investigation culminating with Kepler and Galileo. Following this is the period of quantitative theory (1687-1889) characterized by concomitant developments in mechanics, mathematics, and the philosophy of science that are epitomized by the works of Newton, Euler, Lagrange, Laplace, Hamilton, and Jacobi. (...) For celestial mechanics (...) resolution we owe to the genius of Poincaré, who resurrected the qualitative point of view (...) One advantage of this model is that by suppressing unnecessary coordinates the full generality of the theory becomes evident.”

In this manuscript, we examine simple applications of qualitative methods to continuum mechanics; No differential geometry knowledge is required, except for the tangent space at a point of our affine space, and the tangent bundle, which shed light on the subject. We start with definitions and characterizations (qualitative approach), before quantifying with bases and/or inner dot products.

A fairly long appendix (half of the manuscript) gives standard definitions (qualitative), propositions and proofs, and notations used for calculations (quantification). Discussions with colleagues are a great help.
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Part I
Motions, Eulerian and Lagrangian descriptions, flows

A quantity \( f \) being given, the notation \( g := f \) means: "\( g \) is defined by \( g = f \)". To define Eulerian and Lagrangian functions, we first need to define a motion of an object. The framework is classical mechanics, time being decoupled from space.

1 Motions

1.1 Referential

Let \( \mathbb{R}^3 \) be the classical geometric affine space (space of points), and let \( (\mathbb{R}^3,+,...) \) be the usual associated vector space of bipoint vectors. And we also consider \( \mathbb{R} \) and \( \mathbb{R}^2 \) as subspaces of \( \mathbb{R}^3 \). So we consider \( \mathbb{R}^n, n = 1,2,3 \), and the associated vector space \( \mathbb{R}^n \).

**Origin:** An observer chooses an origin \( O \in \mathbb{R}^n \). Thus a point \( p \in \mathbb{R}^n \) can be located by the observer thanks to the bipoint vector \( \overrightarrow{O}p = \vec{x} \in \mathbb{R}^n \), so that \( p = O + \vec{x} \).

Another observer chooses an origin \( O' \in \mathbb{R}^n \). Thus a point \( p \in \mathbb{R}^n \) can be located by this observer thanks to the bipoint vector \( \overrightarrow{O'}p = \vec{x}' \in \mathbb{R}^n \), so that \( p = O' + \vec{x}' \).

And we have \( \vec{x}' = \overrightarrow{O}O' + \vec{x} \).

**Cartesian coordinate system:** A Cartesian coordinate system in the affine space \( \mathbb{R}^n \) is a set \( \mathcal{R}_c = (O, (\vec{e}_i), i=1,...,n) \) chosen by an observer, where \( O \) is a point called the origin, and \( (\vec{e}_i) := (\vec{e}_i)_{i=1,...,n} \) is a basis in \( \mathbb{R}^n \). Then, quantification of the location of a point \( p \in \mathbb{R}^n \) by the observer who defined \( \mathcal{R}_c \): there exists \( x_1,...,x_n \in \mathbb{R} \) s.t.

\[
p = O + \vec{x} = O + \sum_{i=1}^{n} x_i \vec{e}_i, \quad \text{and} \quad [\overrightarrow{O}p]_\vec{e} = [\vec{x}]_\vec{e} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
\] (1.1)

is the column matrix containing the components of \( \overrightarrow{O}p = \vec{x} \) in the basis \( (\vec{e}_i) \).

Quantification by another observer with his Cartesian referential \( \mathcal{R}'_c = (O', (\vec{e}_i'), i=1,...,n) \):

\[
p = O' + \vec{x}' = O' + \sum_{i=1}^{n} x'_i \vec{e}_i'.
\] (1.2)

**Chronology:** A chronology (or temporal coordinate system) is a set \( \mathcal{R}_t = (t_0, (\Delta t)) \) chosen by an observer, where \( t_0 \in \mathbb{R} \) is a point called the time origin, and \( (\Delta t) \) is called the time unit (a basis in \( \mathbb{R} \)).

**Referential:** A referential \( \mathcal{R} \) is the set

\[
\mathcal{R} = (\mathcal{R}_t, \mathcal{R}_c) = (t_0, (\Delta t), O, (\vec{e}_i), i=1,...,n) = \text{"chronologie","Cartesian coordinate system"}
\] (1.3)

chosen by an observer, made of a chronology and a Cartesian coordinate system.

In the following (framework of classical mechanics), to simplify the writings, the same implicit chronology is used by all observers, and a referential \( \mathcal{R} = (\mathcal{R}_t, \mathcal{R}_c) \) will simply be noted \( \mathcal{R} = \mathcal{R}_c = (O, (\vec{e}_i)) \).

1.2 Einstein’s convention (duality notation)

We will also use Einstein’s convention (duality notation), see § A.10: The components \( x_i \) of \( \vec{x} \) in (1.1) are also named \( x_1 = x^1 \) with Einstein’s convention:

\[
\vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i = \sum_{i=1}^{n} x^i \vec{e}_i, \quad \text{so} \quad [\vec{x}]_\vec{e} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}
\] (1.4)
Moreover Einstein’s convention uses the notation $\sum_{i=1}^{n} x^i e_i =: x^i e_i$, i.e. the sum sign $\sum_{i=1}^{n}$ can be omitted when an index is used twice, once up and once down. However this omission will not be made in this manuscript: The LaTex program makes it easy to print $\sum_{i=1}^{n}$.

**Example 1.1** The height of a child is represented on a wall by a vertical bipoint vector $\vec{x}$ starting from the ground up to a pencil line. The vector $\vec{x}$ is objective = qualitative: It is the same for any observer.

Question: What is the size of the child? (Quantitative = subjective.)

Answer: It depends... on the observer. E.g., an English observer chooses a basis vector $\vec{a}_1$ which length is one English foot (ft). So he writes $\vec{x} = x_1 \vec{a}_1$, and for him the size of the child (size of $\vec{x}$) is $x_1$ in foot. A French observer chooses a basis vector $\vec{b}_1$ which length is one meter (m). So he writes $\vec{x} = y_1 \vec{b}_1$, and for him the size of the child (size of $\vec{x}$) is $y_1$ meter. E.g., if $x_1 = 4$ then $y_1 \simeq 1.22$, since 1 ft = 0.3048 m. The child (the vector $\vec{x}$) is both 4 ft and 1.22 m tall.

With Einstein duality notation: $\vec{x} = x^i a_i = y^i b_i$, and if $x_1 = 4$ then $y_1 \simeq 1.22$.

This manuscript deals with covariant objectivity, thus an English engineer (and his foot) and a French engineer (and his meter) will be able to work together. And they will be able to use the results of Galileo, Descartes, Newton, Euler... who used their own unit of length (and knew nothing about “scalar products” invented in the 19th century).

### 1.3 Motion of an object

Let $\text{Obj}$ be a “real object”, or “material object”, made of particles (e.g., the Moon: Exists independently of an observer).

**Definition 1.2** The motion of $\text{Obj}$ in $\mathbb{R}^n$ is the map

$$\Phi: \left\{ \begin{array}{c} [t_1, t_2] \times \text{Obj} \to \mathbb{R}^n \\ \ (t, P_{\text{Obj}}) \to \begin{array}{c} \vec{p} \\ \text{position at } t \end{array} \end{array} \right. = \Phi(t, P_{\text{Obj}}) = \text{position of } P_{\text{Obj}} \text{ at } t \text{ in } \mathbb{R}^n, \quad (1.5)$$

which describes the motion of the particles $P_{\text{Obj}} \in \text{Obj}$ in the affine space $\mathbb{R}^n$. And $t$ is the time variable, $\vec{p}$ is the space variable, and $(t, \vec{p}) \in \mathbb{R} \times \mathbb{R}^n$ is the time-space variable.

An observer can also choose an origin $O$ and use the bi-point motion vector $\vec{\Phi}(t, P_{\text{Obj}}) := \overrightarrow{OP}(t, P_{\text{Obj}})$ instead of the point $\Phi(t, P_{\text{Obj}})$:

$$\vec{\Phi}: \left\{ \begin{array}{c} [t_1, t_2] \times \text{Obj} \to \mathbb{R}^n \\ \ (t, P_{\text{Obj}}) \to \vec{x} = \vec{\Phi}(t, P_{\text{Obj}}) = \overrightarrow{OP}(t, P_{\text{Obj}}). \end{array} \right. \quad (1.6)$$

But then, two observers with two different origins $O$ and $O'$ have two different bi-point vectors $\vec{x}$ and $\vec{x}'$. Therefore, in the following we won’t use $\vec{\Phi}$. We will exclusively use $\Phi$, cf. (1.5). Moreover, in a non-planar surface considered on its own (a manifold), the notion of bi-point vector is meaningless (it goes “through” the surface: The only available vectors are “tangent vectors”).

**Quantification:** An observer chooses a Cartesian referential $\mathcal{R} = (O, (e_i))$ to describe the motion $\Phi$:

$$p = \Phi(t, P_{\text{Obj}}) = O + \vec{x} = O + \sum_{i=1}^{n} x^i e_i = \text{position of } P_{\text{Obj}} \text{ at } t \text{ in } \mathcal{R}. \quad (1.7)$$

**Remark 1.3** Hypothesis of both Newtonian mechanics (Galileo relativity) and general relativity (Einstein): 1- You can describe a phenomenon only at the actual time $t$ and from the location $p(t)$ you are at (you have neither time nor space ubiquity gift); 2- You don’t know the future; 3- You can use your memory (use the past), or someone else memory if you can communicate objectively.

**Remark 1.4** The motion of an object $\text{Obj}$ (e.g. a planet) has been described before the invention of groups, rings, vector spaces, algebra (19th century) (Copernicus 1473-1543, Descartes 1596-1650).
1.4 Configurations and spatial (Eulerian) variables

Let \( \tilde{\Phi} \) be a motion, cf. (1.5). Let \( t \in [t_1, t_2] \) be fixed, and define

\[
\tilde{\Phi}_t : \bigg\{ \begin{array}{ll}
\text{Obj} & \rightarrow \mathbb{R}^n \\
P_{Obj} & \mapsto p = \tilde{\Phi}_t(P_{Obj}) := \tilde{\Phi}(t, P_{Obj}).
\end{array}
\]  

(1.8)

**Definition 1.5** The “configuration at \( t \)” of \( \text{Obj} \) is the subset of \( \mathbb{R}^n \) (affine space) defined by

\[
\Omega_t = \tilde{\Phi}_t(\text{Obj}) = \text{Im}(\tilde{\Phi}_t) = \text{the range (or image) of} \tilde{\Phi}_t
:= \{ p \in \mathbb{R}^n : \exists P_{Obj} \in \text{Obj} \ s.t. \ p = \tilde{\Phi}_t(P_{Obj}) \}.
\]  

(1.9)

And \( p = \tilde{\Phi}_t(P_{Obj}) \in \Omega_t \) is the spatial variable (at \( t \)), or Eulerian variable, relative to \( P_{Obj} \) at \( t \).

**Hypothesis:** At any time \( t \), the map \( \tilde{\Phi}_t \) is assumed to be one-to-one (= injective): \( \text{Obj} \) does not crash onto itself. And \( \Omega_t \) is supposed to be a “smooth domain” in \( \mathbb{R}^n \), that is, the closure of an open set in \( \mathbb{R}^3 \), or of a 2-D differentiable surface in \( \mathbb{R}^3 \), or of a 1-D differentiable curve in \( \mathbb{R}^3 \) (continuum mechanics).

1.5 Eulerian and Lagrangian variables

**Definition 1.6**

- If \( t \) is the actual time, then \( \Omega_t \) is called the “actual configuration” or “current configuration”, and the spatial variable \( p_t \in \Omega_t = \tilde{\Phi}_t(P_{Obj}) \) is called the Eulerian variable (location of \( P_{Obj} \) at actual time).
- If \( t_0 \) is a time in the past, then \( \Omega_{t_0} \) is called the “initial configuration”, or “reference configuration”, and the spatial variable \( p_{t_0} \in \Omega_{t_0} = \tilde{\Phi}_{t_0}(P_{Obj}) \) is called the Lagrangian variable relative to \( t_0 \).

1.6 Trajectories

Let \( \tilde{\Phi} \) be a motion of \( \text{Obj} \), cf. (1.5). Let \( P_{Obj} \in \text{Obj} \) be a particle (e.g., a particle in the Moon).

**Definition 1.7** The (parametric) trajectory of \( P_{Obj} \) between \( t_1 \) and \( t_2 \) is the function

\[
\tilde{\Phi}_{P_{Obj}} : \bigg\{ [t_1, t_2] \rightarrow \mathbb{R}^n, \quad t \mapsto p(t) = \tilde{\Phi}_{P_{Obj}}(t) := \tilde{\Phi}(t, P_{Obj}) \; (\text{position of} \; P_{Obj} \; \text{at} \; t).\bigg\}
\]  

(1.10)

And its range \( \text{Im}(\tilde{\Phi}_{P_{Obj}}) = \tilde{\Phi}_{P_{Obj}}([t_1, t_2]) \) is the (geometric) trajectory of \( P_{Obj} \).

1.7 Virtual and real motion

**Definition 1.8** A virtual (or possible) motion of \( \text{Obj} \) is a function \( \tilde{\Phi} \) “regular enough for the calculations to be meaningful”: In the following, the parametric trajectories \( \tilde{\Phi}_{P_{Obj}} \) are at least \( C^2 \) for velocities and accelerations to exist. Among all the virtual motions, the observed motion is called the real motion.

1.8 Tangent space, \( \mathbb{R}^n_t \), fiber, bundle

\( \mathbb{R}^n \) is the affine space “of points”, the same at all time (classical mechanics), associated with the vector space \( \mathbb{R}^n \) (made of bipoint vectors). However, to deal with surfaces (manifolds), a vector is considered to be a “tangent vector to the surface at a point”. E.g., on the surface of a sphere \( S \) (e.g., Earth, Moon...) a tangent vector at a point \( p \) cannot be a tangent vector at some other point (a sphere is not flat).

In \( \mathbb{R}^n \), let \( m \in [1, n] \) and let \( S \) be a “regular m-surface” (a m-differentiable manifold in \( \mathbb{R}^n \)). That is, in \( \mathbb{R}^n = \mathbb{R}^3 \), a 3-surface \( S \) is an open set \( \Omega \) in \( \mathbb{R}^3 \), a 2-surface \( S \) is a “usual surface”, a 1-surface \( S \) is a “usual curve”.

**Definition 1.9**

The tangent space at \( p \) noted \( T_pS := \{ \text{tangent vectors } \vec{w}_p \; \text{at} \; S \; \text{at} \; p \} \).

(1.11)

Particular case: If \( S = \Omega \) is an open set in \( \mathbb{R}^n \), then \( T_pS = T_p\Omega = \mathbb{R}^n \) is independent of \( p \).
**Definition 1.10**

The fiber at \( p := \{p\} \times T_pS = \{(p, \vec{w}_p) \} \times T_pS \),
\[
\text{:= pointed vector}
\]
that is, is the set of “pointed vectors at \( p \)” (“a vector equipped with a base point to which it is attached”).

If the context is clear, a pointed vector \((p, \vec{w}_p)\) is simply noted \( \vec{w}_p \).

**Definition 1.11**

The tangent bundle := \( \bigcup_{p \in S} \{(p) \times T_pS\} =\not T S, \)
\[
\text{that is, is the union of the fibers.}
\]

2 Eulerian description (spatial description at actual time \( t \))

2.1 Eulerian function

Let \( \Phi \) be a motion of \( \text{Obj} \), cf. (1.5), and \( \Omega_t = \Phi_t(\text{Obj}) \subset \mathbb{R}^n \) be the configuration at \( t \), cf. (1.9). Let \( \mathcal{C} \) be the set of configurations, that is the subset in the Cartesian “time-space” \( \mathbb{R} \times \mathbb{R}^n \) defined by
\[
\mathcal{C} := \bigcup_{t \in [t_1, t_2]} \{(t) \times \Omega_t \} \subset \mathbb{R} \times \mathbb{R}^n \tag{2.1}
\]

Question: Why don’t we simply use \( \bigcup_{t \in [t_1, t_2]} \Omega_t \) instead of \( \mathcal{C} = \bigcup_{t \in [t_1, t_2]} \{(t) \times \Omega_t \} \)?

Answer: \( \mathcal{C} \) gives the film of the life of \( \text{Obj} \) = the succession of the photos \( \Omega_t \) taken at each \( t \). (And \( \Omega_t \) is obtained from \( \mathcal{C} \) thanks to the pause feature at \( t \).) Whereas \( \bigcup_{t \in [t_1, t_2]} \Omega_t \) is just one photo = the superposition of all the photos on an unique photo: The film is superimposed on one photo... and we do not distinguish the past from the present.

**Definition 2.1** In short, and with (2.1) (relative to \( \text{Obj} \)) and \( m \in \mathbb{N}^* \), a Eulerian function is a function
\[
\mathcal{E}ul : \begin{cases} 
\mathcal{C} \rightarrow \mathbb{R}^m \text{ (or more generally a suitable set of tensors)} \\
(t, p) \rightarrow \mathcal{E}ul(t, p). 
\end{cases} \tag{2.2}
\]

The spatial variable \( p \) is the Eulerian variable.

**Example 2.2** \( \mathcal{E}ul(t, p) = \theta(t, p) \in \mathbb{R} = \text{temperature of the particle } \text{P}_{\text{Obj}} \text{ which is at } t \text{ at } p = \Phi(t, \text{P}_{\text{Obj}}); \)

**Example 2.3** \( \mathcal{E}ul(t, p) = \vec{u}(t, p) \in \mathbb{R}^n = \text{force applied on the particle which is at } t \text{ at } p. \)

**Definition 2.4** In details, a function \( \mathcal{E}ul \) being given as in (2.2), the associated Eulerian function \( \vec{\mathcal{E}}ul \) is the function defined by
\[
\vec{\mathcal{E}}ul : \begin{cases} 
\mathcal{C} \rightarrow \mathbb{R}^m \times \mathbb{R}^n \text{ (or } \mathcal{C} \times \text{ some suitable set of tensors)} \\
(t, p) \rightarrow \vec{\mathcal{E}}ul(t, p) = ((t, p); \mathcal{E}ul(t, p)), 
\end{cases} \tag{2.3}
\]

and is called “a field of functions”; So \( \vec{\mathcal{E}}ul(t, p) \) is the “pointed function” at \((t, p)\) (in time-space).

So, the range \( \text{Im}(\vec{\mathcal{E}}ul) = \vec{\mathcal{E}}ul(\mathcal{C}) \) of an Eulerian function \( \vec{\mathcal{E}}ul \) is the graph of \( \mathcal{E}ul \). (Recall: The graph of a function \( f : x \in A \rightarrow (x, f(x)) \in A \times B \) is the subset \( \{(x, f(x)) \in A \times B \} \subset A \times B \); gives the “drawing of \( f \)”.

And \( \vec{\mathcal{E}}ul \) is written \( \mathcal{E}ul \) for short, if there is no ambiguity.

**Question:** Why introduce \( \vec{\mathcal{E}}ul \)? Isn’t \( \mathcal{E}ul \) sufficient?

**Answer:** With \( p_+ = (t, p) \in \mathbb{R} \times \mathbb{R}^3 \), a value \( y = \mathcal{E}ul(p_+) = \mathcal{E}ul(t, p) \) is drawn on the \( y \)-axis, when the “pointed value” \( \vec{\mathcal{E}}ul(p_+) = (p_+, y = (p_+, \mathcal{E}ul(p_+))) \) is drawn on the graph of \( \vec{\mathcal{E}}ul \).

E.g., a vector \( \vec{v}(p_+) = \overrightarrow{AB} \in \mathbb{R}^3 \) (bipoint vector) can be drawn at any point, while the “pointed vector” \( \vec{\mathcal{E}}ul(p_+) = (p_+, \vec{v}(p_+)) \) is \( \overrightarrow{AB} \) drawn at \( p_+ \).

(Also, (2.3) emphasizes the difference between a Eulerian vector field and a Lagrangian vector function, see (4.2)).
Example 2.5 1. $\dot{\theta}(t,p) = ((t,p);\theta(t,p)) =$ temperature of the particle $P_{Oby}$ which is at $t$ at $p = \Phi(t, P_{Oby}) \in \mathbb{R}^3$. Usually represented by a color at $(t,p)$ (on the graph of $\theta$): On the photo at $t$, the colors gives the different temperatures at different $p$.

2. $\vec{v}(t,p) = ((t,p);\vec{v}(t,p)) =$ a force on the particle $P_{Oby}$ which is at $t$ at $p$. Usually represented by an arrow at $(t,p)$: On the graph of $\vec{v}$. So on the photo at $t$, you see the different vectors at different $p$.

At $t$, with $Eul_t(p) := Eul(t,p)$, the Eulerian field at $t$ is

$$Eul_t : \begin{cases} 
\Omega_t \rightarrow \Omega_t \times L^\infty(\mathbb{R}^3) \\
p \rightarrow Eul_t(p) := (p, Eul_t(p)).
\end{cases}$$

(4.4)

Remark 2.6 E.g., the initial framework of Cauchy (for his description of forces) is Eulerian: The Cauchy stress vector $\vec{t} = \sigma \cdot \vec{a}$ is considered at the actual time $t$ at a point $p_t \in \Omega_t$. (It is not Lagrangian.)

### 2.2 Eulerian velocity (spatial velocity) and speed

Consider a particle $P_{Oby}$ and its (regular) trajectory $\Phi_{P_{Oby}} : t \rightarrow p(t) = \Phi_{P_{Oby}}(t)$, cf. (1.10).

**Definition 2.7** In short, the Eulerian velocity of the particle $P_{Oby}$ which is at $t$ at $p = \Phi(t, P_{Oby})$ is the vectorial valued map defined on $\mathcal{C} = \bigcup_{t \in [t_1, t_2]} \{\{t\} \times \Omega_t\}$ by

$$\vec{v}(t,p) := \Phi_{P_{Oby}}'(t) = \frac{d\Phi_{P_{Oby}}(t)}{dt} = \frac{\Phi_{P_{Oby}}(t+h) - \Phi_{P_{Oby}}(t)}{h} = \lim_{h \rightarrow 0} \frac{\Phi_{P_{Oby}}(t+h) - \Phi_{P_{Oby}}(t)}{h}$$

i.e. $\vec{v}(t,p)$ is the tangent vector at $t$ at $p = \Phi_{P_{Oby}}(t)$ to the trajectory $\Phi_{P_{Oby}}$. (It depends on the chosen unit of time, e.g. per second, or per hour...). Also written

$$\vec{v}(t,p) = \frac{\partial \Phi}{\partial t}(t, P_{Oby}).$$

(2.6)

In details, cf. (2.3), the Eulerian velocity is the function defined with (2.5) by

$$\vec{v}(t,p) = ((t,p), \vec{v}(t,p))$$

(2.7)

(pointed vector), and it is represented by the vector $\vec{v}(t,p)$ drawn at $(t,p)$ (on the graph of $\vec{v}$).

**Remark 2.8** $\frac{d\Phi_{P_{Oby}}(t)}{dt} = \vec{v}(t, \Phi_{P_{Oby}}(t))$, cf. (2.5), is often written

$$\frac{dp}{dt}(t) = \vec{v}(t,p(t)), \text{ or } \frac{d\vec{x}}{dt}(t) = \vec{v}(t,\vec{x}(t)),$$

(2.8)

where $p(t) := \Phi_{P_{Oby}}(t)$, the last equality with a chosen origin $O$ and $\vec{x}(t) = \overrightarrow{OP(t)} = \vec{v}(t,P_{Oby})$, cf. (1.6).

Such an equation is the prototype of an ODE (ordinary differential equation) solved with the Cauchy-Lipschitz theorem, see § 6 and remark 1.3. (A Lagrangian velocity does not produce an ODE, see (4.9).)

**Definition 2.9** If an observer chooses a Euclidean dot product $(\cdot,\cdot)_g$ (e.g. built with the foot or the meter cf. § B.1, the associated norm being $||\cdot||_g$, then the length $||\vec{v}(t,p)||_g$ is named the speed of $P_{Oby}$, or scalar velocity of $P_{Oby}$ (e.g. given in ft/s or in m/s).

And the context must remove the ambiguitities: the “velocity” is either the vector velocity $\vec{v}(t,p) = \Phi_{P_{Oby}}'(t)$ (depends on the time unit), or the speed $||\vec{v}(t,p)||_g$ (also depends on the length unit).

### 2.3 Spatial derivative of the Eulerian velocity

Let $t \in [t_1, t_2]$ and $Eul_t(p) := Eul(t,p)$. Here $Eul_t : \Omega_t \rightarrow \mathbb{R}^n$ is supposed to be regular in $\Omega_t$.

**Definition 2.10** The space derivative $dEul$ of the Eulerian function $Eul$ is the differential $dEul_t$ of the function $Eul_t$, that is, $dEul$ is defined at $t \in \Omega_t$ by, for all $\vec{w} \in \mathbb{R}^n$ vector at $p$,

$$dEul_t(p) \vec{w} := dEul_t(p) \vec{w} = \lim_{h \rightarrow 0} \frac{Eul_t(p + h \vec{w}) - Eul_t(p)}{h} = \lim_{h \rightarrow 0} \frac{Eul_t(p + h \vec{w}) - Eul_t(p)}{h}.$$ 

(2.9)

Thus $dEul_t(p)$ gives in $\Omega_t$ (the photo at $t$) the spatial rate of variations of $Eul$ at $p$.

E.g., the space derivative $dv$ of the Eulerian velocity field is, at $t$ at $p \in \Omega_t$, for all $\vec{w} \in \mathbb{R}^3$,

$$d\vec{v}(t,p) \vec{w} = \lim_{h \rightarrow 0} \frac{\vec{v}(t,p + h \vec{w}) - \vec{v}(t,p)}{h} = \lim_{h \rightarrow 0} \frac{\vec{v}(t,p + h \vec{w}) - \vec{v}(t,p)}{h}.$$ 

(2.10)
2.4 The convective objective term $df, \vec{v}$, written $(\vec{v}, \text{grad})f$ in a basis...

Recall: If $\Omega$ is an open set in $\mathbb{R}^n$ and if $f : \Omega \to \mathbb{R}$ is differentiable at $p$, then its differential at $p$ is the linear map $df(p) : \mathbb{R}^n \to \mathbb{R}$ defined by, for all $\vec{u} \in \mathbb{R}^n$ (vector at $p$),

$$df(p)\vec{u} = \lim_{h \to 0} \frac{f(p + h\vec{u}) - f(p)}{h} \quad (2.11)$$

**Quantification:** Let $(e_i)$ be a Cartesian basis in $\mathbb{R}^n$, and let (usual definition)

$$\frac{\partial f}{\partial x^i}(p) := df(p)e_i, \quad \text{and} \quad [df(p)]_{e^i} = (\frac{\partial f}{\partial x^i}(p)) \quad (2.12)$$

That is, $(e^i)$ being the dual basis of $(e_i)$, cf. (A.33), we have $df(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) e_i$, and the linear form $df(p)$ is represented by a line matrix. And we get, with $\vec{u} = \sum_{i=1}^{n} u^i e_i$,

$$df(p)\vec{u} = [df(p)]_{e^i} [u^i] = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p) u^i = \sum_{i=1}^{n} u^i \frac{\partial f}{\partial x^i}(p) \quad (2.13)$$

We have thus defined the operator (the linear map) relative to a basis $(e_i)$:

$$\vec{u}, \text{grad} = \sum_{i=1}^{n} u^i \frac{\partial f}{\partial x^i} : \left\{ \begin{array}{l} C^1(\Omega; \mathbb{R}) \to C^0(\Omega; \mathbb{R}) \vspace{3pt} \\ f \mapsto (\vec{u}, \text{grad})(f) = \sum_{i=1}^{n} u^i \frac{\partial f}{\partial x^i} \quad (= df(\vec{u})). \end{array} \right. \quad (2.14)$$

For vector valued functions $\vec{f} : \Omega \to \mathbb{R}^m$, the above steps apply to the components of $\vec{f}$ in a basis $(\vec{b}_i)$ in $\mathbb{R}^m$: If $\vec{f} = \sum_{i=1}^{m} f^i \vec{b}_i$, then

$$d\vec{f}, \vec{u} = \sum_{i=1}^{m} (df^i, \vec{u}) \vec{b}_i = \sum_{i=1}^{m} ((\vec{u}, \text{grad}) f^i) \vec{b}_i, \quad \text{and} \quad [d\vec{f}, \vec{b}]_{e^i} = \left[ \frac{\partial f^i}{\partial x^j} \right] \quad \text{(the Jacobian matrix)} \quad (2.15)$$

**Application:** Consider a motion $\Phi_{P_{0t}}$ of a particle $P_{0t} \in \text{Obj}$, cf. (1.10), let $t \in \mathbb{R}$, let $p = \Phi_{P_{0t}}(t)$, let $\vec{v}(t,p) = \dot{\Phi}_{P_{0t}}(t)$ (the Eulerian velocity at $t$ at $p$). And consider a differentiable Eulerian function $\mathcal{E}u_l$, cf. (2.2), and let $\mathcal{E}u_l(t,p) =$ noted $\mathcal{E}u_{l1}(p)$. Then, with $f = \mathcal{E}u_{l1}$ and $\vec{u} = \vec{v}(t,p)$ in (2.11) we define:

**Definition 2.11** The convective derivative of the Eulerian function $\mathcal{E}u_{l1}$ is defined at $t$ at $p_t$ by (derivative along the trajectory)

$$(d\mathcal{E}u_{l1}, \vec{u}_l)(p) = \mathcal{E}u_l(t,p).\vec{v}(t,p) := \mathcal{E}u_{l1}(p)\vec{v}(t,p) = \lim_{h \to 0} \left( \frac{\mathcal{E}u_{l1}(p+h\vec{v}(t,p)) - \mathcal{E}u_{l1}(p)}{h} \right). \quad (2.16)$$

**Quantification:** Let $(e_i)$ be a Cartesian basis in $\mathbb{R}^n$. Then (2.13) gives

$$d\mathcal{E}u_{l1}, \vec{u} := (\vec{v}, \text{grad})\mathcal{E}u_{l1} = \sum_{i=1}^{n} v^i \frac{\partial \mathcal{E}u_{l1}}{\partial x^i} \quad \text{the convective derivative in a basis}. \quad (2.17)$$

2.5 ... and the subjective $\text{grad} f$ (depends on a Euclidean dot product)

An observer chooses a distance unit (foot, meter) and uses the associated Euclidean dot product $(\cdot, \cdot)_g$ in $\mathbb{R}^n$, cf. §B.2 (the following results will depend on $(\cdot, \cdot)_g$, i.e. on the observer).

Let $\Omega$ be an open set in $\mathbb{R}^n$ and $f \in C^1(\Omega; \mathbb{R})$ (scalar valued function, e.g. $f = \mathcal{E}u_{l1} \in C^1(\Omega; \mathbb{R})$). Let $p \in \Omega$.

**Definition 2.12** The $(\cdot, \cdot)_g$-Riesz representation vector $\text{grad}_g f(p) \in \mathbb{R}^n$ of the differential form $df(p)$ is defined by, cf. (C.1),

$$\forall \vec{u} \in \mathbb{R}^n, \quad (\text{grad}_g f(p), \vec{u})_g = df(p).\vec{u}, \quad \text{written} \quad (\text{grad}_g f, \vec{u})_g = df(\vec{u}), \quad (2.18)$$

and is called the gradient of $f$ at $p$ relative to $(\cdot, \cdot)_g$. NB: It depends on $(\cdot, \cdot)_g$, see (C.10).
Quantification with a basis ($\hat{e}_i$): Let $(\hat{e}_i)$ be a Cartesian basis in $\mathbb{R}^n$, and let $\frac{\partial f}{\partial x^i} := df_i, \text{ cf. (2.12.)}$ Then (2.18) gives $[df](p)|_x [\hat{u}] |_x = [\hat{\nabla}_g f(p)]|_x [g]|_x [\hat{u}] |_x$ for all $\hat{u} \in \mathbb{R}^n$, thus (with $[g]|_x$ symmetric)

$$[\hat{\nabla}_g f(p)]|_x = [g]|_x [df(p)]|_x^T \quad \text{(column matrix).}$$

(2.19)

That is $\hat{\nabla}_g f = \sum_{i=1}^n a^i \hat{e}_i$, where $a^i = \sum_{j=1}^n g_{ij} \frac{\partial f}{\partial x^j}$ for all $i$.

Case (2.18) is a $(\cdot, \cdot)_g$-orthonormal basis: Then $[\hat{\nabla}_g f]|_x = [df]|_x^T$ (since $[g]|_x = I$) and $\hat{\nabla}_g f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \hat{e}_i$.

Be careful: The gradient $\hat{\nabla}_g f$ depends on $(\cdot, \cdot)_g$, cf. (2.18)-(2.19), while $\langle \tilde{u}, \hat{\nabla}_g f \rangle$ does not, cf. (2.14): It only depends on the choice of a basis for the definition of the $\frac{\partial f}{\partial x^i}$.

For vector valued functions $\hat{f} : \Omega \to \mathbb{R}^m$, the above steps apply to the components $f^i$ of $\hat{f}$ relative to a basis $(\hat{e}_i)$ in $\mathbb{R}^m$. But there is a notation problem...:

1. For differential $\hat{f}$ there is just one “Jacobian matrix” (relative to a given basis), cf. (2.15), sometimes also called the “gradient matrix” (although no Euclidean dot product is required).

2. But the “gradient of $\hat{f}$”... depends on the authors: It could mean the Jacobian matrix or its transposed, and the use of some Euclidean dot product (which one?) may be required... or not...

3. In the objective setting of this manuscript, we will never talk about the ‘gradient of a vectorial function $\hat{f}$’; only the differential (objective) and the Jacobian of $\hat{f}$ will be used (after a choice of a basis).

Exercise 2.13 A Euclidean setting being chosen, prove

$$(\hat{v}, \hat{\nabla}_g \hat{v}) = \frac{1}{2} \hat{\nabla}(||\hat{v}||^2) + \hat{\text{rot}}\hat{v} \land \hat{v}. $$

Answer. Euclidean basis $(\hat{E}_i)$, Euclidean dot product $(\cdot, \cdot)_g = \text{noted} (\cdot, \cdot)$, associated norm $||.||_g = \text{noted} ||.||$. Thus $\hat{v} = \sum_{i=1}^n v^i \hat{E}_i$ gives $||\hat{v}||^2 = \sum v^i \hat{v}^i$, thus $\frac{\partial ||\hat{v}||^2}{\partial x^i} = 2v^i \frac{\partial v^i}{\partial x^i}$, for any $k = 1, 2, 3$. And, the first component of $\hat{\text{rot}}\hat{v}$ is $\hat{\text{rot}}\hat{v} = \begin{bmatrix} \begin{vmatrix} \frac{\partial v^3}{\partial x^2} & -\frac{\partial v^2}{\partial x^3} \\ \frac{\partial v^2}{\partial x^1} & -\frac{\partial v^1}{\partial x^2} \\ \frac{\partial v^1}{\partial x^3} & -\frac{\partial v^3}{\partial x^1} \end{vmatrix} 
\end{bmatrix}$, idem for $\hat{\text{rot}}\hat{v}^2$ and $\hat{\text{rot}}\hat{v}^3$ (circular permutation). Thus (first component) $\hat{\text{rot}}\hat{v} = (\begin{bmatrix} \begin{vmatrix} \frac{\partial v^3}{\partial x^2} & -\frac{\partial v^2}{\partial x^3} \\ \frac{\partial v^2}{\partial x^1} & -\frac{\partial v^1}{\partial x^2} \\ \frac{\partial v^1}{\partial x^3} & -\frac{\partial v^3}{\partial x^1} \end{vmatrix} 
\end{bmatrix}) = (\hat{\text{rot}}\hat{v} \land \hat{v})$. Thus (first component) $\hat{\text{rot}}\hat{v} \land \hat{v} = \begin{bmatrix} \begin{vmatrix} \frac{\partial v^3}{\partial x^2} & -\frac{\partial v^2}{\partial x^3} \\ \frac{\partial v^2}{\partial x^1} & -\frac{\partial v^1}{\partial x^2} \\ \frac{\partial v^1}{\partial x^3} & -\frac{\partial v^3}{\partial x^1} \end{vmatrix} 
\end{bmatrix}$, idem for the other components.

2.6 Streamline (current line)

Let $t \in \mathbb{R}$. Consider the photo $\Omega_t = \hat{\Phi}_t(\text{Obj})$. Let $p_t \in \Omega_t$, $\varepsilon > 0$, and consider a spatial curve in $\Omega_t$ at $p_t$:

$$c_{p_t} : \begin{cases} 
| - \varepsilon, \varepsilon | \rightarrow \Omega_t 
\begin{aligned}
&| - \varepsilon, \varepsilon | 
&s \rightarrow q(s) = c_{p_t}(s)
\end{aligned}
\end{cases}, \text{ s.t. } c_{p_t}(0) = p_t. \quad (2.20)
$$

So $s$ is a (spatial) curvilinear abscissa (dimension of a length), and $c_{p_t}(| - \varepsilon, \varepsilon |) = \text{Im}(c_{p_t})$ is drawn in $\Omega_t$ (drawn in the photo at $t$), and $\varepsilon$ is small enough for $c_{p_t}(| - \varepsilon, \varepsilon |)$ to be in $\Omega_t$.

Definition 2.14. $\hat{v} \text{ being the Eulerian velocity field of a motion } \hat{\Phi}, \text{ (a parametric) streamline through } p_t \text{ is a curve } c_{p_t} \text{ solution of the differential equation }$

$$\frac{dc_{p_t}}{ds}(s) = \hat{v}(c_{p_t}(s)) \text{ with } c_{p_t}(0) = p_t. \quad (2.21)$$

And $\text{Im}(c_{p_t}) = c_{p_t}(| - \varepsilon, \varepsilon |)$ (in $\Omega_t$) is the geometric associated streamline. And (2.21) is also written

$$\frac{dq}{ds}(s) = \hat{v}(q(s)) \text{ with } q(0) = p_t, \text{ or } \frac{d\bar{x}}{ds}(s) = \hat{v}(\bar{x}(s)) \text{ with } \bar{x}(0) = \hat{\text{Ob}} p_t \quad (2.22)$$

once an origin $O$ has been chosen in $\mathbb{R}^n$. 

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NB: (2.8) cannot be confused with (2.21): the problem (2.8) is time dependent, while, at each \( t \), the problem (2.21) is time independent (the variable is the spatial variable \( s \)).

**Usual notation:** A Cartesian coordinate system \( \mathcal{R} = (O, (e_i)) \) is chosen at \( t \), and \( \vec{x}(s) := \overrightarrow{Oe_i}(s) = \overrightarrow{Oq(s)} = \sum_{i=1}^{n} x_i(s) e_i \), thus \( \frac{d}{ds} \vec{x}(s) = \sum_{i=1}^{n} \frac{dx_i}{ds} e_i \). Thus (2.22) reads as the differential system of equation in \( \mathbb{R}^{n} \)

\[
\forall i = 1, \ldots, n, \quad \frac{dx_i}{ds}(s) = v_i(t, x_1(s), \ldots, x_n(s)) \quad \text{with} \quad x_i(0) = (\overrightarrow{Oe_i}),
\]

(2.23)

(the \( x_i : s \to x_i(s) \) are the \( n \) solutions of the \( n \) equations). Also written

\[
\frac{dx_1}{v_1} = \ldots = \frac{dx_n}{v_n} = ds.
\]

Whatever the notation, it is the differential system of \( n \) equations (2.23) which must be solved.

(With duality notations, \( \frac{dx_i}{v_i}(s) = v^i(t, x^1(s), \ldots, x^n(s)) \).)

### 2.7 Material time derivative (dérivées particulières)

#### 2.7.1 Usual definition

Consider a regular motion \( \vec{\Phi} \), cf. (1.5), and the associated Eulerian velocity field \( \vec{v} \), cf. (2.5).

Goal: to measure the variations of a Eulerian function \( \text{Eul} \) along the trajectory of a particle \( \text{P}_{\text{Obj}} \).

E.g., \( \text{P}_{\text{Obj}} \) being at \( t \) \( p(t) = \vec{\Phi}(t, \text{P}_{\text{Obj}}) \), we look for the variations of the temperature \( \theta(t, p(t)) \) of \( \text{P}_{\text{Obj}} \) through time.

Thus consider a Eulerian function \( \text{Eul} \) and

\[
\text{g}_{\text{P}_{\text{Obj}}} (t) := \text{Eul}(t, \vec{\Phi}_{\text{P}_{\text{Obj}}}(t)) = \text{Eul}(t, p(t)) \quad \text{when} \quad p(t) := \vec{\Phi}_{\text{P}_{\text{Obj}}}(t).
\]

**Definition 2.15** The Material time derivative of \( \text{Eul} \) at \( (t, p(t)) \) is

\[
\frac{D\text{Eul}}{Dt}(t, p(t)) := \text{g}_{\text{P}_{\text{Obj}}}'(t) = \text{the derivative of } g \text{ at } t \quad (= \lim_{h \to 0} \frac{\text{Eul}(t+h, p(t+h)) - \text{Eul}(t, p(t))}{h}).
\]

With (2.25) and \( \vec{\Phi}_{\text{P}_{\text{Obj}}}'(t) = \vec{v}(t, p(t)) \) (Eulerian velocity), we get

\[
(\text{g}_{\text{P}_{\text{Obj}}}'(t)) = \frac{D\text{Eul}}{Dt}(t, p(t)) = \frac{\partial \text{Eul}}{\partial t}(t, p(t)) + d\text{Eul}(t, p(t)) \vec{v}(t, p(t)),
\]

that is, in \( \mathcal{C} = \bigcup_{i \in [1, n]} \{t \times \Omega_t\} \),

\[
\frac{D\text{Eul}}{Dt} := \frac{\partial \text{Eul}}{\partial t} + d\text{Eul} \vec{v}.
\]

**Remark 2.16** • The notation \( \frac{d}{dt} \) (lowercase letters) concerns a function of one variable, e.g. \( \frac{dx_{\text{P}_{\text{Obj}}}}{dt}(t) := \lim_{h \to 0} \frac{g_{\text{P}_{\text{Obj}}}(t+h) - g_{\text{P}_{\text{Obj}}}(t)}{h} \).

• The notation \( \frac{\partial}{\partial t} \) concerns a function of more than one variable, e.g. \( \frac{\partial \text{Eul}}{\partial t}(t, p) = \lim_{h \to 0} \frac{\text{Eul}(t+h, p(t)) - \text{Eul}(t, p(t))}{h} \).

• The notation \( \frac{D}{Dt} \) (capital letters) concerns a Eulerian function when the variables \( t \) and \( p \) are made dependent thanks to a motion, that is \( \frac{D\text{Eul}}{Dt}(t, p(t)) := \lim_{h \to 0} \frac{\text{Eul}(t+h, p(t+h)) - \text{Eul}(t, p(t))}{h} \).

**Other notations** (often practical, but might be ambiguous if composed functions are considered):

\[
\frac{D\text{Eul}}{Dt}(t, p(t)) \text{ noted } \frac{d\text{Eul}(t, p(t))}{dt}, \quad \text{and} \quad \frac{D\text{Eul}}{Dt}(t_0, p(t_0)) \text{ noted } \frac{d\text{Eul}(t, p(t))}{dt} \bigg|_{t=t_0}.
\]

**Quantification.** Let \( (\vec{e}_1, \ldots, \vec{e}_n) \) be a Cartesian basis and \( (dx_i) \) its dual basis (we use duality notations: use classic notations if you prefer). Then

\[
D\text{Eul} = \frac{\partial \text{Eul}}{\partial t} dt + d\text{Eul} = \frac{\partial \text{Eul}}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial \text{Eul}}{\partial x_i} dx_i,
\]

or \( D\text{Eul} = \frac{\partial \text{Eul}}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial \text{Eul}}{\partial x_i} dx_i \), with duality notations.
Further derivations: The variables \( t \) and \( x_i \) are independent (classical mechanics), thus the Schwarz Theorem give the commutativity (as soon as \( \mathcal{E}ul \) is \( C^2 \))
\[
\frac{\partial}{\partial x_i} \frac{\partial \mathcal{E}ul}{\partial t}(t,p) = \frac{\partial}{\partial t} \frac{\partial \mathcal{E}ul}{\partial x_i}(t,p) \text{ noted } \frac{\partial^2 \mathcal{E}ul}{\partial x_i \partial t}(t,p),
\] (2.31)
for all \( i = 1, ..., n \). Which is written
\[
d(\frac{\partial \mathcal{E}ul}{\partial t}) = \frac{\partial (d \mathcal{E}ul)}{\partial t} \text{ which means } \sum_{i=1}^{n} \frac{\partial (d \mathcal{E}ul)}{\partial x_i} dx_i = \sum_{i=1}^{n} \frac{\partial^2 \mathcal{E}ul}{\partial x_i \partial t} dx_i \text{ noted } \sum_{i=1}^{n} \frac{\partial^2 \mathcal{E}ul}{\partial x_i \partial t} dx_i.
\] (2.32)
That is, \( d(\frac{\partial \mathcal{E}ul}{\partial t}) \cdot \vec{w} = \frac{\partial (d \mathcal{E}ul)}{\partial t} \cdot \vec{w} = \sum_{i=1}^{n} \frac{\partial^2 \mathcal{E}ul}{\partial x_i \partial t} w_i \) when \( \vec{w} = \sum_{i=1}^{n} w_i \hat{e}_i \).

2.7.2 Bis: Space-time definition

\( \mathcal{E}ul \) is a "time-space" function (defined on \( C \subset \mathbb{R} \times \mathbb{R}^n \)) supposed to be \( C^1 \). Its differential is thus defined in \( \mathbb{R} \times \mathbb{R}^n \).

**Definition 2.17** and is called the "total differential" (or "total derivative"), and is written \( D \mathcal{E}ul \).

Thus, if \( p_+ = (t, p) \in C \) and \( \vec{w}_+ = (w_0, \vec{w}) \in \mathbb{R} \times \mathbb{R}^n \) then, by definition of a differential,
\[
D \mathcal{E}ul(p_+), \vec{w}_+ := \lim_{h \to 0} \frac{\mathcal{E}ul(p_+ + hw_+) - \mathcal{E}ul(p_+)}{h},
\] (2.33)
that is,
\[
D \mathcal{E}ul(t,p), (w_0, \vec{w}) := \lim_{h \to 0} \frac{\mathcal{E}ul(t+hw_0, p+hw) - \mathcal{E}ul(t,p)}{h}.
\] (2.34)
And we recover (2.28):
\[
D \mathcal{E}ul(t,p) = \frac{\partial \mathcal{E}ul}{\partial t}(t,p) \, dt + d \mathcal{E}ul(t,p) \quad (= \text{the total differential}).
\] (2.35)

**Along a trajectory:** Let \( P_{\mathcal{O}bj} \in \mathcal{O}bj \), and consider its time-space trajectory
\[
\tilde{\Psi}_{P_{\mathcal{O}bj}} : \{ [t_1, t_2] \to \mathbb{R} \times \mathbb{R}^n \}
\] (2.36)
\[
t \to p_+(t) = (t, p(t)) = \tilde{\Psi}_{P_{\mathcal{O}bj}}(t) := (t, \tilde{\Phi}_{P_{\mathcal{O}bj}}(t)).
\] (2.37)
(So \( \text{Im}(\tilde{\Psi}_{P_{\mathcal{O}bj}}) = \text{graph}(\tilde{\Phi}_{P_{\mathcal{O}bj}}) \).) We get, with \( \tilde{\Phi}_{P_{\mathcal{O}bj}}'(t) = \tilde{v}(t,p(t)) \) the Eulerian velocity,
\[
\tilde{\Phi}_{P_{\mathcal{O}bj}}'(t) = (1, \tilde{\Phi}_{P_{\mathcal{O}bj}}'(t)) = (1, \tilde{v}(t,p(t))) \in \mathbb{R} \times \mathbb{R}^n.
\] (2.38)
Thus (2.25) reads \( g(t) = (Eul \circ \tilde{\Psi}_{P_{\mathcal{O}bj}})(t) = \mathcal{E}ul(\tilde{\Psi}_{P_{\mathcal{O}bj}}(t)) \), and (2.35) gives
\[
g'(t) = D \mathcal{E}ul(\tilde{\Psi}(t)), \tilde{\Psi}_{P_{\mathcal{O}bj}}'(t) = \frac{\partial \mathcal{E}ul}{\partial t}(t,p(t)),.1 + d \mathcal{E}ul(t,p(t)),.\tilde{v}(t,p(t)) \text{ noted } \frac{D \mathcal{E}ul}{D t}(t,p(t)),
\] (2.39)
which is (2.28): The material time derivative is the "total derivative" along the time-space trajectory \( \tilde{\Psi}_{P_{\mathcal{O}bj}} \).

**Quantification:** If \( \tilde{\Gamma} \) is a \((time)\) basis in \( \mathbb{R} \) with dual basis \( dt \), and \( \hat{e}_1, ..., \hat{e}_n \) is a Cartesian basis in \( \mathbb{R}^n \) with dual basis \( (dx_1, ..., dx_n) \), then we recover (2.30).

2.7.3 The material time derivative is a derivation

**Proposition 2.18** All the functions are Eulerian and supposed \( C^1 \).

- **Linearity:**
\[
\frac{D(\mathcal{E}ul_1 + \lambda \mathcal{E}ul_2)}{D t} = \frac{D \mathcal{E}ul_1}{D t} + \lambda \frac{D \mathcal{E}ul_2}{D t}.
\] (2.40)
- **Product rules:** If \( \mathcal{E}ul_1, \mathcal{E}ul_2 \) are scalar valued functions then
\[
\frac{D(\mathcal{E}ul_1 \mathcal{E}ul_2)}{D t} = \frac{D \mathcal{E}ul_1}{D t} \mathcal{E}ul_2 + \mathcal{E}ul_1 \frac{D \mathcal{E}ul_2}{D t}.
\] (2.41)
And if \( \vec{w} \) is a vector field and \( T \) a compatible tensor \((so\ that\ T \cdot \vec{w}\ is\ meaningful)\) then
\[
\frac{D(T \cdot \vec{w})}{D t} = \frac{D T}{D t} \cdot \vec{w} + T \cdot \frac{D \vec{w}}{D t}
\] (2.42)
Proof. Let \( i = 1, 2 \), and \( g_i \) defined by \( g_i(t) := \mathcal{E}u_i(t,p(t)) \) where \( p(t) = \tilde{\Phi}(t,p_{0i}) \).

- \((g_1 + g_2)' = g_1' + g_2'\) gives (2.39).
- On the one hand \( \frac{D(T(w),\bar{\omega})}{Dt} = \frac{\partial(T,w)}{\partial t} + d(T,w),\bar{\omega} = \frac{\partial T}{\partial t} w + T, \frac{\partial w}{\partial t} + (d(T,v),\bar{\omega}) + T, (d\bar{w},\bar{\omega}) \) and on the other hand \( \frac{DT}{Dt} w + T, \frac{D\bar{w}}{Dt} = (\frac{\partial T}{\partial t} d + d(T,v),\bar{\omega}) + T, (d\bar{w},\bar{\omega}) \). Thus (2.40)-(2.41).

\[2.7.4\text{ Commutativity issue}\]

**Proposition 2.19** The material time derivative \( \frac{D}{Dt} \) does not commute with the temporal derivation \( \frac{\partial}{\partial t} \) or with the spatial derivation \( \frac{\partial}{\partial x} \). We have

\[
\frac{\partial (\frac{D\mathcal{E}u}{Dt})}{\partial t} = \frac{\partial (\frac{D\mathcal{E}u}{Dt})}{\partial t} + d\mathcal{E}u, \frac{\partial \bar{\omega}}{\partial t} + d\mathcal{E}u \frac{\partial \bar{\omega}}{\partial t} \]
\[
= \frac{\partial^2 \mathcal{E}u}{\partial t^2} d\mathcal{E}u + \frac{\partial d\mathcal{E}u}{\partial t} \bar{\omega} + d\mathcal{E}u \frac{\partial \bar{\omega}}{\partial t} + d\mathcal{E}u \frac{\partial \bar{\omega}}{\partial t}, \quad (2.42)
\]

(The partial derivative \( \frac{\partial d\mathcal{E}u}{\partial t} \) and \( d\mathcal{E}u \) concern the independent variables \( t \) and \( p \), while the total derivative concerns the variables \( t \) and \( p \) when they are linked by \( p = \tilde{\Phi}(t,p_{0i}) \).)

**Proof.** \( \frac{\partial (\frac{D\mathcal{E}u}{Dt})}{\partial t} = \frac{\partial (\frac{D\mathcal{E}u}{Dt})}{\partial t} + d\mathcal{E}u, \frac{\partial \bar{\omega}}{\partial t} + d\mathcal{E}u \frac{\partial \bar{\omega}}{\partial t} \)

\( \text{cf. (2.31), thus (2.42)}. \)

\[d\frac{\partial (\frac{D\mathcal{E}u}{Dt})}{\partial t} = d\frac{\partial (\frac{D\mathcal{E}u}{Dt})}{\partial t} + d\mathcal{E}u, \frac{\partial \bar{\omega}}{\partial t} + d\mathcal{E}u \frac{\partial \bar{\omega}}{\partial t} \]

Thus \( \frac{\partial}{\partial t} \frac{D\mathcal{E}u}{Dt} \neq \frac{D}{Dt} \frac{\partial}{\partial t} \) in general.

\[2.8.\text{ Eulerian acceleration}\]

**Definition 2.21** In short: \( \tilde{\Phi} \) being a \( C^2 \) motion, the Eulerian acceleration of the particle \( p_{0i} \) which is at \( t \) at \( p_t = \tilde{\Phi}(t,p_{0i}) \) is

\[
\tilde{\gamma}(t,p_t) := \tilde{\Phi}_{P_{0i}}(t) = \frac{\partial^2 \tilde{\Phi}}{\partial t^2}(t,p_{0i}).
\]

This defines the Eulerian acceleration (vector) field \( \tilde{\gamma} \), cf. (2.3):

\[
\tilde{\gamma}(t,p_t) = ((t,p_t), \tilde{\gamma}(t,p_t)) \in \mathcal{C} \times \mathbb{R}^1.
\]

**Proposition 2.22** Let \( \vec{v} \) be the Eulerian velocity, that is, \( \vec{v}(t,p(t)) = \tilde{\Phi}_{P_{0i}}(t) \) when \( p(t) = \tilde{\Phi}(t,p_{0i}) \), cf. (2.5). Then

\[
\tilde{\gamma}(t,p_t) = \frac{D\vec{v}}{Dt}(t,p) + d\vec{v}(t,p), \quad \forall(t,p) \in \mathcal{C}.
\]

NB: \( \tilde{\gamma} \) is a non linear function of \( \vec{v} \).

**Proof.** Let \( g(t) = \vec{v}(t,p(t)) = \tilde{\Phi}_{P_{0i}}(t) \); Then \( \tilde{\gamma}(t,p(t)) = \tilde{\Phi}_{P_{0i}}''(t) = (\tilde{\Phi}_{P_{0i}}')(t) = g'(t) = \frac{D\vec{v}}{Dt}(t,p(t)) \).

**Corollary 2.23** If \( \vec{v} \) is \( C^2 \) then

\[
d\tilde{\gamma} = \frac{\partial (d\vec{v})}{\partial t} + d(\vec{v},\vec{v}) + d\vec{v},d\vec{v} = D(d\vec{v}) + d\vec{v},d\vec{v}.
\]
Proof. \[ \frac{D(d\vec{v})}{dt} = \frac{\partial(d\vec{v})}{dt} + d(d\vec{v}), \vec{v}, \text{ and } \vec{v} \text{ is (at least) } C^2, \text{ thus } d(d\vec{v}) = \text{noted } d^2\vec{v}. \]

Exercise 2.24 Prove:

\[ \frac{D(d\text{ul.}\vec{w})}{dt} = d\frac{\partial(d\text{ul.}\vec{w})}{dt} + d\text{ul.}\frac{\partial\vec{w}}{dt} + d^2\text{ul.}(\vec{w}, \vec{v}) + d\text{ul.}d\vec{w}, \vec{v}. \]  

(2.47)

And with \( g_{\text{P}_{\text{obj}}}''(t) = (g_{\text{P}_{\text{obj}}}')'(t) = \frac{D(D\text{ul.})}{dt}t, p(t)) = \text{noted } \frac{D(D\text{ul.})}{dt^2}t, p(t) \), prove:

\[ \frac{D^2\text{ul.}}{dt^2} = \frac{\partial^2\text{ul.}}{dt^2} + 2d\frac{\partial\text{ul.}}{dt} + d\text{ul.}d\frac{\partial\vec{v}}{dt} + d\text{ul.}d\vec{v}, \vec{v} \]  

(2.48)

\[ D\text{ul.} = D\text{ul.}\frac{\partial\vec{w}}{dt} + d\text{ul.}d\frac{\partial\vec{v}}{dt} + d\text{ul.}d\vec{v}, \vec{v} + d\text{ul.}d\vec{w}, \vec{v}. \]

hence (2.48). (Or use (2.42) and (2.47).)

Definition 2.25 If an observer chooses a Euclidean dot product \((\cdot, \cdot)\) (after having chosen a measuring unit), the associated norm being \(\|\cdot\|\), then the length \(\|\vec{v}(t, p)\|\) is the (scalar) acceleration of \(P_{\text{obj}}\).

Quantification: If a basis \((\vec{e}_i)\) is chosen, then (2.14) gives

\[ \vec{\gamma} = \frac{\partial\vec{v}}{dt} + (\vec{v}, \text{grad}) \vec{v} \left( = \frac{\partial\vec{v}}{dt} + d\vec{v}, \vec{v} \right). \]  

(2.49)

2.9 Taylor expansion of \(\tilde{\Phi}\)

Let \(P_{\text{obj}} \in \text{Obj} \) and \(t \in [t_1, t_2] \). Suppose \(\tilde{\Phi}_{P_{\text{obj}}} \in C^2([t_1, t_2]; \mathbb{R}^n) \). Its second-order (time) Taylor expansion reads, in the vicinity of \(t\),

\[ \tilde{\Phi}_{P_{\text{obj}}} (\tau) = \tilde{\Phi}_{P_{\text{obj}}} (t) + (\tau - t) \tilde{\Phi}_{P_{\text{obj}}} '(t) + \frac{(\tau - t)^2}{2} \tilde{\Phi}_{P_{\text{obj}}} ''(t) + o((\tau - t)^2), \]  

(2.50)

that is, with (2.5), (2.43) and \(p(\tau) = \tilde{\Phi}_{P_{\text{obj}}} (\tau)\),

\[ p(\tau) = p(t) + (\tau - t) \vec{v}(t, p(t)) + \frac{(\tau - t)^2}{2} \vec{\gamma}(t, p(t)) + o((\tau - t)^2). \]  

(2.51)

3 Motion on an initial configuration

Instead of working on \(\text{Obj}\), an observer may prefer to work from an initial configuration \(\Omega_{t_0} = \tilde{\Phi}(t_0, \text{Obj})\): This gives the “Lagrangian approach”. (It is not objective: Two observers may choose two different initial times.)

3.1 Definition

Let \(\text{Obj}\) be a material object and let \(\tilde{\Phi}\) be its motion, cf. (1.5). Let \(t_0 \in [t_1, t_2]\) and consider \(\Omega_{t_0} = \tilde{\Phi}_{t_0}(\text{Obj}) = \{ p \in \mathbb{R}^n : \exists P_{\text{obj}} \in \text{Obj} ; p = \tilde{\Phi}(t_0, P_{\text{obj}}) \} \) the configuration at \(t_0\), cf. (1.9), called the initial configuration, or configuration of reference, relative to the observer who chooses \(t_0\).

Definition 3.1 The motion of \(\text{Obj}\) relative to the initial configuration \(\Omega_{t_0} = \tilde{\Phi}(t_0, \text{Obj})\) is

\[ \Phi^{t_0} : \{ [t_1, t_2] \times \Omega_{t_0} \rightarrow \mathbb{R}^n \} \]

\[ (t, p_{t_0}) \mapsto p_t = \Phi^{t_0}(t, p_{t_0}) := \tilde{\Phi}(t, P_{\text{obj}}) \text{ when } p_{t_0} = \tilde{\Phi}(t_0, P_{\text{obj}}). \]  

(3.1)

So, \(p_t = \Phi^{t_0}(t, p_{t_0}) := \tilde{\Phi}(t, P_{\text{obj}})\) is the position at \(t\) of the particle \(P_{\text{obj}}\) (its actual position) which was at \(p_{t_0} = \Phi^{t_0}(t_0, p_{t_0}) := \tilde{\Phi}(t_0, P_{\text{obj}})\) at \(t_0\) (its initial position relative to the observer who chose \(t_0\).)
Notations: Once an initial time $t_0$ has been chosen by an observer, following Marsden and Hughes [12] we can use a capital letter $P$ for positions at $t_0$ and a lowercase letter $p$ for positions at $t$:

$$p_{t_0} \quad \text{named} \quad P = \Phi(t_0, P_{O\theta}) \in \Omega_{t_0}, \quad \text{and} \quad p_t \quad \text{named} \quad p = \tilde{\Phi}(t, P_{O\theta}) \in \Omega_t. \quad (3.2)$$

($P$ depends on $t_0$; When objectivity is under concern, we need to switch back to the notation $p_{t_0}$.)

Remark 3.2 • Talking about the motion of a position $p_{t_0}$ is absurd: A position does not move. Thus $\Phi^{t_0}$ has no existence without the definition, at first, of the motion $\tilde{\Phi}$, then the definition of $\Omega_{t_0} := \tilde{\Phi}(t_0, O\theta)$, and then the definition (3.1).

• The domain of definition of $\Phi^{t_0}$ is $[t_1, t_2] \times \Omega_{t_0}$; it depends on $t_0$, therefore $\Phi^{t_0}$ depends on $t_0$ and the index $t_0$ recalls it. E.g., a late observer with initial time $t_0' > t_0$ defines $\Phi^{t_0'}$ which domain of definition is $[t_1, t_2] \times \Omega_{t_0}$; Thus $\Phi^{t_0'} \neq \Phi^{t_0}$ because $\Omega_{t_0'} \neq \Omega_{t_0}$ in general.

• The following notation is also used:

$$\Phi^{t_0}(t, p_{t_0}) = \Phi(t; t_0, p_{t_0}), \quad (3.3)$$

e.g. for trajectories, the couple $(t_0, p_{t_0})$ being called the initial condition (and $t_0$ and $p_{t_0}$ are the initial conditions).

• And, if an origin $O \in \mathbb{R}^n$ is chosen by the observer, we may also use, with (1.6),

$$\vec{x}_{t_0} \quad \text{named} \quad \vec{x} = \overrightarrow{O P_{t_0}} = \varphi^{t_0}(t_0, \vec{x}_{t_0}) \quad \text{and} \quad \vec{x}_t \quad \text{named} \quad \vec{x} = \overrightarrow{O P_t} = \varphi^{t_0}(t, \vec{x}_{t_0}). \quad (3.4)$$

3.2 Diffeomorphism between configurations

Let $t_0, t \in [t_1, t_2]$. Let $\Omega_{t_0} = \tilde{\Phi}(t_0, O\theta) \quad \text{and} \quad \Omega_t = \tilde{\Phi}(t, O\theta)$ the past and actual configurations. With (3.1), define

$$\Phi^{t_0} : \quad \Omega_{t_0} \to \Omega_t \quad \text{where} \quad p_{t_0} = \tilde{\Phi}(t_0, P_{O\theta}) \quad \to \quad p_t = \Phi^{t_0}(p_{t_0}) := \Phi^{t_0}(t, p_{t_0}) = \tilde{\Phi}(t, P_{O\theta}) \quad (3.5)$$

Hypothesis: For all $t_0, t \in [t_1, t_2]$, the map $\Phi^{t_0} : \Omega_{t_0} \to \Omega_t$ is a $C^k$ diffeomorphism (a $C^k$ invertible function whose inverse is $C^k$), where $k \in \mathbb{N}^*$ depends on the required regularity.

Then $p_{t_0} = \tilde{\Phi}(t_0, P_{O\theta})$ and (3.5) gives $\Phi^{t_0}(\tilde{\Phi}(t_0, P_{O\theta})) = \tilde{\Phi}(t, P_{O\theta})$, true for all $P_{O\theta} \in O\theta$, thus $\Phi^{t_0} \circ \tilde{\Phi} = \Phi_t$, so, in other words $\Phi^{t_0}$ is the diffeomorphism defined by, for all $t_0, t \in [t_1, t_2]$,

$$\Phi^{t_0}_t := \tilde{\Phi}_t \circ (\tilde{\Phi}_{t_0})^{-1}. \quad (3.6)$$

In particular $\Phi^{t_0}_t \circ \Phi^{t_0}_{t_0} = (\tilde{\Phi}_t \circ (\tilde{\Phi}_{t_0})^{-1}) \circ (\tilde{\Phi}_{t_0} \circ (\tilde{\Phi}_t)^{-1}) = I$, thus

$$\Phi^{t_0}_t = (\Phi^{t_0})^{-1}. \quad (3.7)$$

3.3 Trajectories

Let $(t_0, p_{t_0}) \in [t_1, t_2] \times \Omega_{t_0}$ and with (3.1) define

$$\Phi^{t_0}_{p_{t_0}} : \quad [t_1, t_2] \to \mathbb{R}^n \quad t \to p(t) = \Phi^{t_0}_{p_{t_0}} (t) := \Phi^{t_0}(t, p_{t_0}) = \tilde{\Phi}(t, P_{O\theta}) \quad \text{when} \quad p_{t_0} = \tilde{\Phi}(t_0, P_{O\theta}). \quad (3.8)$$

Definition 3.3 $\Phi^{t_0}_{p_{t_0}}$ is called the (parametric) "trajectory of $p_{t_0}$", which means: $\Phi^{t_0}_{p_{t_0}}$ is the trajectory of the particle $P_{O\theta}$ that is located at $p_{t_0} = \tilde{\Phi}(t_0, P_{O\theta})$ at $t_0$. And

$$\text{Im}(\Phi^{t_0}_{p_{t_0}}) = \Phi^{t_0}_{p_{t_0}}([t_1, t_2]) = \bigcup_{t \in [t_1, t_2]} \{ \Phi^{t_0}_{p_{t_0}} (t) \} = \text{Im}(\Phi^{t_0}_{p_{t_0}}). \quad (3.9)$$

is the (geometric) "trajectory of $p_{t_0}$", cf. definition 1.7.

NB: The terminology trajectory for a $p_{t_0}$ is awkward, since a position $p_{t_0}$ does not move: It is indeed the trajectory of a particle $P_{O\theta}$ which is at $p_{t_0}$ at $t_0$ that must be understood.
3.4 Streaklines (lignes d’émission)

Let \( t_0 \) and \( T \) be respectively the start and the end of an observation.

**Definition 3.4** Let \( Q \) be a point in \( \mathbb{R}^n \). The streakline relative to \( Q \) is the set

\[
E_{t_0,T}(Q) = \{ q \in \Omega : \exists \tau \in [t_0, T] : q = \Phi_{\tau}^{-1}(Q) \}
\]

(3.10)

So \( E_{t_0,T}(Q) \) is the set of the positions (a line in \( \mathbb{R}^n \)) of the particles which were at \( Q \) at some \( \tau \in [t_0, T] \).

**Example 3.5** Smoke comes out of a chimney. Let \( Q \) be a point at the top of the chimney from where the particles are colored, and we make a film. At \( T \) we stop filming. Then we superimpose the photos of the film: The colored curve we see is the streakline.

In other words, for \( Q \in \mathbb{R}^n \), the streakline relative to \( Q \) is

\[
E_{t_0,T}(Q) = \bigcup_{\tau \in [t_0, T]} \{ \Phi_{\tau}^{-1}(Q) \} = \bigcup_{u \in [0, T-t_0]} \{ \Phi_{\tau-u}^{-1}(Q) \}.
\]

(3.11)

4 Lagrangian description

4.1 Lagrangian function

4.1.1 Definition

Consider a motion \( \tilde{\Phi} \), cf. (1.5). Choose a \( t_0 \in [t_1, t_2] \) (“in the past”), let \( \Omega_{t_0} = \tilde{\Phi}(t_0, \text{Obj}) \) (an “initial configuration”), and consider \( \Phi_{t_0} : \Omega_{t_0} \to \mathbb{R}^n \) the associated Lagrangian motion, cf. (3.1).

**Definition 4.1** In short, a Lagrangian function, relative to \( \text{Obj} \), \( \tilde{\Phi} \) and \( t_0 \), is a function

\[
\text{Lag}^{t_0} : \left[ [t_1, t_2] \times \Omega_{t_0} \to \text{some suitable set} \right)
\]

(4.1)

and \( p_{t_0} \) is called the Lagrangian variable relative to the choice \( t_0 \).

(To compare with (2.2): A Eulerian function does not depend on \( t_0 \).)

**Example 4.2** Scalar values: \( \text{Lag}^{t_0}(t, p_{t_0}) = \Theta^{t_0}(t, p_{t_0}) = \) temperature at \( t \) at \( p_{t_0} = \Phi_{t_0}^{-1}(p_{t_0}) = \tilde{\Phi}(t, \text{Obj}) \) of the particle \( P_{\text{Obj}} \) that was at \( p_{t_0} = \Phi_{t_0}^{-1}(p_{t_0}) = \tilde{\Phi}(t, \text{Obj}) \) at \( t_0 \).

So, continuing example 2.2, \( \Theta^{t_0}(p_{t_0}) = \theta(t, p_t) \) when \( p_t = \Phi_{t_0}(p_{t_0}) = \tilde{\Phi}(t_0, \text{Obj}) \) (the Eulerian function \( \theta \) does not depend on an initial time \( t_0 \)).

**Example 4.3** Vectorial values: \( \text{Lag}^{t_0}(t, p_{t_0}) = \tilde{U}^{t_0}(t, p_{t_0}) = \) force at \( t \) at \( p_{t_0} = \Phi_{t_0}^{-1}(p_{t_0}) = \tilde{\Phi}(t, \text{Obj}) \) acting on the particle \( P_{\text{Obj}} \) that was at \( p_{t_0} = \Phi_{t_0}^{-1}(p_{t_0}) = \tilde{\Phi}(t, \text{Obj}) \) at \( t_0 \).

So, continuing example 2.3, \( \tilde{U}^{t_0}(t, p_{t_0}) = \tilde{u}(t, p_t) \) when \( p_t = \Phi_{t_0}(p_{t_0}) = \tilde{\Phi}(t_0, \text{Obj}) \) (the Eulerian function \( \tilde{u} \) does not depend on an initial time \( t_0 \)).

**Definition 4.4** In details, a function \( \text{Lag}^{t_0} \) being given as in (4.1), the associated Lagrangian function \( \text{Lag}^{-t_0} \) is the map defined by

\[
\text{Lag}^{-t_0} : (t, p_{t_0}) \to \text{Lag}^{t_0}(t, p_{t_0}) \text{ when } p_t = \Phi_{t_0}(p_{t_0}).
\]

(4.2)

It means that \( \text{Lag}^{t_0}(t, p_{t_0}) \) is represented at \( (t, p_t) \), not at \( (t, p_{t_0}) \). And \( \text{Im}(\text{Lag}^{-t_0}) \neq \text{graph}(\text{Lag}^{t_0}) \). To compare with (2.3).

If \( t \) is fixed then we define \( \text{Lag}^{t_0}_{p_{t_0}} \), and if \( p_{t_0} \in \Omega_{t_0} \) is fixed then we define \( \text{Lag}^{t_0}_{p_{t_0}} \) by

\[
\text{Lag}^{t_0} : \left[ \Omega_{t_0} \to \mathbb{R}^n \right.
\]

(4.3)

and

\[
\text{Lag}^{t_0}_{p_{t_0}} : \left( [t_1, t_2] \to \mathbb{R}^n \right) \text{ (or some suitable set of tensors) }
\]

(4.4)
Definition 4.5 $p_{t_0}$ is also called “material point at $t_0$”, although it is a spatial position at $t_0$, as is the Eulerian variable $p_t = \Phi(t, P_{Oby})$ (which is not called the “material point at $t$”). Recall: The material particle is $P_{Oby}$.

By the way, the Lagrangian variable $p_t$ at $t$ is also called the “updated Lagrangian variable”: Gives $Lag^t(\tau, p_t)$ a value at $\tau$ at $p_t = \Phi_t^\tau(p_t)$).

4.1.2 A Lagrangian function is a two point tensor

With (4.2) and the range $\text{Im}(\tilde{\Phi}^t_{t_0}) = \{(t_0, p_t), Lag^{t_0}(t, p_{t_0}) : p_t = \Phi_t^t(p_{t_0}) \in \Omega_t\}$, and following Marsden and Hughes [12];

Definition 4.6 A Lagrangian function $Lag^{t_0}$ is called a “two point tensor” in reference to the points $p_{t_0}$ (departure) and $p_t$ (arrival).

4.2 Lagrangian function associated with a Eulerian function

4.2.1 Associated Lagrangian function

Let $\tilde{\Phi}$ be a motion, cf. (1.5). Let $\mathcal{E}ul$ be a Eulerian function, cf. (2.3;

Definition 4.7 Let $t_0 \in [t_1, t_2]$. Relative to $t_0$, the Lagrangian function $Lag^{t_0}$ associated with the Eulerian function $\mathcal{E}ul$ is defined by, for all $(t, p_t) \in C$,

$$Lag^{t_0}(t, p_{t_0}) := \mathcal{E}ul(t, p_t) \quad \text{when} \quad p_{t_0} = (\Phi_t^t)^{-1}(p_t).$$

That is, $Lag^{t_0}(t, p_{t_0}) = \mathcal{E}ul(t, \Phi_t^t(p_{t_0}))$ for all $(t, p_t) \in \mathbb{R} \times \Omega_{t_0}$. That is,

$$Lag_t^{t_0} := \mathcal{E}ul_t \circ \Phi_t^t.$$  

So, in terms of particles $P_{Oby}$ of Obj (and the motion $\tilde{\Phi}$),

$$Lag^{t_0}(t, \tilde{\Phi}(t_0, P_{Oby})) := \mathcal{E}ul(t, \tilde{\Phi}(t, P_{Oby})).$$

(Recall: a Eulerian function $\mathcal{E}ul$ does not depend on any initial time $t_0$.)

4.2.2 Remarks

- If you have a Lagrangian function, then you can associate the function

$$\mathcal{E}ul_t := Lag_t^{t_0} \circ (\Phi_t^t)^{-1}.$$  

(4.8)

But, 1- a Eulerian function is independent of any initial time $t_0$, which is not obvious with (4.8). E.g., we also have $\mathcal{E}ul_t := Lag_t^{t_0} \circ (\Phi_t^t)^{-1}$ for any other time $t_0'$, cf. (4.6),

And 2-, the range of $Lag_t^{t_0} \circ (\Phi_t^t)^{-1}$ is not a set of tensors a priori, cf. (4.1).

- A Lagrangian function depends on $t_0$ (while a Eulerian function doesn’t): The Lagrangian function $Lag^{t_0}$ of a late observer who chooses $t_0' > t_0$ will be different from $Lag^{t_0}$ since the domains of definition $\Omega_{t_0}$ and $\Omega_{t_0'}$ are different (in general). However $Lag^{t_0}$ and $Lag^{t_0'}$ define the same result at $t$ at $p_t = \tilde{\Phi}(t, P_{Oby})$, that is, $Lag^{t_0}(t, p_{t_0}) = Lag^{t_0'}(t, p_{t_0'}) = \mathcal{E}ul(t, p_t)$ a value at $t$ at the actual position $p_t = \tilde{\Phi}(t, P_{Oby}) = \Phi_t^t(p_{t_0}) = \Phi_t^t(p_{t_0'}$) of the particle $P_{Oby}$ at $t$.

- For one measurement, there is only one Eulerian function $\mathcal{E}ul$, while there are as many associated Lagrangian function $Lag^{t_0}$ as they are $t_0$ (as many as the number of observers).

4.3 Lagrangian velocity

4.3.1 Definition

Definition 4.8 In short: The Lagrangian velocity at $t$ at $p_t = \tilde{\Phi}(t, P_{Oby})$ of the particle $P_{Oby}$ is

$$\bar{V}^{t_0}(t, p_{t_0}) : \begin{cases} \mathbb{R} \times \Omega_{t_0} & \to \mathbb{R}^n \\ (t, p_{t_0}) & \to \bar{V}^{t_0}(t, p_{t_0}) := \tilde{\Phi}_{P_{Oby}}'(t) \quad \text{when} \quad p_{t_0} = \tilde{\Phi}(t_0, P_{Oby}) \end{cases}$$

(4.9)

Thus $\bar{V}^{t_0}(t, p_{t_0}) = \bar{v}(t, p_t)$, cf. (2.5): It is the tangent vector to the trajectory $\tilde{\Phi}_{P_{Oby}}$ at $p_t = \tilde{\Phi}_{P_{Oby}}(t)$. Thus $\bar{V}^{t_0}$ is a two-point tensor, cf. definition 4.6.
4.3. Lagrangian velocity

So $\vec{V}^t_0(t, p_{t_0})$ is the velocity $\vec{v}(t, p_t)$ at $p_t = \tilde{\Phi}(t, P_{Obj})$ (not at $p_{t_0}$) of the particle $P_{Obj}$ along its trajectory, relative to the particle that was at $p_{t_0}$ at $t_0$.

In details: With (4.9), the Lagrangian velocity is the two point vector field given by

$$\vec{V}^t_0(t, p_{t_0}) : \begin{cases} \mathbb{R} \times \Omega_{t_0} \to (\mathbb{R} \times \Omega_{t_0}) \times \mathbb{R}^3 \\ (t, p_{t_0}) \to \vec{V}^t_0(t, p_{t_0}) := ((t, p_t), \tilde{\Phi}_{P_{Obj}}'(t)), \quad \text{when } p_t = \Phi^t_0(t, p_{t_0}) = \tilde{\Phi}(t, P_{Obj}). \end{cases}$$  \hspace{1cm} (4.10)

If $t$ is fixed, or if $p_{t_0} \in \Omega_{t_0}$ is fixed, then we write

$$\vec{V}^t_0(p_{t_0}) := \vec{V}^t_0(t, p_{t_0}), \quad \text{or} \quad \vec{V}^t_0(t) := \vec{V}^t_0(t, p_{t_0}).$$  \hspace{1cm} (4.11)

Remark: A usual definition is given without explicit reference to a particle; It is, instead of (4.9),

$$\vec{V}^t_0(t, p_{t_0}) := \frac{\partial \Phi^t_0}{\partial t}(t, p_{t_0}), \quad \forall (t, p_{t_0}) \in \mathbb{R} \times \Omega_{t_0}. \hspace{1cm} (4.12)$$

4.3.2 Lagrangian velocity versus Eulerian velocity

(4.9) and (2.5) give, with $p_t = \tilde{\Phi}(t, P_{Obj})$,

$$(\frac{\partial \Phi^t_0}{\partial t} =) \quad \vec{V}^t_0(t, p_{t_0}) = \vec{v}(t, p_t) \quad (= \tilde{\Phi}_{P_{Obj}}'(t) = \text{velocity at } t \text{ of } P_{Obj}). \hspace{1cm} (4.13)$$

In other words, cf. (4.6),

$$\vec{V}^t_0 = \vec{v} \circ \Phi^t_0.$$

4.3.3 Relation between differentials

Thus, for $C^2$ motions,

$$(\frac{\partial d\Phi^t_0}{\partial t}(t, p_{t_0}) =) \quad d\vec{V}^t_0(p_{t_0}) = d\vec{v}_\tau(p_t).d\Phi^t_0(p_{t_0}) \quad \text{when } p_t = \Phi^t_0(p_{t_0}). \hspace{1cm} (4.15)$$

In other words, with

$$F^t_\tau = \frac{\partial \Phi^t_0}{\partial t} \quad \text{the deformation gradient relative to } t_0 \text{ and } t,$$

we have

$$d\vec{V}^t_0(p_{t_0}) = d\vec{v}_\tau(p_t). F^t_\tau(p_{t_0}) \quad \text{when } p_t = \Phi^t_0(p_{t_0}). \hspace{1cm} (4.17)$$

Abusively written (dangerous notation)

$$d\vec{V} = d\vec{v}.F: \quad \text{At what points, what times?} \hspace{1cm} (4.18)$$

4.3.4 Computation of $L = d\vec{V}$ from Lagrangian variables

The Lagrangian approach can be introduced before the Eulerian approach: Then the Eulerian velocities are defined by

$$\vec{v}^t_\tau(t, p_t) := \vec{V}^t_0(t, p_{t_0}), \quad \text{when } p_t = \Phi^t_0(p_{t_0}). \hspace{1cm} (4.19)$$

NB: You can’t see at first glance that the Eulerian velocity $\vec{v}$ is independent of $t_0$; this is why it is called $\vec{v}^t_\tau$.

Then let $\vec{v}^t_\tau(p_t) := \vec{v}(t, p_t)$. Thus $\vec{v}^t_\tau(\Phi^t_0(p_{t_0})) = \frac{\partial \Phi^t_0}{\partial t}(t, p_{t_0})$, and, for $C^2$ motions and with (4.16),

$$d\vec{v}^t_\tau(p_t).d\Phi^t_0(p_{t_0}) = \frac{\partial (d\Phi^t_0)}{\partial t}(t, p_{t_0}) = \frac{\partial F^t_\tau}{\partial t}(t, p_{t_0}) \quad \text{noted } \dot{F}^t_{\tau_{p_{t_0}}}(t), \quad \text{when } p_t = \Phi^t_0(p_{t_0}). \hspace{1cm} (4.20)$$

So

$$d\vec{v}^t_\tau(p_t). F^t_{\tau_{p_{t_0}}}(p_{t_0}) = \dot{F}^t_{\tau_{p_{t_0}}}(t), \quad \text{i.e. } \quad d\vec{v}^t_\tau(p_t) = \dot{F}^t_{\tau_{p_{t_0}}}(t). F^t_{\tau_{p_{t_0}}}(p_{t_0})^{-1}. \hspace{1cm} (4.21)$$

Once again: It is not obvious that $\vec{v}^t_\tau$ does not depend on $t_0$; It may be better to first introduce Eulerian
velocities\textsuperscript{1}, to see that (4.22) is nothing but (4.17) with \( \vec{v} \) independent of \( t_0 \). And in some courses, you find the definition

\[ L^0_t(p_t) := F^0_{t_0}(t) F^{t_0}_t(p_{t_0})^{-1} \]

written in short \( L = \dot{F} F^{-1} \), (4.22)

where it is not obvious that \( L^0_t(p_t) \) does not depend on \( t_0 \). Indeed we have

\[ L_t^0(p_t) = d\vec{\gamma}_t(p_t), \]

which is nothing but (4.17), since \( \vec{V}^0_t(p_{t_0}) = \frac{\partial (\Phi^0_t)}{\partial t}(t, p_{t_0}) = F^0_{t_0}(t) \).

### 4.4 Lagrangian acceleration

Let \( P_{Obj} \in Obj \), \( t_0, t \in \mathbb{R} \), thus \( p_{t_0} = \vec{\Phi}(t_0, P_{Obj}) \) and \( p_t = \vec{\Phi}(t, P_{Obj}) \), the positions of \( P_{Obj} \) at \( t_0 \) and \( t \).

**Definition 4.9** In short, the Lagrangian acceleration at \( t \) at \( p_t \) of the particle \( P_{Obj} \) is

\[
\vec{\Gamma}^{t_0}_t(t, p_{t_0}) := \frac{\partial^2 \vec{\Phi}}{\partial t^2}(t, P_{Obj}) = \vec{\Phi}_{P_{Obj}}''(t) \quad \text{when} \quad p_{t_0} = \vec{\Phi}(t_0, P_{Obj}). \tag{4.24}
\]

Thus, with the Eulerian acceleration \( \vec{\gamma}(t, p_t) = \vec{\Phi}_{P_{Obj}}''(t) \), \( \text{at} \ (t, p_t = \vec{\Phi}(t, P_{Obj})) \), cf. (2.43), we have

\[
\vec{\Gamma}^{t_0}_t(t, p_{t_0}) = \vec{\gamma}(t, p_t) \quad (= \vec{\Phi}_{P_{Obj}}''(t) = \text{acceleration of} \ P_{Obj} \ \text{at} \ t). \tag{4.25}
\]

In details, the Lagrangian acceleration is the "two point vector field" defined on \( \mathbb{R} \times \Omega_{t_0} \) by

\[
\vec{\Gamma}^{t_0}_t(t, p_{t_0}) = ((t, p_t), \vec{\Phi}_{P_{Obj}}''(t)), \quad \text{when} \quad p_t = \Phi^{t_0}(t, p_{t_0}). \tag{4.26}
\]

(To compare with (2.44).) So \( \vec{\Gamma}^{t_0}_t(t, p_{t_0}) \) (a "two point vector") is not drawn on the graph of \( \vec{\Gamma}^{t_0}_t \), but on the graph of \( \vec{\gamma} \) at \( (t, p_t) \).

If \( t \) is fixed, or if \( p_{t_0} \in \Omega_{t_0} \) is fixed, then we denote

\[
\vec{\Gamma}^{t_0}_t(p_{t_0}) := \vec{\Gamma}^{t_0}_t(t, p_{t_0}), \quad \text{or} \quad \vec{\Gamma}^{t_0}_t(t) := \vec{\Gamma}^{t_0}_t(t, p_{t_0}). \tag{4.27}
\]

In other words (4.25) reads

\[
\vec{\Gamma}^{t_0}_t(t) = \vec{\gamma}_t \circ \Phi^{t_0}_t. \tag{4.28}
\]

Thus

\[
d\vec{\Gamma}^{t_0}_t(p_{t_0}) = d\vec{\gamma}_t(p_t).d\Phi^{t_0}_t(p_{t_0}), \quad \text{when} \quad p_t = \Phi^{t_0}_t(p_{t_0}), \tag{4.29}
\]

that is \( d\vec{\Gamma}^{t_0}_t(p_{t_0}) = d\vec{\gamma}_t(p_t).dF^{t_0}_t(p_{t_0}) \) with the deformation gradient \( F^{t_0}_t = d\Phi^{t_0}_t \).

**Dangerous notation:** \( d\vec{\Gamma} = d\vec{\gamma}.F \). At what points? What times?

**Remark:** A usual definition is given without explicit reference to a particle: Instead of (4.24),

\[
\vec{\Gamma}^{t_0}_t(t, p_{t_0}) := \frac{\partial^2 \Phi^{t_0}_t}{\partial t^2}(t, p_{t_0}). \tag{4.30}
\]

\textsuperscript{1}To get Eulerian results from Lagrangian computations can make the understanding of a Lie derivative quite difficult: To introduce the "so-called" Lie derivative in classical mechanics you can find the following steps: 1. At \( t \) consider the Cauchy stress vector \( \mathbf{t} \) (Eulerian), 2. then with a unit normal vector \( \mathbf{n} \), define the associated Cauchy stress tensor \( \mathbf{s} \) (satisfying \( \mathbf{t} = \mathbf{s} \cdot \mathbf{n} \)), 3. then change of variables in integrals to be in \( \Omega_{t_0} \) and to be able to work with Lagrangian variables, 4. then introduce the first Piola-Kirchhoff (two point) tensor \( \mathbf{K} \), 5. then introduce the second Piola-Kirchhoff tensor \( \mathbf{K} \) (endomorphism in \( \Omega_{t_0} \)), 6. then differentiate \( \mathbf{K} \) in \( \Omega_{t_0} \) (although the initials variables are in \( \Omega_t \), 7. then back in \( \Omega_t \) to get back to Eulerian functions (change of variables in integrals), 8. then you get e.g. some Jaumann or Truesdell or Maxwell of Oldroyd or other Lie derivative type terms, the appropriate choice among all these derivatives being quite obscure...

While, with simple Eulerian considerations, it requires a few lines to understand the original Lie derivative (it is a Eulerian concept to begin with) and its simplicity, see § 15, and to deduce linear and non-linear covariant objective results.
4.5 Time Taylor expansion of $\Phi_{t_0}^t$

Let $p_{t_0} \in \Omega_{t_0}$. Then, at second order,
\[
\Phi_{p_{t_0}}^{t_0}(\tau) = \Phi_{p_{t_0}}^{t_0}(t) + (\tau-t)\Phi_{p_{t_0}}^{t_0}(t) + \frac{(\tau-t)^2}{2} \Phi_{p_{t_0}}^{t_0}(t) + o((\tau-t)^2),
\]
that is, with and $p(t) = \tilde{\Phi}_{p_{t_0}}(\tau) = \Phi_{p_{t_0}}^{t_0}(t)$,
\[
p(\tau) = p(t) + (\tau-t)\tilde{v}_{t_0}(t, p_{t_0}) + \frac{(\tau-t)^2}{2} \tilde{\gamma}_{t_0}(t, p_{t_0}) + o((\tau-t)^2).
\]
NB: There are three times involved: $t_0$ (observer dependent), $t$ and $\tau$ (for the Taylor expansion). To compare with (2.50)-(2.51) (independent of $t_0$): $p(\tau) = p(t) + (\tau-t)\tilde{v}(t, p(t)) + \frac{(\tau-t)^2}{2} \tilde{\gamma}(t, p(t)) + o((\tau-t)^2)$.

4.6 A vector field which let itself be deformed by a flow

Let $\tilde{v}_{t_0} : p_{t_0} \in \Omega_{t_0} \to \tilde{w}_{t_0}(p_{t_0}) \in \mathbb{R}^n$ be a vector field in $\Omega_{t_0}$. Then let $t \in \mathbb{R}$ and consider the vector field $\tilde{w}_{t_0*, t}$ in $\Omega_{t}$ which results from the deformation of $\tilde{w}_{t_0}$ by the motion, that is, defined by,
\[
\tilde{w}_{t_0*, t}(p(t)) := d\Phi_{p_{t_0}}^{t_0}(t, p_{t_0}).\tilde{w}_{t_0}(p_{t_0}) \quad \text{when} \quad p(t) = \Phi_{p_{t_0}}^{t_0}(t).
\]
See figure 5.1. ($\tilde{w}_{t_0*}$ looks like a Eulerian vector field, but is not since it depends on $t_0$): $\tilde{w}_{t_0*}$ is the push-forward of $\tilde{w}_{t_0}$ by $\Phi_{p_{t_0}}^{t_0}$.

Proposition 4.10 For $C^2$ motions, we have
\[
\frac{D\tilde{w}_{t_0*, t}}{Dt} = d\tilde{v}_{t_0*, t}, \quad \text{i.e.} \quad \frac{\partial \tilde{w}_{t_0*, t}}{\partial t} + d\tilde{w}_{t_0*, t} = 0, \quad \text{i.e.} \quad \mathcal{L}_{\tilde{v}} \tilde{w}_{t_0*, t} = 0,
\]
where the Eulerian vector field $\mathcal{L}_{\tilde{v}} \tilde{u} := \frac{D\tilde{u}}{Dt} - d\tilde{v} \tilde{u}$ is the Lie derivative of a Eulerian vector field $\tilde{u}$ along the motion $\tilde{v}$.

Interpretation: The Lie derivative $\mathcal{L}_{\tilde{v}} \tilde{u}(t, p_{t_0}) = \lim_{h \to 0} \frac{\tilde{u}(t+h, p_{t_0+(t+h)}) - \tilde{u}(t, p_{t_0})}{h}$ gives a measure of resistance of $\tilde{u}$ to a motion $\tilde{v}$ which velocity is $\tilde{v}$, see § 15.6.1: So, in (4.34), the result $\mathcal{L}_{\tilde{v}} \tilde{w}_{t_0*, t} = 0$ is "obvious" since $\tilde{w}_{t_0*, t}$ is the result of the deformation due to the motion, that is, $\tilde{w}_{t_0*}$ is the vector field that has let itself be deformed by the flow (the push-forward by the motion).

Proof. Let $F_{p_{t_0}}^{t_0}(t) := \Phi_{p_{t_0}}^{t_0}(t, p_{t_0})$, cf. (4.16), let $p(t) = \Phi_{p_{t_0}}^{t_0}(t, p_{t_0})$. Then $\tilde{w}_{t_0*, t}(p(t)) = F_{p_{t_0}}^{t_0}(t).\tilde{w}_{t_0}(p_{t_0})$ gives $\frac{D\tilde{w}_{t_0*, t}}{Dt}(t, p(t)) = F_{p_{t_0}}^{t_0}(t).\tilde{w}_{t_0*}(t, p_{t_0}) = F_{p_{t_0}}^{t_0}(t).F_{t_0}^{\tilde{v}}(p_{t_0})^{-1}.\tilde{w}_{t_0*}(t, p(t)) = \frac{d\tilde{v}}{dt}(p_{t_0}).\tilde{w}_{t_0*}(p(t))$, i.e. (4.34). \(\blacksquare\)

5 Deformation gradient $F$

5.1 Definitions

5.1.1 $F := d\Phi$

Consider a motion $\tilde{\Phi} : \left\{ \mathbb{R} \times \text{Obj} \to \mathbb{R}^n \right\}$ of an object $\text{Obj}$, cf. (1.5), and let $\Omega_{t_0} := \tilde{\Phi}(t, \text{Obj})$ (configuration of $\text{Obj}$ at $t$). Fix $t_0, t \in \mathbb{R}$ and let $\Phi_{t_0}^{t} : \left\{ \Omega_{t_0} \to \Omega_t \right\}$ be the associated motion, cf. (3.5). Suppose $\Phi_{t_0}^t$ is differentiable in $\Omega_{t_0}$: For all $p_{t_0} \in \Omega_{t_0}$, there exists a linear map $L_{p_{t_0}} := d\Phi_{t_0}^{t}(p_{t_0}) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, called the differential of $\Phi_{t_0}^{t}$ at $p_{t_0}$, such that, for all $\tilde{w}_{t_0} \in \mathbb{R}^n$ vector at $p_{t_0}$,
\[
\Phi_{t_0}^{t}(p_{t_0} + h\tilde{w}_{t_0}) = \Phi_{t_0}^{t}(p_{t_0}) + h d\Phi_{t_0}^{t}(p_{t_0}).\tilde{w}_{t_0} + o(h).
\]

Definition 5.1 The differential $d\Phi_{t_0}^{t} : \left\{ \Omega_{t_0} \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \right\}$ is also named $F_{t_0}^{t}$ and called "the coariant deformation gradient between $t_0$ and $t$", and $F_{t_0}^{t}(p_{t_0}) := d\Phi_{t_0}^{t}(p_{t_0}) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is "the coariant deformation gradient at $p_{t_0}$ between $t_0$ and $t$".

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So the linear map \( F_{t_0}^{t}(p_{t_0}) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) is defined by

\[
F_{t_0}^{t}(p_{t_0}) := d\Phi_{t_0}^{t}(p_{t_0}) := \begin{cases} 
\mathbb{R}^n_{t_0} ightarrow \mathbb{R}^n \\
\bar{w}_{t_0} ightarrow F_{t_0}^{t}(p_{t_0}) \cdot \bar{w}_{t_0} = \lim_{h \to 0} \frac{\Phi_{t_0}^{t}(p_{t_0} + h\bar{w}_{t_0}) - \Phi_{t_0}^{t}(p_{t_0})}{h}.
\end{cases} \tag{5.2}
\]

Marsden–Hughes notations, \( t_0 \) being imposed: \( \Phi := \Phi_{t_0}^{t}, P := p_{t_0}, F := F_{t_0}^{t}, \bar{W} := \bar{w}_{t_0} \in \mathbb{R}^n, \) thus

\[
\Phi(P + h\bar{W}) = \Phi(P) + h F(P).\bar{W} + o(h), \quad \text{and} \quad F(P).\bar{W} = \lim_{h \to 0} \frac{\Phi(P + h\bar{W}) - \Phi(P)}{h} \quad (\in \mathbb{R}^n). \tag{5.3}
\]

**Remark 5.2** NB: \( F_{t_0}^{t} = d\Phi_{t_0}^{t} \) is also called, for short, “the deformation gradient between \( t_0 \) and \( t \)”; But \( F_{t_0}^{t} \) is not a “gradient” (its definition does not need a Euclidean dot product); It may lead to confusions as far as objectivity and covariance–contravariance are concerned. So it would be simpler to stick to the name “the differential of \( \Phi_{t_0}^{t} \)”; E.g., in thermodynamics the differential \( dU \) of the internal energy \( U \) is not called “gradient”; It is just called “differential”.

### 5.1.2 Its values: Push-forward

**Definition 5.3** Let \( F_{t_0}^{t} = d\Phi_{t_0}^{t} \). Let \( \bar{w}_{t_0} := \begin{cases} \Omega_{t_0} \rightarrow \mathbb{R}^n_{t_0} \\
p_{t_0} \rightarrow \bar{w}_{t_0}(p_{t_0}) \end{cases} \) be a vector field in \( \Omega_t \). Its push-forward by the motion \( \Phi_{t_0}^{t} \) is the vector field

\[
(\Phi_{t_0}^{t})_{\ast} \bar{w}_{t_0} := \begin{cases} \Omega_t \rightarrow \mathbb{R}^n_t \\
p_t \rightarrow \bar{w}_{t_0}(p_t) := F_{t_0}^{t}(p_{t_0}).\bar{w}_{t_0}(p_{t_0}) \end{cases}, \quad \text{when} \quad p_t = \Phi_{t_0}^{t}(p_{t_0}), \tag{5.4}
\]

defined in \( \Omega_t \) by the values in (5.2).

So \( \bar{w}_{t_0}(p_t) \) is the result of the deformation of \( \bar{w}_{t_0} \) by the motion \( \Phi_{t_0}^{t} \) (the result of the transport of \( \bar{w}_{t_0} \) by the motion \( \Phi_{t_0}^{t} \)), see figure 5.1. Marsden–Hughes notation:

\[
(\Phi, \bar{W})(p) := \bar{W}_s(p) := F(P).\bar{W}(p), \quad \text{when} \quad p = \Phi(P). \tag{5.5}
\]

![Figure 5.1](image_url)

Figure 5.1: \( c_{t_0} : s \rightarrow P = c_{t_0}(s) \) is a (spatial) curve in \( \Omega_{t_0} \) which is deformed by motion into the curve \( c_t = \Phi_{t_0}^{t} \circ c_{t_0} : s \rightarrow P = c_{t_0}(s) = c_{t_0}(s) = \Phi_{t_0}^{t}(P) \) in \( \Omega_t \). Let \( \bar{W}(P) := \bar{W}_P = c_{t_0}(s) \) be the tangent vector at \( P \) at \( c_{t_0} \), and \( \bar{w}(p) = \bar{w}_P := c_{t_0}'(s) \) be the tangent vector at \( p \) at \( c_t \). Then \( c_t = \Phi_{t_0}^{t} \circ c_{t_0} \) gives (relation between the tangent vectors) \( c_t'(s) = d\Phi_{t_0}^{t}(p_{t_0}).c_{t_0}'(s) \), i.e. \( \bar{w}(p) = F_{t_0}^{t}(P).\bar{W}(P) \), and the vector field \( \bar{w} \) thus defined (along \( \text{Im}(c_t) \)) is called the push-forward by \( \Phi_{t_0}^{t} \) of the vector field \( \bar{W} \) (defined along \( \text{Im}(c_{t_0}) \)).

**An application:** If \( \bar{w} \) is a Eulerian vector field (e.g., a “force field”), then, at any time-space point \( (t, p_t) \in \{t\} \times \Omega_t \) we can compare the actual value \( \bar{w}(t, p_t) \) with \( \bar{w}_{t_0}(t, p_t) = \) the value transported by
the motion (virtual value); And the rate
\[
\frac{\bar{w}(t, p(t)) - \bar{w}_{t,0*}(t, p(t))}{t - t_0} = (\text{actual}) - (\text{transported})
\]
(5.6)
gives, as \( t \to t_0 \), the Lie derivative \( \mathcal{L}_v\bar{w}(t_0, p_{t_0}) \) (rate of stress, see § 15.5).

5.1.3 A full definition of \( F \): A two point tensors

**Definition 5.4** (Marsden–Hughes [12] ) The function \( F^t_{t_0} \) is called a two point tensor, referring to the points \( p_{t_0} \in \Omega_{t_0} \) (set of definition of \( \Phi^t_{t_0} \)) and \( p_t = \Phi^t_{t_0}(p_{t_0}) \in \Omega_t \) (arrival set of \( \Phi^t_{t_0} \)).

To be more precise:

**Definition 5.5** The (covariant) deformation gradient between \( t_0 \) and \( t \) is defined to be “the second component” of “the tangent map \( T\Phi^t_{t_0} \)” which is the function
\[
T\Phi^t_{t_0} : \{ \begin{array}{l}
T\Omega_{t_0} \to T\Omega_t \\
(p_{t_0}, \bar{w}(p_{t_0})) \to (p_t, \bar{w}_{t,0*}(p_t))
\end{array} \}
\]
where \( T\Omega \) is the tangent bundle, cf. § 1.8. And the tangent map \( T\Phi^t_{t_0} \) is called a two point tensor, referring to the points \( p_{t_0} \in \Omega_{t_0} \) and \( p_t \in \Omega_t \).

**Remark 5.6** This definition of “a two point tensor” is a shortcut: \( F^t_{t_0} := d\Phi^t_{t_0} \) is not a “tensor”, i.e. is not a bilinear form (which gives scalar results) since \( F^t_{t_0}(p_{t_0}) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) gives vector results cf. (5.4). However the linear map \( F := F^t_{t_0}(p_{t_0}) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) can be naturally and canonically associated with the bilinear form \( \bar{F} := F^t_{t_0}(p_{t_0}) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}) \) defined by, for all \( \bar{u}_{p_{t_0}} \in \mathbb{R}^n \) and \( \ell_p \in \mathbb{R}^n \), with \( p_t = \Phi^t_{t_0}(p_{t_0}) \):
\[
\bar{F}(\ell_p, \bar{u}_{p_{t_0}}) := \ell_p, F_p, \bar{u}_{p_{t_0}}.
\]
See (T.13). And it is \( \bar{F} \) which defines the so-called “two point tensor”.

5.2 The unfortunate notation \( d\bar{x} = F.d\bar{X} \)

Let \( O_{t_0} \) be an origin at \( t_0 \) and \( o_t \) be an origin at \( t \), and let \( \bar{X} = \overline{O_{t_0}P_{t_0}} \) and \( \bar{x} = \overline{o_tP_t} \).

5.2.1 Introduction
\( \bar{w}_{t,0*}(p_t) := F^t_{t_0}(p_{t_0}), \bar{w}_{t,0*}(p_{t_0}) \) in (5.4) is sometimes written
\[
d\bar{x} = F.d\bar{X} : “\text{a very unfortunate notation}”.
\]
(5.9)
But this notation amounts to “confuse a length and a speed”! E.g., you find the sentence “(5.9) is even true if \( ||d\bar{X}|| = 1 ||\)”. Explanations:

5.2.2 Where does this notation come from?

The notation (5.9) comes from the first order Taylor expansion
\[
\Phi^t_{t_0}(Q) = \Phi^t_{t_0}(P) + d\Phi^t_{t_0}(P).\overline{PQ} + o(||\overline{PQ}||),
\]
(5.10)
which reads when \( q := \Phi^t_{t_0}(Q) \) and \( p := \Phi^t_{t_0}(P) \):
\[
\frac{q - p}{\delta \bar{x}} = \frac{d\Phi^t_{t_0}(P)}{F}(Q - P) + o(Q - P) \quad (\text{relation between bi-point vectors}),
\]
(5.11)
And as \( Q \to P \) we get \( 0 = 0 \), quite useless... While (5.10) also gives, when \( Q = P + h\bar{w}_{t,0} \),
\[
\frac{q - p}{h} = d\Phi^t_{t_0}(P).\frac{Q - P}{h} + o(1) \quad (\text{relation between rates}),
\]
(5.12)
And (5.12) is useful: As \( Q \to P \) you get the relation \( \bar{w}_{t,0*}(p) = F^t_{t_0}(P), \bar{w}_{t,0}(P) \) between tangent vectors and \( h \to 0 \) (push-forwards), see figure 5.1 and next § 5.2.3.
5.2.3 Vector approach...

Details for (5.9): Vectorial approach. Consider a $C^1$ spatial curve in $\Omega_{t_0}$, cf. figure 5.1,

$$c_t : \begin{cases} [s_1, s_2] \rightarrow \Omega_{t_0} \\ s \rightarrow P := c_t(s) \end{cases}, \quad \text{and} \quad \vec{X}(s) := \overrightarrow{O_{t_0}c_t(s)} = \overrightarrow{O_{t_0}P}.$$  \hspace{1cm} (5.13)

E.g., consider a spring which particles at $P = c_t(s)$. This spatial curve $c_t$ is push-forwarded (the spring is deformed = transported) by $\Phi_t^{\circ}$ and becomes the spatial curve in $\Omega_t$, cf. figure 5.1,

$$c_t := \Phi_t^{\circ} \circ c_t : \begin{cases} [s_1, s_2] \rightarrow \Omega_t \\ s \rightarrow p := c_t(s) = \Phi_t^{\circ}(c_t(s)) \end{cases}, \quad \text{and} \quad \vec{x}(s) = \overrightarrow{O_{t_0}c_t(s)} = \overrightarrow{O_{t_0}P}.$$  \hspace{1cm} (5.14)

cf. figure 5.1. Thus the relation between the tangent vectors is, see figure 5.1,

$$\frac{d\vec{x}}{ds}(s) = \Phi_t^{\circ}(c_t(s)). \frac{dc_t}{ds}(s), \quad \text{i.e.} \quad \frac{d\vec{x}}{ds}(s) = F_t^{\circ}(P). \frac{d\vec{X}}{ds}(s).$$  \hspace{1cm} (5.15)

But you cannot simplify by $ds$ (!): The notation $dx = F.d\vec{X}$ is absurd since (5.15) refers to derivatives at $s$ (it is absurd to confuse “a value” with “a rate”, as absurd as confusing length and speed).

NB: Now with (5.15), we can choose $||\frac{d\vec{x}}{ds}(s)|| = 1 = ||\overrightarrow{P_0}(s)|| = ||\frac{d\vec{x}}{ds}(s)||$: It means that the parametrization of the curve $c_t$ in $\Omega_{t_0}$ uses an intrinsic curvilinear parameter $s$, i.e., s.t. $||c_t(s)|| = 1$ for all (space variable) $s$, i.e. the length of the tangent vector is 1, i.e. $||\vec{W}_P|| = 1$ in figure 5.1. It is not $||d\vec{X}|| = 1$.

5.2.4 ... and differential approach

Differential approach (as in thermodynamics) to get a kind of (5.9).

Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be Cartesian bases in $\mathbb{R}^n_{t_0}$ in $\mathbb{R}^n_t$. Let

$$\vec{x} = \overrightarrow{O_{t_0}P} = \sum_{i=1}^n \varphi_i(P) \vec{b}_i = \sum_{i=1}^n x_i(P) \vec{b}_i, \quad \text{so} \quad \varphi_i^{\text{noted}} = x_i.$$  \hspace{1cm} (5.16)

With $\frac{\partial \varphi_i}{\partial \vec{X}_j}(P) := d\varphi_i(P) \vec{a}_j$ (usual notation), the Jacobian matrix of $\Phi_t^{\circ}$ relative to the chosen bases is

$$[F_t^{\circ}(P)]_{\vec{a},\vec{b}} = [d\Phi_t^{\circ}(P)]_{\vec{a},\vec{b}} = [\frac{\partial \varphi_i}{\partial \vec{X}_j}(P)], \quad \text{written in short} \quad [F] = \frac{d\vec{x}}{d\vec{X}}.$$  \hspace{1cm} (5.17)

And unfortunately, (5.17) is sometimes also written $F.d\vec{X} = d\vec{x}$. ...

Question. What is the meaning 1- of $[F] = \frac{d\vec{x}}{d\vec{X}}$ in (5.17) and 2- of the notation $F.d\vec{X} = d\vec{x}$?

Answer. 1- The Jacobian matrix $[F_t^{\circ}(P)]_{\vec{a},\vec{b}} = [d\Phi_t^{\circ}(P)]_{\vec{a},\vec{b}} = [\frac{\partial \varphi_i}{\partial \vec{X}_j}(P)]$ only tells us that $\frac{\partial \varphi_i}{\partial \vec{X}_j}(P) := d\varphi_i(P) \vec{a}_j$ for all $i, j$ when (5.16) is in use. Nothing else.

2- With $(d\vec{X}_j)$ the (covariant) dual basis of the basis $(\vec{a}_i)$ and $F_{ij}(P) := \frac{\partial \varphi_i}{\partial \vec{X}_j}(P)$, the differentials of the $\varphi_i$ at $P$ read

$$d\varphi_i(P) = \sum_{j=1}^n F_{ij}(P) \, d\vec{X}_j \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}).$$  \hspace{1cm} (5.18)

NB: (5.18) is an equality between (differential) functions\(^2\): (5.18) is not an equality between components of vectors, even if you write it $d\vec{x} = F.d\vec{X}$. (And there is no other possible meaning in thermodynamics.)

\(^2\)Spivak [7] chapter 4: Classical differential geometres (and classical analysts) did not hesitate to talk about “infinitely small” changes $dx$ of the coordinates $x^i$, just as Leibnitz had. No one wanted to admit that this was nonsense, because true results were obtained when these infinitely small quantities were divided into each other (provided one did it in the right way) $[dx^i = \text{noted } \frac{\partial x}{\partial x^i}$ with duality notations]. Eventually it was realized that the closest one can come to describing an infinitely small change is to describe a direction in which this change is supposed to occur, i.e., a tangent vector [see (5.15)]. Since $df$ is supposed to be the infinitesimal change of $f$ under an infinitesimal change of the point, $df$ must be a function of this change, which means that $df$ should be a function on tangent vectors. The $dx^i$ themselves then metamorphosed into functions, and it became clear that they must be distinguished from the tangent vectors $\partial / \partial x^i$. Once this realization came, it was only a matter of making new definitions, which preserved the old notation, and waiting for everybody to catch up.
5.2.5 The ambiguous notation \( \dot{\mathbf{x}} = \dot{F}.d\mathbf{X} \)

The unfortunate notation \( d\mathbf{x} = F.d\mathbf{X} \), cf. (5.9), gives the unfortunate notations (produces misunderstandings)

\[
\dot{\mathbf{x}} = \dot{F}.d\mathbf{X}, \quad \text{and} \quad \ddot{\mathbf{x}} = \mathbf{L}.d\mathbf{x},
\]

(5.19)

where \( \mathbf{L} = \dot{F}F^{-1} = \dot{d}\mathbf{v} \) is the differential of the Eulerian velocity, see (4.22)-(4.23).

A legitimate notation for (5.19) is deduced from (5.15) when (5.14) is rewritten so that \( t \) explicitly appears as a variable:

\[
c(t, s) = c(t, s) = \Phi^{t_a}(t, c_{t_0}(s)).
\]

(5.20)

(Recall: \( t \) is a time variable, \( s \) is a space variable.) Thus, rewriting of (5.15),

\[
\frac{\partial c}{\partial s}(t, s) = d\Phi^{t_a}(t, c_{t_0}(s)), \quad \frac{dc}{ds}(s), \quad \text{i.e.} \quad \vec{w}_{t_0}(t, p(t)) = \Phi^{t_a}(t, p(t)).
\]

(5.21)

Thus, along the trajectory of the particle which was at \( t_0 \) at \( p_0 = c_{t_0}(s) \) and is now at \( t \) at \( p_t \), the time evolution rate for the tangent vector \( \vec{w}_{t_0}(t, p(t)) \) (= the vector \( \vec{w}_p \) in figure 5.1) is:

\[
\frac{D\vec{w}_{t_0}}{Dt}(t, p(t)) = \frac{\partial \Phi^{t_a}}{\partial t}(t, p_0).\vec{w}_{t_0}(p_0) = \frac{\partial \Phi^{t_a}}{\partial t}(t, p_0).\Phi^{t_a}(p_0)^{-1}.\vec{w}_{t_0}(t, p(t)).
\]

(5.22)

Since \( \frac{\partial \Phi^{t_a}}{\partial t}(t, p_0).F^{t_a}(p_0)^{-1} = d\vec{v}(t, p_t) \), cf. (4.17), we have obtained

\[
\begin{align*}
\frac{D\vec{w}_{t_0}}{Dt} & = d\vec{v}.\vec{w}_{t_0}, \\
\end{align*}
\]

(5.23)

which is what (5.19) means: Gives the evolution over time of the tangent vector \( \vec{w}_{t_0}(t, p(t)) \).

5.3 Quantification with bases

For clarity, we use both the

1- Classic (basic) notations: \((\vec{a}_i)\) is a basis in \( \mathbb{R}^n_t \) chosen by an observer at \( t_0 \) (in the past) and \((\vec{b}_i)\) is a basis in \( \mathbb{R}^n_t \) chosen by an observer at \( t \) (present time), and

2- Duality Marsden-Hughes notations: \((\vec{a}_i)\) named \((\vec{E}_i)\) and \((\vec{b}_i)\) named \((\vec{c}_i)\), and \( p_{t_0} \) named \( P \) and \( p_t \) named \( \vec{p} \), together with duality notations (Einstein convention).

If \( f \in C^1(\Omega_{t_0}; \mathbb{R}) \), then let (usual standard notation), for all \( j = 1, \ldots, n \),

\[
\frac{\partial f}{\partial x_j}(p_0) \overset{\text{class.}}{=} df(p_0).\vec{a}_j, \quad \text{or} \quad \frac{\partial f}{\partial x_j}(P) \overset{\text{dual.}}{=} df(P).\vec{E}_j.
\]

(5.24)

Thus, for any \( \vec{W} \in \mathbb{R}^n_{t_0} \), with \( \vec{W} \overset{\text{class.}}{=} \sum_{j=1}^n W_j \vec{a}_j \) and \( \vec{W} \overset{\text{dual.}}{=} \sum_{j=1}^n W^j \vec{E}_j \),

\[
\begin{align*}
\text{class.} : \quad df(p_0),\vec{W} & = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p_0)W_j = [df(p_0)]\overset{\text{[\vec{W}]}_i}{\vec{a}}, \quad [df]_i = (\frac{\partial f}{\partial x_1} \ldots \frac{\partial f}{\partial x_n}), \\
\text{dual.} : \quad df(P),\vec{W} & = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(P)W^j = [df(P)]\overset{\text{[\vec{W}]}_i}{\vec{a}}, \quad [df]_i = (\frac{\partial f}{\partial x_1} \ldots \frac{\partial f}{\partial x_n}).
\end{align*}
\]

(5.25)

Remark 5.7 \( J, j \) are dummy variables when used in a summation: E.g., \( df,\vec{W} = \sum_{j=1}^n \frac{\partial f}{\partial x_j}W^j = \frac{\partial f}{\partial x_1}W^1 + \frac{\partial f}{\partial x_2}W^2 + \ldots \) (there is no uppercase for 1, 2, ...). And Marsden–Hughes notations (capital letters for the past) are not at all compulsory, classical notations being just as good (and preferable if you hesitate, since they are not be misleading). See § A.
Then let \( \omega_t \) be an origin in \( \mathbb{R}^n \) chosen by an observer at \( t \), and let
\[
p_t = \Phi(p_{t_0}) = \omega_t + \sum_{i=1}^{n} \phi_i(p_{t_0}) \tilde{b}_i.
\]

(Marsden notations: \( p = \Phi(P) = \omega_t + \sum_{i=1}^{n} \phi^i(P) \tilde{e}_i \). Thus \( d\Phi(p_{t_0}).\tilde{W} = \sum_{i=1}^{n} (d\phi_i(p_{t_0})\tilde{W}) \tilde{b}_i \) for all \( \tilde{W} \),
thus, with \( F = d\Phi \) and \( \tilde{W} = \sum_{i=1}^{n} W_i \tilde{a}_i \),
\[
F(p_{t_0}).\tilde{W} = \sum_{i,j=1}^{n} \frac{\partial \phi_i}{\partial x_j}(p_{t_0}) W_i \tilde{a}_j,
\]
i.e. \( [F(p_{t_0}).\tilde{W}]_{\tilde{a},\tilde{b}} = [F(p_{t_0})]_{\tilde{a},\tilde{b}} [\tilde{W}]_{\tilde{a}} \).

\[
[F]_{\tilde{a},\tilde{b}} = \left[ \frac{\partial \phi^j}{\partial x_k} \right] 
\]
being the Jacobian matrix of \( \Phi \) relative to the bases (\( \tilde{a}_i \)) and (\( \tilde{b}_i \)).
(Marsden notations: \( F(P).\tilde{W} = \sum_{i,j=1}^{n} \frac{\partial \phi^j}{\partial x_i} (P) W^i \tilde{e}_i \) and \( [F]_{\tilde{e},\tilde{e}} = \left[ \frac{\partial \phi^j}{\partial x_i} \right] \) and \( [F(P).\tilde{W}]_{\tilde{e}} = [F(P)]_{\tilde{e},\tilde{e}} [\tilde{W}]_{\tilde{e}} \).

Similarly, for the second order derivative \( d^2\Phi = dF \) (when \( \Phi \) is \( C^2 \)), with \( \tilde{U} = \sum_{j=1}^{n} U_j \tilde{a}_j \) and \( \tilde{W} = \sum_{i=1}^{n} W_i \tilde{a}_i \) we get (for \( C^2 \) motions)
\[
d^2\Phi(\tilde{U},\tilde{W}) = \sum_{i=1}^{n} \sum_{j,k=1}^{n} \frac{\partial^2 \phi_i}{\partial x_j \partial x_k} U_j W_k \tilde{b}_i = \sum_{i=1}^{n} \sum_{j,k=1}^{n} \frac{\partial^2 \phi_i}{\partial x_j \partial x_k} U^i W^j \tilde{c}_i.
\]

5.4. **Remark: Tensorial notations**

The linear map \( F := F^t_{\ell}(p_{t_0}) \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_{t_0}) \) can be canonically naturally associated with the bipoint tensor \( \hat{F} \in \mathcal{L}(\mathbb{R}^n_{t_0}, \mathbb{R}^n_{t_0}; \mathbb{R}) \) defined by, for all \( (\ell, \tilde{W}) \in \mathbb{R}^n_{t_0} \times \mathbb{R}^n_{t_0} \),
\[
\hat{F}(\ell, \tilde{W}) := \ell.F.\tilde{W},
\]
see § A.9.4. Then, with (5.26) and (\( \pi_{a_i} \)) the dual basis of (\( \tilde{a}_i \)), we have \( d\phi_i = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j} \pi_{a_j} = \sum_{j=1}^{n} F_{ij} \pi_{a_j} \),
thus
\[
F.\tilde{a}_j = \sum_{i=1}^{n} F_{ij} \tilde{b}_i,
\]
thus \( \hat{F} = \sum_{i=1}^{n} \tilde{b}_i \otimes d\phi_i = \sum_{i,j=1}^{n} F_{ij} \tilde{b}_i \otimes \pi_{a_j} \).

And we do recover, with the contraction rule (see (Q.23)), \( \hat{F}.\tilde{W} = (\sum_{i=1}^{n} \tilde{b}_i \otimes d\phi_i)(\tilde{W}) = \sum_{j=1}^{n} \tilde{b}_j (d\phi_i, \tilde{W}) = \sum_{j=1}^{n} \tilde{b}_j (W_i \frac{\partial \phi_i}{\partial x_j}) = \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j} W_i \tilde{b}_j = \hat{F}.\tilde{W} \).

(Marsden notations: \( dX^\ell \)) is the dual basis of \( (\tilde{E})_j \) and \( F.\tilde{E}_j = \sum_{i=1}^{n} F_{ij} \tilde{c}_i \) and \( \hat{F} = \sum_{i,j=1}^{n} F_{ij} \tilde{c}_i \otimes dX^j \).

And we recover (5.28) since then \( \hat{F} = \sum_{i=1}^{n} \tilde{b}_i \otimes d\phi_i \),
\[
d^2\Phi \sim \hat{dF} = \sum_{i=1}^{n} \tilde{b}_i \otimes d^2\phi_i = \sum_{i,j,k=1}^{n} \frac{\partial^2 \phi_i}{\partial x_j \partial x_k} \tilde{b}_i \otimes \pi_{a_j} \otimes \pi_{a_k},
\]
and we recover (5.28) since then \( d^2\Phi(\tilde{U}, \tilde{W}) = \sum_{i=1}^{n} \tilde{b}_i \otimes d^2\phi_i(\tilde{U}, \tilde{W}) = \sum_{i,j,k=1}^{n} \frac{\partial^2 \phi_i}{\partial x_j \partial x_k} \tilde{b}_i \otimes \pi_{a_j} \otimes \pi_{a_k} \),
(Marsden notations: \( d\tilde{F} = \sum_{i=1}^{n} \tilde{c}_i \otimes d^2\phi_i \)).

**Remark 5.8** In some manuscripts you find the notation \( F = d\Phi \odot \nabla X \). It does not help to understand what \( F \) is (it is the differential \( d\Phi \)), and should not be used as far as objectivity is concerned:

- A differentiation is not a tensorial operation, see remark R.1, so why use the tensor product notation \( \Phi \otimes \nabla X \), when the standard notation \( d\Phi \simeq F = \sum_{i=1}^{n} \tilde{c}_i \otimes d\phi_i \) is legitimate, explicit and easy to manipulate, cf. (5.30)?
- It could be misinterpreted, since, in mechanics, \( \nabla \) is often understood to be the gradient (a gradient needs a Euclidean dot product: Which one?) which is contravariant, while a differential is covariant (a differential is unambiguous in thermodynamics).
- It gives the confusing notation \( \Phi \otimes \nabla_X \otimes \nabla_X \); instead of the legitimate (5.31) which is explicit and easy to manipulate.

\[ \Box \]
5.5 Change of coordinate system at \( t \) for \( F \)

Let \( p_{t_0} \in \Omega_{t_0} \), let \( p_t = \Phi^{t_0}_t(p_{t_0}) \), let \( \vec{W}(p_t) \in \mathbb{R}^n_{t_0} \), let \( \vec{w}(p_t) = F^{t_0}_t(p_{t_0}), \vec{W}(p_{t_0}) \in \mathbb{R}^n_t \) (its push-forward), written \( \vec{w} = F\vec{W} \) for short.

At the actual time \( t \), we are interested in the result seen by two observers, the first using a Cartesian basis \( (\vec{b}_1) \), the second using a Cartesian basis \( (\vec{b}_2) \): Both bases are in \( \mathbb{R}^n_t \). Let \( P = [P_{ij}] \) be the transition matrix from \( (\vec{b}_1) \) to \( (\vec{b}_2) \), that is, \( b_{j2} = \sum_{i=1}^{n} p_{ij} b_{i1} \) for all \( j \).

The change of basis formula for vectors from \( (\vec{b}_1) \) to \( (\vec{b}_2) \) in \( \mathbb{R}^n_{t_0} \) applied to \( \vec{w} \in \mathbb{R}^n_{t_0} \) gives

\[
[\vec{w}]_{\vec{b}_2} = P^{-1} [\vec{w}]_{\vec{b}_1}, \quad \text{thus} \quad [F,\vec{W}]_{\vec{b}_2} = P^{-1} [F,\vec{W}]_{\vec{b}_1}. \tag{5.32}
\]

Thus, with \( (\vec{a}_i) \) a basis in \( \mathbb{R}^n_{t_0} \), chosen by some observer in the past, we get

\[
[F]_{[\vec{a},\vec{b}]} [\vec{W}]_{\vec{a}} = P^{-1} [F]_{[\vec{a},\vec{b}]} [\vec{W}]_{\vec{a}}, \tag{5.33}
\]

true for all \( \vec{W} \), thus

\[
[F]_{[\vec{a},\vec{b}]} = P^{-1} [F]_{[\vec{a},\vec{b}]} \tag{5.34}
\]

NB: \( (5.34) \) is not the change of coordinate system formula for an endomorphism, which would be nonsense since \( F := F^{t_0}_t(p_{t_0}) : \mathbb{R}^n_{t_0} \to \mathbb{R}^n_t \) is not an endomorphism, cf. \((5.7)\); \((5.34)\) is the usual change of basis formula for vectors in \( \mathbb{R}^n_t \), cf. \((5.32)\).

5.6 Spatial Taylor expansion of \( \Phi^{t_0}_t \) and \( F^{t_0}_t \)

\( \Phi^{t_0}_t \) is supposed to be \( C^1 \) for all \( t_0, t \). Let \( p_{t_0} \in \Omega_{t_0} \) and \( \vec{W}_{p_{t_0}} \in \mathbb{R}^n_{t_0} \). Then

\[
\Phi^{t_0}_t(p_{t_0} + h \vec{W}_{p_{t_0}}) = \Phi^{t_0}_t(p_{t_0}) + h d\Phi^{t_0}_t(p_{t_0}).\vec{W}_{p_{t_0}} + o(h) = \Phi^{t_0}_t(p_{t_0}) + h F^{t_0}_t(p_{t_0}).\vec{W}_{p_{t_0}} + o(h). \tag{5.35}
\]

And, with \( \Phi^{t_0}_t \) \( C^2 \),

\[
\Phi^{t_0}_t(p_{t_0} + h \vec{W}_{p_{t_0}}) = \Phi^{t_0}_t(p_{t_0}) + h d\Phi^{t_0}_t(p_{t_0}).\vec{W}_{p_{t_0}} + \frac{h^2}{2} d^2\Phi^{t_0}_t(p_{t_0}).(\vec{W}_{p_{t_0}}, \vec{W}_{p_{t_0}}) + o(h^2) = \Phi^{t_0}_t(p_{t_0}) + h F^{t_0}_t(p_{t_0}).\vec{W}_{p_{t_0}} + \frac{h^2}{2} dF^{t_0}_t(p_{t_0}).(\vec{W}_{p_{t_0}}, \vec{W}_{p_{t_0}}) + o(h^2). \tag{5.36}
\]

And

\[
F^{t_0}_t(p_{t_0} + h \vec{W}_{p_{t_0}}) = F^{t_0}_t(p_{t_0}) + h dF^{t_0}_t(p_{t_0}).\vec{W}_{p_{t_0}} + o(h) \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_t), \tag{5.37}
\]

that is, for any \( \vec{U}_{p_{t_0}} \in \mathbb{R}^n_{t_0} \),

\[
F^{t_0}_t(p_{t_0} + h \vec{W}_{p_{t_0}}).\vec{U}_{p_{t_0}} = F^{t_0}_t(p_{t_0}).\vec{U}_{p_{t_0}} + h (dF^{t_0}_t(p_{t_0}).\vec{W}_{p_{t_0}}).\vec{U}_{p_{t_0}} + o(h) \in \mathbb{R}^n_t, \tag{5.38}
\]

with \( (dF^{t_0}_t(p_{t_0}).\vec{W}_{p_{t_0}}).\vec{U}_{p_{t_0}} = d^2\Phi^{t_0}_t(p_{t_0}).(\vec{W}_{p_{t_0}}, \vec{U}_{p_{t_0}}) \).

5.7 Time Taylor expansion of \( F^{t_0}_t \)

Let \( \vec{\Phi} \) be a \( C^2 \) motion of \( \text{Obj} \), let \( t_0 \in \mathbb{R} \), and let \( \Phi^{t_0}_t \) be the associated motion. The time Taylor expansion of \( \Phi^{t_0}_t \) is given in \((4.31)-(4.32)\). So, with \( p_t = \Phi(t, P_{\text{Obj}}) \) and \( p_{t_0} = \Phi(t_0, P_{\text{Obj}}) \), with \( \vec{v}(t, p_t) = \partial \Phi^{t_0}_t \partial t(t, p_{t_0}) \) the Eulerian velocity and \( \vec{V}^{t_0}_t(t, p_{t_0}) := \frac{\partial \Phi^{t_0}_t}{\partial t}(t, p_{t_0}) = \vec{v}(t, p_t) \) the Lagrangian velocity, and with \( F^{t_0}_t(t, p_{t_0}) = \partial \Phi^{t_0}_t(t, p_{t_0}) \) the \( \vec{\Phi} \)-velocity of \( \Phi^{t_0}_t(t, p_{t_0}) \), we have

\[
(F^{t_0}_t)'(t) = \frac{\partial (d\Phi^{t_0}_t)}{\partial t}(t, p_{t_0}) = d(\Phi^{t_0}_t)(t, p_{t_0}) = \frac{\partial F^{t_0}_t}{\partial t}(t, p_{t_0}) = \frac{\partial (d\Phi^{t_0}_t)}{\partial t}(t, p_{t_0}) \tag{5.39}
\]

Short abusive notation: \( F' = d\vec{v}.F \) (at what points, what times?), or \( \dot{F} = \vec{V}.F = d\vec{V}.F \).
If $\Phi^0$ is $C^3$ then $\frac{\partial^2 \Phi^0}{\partial t^2}(t, p_n) = \tilde{A}^0(t, p_n) = \tilde{\gamma}(t, p(t))$ (Lagrangian and Eulerian accelerations), hence

$$(F_{\tilde{p}_0}^{\tilde{t}})^{\prime}(t) = d\tilde{A}^0(t, p_n) = d\tilde{\gamma}(t, p_t).F_{\tilde{p}_0}^{\tilde{t}}(t).$$

(5.40)

Short abusive notation: $F^\prime = d\tilde{v}_t F$ (at what points, what times?), or $\tilde{F} = d\tilde{\gamma}_t F$ ($= d\tilde{A}$).

Thus, the second order time Taylor expansion of $F_{\tilde{p}_0}^{\tilde{t}}$ is

$$F_{\tilde{p}_0}^{\tilde{t}}(t+h) = \left( F_{\tilde{p}_0}^{\tilde{t}} + h\frac{\partial F_{\tilde{p}_0}^{\tilde{t}}}{\partial t} + \frac{h^2}{2} d\tilde{A}^0_{\tilde{p}_0}(t) \right) + o(h^2)$$

$$= \left( I + h\tilde{\gamma} + \frac{h^2}{2} dt^2 \right) (t, p(t), F_{\tilde{p}_0}^{\tilde{t}}(t) + o(h^2)) \quad \text{when} \quad p(t) = \Phi^0(t, p_n).$$

(5.41)

NB: they are three times are involved: $t_0$ (observer dependent), $t$ and $t+h$, as for (4.31).

In particular, $F_{\tilde{p}_0}^{\tilde{t}}(p_{t_0}) = I$ gives

$$F_{\tilde{p}_0}^{\tilde{t}}(t_0+h) = \left( I + h\tilde{\gamma} + \frac{h^2}{2} dt^2 \right) (t_0, p_{t_0}) + o(h^2)$$

(5.42)

Remark 5.9 $\gamma = \frac{\partial \gamma}{\partial t} + d\tilde{v}_t \tilde{v}$ is not linear in $\tilde{v}$. Idem,

$$d\tilde{\gamma}_t = \frac{D\tilde{v}}{Dt} = \frac{\partial \tilde{\gamma}}{\partial t} + d\tilde{v}_t \tilde{\gamma} = \frac{\partial \tilde{v}}{\partial t} + d\tilde{\gamma}_t \tilde{v} + d\tilde{v}_t \tilde{v} \tilde{\gamma} \quad (= \frac{D(d\tilde{v})}{Dt} + d\tilde{v}_t d\tilde{v})$$

(5.43)

is non linear in $\tilde{v}$, and gives $F_{\tilde{p}_0}^{\tilde{t}}(t) = (\frac{D\tilde{v}}{Dt} + d\tilde{\gamma}_t \tilde{v} + d\tilde{v}_t \tilde{v} \tilde{\gamma})(t, p_t).F_{\tilde{p}_0}^{\tilde{t}}(t)$, non linear in $\tilde{v}$.

Exercise 5.10 Directly check that $F^\prime = d\tilde{v}_t F$ gives $F^\prime = d\tilde{\gamma}_t F$.

Answer: $F^\prime(t) = d\tilde{v}_t(p(t)).F(t)$ gives $F^\prime(t) = \frac{D(d\tilde{v})}{Dt}(t, p(t)).F(t) + d\tilde{v}_t(p(t)).F(t)$ with $\frac{D(d\tilde{v})}{Dt} = d\tilde{\gamma}_t - d\tilde{v}_t \tilde{v}$, cf. (5.43), thus $F^\prime(t) = (d\tilde{\gamma}_t - d\tilde{v}_t \tilde{v} \tilde{\gamma})(t, p(t)).F(t) + d\tilde{v}_t(p(t)).F(t)$.

6 Flow

6.1 Introduction: Motion versus flow

- A motion $\Phi : (t, P_{t_0}) \rightarrow p_t = \Phi(t, P_{t_0})$ locates a particle $P_{t_0}$ at $t$ in the affine space $\mathbb{R}^n$, cf. (1.5), and the Eulerian velocity field $\tilde{\nu}$ of the particle is $\tilde{\nu}(t, p_t) := \frac{\partial \Phi^0}{\partial t}(t, P_{t_0})$ when $p_t = \Phi(t, P_{t_0})$, cf. (2.5).

- A flow starts from an Eulerian velocity field $\tilde{\nu}$ to deduce a motion by solving the ODE (ordinary differential equation) $\frac{dp}{dt}(t) = \tilde{\nu}(t, \phi(t))$ (with Cauchy–Lipschitz theorem). In particular, a flow doesn’t deal with Lagrangian velocities (two point tensors): It deals with ODEs and Eulerian velocities (tensors).

6.2 Definition

Let $\tilde{\nu} : \left\{ \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \right\}$ be a unstationary vector field (e.g., a Eulerian velocity field). We look for maps $\Phi : \left\{ \mathbb{R} \rightarrow \Omega \rightarrow t \rightarrow p = \Phi(t) \right\}$ which are locally (i.e. in the vicinity of some $t$) solutions of the ODE (ordinary differential equation)

$$\frac{d\Phi}{dt}(t) = \tilde{\nu}(t, \Phi(t)), \text{ also written } \frac{dp}{dt}(t) = \tilde{\nu}(t, p(t)).$$

(6.1)

(Also written $\frac{d\Phi}{dt}(t) = \bar{\nu}(t, \bar{x}(t))$ with $\bar{x}(t) = \bar{\nu}^3_t$, cf. (1.6).)

Definition 6.1 A solution $\Phi$ of (6.1) is a flow of $\tilde{\nu}$; Also called an integral curve of $\tilde{\nu}$, since $\Phi(t) = \int_{t_0}^t \tilde{\nu}(\bar{u}, \phi(\bar{u})) \, d\bar{u} + \Phi(t)$.

Remark 6.2 Improper notation for (6.1):

$$\frac{dp}{dt}(t) \quad \text{noted } \frac{dp(t)}{dt} = \frac{d\Phi(t)}{dt} = (\tilde{\nu}(t, p(t))).$$

(6.2)

Question: If the notation $\frac{dp(t)}{dt}$ is used, then what is the meaning of $\frac{dp(d\Phi(t))}{dt}$?

Answer: It means, either $\frac{dp}{dt}(f(t))$, or $\frac{dp(f)}{dt}(t) = \frac{d\Phi}{dt}(f(t)) \frac{df}{dt}(t)$: Ambiguous. So it is better to use $\frac{dp}{dt}(t)$, and to avoid $\frac{dp(t)}{dt}$, unless the context is clear (no composite functions).
Remark 6.3 Another improper notation for (6.1): \( \frac{d\Phi}{dt}(t) = \vec{v}(t, p) \), which does not mean \( \frac{d\Phi}{dt}(u) = \vec{v}(t, p(u)) \) for any \( u \), but \( \frac{d\Phi}{dt}(t) = \vec{v}(t, p(t)) \). (The notation \( \frac{d\Phi}{dt}(u) = \vec{v}(t, p(u)) \) is used for streamlines, cf. (2.21).)

6.3 Cauchy–Lipschitz theorem

Let \((t_0, p_{t_0})\) be in the definition domain of \( \vec{v} \) (and \((t_0, p_{t_0})\) will be called an initial condition). We look for \( \Phi \) solution of

\[
\frac{d\Phi}{dt}(t) = \vec{v}(t, \Phi(t)) \quad \text{and} \quad \Phi(t_0) = p_{t_0},
\]

which is an ODE (Ordinary Differential Equation) with an initial condition.

Definition 6.4 Let \( t_1, t_2 \in \mathbb{R}, t_1 < t_2 \). Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and \( \overline{\Omega} \) its closure. Let \( ||.|| \) be a norm in \( \mathbb{R}^n \). A continuous map \( \vec{v}: [t_1, t_2] \times \overline{\Omega} \to \mathbb{R}^n \) is Lipschitzian iff it is “space Lipschitzian, uniformly in time”, that is, if

\[
\exists k > 0, \forall t \in [t_1, t_2], \forall p, q \in \overline{\Omega}, ||\vec{v}(t, q) - \vec{v}(t, p)|| \leq k||q - p||.
\]

That is, \( \frac{||\vec{v}(q) - \vec{v}(p)||}{||q - p||} \leq k \), for all \( t \) and all \( p \neq q \). That is, the variations of \( \vec{v} \) are bounded in space, uniformly in time.

Theorem 6.5 (and definition) (Cauchy–Lipschitz). Let \( \vec{v}: [t_1, t_2] \times \overline{\Omega} \to \mathbb{R}^n \) be Lipschitzian, cf. (6.4), and let \((t_0, p_{t_0}) \in [t_1, t_2] \times \overline{\Omega} \). Then there exists \( \varepsilon = \varepsilon_{t_0, p_{t_0}} > 0 \) s.t. (6.3) has a unique solution \( \Phi : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^n \), which is called a flow of \( \vec{v} \) (more precisely, is called the flow of \( \vec{v} \) which satisfies \( \Phi(t_0) = p_{t_0} \)).

Moreover, if \( \vec{v} \) is \( C^k \) then \( \Phi \) is \( C^{k+1} \).

Proof. See e.g. Arnold [2], or any ODE course. In particular \( ||\vec{v}||_\infty := \sup_{t \in [t_0 - \varepsilon, t_0 + \varepsilon], p \in \overline{\Omega}} ||\vec{v}(t, p)||_{\mathbb{R}^n} \) (maximum speed) exists since \( \vec{v} \in C^0 \) on the compact \([t_1, t_2] \times \overline{\Omega}\), see definition 6.4, hence we can choose \( \varepsilon = \min(t_0 - t_1, t_2 - t_0, \frac{d(p_{t_0}, p_0)}{||p_0||_{\infty}}) \) (the time needed to reach the border \( \partial \Omega \) from \( p_{t_0} \)).

Notations. The solution \( \Phi \) of (6.3) is noted \( \Phi_{t_0}^{t_{t_0}} : \) So, for all \( t \in [t_0 - \varepsilon, t_0 + \varepsilon] \),

\[
\frac{d\Phi_{t_0}^{t_{t_0}}(t)}{dt} = \vec{v}(t, \Phi_{t_0}^{t_{t_0}}(t)) \quad \text{and} \quad \Phi_{t_0}^{t_{t_0}}(t_0) = p_{t_0}.
\]

We have thus defined for some \( \Omega_{t_0} \subset \Omega \) (see next theorem 6.6),

\[
\Phi_{t_0}^{t_{t_0}} : \left[ [t_0 - \varepsilon, t_0 + \varepsilon] \times \Omega_{t_0} \to \mathbb{R}^n \right]
\]

\[
(t, p_{t_0}) \to p = \Phi_{t_0}^{t_{t_0}}(t, p_{t_0}) := \Phi_{t_0}^{t_{t_0}}(t) : \]

The function \( \Phi_{t_0}^{t_{t_0}} \) is also called a flow. And (6.5) reads

\[
\frac{d\Phi_{t_0}^{t_{t_0}}}{dt}(t, p_{t_0}) = \vec{v}(t, \Phi_{t_0}^{t_{t_0}}(t, p_{t_0})), \quad \text{and} \quad \Phi_{t_0}^{t_{t_0}}(t_0, p_{t_0}) = p_{t_0}.
\]

We have thus defined,

\[
\Phi : \left[ [t_1, t_2] \times [t_1, t_2] \times \Omega_{t_0} \to \Omega \right]
\]

\[
(t, t_0, p_{t_0}) \to p = \Phi(t, t_0, p_{t_0}) := \Phi_{t_0}^{t_{t_0}}(t)_{\text{notat \( \Phi(t, t_0, p_{t_0}) \))} : \]

The function \( \Phi \) is also called a flow. And (6.5) reads

\[
\frac{d\Phi_{t_0}^{t_{t_0}}}{dt}(t, t_0, p_{t_0}) = \vec{v}(t, \Phi_{t_0}^{t_{t_0}}(t, t_0, p_{t_0})), \quad \text{with} \quad \Phi_{t_0}^{t_{t_0}}(t_0, t_0, p_{t_0}) = p_{t_0}.
\]

Other notation: \( \Phi_{t, t_0} := \Phi_{t_0}^{t_{t_0}} : \Omega_{t_0} \to \mathbb{R}^n \), and (6.9) is also written

\[
\frac{d\Phi_{t, t_0}}{dt}(p_{t_0}) = \vec{v}(t, \Phi_{t, t_0}(p_{t_0})), \quad \text{and} \quad \Phi_{t, t_0}(p_{t_0}) = p_{t_0}.
\]

Theorem 6.6 Let \( \vec{v} \) be Lipschitzian, cf. (6.4). Let \( t_0 \in [t_1, t_2] \), and let \( \Omega_{t_0} \) be an open set s.t. \( \Omega_{t_0} \subset \mathbb{R}^n \), that is, there exists a compact set \( K \subset \mathbb{R}^n \) s.t. \( \Omega_{t_0} \subset K \subset \Omega \). Then there exists \( \varepsilon > 0 \) s.t. a flow \( \Phi_{t_0}^{t_{t_0}} \) exists on \( [t_0 - \varepsilon, t_0 + \varepsilon] \times \Omega_{t_0} \), cf. (6.6).
Proof. Let \( d = d(K, \mathbb{R}^n - \Omega) \) (\( d \) distance of \( K \) to the border of \( \Omega \)).

Let \( \|\vec{v}\|_\infty := \sup_{t \in [t_1, t_2], \vec{p} \in \mathbb{R}^n} \|\vec{v}(t, p)\|_\infty \) exists since \( \vec{v} \in C^0 \) on the compact \([t_1, t_2] \times \Omega\).

Let \( \varepsilon = \min(t_0 - t_1, t_2 - t_0, \frac{d}{\|\vec{v}\|_\infty}) \) (less that the minimum time to reach the border from \( K \) at maximum speed \( ||\vec{v}||_\infty \)).

Let \( P \in K \) and \( t \in [t_0 - \varepsilon, t_0 + \varepsilon] \). Then \( \Phi^b_\omega(t) \) exists, cf. theorem 6.5, and \( \|\Phi^b_\omega(t) - \Phi^b_\omega(t_0)\|_\infty \leq |t - t_0| \text{sup}_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} \|\vec{v}(t)\|_\infty \) (mean value theorem since, \( \vec{v} \) being \( C^0 \), \( \Phi^b_\omega \) is \( C^1 \)). Thus \( \|\Phi^b_\omega(t) - \Phi^b_\omega(t_0)\|_\infty \leq |t - t_0| \|\vec{v}\|_\infty \) thus \( \Phi^b_\omega(t) \in \Omega \). Thus \( \Phi^b_\omega \) exists on \([t_0 - \varepsilon, t_0 + \varepsilon]\), for all \( P \in K \). Thus \( \Phi^b_\omega \) exists on \([t_0 - \varepsilon, t_0 + \varepsilon]\).

Remark 6.7 The (Eulerian) approach of a flow starts with a Eulerian type velocity (independent of any initial time), and then, due to the introduction of initial conditions, leads to Lagrangian functions, cf. (6.5). Once again, Lagrangian functions are the result of Eulerian functions: And in this manuscript we were careful to introduce § 2 (Eulerian) before § 4 (Lagrangian).

6.4 Examples

Example 1 \( \mathbb{R}^2 \) with an origin \( O \) and a Euclidean basis \((\vec{e}_1, \vec{e}_2)\). Let \( p \in \mathbb{R}^2 \), \( O \vec{p} = \text{noted } \vec{x} = x \vec{e}_1 + y \vec{e}_2 = \text{noted } (x, y) \). Let \( t_1 = -1, t_2 = 1 \) and \( \Omega = [0, 2] \times [0, 1] \) (observation window). Let \( t_0 \in [t_1, t_2] \), \( a,b \in \mathbb{R}, a \neq 0 \), and consider

\[
v(t, p) = \begin{cases} v_1(t, x, y) = ay, \\ v_2(t, x, y) = b \sin(t - t_0). \end{cases}
\]

(\( b = 0 \) corresponds to the stationary case = shear flow.) So \( \vec{v} \) is a \( C^\infty \) vector field.

Then (6.7) reads, with \( \vec{x}(t_0) = (x_0, y_0) \) and \( \vec{p}(t) = \overrightarrow{O\Phi^b_\omega(t)} = \text{noted } \left( x(t), y(t) \right) \).

\[
\begin{align*}
\frac{dx}{dt}(t) &= v_1(t, x(t), y(t)) = ay(t), \quad x(t_0) = x_0, \\
\frac{dy}{dt}(t) &= v_2(t, x(t), y(t)) = b \sin(t - t_0), \quad y(t_0) = y_0.
\end{align*}
\]

Thus

\[
\overrightarrow{O\Phi^b_\omega(t)} = \frac{\partial \Phi^b_\omega}{\partial t} = \begin{cases} x(t) = x_0 + a(y_0 + b)(t - t_0) - ab \sin(t - t_0), \\
y(t) = y_0 + b - b \cos(t - t_0).
\end{cases}
\]

Example 2 \( \mathbb{R}^2 \) with an origin \( O \) and a Euclidean basis \((\vec{e}_1, \vec{e}_2)\). Let \( \vec{p} = \text{noted } \vec{x} = \left( \begin{array}{c} x \\ y \end{array} \right), \) let \( \omega > 0 \)

and consider

\[
\vec{v}(t, x, y) = \begin{pmatrix} -\omega y \\ \omega x \end{pmatrix} = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \text{noted } \vec{v}(x, y).
\]

(Spin vector field.) With \( \overrightarrow{O\Phi^b_\omega} = \vec{x}_t = \left( \begin{array}{c} x_t \\ y_t \end{array} \right) \), with \( r_t = \sqrt{x_t^2 + y_t^2} \) and with \( \theta_0 \) such that \( \vec{x}_t = \left( \begin{array}{c} x_t = r_t \cos(\omega \theta_0) \\ y_t = r_t \sin(\omega \theta_0) \end{array} \right) \), the solution \( \Phi^b_\omega(t) \) of (6.7) is

\[
\overrightarrow{O\Phi^b_\omega(t)} = \frac{\partial \Phi^b_\omega}{\partial t} = \vec{x}(t) = \begin{cases} x(t) = r_t \cos(\omega t), \\
y(t) = r_t \sin(\omega t). \end{cases}
\]

Indeed, \( \frac{\partial \Phi^b_\omega}{\partial t}(t, \vec{x}_0) = \begin{pmatrix} v_1(t, x(t, \vec{x}_0), y(t, \vec{x}_0)) \\ v_2(t, x(t, \vec{x}_0), y(t, \vec{x}_0)) \end{pmatrix} = \begin{pmatrix} -\omega y(t, \vec{x}_0) \\ \omega x(t, \vec{x}_0) \end{pmatrix}, \) thus \( \frac{\partial \Phi^b_\omega}{\partial t}(t, \vec{x}_0) = -\omega y(t, \vec{x}_0) \) and \( \frac{\partial \Phi^b_\omega}{\partial y}(t, \vec{x}_0) = \omega x(t, \vec{x}_0) \), hence \( y \). Idem for \( x \). Here \( \sqrt{\vec{v}(t, x, y)} = \omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \omega \begin{pmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{pmatrix} \) is the \( \pi/2 \)-rotation composed with the homothety with ratio \( \omega \).

6.5 Composition of flows

Let \( \vec{v} \) be a vector field on \( \mathbb{R} \times \Omega \) and \( \Phi^b_\omega \) solution of (6.5). We use the notations

\[
p_t = \Phi^b_\omega(t) = \Phi_{t_1, t_0}(p_0) \quad \Phi^b_\omega(t) = \Phi(t, t_0, p_0) = \Phi(t; t_0, p_0).
\]
6.5.1 Law of composition of flows

**Proposition 6.8** For all $t_0, t_1, t_2 \in \mathbb{R}$, we have

$$\Phi_{t_2}^{t_1} \circ \Phi_{t_0}^{t_1} = \Phi_{t_2}^{t_0}$$

i.e. $\Phi_{t_2}^{t_1} \circ \Phi_{t_1}^{t_0} = \Phi_{t_2}^{t_0}$. \hspace{1cm} (6.17)

(“The composition of the photos gives the film”). So,

$$p_{t_2} = \Phi_{t_2}^{t_1}(p_{t_1}) = \Phi_{t_2}^{t_0}(p_{t_0})$$

when $p_{t_1} = \Phi_{t_1}^{t_0}(p_{t_0})$, \hspace{1cm} (6.18)

i.e.,

$$p_{t_2} = \Phi_{t_2}^{t_1}(p_{t_1}) = \Phi_{t_2}^{t_0}(p_{t_0})$$

when $p_{t_1} = \Phi_{t_1}^{t_0}(p_{t_0})$. \hspace{1cm} (6.19)

Thus

$$d\Phi_{t_2}^{t_1}(p_{t_1}).d\Phi_{t_2}^{t_0}(p_{t_0}) = d\Phi_{t_2}^{t_0}(p_{t_0}), \quad \text{i.e.} \quad d\Phi_{t_2}^{t_1}(p_{t_1}).d\Phi_{t_2}^{t_0}(p_{t_0}) = d\Phi_{t_2}^{t_0}(p_{t_0}). \hspace{1cm} (6.20)$$

**Proof.** Let $p_{t_1} = \Phi_{t_0}^{t_0}(t_1)$. (6.7) gives

$$\begin{cases}
\frac{d\Phi_{p_0}^{t_0}}{dt}(t) = \vec{v}(t, \Phi_{t_0}^{t_0}(t)), \\
\frac{d\Phi_{p_1}^{t_i}}{dt}(t) = \vec{v}(t, \Phi_{t_0}^{t_i}(t)),
\end{cases} \quad \text{with} \quad p_{t_1} = \Phi_{t_0}^{t_0}(t_1) = \Phi_{p_1}^{t_0}(t_1). \hspace{1cm} \blacksquare$$

Thus $\Phi_{t_0}^{t_0}$ and $\Phi_{t_1}^{t_0}$ satisfy the same ODE with the same value at $t_1$; Thus they are equal (uniqueness thanks to Cauchy–Lipschitz theorem), thus $\Phi_{t_0}^{t_1}(t) = \Phi_{t_0}^{t_0}(t)$ when $p_{t_1} = \Phi_{t_1}^{t_0}(p_{t_0})$, that is, $\Phi_{t_2}^{t_1}(p_{t_1}) = \Phi_{t_2}^{t_0}(p_{t_0})$ when $p_{t_1} = \Phi_{t_1}^{t_0}(p_{t_0})$, which is (6.17) for any $t = t_2$. Thus (6.20).

NB: Then a flow is compatible with the motion of an object $\text{Obj}$. Indeed, (3.6) gives $\Phi_{t_2}^{t_1} \circ \Phi_{t_1}^{t_0} = (\Phi_{t_2}^{t_0} \circ (\Phi_{t_1}^{t_0})^{-1}) \circ (\Phi_{t_1}^{t_0} \circ (\Phi_{t_0}^{t_0})^{-1}) = \Phi_{t_2}^{t_0} \circ (\Phi_{t_0}^{t_0})^{-1} = \Phi_{t_2}^{t_0}$ that is (6.17).

6.5.2 Stationary case

**Definition 6.9** $\vec{v}$ is a stationary vector field iff $\frac{\partial \vec{v}}{\partial t} = 0$; And then we write $\vec{v}(t, p) = \text{noted } \vec{v}(p)$. And the associated flow $\Phi_{t_0}^{t_0}$, that satisfies

$$\frac{\partial \Phi_{t_0}^{t_0}}{\partial t}(t, p_{t_0}) = \vec{v}(p_{t_0}) \quad \text{when} \quad p_{t} = \Phi_{t_0}^{t_0}(t, p_{t_0}), \hspace{1cm} (6.21)$$

is said to be stationary.

**Proposition 6.10** If $\vec{v}$ is a stationary vector field then, for all $t_0, t_1, h$, when meaningful,

$$\Phi_{t_2}^{t_1} = \Phi_{t_0}^{t_0}$$

i.e. $\Phi_{t_2}^{t_1} = \Phi_{t_0}^{t_0}$. \hspace{1cm} (6.22)

In other words, for all $h$ (small enough to be meaningful),

$$\Phi_{t_1}^{t_1+h} = \Phi_{t_0}^{t_0}$$

i.e. $\Phi_{t_1}^{t_1+h} = \Phi_{t_0}^{t_0}$. \hspace{1cm} (6.23)

**Proof.** Let $q \in \Omega_{t_0}$. $\alpha(h) = \Phi_{t_0}^{t_0}(q) = \Phi_{t_0}^{t_0}(t_0+h)$ and $\beta(h) = \Phi_{t_1}^{t_1}(q) = \Phi_{t_2}^{t_2}(t_1+h)$.

Thus $\alpha'(h) = \frac{d\Phi_{t_0}^{t_0}}{dt}(t_0+h) = \vec{v}(t_0+h, \Phi_{t_0}^{t_0}(t_0+h)) = \vec{v}(t_0+h) = \vec{v}(\alpha(h))$ (stationary flow), and

$$\beta'(h) = \frac{d\Phi_{t_1}^{t_1}}{dt}(t_1+h) = \vec{v}(t_1+h, \Phi_{t_1}^{t_1}(t_1+h)) = \vec{v}(t_1+h) = \vec{v}(\beta(h)) \hspace{1cm} (6.21)$$

Thus $\alpha$ and $\beta$ satisfy the same ODE with the same initial condition $\alpha(0) = \beta(0) = q$. Thus $\alpha = \beta$. Hence (6.22). Thus, with $h = t_1-t_0$, i.e. with $t_1 = t_0+h$ and $t_0+h = t_1$, we get (6.23). \hspace{1cm} \blacksquare
Corollary 6.11 If \( \vec{v} \) is a stationary vector field, cf. (6.21), then
\[
d\Phi_t^{t_0}(p_{t_0}, \vec{v}(p_{t_0})) = \vec{v}(p_t) \quad \text{when} \quad p_t = \Phi_t^{t_0}(p_{t_0}),
\] (6.24)
that is, if \( \vec{v} \) is stationary, then \( \vec{v} \) is transported along itself (see the push-forward \$ 10.5.2).

**Proof.** \( \Phi_t^{t_0+\tau} \circ \Phi_0^{\tau} = \Phi_t^{t_0+\tau} \), cf. (6.17). Thus, \( \vec{v} \) being stationary, \( \Phi_t^{t_0} \circ \Phi_0^{\tau} = \Phi_t^{t_0+\tau} \), cf. (6.23). That is, \( \Phi(t; t_0, (\Phi(t_0+s; t_0, p_{t_0})) = \Phi(t+s; t_0, p_{t_0}) \). Thus (s derivative)
\[
d\Phi(t; t_0, (\Phi(t_0+s; t_0, p_{t_0})))\frac{\partial \Phi}{\partial s}(t_0+s; t_0, p_{t_0}) = \vec{v}(t+s, \Phi(t+s; t_0, p_{t_0}))
\]
(see \$ 6.7.1 for non ambiguous calculations), thus (stationary flow):
\[
d\Phi_t^{t_0}(\Phi_0^{t_0+s}(p_{t_0})), \vec{v}(\Phi_0^{t_0+s}(p_{t_0})) = \vec{v}(\Phi_t^{t_0+s}(p_{t_0})).
\]
And \( p_t = \Phi_t^{t_0}(p_{t_0}) \) and \( s = 0 \) give (6.24).

6.6 Velocity on the trajectory traveled in the opposite direction

Let \( t_0, t_1 \in \mathbb{R}, t_1 > t_0, \) and \( p_{t_0} \in \mathbb{R}^n \). Consider the trajectory \( \Phi_t^{t_0}_{p_{t_0}} : \left\{ [t_0, t_1] \to \mathbb{R}^n \right\} \), \( t \to p(t) = \Phi_t^{t_0_{p_{t_0}}}(t) \). So \( p_{t_0} \)
is the beginning of the trajectory, \( p_{t_1} = \Phi_{t_0}^{t_0}(p_{t_0}) \) the end, and \( \vec{v}(t, p(t)) = \frac{d\Phi_{t_0}^{t_0}}{dt}(t) \) the velocity.

Define the trajectory traveled in the opposite direction by
\[
\Psi_{t_0}^{t_1} : \left\{ [t_0, t_1] \right\} \to \mathbb{R}^n
u \to q(u) = \Psi_{t_0}^{t_1}(u) = \Phi_{t_0}^{t_0}(t_0 + t_1 - u) = \Phi_{t_0}^{t_0}(t) = p(t) \quad \text{when} \quad t = t_0 + t_1 - u.
\] (6.25)
In particular \( q(t_0) = \Psi_{t_0}^{t_1}(t_0) = \Phi_{t_0}^{t_0}(t_1) = p(t_1) \) and \( q(t_1) = \Psi_{t_0}^{t_1}(t_1) = \Phi_{t_0}^{t_0}(t_0) = p(t_0) \).

**Proposition 6.12** The velocity on the trajectory traveled in the opposite direction is the opposite of the velocity on the initial trajectory:
\[
\frac{d\Psi_{t_0}^{t_1}}{du}(u) = q'(u) = -p'(t) = -\vec{v}(t, p(t)) \quad \text{when} \quad t = t_0 + t_1 - u,
\] (6.26)

**Proof.** \( \Psi_{t_0}^{t_1}(u) = \Phi_{t_0}^{t_0}(t_0 + t_1 - u) \) gives
\[
\frac{d\Psi_{t_0}^{t_1}}{du}(u) = -\frac{d\Phi_{t_0}^{t_0}}{dt}(t_0 + t_1 - u) = -\vec{v}(t_0 + t_1 - u, \Phi_{t_0}^{t_0}(t_0 + t_1 - u)) = -\vec{v}(t, \Phi_{t_0}^{t_0}(t)) \quad \text{when} \quad t = t_0 + t_1 - u.
\]

6.7 Variation of the flow as a function of the initial time

6.7.1 Ambiguous and non ambiguous notations

Let \( \Phi : (t, u, p) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \Phi(t, u, p) \in \mathbb{R}^n \) be a \( C^1 \) function. The partial derivatives are
\[
\partial_1 \Phi(t, u, p) := \lim_{h \to 0} \frac{\Phi(t+h, u, p) - \Phi(t, u, p)}{h},
\] (6.27)
\[
\partial_2 \Phi(t, u, p) := \lim_{h \to 0} \frac{\Phi(t, u+h, p) - \Phi(t, u, p)}{h},
\] (6.28)
and \( \partial_3 \Phi(t, u, p) \) defined by, for all \( \vec{w} \in \mathbb{R}^n \) (a vector at \( p \),
\[
\partial_3 \Phi(t, u, p).\vec{w} := \lim_{h \to 0} \frac{\Phi(t, u, p+h\vec{w}) - \Phi(t, u, p)}{h} \quad \text{noted} \quad d\Phi(t, u, p).\vec{w},
\] (6.29)
When the name of the first variable is systematically noted \( t \), then
\[
\partial_1 \Phi(t, u, p) \quad \text{noted} \quad \frac{\partial \Phi}{\partial t}(t, u, p) \quad \text{noted} \quad \frac{\partial \Phi(t, u, p)}{\partial t}.
\] (6.30)
NB: This notation can be ambiguous: What is the meaning of \( \frac{\partial \Phi}{\partial t}(t, p) \)? In ambiguous situations, we use the notation \( \partial_1 \Phi \).
When the name of the second variable is systematically noted \( u \), then
\[
\partial_2 \Phi(t, u, p) \text{ noted } \frac{\partial \Phi}{\partial u} (t, u, p) = \frac{\partial \Phi(t, u, p)}{\partial u}.
\] (6.31)

NB: This notation can be ambiguous: What is the meaning of \( \frac{\partial \Phi}{\partial u} (u; u, p) \)? In ambiguous situations, we use the notation \( \partial_2 \Phi \).

When the name of the third variable is systematically noted \( p \), then
\[
\partial_3 \Phi(t, u, p) \text{ noted } \frac{\partial \Phi}{\partial p} (t, u, p) = \frac{\partial \Phi(t, u, p)}{\partial p} = d \Phi(t, u, p).
\] (6.32)

Since \( p \) is the only space variable, the notation \( \frac{\partial \Phi}{\partial p} \) is less ambiguous than \( \frac{\partial \Phi}{\partial p} (t_1, t_2, p) \).

### 6.7.2 Variation of the flow as a function of the initial time

(6.7), that is, \( \frac{\partial \Phi}{\partial t} (t; u, q) = \tilde{v}(t, \Phi(t; u, q)) \) with \( \Phi(u; u, q) = q \) for any \( (u, q) \in \mathbb{R} \times \mathbb{R}^3 \), reads
\[
\partial_t \Phi(t; u, q) = \tilde{v}(t, \Phi(t; u, q)) \quad \text{with} \quad \Phi(u; u, q) = q.
\] (6.33)

The law of composition of the flows give, cf. (6.19),
\[
\Phi(t; u, \Phi(t; u, q)) = \Phi(t; t_0, q).
\] (6.34)

Thus the derivative in \( u \) gives, cf. § 6.7.1 (non ambiguous)
\[
\partial_u \Phi(t; u, \Phi(u; t_0, q)) + d \Phi(t; u, \Phi(u; t_0, q)) \partial_u \Phi(u; t_0, q) = 0,
\]
i.e.
\[
\partial_2 \Phi(t; u, \Phi(u; t_0, q)) + d \Phi(t; u, \Phi(u; t_0, q)) \tilde{v}(t, u, \Phi(u; t_0, q)) = 0,
\] (6.35)

written \( \frac{\partial \Phi}{\partial t} (t; u, p) + d \Phi(t; u, p) \tilde{v}(u, p) = 0 \) with \( p = \Phi(u; t_0, p) = p_0 \).

(Or, ambiguous, \( \frac{\partial \Phi(t; u, p(u))}{\partial u} = -d \Phi(t; u, p(u)) \tilde{v}(u, p(u)) \).) Thus \( u = t_0 \) and \( \Phi(t; t_0, p_{t_0}) = p_{t_0} \) give
\[
\partial_2 \Phi(t; t_0, p_{t_0}) = -d \Phi(t; t_0, p_{t_0}) \tilde{v}(t_0, p_{t_0}),
\]
written \( \frac{\partial \Phi}{\partial t_0} (t; t_0, p_{t_0}) = -d \Phi(t; t_0, p_{t_0}) \tilde{v}(t_0, p_{t_0}).
\) (6.36)

(Or, ambiguous, \( \frac{\partial \Phi(t; t_0, p_{t_0})}{\partial t_0} = -d \Phi(t; t_0, p_{t_0}) \tilde{v}(t_0, p_{t_0}). \)

In particular \( \partial_2 \Phi(t_0; t_0, p_{t_0}) = -\tilde{v}(t_0, p_{t_0}) \), i.e. \( \frac{\partial \Phi}{\partial t_0} (t_0, p_{t_0}) = -\tilde{v}(t_0, p_{t_0}). \) (Or, ambiguous,
\[
\frac{d \Phi(t; t_0, p_{t_0})}{dt_0} \big|_{t_0=t_0} = -\tilde{v}(t_0, p_{t_0}).
\]

### 7 Decomposition of \( d \tilde{v} \)

Let \( \tilde{\Phi} : [t_1, t_2] \times \text{Obj} \to \mathbb{R}^n \) be a regular motion, cf. (1.5), and let \( \tilde{v} : C \to \mathbb{R}^n \) be the Eulerian velocity field, cf. (2.5), that is, \( \tilde{v}(t, p) = \frac{\partial \Phi}{\partial t} (t, P_{t_0}) \) when \( p = \tilde{\Phi}(t, P_{t_0}) \). Its differential \( d \tilde{v} \) is given in (2.10).

A observer chooses a unit of measurement (foot, meter...) and builds the associated Euclidean dot product \( (\cdot, \cdot)_g \), cf. § B.2.

### 7.1 Rate of deformation tensor and spin tensor

Thanks to a chosen Euclidean dot product \( (\cdot, \cdot)_g \), in \( \mathbb{R}^n_1 \), we can consider the transposed endomorphism
\[
d\tilde{v}_T(p)_g := \text{noted } d\tilde{v}_T(p)_g^T \in \mathcal{L}(\tilde{\mathbb{R}}_1^g; \tilde{\mathbb{R}}_1^g);
\]
It is given by, for all \( \tilde{w}_1, \tilde{w}_2 \in \tilde{\mathbb{R}}_1^g \) vectors at \( p \),
\[
(d\tilde{v}_T(p)_g)^T \tilde{w}_1, \tilde{w}_2)_g = (\tilde{w}_1, d\tilde{v}_T(p)_g \tilde{w}_2)_g
\] (7.1)
cf. § A.7. And we use the usual notations (definitions)
\[
d\tilde{v}_T^T(p) := d\tilde{v}_T(p)_g^T \equiv d\tilde{v}_T(t, p)_g \equiv d\tilde{v}(t, p)^T.
\] (7.2)
Definition 7.1 The (Eulerian) rate of deformation tensor, or stretching tensor, is the (Euclidean) symmetrical part of $\overline{d\vec{v}}$, that is, the Eulerian map

$$D = \frac{d\overline{v} + d\overline{v}^T}{2}, \quad \text{i.e.,} \quad \forall (t,p) \in \bigcup_{t \in \mathbb{R}} \{\{t\} \times \Omega_t\}, \quad D(t,p) = \frac{d\overline{v}(t,p) + d\overline{v}(t,p)^T}{2}. \quad (7.3)$$

And the (Eulerian) spin tensor is the antisymmetric part of $d\overline{v}$, that is,

$$\Omega = \frac{d\overline{v} - d\overline{v}^T}{2}, \quad \text{i.e.,} \quad \forall (t,p) \in \bigcup_{t \in \mathbb{R}} \{\{t\} \times \Omega_t\}, \quad \Omega(t,p) = \frac{d\overline{v}(t,p) - d\overline{v}(t,p)^T}{2}. \quad (7.4)$$

So

$$d\overline{v} = D + \Omega. \quad (7.5)$$

NB: the same notation will be used for the set $\Omega_t = \Phi^\sharp_t(\Omega_{t_0}) \subset \mathbb{R}^n$ and for the spin tensor $\Omega_t = \frac{d\overline{v}_t - d\overline{v}_t^T}{2}$, given by $\Omega_t(p) = \frac{d\overline{v}_t(p) - d\overline{v}_t(p)^T}{2}$, cf. (7.4): The context removes ambiguities.

7.2 Quantification with a basis

Consider a basis $(\vec{e}_i)$ in $\mathbb{R}^n$. (7.1) then gives $[g]_\vec{e} [d\overline{v}]_\vec{e} = [d\overline{v}]_\vec{e}$. [Eulerian framework $\vec{e}$]

In particular, if $(\vec{e}_i)$ is a $(\cdot,\cdot)_g$-orthonormal basis (a $(\cdot,\cdot)_g$ Euclidean basis), then $[g]_\vec{e} = I$ and $[d\overline{v}]_\vec{e} = [d\overline{v}]$. That is, if $\vec{v} = \sum i v_i \vec{e}_i$, then $d\overline{v} \vec{e}_i = \sum j = \frac{\partial v_i}{\partial x_j} \vec{e}_j$, then $d\overline{v} \vec{e}_i = \sum j = \frac{\partial v_i}{\partial x_j} \vec{e}_j$, that is,

$$[d\overline{v}]_\vec{e} = \left[ \frac{\partial v_i}{\partial x_j} \right]_{j=1,\ldots,n} \quad \text{Euclidean framework} \quad [d\overline{v}]_\vec{e} = \left[ \frac{\partial v_i}{\partial x_j} \right]_{j=1,\ldots,n} \quad (7.6)$$

Thus for the endomorphisms $D$ and $\Omega$ and with $\mathcal{D} \vec{e}_j = \sum i D_{ij} \vec{e}_i$ and $\Omega \vec{e}_j = \sum i \Omega_{ij} \vec{e}_i$, we get

$$\mathcal{D} \vec{e}_j = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad \text{and} \quad \Omega \vec{e}_j = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right), \quad (7.7)$$

With duality notations, $\vec{v} = \sum i v_i \vec{e}_i$ and $d\overline{v} \vec{e}_i = \sum j = \frac{\partial v_i}{\partial x_j} \vec{e}_j$, thus $d\overline{v} \vec{e}_i = \sum j = \frac{\partial v_i}{\partial x_j} \vec{e}_j$. Indeed the quantity $d\overline{v} \vec{e}_i$ is not objective because of the use of the inner dot product $(\cdot,\cdot)_g$ (observer dependent). If you do want to use Einstein’s convention, you have to use $[g]_\vec{e} [d\overline{v}]_\vec{e} = [d\overline{v}]_\vec{e} [g]_\vec{e}$, i.e. $\sum k = 1 g_{ik} (d\overline{v})_k^i = \sum k = 1 g_{ik} (d\overline{v})_k^i$, even if $g_{ij} = \delta_{ij}$.

8 Interpretation of the rate of deformation tensor

We are interested in the evolution of the deformation gradient along the trajectory of a particle $P_{Obj}$ which was at $p_{t_0}$ at $t_0$. So:

Let $\Phi : [t_1, t_2] \times Obj \to \mathbb{R}^n$ be a regular motion, cf. (1.5), $t_0 \in [t_1, t_2], \Omega_{t_0} = \tilde{\Phi}(t_0, Obj), P_{Obj} \in Obj, p_{t_0} = \tilde{\Phi}(t_0, P_{Obj}), \Phi_0 : t \in [t_1, t_2] \to p(t) = \Phi_{p_0}(t) = \tilde{\Phi}(t, P_{Obj}) \in \mathbb{R}^n$ (trajectory of $P_{Obj}$), cf. (3.8). And name $F(t) := T^0_{p_0}(t) = d\Phi^0(t, p_{t_0}) := d\tilde{\Phi}^0(t, p_{t_0})$, cf. (5.2).

Let $\vec{A} = \vec{a}(t_0, p_{t_0})$ and $\vec{B} = \vec{b}(t_0, p_{t_0})$ be vectors at $t_0$ at $p_{t_0}$ in $\Omega_{t_0}$, and consider their push-forwards by the flow $\Phi^t_{p_0}$ (the transported vectors along a trajectory)

$$\vec{a}(t, p(t)) = F^t_{p_0}(t, p_{t_0}), \vec{A} \quad \text{and} \quad \vec{b}(t, p(t)) = F^t_{p_0}(t, p_{t_0}), \vec{B}, \quad (8.1)$$

see (5.4) and figure 5.1: Define the Eulerian scalar function

$$(\vec{a}, \vec{b}) : \mathcal{C} \to \mathbb{R} \quad (t, p_t) \to (\vec{a}, \vec{b})_t(t, p_t) := (\vec{a}(t, p_t), \vec{b}(t, p_t))_g \quad (= (F(t) \vec{A}, F(t) \vec{B})_g). \quad (8.2)$$

Proposition 8.1 The rate of deformation tensor $D = \frac{d\overline{v} + d\overline{v}^T}{2}$ gives (half) the evolution rate between two vectors deformed by the flow, that is, along trajectories,

$$\frac{D(\vec{a}, \vec{b})}{Dt}_g = 2(D \vec{a}, \vec{b})_g. \quad (8.3)$$
9.1 Ane motions and rigid body motions

Proof. Let \( F^t_{t_0} = F \) and \( f(t) := (\bar{a}, \bar{b})_g(t, p(t)) = (F(t).\bar{A}, F(t).\bar{B})_g \) where \( p(t) = \Phi^t_{t_0}(p_{t_0}) \); Thus

\[
f'(t) = (F'(t).\bar{A}, F(t).\bar{B})_g + (F(t).\bar{A}, F'(t).\bar{B})_g. \tag{8.4}
\]

And \( F'(t) = d\vec{v}(t, p(t)).F(t), \) cf. (4.21). Thus

\[
f'(t) = (d\vec{v}(t, p(t)).F(t).\bar{A}, F(t).\bar{B})_g + (F(t).\bar{A}, d\vec{v}(t, p(t)).F(t).\bar{B})_g
\]

\[
= (d\vec{v}(t, p(t)).\bar{a}(t, p(t)), \bar{b}(t, p(t)))_g + (\bar{a}(t, p(t)), d\vec{v}(t, p(t)).\bar{b}(t, p(t)))_g \tag{8.5}
\]

i.e. (8.3).

9 Rigid body motions and the spin tensor

The need of Euclidean dot products is required to characterize a rigid body motion. So consider a Euclidean dot product \((\cdot, \cdot)_g\).

Result: A rigid body motion can be defined by \( d\vec{v} + d\vec{v}^T = 0 \), i.e., \( D = 0 \) (Eulerian approach independent of any initial time \( t_0 \) chosen by some observer).

But the usual classical introduction to rigid body motion relies on some \( t_0 \) (Lagrangian approach). So, to begin with, let us do it with the Lagrangian approach.

9.1 Ane motions and rigid body motions

Let \( \bar{\Phi} : [t_0, t_2] \times Obj \to \mathbb{R}^n \) be a regular motion, cf. (1.5), let \( \Omega_t = \bar{\Phi}(t, Obj) \), let \( C = \bigcup_{t \in [t_0, t_2]} \{t\} \times \Omega_t \), cf (2.1), and let \( \vec{v} : C \to \mathbb{R}^n \) be the Eulerian velocity field, cf. (2.5), that is, \( \vec{v}(t, p) = \frac{\partial \bar{\Phi}}{\partial t}(t, P_{Obj}) \) when \( p = \bar{\Phi}(t, P_{Obj}) \).

Let \( t_0, t \in [t_1, t_2], \) let \( \Phi^t_{t_0} \) be the motion associated to \( \bar{\Phi} \) relative to \( t_0 \), cf. (3.1) and \( \Phi^t_{t_0} \) the motion associated to \( \bar{\Phi} \) relative to \( t_0 \), cf. (3.5), and let \( \bar{\nu}_t : t \in [t_1, t_2] \times \Omega_t \), then use the non ambiguous notation \( F^t_{t_0}(p_{t_0}) := d\Phi(t, p_{t_0}) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \) at \( p_{t_0} \in \Omega_t \). relative to \( t_0 \) and \( t \), cf. (4.16): For all \( \bar{w}_{t_0} \in \mathbb{R}^n \) vector at \( p_{t_0} \),

\[
\Phi^t_{t_0}(p_{t_0} + h\bar{w}_{t_0}) = \Phi^t_{t_0}(p_{t_0}) + h F^t_{t_0}(p_{t_0}).\bar{w}_{t_0} + o(h), \tag{9.1}
\]

i.e., \( F^t_{t_0}(p_{t_0}).\bar{w}_{t_0} = \lim_{h \to 0} \Phi^t_{t_0}(p_{t_0} + h\bar{w}_{t_0}) - \Phi^t_{t_0}(p_{t_0}) \). (Recall: If you don’t like the notation \( F^t_{t_0} \), then use the non ambiguous notation \( \bar{T}_{p_t}(\Omega_t) \), cf. § 1.8.)

9.1.1 Ane motions

Definition 9.1 \( \Phi^t_{t_0} \) is an “affine motion” (understood is a “space affine motion”) iff \( \Phi^t_{t_0} \) is affine at all times \( t \), that is, iff (9.1) reads, for all \( t \in [t_1, t_2] \) and for all \( p_{t_0}, q_{t_0} \in \Omega_t \) (with \( q_{t_0} \) in an open vicinity of \( p_{t_0} \) such that the segment \( [p_{t_0}, q_{t_0}] \) is in \( \Omega_t \)),

\[
\Phi^t_{t_0}(p_{t_0}) = \Phi^t_{t_0}(q_{t_0}) + F^t_{t_0}(p_{t_0}).(p_{t_0} - q_{t_0}). \tag{9.2}
\]

Proposition 9.2 and definition. If \( \Phi^t_{t_0} \) is an affine motion, then for all \( t \in [t_1, t_2] \) and for all \( p_{t_0}, q_{t_0} \in \Omega_t \) (with \( q_{t_0} \) in an open vicinity of \( p_{t_0} \) such that the segment \( [p_{t_0}, q_{t_0}] \) is in \( \Omega_t \)),

\[
F^t_{t_0}(p_{t_0}) = F^t_{t_0}(q_{t_0}). \tag{9.3}
\]

Then \( F^t_{t_0}(p_{t_0}) = \) noted \( F^t_{t_0} \) and \( dF^t_{t_0}(p_{t_0}) = d^2\Phi^t_{t_0}(p_{t_0}) = 0 \). And then \( \Phi^t \) is an affine motion for all \( t \in [t_1, t_2] \) (updated Lagrangian description): For all \( p_t \in \Omega_t \), all \( q_t \in \) an open vicinity of \( p_t \) and all \( t \in [t_1, t_2] \),

\[
\Phi^t_{t_0}(q_t) = \Phi^t_{t_0}(p_t) + F^t_{t_0}(p_t - q_t). \tag{9.4}
\]

And \( \Phi \) is said to be an “affine motion” (understood is a “space affine motion”).
9.1.2 Rigid body motion

9.1. Ane motions and rigid body motions

Lagrangian velocity

\[ \dot{x}_i(p_0) = \dot{v}_i(p_0) + d\dot{v}_i(p_0) = \dot{v}_i(p_0) + \frac{\partial \Phi^{(o)}_i(p_0)}{\partial t} - \frac{\partial \Phi^{(o)}_i(p_0)}{\partial t} \]

and, similarly,

\[ \dot{\Phi}_i^{(o)}(p_0) = \Phi^{(e)}_i(q_0) + \frac{\partial \Phi^{(e)}_i(q_0)}{\partial t} = \dot{\Phi}_i^{(e)}(q_0) + \frac{\partial \Phi^{(e)}_i(q_0)}{\partial t} \]

Thus (addition) \( \Phi_i^{(o)}(q_0) + \Phi_i^{(e)}(p_0) = \Phi_i^{(o)}(p_0) + \Phi_i^{(e)}(q_0) \)

\[ \frac{\partial \Phi^{(o)}_i(p_0)}{\partial t} = \frac{\partial \Phi^{(o)}_i(p_0)}{\partial \bar{u}_i} = 0, \text{ true for all } p_0, q_0, \text{ thus } \frac{\partial \Phi^{(o)}_i(p_0)}{\partial \bar{u}_i} = 0 \]

Thus \( d^2\Phi_i^{(o)}(p_0), \bar{u}_i = \lim_{h \to 0} \frac{d\Phi_i^{(o)}(p_0 + h \bar{u}_i)}{h} - \frac{d\Phi_i^{(o)}(p_0)}{h} = 0 \) for all \( p_0 \) and all \( \bar{u}_i \), thus \( d^2\Phi_i^{(o)}(p_0) = 0 \).

And (6.17) gives \( \Phi_i^{(e)}(p_0) = \Phi_i^{(e)}(q_0) \), thus, with \( p_t \equiv \Phi_i^{(o)}(p_0) \), we have \( d\Phi_i^{(o)}(p_t).d\Phi_i^{(o)}(p_t) = d\Phi_i^{(o)}(p_t) \), thus \( d\Phi_i^{(o)}(p_t) = d\Phi_i^{(o)}(p_t) \), \( d\Phi_i^{(o)}(p_t) \), and (9.2) gives

\[ d\Phi_i^{(o)}(p_t) = d\Phi_i^{(o)}(p_t) \]

thus \( (9.4) \).

Corollary 9.3 If \( \tilde{\Phi} \) is affine then, for all \( t \), the Eulerian velocity \( \tilde{v}_t \) is affine, and, for all \( t_0, t \), the Lagrangian velocity \( \tilde{V}_t^{(o)} \) is affine

\[ \begin{cases} \tilde{v}_t(q_t) = \tilde{v}_t(p_t) + d\tilde{v}_t(p_t) \quad \text{with} \quad d\tilde{v}_t(p_t) = (F^{(o)}(t))(\dot{F}^{(o)}(t))^{-1}d\tilde{v}_t, \\ \tilde{V}_t^{(o)}(q_t) = \tilde{V}_t^{(o)}(p_t) + d\tilde{V}_t^{(o)}(p_t) \quad \text{with} \quad d\tilde{V}_t^{(o)}(p_t) = (F^{(o)}(t))(\dot{F}^{(o)}(t))^{-1}d\tilde{V}_t^{(o)} \end{cases} \]

\( (d\tilde{v}_t(p_t) \) is independent of \( p_t \) and \( d\tilde{V}_t^{(o)}(p_t) \) is independent of \( p_t \).

Proof. (9.2) gives \( \Phi_i^{(o)}(t, q_0) = \Phi_i^{(o)}(t, p_0) + F^{(o)}(t), \bar{u}_i \), and the derivation in time gives (9.6), then (9.6) \( \text{1 thanks to } p_t = \Phi_i^{(o)}(p_0), \ q_t = \Phi_i^{(o)}(q_0) \) and \( \bar{u}_i \), \( \bar{u}_i \), \( \bar{u}_i \), (9.2).

Example 9.4 In \( \mathbb{R}^2 \), with a basis \((\tilde{E}_1, \tilde{E}_2) \in \mathbb{R}^n \), and a basis \((\tilde{e}_1, \tilde{e}_2) \in \tilde{E}_2^2 \), then \( \tilde{F}^{(o)} \) given by \( \tilde{F}^{(o)} = \begin{pmatrix} 1+t & 2t^2 \\ 3t^3 & e^t \end{pmatrix} \) derives from the affine motion \( \tilde{F}^{(o)}(p_0)\tilde{F}^{(o)}(q_0) \) \( = \begin{pmatrix} 1+t & 2t^2 \\ 3t^3 & e^t \end{pmatrix} \). 

9.1.2 Rigid body motion

A Euclidean dot product \((\cdot, \cdot) \) in \( \mathbb{R}^n \) is required, the same at all time \( t \).

Definition 9.5 A rigid body motion is an affine motion \( \tilde{\Phi} \), cf. (9.3), such that,

\[ \forall t_0, t \in \mathbb{R}, \forall \bar{u}_{t_0}, \bar{w}_{t_0} \in \bar{E}_{t_0}^n, \quad (F^{(o)}(t_0)\bar{u}_{t_0}, F^{(o)}(t_0)\bar{w}_{t_0}) = (\bar{u}_{t_0}, \bar{w}_{t_0}) \text{.} \]

Shorten notation: For all \( \tilde{U}, \tilde{W} \in \bar{E}_{t_0}^n \),

\[ (F, \tilde{U}, \tilde{W}) = (\tilde{U}, \tilde{W}) \text{.} \]

Thus (9.8) tells that \( \tilde{\Phi} \) is a rigid body motion iff

\[ \forall t \in \mathbb{R}, \quad (F^{(o)}(t))^{T}F^{(o)}(t) = I_{t_0}, \text{ written } F^{T}F = I, \text{ or } F^{-1} = F^{T} \text{.} \]

(9.9)

where \( I_{t_0} : \bar{E}_{t_0}^n \to \bar{E}_{t_0}^n \text{, the identity endomorphism in } \bar{E}_{t_0}^n \). (For a rigid body motion, the Cauchy–Green deformation tensor \( C := F^{T}F \text{ satisfies } C = I_{t_0} \).)

Proposition 9.6 If \( \Phi^{(o)} \) is a rigid body motion, if \( \tilde{\Phi} \) is a \((\cdot, \cdot) \)-Euclidean basis in \( \bar{E}_{t_0}^n \) and if \( \bar{u}_j(t) = \tilde{F}^{(o)}(t)\tilde{A}_j(t) \) for all \( j \), then \( (\bar{a}_j(t)) \) is a \((\cdot, \cdot) \)-Euclidean basis with the same orientation than \( (\tilde{A}_j(t)) \).

Proof. \( (\bar{a}_j(t), \bar{a}_j(t)) = (\tilde{F}^{(o)}(t), \tilde{F}^{(o)}(t), \tilde{A}_j(t)) = (\tilde{F}^{(o)}(t), \tilde{F}^{(o)}(t), \tilde{A}_j(t)) = (\tilde{A}_j(t), \tilde{A}_j(t)), \tilde{a}_j(t), \tilde{a}_j(t)) \equiv \delta_{ij} \) for all \( i, j \) and all \( t \), cf. (9.9). So \( (\bar{a}_j(t)) \) is \((\cdot, \cdot) \)-orthonormal basis. And \( \det(\bar{u}_j(t), \cdots, \bar{u}_j(t)) = \det(F^{(o)}(t), \cdots, F^{(o)}(t)) = \det(F^{(o)}(t)) \det(\tilde{A}_j(t), \cdots, \tilde{A}_j(t)) \). And \( \Phi^{(o)} \) is at least \( C^{(1)} \text{ diffeomorphism) } \), thus \( t \to \det(F^{(o)}(t)) = \det(F^{(o)}(t)) \) is continuous and does not vanish, with \( \det(F^{(o)}(t)) = \det(I) = 1 \), thus \( \det(F^{(o)}(t)) > 0 \) for all \( t \); Hence \( \det(\bar{u}_j(t), \cdots, \bar{u}_j(t)) \) as the sign of \( \det(\tilde{A}_j(t), \cdots, \tilde{A}_j(t)) \). The bases have the same orientation.
Example 9.7 In \( \mathbb{R}^2 \), a rigid body motion is given by 
\[
F_t^o = \begin{pmatrix} \cos(\theta(t-t_0)) & -\sin(\theta(t-t_0)) \\ \sin(\theta(t-t_0)) & \cos(\theta(t-t_0)) \end{pmatrix}
\]
with \( \theta \) a regular function s.t. \( \theta(0) = 0 \).

Exercise 9.8 Let \( \widetilde{\Phi} \) be a rigid body motion. Prove 
\[
(F^T)'(t) = (F'(t))^T,
\]
and \( F^T.F' \) is antisymmetric.

Answer. Here \( F \) means \( F^o_{p_0} \). And \( F^T \) is defined by 
\[
((F^o_{p_0})^T(\vec{v}(t,p_1), \vec{u}_0(p_0)))_g = (F^o_{p_0}(\vec{v}(t,p_1), \vec{u}_0(p_0)), \vec{w}(t,p_1))_g,
\]
written \( (F^T(t),\vec{a}(t,p), \vec{u}_0(p_0))_g = (F(t),\vec{a}(t,p), \vec{u}_0(p_0))_g \) and \( (F^T(t),\vec{a}(t,p), \vec{w}(t,p))_g = (F(t),\vec{a}(t,p), \vec{w}(t,p))_g \), which simplifies into \( ((F^T(t),\vec{a}(t,p), \vec{u}_0(p_0))_g = (F(t),\vec{a}(t,p), \vec{u}_0(p_0))_g = (F(t),\vec{a}(t,p), \vec{w}(t,p))_g \), thus \( (F^T)'(t) = (F'(t))^T \).

And (9.9) reads \( F(t) = I_{p_0} \text{, thus } (F^T)'(t).F(t) + F'(t).F(t) = 0 \), thus \( (F^T)'(t).F(t) + F'(t).F(t) = 0, \) thus \( F^T.F' \) is antisymmetric.

9.2.1 Reminder

9.2 Representation of the spin tensor \( \Omega \): vector, and pseudo vector

9.2.1 Reminder

* Let \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\) be a Euclidean basis, cf. § B.1. The associated determinant \( \det_{\vec{e}} \) is the alternating multilinear form defined by \( \det_{\vec{e}}(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1 \). And the (algebraic or signed) volume limited by three vectors \( \vec{a}, \vec{b}, \vec{c} \) is the value \( \det_{\vec{e}}(\vec{a}, \vec{b}, \vec{c}) \). (The positive volume is \( |\det_{\vec{e}}(\vec{a}, \vec{b}, \vec{c})| \), see § D.)

* Let \( A, B \) be two observers (e.g. \( A=\text{English} \) and \( B=\text{French} \)), let \((\vec{a})\) be a Euclidean basis chosen by \( A \) (e.g. based on the foot), and let \((\vec{b})\) be a Euclidean basis chosen by \( B \) (e.g. based on the meter), see § B.1. Let \( \lambda = ||\vec{b}||_a > 0 \). The relation between the determinants is:

\[
\det_{\vec{a}} = \begin{cases} +\lambda^3 \det_{\vec{b}} & \text{if } \det_{\vec{b}}(\vec{b}_i, \vec{b}_j, \vec{b}_k) > 0 \text{ (i.e. if the bases have the same orientation)}, \\ -\lambda^3 \det_{\vec{b}} & \text{if } \det_{\vec{b}}(\vec{b}_i, \vec{b}_j, \vec{b}_k) < 0 \text{ (i.e. if the bases have opposite orientation),} \end{cases}
\]

(9.12)

That is, if \( \vec{u}_1, \vec{u}_2, \vec{u}_3 \) define a parallelepiped, then its algebraic volume \( \det_{\vec{a}}(\vec{u}_1, \vec{u}_2, \vec{u}_3) \) relative to the unit of measure of \( A \) is equal to \( \pm \lambda^3 \) times the algebraic volume \( \det_{\vec{b}}(\vec{u}_1, \vec{u}_2, \vec{u}_3) \), the sign depending on the relative orientation of the bases.

* Euclidean framework: An endomorphism \( L \) is antisymmetric iff

\[
L^T = -L \quad \text{(Euclidean framework).} \tag{9.13}
\]

That is, if \( (\cdot, \cdot)_g \) is a Euclidean dot product, then \( L \) is antisymmetric iff \( (L\vec{u}, \vec{v})_g + (\vec{u}, L\vec{v})_g = 0 \) for all \( \vec{u}, \vec{v} \).
9.2.2 Definition of the vector product (cross product)

Let \( \vec{u}, \vec{v} \in \mathbb{R}^3 \) and let \((\vec{e}_i)\) be a \((\cdot, \cdot)_g\)-Euclidean basis. Let \( \ell_{\vec{e}, \vec{a}, \vec{b}} \in (\mathbb{R}^3)^* \) be the linear form defined by

\[
\ell_{\vec{e}, \vec{a}, \vec{b}} : \begin{cases} 
\mathbb{R}^3 \to \mathbb{R} \\
\vec{z} \to \ell_{\vec{e}, \vec{a}, \vec{b}}(\vec{z}) := \det(\vec{u}, \vec{v}, \vec{z}).
\end{cases}
\]  

(9.14)

**Definition 9.10** The cross product \( \vec{u} \wedge \vec{v} \) of two vectors \( \vec{u} \) and \( \vec{v} \) is the \((\cdot, \cdot)_g\)-Riesz representation vector of \( \ell_{\vec{e}, \vec{a}, \vec{b}} \), that is, \( \vec{u} \wedge \vec{v} \in \mathbb{R}^3 \) is characterized by, cf. (C.3),

\[
\forall \vec{z} \in \mathbb{R}^3, \quad (\vec{u} \wedge \vec{v}, \vec{z})_g = \det(\vec{u}, \vec{v}, \vec{z}) = (\ell_{\vec{e}, \vec{a}, \vec{b}}(\vec{z})).
\]  

(9.15)

We have thus defined the bilinear cross product operator (depends on the chosen Euclidean dot product and on the chosen orientation of the Euclidean basis, cf. (9.12) with \( \lambda = \pm 1 \))

\[
\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (\vec{u}, \vec{v}) \mapsto \wedge_c(\vec{u}, \vec{v}) := \vec{u} \wedge_c \vec{v}.
\]  

(9.16)

(The bilinearity is trivial thanks to the multilinearity of the determinant.) If one Euclidean basis is imposed by one observer to all the other observers, then \( \vec{u} \wedge_c \vec{v} \) is written \( \vec{u} \wedge \vec{v} \).

**Calculation:** \( \vec{u} = \sum_{i=1}^3 u^i \vec{e}_i \) and \( \vec{v} = \sum_{i=1}^3 v^i \vec{e}_i \) give

\[
(\vec{u} \wedge_c \vec{v}, \vec{e}_1)_c = \det(\vec{u}, \vec{v}, \vec{e}_1) = \det \begin{pmatrix} u^1 & v^1 & 1 \\
 u^2 & v^2 & 0 \\
 u^3 & v^3 & 0 \end{pmatrix} = \det \begin{pmatrix} u^2 v^3 - u^3 v^2 \\
 u^3 v^1 - u^1 v^3 \\
 u^1 v^2 - u^2 v^1 \end{pmatrix} = u^2 v^3 - u^3 v^2,
\]  

(9.17)

since (column-linearity) the multi-linearity gives \( \det_{\vec{e}}(\vec{u}, \vec{v}, \vec{e}_1) = \det_{\vec{e}}(u^1 \vec{e}_1 + \sum_{i=2}^3 u^i \vec{e}_i, \sum_{i=1}^3 v^i \vec{e}_i, \vec{e}_1) = u^1 \det_{\vec{e}}(\vec{e}_1, \sum_{i=2}^3 v^i \vec{e}_i, \vec{e}_1) + \det_{\vec{e}}(\sum_{i=2}^3 u^i \vec{e}_i, \sum_{i=1}^3 v^i \vec{e}_i, \vec{e}_1) = 0 + \det_{\vec{e}}(\sum_{i=2}^3 u^i \vec{e}_i, \sum_{i=2}^3 v^i \vec{e}_i, \vec{e}_1) = \cdots = \det_{\vec{e}}(\sum_{i=2}^3 u^i \vec{e}_i, \sum_{i=2}^3 v^i \vec{e}_i, \vec{e}_1). \) Similar calculation for \((\vec{u} \wedge_c \vec{v}, \vec{e}_2)_c \) and \((\vec{u} \wedge_c \vec{v}, \vec{e}_3)_c \), so

\[
[u \wedge_c v]_{\vec{e}} = \begin{pmatrix}
u^2 v^3 - u^3 v^2 \\
u^3 v^1 - u^1 v^3 \\
u^1 v^2 - u^2 v^1 \end{pmatrix}.
\]  

(9.18)

(In particular \( \vec{e}_1 \wedge_c \vec{e}_2 = \vec{e}_3 \), and similar result with “circular permutation”.)

**Proposition 9.11**

1- \( \vec{u} \wedge_c \vec{v} = -\vec{v} \wedge_c \vec{u} \).

2- If \( \vec{u} \parallel \vec{v} \) then \( \vec{u} \wedge_c \vec{v} = 0 \).

3- If \( \vec{u} \) and \( \vec{v} \) are independent then \( \vec{u} \wedge_c \vec{v} \) is orthogonal to the linear space \( \text{Vect}\{\vec{u}, \vec{v}\} \) generated by \( \vec{u} \) and \( \vec{v} \).

4- \( \vec{u} \wedge_c \vec{v} \) depends on the unit of measurement and on the orientation of \((\vec{e}_i)\): If \((\cdot, \cdot)_a\) and \((\cdot, \cdot)_b\) are two Euclidean dot products, then \( \exists \lambda > 0 \) such that \((\cdot, \cdot)_a = \lambda^2 (\cdot, \cdot)_b \) and then

\[
\vec{u} \wedge_a \vec{v} = \pm \lambda \vec{u} \wedge_b \vec{v}.
\]  

(9.19)

**Proof.** 1- \( \det_{\vec{e}}(\vec{u}, \vec{v}, \vec{z}) = -\det_{\vec{e}}(\vec{v}, \vec{u}, \vec{z}) \) (since \( \det_{\vec{e}} \) is alternated).

2- If \( \vec{u} \parallel \vec{v} \) then \( (\vec{u} \wedge_c \vec{v}, \vec{z})_g := \det_{\vec{e}}(\vec{u}, \vec{v}, \vec{z}) = 0 \), so \( \vec{u} \wedge_c \vec{v} \perp_g \vec{z} \), for all \( \vec{z} \).

3- If \( \vec{z} \in \text{Vect}\{\vec{u}, \vec{v}\} \) then \( (\vec{u} \wedge_c \vec{v}, \vec{z})_g := \det_{\vec{e}}(\vec{u}, \vec{v}, \vec{z}) = 0 \) thus \( \vec{u} \wedge_c \vec{v} \perp_g \vec{z} \).

4- \( (\vec{u} \wedge_a \vec{v}, \vec{z})_a = \det(\vec{u}, \vec{v}, \vec{z})_a(9.15) \) \( \pm \lambda^2 \det(\vec{u}, \vec{v}, \vec{z})_b(9.12) \) \( = \pm \lambda^3 \det(\vec{u}, \vec{v}, \vec{z})_b(9.15) \) \( = \pm \lambda^3 (\vec{u} \wedge_b \vec{v}, \vec{z})_b \) = \( \pm \lambda^3 \frac{1}{\lambda^3}(\vec{u} \wedge_b \vec{v}, \vec{z})_a \)

thus (9.19).

**Exercise 9.12** Prove that \( \vec{u} \wedge_c \vec{v} \) is a contravariant vector (change of basis contravariant formula).

**Answer.** It is a vector (Riesz representation vector) in \( \mathbb{R}^3 \), so it is also called a contravariant vector: Satisfies the contravariance formula, see (C.18).
9.2.3 Antisymmetric endomorphism represented by a vector

**Proposition 9.13** Let $\Omega \in \mathcal{L}(\mathbb{R}^3;\mathbb{R}^3)$ be an antisymmetric endomorphism relative to a chosen Euclidean dot product $(\cdot,\cdot)_g$, that is $\Omega^T = -\Omega$. Then, a Euclidean basis $(\vec{e}_i)$ being chosen, there exists a unique vector $\vec{\omega}_c \in \mathbb{R}^3$ s.t.

$$\forall \vec{y} \in \mathbb{R}^3, \quad \Omega \vec{y} = \vec{\omega}_c \wedge \vec{y},$$

that is, such that

$$\forall \vec{y}, \vec{z} \in \mathbb{R}^3, \quad (\Omega \vec{y}, \vec{z})_g = \det(\vec{\omega}_c, \vec{y}, \vec{z}).$$

And:

$$\text{If } [\Omega]_g = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \text{ then } [\vec{\omega}_c]_g = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$  \hspace{1cm} (9.22)

In particular $\Omega \vec{\omega}_c = 0 = \vec{\omega}_c \wedge \vec{\omega}_c$, and 0 is an eigenvalue of $\Omega$ associated to the eigenvector $\vec{\omega}_c$.

**Proof.** $\Omega$ being antisymmetric, its matrix is of the type given in (9.22). Then, if $\Omega$ and $\vec{\omega}_c$ are given by (9.22), then (9.20) is immediately verified, so $\vec{\omega}_c$ exists. Calculation of its components: Let $\vec{\omega} = \omega^1 \vec{e}_1 + \omega^2 \vec{e}_2 + \omega^3 \vec{e}_3$; Then $\Omega \vec{y} = \vec{\omega} \wedge \vec{y}$ for all $\vec{y}$ gives $\Omega \vec{e}_1 = [\Omega]_g [\vec{e}_1]_g = \begin{pmatrix} 0 \\ c \\ -b \end{pmatrix}$ and $[\vec{\omega} \wedge \vec{e}_1]_g = \begin{pmatrix} 0 \\ -\omega^3 \\ -\omega^2 \end{pmatrix}$, cf. (9.18), thus $\omega^3 = c$ and $\omega^2 = b$. Ident with $\vec{e}_2$ so that $\omega^1 = a$. Thus $\vec{\omega} = a \vec{e}_1 + b \vec{e}_2 + c \vec{e}_3$, thus $\vec{\omega}$ is unique and we have (9.22).

**Proposition 9.14** Let $(\vec{a}_i)$ and $(\vec{b}_i)$ be Euclidean bases, let $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ be the associated Euclidean dot products, let $||\vec{b}_1||_a = \lambda$, so $(\cdot,\cdot)_a = \lambda^2 (\cdot,\cdot)_b$. Suppose $[\Omega]_a = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$, so $[\vec{\omega}_a]_a = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, cf. (9.22). Then (change of representation vector for $\Omega$):

1. If $\vec{b}_i = \lambda \vec{a}_i$ for all $i$ (change of unit and same orientation) then

$$[\Omega]_b = [\Omega]_a \quad \text{and} \quad \vec{\omega}_b = \lambda \vec{\omega}_a.$$  \hspace{1cm} (9.23)

2. If $\vec{b}_1 = -\lambda \vec{a}_1$, $\vec{b}_2 = \lambda \vec{a}_2$, $\vec{b}_3 = \lambda \vec{a}_3$ (change of unit and opposite orientation) then

$$[\Omega]_b = \begin{pmatrix} 0 & c & -b \\ -c & 0 & -a \\ b & a & 0 \end{pmatrix} \quad \text{and} \quad \vec{\omega}_b = -\lambda \vec{\omega}_a.$$  \hspace{1cm} (9.24)

3. More generally, if $(\vec{b}_i)$ and $(\vec{a}_i)$ have the same orientation, then $\vec{\omega}_b = \lambda \vec{\omega}_a$, else $\vec{\omega}_b = -\lambda \vec{\omega}_a$.

(NB: the formula $\vec{\omega}_b = \pm \lambda \vec{\omega}_a$ is a change of vector formula, not a change of basis formula. A representation vector of an antisymmetric endomorphism is not unique since it depends on the choice of a Euclidean unit of measure and the relative orientation of a Euclidean basis, cf. 3-.)

**Proof.** Apply (9.19), or:

$\Omega = -c \vec{a}_1 \circ a^2 + b \vec{a}_3 \circ a^3 - a \vec{a}_2 \circ a^3 + \text{antisym}.$

1. If $\vec{b}_1 = \lambda \vec{a}_1$ for all $i$ then $b^1 = \frac{1}{\lambda} a^1$, thus $\Omega = -c \vec{b}_1 \circ b^2 + b \vec{b}_1 \circ b^3 - a \vec{b}_2 \circ b^3 + \text{antisym}$. Thus $\vec{\omega}_b = a \vec{b}_1 + b \vec{b}_2 + c \vec{b}_3 = \lambda(a \vec{a}_1 + b \vec{a}_3 + c \vec{a}_3) = \lambda \vec{\omega}_a$.

2. If $\vec{b}_1 = -\lambda \vec{a}_1$, $\vec{b}_2 = \lambda \vec{a}_2$, $\vec{b}_3 = \lambda \vec{a}_3$ then $b^1 = -\frac{1}{\lambda} a^1$, $b^2 = \frac{1}{\lambda} a^2$, $b^3 = \frac{1}{\lambda} a^3$; Thus $\Omega = c \vec{b}_1 \circ b^2 - b \vec{b}_1 \circ b^3 - a \vec{b}_2 \circ b^3 + \text{antisym}$. Thus $\vec{\omega}_b = a \vec{b}_1 - b \vec{b}_2 - c \vec{b}_3 = \lambda(-a \vec{a}_1 - b \vec{a}_2 - c \vec{a}_3) = -\lambda \vec{\omega}_a$.

3. $\Omega(\vec{v},\vec{z})_b = \det(g(\vec{\omega}_b, \vec{v}, \vec{z}), = \pm \lambda^2 \det(\vec{\omega}_a, \vec{v}, \vec{z})$, and $\Omega(\vec{v},\vec{z})_b = \lambda^2 (\Omega(\vec{v},\vec{z})_a)$, thus $\det(\vec{\omega}_b)_a = \lambda^2 \det(\vec{\omega}_a)_a$, true for all $\vec{v}, \vec{z}$. Thus $\pm \lambda \det(\vec{\omega}_b)_a = [\vec{\omega}_a]_a$, thus $\vec{\omega}_b = \pm \vec{\omega}_a$, with + for same orientation, and with − otherwise, cf. (9.12).

**Interpretation of $\vec{\omega}_c$: Suppoe $[\Omega]_g = \sqrt{a^2 + b^2 + c^2}$.

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $\Omega$ is the rotation with angle $\frac{\pi}{4}$ in the horizontal plane composed with the dilation with ration $\sqrt{a^2 + b^2 + c^2}$. And $[\vec{\omega}_c]_g = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$.
\[
\sqrt{a^2+b^2+c^2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{a^2+b^2+c^2} \vec{e}_3 \quad \text{is orthogonal to the horizontal plane and gives the rotation axis and the dilation coefficient.}
\]

**Exercise 9.15** Let \( \Omega \) s.t. \([\Omega]_{|_\mathcal{E}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \) (see (9.22)). Find a direct orthonormal basis \( (\vec{b}_i) \) (relative to \( (\vec{e}_i) \)) s.t. \([\Omega]_{|_\mathcal{E}} = \sqrt{a^2+b^2+c^2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \). Then choose \( \vec{b}_1 \perp \vec{b}_3 \), e.g., \([\vec{b}_1]_{|_\mathcal{E}} = \begin{pmatrix} -b \\ a \\ 0 \end{pmatrix} \).

Then choose \( \vec{b}_2 = \vec{b}_3 \wedge \vec{b}_1 \), that is, \([\vec{b}_2]_{|_\mathcal{E}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \). Then choose \( \vec{b}_1 = \vec{b}_1 \wedge \vec{b}_2 \), that is, \([\vec{b}_1]_{|_\mathcal{E}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \). \( (\vec{b}_i) \) is a direct orthonormal basis, and the transition matrix from \( (\vec{e}_i) \) to \( (\vec{b}_i) \) is \( P = \begin{pmatrix} [\vec{b}_1]_{|_\mathcal{E}} & [\vec{b}_2]_{|_\mathcal{E}} & [\vec{b}_3]_{|_\mathcal{E}} \end{pmatrix} \). And \([\Omega]_{|_\mathcal{E}} = P^{-1}[\Omega]_{|_\mathcal{E}}P \) (change of basis formula), with \( P^{-1} = PT \) (change of orthonormal basis), thus \([\Omega]_{|_\mathcal{E}} = P^{-1}[\Omega]_{|_\mathcal{E}}P \)

\[
\begin{align*}
\text{With } [\Omega]_{|_\mathcal{E}} & = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} = \begin{pmatrix} -ac & -bc & -ac \\ ac & -bc & ac \\ ab & -bc & -ac \end{pmatrix} = \begin{pmatrix} \sqrt{a^2+b^2+c^2} & \sqrt{a^2+b^2+c^2} & \sqrt{a^2+b^2+c^2} \end{pmatrix} \text{(expected), and } [\Omega]_{|_\mathcal{E}}[\vec{b}_i]_{|_\mathcal{E}} = 0 \text{ (expected since } \vec{b}_3 \parallel \vec{w}_3). \\
\end{align*}
\]

**Remark 9.16** (\( e^1 = (dx^1) \) is the dual basis. (9.22) reads (tensorial notation) \( \Omega = -c(dx^1 \otimes dx^2 - dx^2 \otimes dx^1) + b(dx^1 \otimes dx^3 - dx^3 \otimes dx^1) - a(dx^2 \otimes dx^3 - dx^3 \otimes dx^2) \).

**Exercise 9.17** Prove, with the change of basis formulas, that \( \vec{w}_e \) is contravariant (change of basis contravariant formula).

**Answer.** \( \vec{w}_e \) is a vector in \( \mathbb{R}^3 \), so it is also called a contravariant vector. Change of basis: Let \( P_k \) be the change of basis endomorphism from a basis \( (\vec{e}_i) \) to a basis \( (\vec{a}_i) \). Let \( P_k \) be the change of basis endomorphism from a basis \( (\vec{e}_i) \) to a basis \( (\vec{b}_i) \). Thus \( P_k = P_{k-1}P_{k-1}^{-1} \), for all \( j \), thus, \( P_k = P_{k-1}P_{k-1}^{-1} \) is the change of basis endomorphism from a basis \( (\vec{a}_i) \) to a basis \( (\vec{b}_i) \). We have, \( \det(P_k) = \det(P_{k-1}) \det(P_{k-1}) = \det(P_{k-1}^{-1}) \det(P_{k-1}) \).

**Definition 9.18** If \( \vec{v} \) is a vector field \( C^1 \), if \( (\vec{e}_i) \) is a Euclidean basis in \( \mathbb{R}^3 \), and if \( \vec{v} = \sum_{i=1}^3 v^i \vec{e}_i \), then the curl of \( \vec{v} \) relative to \( (\vec{e}_i) \) is the vector field given by

\[
[\text{rot}_e \vec{v}]_{|_\mathcal{E}} = \begin{pmatrix} \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \\ \frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3} \end{pmatrix}.
\]

**Proposition 9.19** Let \( \Omega = \frac{\partial \vec{v}}{\partial x} \), and let \( \vec{w}_e(t, p_t) \) be the associated vector relative to the Euclidean basis \( (\vec{e}_i) \), cf. (9.20). Then

\[
\vec{w}_e = \frac{1}{2} \text{rot}_e \vec{v},
\]

that is, at \( t \) at \( p_t \), \( \text{rot}_e \vec{v}(t, p_t) = 2\vec{w}_e(t, p_t) \).
This formula is widely used in mechanics, and unfortunately improperly noted \( \Omega \) matrix. Then (9.30) gives, for all \( \vec{y} \) gives (9.27).

### 9.2.5 Pseudo-cross product and pseudo-vector (column matrix)

We leave the vector framework (tensorial) to enter the matrix world (no vector, no objectivity, no basis...).

**Definition 9.20** Let \( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = [\vec{x}] \) and \( \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = [\vec{y}] \) be two column matrices (collection of real numbers). Their pseudo-cross product \( [\vec{x}] \wedge [\vec{y}] \) is the name given to the matrix product defined by

\[
[\vec{x}] \wedge [\vec{y}] = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \wedge \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.
\]

(9.28)

And the column matrix defined by \( [\vec{x}] \wedge [\vec{y}] \) is called a pseudo-vector (or a column-vector).

### 9.2.6 Antisymmetric matrix represented by a pseudo-vector

**Definition 9.21** Let \( A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \) be an antisymmetric matrix. The pseudo-vector \( \hat{\omega} \) associated to \( A \) is the column matrix given by,

\[
\hat{\omega} := \begin{pmatrix} a \\ b \\ c \end{pmatrix}.
\]

(9.29)

Thus, the matrix \( \hat{\omega} \) (the pseudo-vector) satisfies

\[
A[\vec{y}] = \hat{\omega} \wedge [\vec{y}], \text{ i.e. } A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \hat{\omega} \wedge \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \text{ for all column matrix } [\vec{y}] = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.
\]

(9.30)

This is a matrix computation, cf. (9.28): there is no basis, no orientation.

### 9.2.7 Antisymmetric endomorphism and its pseudo-vectors representations

Let \( (\cdot, \cdot)_g \) be a Euclidean dot product. Let \( \Omega \) be an antisymmetric endomorphism relative to \((\cdot, \cdot)_g\), so \( \Omega^T = -\Omega \), cf. (9.13). Let \((\vec{e}_i)\) be a \((\cdot, \cdot)_g\)-Euclidean associated basis, so \( [\Omega]_g = A \) is an antisymmetric matrix. Then (9.30) gives, for all \( \vec{y} \in \mathbb{R}^3 \),

\[
[\Omega]_g[\vec{y}] = A[\vec{y}] = \hat{\omega} \wedge [\vec{y}].
\]

(9.31)

This formula is widely used in mechanics, and unfortunately improperly noted \( \Omega \vec{y} = \omega \wedge \vec{y} \).

**Be careful:** (9.31) is not a vectorial formula: This is just a formula for matrix calculations which gives false result if a change of basis is considered. E.g:

Let \((\vec{a}_1, \vec{a}_2, \vec{a}_3)\) be a \((\cdot, \cdot)_g\)-Euclidean basis, and define \((\vec{b}_1, \vec{b}_2, \vec{b}_3) = (-\vec{a}_1, \vec{a}_2, \vec{a}_3)\). So \((\vec{b}_i)\) is also a \((\cdot, \cdot)_g\)-Euclidean basis, but with a different orientation. Let \( P \) be the transition matrix from \((\vec{a}_i)\) to \((\vec{b}_i)\), that is, \( P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). Let \( [\Omega]_g = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \). Thus

\[
[\Omega]_g = P^{-1}[\Omega]_g, P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} = \begin{pmatrix} 0 & c & -b \\ -c & 0 & -a \\ b & a & 0 \end{pmatrix}.
\]

(9.32)

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• Thus the vectorial approach (9.22) gives

\[
[\vec{\omega}_a]_{\vec{a}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad [\vec{\omega}_b]_{\vec{b}} = \begin{pmatrix} -b \\ -c \\ a \end{pmatrix}, \quad \text{i.e.} \quad \begin{cases} \vec{\omega}_a = a\vec{a}_1 + b\vec{a}_2 + c\vec{a}_3, \\ \vec{\omega}_b = a\vec{b}_1 - b\vec{b}_2 - c\vec{b}_3, \end{cases}
\]

thus \(\vec{\omega}_b = -\vec{\omega}_a, \) (9.33)

since \((\vec{b}_1, \vec{b}_2, \vec{b}_3) = (-\vec{a}_1, \vec{a}_2, \vec{a}_3)\) (calculation already done in the proof of proposition 9.14: Change of vector).

• While the matrix approach (9.30) gives \([\Omega]_{\vec{a}}[\vec{a}] = \vec{\omega}_a \wedge [\vec{a}]\) and \([\Omega]_{\vec{b}}[\vec{b}] = \vec{\omega}_b \wedge [\vec{b}]\), with

\[
\vec{\omega}_a = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \vec{\omega}_b = \begin{pmatrix} -b \\ -c \\ a \end{pmatrix}, \quad \text{so} \quad \vec{\omega}_a \neq -\vec{\omega}_b, \quad (9.34)
\]

to compare with \(\vec{\omega}_b = -\vec{\omega}_a, \) cf. (9.33). And \(\vec{\omega}\) does not represent a single vector either, since it does not satisfy the change of basis formula \([\vec{\omega}]_{\vec{b}} = P^{-1} [\vec{\omega}]_{\vec{a}}\): we have \(\vec{\omega}_b \neq P^{-1} \vec{\omega}_a\). Thus \(\vec{\omega}\) is not a vector (is not tensorial): It is just a matrix (a collection of reals called a “pseudo-vector”). This should not be a surprise: The change of basis formula is \([L]_{\text{new}} = P^{-1} [L]_{\text{old}} P \) for the matrix of an endomorphism, when it is \([\vec{u}]_{\text{new}} = P^{-1} [\vec{u}]_{\text{old}}\) for vectors... and \(P \neq I\) (in general).

9.3 Examples

9.3.1 Rectilinear motion

Let \(\tilde{\Phi} : [t_1, t_2] \times \text{Obj} \to \mathbb{R}^n\) be a \(C^1\) motion. Let \(t_0 \in [t_1, t_2]\) and \(P_{\text{Obj}} \in \text{Obj}\).

**Definition 9.22** The motion of \(P_{\text{Obj}}\) is rectilinear if, for all \(t_0, t \in [t_1, t_2]\),

\[
\frac{\tilde{\Phi}_{P_{\text{Obj}}}(t) - \tilde{\Phi}_{P_{\text{Obj}}}(t_0)}{t-t_0} \parallel \tilde{\Phi}_{P_{\text{Obj}}}'(t_0).
\]

(9.35)

And the motion is rectilinear uniform if, for all \(t_0, t \in [t_1, t_2]\),

\[
\tilde{\Phi}_{P_{\text{Obj}}}(t) = \tilde{\Phi}_{P_{\text{Obj}}}(t_0) + (t-t_0) \tilde{\Phi}_{P_{\text{Obj}}}'(t_0), \quad \text{i.e.} \quad p(t) = p(t_0) + (t-t_0) V_{P_{\text{Obj}}}(t_0, p(t_0))
\]

(9.36)

when \(p(t) = \tilde{\Phi}(t, P_{\text{Obj}})\), that is, the trajectory is traveled at constant velocity.

9.3.2 Circular motion

Let \((\vec{E}_1, \vec{E}_2)\) be a Euclidean basis. Let \(t_0 \in [t_1, t_2]\). A motion \(\Phi^\omega\) is a circular motion if

\[
\overrightarrow{\Phi^\omega_{P_{\text{Obj}}}(t)} = x(t)\vec{E}_1 + y(t)\vec{E}_2, \quad [\overrightarrow{\Phi^{\omega_{P_{\text{Obj}}}}}(t)]_E = \begin{pmatrix} x(t) = a + R \cos(\theta(t)) \\ y(t) = b + R \sin(\theta(t)) \end{pmatrix},
\]

(9.37)

for some \(R > 0\) (called the radius), some \(a, b \in \mathbb{R}\), and some function \(\theta : \mathbb{R} \to \mathbb{R}\) and \(\begin{pmatrix} a \\ b \end{pmatrix} = O_C \in \mathbb{R}^2\) is the center of the circle and \(\theta(t)\) is the angle at \(t\). And the particle \(P_{\text{Obj}}\) (s.t. \(\tilde{\Phi}(t_0, P_{\text{Obj}}) = P\)) stays on the circle with center \(O_C\) and radius \(R\).

The circular motion is uniform if, for all \(t, \theta''(t) = 0\), that is, \(\exists \omega_0 \in \mathbb{R}, \forall t \in [t_1, t_2], \theta(t) = \omega_0 t\).

Notation:

\[
\varphi^\omega_{P_{\text{Obj}}}(t) = R \cos(\theta(t))\vec{E}_1 + R \sin(\theta(t))\vec{E}_2, \quad \text{so} \quad [\varphi^\omega_{P_{\text{Obj}}}(t)]_E = \begin{pmatrix} R \cos(\theta(t)) \\ R \sin(\theta(t)) \end{pmatrix}.
\]

(9.38)

Thus the Lagrangian velocity of a circular motion is

\[
\vec{V}^\omega_{P_{\text{Obj}}}(t) = (\Phi^\omega_{P_{\text{Obj}}})'(t) = (\varphi^\omega_{P_{\text{Obj}}})'(t), \quad \text{so} \quad [\vec{V}^\omega_{P_{\text{Obj}}}(t)]_E = R \theta'(t) \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix},
\]

(9.39)

and \(\vec{V}^\omega_{P_{\text{Obj}}}(t)\) is orthogonal to \(\varphi^\omega_{P_{\text{Obj}}}(t)\) (the radius vector). And the Lagrangian acceleration is

\[
\vec{\Gamma}^\omega_{P_{\text{Obj}}}(t) = R \theta''(t) \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix} + R(\theta'(t))^2 \begin{pmatrix} -\cos(\theta(t)) \\ -\sin(\theta(t)) \end{pmatrix}.
\]

(9.40)
Consider
\[ \vec{e}_r(t) = \frac{\varphi_r^p(t)}{||\varphi_r^p(t)||} = \left( \frac{\cos(\theta(t))}{\sin(\theta(t))} \right), \quad \text{and} \quad \vec{e}_\theta(t) = \left( \frac{-\sin(\theta(t))}{\cos(\theta(t))} \right), \quad (9.41) \]
thus \((\vec{e}_r(t), \vec{e}_\theta(t))\) is an orthonormal basis, and \((p(t), (\vec{e}_r(t), \vec{e}_\theta(t)))\) is the Frénet frame relative to the motion. Then:
\[ \vec{V}^t_{p}(t) = R\theta'(t) \vec{e}_\theta(t), \quad (9.42) \]
and:
\[ \vec{\Gamma}^t_{p}(t) = -R(\theta'(t))^2 \vec{e}_r(t) + R\theta''(t) \vec{e}_\theta(t). \quad (9.43) \]

In \(\mathbb{R}^3\): The plane \((\vec{E}_1, \vec{E}_2)\) is seen in \(\mathbb{R}^3\), and \(\vec{E}_1 = (1, 0, 0)\) and \(\vec{E}_2 = (0, 1, 0)\) are seen as the vectors \(\vec{E}_1 = (1, 0, 0)\) and \(\vec{E}_2 = (0, 1, 0)\). And let \(\vec{E}_3 = (0, 0, 1)\). Thus:
\[ \vec{V}^t_{p}(t) = \vec{w}(t) \wedge \varphi^p_r(t), \quad \text{where} \quad \vec{w}(t) = \omega(t) \vec{e}_3 \quad \text{and} \quad \omega(t) = \theta'(t), \quad (9.44) \]
cf. (9.42). And
\[ \vec{\Gamma}^t_{p}(t) = \frac{d\vec{w}}{dt}(t) \wedge \varphi^p_r(t) + \vec{w}(t) \wedge \vec{V}^t_{p}(t) \quad (= R \frac{d\omega}{dt}(t) \vec{e}_\theta(t) - \omega^2(t) R \vec{e}_r(t)). \quad (9.45) \]

9.3.3 Motion of a planet (centripetal acceleration)

Illustration: Obj is e.g. a planet from the solar system.

Let \((\vec{e}_1, \vec{e}_2, \vec{e}_3)\) be a Euclidean basis (e.g. fixed relative to stars an \((\vec{e}_1, \vec{e}_2)\) define the ecliptic plane), \((\cdot, \cdot)\) be the Euclidean associated dot product, \(||.||\) the Euclidean associated norm, \(\mathcal{O}\) an origin in \(\mathbb{R}^3\) (e.g. the center of the Sun), and \(\mathcal{R} = (\mathcal{O}, (\vec{e}_3))\).

Consider a motion \(\Phi\) of Obj in \(\mathcal{R}\), cf. (1.5). Let \(t_0 \in [t_1, t_2]\), and consider \(\Phi^{t_0} = \text{noted} \Phi\) or \(\varphi^{t_0} = \text{noted} \varphi\), cf. (3.1)-(3.4).

Definition 9.23 The motion of a particle \(P_{\text{Obj}}\) is a centripetal acceleration motion iff the particle is not static and, at all time, its acceleration \(\vec{A}(t)\) points to a fixed point \(F\) (focus).

We will take the focus \(F\) as the origin of the referential, that is, \(\mathcal{O} := F\).

Thus, for all \(t \in [t_1, t_2]\), \(\vec{O}\Phi^{t}(t) \parallel \vec{A}(t)\), that is,
\[ \vec{O}\Phi^{t}(t) \parallel \vec{A}(t) = 0. \quad (9.46) \]

Remark 9.24 A rectilinear motion is a centripetal acceleration motion, but such a motion is usually excluded in the definition 9.23.

Example 9.25 The motion of a planet from the solar system is a centripetal acceleration motion: An elliptical motion of focus the center of the Sun.

Example 9.26 The second Newton’s law of motion \(\sum \vec{f} = m\vec{\gamma}\) (Galilean referential) gives: If \(\sum \vec{f}\) is, at all time, directed to a unique point \(F\), then the motion is a centripetal acceleration motion.

Let \(\Phi\) be a centripetal acceleration motion, let \(\mathcal{O}\) be the focus, and let \(\varphi^p(t) := \vec{O}\Phi^{t}(t)\). So the Lagrangian velocity and acceleration are
\[ \vec{V}_p(t) = \frac{d\varphi^p(t)}{dt}, \quad \text{and} \quad \vec{A}_p(t) = \frac{d^2\varphi^p(t)}{dt^2} = \frac{d^2\varphi^p(t)}{dt^2}(t), \quad (9.47) \]
and \(\varphi^p(t) \wedge \vec{A}_p(t) = 0\), cf. (9.46).

Definition 9.27 The areolar velocity at \(t\) is the vector
\[ \vec{Z}(t) = \frac{1}{2} \varphi^p(t) \wedge \vec{V}_p(t), \quad (9.48) \]
**Proposition 9.28** If $\Phi$ is a centripetal acceleration motion, then the areolar velocity is constant, that is, $\frac{d\vec{Z}}{dt}(t) = \vec{0}$ for all $t$, so
\[ \vec{Z}(t) = \vec{Z}(t_0), \quad \forall t. \tag{9.49} \]
That is, the position vectors sweep equal areas in equal times. And $\vec{Z}(t_0) = \vec{0}$ iff $\Phi$ is a rectilinear motion.

If $\vec{Z}(t_0) \neq \vec{0}$ then:
- $\vec{\varphi}_P(t)$ and $\vec{V}_P(t)$ are orthogonal to $\vec{Z}(t_0)$ at all time $t$,
- The motion of the particle $P_{0\Delta t}$ takes place in the affine plane orthogonal to $\vec{Z}(t_0)$ passing through $O$.
- $\vec{V}_P(t)$ never vanishes.

**Proof.** (9.48) and (9.46) give $2\frac{d\vec{Z}}{dt}(t) = \frac{d\vec{Z}}{dt}(t) \wedge \vec{V}_P(t) + \vec{\varphi}(t) \wedge \vec{V}_P(t) = \vec{V}_P(t) \wedge \vec{V}_P(t) + \vec{\varphi}(t) \wedge \vec{\varphi}_P(t) = \vec{0} + \vec{0}$. Thus $\vec{Z}$ is constant, $\vec{Z}(t) = \vec{Z}(t_0)$ for all $t$. Then, if $\vec{Z}(t_0) \neq \vec{0}$ then $\vec{Z}(t) \neq \vec{0}$ for all $t$, and

- $\vec{Z}(t) = \frac{1}{2} \vec{\varphi}_P(t) \wedge \vec{V}_P(t)$ gives that $\vec{\varphi}_P(t)$ et $\vec{V}_P(t)$ are orthogonal to $\vec{Z}(t_0)$ for all $t$, thus $\vec{\varphi}_P(t)$ is orthogonal to $\vec{Z}(t_0)$, cf. (9.46).

- The Taylor expansion reads $\vec{\varphi}_P(t) = \vec{\varphi}_P(t_0) + \vec{V}_P(t_0)(t-t_0) + \int_{t_0}^t \vec{\varphi}_P(\tau)(t-\tau)^2 d\tau$, with $\vec{V}_P(t_0)$ and $\vec{\varphi}_P(t) \perp \vec{Z}(t_0)$ for all $t$, thus $\vec{\varphi}_P(t) - \vec{\varphi}_P(t_0) \perp \vec{Z}(t_0)$ for all $\tau$, that is $\vec{O}_P(t) - \vec{O}_P(0) = \vec{P}_P(t) \perp \vec{Z}(t_0)$ for all $\tau$. Thus $P(t)$ belongs to the affine plane containing $P$ orthogonal to $\vec{Z}(t_0)$, for all $t$. And $\vec{O}_P = \vec{\varphi}_P(0) \perp \vec{Z}(t_0)$, thus $O$ belong to the same plane.

- $\vec{Z}(t) = \vec{Z}(t_0) \neq \vec{0}$ implies $\vec{V}_P(t) \neq \vec{0}$ for all $t$, and (9.48) gives: $(\vec{\varphi}_P(t), \vec{V}_P(t), \vec{Z}(t_0))$ is a positively-oriented basis. Since $\vec{\varphi}_P$ and $\vec{V}$ are continuous and do not vanish, since $\vec{Z}(t_0) \neq \vec{0}$, we get: $P_{0\Delta t}$ “turns around $\vec{Z}(t_0)$” and keeps its direction.

If $\vec{Z}(t) = \vec{0}$ then $\vec{\varphi}_P(t) \parallel \vec{V}_P(t)$ for all $t$, cf. (9.48), so $\vec{V}_P(t) = f(t)\vec{\varphi}_P(t)$ where $f$ is some scalar function. And $\vec{V}_P(t) = \vec{\varphi}_P(t) + \vec{\varphi}_P^\prime(t) = f(t)\vec{\varphi}_P(t)$, thus $\vec{\varphi}_P(t) = \vec{\varphi}_P(t_0) + f(t)\vec{\varphi}_P^\prime(t)$ where $F$ is a primitive of $f$ s.t. $F(t_0) = 0$, thus $\vec{\varphi}_P(t) \parallel \vec{\varphi}_P(t_0)$, so $\vec{O}_F/\vec{P}(t_0) \parallel \vec{O}_F/\vec{P}(t_0)$, for all $t$: The motion is rectilinear. ☐

**Interpretation.** (Non rectilinear motion.) The area swept by $\vec{\varphi}_P(t)$ is, at first order, the area of the triangle whose sides are $\vec{\varphi}_P(t)$ and $\vec{\varphi}_P(t + \tau)$ (“angular sector”). So, with $\tau$ close to $0$, let
\[ \vec{S}_i(\tau) = \frac{1}{2} \vec{\varphi}_P(t) \wedge \vec{\varphi}_P(t + \tau), \quad \text{and} \quad S_i(\tau) = ||\vec{S}_i(\tau)||, \tag{9.50} \]
the vectorial an scalar area. With $\vec{\varphi}_P(t + \tau) = \vec{\varphi}_P(t) + \vec{V}_P(t)\tau + o(\tau)$ we get
\[ \vec{S}_i(\tau) = \frac{1}{2} \vec{\varphi}_P(t) \wedge (\vec{V}_P(t)\tau + o(\tau)), \tag{9.51} \]
Since $\vec{S}_i(0) = 0$ we get $\frac{\vec{S}_i(\tau) - \vec{S}_i(0)}{\tau} = \frac{1}{2} \vec{\varphi}_P(t) \wedge \vec{V}_P(t) + o(1)$, then
\[ \frac{d\vec{S}_i}{dt}(0) = \frac{1}{2} \vec{\varphi}_P(t) \wedge \vec{V}_P(t) = \vec{Z}(t) = \vec{Z}(t_0), \tag{9.52} \]
thanks to (9.49), thus
\[ \frac{d\vec{S}_i}{dt}(0) = \frac{d\vec{S}_i}{d\tau}(0), \quad \forall t \in [t_0, T], \tag{9.53} \]
that is, the rate of variation of $\vec{S}_i$ is constant. And with $||\vec{S}_i(\tau)||^2 = (\vec{S}_i(\tau), \vec{S}_i(\tau))$ we get
\[ \frac{d||\vec{S}_i||^2}{d\tau}(\tau) = 2(\vec{S}_i(\tau), \vec{\varphi}_P^\prime(\tau)), \tag{9.54} \]
so, since $\vec{S}_i(t_0) = 0$,
\[ \frac{d||\vec{S}_i||^2}{dt}(0) = 0. \tag{9.55} \]
Therefore the function $t \rightarrow ||\vec{S}_i(0)||^2 = S_i(0)^2$ is constant, thus $t \rightarrow S_i(0)$ est constant, and $\frac{dS_i}{dt}(0)$ is constant.
Exercise 9.29 Give a parametrization of the swept area, and redo the calculations.

Answer. Let
\[ r(t) = ||\vec{\varphi}(t)||, \quad \theta(t) = \rho(t)OP \quad \text{(angle)}, \]
then
\[ \vec{\varphi}(t) = \begin{pmatrix} r(t) \cos(\theta(t)) \\ r(t) \sin(\theta(t)) \end{pmatrix}. \]
(9.56)

Thus
\[ V(t) = \begin{pmatrix} r'(t) \cos(\theta(t)) - r(t)\theta'(t) \sin(\theta(t)) \\ r'(t) \sin(\theta(t)) + r(t)\theta'(t) \cos(\theta(t)) \end{pmatrix}. \]
(9.58)
With (9.48) we get
\[ \vec{Z}(t) = \frac{1}{2} \begin{pmatrix} 0 \\ r(2t)\theta'(t) \end{pmatrix}, \quad \text{with} \quad r^2(t)\theta'(t) = r^2(t_0)\theta'(t_0) \quad \text{(constant)}, \]
(9.59)
cf. (9.49). A parametrization of the swept area is then
\[ \mathcal{A}: \{ (0,1) \times [t_0, T] \to \mathbb{R}^2 \}, \quad \vec{A}(\rho, t) = \begin{pmatrix} \rho \cos(\theta(t)) \\ \rho \sin(\theta(t)) \end{pmatrix}. \]
(9.60)

Therefore, the tangent associated vectors are
\[ \frac{\partial \vec{A}}{\partial \rho}(\rho, t) = \begin{pmatrix} r(t) \cos(\theta(t)) \\ r(t) \sin(\theta(t)) \end{pmatrix}, \quad \frac{\partial \vec{A}}{\partial t}(\rho, t) = \begin{pmatrix} \rho r'(t) \cos(\theta(t)) - \rho \theta'(t) \theta'(t) \sin(\theta(t)) \\ \rho r'(t) \sin(\theta(t)) + \rho \theta'(t) \theta'(t) \cos(\theta(t)) \end{pmatrix}. \]
(9.61)

hence the vectorial and scalar element areas are
\[ d\vec{\sigma} = \left( \frac{\partial \vec{A}}{\partial \rho} \wedge \frac{\partial \vec{A}}{\partial t} \right) d\rho dt = \begin{pmatrix} 0 \\ \rho \theta'(t) \end{pmatrix}, \quad d\sigma = \rho^2 \theta' d\rho dt. \]
(9.62)

Therefore the area between \( t_0 \) and \( t \) is
\[ \mathcal{A}(t) = \mathcal{A}(t_0) + \int_{\rho=0}^{1} \int_{\tau=t_0}^{t} \rho^2 \theta'(\tau) \, d\rho d\tau = \frac{1}{2} \int_{t=t_0}^{t} r(\tau)^2 \theta'(\tau) \, d\tau. \]
(9.63)

Hence
\[ \mathcal{A}'(t) = r(t)^2 \theta'(t) = r(t_0)^2 \theta'(t_0) \quad (= \text{constant} = ||\vec{Z}(t_0)||), \]
(9.64)
cf. (9.59).

Exercise 9.30 Prove the Binet formulas (non rectilinear central motion):
\[ V_p(t)^2 = Z_0^2 \left( \frac{1}{r^2} + \left( \frac{d^2}{dt^2} \right)^2 \right)(t), \quad \Phi_p(t) = -Z_0^2 \left( \frac{1}{r^2} + \frac{d^2}{dt^2} \right)(t) \vec{e}_r(t), \]
(9.65)
for the energy and the acceleration.

Answer. Proposition 9.28 tells that \( \Phi \) is a planar motion. With (9.56) and \( \vec{e}_r(t) = \begin{pmatrix} \cos(\theta(t)) \\ \sin(\theta(t)) \end{pmatrix} \) we have
\[ \vec{\varphi}(t) = r(t)\vec{e}_r(t) \quad \text{(in the plane). Let} \quad \vec{e}_\theta(t) = \begin{pmatrix} -\sin(\theta(t)) \\ \cos(\theta(t)) \end{pmatrix}, \quad \text{thus} \]
\[ \vec{V}(t) = \frac{d}{dt}(t)\vec{e}_r(t) + r(t)\frac{d\vec{e}_r}{dt}(t) = r(t)\vec{e}_r(t) + r(t)\theta'(t)\vec{e}_\theta(t). \]
And \( \vec{e}_r(t) \perp \vec{e}_\theta(t) \) gives
\[ V^2(t) = (r'(t))^2 + (r(t)\theta'(t))^2. \]
Since \( \theta'(t) \neq 0 \) for all \( t \) (non rectilinear central motion) Let \( s(\theta(t)) = r(t) \). Let us suppose that \( \theta \) is \( C^1 \), thus
\( \theta' > 0 \) or \( \theta' < 0 \), and \( \theta : t \to \theta(t) \) defines a change of variable. And

\[
r'(t) = s'(\theta(t))\theta'(t).
\]

And (9.64) and \( \theta'(t) = \frac{Z_0}{r(t)} \) give

\[
V^2(t(\theta)) = (s'(\theta))^2 \frac{Z_0^2}{r^2(t)} + r^2(t) \frac{Z_0^2}{r^4(t)} = Z_0^2 \left( \frac{(s'(\theta))^2}{s^2(\theta)} + \frac{1}{s^2(\theta)} \right) = Z_0^2 \left( \frac{d^2}{d\theta^2} (\theta) \right)^2 + \frac{1}{s^2(\theta)}.
\]

Thus \( r(t) = s(\theta) \) and \( \frac{ds}{d\theta} = \frac{dZ_0}{d\theta} \) give the first Binet formula. Then

\[
\vec{\Gamma}(t) = r''(t)\vec{e}_r(t) + r'(t)\frac{d\vec{e}_r}{dt}(t) + (r'(t)\theta'(t) + r(t)\theta''(t))\vec{e}_\theta(t) + r(t)\theta'(t)\frac{d\vec{e}_\theta}{dt}(t),
\]

with \( \frac{d\vec{e}_r}{dt} \parallel \vec{e}_r \), and \( \frac{d\vec{e}_\theta}{dt}(t) = -\theta'(t)\vec{e}_r(t) \), and \( \vec{e}_\theta \perp \vec{\Gamma} \) (central motion), we get

\[
\vec{\Gamma}(t) = (r''(t) - r(t)(\theta'(t))^2)\vec{e}_r(t).
\]

And

\[
r'(t) = s'(\theta)\theta'(t) = s'(\theta) \frac{Z_0}{r^2(t)} = Z_0 \frac{s'(\theta)}{s^2(\theta)} = -Z_0 \frac{d^2}{d\theta^2}(\theta),
\]

thus

\[
r''(t) = -Z_0 \frac{d^2}{d\theta^2}(\theta) \theta'(t) = - \frac{Z_0^2}{r^2(t)} \frac{d^2}{d\theta^2}(\theta),
\]

which is the second Binet formula.
Part II
Push-forward

10 Push-forward

We tackle an essential question: What does “to transport a quantity” mean? The general tool to describe
“transport” is “push-forward” (the “take with you” operator in case of a motion). This has been done
at § 5.1 for a motion, see figure 5.1.

The push-forward also gives the tool needed to understand the velocity addition formula: In that case,
the push-forward is the translator (between two observers). Which then enables to define and understand
the “covariant objectivity” that enables an English observer with his foot and a French observer with his
meter to work together, without causing an accident like the Mars climate orbiter crash.

As usual, we start with qualitative results (no basis, no inner dot product: Observer independent results); Then, quantification will be done (quantitative results).

10.1 Definition

Let $E$ and $F$ be affine spaces (e.g. sub spaces of the affine space $\mathbb{R}^3$). Let $E$ and $F$ be associated
vector spaces equipped with norms $||.||_E$ and $||.||_F$ (we will consider diffeomorphisms). We suppose that $E$ and
$F$ are finite dimensional, and $\dim E = \dim F = n (= 1, 2$ or $3$ in the following). Let $U_E$ and $U_F$ be open
sets in the affine spaces $E$ and $F$ (or eventually in the vector spaces $E$ and $F$). Consider

\[ \Psi : \begin{cases} U_E &\to U_F \\ p_E &\mapsto p_F = \Psi(p_E) \end{cases} \]

that is $\Psi$ is a $C^1$ invertible map which inverse is $C^1$. $\Psi$ will be our push-forward map.

And $\Psi^{-1}$ will be our pull-back map (= the push-forward with $\Psi^{-1}$).

Example: $\Psi = \Phi_{t_0}^t : \Omega_{t_0} \to \Omega_t$ is the motion that transforms $\Omega_{t_0}$ into $\Omega_t$, cf. (3.5).

Example: $\Psi = \Theta_t : \mathcal{R}_B \to \mathcal{R}_A$ is the translator at $t$ for the change of observer (essential for the
velocity addition formula), see § 16.

Example: $\Psi = \text{a coordinate system, see example 10.17.}$

10.2 Push-forward and pull-back of points

Definition 10.1 With (10.1). If $p_E \in U_E$ (a point in $U_E$) then its push-forward by $\Psi$ is the point $p_F \in U_F$
defined by

\[ p_F := \Psi(p_E) = \Psi_*(p_E) \]

(10.2)

The notation used here has been proposed by Spivak; Also see Abraham and Marsden [1] (second
d edition) who adopted this notation; And the last notation $\Psi_*(p_E)$ is used when an explicit reference to $\Psi$
is needed.

Example 10.2 Let $\Psi = \Phi_{t_0}^t : \Omega_{t_0} \to \Omega_t$. Let $(p_E =) p_{t_0} \in \Omega_{t_0}$. Then

\[ (p_F =) p_t = \Phi_{t_0}^t(p_{t_0}) = \Psi_*(p_{t_0}) \]

push-forward of $p_{t_0}$ by $\Phi_{t_0}^t$, see figure 10.1.

Definition 10.3 If $p_F \in U_F$ then its pull-back by $\Psi$ is the point $p_E \in U_E$ defined by

\[ p_E := \Psi^{-1}(p_F) = \Psi^*(p_F) \]

(10.3)
Figure 10.1: $c_t : s \to p_t = c_t(s)$ is a (spatial) curve in $\Omega_t$. It is transported by the motion and becomes the curve $c_t = \Phi^{t_0}_t \circ c_{t_0}$ named $c_{t_0}$ in $\Omega_t$. The tangent vector at $p_t = c_t(s)$ is $\vec{w}_{c_t}(p_t) = c_t'(s)$, and the tangent vector at $p_t = c_t(s) = \Phi^{t_0}(c_{t_0}(s))$ is $\vec{w}_{c_{t_0}}(p_t) = c_t'(s) = F^{t_0}_{t_0}(p_{t_0}).$

10.3 Push-forward and pull-back of scalar functions

10.3.1 Definitions

**Definition 10.4** Let $f_{\varepsilon} : \begin{cases} U_{\varepsilon} \to \mathbb{R} \\ p_{\varepsilon} \to f_{\varepsilon}(p_{\varepsilon}) \end{cases}$ be a scalar function. Its push-forward by $\Phi$ is the scalar function

$$f_{\varepsilon} : \begin{cases} U_{\varepsilon} \to \mathbb{R} \\ p_{\varepsilon} \to f_{\varepsilon}(p_{\varepsilon}) := f_{\varepsilon}(p_{\varepsilon}) \quad \text{when} \quad p_{\varepsilon} = \Phi(p_{\varepsilon}). \quad (10.5) \end{cases}$$

So

$$f_{\varepsilon} := f_{\varepsilon} \circ \Psi^{-1}. \quad (10.6)$$

And

$$f_{\varepsilon} := f_{\varepsilon} \circ \Psi^{-1} = \Psi(f_{\varepsilon}). \quad (10.7)$$

so $f_{\varepsilon}(p_{\varepsilon}) = f_{\varepsilon}(p_{\varepsilon}) = (\Psi \circ f_{\varepsilon})(p_{\varepsilon}) = \Psi \times f_{\varepsilon}(p_{\varepsilon}) = \Psi \circ f_{\varepsilon}(p_{\varepsilon}) = \Psi \circ f_{\varepsilon} = \Psi_{\varepsilon} f_{\varepsilon}$. And we have defined

$$\Psi_{\varepsilon} : \begin{cases} F(U_{\varepsilon}; \mathbb{R}) \to F(U_{\varepsilon}; \mathbb{R}) \\ f_{\varepsilon} \to f_{\varepsilon} := \Psi(f_{\varepsilon}) = \Psi_{\varepsilon} f_{\varepsilon} = f_{\varepsilon} := f_{\varepsilon} \circ \Psi^{-1}, \quad (10.8) \end{cases}$$

the notation $\Psi_{\varepsilon}(f_{\varepsilon}) = \Psi_{\varepsilon} f_{\varepsilon}$ since $\Psi_{\varepsilon}$ is trivially linear since $(f_{\varepsilon} + \lambda g_{\varepsilon}) \circ \Psi^{-1} = f_{\varepsilon} \circ \Psi^{-1} + \lambda g_{\varepsilon} \circ \Psi^{-1}$.

**Example 10.5** Let $t_0 = \theta_{t_0} : p_{t_0} \in \Omega_{t_0} \to f_{t_0}(p_{t_0})$ be the temperature of the particle $P_{\Omega_{t_0}}$ which is at $t_0$ at $p_{t_0} = \tilde{\Phi}(t_0, P_{\Omega_{t_0}})$. Let $\Psi = \Phi^{t_0}_{t} : \Omega_{t_0} \to \Omega_t$, and $\Psi_{t_0} = \Phi^{t_0}_{t_0}(p_{t_0})$. Then $f_{t_0} = \Phi^{t_0}_{t} \circ f_{t_0} : \Omega \to \mathbb{R}$ is given by

$$\Phi^{t_0}_{t} f_{t_0}(p_t) = \Phi^{t_0}_{t_0}(p_t):= f_{t_0}(p_{t_0}). \quad (10.9)$$

So $\Phi^{t_0}_{t} f_{t_0}(p_t)$ gives at $t$ at $p_t$ the value that $f_{t_0}$ had at $p_{t_0}$: So $f_{t_0}$ is the “memory function”. In other words, $\Phi^{t_0}_{t} : \begin{cases} F(\Omega_{t_0}; \mathbb{R}) \to F(\Omega_t; \mathbb{R}) \\ f_{t_0} \to f_{t_0} \end{cases}$ “transports the memory”, see next § 10.3.2.
Definition 10.6 Let \( f : \begin{cases} U \rightarrow \mathbb{R} \\
 p \rightarrow f(p) \end{cases} \) be a scalar function. Its pull-back by \( \Psi \) is the push-forward by \( \Psi^{-1} \), i.e., is the scalar function

\[
\Psi^* f = f^{\text{named}} = \Psi^* \circ f : \begin{cases} U \rightarrow \mathbb{R} \\
p \rightarrow \Psi^* f(p) = f(\Psi(p)) \quad \text{when} \quad p \in \Psi(\mathbb{R}). \tag{10.10} \end{cases}
\]

And the pull-back \( f^* \) of \( f \) by \( \Psi \) is also named \( \Psi^* f \), so

\[
f^* = \Psi^* f := f \circ \Psi, \quad \text{and} \quad \Psi^* f(\Psi(p)) = f(p) \quad \text{when} \quad p \in \Psi(\mathbb{R}). \tag{10.11} \]

Proposition 10.7

\[
\Psi^* \circ \Psi_* = I \quad \text{and} \quad \Psi_* \circ \Psi^* = I. \tag{10.12} \]

Proof. \( \Psi^*(\Psi_* f)(p) = \Psi_* f(\Psi(p)) = f(\Psi(p)) \), thus \( \Psi^* \circ \Psi_* : \mathcal{F}(U_\mathbb{R}; \mathbb{R}) \rightarrow \mathcal{F}(U_\mathbb{R}; \mathbb{R}) \) is the identity. Since \( \Psi_* (\Psi^* f)(p) = \Psi^* f(\Psi(p)) = f(\Psi(p)) \), thus \( \Psi^* \circ (\Psi_* f)(p) = f(\Psi(p)) \), which is the identity. \( \blacksquare \)

10.3.2 Interpretation: Why is it useful?

Following example 10.5. We record the temperature \( \theta(t, p) \) at all \( t \) and \( p \). So \( \theta : \begin{cases} \mathcal{C} = \bigcup_t (t \times \Omega_t) \rightarrow \mathbb{R} \\
 (t, p) \rightarrow \theta(t, p) \end{cases} \) is a Eulerian scalar valued function, cf. (2.2). Then we consider its restriction \( \theta_{t_0} : \begin{cases} \Omega_{t_0} \rightarrow \mathbb{R} \\
p_{t_0} \rightarrow \theta_{t_0}(p_{t_0}) := \theta(t_0, p_{t_0}) \end{cases} \) which gives the temperatures at \( t_0 \) in \( \Omega_{t_0} \). The push-forward of the function \( \theta_{t_0} \) by \( \Phi^*_{t_0} \) is the “memory function”, cf. (10.5),

\[
\theta_{t_0}^* : \begin{cases} \Omega_t \rightarrow \mathbb{R} \\
p_t \rightarrow \theta_{t_0}^*(p_t) := \theta_{t_0}(p_{t_0}) \quad \text{when} \quad p_t = \Phi^*_{t_0}(p_{t_0}) \tag{10.13} \end{cases}
\]

Question: Why do we introduce \( \theta_{t_0}^* \) since we have \( \theta_{t_0} \)?

Answer: An observer does not have the gift of temporal and/or spatial ubiquity; he can only consider values at the actual \( t \) at the actual \( p \) where he is (time and space). See remark 1.3.

E.g., at \( t_0 \) at \( p_{t_0} \) the observer writes the value \( \theta_{t_0}(p_{t_0}) \) on a piece of paper (for memory), then put the piece of paper is his pocket; and once arrived at \( t_0 \) at \( p_0 \), he reads the value on the paper and name it \( \theta_{t_0}(p_0) \), or more explicitly \( \Phi^*_{t_0}(\theta_{t_0}(p_0)) \), because he is now at \( t_0 \) at \( p_0 \), so with \( \theta_{t_0}^*(p) := \theta_{t_0}(p_{t_0}) \).

Application: Then, at \( p_t \) at \( t_0 \), the observer can compare the actual value \( \theta_t(p_t) \) with the transported value \( \theta_{t_0}^*(p_t) = \theta_{t_0}(p_{t_0}) \) the memory (no need of any ubiquity gift). Here, for scalar valued functions, we simply get

\[
\theta_t(p_t) - \theta_{t_0}^*(p_t) = \theta_t(p_t) - \theta_{t_0}(p_{t_0}). \tag{10.14} \]

With (10.14), the Lie derivative will be easy to understand.

10.4 Push-forward and pull-back of curves

With (10.1). Let

\[
c_\mathcal{E} : [\mathbb{R}] \rightarrow U_\mathcal{E} \\
s \rightarrow p_\mathcal{E} = c_\mathcal{E}(s) \tag{10.15} \]

be a curve in \( U_\mathcal{E} \), see figure 10.1.

Definition 10.8 The push-forward of \( c_\mathcal{E} : [\mathbb{R}] \rightarrow U_\mathcal{E} \) by \( \Psi \) is the curve

\[
c_\mathcal{F} := \Psi \circ c_\mathcal{E} : [\mathbb{R}] \rightarrow U_\mathcal{F} \\
s \rightarrow c_\mathcal{F}(s) := \Psi(c_\mathcal{E}(s)) = \Psi(p_\mathcal{E}(s)), \tag{10.16} \]

so \( \text{Im}(c_\mathcal{F}) = \Psi_*(c_\mathcal{E}) \), that is, \( \text{Im}(c_\mathcal{F}) \) is made of the push-forwards of the points of \( \text{Im}(c_\mathcal{E}) \), see figure 10.1.
Example 10.9 See figure 10.1. \( \Psi = \Phi^{t_0}_t \), and \( c_\varepsilon = c_{t_0} : \left\{ \begin{array}{l} ]-\varepsilon,\varepsilon[ \to \Omega_{t_0} \\ s \to p_t = c_{t_0}(s) \end{array} \right\} \) (so \( s \) is a space variable). And

\[
c_t = c_{t_0*} = \Phi^{t_0}_t \circ c_{t_0} : \left\{ \begin{array}{l} ]-\varepsilon,\varepsilon[ \to \Omega_t \\ s \to p_t = c_t(s) := \Phi^{t_0}_t(p_{t_0}) \end{array} \right\} \ \text{when} \quad p_t = \Phi^{t_0}_t(p_{t_0}) = p_{t_0*}. \tag{10.17}
\]

And \( \text{Im}(c_{t_0*}) = \Phi^{t_0}_t(\text{Im}(c_{t_0})) \).

**Definition 10.10** If

\[
c_F : \left\{ \begin{array}{l} ]-\varepsilon,\varepsilon[ \to U_F \\ s \to p_F = c_F(s) \end{array} \right\}
\]

is a curve in \( U_F \) then its pull-back \( c_\varepsilon = c_F^* \) by \( \Psi \) is its push-forward by \( \Psi^{-1} \):

\[
c_\varepsilon = c_F^* := \Psi^{-1} \circ c_F : \left\{ \begin{array}{l} ]-\varepsilon,\varepsilon[ \to U_F \\ s \to p_\varepsilon = c_F(s) = c_F^*(s) := \Psi^{-1}(c_F(s)) = \Psi^{-1}(p_F). \tag{10.19}
\end{array} \right\}
\]

10.5 Push-forward and pull-back of vector fields

This is one of the most important concepts for mechanical engineers. We need a norm \( ||.||_E \) in \( E \) (we need first-order Taylor expansions).

10.5.1 Approximate description: Transport of a “small bipoint vector”

Let \( p_\varepsilon \) and \( q_\varepsilon \) be points in \( U_E \), and let \( p_F = p_\varepsilon* = \Psi(p_\varepsilon) \) and \( q_F = q_\varepsilon* = \Psi(q_\varepsilon) \) in \( U_F \) be the push-forwards by \( \Psi \) cf. (10.1). The first order Taylor expansion gives

\[
(\Psi(q_\varepsilon) - \Psi(p_\varepsilon)) = q_F - p_F = d\Psi(p_\varepsilon).(q_\varepsilon - p_\varepsilon) + o(||q_\varepsilon - p_\varepsilon||_E), \tag{10.20}
\]

that is,

\[
\tilde{p}_F \tilde{q}_F = d\Psi(p_\varepsilon) \tilde{p}_E \tilde{q}_E + o(||\tilde{p}_E \tilde{q}_E||_E). \tag{10.21}
\]

“By neglecting” the \( o(||\tilde{p}_E \tilde{q}_E||_E) \) we get:

**Definition 10.11** Let \( \vec{w}_E(p_\varepsilon) \in E \) be a vector at \( p_\varepsilon \in U_F \); Its push-forward by \( \Psi \) is the vector \( \vec{w}_F(p_\varepsilon) = \Psi(s) \vec{w}_E(p_\varepsilon) = \Psi_\varepsilon* \vec{w}_E(p_\varepsilon) \in F \) defined at \( p_F = p_\varepsilon = \Psi(p_\varepsilon) \in U_F \) by

\[
\Psi_\varepsilon* \vec{w}_E(p_\varepsilon) = \vec{w}_\varepsilon*(p_\varepsilon) := d\Psi(p_\varepsilon) \vec{w}_E(p_\varepsilon) \overset{\text{noted}}{=} \Psi_\varepsilon* \vec{w}_E(p_\varepsilon). \tag{10.22}
\]

This definition obtained with (10.21) is not really satisfactory for an unambiguous use in classical mechanics (the bad understanding is made obvious at § 5.2). A satisfactory approach consists in rewriting (10.20) with \( q_\varepsilon = p_\varepsilon + h \vec{w}_E \) (with \( h \) small, e.g., \( h \simeq ||q_\varepsilon - p_\varepsilon||_E \) and \( ||\vec{w}_E||_E \simeq 1 \) as

\[
\Psi(p_\varepsilon + h \vec{w}_E) - \Psi(p_\varepsilon) = h d\Psi(p_\varepsilon) \vec{w}_E + o(h), \tag{10.23}
\]

thus, for all \( \vec{w}_E \in E \) (whatever its length, in particular for all \( \vec{w}_E \) s.t. \( ||\vec{w}_E||_E = 1 \), \( \Psi(p_\varepsilon + h \vec{w}_E) - \Psi(p_\varepsilon) = d\Psi(p_\varepsilon) \vec{w}_E + o(1) \) as \( h \to 0 \). We recover the definition of the differential of \( \Psi \) at \( p_\varepsilon \) in any direction \( \vec{w}_E \):

\[
d\Psi(p_\varepsilon) \vec{w}_E = \lim_{h \to 0} \Psi(p_\varepsilon + h \vec{w}_E) - \Psi(p_\varepsilon) = \vec{w}_\varepsilon*(p_\varepsilon) \quad \text{at} \quad p_F = \Psi(p_\varepsilon). \tag{10.24}
\]

(Recall: The differential \( d\Psi(p_\varepsilon) : E \to F \) is linear.)

Quantification (with components): Let \( (\vec{a}_i) \) and \( (\vec{b}_i) \) be bases in \( E \) and \( F \). Then (10.24) gives

\[
[d\Psi(p_\varepsilon)]_{\vec{b}_i} = [d\Psi(p_\varepsilon)]_{\vec{a}_i} [\vec{w}_E(p_\varepsilon)]_{\vec{a}_i}. \tag{10.25}
\]
10.5.2 Definition of the push-forward of a vector field

To fully grasp the definition 10.11, cf. (10.22) or (10.24), we use an unambiguous definition of a “vector field”: A function made of “tangent vectors to a curve” (this approach gives the push-forward on any manifold, e.g., on any non-planar surface considered alone, that is, as not immersed in an affine space). So, cf. figure 10.1:

- Let \( c_\varepsilon \) be a regular curve in \( U_\varepsilon \), cf. (10.15). The tangent vector at \( \text{Im}(c_\varepsilon) \) at \( p_\varepsilon = c_\varepsilon(s) \) is
  \[
  \vec{w}_\varepsilon(p_\varepsilon) := c_\varepsilon'(s) \quad (= \lim_{h \to 0} \frac{c_\varepsilon(s + h) - c_\varepsilon(s)}{h} \in E). \tag{10.26}
  \]

See figure 10.1. This defines the vector field \( \vec{w}_\varepsilon \) :

\[
\left\{ \begin{array}{l}
U_\varepsilon \to E \\
p_\varepsilon \to \vec{w}_\varepsilon(p_\varepsilon)
\end{array} \right\}
\]

- The push-forward curve by \( \Psi \) is \( c_{\varepsilon*} = \Psi \circ c_\varepsilon \), cf. (10.16) (the curve transformed by \( \Psi \)). Thus the tangent vector at \( \text{Im}(c_{\varepsilon*}) \) at \( p_\varepsilon = c_{\varepsilon*}(s) \) is
  \[
  (\Psi_*\vec{w}_\varepsilon)(p_\varepsilon) = \vec{w}_{c_{\varepsilon*}}(p_\varepsilon) := c_{\varepsilon*}'(s) \quad (= \lim_{h \to 0} \frac{c_{\varepsilon*}(s + h) - c_{\varepsilon*}(s)}{h} \in F). \tag{10.27}
  \]

See figure 10.1. This defines the vector field \( \vec{w}_{c_{\varepsilon*}} \) :

\[
\left\{ \begin{array}{l}
U_\varepsilon \to F \\
p_\varepsilon \to \vec{w}_{c_{\varepsilon*}}(p_\varepsilon)
\end{array} \right\}
\]

- And \( c_{\varepsilon*} = \Psi \circ c_\varepsilon \) gives \( c_{\varepsilon*}'(s) = d\Psi(c_\varepsilon(s)), c_{\varepsilon*}''(s) \) for all \( s \in [\varepsilon, 1] \), thus
  \[
  (\Psi_*\vec{w}_\varepsilon)(p_\varepsilon) = \vec{w}_{c_{\varepsilon*}}(p_\varepsilon) = d\Psi(p_\varepsilon) \cdot \vec{w}_\varepsilon(p_\varepsilon) \quad \text{when} \quad p_\varepsilon = \Psi(p_\varepsilon). \tag{10.28}
  \]

- In particular, with rectilinear curves, we recover (10.24) (definition of the differential).

**Definition 10.12** Let \( \vec{w}_\varepsilon \) :

\[
\left\{ \begin{array}{l}
U_\varepsilon \to E \\
p_\varepsilon \to \vec{w}_\varepsilon(p_\varepsilon)
\end{array} \right\}
\]

be a regular vector field in \( U_\varepsilon \) (so it has integral curves \( c_\varepsilon \)). Its push-forward by \( \Psi \) is the vector field \( \Psi_*\vec{w}_\varepsilon = \vec{w}_{c_{\varepsilon*}} \) :

\[
\left\{ \begin{array}{l}
U_\varepsilon \to F \\
p_\varepsilon \to \vec{w}_{c_{\varepsilon*}}(p_\varepsilon)
\end{array} \right\}
\]

defined by (10.28) (its integral curves are the \( c_{\varepsilon*} = \Psi \circ c_\varepsilon \)). That is,

\[
\Psi_*\vec{w}_\varepsilon = \vec{w}_{c_{\varepsilon*}} := (d\Psi, \vec{w}_\varepsilon) \circ \Psi^{-1}, \tag{10.29}
\]

the notation \( \vec{w}_{c_{\varepsilon*}} \) being used if \( \Psi \) is implicit.

This defines the map \( \Psi_* : \mathcal{F}(U_\varepsilon; E) \to \mathcal{F}(U_\varepsilon; F) \)

\[
\vec{w}_\varepsilon \to \Psi_*\vec{w}_\varepsilon := \vec{w}_{c_{\varepsilon*}} \in \mathcal{F}(U_\varepsilon; F).
\]

(We use the same notation \( \Psi_* \) as in definition 10.4 for scalar valued functions: The context removes ambiguity.)

10.5.3 Interpretation: Essential to continuum mechanics

Continuing example 10.9, see figure 10.1.

Consider a Eulerian vector valued function \( \vec{w} : \mathcal{C} = \bigcup_t \{t\} \times \Omega_t \to \mathbb{R}^n \)

\[
(t, p) \to \vec{w}(t, p)
\]

(cf. (2.2), e.g. a force field. Then, at \( t_0 \) consider \( \vec{w}_{t_0} \) :

\[
\Omega_{t_0} \to \mathbb{R}^n \\
p_{t_0} \to \vec{w}_{t_0}(p_{t_0}) := \vec{w}(t_0, p_{t_0})
\]

(= the force field at \( t_0 \)). And (10.28) reads

\[
\vec{w}_{t_0*}(p_t) = \Phi_t^{t_0}(p_{t_0}) \cdot \vec{w}_{t_0}(p_{t_0}) = \text{the push-forward of} \vec{w}_{t_0}(p_{t_0}) \text{by} \Phi_t^{t_0}, \tag{10.30}
\]

see figure 10.1, with \( \vec{w}_{t_0*}(p_t) \in \mathbb{R}^n \).

**Application:** At \( p_t \) at \( t \), the observer can compare the actual value \( \vec{w}(p_t) \) (e.g. the actual force) with the transported value \( \vec{w}_{t_0*}(p_t) = (\Psi_*\vec{w}_{t_0})(p_t) \) (= the transported memory see next remark 10.13 = the vector that has let itself be deformed by the flow): The difference

\[
\vec{w}(p_t) - \vec{w}_{t_0*}(p_t) \text{ is meaningful at} \ t \text{ at} \ p_t \text{ for one observer} \tag{10.31}
\]

without any ubiquity gift (time or space). This difference gives a measure of the stress, and the rate

\[
\frac{\vec{w}(p_t) - \vec{w}_{t_0*}(p_t)}{h}
\]

defines the Lie derivative, see § 15.5.
Remark 10.13 Unlike scalar functions, cf. §10.3.2: At $t_0$ at $p_{t_0}$ you can’t just draw a vector $\vec{w}_{t_0}(p_{t_0})$ on a piece of paper, put the paper in your pocket, then let yourself be carried by the flow $\Psi = \Phi^t_{p_0}$ (push-forward), then, once arrived at $t$ at $p_t$, take the paper out of your pocket at $t$ at $p_t$ and read it: The direction and the length of the vector have been modified by the flow, see (10.30) and figure 10.1: $\vec{w}_{t_0}(p_t)$ is result of the deformation of $\vec{w}_{t_0}$ by the flow. (A vector is not just a collection of scalar components.)

Exercise 10.14 Prove:
\[ c''_x(s) = d\vec{u}_E(p_E).\vec{w}_E(p_E), \tag{10.32} \]
and
\[ d\vec{w}_E.(p_F).d\Psi(p_E) = d\Psi(p_E).d\vec{w}_E(p_E) + d^2\Psi(p_E).\vec{w}_E(p_E), \tag{10.33} \]
and
\[ c''_E(s) = d\vec{w}_E.(p_F).\vec{w}_E(p_F) = (d\Psi(p_E).c''_E(s) + d^2\Psi(p_E).c'_E(s),c'_E(s)). \tag{10.34} \]

Answer. $c''_E(s) = \vec{w}_E(c_2(s))$ gives $c''_E(s) = d\vec{u}_E(c_2(s)).c'_E(s)$, hence (10.32).

$\vec{w}_E.(\Psi(p_E)) = d\Psi(p_E).\vec{w}_E(p_E)$ by definition of $\vec{w}_E$, hence (10.33).

$c_E(s) = \Psi(c_2(s))$ gives $c''_E(s) = d\Psi(c_2(s)).c'_E(s) = d\Psi(c_2(s)).\vec{w}_E(c_2(s)) = \vec{w}_E(c_2(s)).$ Thus $c''_E(s) = (d^2\Psi(c_2(s)).c'_E(s),c'_E(s) + d\Psi(c_2(s)).c''_E(s) = d\vec{w}_E(c_2(s)).c'_E(s)$, hence (10.34).

10.5.4 Pull-back of a vector field

Definition 10.15 The pull-back is the push-forward by $\Psi^{-1}$. So: Let $\vec{w}_F = \left\{ U_F \rightarrow \Phi \right\} p_F \rightarrow \vec{w}_F(p_F)$ be a vector field on $U_F$. Its pull-back by $\Psi$ is the vector field $\vec{w}_E = \Psi^*\vec{w}_F$ defined on $U_E$ by
\[ \Psi^*\vec{w}_F = \vec{w}_F^*: \left\{ U_E \rightarrow E \right\} p_E \rightarrow \vec{w}_F(p_E) := d\Psi^{-1}(p_E).\vec{w}_F(p_F), \tag{10.35} \]
In other words,
\[ \Psi^*\vec{w}_F = \vec{w}_F^* = (d\Psi^{-1},\vec{w}_F) \circ \Psi. \tag{10.36} \]

Proposition 10.16
\[ \Psi_* \circ \Psi_* = I \quad \text{and} \quad \Psi_* \circ \Psi^* = I, \tag{10.37} \]
that is, $\Psi^*(\Psi_*\vec{w}_E) = \vec{w}_E$ or $\Psi_*\vec{w}_E = (\Psi^*)^{-1}(\vec{w}_E)$, and $\Psi_*(\Psi^*\vec{w}_E) = \vec{w}_E$ or $\Psi^*\vec{w}_E = (\Psi_*)^{-1}(\vec{w}_E)$.

Proof. $\Psi^*(\Psi_*\vec{w}_E)(p_E) = d\Psi^{-1}(p_E).\Psi_*\vec{w}_E(p_F) = d\Psi^{-1}(p_E).d\Psi(p_F).\vec{w}_E(p_E) = \vec{w}_E(p_E)$, thus $\Psi^* \circ \Psi_* : F(U_E; E) \rightarrow F(U_E; E)$ is the identity. Idem for the second equality. (Pull-back = inverse of push-forward).

10.6 Quantification with bases

The presentation of push-forwards were, up to now, qualitative (observer independent). Now we quantify (observer dependent). Let $O_F$ be a origin in $F$ and $(\vec{b}_i)$ be a Cartesian basis in $F$. Let $p_E \in U_E$, let
\[ p_F = \Psi(p_E) = O_F + \sum_{i=1}^{n} \psi^i(p_E) \vec{b}_i, \quad \text{i.e.} \quad [O_F p_F]_{\vec{b}} = \begin{pmatrix} \psi^1(p_E) \\ \vdots \\ \psi^n(p_E) \end{pmatrix}. \tag{10.38} \]

Thus,
\[ d\Psi(p_E) = \sum_{i=1}^{n} \vec{b}_i \otimes d\psi^i(p_E), \tag{10.39} \]

Thus, with the generic notation
\[ \frac{\partial f}{\partial x^j}(p_E) := df(p_E).\vec{a}_j = \lim_{h \rightarrow 0} \frac{f(p_E + \vec{a}_j) - f(p_E)}{h}, \tag{10.40} \]
we get, for all \( j \in [1, n] \),
\[
d\Psi(p_{\vec{c}}) \vec{a}_j = \sum_{i=1}^{n} (d\psi^i(p_{\vec{c}})) \vec{b}_i = \sum_{i=1}^{n} \frac{\partial \psi^i}{\partial x^j}(p_{\vec{c}}) \vec{b}_i, \quad \text{and} \quad [d\Psi(p_{\vec{c}})]_{\vec{a}, \vec{b}} = \frac{\partial \psi^i}{\partial x^j}(p_{\vec{c}}), \tag{10.41}
\]

\([d\Psi(p_{\vec{c}})]_{\vec{a}, \vec{b}}\) being the Jacobian matrix of \( \Psi \) at \( p_{\vec{c}} \) relative to the chosen bases. Thus, for any vector \( \vec{w}_E(p_{\vec{c}}) \in E \) at \( p_{\vec{c}} \), its push-forward is, with \( p_{\vec{x}} = \Psi(p_{\vec{c}}) \),
\[
\vec{w}_E(p_{\vec{x}}) = d\Psi(p_{\vec{c}}) \vec{w}_E(p_{\vec{c}}) = \sum_{i=1}^{n} (d\psi^i(p_{\vec{c}})) \vec{w}_E(p_{\vec{c}}) \vec{b}_i, \quad \text{where} \quad [\vec{w}_E(p_{\vec{x}})]_{\vec{b}} = [d\Psi(p_{\vec{c}})]_{\vec{a}, \vec{b}} [\vec{w}_E(p_{\vec{c}})]_{\vec{a}}. \tag{10.42}
\]

(With tensorial expression that shows the bases is use, \( d\Psi(p_{\vec{c}}) = \sum_{i=1}^{n} \frac{\partial \psi^i}{\partial x^j}(p_{\vec{c}}) \vec{b}_i \otimes a^j \), where \( (a^j) \) is the dual basis of \((\vec{a}_i)\).)

**Example 10.17** Change of coordinate system interpreted as a push-forward: Paradigmatic example of the polar coordinate system. Consider the Cartesian vector space \( \mathbb{R} \times \mathbb{R}^n \) \( \vec{c}(\vec{r}) = \{ \vec{c}(r, \theta) \} \) (parametric space), with its canonical basis \((\vec{A}_1, \vec{A}_2)\), and \( \vec{q} = r \vec{A}_1 + \theta \vec{A}_2 \) named \((r, \theta)\). And consider the geometric affine space \( \mathbb{R}^2 \) (of positions) with a point \( O \) (origin), together with its associated vector space \( \mathbb{R}^2 \) and with a Euclidean basis \((\vec{E}_1, \vec{E}_2)\). Then consider the "polar coordinate system", that is, the map
\[
\vec{r} : \{ \mathbb{R}^2\} \rightarrow \{ \mathbb{R} \times \mathbb{R} \} \rightarrow \mathbb{R}^2,
\]
where
\[
\vec{r}(r, \theta) = \vec{p} = p(\vec{q}) = \Psi(q) = \Psi(r, \theta), \quad \text{and} \quad \vec{O}_p = \vec{x}, \tag{10.43}
\]
\[
\vec{O}_p = \vec{x} = \Psi^1(r, \theta) \vec{E}_1 + \Psi^2(r, \theta) \vec{E}_2 = r \cos \theta \vec{E}_1 + r \sin \theta \vec{E}_2, \tag{10.44}
\]
i.e.,
\[
[\vec{r}]_{\vec{E}} = [\vec{O} \Psi(r,q)]_{\vec{E}} = \begin{pmatrix} \Psi^1(r, \theta) = x = r \cos \theta \\ \Psi^2(r, \theta) = y = r \sin \theta \end{pmatrix}. \tag{10.45}
\]
The need to use \( p \) instead of \( \vec{x} \) appears when a surface in \( \mathbb{R}^3 \) is considered.

The \( i \)-th coordinate line at \( \vec{q} = (r, \theta) \) in the parametric space \( \mathbb{R}^2 \) is the straight line \( \vec{c}_{\vec{q},i} : s \in \mathbb{R} \rightarrow \vec{c}_{\vec{q},i}(s) = \vec{q} + s \vec{A}_i \), so with \( \vec{c}_{\vec{q},i}(0) = \vec{q} \) and its tangent vector at \( \vec{c}_{\vec{q},i}(s) = \vec{c}_{\vec{q},i}'(s) = \vec{A}_i \) (independent of \( s \): Cartesian basis).

This line is transformed by \( \Psi \) into the polar coordinate line at \( p \) which is the curve \( c_{\vec{p},i}(s) = \Psi(q) + s \vec{A}_i \), so \( c_{\vec{p},i}(0) = p \) and \( [c_{\vec{p},i}(s)]_{\vec{E}} = \frac{(r + s \cos \theta)}{(r + s \sin \theta)} \) (straight line) and
\[
[c_{\vec{p},i}(s)]_{\vec{E}} = \begin{pmatrix} r \cos(\theta + s) \\ r \sin(\theta + s) \end{pmatrix} \text{ (circle). And the tangent vector at } c_{\vec{p},i}(s) \text{ is } c_{\vec{p},i}'(s) = \text{named } \vec{a}_{i,p}(p), \text{ thus, with } \vec{O}_{\vec{p}} = \vec{p} = \Psi(q) = \Psi(r, \theta),
\]
\[
\begin{cases}
\vec{a}_{1,p}(p) = d\Psi(q) \vec{A}_1 = \lim_{h \rightarrow 0} \frac{\Psi(q + \vec{A}_1) - \Psi(q)}{h} = \frac{\partial \Psi}{\partial r} \vec{q}, \quad \text{and} \quad [\vec{a}_{1,p}(p)]_{\vec{E}} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} ;
\end{cases}
\]
\[
\begin{cases}
\vec{a}_{2,p}(p) = d\Psi(q) \vec{A}_2 = \lim_{h \rightarrow 0} \frac{\Psi(q + \vec{A}_2) - \Psi(q)}{h} = \frac{\partial \Psi}{\partial \theta} \vec{q}, \quad \text{and} \quad [\vec{a}_{2,p}(p)]_{\vec{E}} = \begin{pmatrix} -r \sin \theta \\ r \cos \theta \end{pmatrix} .
\end{cases} \tag{10.46}
\]
The basis \((\vec{a}_{1,p}(p), \vec{a}_{2,p}(p))\) is the polar coordinate system basis at \( p \). In other words, the Jacobian matrix of \( \Psi \) at \( \vec{q} \) is \([d\Psi(q)]_{\vec{E}} = \begin{pmatrix} [\vec{a}_{1,p}(p)]_{\vec{E}} \\ [\vec{a}_{2,p}(p)]_{\vec{E}} \end{pmatrix} = \begin{pmatrix} [\frac{\partial \Psi}{\partial r} \vec{q}]_{\vec{E}} \\ [\frac{\partial \Psi}{\partial \theta} \vec{q}]_{\vec{E}} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ -r \sin \theta \\ r \cos \theta \end{pmatrix} = \frac{\partial \psi^i}{\partial x^j}(\vec{q}) \text{ when } p = \Psi(q).
\]

Toward derivation: The Christoffel symbols \( \gamma^l_j \) \( p \in \mathbb{R} \) at \( p = \Psi(q) \) are the components of \( d\vec{a}_{i,p}(p) \vec{a}_{i,p}(p) \) in the coordinate basis \((\vec{a}_{k,p}(p))\):
\[
d\vec{a}_{i,p}(p) \vec{a}_{i,p}(p) = \sum_{k=1}^{n} \gamma^l_j \vec{a}_{k,p}(p).
\tag{10.47}
\]
And \( \vec{a}_{i,p}(p) = d\Psi(q) \vec{A}_i \) at \( p = \Psi(q) \) gives \((\vec{a}_{i,p} \circ \Psi)(q) = d\Psi(q) \vec{A}_i \), thus \( d\vec{a}_{i,p}(p) \circ \Psi(q) = d\Psi(q) \vec{A}_i = \vec{E} \).
\[ d^2 \Psi(q)(A_i, A_j) \text{, thus} \]
\[ d\bar{a}_{ij}(p) \bar{a}_{rs}(p) = \frac{\partial^2 \Psi}{\partial q^i \partial q^r}(q), \quad (10.48) \]
thus \( \Psi C^2 \) gives \( \gamma^k_{ij}(p) = \gamma^k_{ij}(p) \) for all \( i, j, k \) (symmetry of the bottom indices for a holonomic basis = the basis of a coordinate system).

For the polar coordinates, \( \frac{\partial^2 \Psi}{\partial q^r \partial q^r}(q) = \cos \theta \hat{E}_1 + \sin \theta \hat{E}_2 \) gives \( \frac{\partial \Psi}{\partial q^r}(q) = 0 \), thus \( \gamma^1_{11} = \gamma^2_{11} = 0 \), and \( \partial^2 \Psi \partial q^r \partial q^r(q) = -\sin \theta \hat{E}_1 - \cos \theta \hat{E}_2 \) gives \( \frac{\partial^2 \Psi}{\partial q^r \partial q^r}(q) = -\cos \theta \hat{E}_1 - \sin \theta \hat{E}_2 = -\hat{e}_1(p, q) \), thus \( \gamma^2_{22} = 0 \).

Application with \( f \) a \( \mathbb{C}^2(\mathbb{R}^n; \mathbb{R}) \) function, when \( f(p) = g(q) \) at \( p = \Psi(q) \), that is, when \( f \) is given by \( (f \circ \Psi)(q) = \text{named } g(q) \). Then \( \frac{\partial f}{\partial q^i}(p) = df(p) \hat{E}_i \) is the usual notation, as well as \( \frac{\partial f}{\partial q^i}(q) = df(q) \hat{A}_i \); Then by definition \( \frac{\partial f}{\partial q^i}(p) := \frac{\partial f}{\partial q^i}(q) \) (recall \( f \) is a function of \( p \), not a function of \( q \), so this is the definition).

Thus, \[
\frac{\partial f}{\partial q^i}(p) = df(q).\bar{A}_i = df(\Psi(q)).\bar{A}_i = df(p).\bar{a}_i(p) = (df.\bar{a}_i)(p). \quad (10.49)
\]

Thus for second order derivation, with \( (df.\bar{a}_i)(p) = (df.\bar{a}_i)(\Psi(q)) \) we get
\[ \frac{\partial \frac{\partial f}{\partial q^i}}{\partial q^j}(p) = \frac{\partial (df.\bar{a}_i)}{\partial q^j}(p).\bar{a}_s(p) = \frac{\partial df}{\partial q^i}(p).\bar{a}_s(p) + \frac{\partial df}{\partial q^i}(p).\bar{a}_s(p).\bar{a}_j(p), \quad (10.50) \]
thus
\[ \frac{\partial^2 f}{\partial q^i \partial q^j}(p) = \partial^2 f(p)(\bar{a}_i(p), \bar{a}_j(p) + \sum_{k=1}^n \frac{\partial f}{\partial q^k}(p) \gamma^k_{ij}(p) \bar{a}_k(p). \quad (10.51) \]

So first order derivatives for \( f \) are still alive: The Christoffel symbols have appeared.

NB: The independent variables \( r \) and \( \theta \) don’t have the same dimension (a length and an angle): There is no physical meaningful inner dot product in the parameter space \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(r, \theta)\} \), but this space is however very useful! (As in thermodynamics: No inner dot product in the \( (T, V) \) space.)

NB: \( \bar{a}_i(p) \) is also noted \( \bar{a}_i(p) \). Here we are in the flat geometric space \( \mathbb{R}^2 \), and a point \( p \) can be spotted thanks to a bi-point vector \( \bar{x} = \bar{O}_p \).

NB: The normalized basis \( (\bar{a}_i(p), \frac{1}{\sqrt{2}} \bar{a}_2(p)) \) is not holonomic, that is, it is not the basis of a coordinate system, and its use makes derivations quite complicated (no symmetry for the Christoffel symbols).

11 Homogeneous and isotropic material

Framework: linear modelization and elastic materials.

Let \( P \in \Omega_t \) and suppose that the “Cauchy stress vector” \( \bar{f}_i(p) \) at \( p = \Phi^p(P) \) only depends on \( P \) and on \( F_t^{0}(P) = d\Phi^t_0(P) \) is the differential of the motion, or “first (covariant) gradient”, that is,
\[ \bar{f}_i(p) = \text{function}(P, F_t^{0}(P)). \quad (11.1) \]

**Definition 11.1** A material is homogeneous if function doesn’t depend on the first variable \( P \), i.e., iff, for all \( P \in \Omega_t \),
\[ (\bar{f}_i(p) = \text{function}(P, F_t^{0}(P)) = \text{function}(F_t^{0}(P)). \quad (11.2) \]

**Definition 11.2** (Isotropy.) Consider an inner dot product, the same at any time. A material is isotropic at a point \( P \in \Omega_t \) iff function doesn’t depend on the direction you consider, i.e., iff, for any rotation \( R_{t_0}(P) \),
\[ (\bar{f}_i(p) = \text{function}(P, F_t^{0}(P)) = \text{function}(P, R_{t_0}(P), F_t^{0}(P)). \quad (11.3) \]

**Definition 11.3** A material is isotropic homogeneous iff, for all \( P \in \Omega_t \) and for all rotation \( R_{t_0}(P) \):
\[ (\bar{f}_i(p) = \text{function}(P, F_t^{0}(P), R_{t_0}(P)) = \text{function}(F_t^{0}(P)). \quad (11.4) \]
12 The inverse of the deformation gradient

12.1 Definition of $H = F^{-1}$

Let $t_0, t \in \mathbb{R}$, $\Phi^{t_0}_t : \begin{cases} \Omega_{t_0} \to \Omega_t \\ \{ p \} \to \Phi^{t_0}_t(p) \end{cases}$ and $F^{t_0}_t = d\Phi^{t_0}_t : \begin{cases} \tilde{\Omega}_{t_0} \to \tilde{\Omega}_t \\ \tilde{W}(P) \to \tilde{w}(P) = F^{t_0}_t(P).\tilde{W}(P) \end{cases}$ With $p = \Phi^{t_0}_t(P)$ and $((\Phi^{t_0}_t)^{-1} \circ \Phi^{t_0}_t)(P) = P$ we get

$$d(\Phi^{t_0}_t)^{-1}(p).d\Phi^{t_0}_t(P) = I_{t_0}, \text{ thus } d(\Phi^{t_0}_t)^{-1}(p) = d\Phi(P)^{-1}. \tag{12.1}$$

Thus

$$H^{t_0}_t := (F^{t_0}_t)^{-1} : \begin{cases} \Omega_t \to L(\tilde{\Omega}_t; \tilde{\Omega}_{t_0}) \\ p \to H^{t_0}_t(p) := (F^{t_0}_t)^{-1}(p) = F^{t_0}_t(p) \text{ when } p = \Phi^{t_0}_t(P). \tag{12.2} \end{cases}$$

(And $H^{t_0}_t$ is a two point tensor.)

Then let

$$H^{t_0}_t : \begin{cases} \mathcal{C} = \bigcup_t \{(t) \times \Omega_t \} \to L(\tilde{\Omega}_t; \tilde{\Omega}_{t_0}) \\ (t, p_t) \to H^{t_0}_t(p_t) := H^{t_0}_t(p_t) = (F^{t_0}_t,t_0) \text{ when } p_t = \Phi^{t_0}_t(t_0). \tag{12.3} \end{cases}$$

(12.7) time derivative $g'(t)$ of the function $g : t \to g(t) = H^{t_0}_t(t, \Phi^{t_0}_t(t, P)) = H^{t_0}_t(t, \tilde{F}(t, P_{\text{Obj}})).$

**Exercise 12.1** Prove that $\frac{D^{t_0}_t}{D t}(t, p(t)) = H^{t_0}_t(p(t)). \frac{D^{t_1}_t}{D t}(t, p(t)).$

**Answer:** (6.18) gives $p(t) = \Phi^{t_0}_t(p_0) = \Phi^{t_0}_t(\Phi^{t_0}_t(p_0))$, and let $p_t = \Phi^{t_0}_t(p_0)$.

Hence $F^{t_0}_t(p_0) = F^{t_0}_t(p_1), \Phi^{t_0}_t(p_0)$, thus $F^{t_0}_t(p_0) = F^{t_0}_t(p_1), \Phi^{t_0}_t(p_0) = F^{t_0}_t(p_1), \Phi^{t_0}_t(p_0),$ thus $H^{t_0}_t(p_0) = H^{t_0}_t(p_1), \Phi^{t_0}_t(p_0),$ that is, $H^{t_0}_t(t, p(t)) = H^{t_0}_t(t, p(t), H^{t_0}_t(t, p(t))).$ Thus the result. \(\blacksquare\)

12.2 Time derivatives of $H$

Let $p(t) = \Phi^{t_0}_t(t, P).$ Since $H^{t_0}_t(t, p(t), F^{t_0}_t(t, P) = I_{t_0}$, we get, with the shortened notation $H.F = I,$

$$\frac{D H}{D t} = I + H.\frac{D F}{D t} = 0, \text{ and } \frac{D^2 H}{D t^2} = I + H.\frac{D^2 F}{D t^2} = 0. \tag{12.6}$$

(Full notations: $\frac{D^2 H}{D t^2}(t, p(t)). F^{t_0}_t(t, P) + \frac{D H}{D t}(t, p(t)). \frac{D^2 F}{D t^2}(t, P) = 0,$ and

$$\frac{D^2 H}{D t^2}(t, p(t)). F^{t_0}_t(t, P) + \frac{D H}{D t}(t, p(t)). \frac{D^2 F}{D t^2}(t, P) = 0.)$$

Thus, with (5.39),

$$\frac{D H}{D t} = -H.\frac{D F}{D t}, \tag{12.7}$$

Thus, with (5.43):

$$\frac{D^2 H}{D t^2} = -(\frac{D H}{D t} + H.\frac{D F}{D t} = H.d\tilde{v}.d\tilde{v} - H.\frac{D(d\tilde{v})}{D t} = -H.\frac{D(d\tilde{v})}{D t}. \tag{12.8}$$

So the second order Taylor expansion in time (along a trajectory) is

$$H^{t_0}_t(t+h, p(t+h)) = H^{t_0}_t(I - h \tilde{v} - h^2 \frac{D(d\tilde{v})}{D t}(t, p(t)) + o(h^2)$$

$$= H^{t_0}_t(I - h \tilde{v} - h^2 (d\tilde{v} - d\tilde{v}.d\tilde{v}))(t, p(t)) + o(h^2). \tag{12.9}$$

In particular:

$$H^{t_0}_t(t_0+h, p(t_0+h)) = (I - h \tilde{v} - h^2 (d\tilde{v} - d\tilde{v}.d\tilde{v}))(t_0, p_0) + o(h^2). \tag{12.10}$$
Exercise 12.2 Check that (12.9) and (5.41) give \( F^{-1}F = I \), up to the second order.

Answer.

\[
F^h(t+h)^{-1}F^h(t+h) = [F^{-1}(I - hDv - h^2/2 (d\gamma - dv.dv)) + o(h^2)](I + hDv + h^2/2 d\gamma).F + o(h^2)]
\]

\[
= F^{-1}(I + h(d\gamma - dv) + h^2/2 (d\gamma - dv.dv) + o(h^2)).F + I + o(h^2).
\]

Exercise 12.3 Prove (12.7), using the derivation of \( H^\alpha(t, p(t)) \delta w_\alpha(t, p(t)) = \tilde{W}(P) \) when \( p(t) = \Phi^\alpha(t, P) \) and \( \delta w_\alpha(t, p(t)) = F^\alpha(t, P), \tilde{W}(P) \).

Answer. \( \frac{dH^\alpha}{dt} \delta w_\alpha + H^\alpha \frac{dH^\alpha}{dt} \delta w_\alpha = 0 \), and \( \frac{dH^\alpha}{dp} \delta w_\alpha \delta w_\alpha = \frac{\partial H^\alpha}{\partial p} \tilde{W}(P) = d\gamma(t, p(t)).F^\alpha(t, P), \tilde{W}(P) = d\gamma(t, p(t)).\tilde{w}_\alpha(t, p(t)), \) thus \( \frac{dH^\alpha}{dt} \delta w_\alpha + H^\alpha d\gamma.\tilde{w}_\alpha = 0 \), thus \( \frac{dH}{dt} = -H.d\gamma \).

13 Push-forward and pull-back of differential forms

13.1 Definition

Setting of § 10.1: \( U_\mathcal{E} \) is an open set in the affine space \( \mathcal{E} \) associated to the normed vector space \( E \).

Let \( \alpha_\mathcal{E} : \left\{ \begin{array}{l}
U_\mathcal{E} \to E^* = \mathcal{L}(E; \mathbb{R}) \\
p_\mathcal{E} \to \alpha_\mathcal{E}(p_\mathcal{E})
\end{array} \right. \) be a differential form on \( U_\mathcal{E} \) (a field of linear forms), that is \( \alpha_\mathcal{E} \in \Omega^1(U_\mathcal{E}) = T^*_1(U_\mathcal{E}) \), cf. (R.11). If \( \tilde{w}_\mathcal{E} : \left\{ \begin{array}{l}
U_\mathcal{E} \to E \\
p_\mathcal{E} \to \tilde{w}_\mathcal{E}(p_\mathcal{E})
\end{array} \right. \in \Gamma(U_\mathcal{E}) \) (a field of vectors in \( U_\mathcal{E} \)) then

\[
f_\mathcal{E} = \alpha_\mathcal{E}.\tilde{w}_\mathcal{E} : \left\{ \begin{array}{l}
U_\mathcal{E} \to \mathbb{R} \\
p_\mathcal{E} \to f_\mathcal{E}(p_\mathcal{E}) = (\alpha_\mathcal{E}.\tilde{w}_\mathcal{E})(p_\mathcal{E}) = \alpha_\mathcal{E}(p_\mathcal{E}).\tilde{w}_\mathcal{E}(p_\mathcal{E})
\end{array} \right.
\]
is a scalar function which gives the value of \( \alpha_\mathcal{E} \) on \( \tilde{w}_\mathcal{E} \) (a field of scalar functions which gives the measure of \( \tilde{w}_\mathcal{E} \) by \( \alpha_\mathcal{E} \)).

Consider a diffeomorphism \( \Psi \), cf. (10.1). The push-forward of \( f_\mathcal{E} = \alpha_\mathcal{E}.\tilde{w}_\mathcal{E} \) by \( \Psi \) is \( f_\mathcal{E} = \Psi_*(f_\mathcal{E}) = \Psi_*(\alpha_\mathcal{E}.\tilde{w}_\mathcal{E}) \), and is given by, with \( p_\mathcal{F} = \Psi(p_\mathcal{E}) \), cf. (10.5),

\[
\Psi_*(\alpha_\mathcal{E}.\tilde{w}_\mathcal{E})(p_\mathcal{F}) = (\alpha_\mathcal{E}.\tilde{w}_\mathcal{E})(p_\mathcal{E}) = \alpha_\mathcal{E}(p_\mathcal{E}).\tilde{w}_\mathcal{E}(p_\mathcal{E}).
\]

(13.1)

Since the push-forward of a vector field \( \tilde{w}_\mathcal{E} \) is given by \( \tilde{w}_\mathcal{E} = d\Psi(p_\mathcal{E}).\tilde{w}_\mathcal{F}(p_\mathcal{E}) \), cf. (10.28), we get

\[
\Psi_*(\alpha_\mathcal{E}.\tilde{w}_\mathcal{E})(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).d\Psi(p_\mathcal{E})^{-1}.\tilde{w}_\mathcal{F}(p_\mathcal{F}),
\]

(13.2)

so (compatibility hypothesis):

**Definition 13.1** Let \( \alpha_\mathcal{E} \in \Omega^1(U_\mathcal{E}) \) a differential form (a regular field of linear forms on \( U_\mathcal{E} \)). Its push-forward by \( \Psi \) is the differential form \( \alpha_\mathcal{F} \in \Omega^1(U_\mathcal{F}) \) defined by, with \( p_\mathcal{F} = \Psi(p_\mathcal{E}) \),

\[
\alpha_\mathcal{F}(p_\mathcal{F}) := \alpha_\mathcal{E}(p_\mathcal{E}).d\Psi(p_\mathcal{E})^{-1} \text{ (hamed)} = (\Psi_*(\alpha_\mathcal{E}))(p_\mathcal{F}).
\]

(13.3)

In other words,

\[
\alpha_\mathcal{F} := (\alpha_\mathcal{E} \circ \Psi^{-1}).d\Psi^{-1} \text{ (hamed)} = (\Psi_*(\alpha_\mathcal{E})).
\]

(13.4)

(See also remark 13.6.)

**Proposition 13.2** For all \( \alpha_\mathcal{E} \in \Omega^1(U_\mathcal{E}) \), all \( \alpha_\mathcal{F} \in \Omega^1(U_\mathcal{F}) \), all \( \tilde{w}_\mathcal{F} \in \Gamma(U_\mathcal{F}) \), all \( \tilde{w}_\mathcal{E} \in \Gamma(U_\mathcal{E}) \), in short

\[
\alpha_\mathcal{F}.\tilde{w}_\mathcal{F} = \alpha_\mathcal{E}.\tilde{w}_\mathcal{E}^*, \text{ and } \alpha_\mathcal{F}.\tilde{w}_\mathcal{E} = \alpha_\mathcal{E}.\tilde{w}_\mathcal{F}^*.
\]

(13.5)

**Full notation:** \( \alpha_\mathcal{F}(p_\mathcal{F}).\tilde{w}_\mathcal{F}(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).\tilde{w}_\mathcal{E}^*(p_\mathcal{E}) \), and \( \alpha_\mathcal{F}^*(p_\mathcal{F}).\tilde{w}_\mathcal{E}(p_\mathcal{F}) = \alpha_\mathcal{E}(p_\mathcal{E}).\tilde{w}_\mathcal{F}^* (p_\mathcal{F}), \) for all \( p_\mathcal{F} = \Psi(p_\mathcal{E}) \in U_\mathcal{F} \).
**Proposition 13.3** The differential of a scalar function commutes with the push-forward, that is, if \( f \in C^1(U_E; \mathbb{R}) \) then

\[
d(\Psi_\ast f) = \Psi_\ast (df),
\]

that is, \( d(\Psi_\ast f)(p_F) = \Psi_\ast (df)(p_F) \) for all \( p_F \in U_F \).

**Proof.** Let \( \Psi = \Psi(p_E) \). We have \( \Psi_\ast f(p_F) = f(p_E) = f(\Psi^{-1}(p_F)) \), cf. (10.5). Thus \( d(\Psi_\ast f)(p_F) = df(p_E).d\Psi^{-1}(p_F) \).

And (13.3) gives \( (\Psi_\ast (df))(p_F) = df(p_E).d\Psi(p_E)^{-1} \). With \( d\Psi(p_E)^{-1} = d\Psi^{-1}(p_F) \) we get (13.6). \( \blacksquare \)

**Definition 13.4** Let \( \alpha_F : U_F \to F^* \). Its pull-back by \( \Psi \) is the differential form \( \alpha_F^* = \Psi^* \alpha_F : U_E \to E^* \) defined by, with \( p_E = \Psi^{-1}(p_F) \),

\[
\alpha_F^*(p_E) = \Psi^* \alpha_F(p_E) := \alpha_F(p_F).d\Psi(p_E).
\]

In other words,

\[
\alpha_F^* = \Psi^* \alpha_F := (\alpha_F \circ \Psi).d\Psi.
\]

(See also remark 13.6.)

**Proposition 13.5**

\[
\Psi^* \circ \Psi_\ast = I \quad \text{and} \quad \Psi_\ast \circ \Psi^* = I.
\]

**Proof.** \( \Psi^*(\Psi_\ast \alpha_E)(p_E) = \Psi_\ast \alpha_E(p_F).d\Psi(p_E) = \alpha_E(p_E).d\Psi^{-1}(p_F).d\Psi(p_E) = \alpha_E(p_E).d\Psi(p_E) = \alpha_E(p_F) \).

\[
\Psi_\ast (\Psi^* \alpha_F)(p_E) = \Psi_\ast \alpha_F(p_E).d\Psi^{-1}(p_F).d\Psi(p_E) = \alpha_F(p_F).d\Psi(p_E).d\Psi^{-1}(p_F) = \alpha_F(p_F).
\]

\( \blacksquare \)

**Remark 13.6** The pull-back \( \Psi^* \alpha_F \) can also be defined thanks to the natural canonical isomorphism

\[
\left\{ \begin{array}{c}
L(E; F) \to L(F^* ; E^*) \\
L \to L^*
\end{array} \right\}
\]

given by \( L^*(\ell_F).\bar{\alpha}_E = \ell_F.(L.\bar{\alpha}_E) \) for all \( \bar{\alpha}_E \in E \). That is, \( L^*(\ell_F) = \ell_F.L \).

And \( L^*(\ell_F) \) is called the pull-back of \( \ell_F \) by \( L \). In particular, if \( \ell_F = \alpha_F(p_F) \) and \( L = d\Psi(p_E) \) then \( d\Psi(p_E)^*(\alpha_F(p_F)) = \alpha_F(p_F).d\Psi(p_E) \), that is (13.7). And this approach gives: \( \alpha_F^* = \Psi^* \alpha_F \) is “covariant objective” (independent of any observer).

And, if \( \Psi \) is invertible, then the push-forward is the pull-back considered with \( \Psi^{-1} \) instead of \( \Psi \). \( \blacksquare \)

### 13.2 Incompatibility: Riesz representation and push-forward

A push-forward is independent of any inner dot product. Now we quantify with inner dot products (observer dependent). Let \( \langle \cdot, \cdot \rangle_g \) be an inner dot product in \( E \), and let \( \langle \cdot, \cdot \rangle_h \) be an inner dot product in \( F \).

Let \( \alpha_E \in \Omega^1(U_E) \) and \( \alpha_F \in \Omega^1(U_F) \). Thus \( \alpha_E(p_E) \in E^* \) for all \( p_E \in U_E \), and \( \alpha_F(p_F) \in F^* \) for all \( p_F \in U_F \).

Let \( \bar{\alpha}_E(g)(p_E) \in E \) be the \( \langle \cdot, \cdot \rangle_g \) Riesz representation vector of \( \alpha_E(p_E) \), and let \( \bar{\alpha}_F(h)(p_F) \in F \) be the \( \langle \cdot, \cdot \rangle_h \) Riesz representation vector of \( \alpha_F(p_F) \), that is, for all \( \bar{\alpha}_E(p_E) \in E \) and all \( \bar{\alpha}_F(p_F) \in F \),

\[
\alpha_E(p_E).\bar{\alpha}_E(p_E) = (\bar{\alpha}_E(g)(p_E), \bar{\alpha}_E(g)(p_E))_g, \quad \text{and} \quad \alpha_F(p_F).\bar{\alpha}_F(p_F) = (\bar{\alpha}_F(h)(p_F), \bar{\alpha}_F(h)(p_F))_h.
\]

This defines the vector fields \( \bar{\alpha}_E(g) \) in \( U_E \) and \( \bar{\alpha}_F(h) \) in \( U_F \) (dependent on \( \langle \cdot, \cdot \rangle_g \) and \( \langle \cdot, \cdot \rangle_h \)).

Recall: The transposed relative to \( \langle \cdot, \cdot \rangle_g \) and \( \langle \cdot, \cdot \rangle_h \) of the linear map \( d\Psi(p_E) \in L(E; F) \) is the linear map \( d\Psi(p_E)^T \) in \( L(F; E) \) defined by

\[
(d\Psi(p_E)^T)_{gh} = (\bar{\alpha}_F(h)(p_F), \bar{\alpha}_F(h)(p_F))_h
\]

for all \( \bar{\alpha}_E \in E \) and \( \bar{\alpha}_F \in F \), cf. (A.24).
Proposition 13.7 If \( \alpha_F = \Psi_*, \alpha_E \) (= the push-forward of \( \alpha_E \) by \( \Psi \)), then

\[
\tilde{\alpha}_{Fh}(p_F) = d\Psi(p_F)^{-T}\tilde{\alpha}_{Eg}(p_E), \quad \text{so} \quad \tilde{\alpha}_{Fh} \neq \Psi_*\tilde{\alpha}_{Eg} \text{ in general,}
\]

(13.12)
that is, \( \tilde{\alpha}_{Fh} \) is not the push-forward of \( \tilde{\alpha}_{Eg} \) unless \( d\Psi(p_E)^{-T} = d\Psi(p_F), \) i.e., unless \( d\Psi(p_E)^{-T} = I_E \) (a "rigid like body motion").

So the Riesz representation vector of the push-forwarded linear form is not the push-forwarded representative vector of the linear form push-forwarded. (This is expected: a push-forward is independent of any inner dot product, when a Riesz representation vector has no existence without an inner dot product.)

Proof. \( p_F = \Psi(p_E) \) and \( \alpha_F = \Psi_*, \alpha_E \) give \( \alpha_F(p_F) = \alpha_E(p_E).d\Psi^{-1}(p_F) \), cf. (13.3). And (13.10) gives

\[
(\tilde{\alpha}_{Fh}(p_F), \bar{w}_F(p_F))_h = \alpha_F(p_F).\bar{w}_F(p_F) = (\alpha_E(p_E).d\Psi(p_E)^{-1}).\bar{w}_F(p_F) = \alpha_E(p_E).d\Psi(p_E)^{-1}.\bar{w}_F(p_F)
\]

\[
= (\tilde{\alpha}_{Eg}(p_E), d\Psi(p_E)^{-1}.\bar{w}_F(p_F))_g = (d\Psi(p_E)^{-T}.\tilde{\alpha}_{Eg}(p_E), \bar{w}_F(p_F))_h.
\]

(13.13)

True for all \( \bar{w}_F \), hence (13.12).

So it is better to avoid using a representation vector (not intrinsic to the differential form) when using push-forwards, cf. (B.8). That is, it is better to consider the original (the differential form) rather than a representative (a vector of representation: Which one, cf. (B.8)?)

Reminder: There is no natural canonical isomorphism between \( E \) and \( E^* \), see § T.2, and covariance cannot be confused with contravariance.

14 Push-forward and pull-back of tensors

To lighten the presentation, we deal with order 1 and 2 tensors. Similar approach for any tensor.

14.1 Push-forward and pull-back of order 1 tensors

We write (10.28) with \( T \simeq \bar{w}_E \in T_0^1(U_E) \simeq \Gamma(U_E) \) and (13.3) with \( T = \alpha \in T_0^1(U_E) \) as, in short,

\[
(T_* =) \quad \Psi_*(T) = T(\Psi^*),
\]

(14.1)
that is, for all \( \xi \) a vector field or a differential form in \( U_F \) when required, in short,

\[
(\Psi_*T)(\xi) = T(\Psi^*\xi),
\]

(14.2)
that is \( (\Psi_*T)(p_F)(\xi(p_F)) = T(p_E)(\Psi^*\xi(p_E)) \) when \( p_F = \Psi(p_E) \). Indeed:

- If \( T = \alpha_E \in T_0^1(U_E) = \Omega^1(U_E) \) then \( \xi = \bar{w}_F \in \Gamma(U_F) \) gives, \( (\Psi_*\alpha_E)(p_F) \) and \( \alpha_E(p_E) \) being linear,

\[
(\Psi_*\alpha_E)(p_F).\bar{w}_F(p_F)^{(14.2)} = \alpha_E(p_E).\Psi^*\bar{w}_F(p_E)^{(10.35)} = \alpha_E(p_E).d\Psi^{-1}(p_E).\bar{w}_F(p_F).
\]

(14.3)
True for all \( \bar{w}_F \): We do get \( \Psi_*\alpha_E(p_F) = \alpha_E(p_E).d\Psi^{-1}(p_E) \), i.e. (13.3).

- If \( T \in T_0^1(U_E) \), then \( \xi = \alpha_E \in \Omega^1(U_E) \) gives

\[
(\Psi_*T)(p_F)(\alpha_F(p_F))^{(14.2)} = T(p_E)(\Psi^*\alpha_F(p_E))^{(13.7)} = T(p_E)(\alpha_F(p_F).d\Psi(p_E)),
\]

(14.4)
and the natural canonical isomorphism \( \mathcal{J} : \{ \begin{array}{c} E \\ \bar{w} \end{array} \rightarrow E^{**} \bar{w} \rightarrow L_{\bar{w}} \} \) defined by \( L_{\bar{w}}(\ell) = \ell.\bar{w} \) for all \( \ell \in E^* \),

\[
\text{cf. (Q.11), gives}
\]

\[
\alpha_F(p_F).\Psi^*\bar{w}_E(p_F) = \Psi^*\alpha_E(p_E).\bar{w}_E(p_E) = \alpha_F(p_F).d\Psi(p_E).\bar{w}_E(p_E).
\]

(14.5)
True for all \( \alpha_F \): We do get \( \Psi_*\bar{w}_E(p_F) = d\Psi(p_E).\bar{w}_E(p_E) \), i.e. (10.28).

Idem for pull-backs:

\[
(T^* =) \quad \Psi^*(T) = T(\Psi_*),
\]

(14.6)
that is \( \Psi^*(p_E)(\beta_E(p_E)) = T(p_F)(\Psi_*\beta_E(p_F)) \) when \( p_F = \Psi(p_E) \).
14.2 Push-forward and pull-back of order 2 tensors

Definition 14.1 Let $T$ be an order 2 tensor on $U_E$. Its push-forward by $\Psi$ is the order 2 tensor on $U_{F}$ defined by, in short,

$$T_{*}(\cdot, \cdot) := \Psi_{*} T(\Psi^{*} \cdot, \Psi^{*} \cdot).$$

(14.7)

Full notation: $T_{*}(p_F)(\xi_1(p_F), \xi_2(p_F)) := T(p_E)(\xi_1^{*}(p_E), \xi_2^{*}(p_E))$ when $p_F = \Psi(p_E)$, for all $\xi_1, \xi_2$ vector fields or differential forms in $U_F$ as required.

Let $T$ be an order 2 tensor on $U_E$. Its pull-back by $\Psi$ is the order 2 tensor on $U_E$ defined by,

$$(T^{*}(\cdot, \cdot) := \Psi^{*} T(\cdot, \cdot) = T(\Psi^{*}, \Psi^{*}).$$

(14.8)

Full notation: $T^{*}(p_F)(\xi_1(p_F), \xi_2(p_F)) := T(p_E)(\xi_1^{*}(p_E), \xi_2^{*}(p_E))$ when $p_F = \Psi(p_E)$, for all $\xi_1, \xi_2$ vector fields or differential forms in $U_F$ as required.

(Definitions generalized to any $r s$ tensors.)

Example 14.2 For an elementary tensor $T = \alpha_1 \otimes \alpha_2 \in \mathcal{A}^{1,2}(U_F)$, made of differential forms $\alpha_1, \alpha_2 \in \Omega^1(U_F)$, for all $\vec{w}_1, \vec{w}_2 \in \Gamma(U_E)$, in short,

$$(\alpha_1 \otimes \alpha_2)_{*}(\vec{w}_1, \vec{w}_2) = (\alpha_1 \otimes \alpha_2)(\vec{w}_1^{*}, \vec{w}_2^{*}) = (\alpha_1, \vec{w}_1^{*})(\alpha_2, \vec{w}_2^{*}) = (\alpha_1 \otimes \alpha_2)(\vec{w}_1^{*}, \vec{w}_2^{*}),$$

(14.9)

thus

$$(\alpha_1 \otimes \alpha_2)_{*} = \alpha_{1*} \otimes \alpha_{2*}.$$  (14.10)

(And any tensor is a finite sum of elementary tensors.)

More generally, if $T \in \mathcal{A}^{r,s}(U_F)$ then for all vector fields $\vec{w}_1, \vec{w}_2$ in $U_{F}$, in short,

$$T_{*}(\vec{w}_1, \vec{w}_2) = T(\vec{w}_1^{*}, \vec{w}_2^{*}) = (T(\vec{w}_1^{*}, \vec{w}_2^{*}) = (T^{*}(\vec{w}_1, \vec{w}_2)),$$

(14.11)

Full notation: $T_{*}(p_F)(\vec{w}_1(p_F), \vec{w}_2(p_F)) = T(p_E)(\vec{w}_1(p_E), \vec{w}_2(p_E), \vec{w}_1^{*}(p_E), \vec{w}_2^{*}(p_E))$ when $p_F = \Psi(p_E)$.

Expression with bases $(\vec{a})$ in $E$ and $(\vec{b})$ in $F$, in short: $T_{*}(\vec{w}_1, \vec{w}_2) = [\vec{w}_1]_{\vec{b}}^{\vec{a}}[T_{*}]_{\vec{a}}^{\vec{b}}[\vec{w}_2]_{\vec{b}}$ and

$$[T_{*}]_{\vec{a}}^{\vec{b}} = [T]_{\vec{a}}^{\vec{b}}, [\vec{w}_1]_{\vec{b}}^{\vec{a}} = [\vec{w}_1]_{\vec{b}}^{\vec{a}}, [\vec{w}^{*}]_{\vec{b}}^{\vec{a}} = [\vec{w}^{*}]_{\vec{b}}^{\vec{a}}, [\psi]_{\vec{b}}^{\vec{a}}, [\psi]_{\vec{b}}^{\vec{a}}$, thus, in short,

$$[T_{*}]_{\vec{a}}^{\vec{b}} = [\psi]^{T*} [T]_{\vec{a}}^{\vec{b}}, [\psi]^{T*} [\psi^{T*}]_{\vec{a}}^{\vec{b}}, [\psi^{T*}]_{\vec{a}}^{\vec{b}}.$$  (14.12)

Full notation: $[(\Psi^{*}, T)(p_F)]_{\vec{b}}^{\vec{a}} = [\psi]^{T*} [T]_{\vec{a}}^{\vec{b}}, [\psi]^{T*} [\psi^{T*}]_{\vec{a}}^{\vec{b}}$, when $p_F = \Psi(p_E)$.

And for the pull-back: For all vector fields $\vec{w}_1, \vec{w}_2$ in $U_E$,

$$T^{*}(\vec{w}_1, \vec{w}_2) = T^{*}(\vec{w}_1^{*}, \vec{w}_2^{*}) = T^{*}(\vec{w}_1^{*}, \vec{w}_2^{*}).$$

(14.13)

Example 14.3 For the elementary tensor $T = \vec{u} \otimes \alpha \in \mathcal{A}^{1,1}(U_F)$, made of the vector field $\vec{u} \in \Gamma(U_F)$ and of the differential form $\alpha \in \Omega^1(U_F)$, for all $\beta, \vec{w} \in \Gamma(U_F)$, in short,

$$(\vec{u} \otimes \alpha)_{*}(\beta, \vec{w}) = (\vec{u} \otimes \alpha)(\beta^{*}, \vec{w}^{*}) = (\vec{u}, \beta^{*})(\alpha, \vec{w}^{*}) = (\vec{u} \otimes \alpha)(\beta^{*}, \vec{w}^{*}) = (\vec{u} \otimes \alpha)(\beta^{*}, \vec{w}^{*}),$$

(14.14)

thus

$$(\vec{u} \otimes \alpha)_{*} = \vec{u}_{*} \otimes \alpha_{*}.$$  (14.15)

More generally, if $T \in \mathcal{A}^{r,1}(U_F)$ then for all vector fields $\vec{w} \in \Gamma(U_F)$ and differential forms $\beta \in \Omega^1(U_F)$,

$$T_{*}(\beta, \vec{w}) = T^{*}(\beta^{*}, \vec{w}^{*}) = T^{*}(\beta \otimes \vec{w}^{*}).$$

(14.16)

Full notation: With $p_F = \Psi(p_E)$, $T_{*}(p_F)(\beta(p_F), \vec{w}(p_F)) = T(p_E)(\beta(p_E), \vec{w}(p_E), \beta^{*}(p_E), \vec{w}^{*}(p_E))$.

Expression with bases $(\vec{a})$ in $E$ and $(\vec{b})$ in $F$, in short: $T_{*}(\beta, \vec{w}) = [\beta]_{\vec{b}}^{\vec{a}}[T_{*}]_{\vec{a}}^{\vec{b}}[\vec{w}]_{\vec{b}}$ and $T_{*}(\beta, \vec{w}) = T(\beta^{*}, \vec{w}^{*}) = [\beta]_{\vec{b}}^{\vec{a}}[T]_{\vec{a}}^{\vec{b}}[\vec{w}^{*}]_{\vec{b}}$, thus

$$[T_{*}]_{\vec{a}}^{\vec{b}} = [\psi]^{T*} [T]_{\vec{a}}^{\vec{b}}, [\psi]^{T*} [\psi^{T*}]_{\vec{a}}^{\vec{b}}, [\psi^{T*}]_{\vec{a}}^{\vec{b}}.$$  (14.17)

Full notation: $[(\Psi^{*}, T)(p_F)]_{\vec{b}}^{\vec{a}} = [\psi]^{T*} [T]_{\vec{a}}^{\vec{b}}, [\psi]^{T*} [\psi^{T*}]_{\vec{a}}^{\vec{b}}$, when $p_F = \Psi(p_E)$.
14.3 Push-forward and pull-back of endomorphisms

Consider the natural canonical isomorphism, cf. (Q.44),

$$J_2 : \begin{cases} \mathcal{L}(E; E^*) \to \mathcal{L}(E^*, E; \mathbb{R}) \\ L \to T_L = J_2(L) \end{cases} \quad \text{where} \quad T_L(\alpha, \vec{u}) := \alpha.\vec{L} \vec{u}, \quad \forall (\alpha, \vec{u}) \in E^* \times E. \quad (14.18)$$

**Definition 14.4** The push-forward by \( \Psi \) of a field of endomorphisms \( L \) on \( U_{E} \) is the field of endomorphisms \( \Psi_*L = L_* \) on \( U_{F} \) defined by

$$\Psi_*L = \left( J_2 \right)^{-1}(((T_L)_*)). \quad (14.19)$$

So, in short,

$$\Psi_*L = \left( L_* = d\Psi_*L.d\Psi^{-1} \right) \quad (14.20)$$

Full notation:

$$L_*(p_{F}) = d\Psi(p_{E}).L(p_{E}).d\Psi^{-1}(p_{F}) \quad \text{when} \quad p_{F} = \Psi(p_{E}).$$

Indeed, for all \((\beta, \vec{w}) \in \Omega^1(U_{F}) \times \Gamma(U_{F})\),

$$(T_L)_*(\beta, \vec{w}) = (T_L(\beta.\vec{L} \vec{w}, \vec{L} \vec{w})). \quad (14.16)$$

And with bases we get \([L_*]_{\vec{b}} = [d\Psi]_{\vec{a}, \vec{b}}[L]_{\vec{a}, \vec{b}}^{-1} \vec{a} \vec{b} \text{ “as in (14.17)”.} \]

**Example 14.5** Elementary field of endomorphisms \( L = (J_2)^{-1}(\vec{u} \odot \alpha) \), i.e. \( L_{\vec{a}} = (\alpha.\vec{a})\vec{u} \) for all \( \vec{a} \in \Gamma(U_{E}) \). Thus \( L_* = (J_2)^{-1}((\vec{u} \odot \alpha)_*) \) \( (14.15) \) \( (J_2)^{-1}(\vec{w} \odot \alpha_*), \) and \( L_* \vec{b} = (\alpha.\vec{b})\vec{a} \) for all \( \vec{b} \in \Gamma(U_{E}). \)

**Definition 14.6** Let \( L \) be a field of endomorphisms on \( U_{F} \). Its pull-back by \( \Psi \) is the field of endomorphisms \( \Psi^*L = L^* \) on \( U_{E} \) defined by, in short

$$\Psi^*L = \left( L^* = d\Psi^{-1}.L.d\Psi \right) \quad (14.22)$$

Full notation:

$$L^*(p_{E}) = d\Psi^{-1}(p_{F}).L(p_{F}).d\Psi(p_{E}) \quad \text{when} \quad p_{F} = \Psi(p_{E}).$$

14.4 Application to derivatives of vector fields

Let \( \vec{w} \in \Gamma(U_{E}) \) be a vector field on \( U_{E} \). Its derivative \( d\vec{w} \) is a field of endomorphisms on \( E \), cf. example R.8. So its push-forward \( \Psi_*\vec{w} \) is a field of endomorphisms on \( E \), given by, in short cf. (14.20),

$$(d\vec{w})_* = d\Psi_*d\vec{w}d\Psi^{-1}. \quad (14.23)$$

Full notation:

$$(d\vec{w})_*(p_{F}) = d\Psi(p_{E}).d\vec{w}(p_{E}).d\Psi(p_{E})^{-1} \quad \text{when} \quad p_{F} = \Psi(p_{E}).$$

14.5 Application to derivative of differential forms

The map

$$J : \begin{cases} \mathcal{L}(E; E^*) \to \mathcal{L}(E, E; \mathbb{R}) \\ L \to J(L), \quad J(L)(\vec{u}_1, \vec{u}_2) := (L.\vec{u}_1).\vec{u}_2, \end{cases} \quad (14.24)$$

is a natural canonical isomorphism. Thus we can “identify” \( L \) with \( J(L) \). So, if \( T \in T^2_0(U_{E}) \) then its push-forward \( \Psi_*T \) being given by (14.11):

**Definition 14.7** Let \( L : U_{E} \to \mathcal{L}(E; E^*) \) (a field of linear maps). Its push-forward by \( \Psi \) is the field of linear maps \( \Psi_*L = L_* : U_{F} \to \mathcal{L}(F; F^*) \) given by, in short,

$$L_*((\cdot)) = (L.d\Psi^{-1}.(\cdot)).d\Psi^{-1}. \quad (14.25)$$

Full notation:

$$L_*(p_{F}.(\cdot)) = (L(p_{E}).d\Psi^{-1}(p_{F}):(\cdot)).d\Psi^{-1}(p_{F}) \quad \text{when} \quad p_{F} = \Psi(p_{E}).$$

That is, in short, for all \( \vec{w} \in T^2_0(U_{E}), \)

$$L_*\vec{w} = (L.d\Psi^{-1}.\vec{w}).d\Psi^{-1} \in F^*. \quad (14.26)$$

Full notation:

$$L_*(p_{F}.)\vec{w}(p_{F}) = (L(p_{E}).d\Psi^{-1}(p_{F}).\vec{w}(p_{F})).d\Psi^{-1}(p_{F}) \in F^*. \quad \text{That is, in short, for all} \quad \vec{w}_1, \vec{w}_2 \in \Gamma(U_{E}), \quad (L_*\vec{w}_1).\vec{w}_2 = (L.d\Psi^{-1}.\vec{w}_1).d\Psi^{-1}.\vec{w}_2 \in \mathbb{R}, \quad (14.27)$$

cf. (14.11). Full notation:

$$(L_*(p_{F}).\vec{w}_1(p_{F})).\vec{w}_2(p_{F}) = (L(p_{E}).d\Psi^{-1}(p_{F}).\vec{w}_1(p_{F})).d\Psi^{-1}(p_{F}).\vec{w}_2(p_{F}).$$

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Application: Let $\alpha \in T^1_0(U_\mathcal{E})$ be a differential form on $U_\mathcal{E}$. Its derivative $d\alpha : U_\mathcal{E} \rightarrow \mathcal{L}(E; E^*)$ is given by $d\alpha(p_\mathcal{E}) . \bar{u}(p_\mathcal{E}) = \lim_{h \to 0} \frac{\alpha(p_\mathcal{E} + h \bar{u}(p_\mathcal{E}))) - \alpha(p_\mathcal{E}))}{h} \in E^*$, for all $\bar{u} \in \Gamma(U_\mathcal{E})$; That is, for all $\bar{u}_1, \bar{u}_2 \in \Gamma(U_\mathcal{E})$,

$$(d\alpha(p_\mathcal{E}) . \bar{u}_1(p_\mathcal{E}))) \bar{u}_2(p_\mathcal{E}) = \lim_{h \to 0} \frac{\alpha(p_\mathcal{E} + h \bar{u}_1(p_\mathcal{E}))) \bar{u}_2(p_\mathcal{E}) - (\alpha(p_\mathcal{E}) . \bar{u}_1(p_\mathcal{E}))) \bar{u}_2(p_\mathcal{E})}{h} \in \mathbb{R}. \quad (14.28)$$

With (14.24), we write $d^2\alpha(\bar{u}_1), \bar{u}_2 = d^2\alpha(\bar{u}_2, \bar{u}_1)$. Recall: If $\alpha = df$ (case $\alpha$ is exact) then Schwarz theorem tells that $d^2f(\bar{u}_1), \bar{u}_2 = d^2f(\bar{u}_2), \bar{u}_1$, that is, with (14.24), $d\alpha = d^2f$ is “a symmetric bilinear form”: $d^2 f(\bar{u}_1, \bar{u}_2) = d^2 f(\bar{u}_2, \bar{u}_1)$.

And $\Psi_*(d\alpha) = \Psi_*(d\alpha) = \Psi_*(d\alpha)$, the push-forward of $d\alpha$, is given by, cf. (14.27), for all $\bar{w}_1, \bar{w}_2 \in \Gamma(U_\mathcal{F})$,

$$(d\alpha)_*(\bar{w}_1, \bar{w}_2) = d\alpha(d\Psi^{-1}.\bar{w}_1, d\Psi^{-1}.\bar{w}_2) \quad (14.29)$$

that is $(d\alpha)_*(\bar{w}_1, \bar{w}_2(p_\mathcal{F})), \bar{w}_2(p_\mathcal{F}) = (d\alpha)(p_\mathcal{E})*d\Psi^{-1}(p_\mathcal{F}), \bar{w}_1(p_\mathcal{F}))d\Psi^{-1}(p_\mathcal{F}), \bar{w}_2(p_\mathcal{F})$, when $p_\mathcal{F} = \Psi(p_\mathcal{E})$.

In particular, $(d^2 f)_*(\bar{w}_1, \bar{w}_2) = d^2 f(d\Psi^{-1}.\bar{w}_1, d\Psi^{-1}.\bar{w}_2) = d^2 f(\bar{w}_1, \bar{w}_2))$.

**Definition 14.8** Let $L : U_\mathcal{F} \rightarrow \mathcal{L}(F; F^*)$ (a field of linear maps). Its pull-back by $\Psi$ is the field of linear maps $\Psi^*L : U_\mathcal{E} \rightarrow \mathcal{L}(E; E^*)$ given by, in short,

$$(\Psi^*L)(.) = (L, d\Psi(.))(.)d\Psi. \quad (14.30)$$

### 14.6 $\Psi_*(d\bar{w})$ versus $d(\Psi_*\bar{w})$: No commutativity

Let $\Psi$ be a $C^2$ diffeomorphism. Let $\bar{u} \in \Gamma(U_\mathcal{E})$; Its push-forward by $\Psi$ is $\bar{u}_* = \Psi_*\bar{u} \in \Gamma(U_\mathcal{F})$, so, with $p_\mathcal{F} = \Psi(p_\mathcal{E})$ we have $\bar{u}_*(p_\mathcal{F}) = d\Psi(p_\mathcal{E}).\bar{u}(p_\mathcal{E}) = (d\Psi \circ \Psi^{-1})(p_\mathcal{F}))(\bar{u} \circ \Psi^{-1})(p_\mathcal{F})$, cf. (10.29). Thus

$$d\bar{u}_*(p_\mathcal{F})*\bar{w}(p_\mathcal{F}) = (d^2 \Psi(p_\mathcal{E})), (d\Psi^{-1}(p_\mathcal{F}), \bar{w}(p_\mathcal{F})), \bar{w}(p_\mathcal{E}) + d\Psi(p_\mathcal{E}).d\bar{u}(p_\mathcal{E}).d\Psi^{-1}(p_\mathcal{F}).\bar{w}(p_\mathcal{F}) \in \mathcal{L}(F; F) \quad (14.31)$$

where $\bar{w} \in \Gamma(U_\mathcal{F})$ In short:

$$d\bar{u}_* \bar{w} = d\Psi .d\bar{u} .d\Psi^{-1} \bar{w} + (d^2 \Psi .(d\Psi^{-1} \bar{w}))(\bar{w}) \quad (14.32)$$

Thus, with (14.23), the differentiation $d$ and the push-forward $\Psi_*$ do not commute on $\Gamma(U_\mathcal{E})$ (it does if $\Psi$ is affine).

### 14.7 $\Psi_*(d\alpha)$ versus $d(\Psi_*\alpha)$: No commutativity

Let $\Psi$ be a $C^2$ diffeomorphism. Let $\alpha \in \Omega^1(U_\mathcal{E})$; Its push-forward by $\Psi$ is $\alpha_* = \Psi_* \alpha \in \Omega^1(U_\mathcal{F})$, so, with $p_\mathcal{F} = \Psi(p_\mathcal{E})$ we have $\alpha_*(p_\mathcal{F}) = \alpha(p_\mathcal{E})*d\Psi^{-1}(p_\mathcal{F}) = (d\Psi \circ \Psi^{-1})(p_\mathcal{F}))(\bar{u} \circ \Psi^{-1})(p_\mathcal{F})$, cf. (10.29). Thus

$$d\alpha_*(p_\mathcal{F})*\bar{w}_1(p_\mathcal{F}) = (d\alpha(p_\mathcal{E})), (d\Psi^{-1}(p_\mathcal{F}), \bar{w}_1(p_\mathcal{F})), \bar{w}_1(p_\mathcal{F}) + (d\alpha(p_\mathcal{E})), \alpha(p_\mathcal{F})*(d\Psi^{-1}(p_\mathcal{F}), \bar{w}_1(p_\mathcal{F}) \in F^*, \quad (14.33)$$

where $\bar{w}_1 \in \Gamma(U_\mathcal{F})$. That is, with $\bar{w}_1, \bar{w}_2 \in \Gamma(U_\mathcal{F})$, in short,

$$((d\alpha_*, \bar{w}_1), \bar{w}_2) = (d\alpha, d\Psi^{-1}(\bar{w}_1)), d\Psi^{-1}(\bar{w}_2) + \alpha, d^2 \Psi^{-1}(\bar{w}_1, \bar{w}_2) \in \mathbb{R}. \quad (14.34)$$

Thus, with (14.27), the differentiation $d$ and the push-forward $\Psi_*$ do not commute on $\Gamma(U_\mathcal{E})$ (it does if $\Psi$ is affine).
Part III
Lie derivative

15 Lie derivative

15.1 Introduction

The main results are:

1- The Lie derivative $\mathcal{L}_{\vec{v}} f$ of a scalar valued function $f$ is reduced to the material derivative:

$$\mathcal{L}_{\vec{v}} f = \frac{D f}{Dt},$$

see (15.1).

2- The Lie derivative $\mathcal{L}_{\vec{v}} \vec{w}$ of a vector field $\vec{w}$ is more than just the material derivative (a vector is not just a collection of scalar components):

$$\mathcal{L}_{\vec{v}} \vec{w} = \frac{D \vec{w}}{Dt} - d \vec{v}. \vec{w},$$

see (15.2). In particular, the spatial variations $d \vec{v}$ of $\vec{v}$ act on the evolution of $\vec{w}$ (expected), cf. the $-d \vec{v}. \vec{w}$ term. It gives the rate of stress on a vector field due to the flow.

3- The use of the Lie derivative enables to overpass the limitation of the Cauchy’s method which imposes the use of a Euclidean dot product to compare two vectors and their relative deformation to evaluate a stress. (Cauchy died in 1857, and Lie was born in 1842: Unfortunately Cauchy could not use the Lie derivative.)

Remark: In classical continuum mechanic books, the Lie derivative is often introduced for “second order tensors”. However a tensor needs vector fields to be defined, and to understand the Lie derivative of tensors, we first have to understand the Lie derivative of vector fields. Also see footnote page 26, and § E.1.

15.2 Ubiquity gift not required

Let $\tilde{\Phi} : (t, P_{Ob}) \in [t_1, t_2] \times Ob \to P(t) = \tilde{\Phi}(t, P_{Ob}) \in \mathcal{R}$ be a motion relative to a referential $\mathcal{R}$, cf. (1.5). Let $\mathcal{E}ul$ be a Eulerian function, cf. (2.2). Its material time derivative is usually given by, cf. (2.27),

$$\frac{D \mathcal{E}ul}{Dt}(t, p(t)) = \lim_{h \to 0} \frac{\mathcal{E}ul(t+h, p(t+h)) - \mathcal{E}ul(t, p(t))}{h} = (\frac{\partial \mathcal{E}ul}{\partial t}(t, p(t)) + d \mathcal{E}ul(t, p(t)).\vec{v}(t, p(t))).$$

(15.3)

Issues: The computed rate $\mathcal{E}ul(t+h, p(t+h)) - \mathcal{E}ul(t, p(t))$ raises issues:

- The difference $\mathcal{E}ul(t+h, p(t+h)) - \mathcal{E}ul(t, p(t))$ requires the time and space ubiquity gift for an observer, since $\mathcal{E}ul(t+h, p(t+h)) - \mathcal{E}ul(t, p(t))$ mixes to distinct times $t$ and $t+h$ and to distinct locations $p(t)$ and $p(t+h)$. See remark 1.3.

- The difference $\mathcal{E}ul(t+h, p(t+h)) - \mathcal{E}ul(t, p(t))$ can be impossible, if, e.g., $\mathcal{E}ul = \vec{w}$ is a vector field in a “non planar surface” (manifold) since then $\mathcal{E}ul(t+h, p(t+h))$ and $\mathcal{E}ul(t, p(t))$ do not belong to the same (tangent) vector space.

Consequence: If you want to compare $\mathcal{E}ul(t+h, p(t+h))$ and $\mathcal{E}ul(t, p(t))$, you need the duration $h$ to get from $t$ to $t+h$, and you need to move from $p(t)$ to $p(t+h)$. So, to compare $\mathcal{E}ul(t+h, p(t+h))$ and $\mathcal{E}ul(t, p(t))$ you must (without any gift of ubiquity):

- take the value $\mathcal{E}ul(t, p(t))$ (for memory),
- moves with it along the trajectory $\tilde{\Phi}_{P_{Ob}}$,
- and then, the value $\mathcal{E}ul(t, p(t))$ (the memory) has (eventually) been transformed by the flow into

$$(\Phi_{t+h})^\ast \mathcal{E}ul_\ast(p(t+h)) = \mathcal{E}ul_\ast(p(t+h))^{\text{noed}} = \mathcal{E}ul_\ast(t+h, p(t+h)),$$

(15.4)

cf. (10.28) for vector fields (push-forward).
And finally, you can now compare the actual value $\mathcal{E}_t(t+h, p(t+h))$ you see at $t+h$ at $p(t+h)$, with the value $\mathcal{E}_t(t+h, p(t+h))$ you arrived with (the transported memory) (no gift of ubiquity required): So the rate

$$\text{Rate} = \frac{\mathcal{E}_t(t+h, p(t+h)) - \mathcal{E}_t(t+h, p(t+h))}{h}$$

is meaningful (15.5)
since it is computed at a unique time $t+h$ and at a unique point $p(t+h)$ (no gift of ubiquity required).

NB: In a non planar surface (in a non planar manifold) (15.5) is the only meaningful rate to consider (to compute a “rate of evolution”).

**Definition 15.1** Let $\vec{v}(t, p) = \frac{\partial}{\partial t} \mathcal{E}_t(t, P_{Obg})$ be the Eulerian velocity at $t$ at $p = \Phi(t, P_{Obg})$, cf. (2.5). In $\mathbb{R}^n$
(affine space), the Lie derivative $\mathcal{L}_v \mathcal{E}_t$ along $\vec{v}$ of an Eulerian function $\mathcal{E}_t\mathcal{E}_t$ is the Eulerian function $\mathcal{L}_v \mathcal{E}_t$ defined by, at $t$ at $p_t = \Phi(t, P_{Obg})$,

$$\mathcal{L}_v \mathcal{E}_t(t, p_t) := \lim_{h \to 0} \frac{\mathcal{E}_t(t+h, p(t+h)) - ((\Phi_{t+h})_*, \mathcal{E}_t)(t+h, p(t+h))}{h}.$$ (15.6)

**Remark 15.2** $(\Phi_{t+h})_* \mathcal{E}_t(t+h, p(t+h))$ is exclusively strain related (the memory transport along a flow), when $\mathcal{E}_t(t+h, p(t+h))$ is the (true) value of $\mathcal{E}_t$ at $t+h$ at $p(t+h)$. See § 10.5.3.

**15.3 Definition rewritten**

The rate in (15.6) has to be slightly modified to be adequate in all situations: $\mathcal{E}_t(t+h, p(t+h)) - \mathcal{E}_t(t+h, p(t+h))$ is computed at $(t+h, p(t+h))$ which moves as $h \to 0$ (on a “non-planar manifold” this is problematic). The “natural” definition, as far as mechanics and physics are concerned, is to arrive with the memory, so, instead of (15.6):

**Definition 15.3** The Lie derivative $\mathcal{L}_v \mathcal{E}_t$ along $\vec{v}$ of an Eulerian function $\mathcal{E}_t$ is the Eulerian function $\mathcal{L}_v \mathcal{E}_t$ defined by, at $t$ at $p_t = \Phi(t, P_{Obg})$,

$$\mathcal{L}_v \mathcal{E}_t(t, p_t) := \lim_{h \to 0} \frac{\mathcal{E}_t(t, p_t) - ((\Phi_{t+h})_*, \mathcal{E}_t)(t+h, p(t+h))}{h}, \quad \text{rate in } \Omega_t.$$ (15.7)

Here the observer must:

- At $t+h$ at $p_{t+h} = \Phi(t+h, P_{Obg})$, take the value $\mathcal{E}_t(t-h, p_{t-h})$,
- travel along the trajectory $\Phi_{t+h}$,
- once at $t$ at $p_t$, this value has become $((\Phi_{t-h})_*, \mathcal{E}_t)(t-h, p_t)$,
- and then the comparison with $\mathcal{E}_t(t, p_t)$ can be done (no ubiquity gift required).

The alternative definition found in differential geometry books uses the pull-back:
Definition 15.4 The Lie derivative of a Eulerian function $\mathcal{E}ul$ along a flow of Eulerian velocity $\vec{v}$ is the Eulerian function $\mathcal{L}_v\mathcal{E}ul$ defined at $(t, p_t)$ by

$$\mathcal{L}_v\mathcal{E}ul(t, p_t) := \lim_{h \to 0} \frac{((\Phi_{t+h})^*\mathcal{E}ul_t+h)(p_t) - \mathcal{E}ul_t(p_t)}{h}, \quad \text{rate in } \Omega_t, \quad (15.8)$$

In other words, if $g$ is a a function defined by

$$g(\tau) = ((\Phi_{t})^*\mathcal{E}ul_t)(p_t)$$

(15.9)

(in particular $g(t) = \mathcal{E}ul_t(p_t)$), then (15.8) reads

$$\mathcal{L}_v\mathcal{E}ul(t, p_t) := g'(t) = \lim_{\tau \to t} \frac{g(\tau) - g(t)}{\tau - t} \quad \text{also written} \quad \frac{d((\Phi_t)^*\mathcal{E}ul_t)(p_t)}{dt} \mid_{\tau = t} = (15.10)$$

NB: (15.7) and (15.8) are equivalent since (15.8) also reads (with $h \to -h$)

$$\mathcal{L}_v\mathcal{E}ul(t, p_t) = \lim_{h \to 0} \frac{((\Phi_{t-h})^*\mathcal{E}ul_{t-h})(p_t) - \mathcal{E}ul_t(p_t)}{-h} = \lim_{h \to 0} \frac{\mathcal{E}ul_t(p_t) - ((\Phi_{t-h})^*\mathcal{E}ul_{t-h})(p_t)}{h}, \quad (15.11)$$

and $(\Phi_{t-h})^* = ((\Phi_t^{-1})^*)_* = (\Phi_t^t)^*$, cf. e.g. (10.37) and (3.7).

Remark 15.5 More precise definition, as in (2.3): E.g. with (15.8), the Lie derivative $\tilde{\mathcal{L}}_v\mathcal{E}ul$ of a Eulerian function $\mathcal{E}ul$ along a flow of Eulerian velocity $\vec{v}$ is defined by, at $t$ at $p_t = \tilde{\Phi}(t, P_{\Phi_0})$,

$$\tilde{\mathcal{L}}_v\mathcal{E}ul(t, p_t) := \frac{\mathcal{E}ul_t(p_t)}{\text{function at } t \text{ at } p_t}. \quad (15.12)$$

(And, to lighten the notation, $\tilde{\mathcal{L}}_v\mathcal{E}ul(t, p_t) = \text{named } \mathcal{L}_v\mathcal{E}ul(t, p_t)$.)

Remark 15.6 In the affine manifold $\mathbb{R}^n$, the tangent spaces are all identified with the vector space $\mathbb{R}^n$, and the definition of $\mathcal{L}_v\mathcal{E}ul$ given by (15.6) is equivalent to the definition given by (15.8): E.g. for vector fields $\tilde{v}$, with $\vec{p}(t+h) = \Phi_{t,h}(p_t)$ we have

$$\tilde{w}_{t+h}(\vec{p}(t+h)) - (\Phi_{t,h}(\tilde{w}_t))(p_t) = \tilde{w}_{t+h}(\vec{p}(t+h)) - \tilde{w}_t(p_t) = d\Phi_{t,h}(p_t).((\Phi_{t,h})^*\tilde{v}_{t+h}) - \tilde{w}_t(p_t),$$

and $d\Phi_{t,h}(p_t) \to h \to 0 I$.

15.4 Lie derivative of a scalar function

Let $f$ be a $C^1$ Eulerian scalar function, let $p(\tau) = \tilde{\Phi}(\tau, P_{\Phi_0}) = p_\tau$. Then (10.10) gives $(\Phi_t^t)_* f_{t-h}(p_t) = f_{t-h}(p_{t-h})$, (15.18) gives

$$\mathcal{L}_v f(t, p_t) = \lim_{h \to 0} \frac{f(t, p(t)) - f(t-h, p(t-h))}{h} = \lim_{h \to 0} \frac{f(t+h, p(t+h)) - f(t, p(t))}{h} = \frac{Df}{Dt}(t, p_t), \quad (15.13)$$

hence, cf. (2.27),

$$\mathcal{L}_v f = \frac{Df}{Dt} = \frac{\partial f}{\partial t} + df.\vec{v}. \quad (15.14)$$

Thus for scalar functions, the Lie derivative is the material derivative.

Interpretation: $f(t, p(t))$ is the value of $f$ at the actual time $t$ at $p(t)$, whereas $f(t-h, p(t-h))$ is the transported memory (the value $f$ had at $t-h$ at $p_{t-h}$). Thus $\mathcal{L}_v f$ measures the evolution of $f$ along a trajectory.

Proposition 15.7

$$\frac{Df}{Dt} = 0 \text{ in } C \iff \forall t, \tau \in [t_0, T], \, f_{\tau} = (\Phi_t^t)_* f_t. \quad (15.15)$$

That is, $\frac{Df}{Dt} = 0$ iff $f$ let itself be carried by the flow, i.e., if $f(t, p(t)) = f(t_0, p(t_0))$ for all $t$.

Proof. Let $p(t) = \tilde{\Phi}(t, P_{\Phi_0}) = p_t$ and $p(\tau) = \tilde{\Phi}(\tau, P_{\Phi_0}) = p_\tau$, then $p_\tau = \Phi_{\tau}^\tau(p_t) = \Phi_{t}^\tau(p_t)$.

$
\Leftarrow$: If $f_{\tau} = (\Phi_t^\tau)_* f_t$, then $f_{\tau}(p_\tau) = f_t(p_t)$, thus $\lim_{\tau \to t} \frac{f_t(p(\tau)) - f_t(p(t))}{\tau - t} = 0$, that is, $\frac{Df}{Dt} = 0$.

$\Rightarrow$: If $\frac{Df}{Dt} = 0$ then $f(t, p(t))$ is a constant function on the trajectory $t \to \tilde{\Phi}(t, P_{\Phi_0})$, for any particle $P_{\Phi_0}$, so $f(t, p(t)) = f(t, p_0)$ when $p(\tau) = \Phi_{\tau}^\tau(p_t)$, that is, $f(t, p_\tau) = (\Phi_t^\tau)_* f_t(p_\tau)$. 

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Exercise 15.8 Prove:
\[
\mathcal{L}_v(D^2f) = \frac{D^2f}{Dt^2} = \frac{\partial^2 f}{\partial t^2} + 2\Bigl(\frac{\partial f}{\partial t}\Bigr)\ddot{v} + d^2f(\dddot{v}, \ddot{v}) + df(\frac{\partial^2 \varphi}{\partial t^2} + d\ddot{v}).
\] (15.16)

Answer. See (2.48). 

15.5 Lie derivative of a vector field

Let \(\vec{w}\) be a \(C^1\) (Eulerian) vector field, e.g. interpreted as a "force field".

**Proposition 15.9**

\[
\mathcal{L}_v\vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w} = \frac{\partial \vec{w}}{\partial t} + d\vec{v}.\dot{\vec{v}} - d\vec{v}.\vec{w}.
\] (15.17)

Thus the Lie derivative is not reduced to the material derivative \(\frac{D\vec{w}}{Dt}\) (unless \(d\vec{v} = 0\), i.e. unless \(\vec{v}\) is uniform): The spatial variations of \(\vec{v}\) influence the rate of stress, which is expected.

**Proof.** E.g., with (15.8): Let \(\vec{g}(\tau) = (\Phi^\tau_* \vec{w})(t, p(t)) = d\Phi^\tau(t, p(t))^{-1}.\vec{w}(\tau, p(\tau))\) when \(p(\tau) = \Phi^\tau(t, p(t))\). Then (15.8) reads:
\[
\mathcal{L}_v\vec{w}(t, p(t)) = \vec{g}'(t).
\] (15.18)

Since \(\vec{z}(\tau) := \vec{w}(\tau, p(\tau)) = d\Phi^\tau(t, p(t)).\vec{g}(\tau)\) we get
\[
\vec{z}'(\tau) = \frac{D\vec{w}}{Dt}(t, p(t)) = \frac{\partial (d\Phi^\tau(\tau, p(t)).\vec{g}(\tau))}{\partial \tau} + d\Phi^\tau(\tau, p(t)).\vec{g}'(\tau)
\]
\[
= (d\vec{w}(\tau, p(\tau)).\vec{F}'(\tau, p(t))).(F^\tau(\tau, p(t))^{-1}.\vec{w}(\tau, p(\tau)) + F^\tau(\tau, p(t)).\vec{g}'(\tau)
\]
\[
= d\vec{w}(\tau, p(\tau)).\vec{w}(\tau, p(\tau)) + F^\tau(\tau, p(t)).\vec{g}'(\tau).
\] (15.19)

Thus \(\frac{D\vec{w}}{Dt}(t, p(t)) = d\vec{w}(t, p(t)).\vec{w}(t, p(t)) + I.\vec{g}'(t) = d\vec{w}(t, p(t)).\vec{w}(t, p(t)) + \mathcal{L}_v\vec{w}(t, p(t))\), i.e. (15.17).

**Remark 15.10** A function \(f : p \to f(p) \in \mathbb{R}\) gives scalar values \(f(p)\), and a scalar \(f(p)\) has no "direction". Thus \(f(p)\) is not sensitive to the variations \(d\vec{v}\) of \(\vec{v}\). There is no \(d\vec{v}\) term in (15.14).

While a vector valued function \(\vec{w} : p \to \vec{w}(p) \in \mathbb{R}^n\) gives vector values \(\vec{w}(p)\), and a vector \(\vec{w}(p)\) has a direction. And, submitted to a flow, \(\vec{w}(p)\) is sensitive to the variations of the flow. The \(-d\vec{v}.\vec{w}\) term attests it. (A vector is not just a collection of scalar components.)

**Remark 15.11** The Cauchy expression \(\vec{r} = \vec{g} \cdot \vec{n}\) states that the Cauchy stress vector is a linear function of the normal vector \(\vec{n}\) to a surface, meaning that a surface is considered to be locally planar (first order Taylor expansion), thus excludes second order effects (curvature effect). While the Lie approach does not make such hypothesis (and we can use \(\mathcal{L}_v(\mathcal{L}_\vec{w}\vec{v})\) for second order terms).

**Remark 15.12** The Lie bracket of two vector fields \(\vec{v}\) and \(\vec{w}\) is
\[
[\vec{v}, \vec{w}] := d\vec{w}.\vec{v} - d\vec{v}.\vec{w} =: \mathcal{L}_v^0\vec{w}.
\] (15.20)

And \(\mathcal{L}_v^0\vec{w}\) is called the autonomous Lie derivative of \(\vec{w}\) along \(\vec{v}\). Thus
\[
\mathcal{L}_v\vec{w} = \frac{\partial \vec{w}}{\partial t} + [\vec{v}, \vec{w}] = \frac{\partial \vec{w}}{\partial t} + \mathcal{L}_v^0\vec{w}.
\] (15.21)

NB: \(\mathcal{L}_v^0\vec{w}\) is used when \(\vec{v}\) et \(\vec{w}\) are stationary vector fields, thus does not concern objectivity: A stationary vector field in a referential is not necessary stationary in another (moving) referential.
15.6 Examples and interpretations

15.6.1 Flow resistance measurement

Proposition 15.13

\[ \mathcal{L}_\phi \vec{w} = 0 \iff \frac{D \vec{w}}{Dt} = d\vec{v} \cdot \vec{w} \iff \forall t, \tau \in [t_0, T], \vec{w}_\tau = (\Phi^\tau_t)_* \vec{w}_t. \]  \tag{15.22}

Interpretation with (15.7):

- \( \vec{w}(t, p(t)) \) is the “actual” value of the force field \( \vec{w} \) at \( t \) at \( p(t) \).
- whereas the push-forward \( \vec{w}_{t+h}(t, p(t)) \) is the “transported memory” = the “virtual” value that \( \vec{w} \) would have had at \( t \) at \( p(t) \) if it had let itself be carried by the flow (strain dependent, see (10.26)-(10.28)).
- So (15.22) means: The Lie derivative \( \mathcal{L}_\phi \vec{w} \) vanishes if \( \vec{w} \) does not resist the flow (let itself be deformed by the flow).

Proof. \( \Leftarrow \) If \( \vec{w}_\tau(p(\tau)) = F^\tau_t(p(t)) \vec{w}(t, p(t)) \) then \( \vec{w}_\tau(p(\tau)) = F^\tau_t(p(t)), \vec{w}(t, p(t)) \) thus \( \frac{D \vec{w}}{Dt} = \partial \vec{w}/\partial \tau \). (See proposition 4.10.)

\[ \Rightarrow \: \text{Let } \vec{f}(\tau, t, p(t)) = F^\tau_t(p(t))^{-1} \vec{w}_{\tau}(p(\tau)) \text{ be the pull-back of } \vec{w}_\tau \text{ by } \Phi^\tau_t \text{ at } p(t) = (\Phi^\tau_t)^{-1}(p(\tau)). \text{ Then } \vec{w}_\tau(p(\tau)) = F^\tau_t(p(t)), \vec{f}(\tau, t, p(t)) \text{ when } p(\tau) = \Phi^\tau_t(p(t)). \text{ Thus } \frac{D \vec{w}}{Dt} = \partial \vec{w}/\partial \tau = (d\vec{v}(\tau, p(\tau)), F^\tau_t(p(t)), \vec{f}(\tau, t, p(t))) \text{ and } \frac{D \vec{w}}{Dt} = \partial \vec{w}/\partial \tau = d\vec{v}(\tau, p(\tau), \vec{w}(\tau, p(\tau)) + F^\tau_t(p(t)), \frac{\partial \vec{f}}{\partial \tau}(\tau, t, p(t))) \text{. Thus } \mathcal{L}_\phi \vec{w} = 0 \text{ gives } F^\tau_t(p(t)), \frac{\partial \vec{f}}{\partial \tau}(\tau, t, p(t)) = 0, \text{ thus } \frac{\partial \vec{f}}{\partial \tau}(\tau, t, p(t)) = 0 \text{ since } \Phi^\tau_{t+h} \text{ is a diffeomorphism. Thus } \frac{\partial \vec{f}}{\partial \tau}(\tau, t, p(t)) = 0 \text{ is independent of } \tau, \text{ thus } \vec{f}(\tau, t, p(t)) = \vec{f}(t, t, p(t)) = \vec{w}_t(p(t)). \text{ Thus } \vec{w}_t = \text{ the pull-back of } \vec{w}_\tau. \]

15.6.2 Lie Derivative of a vector field along itself

(15.17) gives

\[ \mathcal{L}_\phi \vec{v} = \frac{\partial \vec{v}}{\partial t}. \]  \tag{15.23}

In particular, if \( \vec{v} \) is a stationary vector field, then

\[ \mathcal{L}_\phi \vec{v} = \vec{0}. \]  \tag{15.24}

This is also a direct consequence of (6.24).

15.6.3 Lie derivative along a uniform flow

Suppose \( d\vec{v} = 0 \). Then note \( \vec{v}(t, p) = \vec{v}(t) \) (e.g., \( \vec{v}(t, p) = \lambda(t) \vec{e}_t \) where \( \vec{e}_t \) is a Cartesian basis). Then

\[ \mathcal{L}_\phi \vec{w} = \frac{\partial \vec{w}}{\partial t} + d\vec{v} \cdot \vec{w} = \frac{D \vec{w}}{Dt} \quad \text{(here } \vec{v} = 0). \]  \tag{15.25}

In this case the flow is rectilinear (\( d\vec{v} = 0 \)): there is no curvature (of the flow) to influence the stress on \( \vec{w} \).

Moreover, if \( \vec{w} \) is stationary, that is \( \frac{\partial \vec{w}}{\partial t} = 0 \), then

\[ \mathcal{L}_\phi \vec{w} = d\vec{v} \cdot \vec{w}. \]  \tag{15.26}

= the directional derivative of the vector field \( \vec{w} \) in the direction \( \vec{v} \).

15.6.4 Lie derivative of a uniform vector field

Suppose \( d\vec{w}(t, p) = 0 \). Then

\[ \mathcal{L}_\phi \vec{w} = \frac{\partial \vec{w}}{\partial t} - d\vec{v} \cdot \vec{w} \quad \text{(here } d\vec{w} = 0), \]  \tag{15.27}

thus the stress on \( \vec{w} \) due to the flow depends on \( \vec{v} \) (due to the curvature of the flow). Moreover, is \( \vec{w} \) is stationary then

\[ \mathcal{L}_\phi \vec{w} = -d\vec{v} \cdot \vec{w}. \]  \tag{15.28}

(See remark 15.11.)
15.6.5 Uniaxial stretch of an elastic material

- Strain. With \(|\overrightarrow{OP}|_\varepsilon = |\overrightarrow{OP}(t, P_{\text{obj}})|_\varepsilon = |\overrightarrow{X}|_\varepsilon = \begin{pmatrix} X \\ Y \end{pmatrix} \), with \( \xi > 0, t \geq t_0 \) and \( p(t) = \Phi^t P = \Phi(t, P) \):

\[
\begin{pmatrix} x \\ y \end{pmatrix} = |\overrightarrow{X}|_\varepsilon = |\overrightarrow{OP}(t)|_\varepsilon = |\overrightarrow{OP}^t(t, P)|_\varepsilon = \begin{pmatrix} X \\ Y \end{pmatrix} + \xi(t-t_0) \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X(1+\xi(t-t_0)) \\ Y \end{pmatrix}.
\]

(15.29)

- Eulerian velocity \( \overrightarrow{v}(t, p) = \left( \frac{\xi X}{0}, \frac{\xi Y}{0}, 0 \right) \), thus \( \overrightarrow{d\overrightarrow{v}}(t, p) = \left( \frac{\xi X}{0}, \frac{\xi Y}{0}, 0 \right) \) for \( \xi \neq 0 \).

- Deformation gradient (independent of \( P \)), with \( \kappa_t = \xi(t-t_0) \):

\[
F_t = d\Phi^t(t, P) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I + \kappa_t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(15.30)

In particular \( ||F_t, e_1|| = 1 + \kappa_t \) (stretch rate at \( p_t \)). Infinitesimal strain tensor, since \( F_t^T = F_t \) here:

\[
\varepsilon^{e_1}(P) = \varepsilon = F_t - I = \kappa_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

(15.31)

- Stress. Constitutive law: Linear isotropic elasticity (requires a Euclidean dot product)

\[
\sigma^{e_1}(p_t) = \sigma = \lambda \text{Tr}(\varepsilon)I + 2\mu\varepsilon = \kappa_t \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \lambda \end{pmatrix}.
\]

(15.32)

Cauchy stress vector \( \overrightarrow{T} \) on a surface around \( p \) with normal \( n_t(p) = e_1 \) is \( \overrightarrow{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = n = \frac{1}{n_2} \overrightarrow{e}_1 = \overrightarrow{e}_1 + \overrightarrow{n} = \overrightarrow{e}_1 + \kappa_t \left( \begin{pmatrix} \lambda + 2\mu \\ 0 \end{pmatrix} \right) \xi(t-t_0) \begin{pmatrix} \lambda + 2\mu \\ 0 \end{pmatrix}.
\]

(15.33)

- Push-forwards: \( \mathcal{T}_t^0(p_{t_0}) = 0 \), thus \( F_{t_0+p_{t_0}}^0(p_{t_0})T_{t_0}(p_{t_0}) = \overrightarrow{0} \).

- Lie derivative:

\[
\mathcal{L}_e \overrightarrow{T}(t_0, p_{t_0}) = \lim_{t-t_0} \frac{\overrightarrow{T}(t_0, p_{t_0}) - F_t^0(p_{t_0})T_{t_0}(p_{t_0})}{t-t_0} = \xi \left( \begin{pmatrix} \lambda + 2\mu \\ 0 \end{pmatrix} \right) \text{ (rate of stress at } (t_0, p_{t_0})). \]

(15.34)

If we suppose that this rate of stress is constant over time, then we recover \( T(t, p_t) = T(t_0, p_{t_0}) + \int_{t_0}^t \mathcal{L}_e \overrightarrow{T}(t, p_t)\,dt = \int_{t_0}^t \mathcal{L}_e \overrightarrow{T}(t_0, p_{t_0})\,dt = \mu \xi(t-t_0) \left( \begin{pmatrix} \lambda + 2\mu \\ 0 \end{pmatrix} \right) \).

- Generic computation: \( \mathcal{L}_e \overrightarrow{T} = \frac{\partial T}{\partial t} + \overrightarrow{d\overrightarrow{T}} = \xi \left( \begin{pmatrix} \lambda + 2\mu \\ 0 \end{pmatrix} \right) \xi(t-t_0) \left( \begin{pmatrix} \lambda + 2\mu \\ 0 \end{pmatrix} \right) \).

In particular, \( \overrightarrow{d\overrightarrow{T}}(t_0, p_{t_0})\overrightarrow{T}(t_0, p_{t_0}) = \overrightarrow{0} \). Thus \( \mathcal{L}_e \overrightarrow{T}(t_0, p_{t_0}) = \xi \left( \begin{pmatrix} \lambda + 2\mu \\ 0 \end{pmatrix} \right) \text{ rate of stress at } (t_0, p_{t_0}).
\]

15.6.6 Simple shear of an elastic material

Euclidean basis \( \{e_1, e_2\} \) in \( \mathbb{R}^2 \), the same basis at any time. Initial configuration \( \Omega_{t_0} = \{0, L_1\} \otimes \{0, L_2\} \).

Initial position \( [\overrightarrow{OP}]_\varepsilon = [\overrightarrow{OP}(t_0, P_{\text{obj}})]_\varepsilon = [\overrightarrow{X}]_\varepsilon = \begin{pmatrix} X \\ Y \end{pmatrix} \). Let \( \xi \in \mathbb{R}^n \) and

\[
[\overrightarrow{OP}]_\varepsilon = \Phi(t_0, P_{\text{obj}})]_\varepsilon = [\overrightarrow{X}]_\varepsilon = \begin{pmatrix} x = \varphi_1(t, X, Y) = X + \xi(t-t_0)Y \\ y = \varphi_2(t, X, Y) = Y \end{pmatrix}.
\]

(15.35)

- Eulerian velocity \( \overrightarrow{v}(t, p_t) = \frac{\partial \overrightarrow{X}}{\partial t}(t, P_{\text{obj}}) = \begin{pmatrix} \xi X \\ \xi Y \\ 0 \end{pmatrix} = \begin{pmatrix} \xi X \\ \xi Y \\ 0 \end{pmatrix} \) at \( p_t = \Phi(t, P_{\text{obj}}) \), thus \( \overrightarrow{d\overrightarrow{v}}(p_t) = \begin{pmatrix} 1 & \xi \\ 0 & 0 \end{pmatrix} \).
• Strain. Deformation gradient $F_t = d\Phi^0_t(P)$ (independent of $P$), with $\kappa_t = \xi(t-t_0)$:

$F_t^{\alpha_0} = \begin{pmatrix} 1 & \kappa_t \\ 0 & 1 \end{pmatrix} = F_t^{\alpha_0}$, thus $F_t^{\alpha_0} - I = \kappa_t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. (15.36)

• Strain. Infinitesimal strain tensor:

$\varepsilon^{\alpha_0}(P) = \frac{F_t^{\alpha_0}(P) - I + (F_t^{\alpha_0}(P)-I)^T}{2} = \frac{\kappa_t}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \varepsilon$. (15.37)

• Stress. Constitutive law, usual linear isotropic elasticity (requires a Euclidean dot product):

$\sigma(t,p) = \lambda \text{Tr}(\varepsilon(t,p)) I + 2\mu \varepsilon(t,p) = \mu \kappa_t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma$. (15.38)

Cauchy stress vector $\bar{T}(t,p_t)$ (at $t$ at $p_t$) on a surface around $p$ with normal $\bar{n}_t(p) = \bar{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ chosen independent of $t$ and $p$:

$\bar{T}_t = \bar{\sigma} \cdot \bar{n} = \mu \kappa_t \begin{pmatrix} n_1^2 \\ n_2 \end{pmatrix} = \mu \xi (t-t_0) \begin{pmatrix} n_1^2 \\ n_2 \end{pmatrix} = \bar{T}(t)$ (stress independent of $p_t$). (15.39)

• Push-forward

$d\bar{w} = F_t^{\alpha_0} \bar{W} = \begin{pmatrix} W_1 + \kappa_t W_2 \\ W_2 \end{pmatrix}$. (15.40)

• Lie derivative, with $\bar{T}_{t_0} = \bar{0}$:

$L_{\bar{v}} \bar{T}(t_0,p_{t_0}) = \lim_{t \to t_0} \frac{\bar{T}_t(p_t) - F_t^{\alpha_0}(p_{t_0}) \bar{T}_{t_0}(p_{t_0})}{t - t_0} = \mu \xi \begin{pmatrix} n_1^2 \\ n_2 \end{pmatrix}$ (rate of stress at $(t_0,p_{t_0})$). (15.41)

If we suppose that this rate of stress is constant over time, then we recover $\bar{T}(t,p_t) = \bar{T}(t_0,p_{t_0}) + \int_{t_0}^t L_{\bar{v}} \bar{T}(\tau,p_{\tau_0}) d\tau = \int_{t_0}^t L_{\bar{v}} \bar{T}(t_0,p_{t_0}) d\tau = \mu \xi (t-t_0) \begin{pmatrix} n_1^2 \\ n_2 \end{pmatrix}$.

• Generic computation: $L_{\bar{v}} \bar{T} = \frac{\partial^2}{\partial p^2} + d\bar{T} \cdot \bar{v} - \bar{d} \bar{v} \cdot \bar{T}$. (15.39) gives $\frac{\partial^2}{\partial p^2} (t,p) = \mu \xi \begin{pmatrix} n_1^2 \\ n_2 \end{pmatrix}$ and $d\bar{T} = 0$. With $d\bar{v}_{t_0}, \bar{T}_{t_0} = \bar{0}$. Thus $L_{\bar{v}} \bar{T}(t_0,p_{t_0}) = \mu \xi \begin{pmatrix} n_1^2 \\ n_2 \end{pmatrix}$.

**15.6.7 Shear flow**

![Figure 15.2: Shear flow, cf. (15.42), with $\bar{w}$ constant and uniform. The Lie derivative measures the resistance to the deformation.](image)

Stationary shear field, see (6.11) with $\alpha = 0$ and $t_0 = 0$:

$\bar{v}(x,y) = \begin{pmatrix} v^1(x,y) = \lambda y \\ v^2(x,y) = 0 \end{pmatrix}$, $\bar{d} \bar{v}(x,y) = \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$. (15.42)

Let $\bar{w}(t,p) = \begin{pmatrix} 0 \\ b \end{pmatrix} = \bar{w}(t_0,p_{t_0})$ (constant in time and uniform in space). Then $L_{\bar{v}} \bar{w} = - \bar{d} \bar{v} \bar{w} = \begin{pmatrix} -\lambda b \\ 0 \end{pmatrix}$

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measures “the resistance to deformation due to the flow”. See figure 15.2, the virtual vector \( \vec{w}_v(t, p) = d\Phi(t, p_\alpha)\vec{w}_v(t_0, p_\alpha) \) being the vector that would have let itself be carried by the flow (the push-forward).

### 15.6.8 Spin

**Example 15.14** Rotating flow: Continuing (6.14):

\[
\vec{v}(x, y) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad d\vec{v}(x, y) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \omega R(\pi/2).
\]

(15.43)

In particular \( d^2\vec{v} = 0 \). With \( \vec{w} = \vec{w}_0 \) constant and uniform we get

\[
L_v\vec{w}_0 = -d\vec{v}(p)\vec{w}_0 = -\omega R(\pi/2)\vec{w}_0 \quad (\perp \begin{pmatrix} a \\ b \end{pmatrix} = \vec{w}_0).
\]

(15.44)

It is “the force at which \( \vec{w} \) refuses to turn with the flow”.

### 15.6.9 Second order Lie derivative

**Exercise 15.15** Let \( \vec{v}, \vec{w} \) be \( C^2 \). Prove:

\[
L_v(L_v\vec{w}) = \frac{D^2\vec{w}}{Dt^2} - 2d\vec{v}.\frac{D\vec{w}}{Dt} - \frac{D(d\vec{v})}{Dt}\vec{w} + d\vec{v}.d\vec{v}.\vec{w},
\]

(15.45)

that is,

\[
L_v(L_v\vec{w}) = \frac{\partial^2\vec{w}}{\partial t^2} + 2\frac{\partial\vec{w}}{\partial t} \vec{v} - 2d\vec{v} \frac{\partial\vec{w}}{\partial t} + 2\frac{\partial(d\vec{v})}{\partial t} \vec{w} - \frac{d}{dt} \frac{\partial^2\vec{w}}{\partial t^2} + \left(d^2\vec{v} \vec{v}\right) \vec{v} + d\vec{v}.d\vec{v}.\vec{v} - 2d\vec{v}.d\vec{v}.\vec{v} - \left(d^2\vec{v} \vec{v}\right) \vec{w} + d\vec{v}.d\vec{v}.\vec{w}.
\]

(15.46)

**Answer.** (15.17) gives:

\[
L_v(L_v\vec{w}) = L_v\left(\frac{\partial\vec{w}}{\partial t}\right) + L_v(d\vec{v} \vec{v}) - L_v(d\vec{v} \vec{w})
\]

\[
= \frac{\partial^2\vec{w}}{\partial t^2} + \frac{\partial d\vec{v}}{\partial t} \vec{v} - 2d\vec{v} \frac{\partial\vec{w}}{\partial t} + \left(d\vec{v} \vec{v}\right) \vec{v} - \frac{d}{dt} \frac{\partial^2\vec{w}}{\partial t^2} + \left(d^2\vec{v} \vec{v}\right) \vec{v} + d\vec{v}.d\vec{v}.\vec{v} - 2d\vec{v}.d\vec{v}.\vec{v} - \left(d^2\vec{v} \vec{v}\right) \vec{w} + d\vec{v}.d\vec{v}.\vec{w}
\]

thus (15.46). With (15.17), \(2\) we get \( \frac{d^2\vec{w}}{dt^2} = \frac{\partial^2\vec{w}}{\partial t^2} + \frac{\partial d\vec{v}}{\partial t} \vec{v} + d\vec{v}.\frac{\partial\vec{w}}{\partial t} + \left(d^2\vec{v} \vec{v}\right) \vec{v} + d\vec{v}.d\vec{v}.\vec{v} \), \( \vec{v} \) and \( \vec{w} \) are constant and uniform.

\[
\frac{d^2\vec{w}}{dt^2} = \frac{\partial^2\vec{w}}{\partial t^2} + d\vec{v}.d\vec{v}.\vec{v} + d\vec{v}.d\vec{v}.\vec{w} \]

where (15.46).

### 15.7 Lie derivative of a differential form

Let \( \vec{w} \) be a vector field and \( \alpha \) be a differential form. Then \( f = \alpha.\vec{v} \) is a scalar function, thus, cf. (15.14),

\[
L_v(\alpha.\vec{w}) = \frac{D(\alpha.\vec{w})}{Dt} \quad \text{thus} \quad L_v(\alpha.\vec{w}) = \frac{D\alpha}{Dt} \vec{w} + \alpha. \frac{D\vec{w}}{Dt}.
\]

(15.47)

And \( L_v\vec{w} = \frac{D\vec{w}}{Dt} - d\vec{v}.\vec{w} \), cf. (15.17). Thus

\[
L_v(\alpha.\vec{w}) = \alpha. L_v\vec{w} + \frac{D\alpha}{Dt} \vec{w} + \alpha. d\vec{v}.\vec{w}.
\]

(15.48)
Definition 15.16 Let $\alpha$ be a differential form. The Lie derivative of $\alpha$ along $\vec{v}$ is the differential form defined by
\[
\mathcal{L}_{\vec{v}}\alpha := \frac{D\alpha}{Dt} + \alpha.d\vec{v} = \frac{\partial\alpha}{\partial t} + da.\vec{v} + \alpha.d\vec{v}.
\] (15.49)
(An equivalent definition will be given at (15.55).) Which means, for all vector field $\vec{w}$,
\[
\mathcal{L}_{\vec{v}}\alpha.\vec{w} = \frac{\partial\alpha}{\partial t}.\vec{w} + (da.\vec{v}).\vec{w} + \alpha.(d\vec{v}.\vec{w}).
\] (15.50)
This definition enables to get the “usual derivation property”
\[
\mathcal{L}_{\vec{v}}(\alpha.\vec{w}) = (\mathcal{L}_{\vec{v}}\alpha).\vec{w} + \alpha.(\mathcal{L}_{\vec{v}}\vec{w}).
\] (15.51)
(That is, $\mathcal{L}_{\vec{v}}$ is a derivation.)

Quantification with a basis $(\vec{e}_i)$: Let $\alpha = \sum_{i=1}^n \alpha_i e^i$, $\vec{v} = \sum_{i=1}^n v^i \vec{e}_i$, $\vec{w} = \sum_{i=1}^n w^i \vec{e}_i$, $da = \sum_{i,j=1}^n \alpha_{ij} e^i \otimes e^j$, $\vec{v} = \sum_{i,j=1}^n \alpha_{ij} v^j e_i \otimes e^j$. Then
\[
\mathcal{L}_{\vec{v}}\alpha = \sum_{i=1}^n \frac{\partial\alpha_i}{\partial t} e^i + \sum_{i,j=1}^n \alpha_{ij} v^j e^i + \sum_{i,j=1}^n \alpha_{ij} v^j e^i = \sum_{i=1}^n \frac{\partial\alpha_i}{\partial t} e^i + \sum_{j=1}^n \left( \frac{\partial\alpha_j}{\partial t} + (\alpha_{ij} v^j + \alpha_{ij} v^j \vec{v}) \right) e^i,
\] (15.52) and
\[
\mathcal{L}_{\vec{v}}\alpha.\vec{w} = \sum_{i=1}^n \left( \frac{\partial\alpha_i}{\partial t} + \alpha_{ij} v^j \vec{v} \right) \vec{e}_i = 0.
\] (15.53)

Proposition 15.17 Let $\alpha$ be a differential form, and let $\alpha_t(p) := \alpha(t,p)$. Then
\[
\mathcal{L}_{\vec{v}}\alpha = 0 \iff \frac{D\alpha}{Dt} = -\alpha.d\vec{v} \iff \forall t, \tau \in [t_0,T], \alpha_{\tau} = (\Phi_{\tau}^t)*\alpha_t.
\] (15.54)

Proof. Let $p(\tau) = \Phi(\tau,p_t)$
\[
\Rightarrow: \text{ If } \alpha_{\tau}(p(\tau)) = \alpha_t(p_t).F^t(p_t)^{-1}, \text{ then } \alpha(\tau,p(\tau)).F^t(\tau,p_t) = \alpha_t(p_t), \text{ thus } \frac{D\alpha}{Dt}(\tau,p(\tau)).F^t(\tau,p_t) + \alpha_\tau(p(\tau)) = 0 \text{, thus } \frac{D\alpha}{Dt}(\tau,p(\tau)).F^t(\tau,p_t) = 0 \text{, thus } \mathcal{L}_{\vec{v}}\alpha = 0 \text{ since } F^t_{\tau} \text{ is a diffeomorphism}.
\]
\[
\Rightarrow: \text{ If } \beta(\tau,t,p(t)) = \alpha_t(p(t)).F^t(p(t)) \text{ (pull-back), then } \beta(\tau,t,p(t)) = \alpha(\tau,p(t)).F^t(\tau,p_t). \text{ Thus } \frac{D\alpha}{Dt}(\tau,t,p_t) = 0 \text{, thus } \mathcal{L}_{\vec{v}}\alpha = 0 \text{ since } F^t_{\tau} \text{ is a diffeomorphism}. \text{ Thus } \frac{D\alpha}{Dt}(\tau,t,p_t) = 0 \text{ since } \Phi_{\tau}^t \text{ is a diffeomorphism}. \text{ Thus } \frac{D\alpha}{Dt}(\tau,t,p_t) = \frac{D\alpha}{Dt}(t,t,p_t) \text{, thus } \alpha_t \text{ is the pull-back of } \alpha_\tau.
\]

Remark 15.18 A definition equivalent to (15.9) is
\[
\mathcal{L}_{\vec{v}}\alpha(t_0) := \lim_{\tau \to t} \frac{(\Phi_{\tau}^t)^{-1}_* \alpha_t(p_t) - \alpha_t(p_t)}{\tau - t} = \frac{D(\Phi_{\tau}^t)_*\alpha_t(p_t)}{D\tau} \bigg|_{\tau = t} = \frac{D(\alpha_t(p_t))}{D\tau} \bigg|_{\tau = t} = \frac{D(\alpha_t(p_t)).F^t(p_t)}{D\tau} \bigg|_{\tau = t}.
\] (15.55)
Indeed, if $\beta(\tau) = (\Phi_{\tau}^t)^*\alpha_t(p_t) = \alpha_t(p_t).d\Phi_{\tau}^t(p_t)$, then $\beta'(\tau)$ and then $\tau = t$ give (15.49).

Exercice 15.19 Let $\vec{v}, \alpha C^2$. Prove:
\[
\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\alpha) = \frac{\partial^2\alpha}{\partial t^2} + 2d\frac{\partial\alpha}{\partial t}.\vec{v} + 2d\frac{\partial\alpha}{\partial \vec{v}}.\vec{v} + da.\frac{\partial\alpha}{\partial \vec{v}} + da.\vec{v}.\frac{\partial\alpha}{\partial \vec{v}} + \alpha.(d^2\vec{v}.\vec{v}) + (\alpha.d\vec{v}).\vec{v}.
\] (15.56)

Answer. (15.49) gives
\[
\mathcal{L}_{\vec{v}}(\mathcal{L}_{\vec{v}}\alpha) = \mathcal{L}_{\vec{v}}\left(\frac{\partial\alpha}{\partial \vec{v}}\right) + \mathcal{L}_{\vec{v}}(da.\vec{v}) + \mathcal{L}_{\vec{v}}(\alpha.d\vec{v}) = \frac{\partial^2\alpha}{\partial t^2} + d\left(\frac{\partial\alpha}{\partial \vec{v}}\right).\vec{v} + \frac{\partial\alpha}{\partial \vec{v}}.d\vec{v} + \frac{\partial(\alpha.d\vec{v})}{\partial \vec{v}} + d(da.\vec{v}).\vec{v} + (da.\vec{v}).d\vec{v} + \frac{\partial(\alpha.d\vec{v})}{\partial \vec{v}} + d(da.\vec{v})\vec{v} + (\alpha.d\vec{v}).\vec{v} + d\alpha.\frac{\partial\vec{v}}{\partial \vec{v}} + (d^2\alpha.\vec{v}) + da.\vec{v} + (d^2\alpha.\vec{v})\vec{v} + (\alpha.d\vec{v}).\vec{v} + d\left(\frac{\partial\alpha}{\partial \vec{v}}\right).\vec{v} + \frac{\partial\alpha}{\partial \vec{v}}.d\vec{v} + \alpha.d\vec{v} + (\alpha.d\vec{v})\vec{v} + (\alpha.d\vec{v}).\vec{v} = \frac{\partial^2\alpha}{\partial t^2} + 2d\frac{\partial\alpha}{\partial \vec{v}}.\vec{v} + 2d\frac{\partial\alpha}{\partial \vec{v}}.\vec{v} + da.\frac{\partial\alpha}{\partial \vec{v}} + da.\vec{v}.\frac{\partial\alpha}{\partial \vec{v}} + (d^2\alpha.\vec{v}) + da.(d\vec{v}.\vec{v}) + 2(da.\vec{v}).d\vec{v} + \alpha.(d^2\vec{v}.\vec{v}) + (\alpha.d\vec{v}).\vec{v}.
\] thus (15.56).
15.8 Incompatibility with the representation vector

The Lie derivative has nothing to do with any inner dot product (the Lie derivative does not compare two vectors to get a stress, contrary to a Cauchy type approach). But here, to deal with the classic literature, we consider a Euclidean dot product $g(\cdot,\cdot) = \langle \cdot, \cdot \rangle_g$.

Let $\alpha$ a differential form. Let $\tilde{\alpha}_g(t,p) \in \mathbb{R}^n$ be the $(\cdot,\cdot)_g$-Riesz representation vector of the linear form $\alpha(t,p)$, that is,

$$\forall \tilde{w} \in \mathbb{R}^n, \quad (\alpha(t,p),\tilde{w}) = (\tilde{\alpha}_g(t,p),\tilde{w})_g.$$  \hfill (15.57)

This defines the (Eulerian) vector field $\tilde{\alpha}_g$ (observer dependent since it depends on the choice of a Euclidean dot product, cf. (B.8): $\tilde{\alpha}_g$ is not intrinsic to $\alpha$).

**Proposition 15.20** For all $\tilde{v}, \tilde{w} \in \mathbb{R}^n$,

$$\frac{\partial \alpha}{\partial t}(\tilde{v}, \tilde{w}) + (\partial_x \tilde{v})(\tilde{w})_g = (\alpha, d\tilde{v}, \tilde{w})_g, \quad (\partial_t \tilde{v})(\tilde{w})_g = (\tilde{\alpha}_g(t,p), \tilde{w})_g.$$  \hfill (15.58)

Thus

$$L_{\partial \alpha}(\tilde{v}, \tilde{w}) = (L_{\partial \tilde{\alpha}_g} \tilde{w})_g + (\partial_t \tilde{v})(\tilde{w})_g + (\tilde{\alpha}_g, d\tilde{v}^T, \tilde{w})_g.$$  \hfill (15.59)

So $L_{\partial \tilde{\alpha}_g}$ isn’t the Riesz representation vector of $L_{\partial \alpha}$. (This is expected: A Lie derivative is covariant objective, see § 18.7, which is incompatible with the use of an inner dot product which ruins objectivity.)

**Proof.** $g(\cdot,\cdot)$ being constant and uniform (it is a Euclidean dot product: It is a metric independent of $t$ and $p$), (15.57) immediately gives (15.58) \textit{1,2}, thus (15.58) \textit{3}.

**Remark 15.21** So, to confuse “differential forms” (covariance) and “vector fields” (contravariance), because of the use of an inner dot product (which one?) and Riesz’s representation theorem, creates errors when using the Lie derivative. See also § C.3. It has consequences for constitutive laws of fluids (usually modeled with vector fields) and for constitutive laws of solids (usually modeled with differential forms).

**Exercise 15.22** In (15.59) when do we have $L_{\partial \alpha}(\tilde{v}, \tilde{w}) = (L_{\partial \tilde{\alpha}_g} \tilde{w})_g$?

**Answer.** When, for all $\tilde{w} \in \mathbb{R}^n$,

$$\frac{\partial \alpha}{\partial t}(\tilde{v}, \tilde{w}) + (\partial_x \tilde{v})(\tilde{w})_g + (\partial_t \tilde{v})(\tilde{w})_g = (\tilde{\alpha}_g, d\tilde{v}, \tilde{w})_g.$$  \hfill (15.60)

So when $\alpha.d\tilde{v} = -(d\tilde{v}, \tilde{w})_g$, i.e. $\alpha.d\tilde{v} = -(\tilde{\alpha}_g, d\tilde{v}^T, \tilde{w})_g = -\alpha.\tilde{v}$ for all $\tilde{w}$, i.e. $\alpha."d\tilde{v} = 0$, e.g. for a solid body motion.

**Exercise 15.23** Prove (15.58) \textit{2} with a $(\cdot,\cdot)_g$-orthonormal basis $\langle \tilde{e}_i \rangle$.

**Answer.** Let $(e^i)$ be the dual basis. Let $\alpha = \sum_i \alpha_i e^i, \quad \tilde{\alpha}_g = \sum_i \tilde{\alpha}_i e_i, \quad \tilde{v} = \sum_i v^i e_i, \quad \tilde{w} = \sum_i w^i e_i$. Thus

$$\partial_x \tilde{v} = \sum_i \partial v^i e^i, \quad \partial \tilde{v}_g = \sum_i \partial \tilde{v}_i e^i, \quad d\tilde{v} = \sum_i \partial v^i e^i. \quad \text{Thus } d\tilde{v} = \sum_i \frac{\partial v^i}{\partial x^j} v^j e^i.$$  \hfill (15.58) \textit{2}

Thus $\alpha.d\tilde{v} = \sum_i \partial v^i \tilde{v}^i = \sum_i \partial \tilde{v}_i \tilde{v}^i = \sum_i \partial v^i \tilde{v}^i.$

15.9 Lie derivative of a tensor

15.9.1 Formula

A tensor is defined with vector fields and differential forms. That makes it possible to define the Lie derivative of any tensor with one of the equivalent definitions:

$$L_{\partial} (T \otimes S) = (L_{\partial} T) \otimes S + T \otimes (L_{\partial} S), \quad \text{or } L_{\partial} T(t_0,p_0) = \left. \frac{D((\Phi^t)^* T)(p_0)}{Dt} \right|_{t=t_0}. \hfill (15.61)$$

15.9.2 Lie derivative of a mixed tensor

Mixed tensor $T_m \in T_1^1(\Omega)$: We get the Lie derivative called the Jaumann derivative:

$$L_{\partial} T_m = \left. \frac{DT_m}{Dt} \right|_{t=t_0} = \frac{\partial T_m}{\partial t} + d\tilde{v}.T_m + T_m.d\tilde{v} = \frac{\partial T_m}{\partial t} + d\tilde{v}.T_m + T_m.d\tilde{v}. \hfill (15.62)$$

Can be checked with an elementary tensor $T = \tilde{w} \otimes \alpha$ and $L_{\partial} (\tilde{w} \otimes \alpha) = (L_{\partial} \tilde{w}) \otimes \alpha + \tilde{w} \otimes (L_{\partial} \alpha)$. 77
15.9.3 For a non mixed tensor

For non mixed tensor, we recall:

**Definition 15.24** If $bli ∈ ℒ(E, F; ℝ)$ (bilinear), then its transposed $bli^T ∈ ℒ(F, E; ℝ)$ is defined by

$$∀(v, u) ∈ E × F, \quad bli^T(v, u) = bli(u, v)$$

(15.63)

**Definition 15.25** Let $L ∈ ℒ(E; F)$ (a linear map). Its adjoint $L^* ∈ ℒ(F^*; E^*)$ is defined by, cf. § A.13,

$$∀m ∈ F^*, \quad L^*m = mL, \quad \text{i.e.} \quad ∀m, u ∈ (F^* × E), \quad (L^*m).u = m.(L ⋅ u).$$

(15.64)

(There is no inner dot product involved.) (E.g., with $L = d⃗v_l ∈ ℒ(R^n_l; R^n_l)$, then $(d⃗v_l)^* ∈ ℒ(R^n_l^*; R^n_l^*)$)

15.9.4 Lie derivative of a up-tensor

Up tensor: $T_u ∈ T^2_0(Ω)$: The Lie derivative is called the Oldroyd or the upper-convected Maxwell derivative:

$$[L_vT_u] = \frac{DT_u}{Dt} - d⃗v.T_u - T_u.d⃗v^∗ = \frac{dT_u}{dt} + d⃗v,u - d⃗v.T_u - T_u.d⃗v^∗.$$  (15.65)

Can be checked with an elementary tensor $T = 1 ⋅ 1$ and $L_v(1 ⋅ 1) = (L_v1) ⋅ 1 + 1 ⋅ (L_v1)$.

The “up” refers to positions of indices when a basis $(e_i)$ in $R^n$ is used: $[T_u]_i = [T^2]_i$, that is,

$$T_u = \sum_{j=1}^n T^i_j e_i ⋅ e_j.$$ And here $d⃗v_l(p) ∈ ℒ(R^n_l; R^n_l)$ (Lie transposed $e_i ⋅ e_j$): We have $d⃗v_l(p) = \sum_i^n v^i_p e_i ⋅ e_i$. And $d⃗v_l(p) = \sum_{j=1}^n v^i_p e_i ⋅ e_j ∈ ℒ(R^n_l; R^n_l)$: $d⃗v_l(p) = \sum_i^n v^i_p T_k^j e_i ⋅ e_j$, and $T_u.d⃗v = \sum_{j,k=1}^n T^i_j k^j e_i ⋅ e_j$.

$$[L_vT_u]_i = \frac{DT_u}{Dt}_i - [d⃗v]_i[T_u]_i = [T_u]_i.[d⃗v]^T.$$  (15.66)

15.9.5 Lie derivative of a down-tensor

Down tensor: $T_d ∈ T^2_0(Ω)$: The Lie derivative is called the lower-convected Maxwell derivative:

$$[L_vT_d] = \frac{DT_d}{Dt} + d⃗v.T_d + d⃗v^∗.T_d = \frac{dT_d}{dt} + d⃗v,u + T_d.d⃗v + d⃗v^∗.T_d.$$  (15.67)

Can be checked with an elementary tensor $T = 1 ⋅ 1$ and $L_v(1 ⋅ 1) = (L_v1) ⋅ 1 + 1 ⋅ (L_v1)$.

The “down” refers to positions of indices when a basis $(e_i)$ in $R^n$ is used: $[T_d]_i = [T^2]_i$, that is,

$$T_d = \sum_{j=1}^n T^i_j e_i ⋅ e_j.$$ Thus $T_d.d⃗v = \sum_{j,k=1}^n T^i_j k^j e_i ⋅ e_j$ and $d⃗v^∗.T_d = \sum_{j,k=1}^n T^i_j k^j e_i ⋅ e_j$. Thus

$$[L_vT_d]_i = \frac{DT_d}{Dt}_i + [d⃗v]_i[T_d]_i + [d⃗v]^∗_i[T_d].$$  (15.68)

**Exercise 15.26** Let $g = (., .)_g$ be a constant and uniform metric (a unique inner dot product for all $t, p$, e.g., a Euclidean dot product at all $t$). And with the Riesz representation theorem define $d⃗v^g ∈ T^2_0(Ω)$ from $d⃗v ∈ T^2_0(Ω)$ by:

$$d⃗v^g(u, w) = (d⃗v(u), w) = (d⃗w^T(u), w) = (d⃗v^T(u), w).$$

(15.69)

**Prove:**

$$d⃗v^g = g.d⃗v.$$  (15.70)

**Prove:**

$$L_vg = d⃗v^g + (d⃗v^g)^T$$ and $L_αg = g.d⃗v + d⃗v^∗.g.$

(15.71)

If $(e_i)$ is a $(., .)_g$-orthonormal basis, prove:

$$[L_vg]_e = [d⃗v]^T_e + [d⃗v^g]^T_e.$$  (15.72)

**Answer.** Let $(e_i)$ be a basis, $d⃗v = \sum_{j=1}^n v^i_j e_i ⋅ e_i$, $d⃗v^g = \sum_{j=1}^n T^i_j e_i ⋅ e_i$, $g = \sum_{j=1}^n g^i_j e_i ⋅ e_i$. Thus $g.d⃗v = \sum_{j=1}^n g^i_j v^i_j e_i ⋅ e_i$, that is $(g.d⃗v)(e_i, e_j) = \sum_{j=1}^n g^i_j v^i_j$ for all $i, j$. And (15.69) gives $d⃗v^g(e_i, e_j) = g.(d⃗v^g)(e_i, e_j) = g^i_j (d⃗v^g)(e_i, e_j) = (g.d⃗v)(e_i, e_j)$ for all $i, j$, thus (15.70).

Then $d⃗v(i, j) = (d⃗v^g)(e_i, e_j) = (g.d⃗v)(e_i, e_j) = \sum_{j=1}^n [g^i_j [d⃗v]^T_j ⋅ [g^i_j]_j]_k (since g(., .) is symmetric) = (d⃗v^g)(e_i, e_j)$, thus (15.70).

(15.67) and (15.70) give $L_vg = 0 + g.d⃗v + d⃗v^∗ = d⃗v^g + (d⃗v^g)^T$, i.e. (15.71).

In particular, if $(e_i)$ is a $(., .)_g$-orthonormal basis, then $[g]_e = I$, thus (15.72).
Part IV
Velocity-addition formula and Objectivity

16 Change of referential

16.1 Introduction and problem

Let \( \text{Obj} \) be a material object. Let \( P_{\text{Obj}} \in \text{Obj} \) be a particle, and consider its Eulerian velocities \( \vec{v}_A \) in a referential \( \mathcal{R}_A \) and \( \vec{v}_B \) in a referential \( \mathcal{R}_B \), and let \( \vec{v}_D \) be the velocity of \( \text{Obj}_B \) in \( \mathcal{R}_A \) (the drive speed = vitesse d’entrainement). The classical expression of the velocity-addition formula is:

\[
(\vec{v}_A \text{ the absolute velocity}) = (\vec{v}_B \text{ the relative velocity}) + (\vec{v}_D \text{ the drive speed of } \mathcal{R}_B \text{ in } \mathcal{R}_A), \tag{16.1}
\]

But (16.1) is problematic (self contradictory):
1. The velocities \( \vec{v}_A \) and \( \vec{v}_B \) are velocities in \( \mathcal{R}_A \); e.g., expressed in foot/s by the absolute observer.
2. The velocity \( \vec{v}_B \) is a velocity in \( \mathcal{R}_B \); e.g., expressed in meter/s by the relative observer.
3. Thus (16.1) seems to indicate that the right hand side adds meter/s with foot/s, which is absurd.

Thus \( \vec{v}_B \) cannot just be the velocity in the relative referential. It has to be the “translated velocity for observer \( A \)”.

Details and explanations:

16.2 Framework

Classical mechanics. The same universe for all observers, modeled as an affine space \( \mathbb{R}^n \) with the associated vector space \( \mathbb{R}^n \) (made of bipoint vectors), with \( n = 3 \), or with \( n = 1 \) or 2 if 1-D or 2-D problems are studied. To lighten the writing, the observers use the same time scale and same time origin (otherwise we need to define referentials in \( \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^n = \text{time} \times \text{space} \).

An observer \( A \) chooses a point \( O_A \in \mathbb{R}^n \) (origin), a basis \( (\vec{A}_i) \in \mathbb{R}^n \) to build his referential \( \mathcal{R}_A = (O_A, (\vec{A}_i)) \), the same at all time. And an observer \( B \) chooses a point \( O_B \in \mathbb{R}^n \) (origin), a basis \( (\vec{B}_i) \in \mathbb{R}^n \) to build his referential \( \mathcal{R}_B = (O_B, (\vec{B}_i)) \), the same at all time. Illustration: They both live on “rigid like bodies”, e.g., with \( O_A \) the center of the sun and \( (\vec{A}_i) \) fixed relative to stars, and \( O_B \) some point on Earth and \( (\vec{B}_i) \) fixed on Earth.

Observers \( A \) and \( B \) observe the same “deformable material object” \( \text{Obj} \). The position of a particle \( P_{\text{Obj}} \in \text{Obj} \) in the affine space \( \mathbb{R}^n \) is given thanks to motions, cf. (1.2): The motion of \( \text{Obj} \) is described by observer \( A \) as being \( \Phi_A : [t_1, t_2] \times \text{Obj} \to \mathcal{R}_A \), and is called the absolute motion, and the associated Eulerian velocity field \( \vec{v}_A \) is called the absolute velocity field. And the motion of \( \text{Obj} \) is described by observer \( B \) as being \( \Phi_B : [t_1, t_2] \times \text{Obj} \to \mathcal{R}_B \), and is called the relative motion, and the associated Eulerian velocity field \( \vec{v}_B \) is called the relative velocity field.

So, (1.2) and (2.5) give, for any particle \( P_{\text{Obj}} \in \text{Obj} \) and any \( t \in [t_1, t_2] \),

\[
\begin{align*}
\vec{v}_A(t, p_{\text{Obj}}) &= \Phi_A(t, P_{\text{Obj}}) = O_A + \sum_{i=1}^{n} x_{\text{Obj}}^{i} \vec{A}_i = \text{position of } P_{\text{Obj}} \text{ at } t \text{ located by } A \text{ in } \mathcal{R}_A, \\
\vec{v}_B(t, p_{\text{Obj}}) &= \frac{\partial \Phi_B}{\partial t}(t, P_{\text{Obj}}) = \text{velocity of } P_{\text{Obj}} \text{ at } t \text{ at } p_{\text{Obj}} \text{ in } \mathcal{R}_B, \\
\vec{D}_B(t, p_{\text{Obj}}) &= \frac{\partial \Phi_B}{\partial t}(t, P_{\text{Obj}}) = \text{velocity of } P_{\text{Obj}} \text{ at } t \text{ at } p_{\text{Obj}} \text{ in } \mathcal{R}_B.
\end{align*}
\tag{16.2}
\]

Let \( \text{Obj}_B \) be a material rigid body on which \( \mathcal{R}_B \) is fixed (e.g. \( \text{Obj}_B = \text{the Earth} \)). The rigid body motion of \( \text{Obj}_B \) is described by observer \( A \) as being \( \Phi_D : [t_1, t_2] \times \text{Obj}_B \to \mathcal{R}_A \), and is called the drive motion (mouvement d’entrainement), and the associated Eulerian velocity field \( \vec{v}_D \) is called the drive velocity field. The rigid body motion of \( \text{Obj}_B \) is described by observer \( B \) as being \( \Psi : \text{Obj}_B \to \mathcal{R}_B \) (static function which locates the particles \( Q_{\text{Obj}_B} \in \text{Obj}_B \) in the referential chosen by \( B \)), and the associated
Eulerian velocity field vanishes. So, (1.2) and (2.5) give, for any particle \( Q_{Obj} \in Obj \) and any \( t \in [t_1, t_2] \),

\[
\begin{align*}
q_A &= \tilde{\Phi}_D(t, Q_{Obj}) = O_A + \sum_{i=1}^{n} y_{At}^i \tilde{A}_i = \text{position of } Q_{Obj} \text{ at } t \text{ located by } A \text{ in } \mathcal{R}_A, \\
\vec{v}_D(t, q_A) &= \frac{\partial \tilde{\Phi}_D}{\partial t}(t, Q_{Obj}) = \text{velocity of } Q_{Obj} \text{ at } t \text{ at } q_A \text{ in } \mathcal{R}_A, \\
q_B &= \tilde{\Psi}_B(Q_{Obj}) = O_B + \sum_{i=1}^{n} y_B^i \tilde{B}_i = (\text{static}) \text{ position of } Q_{Obj} \text{ located by } B \text{ in } \mathcal{R}_B, \\
\vec{0} &= \frac{\partial \tilde{\Psi}_B}{\partial t}(Q_{Obj}) = \text{null velocity since } Q_{Obj} \text{ does not move in } \mathcal{R}_B.
\end{align*}
\]

If \( t \) is fixed, then let \( \tilde{\Phi}_{Z \rightarrow t}(P_{Obj}) := \tilde{\Phi}_Z(t, P_{Obj}) \) and \( \tilde{\Phi}_{Z \rightarrow t}(p_{Zt}) := \tilde{\Phi}_Z(t, p_{Zt}) \) and \( \Omega_{Zt} := \tilde{\Phi}_Z(t, P_{Obj}) \), for \( Z = A, B, D \).

The drive motion \( \tilde{\Phi}_D \) being a rigid body motion, we have, for all \( q_{At} = \tilde{\Phi}_{Dt}(Q_{Obj}B_i) \),

\[
\begin{align*}
\tilde{\Phi}_{Dt}(q_{At}) &= \tilde{\Phi}_{Dt}(q_{At}) + \vec{d}_{Dt} \cdot \frac{\vec{q}_{At} \vec{q}_{At}^2}{|\vec{q}_{At} \vec{q}_{At}^2|} = \tilde{\Phi}_{Dt}(q_{At}) + \vec{\omega}_t \wedge \frac{\vec{q}_{At} \vec{q}_{At}^2}{|\vec{q}_{At} \vec{q}_{At}^2|} \in \mathcal{R}_A,
\end{align*}
\]

\( \text{cf. (9.20)}. \)

**Remark 16.1** Let \( P_{Obj} \) be a particle in \( Obj \) followed by the observers \( A \) and \( B \). Let \( p_t \in \mathbb{R}^n \) be its position at \( t \): In the Universe, a particle has just one position since there is no ubiquity in classical mechanics. E.g., \( P_{Obj} \) is the Eiffel tower and its position at \( t \) is \( p_t \) (the Earth moves in the Universe), regardless of the observer (qualitative approach). And, for quantification purposes, the position \( p_t \) is called \( p_{At} \in \mathcal{R}_A \) by observer \( A \) and is called \( p_{Bt} \in \mathcal{R}_B \) by observer \( B \),

\[
p_t = p_{At} = p_{Bt} \in \mathbb{R}^n, \tag{16.5}
\]

\( \text{cf. (16.2)}_{1,3} \) (quantification: Observer dependent). Thus, if you consider two particles \( P_{Obj}1, P_{Obj}2 \), then

\[
\vec{p}_{AtBt} = p_{At} - p_{Bt} \in \mathbb{R}^n \tag{16.6}
\]

is the same bipoint vector for all observers (qualitative approach, see e.g. example 1.1), but its description in a referential (= quantification) depends on the observer, \( \text{cf. (16.2)}_{1,3} \).

\[
\bullet
\]

### 16.3 The translator \( \Theta_t \) for positions

#### 16.3.1 Definition

A motion is an “intra-referential” mapping connecting a particle and a position, \( \text{cf. (1.2)} \): \( \tilde{\Phi}_A \) and \( \tilde{\Phi}_D \) are defined by the observer \( A \) in his referential, \( \tilde{\Phi}_D \) and \( \tilde{\Psi}_B \) are defined by the observer \( B \) in his referential.

**Definition 16.2** At any \( t \), the translation operator (the translator) \( \Theta_t \) from \( B \) to \( A \) is the “inter-referential” diffeomorphism which links at \( t \) the positions of particles of \( ObjB \) as referred to by observers \( A \) and \( B \): That is, \( \Theta_t \) is defined by, for any \( Q_{Obj} \in ObjB \),

\[
\begin{align*}
\left\{ \begin{array}{l}
q_B = \tilde{\Psi}_B(Q_{Obj}) \text{ (static) position of } Q_{Obj} \\
q_A = \tilde{\Phi}_D(Q_{Obj}) \text{ position of } Q_{Obj} \text{ at } t \end{array} \right\} \implies \Theta_t(q_B) = q_A. \tag{16.7}
\end{align*}
\]

That is, at any \( t \), the translator \( \Theta_t \) is defined by

\[
\Theta_t := \tilde{\Phi}_D \circ \tilde{\Psi}_B^{-1} : \quad \mathcal{R}_B \rightarrow \mathcal{R}_A \quad \frac{q_B \text{ position}}{q_B} \rightarrow \frac{q_A \text{ position}}{q_A} = \Theta_t(q_B) := \tilde{\Phi}_D(\tilde{\Psi}_B^{-1}(q_B)) \text{ named } q_{Bt}. \tag{16.8}
\]

the notation \( q_{Bt} := \Theta_t(q_B) \) being the notation of the push-forward by \( \Theta_t \), \( \text{cf. (10.2)} \).
So, at $t$, the link between the positions $q_A(t)$ and $q_B(t)$ of a particle $Q_{ObjB} \in \text{ObjB}$ is (translation for $A$):

$$q_A = \Theta_t(q_B) = \Theta_t(\tilde{\Psi}_B(Q_{ObjB})) := \tilde{\Phi}_D(t, \tilde{\Psi}_B^{-1}(q_B)) = \tilde{\Phi}_D(t, Q_{ObjB}) \text{ named } q_{B*} \in \mathcal{R}_A. \quad (16.9)$$

In other words, $\Theta_t$ is characterized by,

$$\Theta_t \circ \tilde{\Psi}_B = \tilde{\Phi}_D, \quad \text{i.e. } \Theta_t(\tilde{\Psi}_B(Q_{ObjB})) = \tilde{\Phi}_D(t, Q_{ObjB}) \text{ of a particle } Q_{ObjB} \text{ at } t. \quad (16.10)$$

With (16.8) we have defined

$$\Theta : \{[t_1, t_2] \times \mathcal{R}_B \rightarrow \mathcal{R}_A \}$$

$$\Theta(t, q_B) := \Theta_t(q_B) = \tilde{\Phi}_D(t, \tilde{\Psi}_B^{-1}(q_B)) \text{ named } q_{B*}(t), \quad (16.11)$$

called the translator from $B$ and $A$.

**Remark 16.3** NB: The translator $\Theta$ is not a motion: A motion is defined by only one observer and connects a particle and a position, cf. (1.2), when the translator connects two positions as spotted by two observers relative to their referentials, cf. (16.10).

**Exercise 16.4** $\tilde{\Psi}_B$ being a rigid body motion, prove that a straight line in $\mathcal{R}_B$ is seen at $t$ as a straight line in $\mathcal{R}_A$. Deduce that $\Theta_t$ is affine, that is, for all $q_{B1}, q_{B2}$

$$d\Theta_t(q_{B1}) = d\Theta_t(q_{B2}) \text{ named } d\Theta_t, \quad \text{and } \Theta_t(q_{B1} + \lambda q_{B2}) = \Theta_t(q_{B1}) + \lambda d\Theta_t(q_{B2}). \quad (16.12)$$

**Answer.** Straight line as described by observer $B$: Let $p \in \mathbb{R}^n, \vec{v} \in \mathbb{R}^n$, and $c : s \in \mathbb{R} \rightarrow c(s) = p + s\vec{v} \in \mathbb{R}^n$ (straight line in $\mathcal{R}_B$, in particular $c(0) = p$). Then (16.6) and (16.10) give $\tilde{\Phi}_D(c(s)) = c(0)c(s) = s\vec{v} \in \mathbb{R}^n$, thus $\tilde{\Phi}_D(c(s)) = \Theta_t(c(0)) + s\vec{v}$, a straight line in $\mathcal{R}_A$. True for all $\vec{v}$. Thus $\Theta_t$ is affine.

**Exercise 16.5** Setting of exercise 16.4. Observer $B$ chooses a Euclidean basis $(\vec{B}_i)$ at $t_0$ and, calls $(\cdot, \cdot)_a$ the associated Euclidean dot product. Let $q_{B1}$ be the points defined by $\vec{B}_i = \tilde{O}_B q_{B1}$ as seen by $B$. Then let $O_{B*} := \Theta_t(O_{B})$ and $q_{B*} := \Theta_t(q_{B})$ and $\vec{B}_{it*} := \tilde{O}_B q_{B*}$, the positions and bispot vectors as seen by $A$. Observer $A$ chooses a Euclidean dot product $(\cdot, \cdot)_a$ at $t$. Let $\lambda > 0$ such that $(\cdot, \cdot)_a = \lambda^2 (\cdot, \cdot)_a$. Deduce from (16.6) and (16.12) that

$$(\vec{B}_{it*}, \vec{B}_{jt*})_a = \lambda^2 \delta_{ij}, \quad \text{and } \vec{B}_{it*} = d\Theta_t(\vec{B}_i), \quad \text{and } (d\Theta_t)_{ab} d\Theta_t = \lambda^2 I. \quad (16.13)$$

(In particular, $(\vec{B}_{it*})_a$ is a $(\cdot, \cdot)_a$-Euclidean basis at $t$.)

**Answer.** (16.6) gives $\tilde{\Phi}_D(O_{B*}) = \tilde{O}_B q_{B*}$, thus $\vec{B}_{it*} = \vec{B}_i$, thus $(\vec{B}_{it*}, \vec{B}_{jt*})_a = (\vec{B}_i, \vec{B}_j)_a = \lambda^2 (\vec{B}_i, \vec{B}_j)_a$. Thus (16.13).

Then (16.12) gives $\vec{B}_{it*} = d\Theta_t(\vec{B}_i)$. Thus (16.13), since $\Theta_t$ is affine, cf. (16.12). Then, $d\Theta_t(q_{B}) = d\Theta_t \in \mathcal{C}(\mathcal{R}_A; \mathcal{R}_B)$ being a linear map, its transposed $(d\Theta_t)^T$ relative to the chosen inner dot products is given by $( (d\Theta_t)^T \vec{x}, \vec{B}_i)_a = (d\Theta_t, \vec{B}_j, \vec{x})_a$ for all $\vec{x} \in \mathcal{R}_A$ and all $j$, cf. (A.24). Thus (16.13) gives $(\vec{B}_{it*})_a = (d\Theta_t(\vec{B}_i, \vec{B}_j)_a = (d\Theta_t(\vec{B}_i, \vec{B}_j)_a = (d\Theta_t)_{ab} (\vec{B}_i, \vec{B}_j)_a$ for all $i, j$, with $\delta_{ij} = (\vec{B}_i, \vec{B}_j)_a = (\vec{I}_B, \vec{B}_j)_a$. Thus $(d\Theta_t)_{ab} = \lambda^2 I$.)

**16.3.2 With an initial time**

Consider a $t_0 \in [t_1, t_2]$ (e.g. if you want to consider Lagrangian variables). The motion $\Phi_{D}^t$ associated to $\tilde{\Phi}_D$ is defined by, cf. (3.5),

$$q_{At0} = \tilde{\Phi}_D(t_0, Q_{ObjB}) \text{ and } q_{At} = \tilde{\Phi}_D(t, Q_{ObjB}) \implies \Phi_{D}^t(q_{At0}) := q_{At}. \quad (16.14)$$

Thus (16.7) gives $\Theta_t(q_B) = \Phi_{D}^t (q_{At0}) = \Phi_{D}^t (\Theta_{t0}(q_B))$, so

$$\Theta_t = \Phi_{D}^t \circ \Theta_{t0}. \quad (16.15)$$

**Interpretation:** The motion $\tilde{\Phi}_D$ being known by $A$, if $\Theta_{t0}$ is known (there has been an initial communication between the observers $B$ and $A$), then $\Theta_t$ is known at all $t$ (e.g. if $B$ does the measurements for $A$, and if $B$ gives to $A$ its referential at $t_0$, then $A$ knows the referential of $B$ at all times).

**Remark 16.6** (16.7) and (16.14) look alike, but (16.7) has no $t_0$ (recall: $\Theta$ is not a motion). And a space differentiation $d\Theta$ of $\Theta$ is meaningful for the translator $\Theta$ which acts in $\mathbb{R}^n$ (acts on positions): When a space differentiation for a motion $\tilde{\Phi}$ cannot be done directly, since $\tilde{\Phi}$ acts on a material object $\text{Obj}$ (acts on particles), and there is no differential structure in $\text{Obj}$, cf. remark 1.4. In fact, we use a motion to associate to $\text{Obj}$ a differential structure at any $t$: In $\Omega_t = \Phi(t, \text{Obj})$.
16.4 The translator for vector fields

Usual steps with push-forwards. We recall the steps (a vector field is a field of tangent vectors).

Let \( t \) be fixed, let \( c_{Bt} \) be a curve in \( \mathbb{R}^n \) as described by \( B \) at \( t \), and let \( \tilde{w}_{Bt} \) the vector field of tangent vectors to \( \text{Im}(c_{Bt}) \):

\[
c_{Bt} : \left\{ \begin{array}{l}
\mathbb{R} \rightarrow \mathcal{R}_B \\
s \rightarrow q_{Bt} = c_{Bt}(s)
\end{array} \right. \quad \text{and} \quad \tilde{w}_{Bt}(q_{Bt}) = \frac{dc_{Bt}}{ds}(s). \quad (16.16)
\]

See figure 10.1. (The variable \( s \) is a spatial coordinate in the photo taken by \( B \) at \( t \).)

Still at \( t \), the curve \( c_{Bt} \) is spotted in \( \mathcal{R}_A \), by the observer \( A \), as being the (translated) curve

\[
c_{Bt^*} = \Theta_t \circ c_{Bt} : \left\{ \begin{array}{l}
\mathbb{R} \rightarrow \mathcal{R}_A \\
s \rightarrow q_{Bt^*} = c_{Bt^*}(s) := \Theta_t(q_{Bt}), \quad \text{when} \ q_{Bt} = c_{Bt}(s),
\end{array} \right. \quad (16.17)
\]

thus its tangent vectors at \( q_{Bt^*} = c_{Bt^*}(s) := \Theta_t(c_{Bt}(s)) = \Theta_t(q_{Bt}) \) are given by

\[
\tilde{w}_{Bt^*}(q_{Bt^*}) = \tilde{w}_{Bt^*}(c_{Bt^*}(s)) = \frac{dc_{Bt^*}}{ds}(s) = d\Theta_t(c_{Bt}(s)). \frac{dc_{Bt}}{ds}(s) = d\Theta_t(q_{Bt}).\tilde{w}_{Bt}(q_{Bt}). \quad (16.18)
\]

So (push-forward by \( \Theta_t \), for vector fields formula (10.28), see figure 10.1),

\[
\tilde{w}_{Bt^*}(q_{Bt^*}) = d\Theta_t(q_{Bt}).\tilde{w}_{Bt}(q_{Bt}) \in \mathcal{R}_A \quad \text{when} \ q_{Bt^*} = \Theta_t(q_{Bt}). \quad (16.19)
\]

That is, the vector \( \tilde{w}_{Bt}(q_{Bt}) \) seen by \( B \) is seen by \( A \) as being the vector \( \tilde{w}_{Bt^*}(q_{Bt^*}) \).

And the pull-back of a vector field \( \tilde{w}_A \) by \( \Theta_t \) (= the push-forward by \( \Theta_t^{-1} \)) is the translation at \( t \) from \( A \) to \( B \): It is the vector field \( \tilde{w}_{A^*} \) defined in \( \mathcal{R}_B \) by

\[
\tilde{w}_{A^*}(q_A^*) = d\Theta_t^{-1}(q_A^*).\tilde{w}_A(q_A) \in \mathcal{R}_B, \quad \text{when} \ q_A^* = \Theta_t^{-1}(q_A). \quad (16.20)
\]

Exercise 16.7 Prove that, \( \Theta_t \) being affine, if \( p_A \) = \( \Theta_t(p_{Bt}) \) then

\[
d\tilde{w}_{Bt^*}(p_{Bt}) = d\Theta_t^* . d\tilde{w}_{Bt}(p_{Bt}) . d\Theta_t^{-1} = (d\tilde{w}_{Bt^*})(p_{Bt}), \quad (16.21)
\]

push-forward of the endomorphism \( d\tilde{w}_{Bt}(p_{Bt}) \) by \( \Theta_t \), cf. (14.23).

Answer. \( \tilde{w}_{Bt^*}(\Theta_t(p_{Bt})) = d\Theta_t(p_{Bt^*}).\tilde{w}_{Bt}(p_{Bt}) \) gives \( d\tilde{w}_{Bt^*}(p_{Bt}).d\Theta_t(p_{Bt}) = d\Theta_t(p_{Bt}).\tilde{w}_{Bt}(p_{Bt}) + d\Theta_t(p_{Bt}).d\tilde{w}_{Bt}(p_{Bt}). \) And, \( \Theta_t \) affine gives \( d^2\Theta_t = 0 \) and \( d\tilde{w}_{Bt^*}(p_{Bt}) = d\Theta_t . d\tilde{w}_{Bt}(p_{Bt}) . d\Theta_t^{-1}. \)

16.5 The “\( \Theta \)-velocity” = the drive velocity

Definition 16.8 The vector defined at \( t \) at \( q_A \) in \( \mathcal{R}_A \) by:

\[
\tilde{v}_\Theta(t, q_A) = \frac{\partial \Theta}{\partial t} (t, q_A), \quad \text{when} \ q_A = \Theta_t(q_B), \quad (16.22)
\]

is called the “\( \Theta \)-velocity” at \( t \) at \( q_A \) in \( \mathcal{R}_A \).

NB: \( \tilde{v}_\Theta \) looks like a Eulerian velocity, since \( \tilde{v}_\Theta(t, \Theta(t, q_B)) = \frac{\partial \Theta}{\partial t}(t, q_B) \), but is not, because \( \Theta \) is not a motion: \( \Theta \) is an inter-referential mapping, see remark 16.3.

With (16.11) we get, in \( \mathcal{R}_A \),

\[
\frac{\partial \Theta}{\partial t}(t, q_B) = \frac{\partial \tilde{v}_D}{\partial t}(t, Q_{O_{Bt}}) \quad \text{when} \ q_B = \tilde{v}_D(Q_{O_{Bt}}), \quad (16.23)
\]

that is, with (16.22) and (16.3),

\[
\tilde{v}_\Theta(t, q_A) = \tilde{v}_D(t, q_A), \quad \text{so} \quad \tilde{v}_\Theta = \tilde{v}_D \quad (16.24)
\]

Thus the “\( \Theta \)-velocity” \( \tilde{v}_\Theta(t, q_A) \) in \( \mathcal{R}_A \) is equal to the (Eulerian) drive velocity \( \tilde{v}_D(t, q_A) \) in \( \mathcal{R}_A \), which is the velocity in \( \mathcal{R}_A \) of the particle \( Q_{O_{Bt}} \) in \( \mathcal{O}_{Bt} \) which is at \( t \) at \( q_A \).

E.g., if \( \tilde{v}_D \) is a rigid body motion, then \( \tilde{v}_\Theta(t, q_A) = \tilde{v}_D(q_A) = \tilde{v}_D(O_A) + \tilde{w}(t) \wedge \tilde{O}_Aq_A \), cf. (16.4).

Remark 16.9 We could formally define \( \tilde{v}_\Theta(t, q_B) := \frac{\partial \Theta}{\partial t}(t, q_B) \) which looks like a “Lagrangian velocity”, but is not: A Lagrangian function depends on some motion and on some initial time \( t_0 \), and \( \tilde{v}_\Theta := \frac{\partial \Theta}{\partial t} \) does not. See remark 16.3.
16.6 The velocity-addition formula

Let \( \mathcal{O}_B \in \mathcal{O}_A \) (a particle of \( \mathcal{O}_B \)), spotted at \( t \) by \( p_A(t) = \tilde{\Phi}_A(t, \mathcal{O}_B) \in \mathcal{R}_A \), and by \( B \) at \( p_B(t) = \tilde{\Phi}_B(t, \mathcal{O}_B) \in \mathcal{R}_B \). We have \( p_A(t) = \Theta(t, p_B(t)) \), cf. (16.8), that is,

\[
\tilde{\Phi}_A(t, \mathcal{O}_B) = \Theta(t, \tilde{\Phi}_B(t, \mathcal{O}_B)).
\]

Thus

\[
\frac{\partial \tilde{\Phi}_A}{\partial t}(t, \mathcal{O}_B) = \frac{\partial \Theta}{\partial t}(t, \tilde{\Phi}_B(t, \mathcal{O}_B)) + d\Theta(t, \tilde{\Phi}_B(t, \mathcal{O}_B)) \frac{\partial \tilde{\Phi}_B}{\partial t}(t, \mathcal{O}_B),
\]

and \( \tilde{v}_{B^*}(p_A(t)) = d\Theta(p_B(t)).\tilde{v}_B(p_B(t)) \) is the translated velocity at \( t \) for observer \( A \) at \( p_A(t) = \Theta(t, p_B(t)) \), cf. (16.19) (the push-forward vector by \( \Theta(t) \)). Thus, for all \( t \) and all \( p_A(t) = \tilde{\Phi}_A(t, \mathcal{O}_B) \),

\[
\tilde{v}_{A^*}(p_A(t)) = \tilde{v}_{B^*}(p_A(t)) + \tilde{v}_{B^*} - \tilde{v}_{B^*}.
\]

This is “the velocity-addition formula” in \( \mathcal{R}_A \), which reads with (16.24),

\[
(\tilde{v}_{A^*}(t) \text{ the absolute velocity}) = (\tilde{v}_{B^*}(t) \text{ the relative velocity translated for } A) + (\tilde{v}_{B^*}(t) \text{ the drive velocity}).
\]

16.7 The acceleration-addition formula

At \( t \), with \( p_A(t) = \tilde{\Phi}_A(t, \mathcal{O}_B) = \Theta(t, p_B) = \tilde{\Phi}_B(Q_{O_B}) = \Theta(t, q_B) \), define

\[
\begin{align*}
\tilde{\gamma}_A(t, p_A) &:= \frac{\partial^2 \tilde{\Phi}_A}{\partial t^2}(t, \mathcal{O}_B) \quad \text{(acceleration of } \mathcal{O}_B \text{ in } \mathcal{R}_A \text{ at } t), \\
\tilde{\gamma}_B(t, p_B) &:= \frac{\partial^2 \tilde{\Phi}_B}{\partial t^2}(t, \mathcal{O}_B) \quad \text{(acceleration of } \mathcal{O}_B \text{ in } \mathcal{R}_B \text{ at } t), \\
\tilde{\gamma}_\psi(t, q_A) &:= \frac{\partial^2 \Theta}{\partial t^2}(t, q_B) = \frac{\partial^2 \tilde{\Phi}_D}{\partial t^2}(t, \mathcal{O}_{Q_{O_B}}) = \tilde{\gamma}_D(t, p_A) \quad \text{(acceleration of } Q_{O_B} \text{ in } \mathcal{R}_A \text{ at } t),
\end{align*}
\]

the last equation thanks to (16.23). Then (16.26) gives, with \( p_B = \tilde{\Phi}_B(t, \mathcal{O}_B) \),

\[
\begin{align*}
\frac{\partial^2 \tilde{\Phi}_A}{\partial t^2}(t, \mathcal{O}_B) = & \frac{\partial^2 \Theta}{\partial t^2}(t, p_B) + \frac{\partial \Theta}{\partial t}(t, p_B). \frac{\partial \tilde{\Phi}_B}{\partial t}(t, \mathcal{O}_B) \\
& + \left( \frac{d\Theta}{dt}(t, p_B) + d^2 \Theta(t, p_B). \frac{\partial \tilde{\Phi}_B}{\partial t}(t, \mathcal{O}_B) \right) + d\Theta(t, p_B) \frac{\partial^2 \tilde{\Phi}_B}{\partial t^2}(t, \mathcal{O}_B).
\end{align*}
\]

That is, with (16.22), that is \( \frac{\partial \Theta}{\partial t}(t, q_B) = \tilde{v}_B(t, \Theta(t, q_B)) \),

\[
\tilde{\gamma}_A(t, p_A) = \tilde{\gamma}_A(t, p_A) + (d\tilde{v}_A(t, p_A).d\Theta(t, p_B)).\tilde{v}_B(t, p_B) \\
+ \left( (d\tilde{v}_A(t, p_A).d\Theta(t, p_B)) + d^2 \Theta(t, p_B). \tilde{v}_B(t, p_B) \right) + d\Theta(t, p_B) \tilde{\gamma}_B(t, p_B).
\]

(With the classical setting, \( \Theta(t) \) is affine thus \( d^2 \Theta(t) = 0 \). Thus, with (push-forward by the translator \( \Theta(t) \))

\[
\begin{align*}
\tilde{v}(p_A(t)) = & d\Theta(t, p_B).\tilde{v}_B(t, p_B) \quad \text{(the velocity } \tilde{v}(p_B(t)) \text{ translated for } A), \\
\tilde{\gamma}(p_A(t)) = & d\Theta(t, p_B).\tilde{\gamma}_B(t, p_B) \quad \text{(the acceleration } \tilde{\gamma}(p_B(t)) \text{ translated for } A),
\end{align*}
\]

we get

\[
\tilde{\gamma}_A(t, p_A) = \tilde{\gamma}_B(t, p_B) + \tilde{\gamma}_B(t, p_B) + \left( (d\tilde{v}(p_A(t)).d\Theta(t, p_B)).\tilde{v}(p_B(t)) + (d^2 \Theta(t, p_B). \tilde{v}_B(t, p_B). \tilde{v}_B(t, p_B)) \right). \tag{16.34}
\]

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Definition 16.10 The Coriolis acceleration \( \gamma_C \) at \( t \) and \( p_{At} = \Theta_t(p_Bt) \) is the Eulerian vector field defined by
\[
\gamma_C(t, p_{At}) = 2d\bar{\Omega}(p_{At}).\bar{e}_{Bt}(p_{At}) + (d^2\Theta_t(p_Bt).\bar{e}_{Bt}(p_Bt)).\bar{e}_{Bt}(p_Bt).
\]
(With \( d\bar{\Omega}_t = d\bar{\Omega}_{dt} \), cf. (16.24), and with \( d^2\Theta_t = 0 \) if \( \tilde{\Phi}_D \) is a rigid body motion, cf. (16.12).)

Thus (16.34) reads
\[
[\gamma_{At}(p_{At}) = \gamma_{Bt}(p_{At}) + \gamma_{Dt}(p_{At}) + \gamma_{Ct}(p_{At})] \tag{16.36}
\]

Reading: At \( t \), if \( A \) is the “absolute observer” and if \( B \) is the “relative observer”, then in \( \mathcal{R}_A \):
\[
\gamma_{At} \quad \text{the absolute acceleration =} \quad \gamma_{Bt}, \quad \text{the relative acceleration translated for} \ A
+ \gamma_{Dt} \quad \text{the drive acceleration}
+ \gamma_{Ct} \quad \text{the Coriolis acceleration}.
\]

Example 16.11 Classical setting: \( \tilde{\Phi}_D \) is a rigid body motion, thus \( d\bar{\Omega}_{Dt}(p_{At}) = d\bar{\Omega}_{Dt} = \bar{\omega}(t)\wedge \), cf. (16.4), and \( \Theta_t \) is affine, cf. (16.12), thus \( d^2\Theta_t = 0 \), thus
\[
\gamma_{Ct}(p_{At}) = 2\bar{\omega} \wedge \bar{e}_{Bt}(p_{At}), \tag{16.38}
\]
usual expression of the Coriolis expression on Earth.

16.8 Inter-referential change of basis formula
\( \tilde{\Phi}_D \) is supposed to be a rigid body motion. So \( \Theta_t \) is affine, cf. (16.12), (16.19) and (16.13) gives in \( \mathcal{R}_A \) (inter-referential relation and classical setting)
\[
\tilde{B}_{jts} := d\Theta_t.\tilde{B}_j \quad \text{(the basis of} \ B \ \text{at any} \ q_B \ \text{as seen by} \ A \ \text{at} \ t). \tag{16.39}
\]
Let \( \mathcal{P}_t \in \mathcal{L}(\mathcal{R}_A;\mathcal{R}_A) \) be the change of basis endomorphism from \( (\tilde{A}_t) \) to \( (\tilde{B}_{jts}) \) in \( \mathcal{R}_A \). For all \( j \).
\[
\mathcal{P}_t.\tilde{A}_j := \tilde{B}_{jts} \quad \text{named} \sum_{i=1}^n (P_t)_j^i \tilde{A}_i, \quad \text{and} \quad [\mathcal{P}_t]_{\tilde{A}} = [(P_t)_j^i]. \tag{16.40}
\]
thus \( (P_t)_j^i \) is the \( i \)-th component of \( \tilde{B}_{jts} \) in the basis \( (\tilde{A}_t) \). Let \( Q_t := \mathcal{P}_t^{-1} \) (inverse endomorphism in \( \mathcal{R}_A \)), thus
\[
\tilde{A}_j = Q_t.\tilde{B}_{jts} = Q_t.d\Theta_t.\tilde{B}_j. \tag{16.41}
\]
and \( Q_t.d\Theta_t \in \mathcal{L}(\mathcal{R}_B;\mathcal{R}_A) \) is the inter-referential change of basis linear map from \( (\tilde{B}_i) \) to \( (\tilde{A}_t) \).

Let \( \tilde{u}_B \) be a vector field in \( \mathcal{R}_B \), and \( \tilde{u}_{Bts} \) be its translation in \( \mathcal{R}_A \), that is, with \( q_{Bts} = \Theta_t(q_B) \),
\[
\tilde{u}_B(q_B) \quad \text{named} \quad \sum_{i=1}^n u^j_B(q_B)\tilde{B}_j \quad \text{and} \quad \tilde{u}_{Bts}(q_{Bts}) = d\Theta_t.\tilde{u}_B(q_B) \quad \text{named} \quad \sum_{i=1}^n u^j_{Bts}(q_{Bts})\tilde{A}_i. \tag{16.42}
\]

Proposition 16.12 Then,
\[
\sum_{i=1}^n u^j_{Bts}(q_{Bts})\tilde{A}_i = \sum_{j=1}^n u^j_B(q_B)\tilde{B}_{jts} = \sum_{i,j=1}^n (P_t)_j^i u^j_B(q_B)\tilde{A}_i, \tag{16.43}
\]
thus
\[
[\tilde{u}_{Bts}(q_{Bts})]_{\tilde{A}} = [\mathcal{P}_t]_{\tilde{A}}[\tilde{u}_B(q_B)]_{\tilde{B}}, \quad \text{inter-referential change of basis at} \ t. \tag{16.44}
\]

Proof. \( \sum_{i=1}^n u^j_{Bts}(q_{Bts})\tilde{A}_i = \tilde{u}_{Bts}(q_{Bts}) = d\Theta_t.\tilde{u}_B = d\Theta_t.(\sum_{j=1}^n u^j_B(\tilde{B}_j)\tilde{B}_j) = \sum_{j=1}^n u^j_B(\tilde{B}_j)d\Theta_t.\tilde{B}_j = \sum_{j=1}^n u^j_B(q_B)\tilde{B}_{jts} = \sum_{i,j=1}^n u^j_B(q_B)(P_t)_j^i \tilde{A}_i, \text{thus} \ u^j_{Bts} = \sum_{i,j=1}^n (P_t)_j^i u^j_B(q_B). \tag*{\textnormal{\blacksquare}}
\]

Exercise 16.13 Suppose: \( \tilde{\Phi}_D \) is a rigid body motion, \( \Theta_t \) is affine, and \( (\cdot,\cdot)_a \) and \( (\cdot,\cdot)_b \) are Euclidean dot products designed by \( A \) and \( B \), and \( (\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b \). Prove: \( \mathcal{P}_t^T.\mathcal{P}_t = \lambda^2 I \).

Answer. \( (\mathcal{P}_t.\tilde{A}_t, \mathcal{P}_t.\tilde{A}_j, \mathcal{P}_t.\tilde{A}_a) = (\tilde{B}_{jts}, \tilde{B}_j, \tilde{B}_j)_a = (d\Theta_t.\tilde{B}, d\Theta_t.\tilde{B}_j)_a = \lambda^2(\tilde{B}, \tilde{B}_j)_a = \lambda^2 \delta_{ij}, \text{cf. (16.13)}. \tag*{\textnormal{\blacksquare}}
\]
16.9 A summary: Commutative diagrams

16.9.1 Motions and translator, Eulerian

(16.8) gives \( \Theta_t \circ \Psi_t = \Phi_{Dt} \) for any \( t \), thus the following diagram commutes:

\[
\begin{array}{c}
Q_{Obj} \in ObjB \\
\Phi_{Dt} \\
\Theta_t \\
q_{At} = \tilde{\Phi}_D(t, Q_{Obj}) = \Theta_t(q_B) \in R_A.
\end{array}
\]

16.9.2 Motions and translator, Lagrangian

With an initial time \( t_0 \). Consider \( ObjB \), its motion \( \tilde{\Psi}_B \) in \( R_B \) (fixed motion that gives the positions or the particles in \( R_B \)) and its motion \( \tilde{\Phi}_D \) in \( R_A \). We have, cf. (16.15),

\[
\Phi_{Dt}^{t_0} \circ \Theta_{t_0} = \Theta_t. \tag{16.46}
\]

Slight correction: Observer \( B \) is not supposed to have a time ubiquity gift. Thus introduce the time-shift operator \( S_{t_0}^t \) in \( R_B \) given by \( S_{t_0}^t(q_B) = q_B \). Thus (16.46) in fact reads:

\[
\Phi_{Dt}^{t_0} \circ \Theta_{t_0} = \Theta_t \circ S_{t_0}^t. \tag{16.47}
\]

Thus the following diagram commutes:

\[
\begin{array}{c}
Q_{Obj} \in ObjB \\
\tilde{\Psi}_B \\
\Theta_{t_0} \\
q_{At_0} = \Phi_{Dt_0}(Q_{Obj}) = \Theta_{t_0}(q_B) \\
\Phi_{Dt_0}^{t_0} \\
q_{At} = \tilde{\Phi}_D(P_{Obj}) = \Phi_{Dt_0}^{t_0}(P_{At_0}) = \Theta_t(q_B)
\end{array}
\]

(Top line in \( R_B \) with \( S_{t_0}^t \) the time shift, bottom line in \( R_A \), and translation between lines.)

Consider \( Obj \), its motion \( \tilde{\Phi}_B \) in \( R_B \) and its motion \( \tilde{\Phi}_A \) in \( R_A \). Let \( \Phi_{Bt}^{t_0} : R_B \to R_B \) and \( \Phi_{At}^{t_0} : R_A \to R_A \) be the associated motions: Defined by \( \Phi_{Bt}^{t_0} \circ \tilde{\Phi}_B = \tilde{\Phi}_B \) and \( \Phi_{At}^{t_0} \circ \tilde{\Phi}_A = \tilde{\Phi}_A \), cf. (3.6), that is, \( p_{Bt} = \Phi_{Bt}^{t_0}(p_{B_{t_0}}) = \tilde{\Phi}_B(P_{At}) \) when \( p_{B_{t_0}} = \Phi_{B{t_0}}(P_{Obj}) \) and \( p_{At} = \Phi_{At}^{t_0}(P_{At_0}) = \tilde{\Phi}_A(P_{Obj}) \) when \( p_{At_0} = \tilde{\Phi}_A(P_{Obj}) \). So we have \( p_{Bt_0} = \Theta_{t_0}(p_{B_{t_0}}) \) and \( p_{At} = \Theta_{t_0}(p_{At_0}) \), thus \( \Phi_{At}^{t_0}(\Theta_{t_0}(p_{B_{t_0}})) = \Theta_t(p_{Bt_0}) \), so

\[
\Phi_{At}^{t_0} \circ \Theta_{t_0} = \Theta_t \circ \Phi_{Bt}^{t_0}. \tag{16.49}
\]

Thus the following diagram commutes:

\[
\begin{array}{c}
P_{Obj} \in Obj \\
\tilde{\Phi}_B \circ \tilde{\Phi}_A \\
\Theta_{t_0} \\
p_{At_0} = \Phi_{At_0}(P_{Obj}) = \Theta_{t_0}(p_{B_{t_0}}) \\
\Phi_{At_0}^{t_0} \\
p_{At} = \tilde{\Phi}_A(P_{Obj}) = \Phi_{At_0}^{t_0}(P_{At_0}) = \Theta_t(p_{Bt})
\end{array}
\]

(Top line in \( R_B \), bottom line in \( R_A \), and translation between lines.)
\[ \vec{\gamma} \]

16.9.3 Differentials

(16.46) gives, for \( q_B \in \mathcal{R}_B \) and \( q_{B*} = \Theta_{t_0}(q_B) \in \mathcal{R}_A \),

\[ d\Phi^{t_0}_{\mathcal{D}_A}(q_{B*}) \cdot d\Theta_{t_0}(q_B) = d\Theta_{t}(q_B). \quad (16.51) \]

Let \( \vec{u}_B \) be a (stationary) vector field in \( \mathcal{R}_B \). Its push-forwards by \( \Theta_{t_0} \) and \( \Theta_t \) (translations from \( B \) to \( A \)) are, cf. (16.19),

\[
\begin{align*}
\vec{u}_{B*}(q_{B*}) = d\Theta_{t_0}(q_B) \cdot \vec{u}_B(q_B), & \quad \text{when } q_{B*} = \Theta_{t_0}(q_B), \\
\vec{u}_{B*}(q_{B*}) = d\Theta_t(q_B) \cdot \vec{u}_B(q_B), & \quad \text{when } q_{B*} = \Theta_t(q_B).
\end{align*}
\]

Then (16.51) gives:

\[ d\Phi^{t_0}_{\mathcal{D}_A}(q_{B*}) \cdot \vec{u}_{B*}(q_{B*}) = \vec{u}_{B*}(q_{B*}). \quad (16.53) \]

and then \( \vec{u}_{B*} \) is the push-forward of \( \vec{u}_{B*} \) by \( \Phi^{t_0}_{\mathcal{D}_A} \). Thus the commutative diagram (16.51) reads

\[ \begin{array}{ccc}
\vec{u}_B(q_B) \in \mathcal{R}_B & \xrightarrow{d\Theta_{t_0}(q_B)} & \vec{u}_{B*}(q_{B*}) \in \mathcal{R}_A \\
\xrightarrow{d\Theta_t(q_B)} & & \xrightarrow{d\Phi^{t_0}_{\mathcal{D}_A}(q_{B*})} \vec{u}_{B*}(q_{B*}) = (d\Theta_t, \vec{u}_B)(q_B) = d\Phi^{t_0}_{\mathcal{D}_A}(q_{B*}) \cdot \vec{u}_{B*}(q_{B*}) \in \mathcal{R}_A.
\end{array} \]

17 Coriolis force

17.1 Fundamental principal: In a Galilean referential

The second Newton’s law of motion (fundamental principle of dynamics) tells: If you are in a Galilean referential and you quantify vectors (you measure the components relative to your basis), then the sum of the external forces \( \vec{f} \) on an object is equal to the mass of this object multiplied by its acceleration:

\[ \sum_{\text{external}} \vec{f} = m\vec{\gamma} \quad \text{(Galilean referential)}. \quad (17.1) \]

**Remark 17.1** This result is observer dependent (quantification that requires a Galilean setup); The acceleration depends on the observer, cf. (16.36), while usually \( \vec{f} \) can is an objective vector field (forces independent of an observer), see next § 17.2. (The explicit definition of objectivity is given at § 18).

In a Non Galilean referential? Then you have to add “observer dependent forces” = the inertial and Coriolis forces (“apparent forces”). The following § 17.2 details this.

E.g., your referential \( \mathcal{R}_B \) is fixed on a carousel (a spinning merry-go-round); If you sit still in \( \mathcal{R}_B \) then \( \vec{\gamma}_B = \vec{0} \); Your acceleration relative to \( \mathcal{R}_B \) vanishes; But you feel an external force \( \vec{f}_B \neq \vec{0} \) (a “centrifugal force”), hence \( \vec{f}_B \neq m\vec{\gamma}_B \); (17.1) does not apply. But here \( \mathcal{R}_B \) isn’t Galilean. Whereas after a change toward a Galilean referential \( \mathcal{R}_A \) you get \( \vec{f}_A = m\vec{\gamma}_A \), that is \( \vec{f}_A + \vec{f}_{\text{ref, dep}} = m\vec{\gamma}_B \in \mathcal{R}_A \) where \( \vec{f}_{\text{ref, dep}} = -m(\vec{\gamma}_{\Theta_B} + \vec{\gamma}_{C}) \) (referential dependent) and \( \vec{\gamma}_{B*} = \Theta_t \vec{\gamma}_B \) (the translated acceleration), cf. (16.36).

17.2 Inertial and Coriolis forces, and Fundamental Principle

Let \( \mathcal{R}_A \) be a Galilean referential and \( \mathcal{R}_B \) be any referential.

Let \( \vec{f}_{\mathcal{A}}(p_{\mathcal{A}}) \) be the sum, as measured in \( \mathcal{R}_A \), of the external forces acting on a particle \( P_{\mathcal{O}} \in \mathcal{O} \). Thus (17.1) (Newton) gives, at all \( t \),

\[ \vec{f}_{\mathcal{A}} = m\vec{\gamma}_{\mathcal{A}} \in \mathcal{R}_A \quad \text{(Galilean referential)}. \quad (17.2) \]

Then let \( \Theta_t \) be the translator from \( \mathcal{R}_A \) to \( \mathcal{R}_B \), cf. (16.7). The addition-acceleration formula (16.36) gives

\[ \vec{f}_{\mathcal{A}} = m(\vec{\gamma}_B + \vec{\gamma}_{\Theta_t} + \vec{\gamma}_{C}) \in \mathcal{R}_A. \quad (17.3) \]

Notation: Then let \( \tilde{\vec{f}}_B := \vec{f}_{\mathcal{A}}^* \) be the pull-back of \( \vec{f}_{\mathcal{A}} \) by the translator \( \Theta_t \), cf. (16.20), that is, at
any time $t$ and with $p_{At} = \Theta_t(p_{Bt})$,

$$f_{Bt}(p_{Bt}) := d\Theta_t(p_{Bt})^{-1}f_{At}(p_{At}) \in \mathcal{R}_B, \tag{17.4}$$
cf. (16.19)-(16.20). Thus (17.3) gives

$$f_{Bt}(p_{Bt}) = md\Theta_t(p_{Bt})^{-1} \left( \gamma^*(p_{At}) - \gamma^*(q_{At}) \right) \tag{17.5}$$

with the pull-backs notation, that is, $\gamma^*(p_{At}) = d\Theta_t(p_{Bt})^{-1}\gamma(q_{At})$ (translation from $A$ to $B$ for vector fields, cf. (16.20)). Thus

$$f_{Bt}(p_{Bt}) = m\gamma^*(p_{At}) - m\gamma^*(q_{At}) = m\gamma^*_{Bt}(p_{Bt}) \in \mathcal{R}_B. \tag{17.6}$$

**Definition 17.2**

$$\begin{cases} f_{At}(p_{Bt}) = -m\gamma^*_{At}(p_{At}) = \text{inertial force in } \mathcal{R}_B \text{ at } t, \\ f_{Ct}(p_{Bt}) = -m\gamma^*_{Ct}(p_{Bt}) = \text{Coriolis force in } \mathcal{R}_B \text{ at } t. \end{cases} \tag{17.7}$$

They are called “fictitious forces” (do not exist in a Galilean referential, are not objective).

Then (17.6) gives the fundamental principle in a non Galilean referential:

$$f_{Bt}(p_{Bt}) + f_{At}(p_{Bt}) + f_{Ct}(p_{Bt}) = m\gamma^*_{Bt}(p_{Bt}) \quad \text{(non Galilean referential).} \tag{17.8}$$

See e.g. villemin.gerard.free.fr/Scienmod/Coriolis.htm, planet-terre.ens-lyon.fr/article/force-de-coriolis.xml.

### 18 Objectivities

Framework of § 16 (classical mechanics: The time scale is the same for all users).

The goal is to give an objective expression of the laws of mechanics (observer independent), so that two observers, quantifying relative to their referentials, will obtain results which they can communicate to each other (thanks to the translator).

Illustration: observer $A$ chooses a referential $\mathcal{R}_A = (O_A, (\vec{A}_i))$ fixed on the Sun with $(\vec{A}_i)$ a Euclidean basis in foot, observer $B$ chooses a referential $\mathcal{R}_B = (O_B, (\vec{B}_i))$ fixed on the Earth with $(\vec{B}_i)$ a Euclidean basis in meter, and $\Theta_t: \mathcal{R}_B \to \mathcal{R}_A$ is the translator cf. (16.8).

Remark: The functions involved are Eulerian functions (objectivity cannot depend on the choice of an initial time $t_0$ which depends on an observer); If Lagrangian associated expressions are needed, they are deduced a posteriori.

#### 18.1 Covariant objectivity of a scalar function

Let $f$ be a Eulerian scalar function corresponding to a “physical quantity” (e.g., temperature).

The observers $A$ and $B$ describe $f$ in their referential as being the functions $f_A : \mathbb{R} \times \mathcal{R}_A \to \mathbb{R}$ and $f_B : \mathbb{R} \times \mathcal{R}_B \to \mathbb{R}$. At $t$ fixed, let $f_{A\, t}(p) := f_A(t, p)$ and $f_{B\, t}(p) := f_B(t, p)$.

**Definition 18.1** The “physical quantity $f$” is objective covariant iff, for all referentials $\mathcal{R}_A$ and $\mathcal{R}_B$ and for all $t$,

$$f_{At} = \Theta_{At}f_{Bt} = \text{push-forward by } \Theta_{At}, \text{ cf. (10.5)}, \tag{18.1}$$

that is,

$$f_{At}(p_{At}) = f_{Bt}(p_{Bt}) \quad \text{when } p_{At} = \Theta_{At}(p_{Bt}). \tag{18.2}$$

Or iff $f_{Bt}$ is the pull-back of $f_{At}$ by $\Theta_{At}$, cf. (10.10), that is, $f_{Bt} = \Theta_{At}^*f_{At}$.

And then $f_A$ and $f_B$ are denoted $f$. 

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18.2 Covariant objectivity of a vector field

Let \( \vec{w} \) be a Eulerian vector valued function corresponding to a “physical quantity” (e.g., a force).

The observers \( A \) and \( B \) describe \( \vec{w} \) in their referential as being the functions \( \vec{w}_A : \mathbb{R} \times \mathcal{R}_A \to \mathbb{R}^n \) and \( \vec{w}_B : \mathbb{R} \times \mathcal{R}_B \to \mathbb{R}^n \). At \( t \) fixed, let \( \vec{w}_{At}(p) := \vec{w}_A(t,p) \) and \( \vec{w}_{Bt}(p) := \vec{w}_B(t,p) \).

**Definition 18.2** The “physical quantity \( \vec{w} \)” is objective covariant iff, for all referentials \( \mathcal{R}_A \) and \( \mathcal{R}_B \) and for all \( t \),

\[
\vec{w}_{At} = \Theta_{At} \vec{w}_{Bt} = \text{push-forward by } \Theta_t \tag{18.3}
\]

that is,

\[
\vec{w}_{At}(p_{At}) = d\Theta_t(p_{Bt}).\vec{w}_{Bt}(p_{Bt}) \quad \text{when} \quad p_{At} = \Theta_t(p_{Bt}). \tag{18.4}
\]

Or iff \( \vec{w}_{Bt} \) is the pull-back of \( \vec{w}_{At} \) by \( \Theta_t \), cf. (10.35), that is, \( \vec{w}_{Bt} = \Theta_{t\ast} \vec{w}_{At} \).

And then \( \vec{w}_A \) and \( \vec{w}_B \) are denoted \( \vec{w} \).

**Example 18.3** Fundamental counter-example: A (Eulerian) velocity field is not objective, cf. (16.27), because the drive velocity \( \vec{v}_D = \vec{D} \neq 0 \) in general. Neither is a (Eulerian) acceleration field, cf. (16.36).

*In fact:* An objective quantity is independent of a motion.

**Example 18.4** The field of gravitational forces \( \vec{g} \) is objective.

Remark: Compare \( \vec{g}_{At} = m\vec{g}_{At} \) in a Galilean referential, cf. (17.1), and \( \vec{g}_{Bt} = \vec{A}_t + \vec{f}_t + \vec{f}_{Ct} = m\vec{g}_{Bt} \) in a non Galilean referential, cf § 17: \( \vec{g} \) is objective, so \( \vec{g}_{At}(p_{At}) = d\Theta_t(p_{Bt}).\vec{g}_{Bt}(p_{Bt}) \), but the acceleration \( \vec{g} \) is not objective.

18.3 “Isometric objectivity” and “Frame Invariance Principle”

This manuscript is not intended to describe “isometric objectivity”:

“Isometric objectivity” is the framework in which the “principle of material frame-indifference” (“frame invariance principle”) is settled: “Rigid body motions should not affect the stress constitutive law of a material”. E.g., Truesdell—Noll [19] p. 41:

« Constitutive equations must be invariant under changes of frame of reference. »

Or Germain [9]:

« Axiom of power of internal forces. The virtual power of the "internal forces" acting on a system \( S \) for a given virtual motion is an objective quantity; i.e., it has the same value whatever be the frame in which the motion is observed. »

**NB:** Both of these affirmations are limited to “isometric changes of frame” (the same metric for all), as Truesdell—Noll [19] page 42-43 explain.

**NB:** The “isometric objectivity” (defined by one observer) has nothing to do with the covariant objectivity (relation between two observers). The “isometric objectivity” is an intra-referential concept, which concerns one observer who defined his Euclidean dot product (to give a meaning to a rigid body motion) and who looks at change of orthonormal bases to validate a constitutive law.

If we want to interpret “isometric objectivity” in the “covariant objectivity” setting, then “isometric objectivity” corresponds to a dictatorial management: One observer, e.g. A with his Euclidean referential \( \mathcal{R}_A \) based on the English the foot, imposes his unit of length (the foot) to all other users (isometry); And only a change of orthonormal bases is allowed. This is a “subjective” approach: An English observer (foot) and a French observer (meter) cannot work together (they don’t use the same metric).

Moreover, “isometric objectivity” leads to despise the difference between covariance and contravariance, because of the uncontrolled generalized use of the Riesz representation theorem, see e.g. § C.3 and § T.2, and leads to the misunderstanding of the Lie derivative of order 2 tensors (Jaumann, Truesdell, Oldroyd, lower-Maxwell, upper-Maxwell...).

**Remark 18.5** Marsden et Hughes [12] p. 8 use the same setting at first. But, pages 22 and 163, they say that a “good modelization” has to be “covariant objective”: And they do propose a covariant modelization for elasticity at § 3.3, without mixing linear forms (“covariant vectors”) and vectors (“contravariant vectors”). (They use metrics but pay attention to the use of the Riesz representation theorem.)
18.4 Objectivity of differential forms

A differential form acts on vector fields, and the objectivity of a differential form is deduced from the previous §.

Let \( \alpha \) be a (Eulerian) differential form corresponding to a “physical quantity”. The observers \( A \) and \( B \) describe \( \alpha \) as being the functions \( \alpha_A: \mathbb{R} \times \mathcal{R}_A \to \mathcal{R}_A \) and \( \alpha_B: \mathbb{R} \times \mathcal{R}_B \to \mathcal{R}_B \).

**Definition 18.6** The “physical quantity \( \alpha \)” is objective covariante iff, for all referentials \( \mathcal{R}_A \) and \( \mathcal{R}_B \) and for all \( t \),
\[
\alpha_{At} = \Theta_t \alpha_{Bt} = \text{push-forward by } \Theta_t, \text{ cf. (10.28)}
\]
that is,
\[
\alpha_{At}(p_{At}) = \alpha_{Bt}(p_{Bt}) \cdot \Theta_t(p_{Bt})^{-1} \quad \text{when } p_{At} = \Theta_t(p_{Bt}).
\]
Or iff \( \alpha_{Bt} \) is the pull-back of \( \alpha_{At} \) by \( \Theta_t \), that is, \( \alpha_{Bt} = \Theta^* \alpha_{At} \).

And then \( \alpha_A \) and \( \alpha_B \) are denoted \( \alpha \).

Thus, if \( \vec{w} \) is an objective vector field and \( \alpha \) is an objective differential form, then the scalar function \( \alpha \vec{w} \) is objective:
\[
\alpha_{At}(\vec{w}_{At}) = \alpha_{Bt}(\vec{w}_{Bt}) \cdot \Theta_t(\vec{w}_{Bt}),
\]
since \( \alpha_{At}(\vec{w}_{At}) = (\alpha_{Bt}(\vec{w}_{Bt}) \cdot \Theta_t(\vec{w}_{Bt}))^{-1} \cdot (\Theta_t(\vec{w}_{Bt}) \cdot \vec{w}_{Bt}(\vec{w}_{Bt})) = \alpha_{Bt}(\vec{w}_{Bt}) \cdot \vec{w}_{Bt}(\vec{w}_{Bt}). \)

Matrix expression. With proposition 16.12 and (16.44), we get: If \( \alpha \) is objective, then
\[
[\alpha_{At}(\vec{w}_{At})]_{ij} = [\alpha_{Bt}(\vec{w}_{Bt})]_{ij}. \quad \text{A matrix, cf. (18.8), and one of the traps of isometric objectivity is that}
\]
\[
P^{-1} = P^T \text{ (change of orthonormal basis) so that } [\alpha]_{ij}^T = P^{-1}.[\alpha]_{ij}^T \quad \text{cf. (18.8), equality which looks like a contravariant expression, cf. (16.44),... except for the transpose!...: Thus the temptation is great to identify a linear form with a vector...}
\]
But recall: It is impossible to identify a linear form with a vector. There is no natural canonical isomorphism between a vector space and its dual space (the space of linear functionals), see § T.2. In particular the Riesz representation theorem is not an identification: It is just a representation depending on a measuring tool, cf. (C.11), that is, which depends on one observer.

Recall: Covariant objectivity respects the nature (vectorial or differential) of the mathematical object.

In particular a covariant object framework does not use any inner dot product to begin with, and therefore a linear form cannot be confused with a vector (there is no use of the Riesz’s representation theorem). E.g., the respect of covariance do not raise the problems seen at § 13.2, and the problem at § 15.8, problems due to the use of an inner dot product.

18.5 Objectivity of tensors

A tensor acts on both vector fields and differential forms, and its objectivity is deduced from the previous §.

Let \( T \) be a (Eulerian) tensor corresponding to a “physical quantity”. The observers \( A \) and \( B \) describe \( T \) as being the functions \( T_A: \mathbb{R} \times \mathcal{R}_A \to \mathcal{R}_A \) and \( T_B: \mathbb{R} \times \mathcal{R}_B \to \mathcal{R}_B \).

**Definition 18.8** The (Eulerian) “physical quantity” is objective covariante iff, for all referentials \( \mathcal{R}_A \) and \( \mathcal{R}_B \) and for all \( t \), \( T_{At} \) is the push-forward of \( T_{Bt} \) by \( \Theta_t \), cf. (14.7), that is, with \( p_{At} = \Theta_t(p_{Bt}) \).
\[
T_{At}(q_{At})(...,\ldots) = T(q_{Bt})(\Theta_t q_{At}^*,...,\Theta_t q_{At}^*), \quad \text{i.e. } T_{At} = \Theta_t^* T_{Bt} \text{ noted } T_{Bt}.
\]
And then \( T_A \) and \( T_B \) are denoted \( T \).

**Example 18.9 (Objectivity of a differential \( d\vec{w} \))** Let \( \vec{w} \) be an objective vector field, seen as \( \vec{w}_A \) in \( \mathcal{R}_A \) and \( \vec{w}_B \in \mathcal{R}_B \) by \( \vec{w}_{At} = \Theta_t(\vec{w}_{Bt}) = \alpha_{Bt}(p_{Bt}) \cdot \vec{w}_{Bt}(p_{Bt}) \) when \( p_{At} = \Theta_t(p_{Bt}) \), cf. (18.4).
\[
d\vec{w}_{At}(p_{At}) = d\Theta_t(p_{Bt}) \cdot d\vec{w}_{Bt}(p_{Bt}) \cdot d\Theta_t(p_{Bt})^{-1} + (d^2\Theta_t(p_{Bt}) \cdot \vec{w}_{Bt}(p_{Bt})) \cdot d\Theta_t(p_{Bt})^{-1} \neq d\Theta_t(p_{Bt}) \cdot d\vec{w}_{Bt}(p_{Bt}) \cdot d\Theta_t(p_{Bt})^{-1} \text{ when } d^2\Theta_t \neq 0.
\]

Thus \( d\vec{w} \) is not objective in general (unless \( \Theta_t \) is affine). However in classical mechanics, \( \Theta_t \) is affine, and
thus $d\vec{w}$ is considered to be objective when $\vec{w}$ is. And

$$d^2\vec{w}_{M}(q_M).d\Theta_t(q_M).d\Theta_t(q_B) + d\vec{w}_M(q_M).d\Theta_t(q_M)$$

$$= d^2\Theta_t(q_M).\vec{w}_{B_t}(q_B) + 2d^2\Theta_t(q_M).d\vec{w}_{B_t}(q_B) + d\Theta_t(q_M).d^2\vec{w}_{B_t}(q_B).$$  \hspace{2cm} (18.11)

Thus

$$d^2\vec{w} \text{ is objective iff } d^2\Theta_t = 0 \text{ (i.e. } \Theta_t \text{ is affine}).$$  \hspace{2cm} (18.12)

18.6 Non objectivity of the velocities

18.6.1 Eulerian velocities

The velocity addition formal is, cf. (16.28), with

$$\vec{v}_{B_t}(p_M) = d\Theta_t(p_B).\vec{v}(p_B) \text{ when } p_M = \Theta_t(p_B).$$

Thus

$$\vec{v}_{M}(p_M) = \vec{v}_{B_t}(p_M) + \vec{v}_{D_t}(p_M)$$

$$\neq \vec{v}_{B_t}(p_M) \text{ when } \vec{v}_{D_t}(p_M) \neq 0,$$  \hspace{2cm} (18.13)

thus a Eulerian velocity field is not objective.

18.6.2 Lagrangian velocities

The Lagrangian velocities do not define a vector field, cf. § 4.1.2. Thus asking about the objectivity of Lagrangian velocities is meaningless.

18.6.3 $d\vec{v}$ is not objective

d$\vec{v}_{B_t}$ is a field of endomorphisms in $R_B$. Its push-forward by $\Theta_t$ into $R_A$ is given by, cf. (14.23),

$$(d\vec{v}_{B_t})(p_M) = d\Theta_t(p_B).d\vec{v}_{B_t}(p_B).d\Theta_t(p_B)^{-1},$$  \hspace{2cm} (18.14)

when $p_M = \Theta_t(p_B)$. And $(\vec{v}_M - \vec{v}_{D_t})(p_M) = d\Theta_t(p_B).\vec{v}_{B_t}(p_B)$ (velocity-addition formula), thus

$$d(\vec{v}_M - \vec{v}_{D_t})(p_M).d\Theta_t(p_B) = d^2\Theta_t(p_B).\vec{v}_{B_t}(p_B) + d\Theta_t(p_B).d\vec{v}_{B_t}(p_B).$$  \hspace{2cm} (18.15)

Hence, with (18.14),

$$d(\vec{v}_M - \vec{v}_{D_t})(p_M) = (d\vec{v}_{B_t})(p_M) + (d^2\Theta_t(p_B).\vec{v}_{B_t}(p_B)).d\Theta_t(p_B)^{-1}.$$  \hspace{2cm} (18.16)

Thus $d\vec{v}_M \neq \Theta_t(d\vec{v}_{B_t})$, hence $d\vec{v}$ is not objective, even if $d^2\Theta_t = 0$ (because of $\vec{v}_{D_t}$). So, with $\vec{v}_{B_t}(p_M) = d\Theta_t(p_B).\vec{v}_{B_t}$ we get in $R_A$:

$$d(\vec{v}_M - \vec{v}_{D_t})(p_M).d\vec{v}_{B_t}(p_M) = d\Theta_t(p_B).d\vec{v}_{B_t}(p_B).\vec{v}_{B_t}(p_B) + d^2\Theta_t(p_B)(\vec{v}_{B_t}(p_B), \vec{v}_{B_t}(p_B)).$$  \hspace{2cm} (18.17)

18.6.4 $d\vec{v}$ is not “isometric objective”

“Isometric objective” implies

- The use of the same Euclidean metric in $R_B$ and $R_A$, i.e. $(\cdot, \cdot)_a = (\cdot, \cdot)_b$.
- $\Phi_0^{\nu}$ is a solid body motion, i.e. $d\Phi_0^{\nu}.d\Phi_0^{\nu} = I$ and $\Theta_t$ is affine (so $d^2\Theta_t = 0$ for all $t$).

Thus (18.16) gives

$$d\vec{v}_M = d\vec{v}_{B_t} + d\vec{v}_{D_t}.$$  \hspace{2cm} (18.18)

So $d\vec{v}_M$ is not the push-forward $d\vec{v}_{B_t} = \Theta_t(d\vec{v}_{B_t})$ of $\vec{v}_{B_t}$ when $d\vec{v}_{D_t} \neq 0$, that is, when $R_B$ is not animated with a uniform translation motion (e.g. for the Earth around the Sun). Thus $d\vec{v}$ is not “isometric objective”.

Remark 18.10 In classical mechanics courses, to establish that $d\vec{v}$ isn’t “isometric objective”, it is shown that (change of coordinate system for differentials of velocities for moving Euclidean referentials)

$$[d\vec{v}_M]_{\vec{B}_{t}} = Q_t.[d\vec{v}_M]_{\vec{A}}Q_t^{-1} + Q'(t).Q_t^{-1}$$

written $[L]_{\vec{B}_{t}} = Q.[L]_{\vec{A}}Q^T + \dot{Q}.Q^T.$  \hspace{2cm} (18.19)

Thus $d\vec{v}_M(p_M)$ does not satisfy the change of coordinate rules for endomorphisms, because of $\dot{Q}.Q^T$. So $d\vec{v}$ is not “isometric objective”.

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Remark: (18.19) is not a change of (inter-)referential formula: (18.19) is given in one (intra-)referential \( \mathcal{R}_A \) with a moving coordinate basis \((B_{t*}, \vec{A}_t) = (d\Phi_{t*}^0, \vec{A}_t)\) in \( \mathcal{R}_A \), cf. (16.40):

\[
\vec{B}_{t*} = P_t \vec{A}_t = \sum_{i=1}^{n} (P_t i)_j \vec{A}_i, \quad P_t = [(P_t i)_j] = [P_t]_{\vec{A}}, \quad Q_t = P_t^{-1}.
\]  

(18.20)

And with some \( t_0 \in \mathbb{R} \), \( p_{A_*} = \Phi_{A_*}^0(p_{A_0}) \), and \( d\Phi^0_{A_*}(t, p_{A_0}) = F(t) \), we have, cf. (4.22),

\[
d\vec{v}_{A_*}(p_{A_*}) = F(t).F(t)^{-1}.
\]

(18.21)

And with (5.34) and \( Q_t := P_t^{-1} \) we have

\[
[F(t)]_{\vec{B}_{t*}} = Q(t).[F(t)]_{\vec{A}}, \quad \text{then} \quad [F'(t)]_{\vec{B}_{t*}} = Q'(t).[F(t)]_{\vec{A}} + Q(t).[F'(t)]_{\vec{A}}
\]

(18.22)

Then (18.21) and (18.22) give

\[
[d\vec{v}_{A_*}(p_{A_*})]_{\vec{B}_{t*}} = Q'(t).[F(t)]_{\vec{A}}.F(t)^{-1}][\vec{B}_{t*}] + Q(t).[F'(t)]_{\vec{A}}.[F(t)^{-1}][\vec{B}_{t*}]
\]

\[
= Q'(t).[F(t)]_{\vec{A}}.F(t)^{-1} + Q(t).[F'(t)]_{\vec{A}}.[F(t)^{-1}].Q(t)^{-1}
\]

\[
= Q'(t).Q^{-1}(t) + Q(t).[d\vec{v}_{A_*}(p_{A_*})][\vec{A}].Q(t)^{-1},
\]

(18.23)

that is, (18.19). And, with \( L = \vec{d} \vec{v} \) cf. (4.22), this result is written \( [L]_{\vec{B}_{t*}} = \dot{Q}.Q^T + Q.[L]_{\vec{A}}.Q^T. \) (And with \( Q.Q^T = I \), we get \( \dot{Q}.Q^T + (\dot{Q}.Q^T)^T = 0, \) thus \( L + L^T \) is “isometric objective.”)

\section*{Exercice 18.11}

Prove that \( d^2 \vec{v} \) is “isometric objective”.

\section*{Answer.}

(18.11) with \( \vec{v}_{A} - \vec{d} \vec{v} \) instead of \( \vec{w}_A \) and \( \vec{v}_B \) instead of \( \vec{w}_B \) give, in an “isometric objective” framework,

\[
d^2(\vec{v}_{A_*}(p_{A_*})).(\vec{u}_{B_*}, \vec{w}_{B_*}) = d\Theta_{(p_{A_*})}.d^2 \vec{v}_{B_*}(p_{B_*})(\vec{u}_{B_*}, \vec{w}_{B_*}).
\]

(18.24)

Thus if \( \Phi_{D_*} \) is affine (which is the case of a rigid body motion) then \( d^2 \vec{v}_{D_*} = 0, \) cf. (9.6), and then \( d^2 \vec{v} \) is “isometric objective”:

\[
d^2 \vec{v}_{A_*}(p_{A_*}).(\vec{u}_{B_*}, \vec{w}_{B_*}) = d\Theta_{(p_{A_*})}.d^2 \vec{v}_{B_*}(p_{B_*})(\vec{u}_{B_*}, \vec{w}_{B_*}).
\]

\section*{18.6.5 \( d\vec{v} + d\vec{v}^T \) is “isometric objective”}

\section*{Proposition 18.12}

If \( \Phi^0_{D_*} \) is a solid body motion then \( d\vec{v} + d\vec{v}^T \) is “isometric objective”

\[
d\vec{v}_{A_*} + d\vec{v}_{A_*}^T = (d\vec{v}_{B_*} + d\vec{v}_{B_*}^T)_*.
\]

(18.25)

(That is, the rate of deformation tensor is independent of a eventual added rigid motion.)

\section*{Proof.}

(18.18) gives \( d\vec{v}_{A_*} + d\vec{v}_{A_*}^T = d(\vec{v}_{B_*}) + d\vec{v}_{D_*} + d\vec{v}_{B_*}^T + d\vec{v}_{D_*}^T \), and \( \Phi_{D_*}^0 \) is a solid body motion, so

\[
d\vec{v}_{D_*} + d\vec{v}_{D_*}^T = 0, \quad \text{cf. (9.11)}, \quad \text{thus} \quad d\vec{v}_{A_*} + d\vec{v}_{A_*}^T = d(\vec{v}_{B_*}) + d(\vec{v}_{B_*}^T).
\]

And \( \vec{v}_{B_*}(p_{A_*}) = d\Theta_{(p_{B_*})}.\vec{B}_{B_*}(p_{B_*}) \) and \( p_{A_*} = \Theta_{(p_{B_*})} \) and “isometric objectivity” \( (d^2 \Theta = 0) \) give

\[
d(\vec{v}_{B_*})(p_{A_*}).d\Theta_{(p_{B_*})} = d\Theta_{(p_{B_*})}.d\vec{v}_{B_*}(p_{B_*}). \text{Thus} \ d(\vec{v}_{B_*})(p_{A_*}) = d\Theta_{(p_{B_*})}.d\vec{v}_{B_*}(p_{B_*}).d\Theta_{(p_{B_*})}^{-1} = (d\vec{B}_{B_*})(p_{A_*}).
\]

\section*{Exercice 18.13}

Prove that \( \Omega = \frac{d\vec{v} - d\vec{v}^T}{2} \) is not isometric objective.

\section*{Answer.}

(18.18) gives \( d\vec{v}_{A_*} = d\vec{v}_{B_*} + d\vec{v}_{D_*} \) thus \( \frac{d\vec{v}_{A_*} - d\vec{v}_{A_*}^T}{2} = \frac{d\vec{v}_{B_*} - d\vec{v}_{B_*}^T}{2} + \frac{d\vec{v}_{D_*} - d\vec{v}_{D_*}^T}{2} \neq \frac{d\vec{v}_{B_*} - d\vec{v}_{B_*}^T}{2} \), even if \( \Phi^0_{D_*} \) is a solid body motion (apart from a uniform translation).

\section*{18.7 The Lie derivative are covariant objective}

The objectivity under concern is the covariant objectivity (no inner dot product is required to define a Lie derivative). The Lie derivatives are also called “objective rates” since they are covariant objective.

Framework of \( \S \ 16 \). We have the velocity-addition formula, cf. (16.28), that is, \( \vec{v}_{A_*} = \vec{v}_{B_*} + \vec{v}_{B_*} \) in \( \mathcal{R}_A \).
18.7.1 Scalar functions

**Proposition 18.14** Let \( f \) be a covariant objective function, cf. (18.2). Then its Lie derivative \( \mathcal{L}_\Theta f \) is covariant objective

\[
\mathcal{L}_\Theta f_A = \Theta_*(\mathcal{L}_{\tilde{\Theta}} f_B), \quad \text{i.e.} \quad \mathcal{L}_\Theta f_A(t,p_M) = \mathcal{L}_{\tilde{\Theta}} f_B(t,p_B) \quad \text{when} \quad p_M = \Theta_t(p_B),
\]

(18.26)

i.e., \( \frac{df}{dt}(t,p_M) = \frac{df}{dt}(t,p_B) \).

But the differential \( df \) and the partial derivative \( \frac{df}{dt} \) are not objective in general.

**Proof.** Let \( p_M = p_M(t) = \tilde{\Phi}_A(t, P_{OB}) \) and \( p_B = p_B(t) = \tilde{\Phi}_B(t, P_{OB}) \), so \( p_M(t) = \Theta(t, p_B(t)) \). (16.2) gives \( \mathcal{L}_\Theta f_A(t,p_M) = \frac{df}{dt}(t,p_M) + dfA(p_M).\tilde{\Theta}_A(t,p_M), \) and \( \mathcal{L}_{\tilde{\Theta}} f_B(t,p_B) = \frac{df}{dt}(t,p_B) + dfB(p_B).\tilde{\Theta}_B(t,p_B) \).

And, \( f \) being objective, (18.2) gives \( f_B(t,p_B) = f_A(t,p_M) = f_A(t,\Theta(t,p_B)) \).

Thus \( \frac{df}{dt}(t,p_B) = \frac{df}{dt}(t,\Theta(t,p_B)) + df_A(p_M).\tilde{\Theta}_A(t,p_M), \) and \( df_B(p_B) = df_A(p_M).d\Theta_t(p_B) \).

Thus \( \frac{df}{dt}(t,p_B) + df_B(p_B).\tilde{\Theta}_B(t,p_B) = \frac{df}{dt}(t,p_B) + df_A(p_M).\tilde{\Theta}_A(t,p_M) + df_A(p_M).d\Theta_t(p_B).\tilde{\Theta}_B(p_B) = \frac{df}{dt}(t,p_M) + df_A(p_M).\tilde{\Theta}_A(p_M), \) since \( \tilde{\Theta}_A(p_M) + d\Theta_t(p_B).\tilde{\Theta}_B(p_B) = \tilde{\Theta}_A(p_B) \) is the velocity addition formula (16.28).

18.7.2 Vector fields

**Proposition 18.15** Let \( \tilde{w} \) be a covariant objective vector field, cf. (18.4). Then its Lie derivative \( \mathcal{L}_\Theta \tilde{w} \) is covariant objective:

\[
\mathcal{L}_\Theta \tilde{w}_A = \Theta_*(\mathcal{L}_{\tilde{\Theta}} \tilde{w}_B),
\]

(18.27)

i.e., when \( p_M = \Theta_t(p_B) \),

\[
\mathcal{L}_\Theta \tilde{w}_A(t,p_M) = d\Theta_t(p_B).\mathcal{L}_{\tilde{\Theta}} \tilde{w}_B(t,p_B),
\]

(18.28)

i.e.,

\[
(\frac{D\tilde{w}_A}{dt} - d\tilde{w}_A.d\tilde{w}_A)(t,p_M) = d\Theta_t(p_B).\frac{D\tilde{w}_B}{dt} - d\tilde{w}_B.d\tilde{w}_B)(t,p_B),
\]

(18.29)

i.e.,

\[
(\frac{\partial\tilde{w}_A}{\partial t} + \partial\tilde{w}_A.\tilde{w}_A - \partial\tilde{w}_A.d\tilde{w}_A)(t,p_M) = d\Theta_t(p_B).\frac{\partial\tilde{w}_B}{\partial t} + \partial\tilde{w}_B.d\tilde{w}_B)(t,p_B).
\]

(18.30)

But the convected, Lie autonomous, partial, and material derivatives are not objective (recall the velocity addition formula \( \tilde{v}_M - \tilde{v}_D = \tilde{v}_B \)), we have

\[
(d\tilde{w}_A.\tilde{v}_M - \tilde{v}_D)(p_M) = (d\Theta_t.\tilde{v}_M)(p_B),
\]

(18.31)

\[
(d\tilde{w}_A.\tilde{v}_M - \tilde{v}_D)(p_M) = (d\Theta_t.\tilde{v}_M)(p_B),
\]

(18.32)

\[
(d\tilde{w}_A - \tilde{w}_D)(p_M) = (d\Theta_t.\tilde{w}_B)(p_B),
\]

(18.33)

\[
\frac{\partial\tilde{w}_A}{\partial t}(t,p_M) + \frac{\partial\tilde{w}_A}{\partial t}(p_M) = d\Theta_t(p_B).\frac{\partial\tilde{w}_B}{\partial t}(t,p_B),
\]

(18.34)

\[
\frac{D\tilde{w}_A}{DT}(t,p_M) - d\tilde{w}_A.\tilde{v}_B(p_M) = d\Theta_t(p_B).\frac{D\tilde{w}_B}{DT}(t,p_B) + \frac{\partial\tilde{w}_B}{\partial t}(t,p_B).
\]

(18.35)

\[
\frac{\partial\tilde{v}_M}{\partial t}(t,p_M) = d\Theta_t(p_B).\frac{\partial\tilde{v}_B}{\partial t}(t,p_B).
\]

(18.36)

\[
\frac{\partial\tilde{v}_M}{\partial t}(t,p_M) = d\Theta_t(p_B).\frac{\partial\tilde{v}_B}{\partial t}(t,p_B).
\]

(18.37)

**Proof.** \( \tilde{w}_A(\Theta_t(p_B)) = d\Theta_t(p_B).\tilde{w}_B(p_B) \) gives

\[
d\tilde{w}_A(p_M).d\Theta_t(p_B) = \frac{d^2\Theta_t(p_B).\tilde{w}_B(p_B) + d\Theta_t(p_B).d\tilde{w}_B(p_B),}
\]

(18.38)

with \( d\Theta_t(p_B).\tilde{w}_B(p_B) = \tilde{v}_B(t,p_M) \) (velocity-addition formula), thus

\[
d\tilde{w}_A(p_M).d\Theta_t(p_B).\tilde{w}_B(p_B) = \tilde{v}_B(t,p_M) + d\Theta_t(p_B).d\tilde{w}_B(p_B).\tilde{w}_B(p_B),
\]

hence (18.31). In particular \( d\tilde{w}_A(p_M).\tilde{w}_A(p_M) \neq d\Theta_t(p_B).d\tilde{w}_B(p_B).\tilde{w}_B(p_B) \) (the vector field \( \tilde{w} \) is not objective).
\[ (\vec{v}_A - \vec{v}_D) (\Theta_t(p_B)) = d\Theta_t(p_B) . \vec{v}_B (p_B) \]

\[ d(\vec{v}_A - \vec{v}_D)(p_A).d\Theta_t(p_B) = d^2\Theta_t(p_B).\vec{v}_B (p_B) + d\Theta_t(p_B).d\vec{v}_B (p_B), \]

so, applied to \( \vec{w}_B \) (resp. \( \vec{v}_B \)), we get (18.32) (resp. (18.33)). Hence (18.34).

- If \( q_M = \Theta_t(q_B) \), then \( \vec{w}_A (t, \Theta_t(q_B)) = \Theta_t (t, q_B) . \vec{w}_B (t, q_B) \), so, with \( \frac{\partial \Theta}{\partial t} (t, q_B) = \vec{v}_B (\Theta_t(q_B)) \), we get

\[ \frac{\partial \vec{w}_A}{\partial t} (t, q_B) + d\vec{w}_A (q_M) . \vec{v}_B (q_M) = \frac{d\Theta}{\partial t} (t, q_B) . \vec{w}_B (q_B) + d\Theta(t, q_B) . \frac{\partial \vec{w}_B}{\partial t} (t, q_B) \]

\[ = (d\vec{v}_B (q_M) . d\Theta_t(q_B)) . \vec{w}_B (q_B) + d\Theta_t(q_B) . \frac{\partial \vec{w}_B}{\partial t} (t, q_B). \]

Thus (18.35) since \( \vec{v}_B = \vec{v}_D \); Then with (18.31) we get (18.36).

- \( \vec{v}_B^* (t, \Theta_t(q_B)) = d\Theta_t(q_B) . \vec{v}_B (t, q_B) \) gives

\[ \frac{\partial \vec{v}_B^*}{\partial t} (t, q_M) + d\vec{v}_B^* (q_M) . \vec{v}_B^* (q_M) = \frac{d\Theta}{\partial t} (t, p_B) . \vec{v}_B (p_B) + d\Theta(t, p_B) . \frac{\partial \vec{v}_B^*}{\partial t} (t, p_B) \]

\[ = d\vec{v}_B^* (p_M) . d\Theta_t(p_B) . \vec{v}_B (p_B) + d\Theta_t(p_B) . \frac{\partial \vec{v}_B^*}{\partial t} (t, p_B). \]

hence (18.37).

**18.7.3 Tensors**

**Proposition 18.16** It \( T \) is an objective tensor, then:

\[ \mathcal{L}_{\vec{v}} T_A = \Theta_*(\mathcal{L}_{\vec{v}} T_B). \]

**Proof.** Corollary of (18.26) and (18.28) to get \( \mathcal{L}_{\vec{v}}(\alpha, \vec{w}) = (\mathcal{L}_{\vec{v}} \alpha) . \vec{w} + \alpha.(\mathcal{L}_{\vec{v}} \vec{w}) \); Then use \( \mathcal{L}_{\vec{v}} (t_1 \otimes t_2) = (\mathcal{L}_{\vec{v}} t_1) \otimes t_2 + t_1 \otimes (\mathcal{L}_{\vec{v}} t_2). \)

**18.8 Taylor expansions and ubiquity gift**

**18.8.1 In \( \mathbb{R}^n \) with ubiquity**

**Reminder:**

\[ f(t) = f(t_0) + (t-t_0) f'(t_0) + \frac{(t-t_0)^2}{2} f''(t_0) + o((t-t_0)^2). \]

In particular \( f(t) = \vec{w}(t, p(t)) \) gives

\[ \vec{w}(t, p(t)) = \vec{w}(t_0, p(t_0)) + (t-t_0) \frac{D\vec{w}}{Dt}(t_0, p(t_0)) + \frac{(t-t_0)^2}{2} \frac{D^2\vec{w}}{Dt^2}(t_0, p(t_0))^2 + o((t-t_0)^2) \]

**Problem:** \( \vec{w}(t, p(t)) \) is a vector at \( t \) at \( p(t) \) while \( \vec{w}(t_0, p(t_0)) \) is a vector at \( t_0 \) at \( p(t_0) \), so (18.41) cannot be written

\[ \vec{w}(t, p(t)) - \vec{w}(t_0, p(t_0)) + (t-t_0) \frac{D\vec{w}}{Dt}(t_0, p(t_0)) + \frac{(t-t_0)^2}{2} \frac{D^2\vec{w}}{Dt^2}(t_0, p(t_0))^2 = o((t-t_0)^2), \]

since the left-hand side supposes the ubiquity gift.

And in a non-planar manifold \( \vec{w}(t, p(t)) \in T_{p(t)} \Omega = \) the linear tangent space at \( p(t) \), whereas \( \vec{w}(t_0, p(t_0)) \in T_{p(t_0)} \Omega = \) the linear tangent space at \( p(t_0) \), and the tangent spaces are distinct at two distinct points in general; Thus the left-hand side of (18.42) is meaningless.

In \( \mathbb{R}^n \) the linear space \( \mathbb{R}^n_t \) and \( \mathbb{R}^n_{p(t)} \) are identified, and (18.41) is well defined and very useful.
18.8.2 General case

By definition, cf. (15.8),
\[
L_v \bar{w}(t_0, p(t_0)) = \frac{d \Phi^v_s(p(t_0))^{-1} \bar{w}(t, p(t)) - \bar{w}(t_0, p(t_0))}{t - t_0} + o(1). \tag{18.43}
\]
Thus,
\[
d \Phi^v_s(p(t_0))^{-1} \bar{w}(t, p(t)) = \bar{w}(t_0, p(t_0)) + (t - t_0) L_v \bar{w}(t_0, p(t_0)) + o(t - t_0), \tag{18.44}
\]
or
\[
\bar{w}(t, p(t)) = d \Phi^v_s(p(t_0)) \left( \bar{w}(t, p(t_0)) + (t - t_0) L_v \bar{w}(t_0, p(t_0)) + o(t - t_0) \right). \tag{18.45}
\]
This is the first order Taylor expansion without ubiquity gift.

**Interpretation:** At first order, the estimation of \( \bar{w}(t, p(t)) \) is given by the push-forward \( d \Phi^v_s(p(t_0)) \left( \bar{w}(t_0, p(t_0)) + (t - t_0) L_v \bar{w}(t_0, p(t_0)) \right) \). And here the right and left members of (18.45) are at \( t \) at \( p(t) \). And as for a usual Taylor expansion, to estimate the value at \( t \) at \( p(t) \), we take the value at \( t_0 \) at \( p(t_0) \), here \( \left( \bar{w} + (t - t_0) L_v \bar{w} \right)(t_0, p_0) \), but now we push-forward this value to be at \( t \) at \( p(t) \) (we do not have the ubiquity gift). And (18.45) is meaningful on a (non-planar) manifold.

**Proposition 18.17** In \( \mathbb{R}^n \), with the gift of ubiquity, (18.45) gives (18.41) at first order.

**Proof.** Let \( d \Phi^v_s(t, p_0) = F^v_{p_0}(t) \) and \( h = t - t_0 \); And in \( \mathbb{R}^n \) (5.41) gives \( F^v_{p_0}(t_0 + h) = I_{t_0} + h d \tilde{v}_t(t_0, p_0) + o(h) \), so \( F^v_{p_0}(t_0 + h) \tilde{W}_{p_0} = \tilde{W}_{p_0} + h d \tilde{v}_t(t_0, p_0), \tilde{W}_{p_0} + o(h) \). Thus (18.45) gives:
\[
\bar{w}(t, p(t)) = (I + h d \tilde{v}_t(t, p_0, o(h))) \cdot \left( \bar{w} + h L_v \bar{w} + \frac{h^2}{2} L_v L_v \bar{w} \right)(t_0, p_0) + o(h) = \bar{w}(t, p(t)) + o(h),
\]
which is (18.41).

**Proposition 18.18** In \( \mathbb{R}^n \), at second order we have
\[
\bar{w}(t, p(p(t))) = d^2 \Phi^v_s(p(t_0)), \left( \bar{w} + h L_v \bar{w} + \frac{h^2}{2} L_v L_v \bar{w} + o(h^2) \right). \tag{18.46}
\]
So if the values \( \bar{w}(t_0, p_0), L_v \bar{w}(t_0, p_0) \) and \( L_v(L_v \bar{w})(t_0, p_0) \) are known, then \( \bar{w}(t, p(t)) \) is estimated at second order thanks to the push-forward of \( (\bar{w} + h L_v \bar{w} + \frac{h^2}{2} L_v L_v \bar{w} + o(h^2)) \) by \( \Phi^v_s \).

**Proof.** Let \( d \Phi^v_s(t, p_0) = F^v_{p_0}(t) \) and \( h = t - t_0 \); (5.41) gives \( F^v_{p_0}(t) = I_{t_0} + h d \tilde{v}_t(t_0, p_0) + \frac{h^2}{2} d^2 \tilde{v}_t(t_0, p_0) + o(h^2) \). Thus, omitting the reference to \( (t_0, p_0) \) to lighten the writing,
\[
d \Phi^v_s(p(t_0)), \left( \bar{w} + h L_v \bar{w} + \frac{h^2}{2} L_v L_v \bar{w} + o(h^2) \right) = \left( I + h d \tilde{v} + \frac{h^2}{2} d^2 \tilde{v} \right) \left( \bar{w} + h L_v \bar{w} + \frac{h^2}{2} L_v L_v \bar{w} + o(h^2) \right). \tag{18.47}
\]
The \( h^0 \) term is \( I \bar{w} = \bar{w} \). The \( h^1 \) term is \( L_v \bar{w} \bar{w} = L^2 \bar{w} \). The \( h^2 \) term is the sum of
\[
\bullet \quad \frac{1}{2} L_v L_v \bar{w} = \frac{1}{2} \left( \frac{D^2 \bar{w}}{Dt^2} - 2 \frac{D \bar{w}}{Dt} \frac{d \bar{w}}{Dt} + \frac{D(d \bar{w})}{Dt} + \frac{D \bar{w}}{Dt} \right), \text{cf.}(15.45),
\]
\[
\bullet \quad \frac{d}{dt} L_v \bar{w} = \frac{d \bar{w}}{Dt} + \frac{D \bar{w}}{Dt} \frac{d \bar{w}}{Dt} - 2 \frac{d \bar{w}}{Dt} \frac{d \bar{w}}{Dt} \left( \frac{D \bar{w}}{Dt} - 2 d \bar{w} \frac{d \bar{w}}{Dt} \right),
\]
\[
\bullet \quad \frac{1}{2} \frac{d}{dt} \frac{D \bar{w}}{Dt} \bar{w} = \frac{1}{2} \left( \frac{D \bar{w}}{Dt} \bar{w} + d \bar{w} \frac{d \bar{w}}{Dt} \right), \text{cf.}(2.42).
\]
And the sum gives \( \frac{D^2 \bar{w}}{Dt^2} \).

Alternate proof: Let \( \tilde{g}(t) = d \Phi^v_s(t, p_0)^{-1} \bar{w}(t, p(t)) \) when \( p(t) = \Phi^v_s(t, p_0) \). So \( L_v \tilde{w}(t_0, p(t_0)) = \tilde{g}^{\prime}(t_0), \) cf. (15.18). And \( \frac{d}{dt}(u, u(p)) = d \bar{w}(u, u(p)), \bar{w}(u, u(p)) + F^v_{p_0}(p), \tilde{g}^{\prime}(u), \) cf. (15.19). Thus \( \frac{d}{dt} \tilde{w}(u, u(p)) = \frac{D \bar{w}}{Dt}(u, u(p)), \tilde{w}(u, u(p)) + d \bar{w}(u, u(p)), \frac{D \bar{w}}{Dt}(u, u(p)) + d \bar{w}(u, u(p)), \tilde{w}(u, u(p)) + F^v_{p_0}(p), \tilde{g}^{\prime}(u) \). Thus \( \frac{D^2 \bar{w}}{Dt^2}(t, p(t)) = \left( \frac{D \bar{w}}{Dt} \right) \frac{D \bar{w}}{Dt} + d \bar{w} \frac{D \bar{w}}{Dt} + d \bar{w} L_v \bar{w}(t, p(t)) + \tilde{g}^{\prime\prime}(t) \). With (15.45) we get \( \tilde{g}^{\prime\prime}(t) = L_v(L_v \bar{w})(t, p(t)), \) cf. (15.45).
Part V
Appendix

The definitions, notations and results are detailed, so that no ambiguity is possible. (Some notations can be nightmarish when not understood, or arrive like a bull in a china-shop, or misused.) In particular, all the basic definitions and notations apply to solids, fluids, thermodynamics, general relativity, chemistry... (the same mathematics applies to everyone).

A Classical and duality notations

A.1 Contravariant vector and basis

A.1.1 Contravariant vector

Let \((E, +, \cdot)\) be a real vector space (= a linear space on the field \(\mathbb{R}\)).

Definition A.1 An element \(\vec{x} \in E\) is called a vector, or a contravariant vector.

A vector is a vector... For quantification purposes, a basis is introduced by an observer, and then a vector is expressed in the chosen basis with a set of components. When a second basis is chosen, the same vector is represented by a second set of components, the two set of components being linked by the contravariant formula \(\vec{x}|_{\vec{e}_{\text{new}}}=P^{-1}[\vec{x}]|_{\vec{e}_{\text{old}}},\) see (A.68); Hence the name “contravariant vector” due to the use of the inverse of \(P\).

A.1.2 Basis

Let \(\vec{e}_1, ..., \vec{e}_n \in E\). The vectors \(\vec{e}_1, ..., \vec{e}_n\) are linearly independent iff, for all \(\lambda_1, ..., \lambda_n \in \mathbb{R}\), the equality \(\sum_{i=1}^{n} \lambda_i \vec{e}_i = \vec{0}\) implies \(\lambda_i = 0\) for all \(i = 1, ..., n\). And the vectors \(\vec{e}_1, ..., \vec{e}_n\) span \(E\) iff, for all \(\vec{x} \in E\), \(\exists \lambda_1, ..., \lambda_n \in \mathbb{R}\) such that \(\vec{x} = \sum_{i=1}^{n} \lambda_i \vec{e}_i\). A basis in \(E\) is a finite set \(\{\vec{e}_1, ..., \vec{e}_n\} \subset E\) made of linearly independent vectors which span \(E\), in which case the dimension of \(E\) is \(n\).

A.1.3 Canonical basis

Consider the field \(\mathbb{R}\) and the Cartesian product \(\mathbb{R}^n = \mathbb{R} \times ... \times \mathbb{R}\), \(n\) times. The canonical basis is

\[
\vec{e}_1 = (1, 0, ..., 0), ..., \vec{e}_n = (0, ..., 0, 1),
\]

with 0 the addition identity element used \(n-1\) times, and 1 the multiplication identity element used once.

Remark A.2 The 3-D geometric space (we live in) has no canonical basis: What would the identity element 1 mean? 1 meter? 1 foot? 1 light-year? And there is no “intrinsic” preferred direction to define \(\vec{e}_1\).

A.1.4 Cartesian basis

(René Descartes 1596-1650.) Let \(n = 1, 2, 3\), let \(\mathbb{R}^n\) be the usual affine space (space of points), and let \(\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)\) be the usual real vector space of bipoint vectors with its usual algebraic operations.

Let \(p \in \mathbb{R}^n\), and let \((\vec{e}_i(p))\) be a basis at \(p\).

A Cartesian basis in \(\mathbb{R}^n\) is a basis independent of \(p\) (the same at all \(p\)), and then \((\vec{e}_i(p)) = \text{named}(\vec{e}_i)\).

Example of a non Cartesian basis: The polar basis, see example 10.17 (polar coordinate system).

And a Euclidean basis is a particular Cartesian basis described in § B.1.

A.2 Representation of a vector relative to a basis

We give the classical notation, e.g. used by Arnold [8] and Germain [8], and the duality notation, e.g. used by Marsden and Hughes [12]. Both classical and duality notation are equally good, but if you have any doubts about using duality notations, go back to the unambiguous classical notations.
Definition A.3 Let \( \vec{x} \in E \). Let \((\vec{e}_i)\) be a basis in \( E \). The components of \( \vec{x} \) relative to the basis \((\vec{e}_i)\) are the \( n \) real numbers \( x_1, \ldots, x_n \) (classical notation) also named \( x^1, \ldots, x^n \) (duality notation) such that

\[
[\vec{x}]_{\vec{e}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.
\]

\[\text{class.}\] \[\text{dual}\]

\((\vec{e}_i)\) being the column matrix representing \( \vec{x} \) relative to the basis \((\vec{e}_i)\). (Of course \( x_i = x^i \) for all \( i \).) And the column matrix \([\vec{x}]_{\vec{e}}\) is simply named \([\vec{x}]\) if one chosen basis is implicitly used. With the sum sign:

\[
\vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i = \sum_{j=1}^{n} x^j \vec{e}_j = \sum_{\alpha=1}^{n} x^\alpha \vec{e}_\alpha).
\]

\(\text{class.}\) \(\text{dual}\)

(The index in a summation is a dummy index, even if you do not write the sum sign \(\sum\) to follow Einsteins convention: \(\vec{x} = x^1 \vec{e}_1 = x^2 \vec{e}_2 = x^n \vec{e}_n\).

Example A.4 \( \mathbb{R}^2 \) and \( \vec{x} = 3 \vec{e}_1 + 4 \vec{e}_2 = \sum_{i=1}^{2} x_i \vec{e}_i = \sum_{i=1}^{2} x^i \vec{e}_i \): We have \( x_1 = x^1 = 3 \) and \( x_2 = x^2 = 4 \) with classical and duality notations. And \([\vec{x}]_{\vec{e}} = 3[\vec{e}_1]_{\vec{e}} + 4[\vec{e}_2]_{\vec{e}} = \sum_{i=1}^{2} x_i [\vec{e}_i]_{\vec{e}} = \sum_{i=1}^{2} x^i [\vec{e}_i]_{\vec{e}} \). In particular, with \( \delta_{ij} = \delta_{ij} : = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \) the Kronecker symbols,

\[
\vec{e}_j = \sum_{i=1}^{n} \delta_{ij} \vec{e}_i,
\]

\(\text{class.}\) \(\text{dual}\)

i.e. \([\vec{e}_1]_{\vec{e}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, [\vec{e}_n]_{\vec{e}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \)

that is, relative to the basis \(\vec{e}_i\), the components of \(\vec{e}_j\) are the \(\delta_{ij}\) with classical notations, i.e., and are the \(\delta_{ij}\) with duality notations, for \( i = 1, \ldots, n \). And the matrices \([\vec{e}_j]_{\vec{e}}\) mimic the use of \(\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}\) and its canonical basis.

Remark A.5 The column matrix \([\vec{x}]_{\vec{e}}\) is also called a “column vector”. NB: A “column vector” is not a vector, but just a matrix (a collection of real numbers). See the change of basis formula (A.68) where the same vector is represented by two “column vectors” (two column matrices).

A.3 Bilinear forms

Let \( A \) and \( B \) be vector spaces and \( (F(A; B), +, \cdot) =^\text{noted} F(A; B) \) be the usual real vector space of functions with the internal addition \((f, g) \rightarrow f + g\) defined by \((f + g)(x) = f(x) + g(x)\) and the external multiplication \((\lambda, f) \rightarrow \lambda f\) defined by \((\lambda, f)(x) = \lambda f(x)\), for all \( f, g \in F(A; B), x \in A, \lambda \in \mathbb{R} \). And \( \lambda, f =^\text{noted} \lambda f \).

A.3.1 Definition

Definition A.6 Let \( E \) and \( F \) be vector spaces. A bilinear form on \( E \times F \) is a function \( g(\cdot, \cdot) : \)

\[
\begin{cases}
E \times F \to \mathbb{R} \\
(u, w) \to g(u, w)
\end{cases}
\]

which is linear for each variable, i.e., \( g(\cdot, \cdot) \in F(E \times F; \mathbb{R}) \) satisfies \( g(u_1 + \lambda u_2, w) = g(u_1, w) + \lambda g(u_2, w) \) and \( (u, v_1 + \lambda v_2) = g(u, v_1) + \lambda g(u, v_2) \) for all \( u, u_1, u_2 \in E, v, v_1, v_2 \in F, \lambda \in \mathbb{R} \).

Let \( \mathcal{L}(E, F; \mathbb{R}) \) be the set of bilinear forms \( E \times F \to \mathbb{R} \): It is a vector space, sub-space of \( F(E \times F; \mathbb{R}) \).

A.3.2 Inner dot product, and metric

Definition A.7 In a vector space \( E \), an “inner dot product” (or “inner product”, or “inner scalar product”) is a bilinear form \( g(\cdot, \cdot) : E \times E \to \mathbb{R} \) which is symmetric, i.e. \( g(u, \bar{w}) = g(\bar{w}, u) \) for all \( u, \bar{w} \in E \), and definite positive, i.e. \( g(u, u) > 0 \) for all \( u \neq \bar{0} \). And then (for inner dot products)

\[
g(\cdot, \cdot) =^\text{noted} (\cdot, \cdot)_g, \text{ and } g(u, \bar{w}) = (\bar{u}, \bar{w}) =^\text{noted} \bar{u} \cdot \bar{w} =^\text{noted} \bar{u} \cdot \bar{w},
\]

the last notation if one inner dot product \( (\cdot, \cdot)_g \) is imposed by one observer to all observers (if it is possible... which is not possible in general...). And the associated norm is the function \( ||\cdot||_g : E \to \mathbb{R}_+ \).
given by, for all \( \vec{u}, \vec{v} \in E \),
\[
||\vec{u}||_g = \sqrt{(\vec{u}, \vec{u})_g}.
\] (A.6)

To prove that \( ||.||_g \) is indeed a norm, we use the Cauchy–Schwarz inequality:
\[
\forall \vec{u}, \vec{v} \in E, \quad ||(\vec{u}, \vec{v})_g|| \leq ||\vec{u}||_g ||\vec{v}||_g,
\] (A.7)
which translates the property: “The second order polynomial \( p(\lambda) = ||\vec{u} + \lambda \vec{v}||_g^2 = (\vec{u} + \lambda \vec{v}, \vec{u} + \lambda \vec{w})_g \) is non negative”, which also gives:
\[
||((\vec{u}, \vec{w})_g) = ||\vec{u}||_g ||\vec{w}||_g \text{ iff } \vec{u} = \vec{w}.
\]

**Definition A.8**
Two vectors \( \vec{u}, \vec{w} \in E \) are \((\cdot, \cdot)_g\)-orthogonal iff \( (\vec{u}, \vec{w})_g = 0 \).

**Definition A.9**
With \( \mathbb{R}^n \) our usual affine geometric space, \( n = 1, 2 \) or \( 3 \), and \( \mathbb{R}^n \) is the usual associated vector space made of bipoins vectors. Let \( \Omega \subset \mathbb{R}^n \) be open in \( \mathbb{R}^n \). A metric in \( \Omega \) is a function \( g : \{ \Omega \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}) \} \) such that, at each \( p \in \Omega \), \( g_p \) is an inner dot product in \( \mathbb{R}^n \). (That is, a metric in \( \Omega \) is a \( (\cdot, \cdot)_g \) tensor \( g \) in \( \Omega \) s.t. \( g_p \) is an inner dot product in \( \mathbb{R}^n \) at each \( p \in \Omega \).)

### A.3.3 Quantification: Matrices \([g_{ij}]\)

\( E \) and \( F \) are finite dimensional vector spaces, \( \dim E = n \), \( \dim F = m \), \((\vec{a}_i)\) is a basis in \( E \) and \((\vec{b}_i)\) is a basis in \( F \).

**Definition A.10**
Let \( g \in \mathcal{L}(E, F; \mathbb{R}) \). The components of \( g \) relative to the bases \((\vec{a}_i)\) and \((\vec{b}_i)\) are the \( nm \) scalars
\[
g_{ij} = g(\vec{a}_i, \vec{b}_j),
\] (A.8)
and \( [g]_{\vec{a}, \vec{b}} = [g_{ij}]_{i,j=1,...,n} \) is the matrix of \( g \) relative to the bases \((\vec{a}_i)\) and \((\vec{b}_i)\).

**Proposition A.11**
A bilinear form \( g \in \mathcal{L}(E, F; \mathbb{R}) \) is known as soon as the \( nm \) scalars \( g_{ij} = g(\vec{a}_i, \vec{b}_j) \) are known. Thus \( \dim \mathcal{L}(E, F; \mathbb{R}) = nm \).

**Proof.**
Let \( b_{ij} \in \mathcal{L}(E, F; \mathbb{R}) \) be defined by \( b_{ij}(\vec{a}_k, \vec{b}_\ell) = \delta_{ik}\delta_{j\ell} \) (all the elements the matrix \( [b_{ij}]_{\vec{a}, \vec{b}} \) are zero except the element at the intersection of row \( i \) and column \( j \) which is equal to 1). Then \( \sum_{i,j} \lambda_{ij} b_{ij} = 0 \) implies \( \sum_{i,j} \lambda_{ij} b_{ij}(\vec{a}_k, \vec{b}_\ell) = 0 \) for all \( k, \ell \), thus the \( b_{ij} \) are independent. Then let \( g \in \mathcal{L}(E, F; \mathbb{R}) \). We have \( g = \sum_{i,j} g(\vec{a}_i, \vec{b}_j) b_{ij} \) (trivial), thus the \( b_{ij} \) span \( \mathcal{L}(E, F; \mathbb{R}) \). 

**Example A.12**
\( \dim E = \dim F = 2 \). \( [g]_{\vec{a}, \vec{b}} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \) means \( g(\vec{a}_1, \vec{b}_1) = g_{11} = 1, g(\vec{a}_1, \vec{b}_2) = g_{12} = 2, g(\vec{a}_2, \vec{b}_1) = g_{21} = 0, g(\vec{a}_2, \vec{b}_2) = g_{22} = 3 \). E.g.,
\[
g_{12} = [\vec{a}_1]_{\vec{b}, \vec{b}} = [g]_{\vec{a}, \vec{b}} [\vec{b}_2]_{\vec{a}} = (1, 0) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2.
\]

**Exercise A.13**
If \( g \in \mathcal{L}(E, E; \mathbb{R}) \), if \((\vec{a}_i)\) is a basis, if \( \lambda \in \mathbb{R}^* \), prove:
\[
g(\lambda \vec{a}_i, \vec{v}) = \lambda^2 g(\vec{a}_i, \vec{v}) \forall i \in [1, n][s], \quad \text{then } [g]_{\vec{a}, \vec{b}} = \lambda^2 [g]_{\vec{a}, \vec{b}}.
\] (A.10)

(A change of unit of measurement has a great influence on the matrix of \( g \).) And check:
\[
g(\vec{u}, \vec{w}) = [\vec{u}]_{\vec{a}}^T [g]_{\vec{a}, \vec{b}} [\vec{w}]_{\vec{b}} = [\vec{u}]_{\vec{a}}^T [g]_{\vec{a}, \vec{b}} [\vec{w}]_{\vec{b}}
\] (A.11)

In particular if \( (\cdot, \cdot)_g \) is a Euclidean dot product then the length \( ||\vec{u}||_g = \sqrt{[\vec{u}]_{\vec{a}}^T [g]_{\vec{a}, \vec{b}} [\vec{u}]_{\vec{b}}} \) depends only on the unit of measure used to define \((\cdot, \cdot)_g\). Can also be checked with change of basis formuals.

**Answer.**
\( g(\vec{b}_i, \vec{b}_j) = g(\lambda \vec{a}_i, \lambda \vec{a}_j) = \lambda^2 g(\vec{a}_i, \vec{a}_j) \) (bilinearity) reads \( [g]_{\vec{a}, \vec{b}} = \lambda^2 [g]_{\vec{a}, \vec{b}} \).

Then \( \vec{u} = \sum_{i=1}^n u_i \vec{a}_i \) gives \( u_i = \lambda w_i \) gives \( w_i = \lambda u_i \), thus \( [\vec{u}]_{\vec{a}}^T [g]_{\vec{a}, \vec{b}} [\vec{w}]_{\vec{b}} = \lambda^2 [\vec{u}]_{\vec{a}}^T [g]_{\vec{a}, \vec{b}} [\vec{w}]_{\vec{b}} \).

Change of basis formuals:
\[
[g]_{\vec{a}, \vec{b}} = (P^{-1} [g]_{\vec{a}, \vec{b}} P)^T,(P^{-1} [g]_{\vec{a}, \vec{b}} P),(P^{-1} [g]_{\vec{a}, \vec{b}} P), (P^{-1} [g]_{\vec{a}, \vec{b}} P)
\]
A.4 Linear maps

A.4.1 Definition

**Definition A.14** Let $E$ and $F$ be vector spaces. A function $L : E \to F$ is linear iff $L(\vec{u}_1 + \lambda \vec{u}_2) = L(\vec{u}_1) + \lambda L(\vec{u}_2)$ for all $\vec{u}_1, \vec{u}_2 \in E$ and all $\lambda \in \mathbb{R}$ (distributivity type relation). And (distributivity notation):

$$L(\vec{a}) \in \text{noted} L.\vec{u}, \quad \text{so} \quad L(\vec{u}_1 + \lambda \vec{u}_2) = L.\vec{u}_1 + \lambda L.\vec{u}_2.$$  \hfill (A.12)

NB: This dot notation $L.\vec{u}$ is not a matrix product: No bases have been introduced yet.

Let $L(E;F)$ be the set of linear maps $E \to F$: It is a vector space, subspace of $(F;E,+,.).$

**Vocabulary:** Let $L_r(E;F)$ be the space of invertible linear maps. If $E$ is a finite dimensional vector space, dim $E = n$, then, in algebra, the set $(L_r(E;E),\circ)$ of linear maps equipped with the composition rule is named $GL_n(E)$ = "the linear group" (it is indeed a group). And in elementary courses, the "linear group" $GL_n(M_n)$ refers to $n \times n$ invertible matrices with the matrix product rule.

**Exercise A.15** Let $E = (E,||.||_E)$ and $F = (F,||.||_F)$ be Banach spaces, and let $L_{ic}(E;F)$ be the space of invertible linear continuous maps $E \to F$, with its usual norm $||L|| = \sup_{||\vec{z}||_E = 1} ||L.\vec{z}||_F$. Let $Z : \{L_{ic}(E;F) \to L_{ic}(E;F); \ \ L \to L^{-1}\}$. Prove $dZ(L).M = -L^{-1} \circ M \circ L^{-1}$, for all $M \in L_{ic}(E;F)$. (Remark: In finite dimensions, a linear map is always continuous.)

**Answer.** We have to consider $\lim_{h \to 0} \frac{Z((L+hM) - Z(L))}{h} = \lim_{h \to 0} \frac{(L+hM)^{-1} - L^{-1}}{h}$ = $\frac{dZ(L)M}{h}$ if the limit exists. With $N = L^{-1}.M$ we have $L + hM = L(I + hN)$. Thus $(L + hM)^{-1} = (I + hN)^{-1} = (I - hN + o(h)).L^{-1}$ as soon as $||hN|| < 1$, i.e., $h < \frac{1}{||N||}$, true as soon as $h < \frac{1}{\|L^{-1}\| \cdot ||M||}$. Thus $(L + hM)^{-1} - L^{-1} = -L^{-1}.hN.L^{-1} + o(h) - L^{-1} = h(-N.L^{-1} + o(1))$, thus $\lim_{h \to 0} \frac{(L+hM)^{-1} - L^{-1}}{h} = -N.L^{-1}$. \hfill \Box

A.4.2 Quantification: Matrices $[L_{ij}] = [L^j_i]$

Let $E$ and $F$ be finite dimensional vector spaces (to be able to use matrix representations), dim $E = n$ and dim $F = m$. Let $(\vec{a}_i)$ be a basis in $E$ and $(\vec{b}_i)$ be a basis in $F$.

**Definition A.16** The components of a linear map $L \in L(E;F)$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$ are the $nm$ reals named $L_{ij}$ (classical notation), or $L^j_i$ (duality notation), defined by: For all $i,j = 1,...,n$, the reals $L_{ij} = L.\vec{a}_j$ are the components of the vectors $L.\vec{a}_j$ relative to the basis $(\vec{b}_i)$. That is:

$$\begin{cases}
\text{clas.} : L.\vec{a}_j = \sum_{i=1}^{m} L_{ij} \vec{b}_i, \\
\text{dual} : L.\vec{a}_j = \sum_{i=1}^{m} L^j_i \vec{b}_i,
\end{cases}$$

so $[L.\vec{a}_j]_\vec{b} = \left( \begin{array}{c} L_{ij} \\ \vdots \\ L_{mj} \end{array} \right)$ and $[L^j_i]_\vec{b} = \left( \begin{array}{c} L^j_i \\ \vdots \\ L^j_m \end{array} \right).$  \hfill (A.13)

And

$$[L]_{\vec{a},\vec{b}} = \left[ L_{ij} \right]_{i=1,\ldots,n \atop j=1,\ldots,m} = \left[ L^j_i \right]_{i=1,\ldots,m \atop j=1,\ldots,n}$$

is the matrix of $L$ relative to the bases $(\vec{a}_i)$ and $(\vec{b}_i)$. In particular $[L.\vec{a}_j]_\vec{b}$ is the $j$-th column of $[L]_{\vec{a},\vec{b}}$.

**Particular case:** If $E = F$ (so $L$ is an endomorphism) and if $(\vec{b}_i) = (\vec{a}_i)$ then $[L]_{\vec{a},\vec{a}}$ is named $[L]_{\vec{a}}$.

**Example A.17** $n = m = 2$. $[L]_{\vec{a},\vec{b}} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ means $L.\vec{a}_1 = \vec{b}_1$ and $L.\vec{a}_2 = 2\vec{b}_1 + 3\vec{b}_2$ (column reading).

Thus, $L_{11} = L^1_1 = 1$, $L_{12} = L^1_2 = 2$, $L_{21} = L^2_1 = 0$, $L_{22} = L^2_2 = 3$, with classical and duality notations.

And $L$ being linear, for all $\vec{u} \in E$, $\vec{u} = \sum_{j=1}^{n} u_j \vec{a}_j = \sum_{j=1}^{n} w^j \vec{a}_j$, we get, thanks to linearity,

$$L.\vec{u} = \sum_{i=1}^{m} \sum_{j=1}^{n} L_{ij} u_j \vec{b}_i = \sum_{i=1}^{m} \sum_{j=1}^{n} L^j_i w^j \vec{b}_i,$$

i.e. $[L.\vec{u}]_\vec{b} = [L]_{\vec{a},\vec{b}}[\vec{u}]_{\vec{a}}$.  \hfill (A.15)

Shortened notation: $[L.\vec{u}] = [L]_{\vec{a},\vec{b}}[\vec{u}]_{\vec{a}}$ (implicit bases).
Proposition A.18 A linear map \( L \in \mathcal{L}(E; F) \) is known iff the \( n \) vectors \( L(\bar{a}_j) \) are known. Thus \( \dim(\mathcal{L}(E; F)) = nm \).

Proof. \( F \in E \) and \( \bar{a} = \sum_{i=1}^{n} u_i \bar{a}_i \) give \( L(\bar{a}) = \sum_{i=1}^{n} u_i L(\bar{a}_i) \), thanks to the linearity of \( L \). Thus \( L \) is known iff all the \( n \) vectors \( L(\bar{a}_j) \), \( j = 1, \ldots, n \), are known. And a vector \( L(\bar{a}_j) \) is known iff its \( m \) components \( L_{ij} \) are known. (Use duality notations if you prefer.)

Exercise A.19 If \( L \in \mathcal{L}(E; E) \) (endomorphism), if \( (\bar{a}_i) \) is a basis, prove:

\[
\text{if } \lambda \in \mathbb{R}^* \text{ and if } \bar{b}_i = \lambda \bar{a}_i \forall i \in [1, n], \text{ then } [L]_{\bar{b}} = [L]_{\bar{a}} : (A.16)
\]

A change of unit has not influence on the matrix of an endomorphism. Check this result with the change of basis formula. To compare with (A.10): Covariance and contravariance should not be confused.

Answer. Let \( L \bar{a}_j = \sum_{i=1}^{n} L_{ij} \bar{a}_i \). Then \( L \bar{b}_j = L (\lambda \bar{a}_j) = \lambda L \bar{a}_j = \lambda \sum_{i=1}^{n} L_{ij} \bar{a}_i = \lambda \sum_{i=1}^{n} L_{ij} \frac{\bar{b}_i}{\bar{a}_i} = \sum_{i=1}^{n} L_{ij} \bar{b}_i \).

Change of basis formula: \([L]_{\bar{b}} = P^{-1} [L]_{\bar{a}} P \) with \( P = \lambda I \) here.

A.5 Transposed matrix

The definition can be found in any elementary books, e.g., Strang [18]:

Definition A.20 Let \( M = [M_{ij}]_{i=1, \ldots, n} \) be an \( m \times n \) matrix. Its transposed is the \( n \times m \) matrix \( M^T = [(M^T)_{ij}]_{i=1, \ldots, n} \) defined by \( (M^T)_{ij} := M_{ji} \) (exchange rows and columns).

E.g., if \( M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \) then \( M^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \); E.g., \( (M^T)_{12} = M_{21} = 3 \).

A.6 The transposed endomorphisms of an endomorphism

Let \( (E, (\cdot, \cdot)_g) \) be a finite dimensional vector space equipped with an inner dot product \( g(\cdot, \cdot) = (\cdot, \cdot)_g \).

A.6.1 Definition (requires an inner dot product)

Definition A.21 The transpose of an endomorphism \( L \in \mathcal{L}(E; E) \) relative to the inner dot product \( (\cdot, \cdot)_g \) is the endomorphism \( L^T_g \in \mathcal{L}(E; E) \) defined by

\[
\forall \bar{x}, \bar{y} \in E, \quad (L^T_g (\bar{y}), \bar{x})_g = (\bar{y}, L(\bar{x}))_g, \quad \text{written} \quad (L^T_g \bar{y}, \bar{x})_g = (\bar{y}, L \bar{x})_g: (A.17)
\]

the dot notations since \( L^T_g \) and \( L \) are linear maps (it is not a matrix product: No basis has been introduced yet). This defines the map \( (\cdot)^T_g : \mathcal{L}(E; E) \rightarrow \mathcal{L}(E; E) \) \( L \rightarrow L^T_g \).

We immediately get \( (L^T_g)^T_g = L \).

If \( (\cdot, \cdot)_g \) is imposed and non ambiguous, then \( L^T_g \) is named \( L^T \).

Remark A.22 If \( E \) is finite dimensional, then the existence and uniqueness of \( L^T_g \) is proved thanks to a basis in \( E \). If \( E \) infinite dimensional and \( (E, (\cdot, \cdot)_g) \) a Hilbert space, and if \( L \) is continuous, then \( L^T_g \) exists and is unique by application of the Riesz representation theorem.

A.6.2 Quantification with bases

Relative to a basis \( (\bar{e}_i) \) \( E \), (A.17) gives \( [\bar{x}]^T_g [\cdot].[\bar{y}] = [L \bar{x}]^T_g [\cdot].[\bar{y}] = [\bar{x}]^T_g [L]^T_g [\cdot].[\bar{y}] \) for all \( \bar{x}, \bar{y} \), thus

\[
[g].[L^T_g] = [L]^T_g [g], \text{ i.e. } [L^T_g] = [g]^{-1}.[L]^T_g [g]: (A.18)
\]

Full notation: \( g(\cdot, \cdot)_g [L^T_g]_{\bar{e}} = ([L]_{\bar{e}})^T_{\bar{e}} [\cdot]_{\bar{e}} [\cdot]_{\bar{e}} \), and \( [L^T_g]_{\bar{e}} = [g]_{\bar{e}}^{-1}.([L]_{\bar{e}})^T_{\bar{e}} [\cdot]_{\bar{e}} \). I.e., with classical notations,

\[
g(\bar{e}_i, \bar{e}_j) = g_{ij}, \quad L \bar{e}_j = \sum_{i=1}^{n} L_{ij} \bar{e}_i, \quad L^T_g \bar{e}_j = \sum_{i=1}^{n} (L^T_g)_{ij} \bar{e}_i, (A.19)
\]
i.e., if \([g|_{\mathcal{E}} = [g_{ij}], [L|_{\mathcal{E}} = [L_{ij}], [L^T|_{\mathcal{E}} = [(L^T_{ij})]]],\)

\[
\text{then } \sum_{k=1}^{n} g_{ik}(L^T_{ij})_{kj} = \sum_{k=1}^{n} L_{ki} g_{kj} \quad \text{and} \quad (L^T_{ij})_{ij} = \sum_{k,\ell=1}^{n} ([g]^{-1})_{ik} L_{\ell k} g_{\ell j}. \tag{A.20}
\]

(If \((\vec{c}_g)\) is \((\cdot, \cdot)_g\)-orthonormal basis then \([g] = [\delta_{ij}]\) and \((L^T_{ij})_{ij} = L_{ij}).\)

And, with duality notations, if \(L.\vec{e}_j = \sum_{i=1}^{n} L_{ij} \vec{e}_i\) and \(L^T.\vec{e}_j = \sum_{i=1}^{n} (L^T_{ij})_{ij} \vec{e}_i\), i.e., if \([L|_{\mathcal{E}} = [L^T]_j\) and \([L^T|_{\mathcal{E}} = [(L^T)|_{\mathcal{E}}]_j],\) then

\[
\sum_{k=1}^{n} g_{ik}(L^T_{ij})_{kj} = \sum_{k=1}^{n} L^T_{ki} g_{kj} \quad \text{and} \quad (L^T_{ij})^T_{ij} = \sum_{k,\ell=1}^{n} ([g]^{-1})_{ik} L^T_{\ell k} g_{\ell j}. \tag{A.21}
\]

Remark A.23 Warning: The last equation (A.21)\(_2\) is also written

\[
(L^T_{ij})^T_{ij} = \sum_{k,\ell=1}^{n} ([g]^{-1})_{ik} L^T_{\ell k} g_{\ell j}, \quad \text{when} \quad ([g]^{-1})_{ik} = |g_{ij}|^{-1}, \tag{A.22}
\]

Don’t be fooled by the notation \([g^{ij}]\) here: It is not a duality notation here: It is just the inverse matrix of the matrix \([g_{ij}]\). E.g., if \([g|_{\mathcal{E}} = [g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\) then \([g^{ij}] := ([g]^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\). Return to the classical notations if in doubt. (To understand the origin of the notation \(g^{ij}\), go to §A.1.2.7: \(g^{ij}\) is related with an inner dot product in \(E^*\).

Exercise A.24 In \(\mathbb{R}^2\), let \((\vec{e}_1, \vec{e}_2)\) be a basis. Let \(L \in \mathcal{L}(\mathbb{R}^2; \mathbb{R}^2)\) be defined by \([L|_{\mathcal{E}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). Find two inner dot products \((\cdot, \cdot)_g\) and \((\cdot, \cdot)_h\) in \(\mathbb{R}^2\) such that \(L^T_g \neq L^T_h\) (a transposed endomorphism is not unique.

It is not intrinsic to \(L\) since it depends on a choice of an inner dot product).

Answer. Calculations with (A.18):

Choose \((\cdot, \cdot)_g\) given by \([g|_{\mathcal{E}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\). Thus \([L^T|_{\mathcal{E}} = [I], [L|_{\mathcal{E}} = [I] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\); So \(L^T_g = L\).

Choose \((\cdot, \cdot)_h\) given by \([h|_{\mathcal{E}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\). Thus \([L^T|_{\mathcal{E}} = [h], [L|_{\mathcal{E}} = [h]^{-1} = [h]^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\); So \(L^T_h \neq L\).

Thus \(L^T_g \neq L^T_h\), e.g., \(\vec{e}_2 = L^T_g \vec{e}_1 \neq L^T_h \vec{e}_1 = \frac{1}{2} \vec{e}_2\).

Exercise A.25 Prove: If \(L\) is invertible then \(L^T_g\) is invertible, and \((L^T_g)^{-1} = (L^{-1})^T_g\) (called \(L^{-T}_g\)).

Answer. If the endomorphism \(L^T_g\) was not invertible, then \(\exists \vec{y} \in E \text{ s.t. } \vec{y} \neq \vec{0} \text{ and } L^T_g \vec{y} = \vec{0} = 0, \text{ thus } (\vec{y}, L.\vec{x})_{\mathcal{E}} = 0, \text{ for all } \vec{x}. \text{ And } L \text{ being invertible we can choose } \vec{x} \text{ s.t. } L.\vec{x} = \vec{y} \text{ to get } ||\vec{y}||_{\mathcal{E}} = 0, \text{ thus } \vec{y} = \vec{0}. \text{ Absurd since } \vec{y} \neq \vec{0}. \text{ Thus } L^T_g \text{ is invertible.}

Then \((L^T_g)^{-1} = \begin{pmatrix} \vec{x} & \vec{y} \end{pmatrix} = (\vec{x}, L.\vec{y})_{\mathcal{E}} = \begin{pmatrix} \vec{y} \end{pmatrix} = (L.\vec{x}, \vec{y})_{\mathcal{E}} = (L.\vec{x}, \vec{y})_{\mathcal{E}} = (L.\vec{x}, \vec{y})_{\mathcal{E}}, \text{ true } \forall \vec{x}, \vec{y}, \text{ thus } L^T_g(L^{-1})^T_g = L^T_g(L^{-1})^T_g = (L^T_g)^{-1}, \text{ thus } (L^{-1})^T_g = (L^T_g)^{-1}\).

Exercise A.26 Special case of proportional inner dot products: Let \(\lambda > 0\) and suppose \((\cdot, \cdot)_{gh} = \lambda^2(\cdot, \cdot)_{gh}\).

Prove: \(L^T_a = L^T_b\): Two proportional inner dot products give the same transposed endomorphism.

Answer. \((L^T_a, \vec{y}, \vec{h}) = (\lambda^2(L^T_a, \vec{y}, \vec{h}) = \lambda^2(L^T_b, \vec{y}, \vec{h}) = (L^T_b, \vec{y}, \vec{h}), \text{ for all } \vec{y}, \vec{h}, \text{ so } L^T_a = L^T_b\).

A.6.3 Symmetric endomorphism

Definition A.27 An endomorphism \(L \in \mathcal{L}(E; E)\) is \((\cdot, \cdot)_g\)-symmetric iff \(L_g = L\), that is,

\[
L(\cdot, \cdot)_g\text{-symmetric } \iff L^T_g = L \iff (L.\vec{x}, \vec{y})_g = (\vec{x}, L.\vec{y})_g, \quad \forall \vec{x}, \vec{y} \in E. \tag{A.23}
\]

Remark A.28 The symmetric character of an endomorphism \(L\) is not intrinsic to the endomorphism: It depends on \((\cdot, \cdot)_g; \text{ See exercise A.24: } L^T_g \text{ is } (\cdot, \cdot)_g\text{-symmetric, and } L^T_h \text{ isn’t } (\cdot, \cdot)_g\text{-symmetric.}

Quantification: \(L \in \mathcal{L}(E; E) \text{ is } (\cdot, \cdot)_g\text{-symmetric iff } L^T_g = L \text{ if } [L^T|_{\mathcal{E}} = [L|_{\mathcal{E}} \text{ i.e. if } ([g]_{\mathcal{E}}^{-1}.L^T|_{\mathcal{E}}[g]|_{\mathcal{E}} = [L|_{\mathcal{E}} \text{ cf. (A.18), i.e. if } [L^T|_{\mathcal{E}}[g]|_{\mathcal{E}} = [g]|_{\mathcal{E}}.\text{for all basis } (\vec{e}_i). \text{ E.g., with exercise A.24 and } (\cdot, \cdot)_h, \text{ we have } [L^T|_{\mathcal{E}}[h]|_{\mathcal{E}} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \neq [h]|_{\mathcal{E}}.\text{Thus } L \text{ is not } (\cdot, \cdot)_h \text{ symmetric.}

Particular case \((\vec{e}_i)\) is \((\cdot, \cdot)_g\)-orthonormal basis: \(L \in \mathcal{L}(E; E) \text{ is } (\cdot, \cdot)_g\text{-symmetric iff } [L^T|_{\mathcal{E}} = [L|_{\mathcal{E}}\))


A.7 The transposed of a linear map

A.7.1 Definition (needs two inner dot products)

$E$ and $F$ are finite dimensional vector spaces, $(\cdot, \cdot)_E$ and $(\cdot, \cdot)_F$ are an inner dot products in $E$ and $F$. (The case $E = F$ and $(\cdot, \cdot)_E = (\cdot, \cdot)_F$ is treated in § A.6.)

Let $L \in \mathcal{L}(E; F)$. E.g.: $L = d\Phi^0_t(P) : \mathbb{R}^n \to \mathbb{R}^n$ is the deformation gradient, cf. (5.2), and $(\cdot, \cdot)_E$ is the inner dot product chosen by the observer who made the measurements at $t_0$ (e.g. the foot built Euclidean dot product), and $(\cdot, \cdot)_F$ is the inner dot product chosen by the observer who makes the measurements at $t$ (e.g. the meter built Euclidean dot product).

**Definition A.29** The transposed of $L \in \mathcal{L}(E; F)$ relative to $(\cdot, \cdot)_E$ and $(\cdot, \cdot)_F$ is the linear map $L^T_{gh} \in \mathcal{L}(F; E)$ defined by, for all $(\vec{x}, \vec{y}) \in E \times F$,

$$
(L^T_{gh}(\vec{y}), \vec{x})_F = (\vec{y}, L(\vec{x}))_E,
$$

written $(L^T_{gh}(\vec{y}), \vec{x})_F = (\vec{y}, L(\vec{x}))_E$.

We immediately get $(L^T_{gh})^T_{hy} = L_{yg}$.

If $F = E$ and $(\cdot, \cdot)_E = (\cdot, \cdot)_F$ then $L^T_{gh} = L^T$, see § A.6.

A.7.2 Quantification with bases

Let $(\vec{a}_i)$ be a basis in $E$, $(\vec{b}_j)$ be a basis in $F$. (A.24) gives $(L^T_{gh})_{ij} = (\vec{b}_j, L(\vec{a}_i))_F$.

i.e., $[\vec{a}_i]^T|_E g[\vec{a}, L(T_{gh})_{ij}]|_G = ([L(\vec{a}_i)]|_G)^T |_E [h]|_E [\vec{b}_j]|_G = ([\vec{a}_i]|_G)^T |_E [g]|_E [\vec{b}_j]|_G$ for all $i, j$, thus,

$[g]|_E [L^T_{gh}]_{ij} = ([L(\vec{a}_i)]|_F)^T |_E [h]|_E [\vec{b}_j]|_F$.

Shortened notations:

$$
[g]|_E [L^T_{gh}] = [L]^T |_E [h], \quad \text{i.e.} \quad [L^T_{gh}] = [g]^{-1} |_E [L]^T |_E [h].
$$

(A.25)

With components, classical and duality notations:

$$
\begin{aligned}
\text{class. : } L\vec{a}_j = \sum_{i=1}^n L_{ij}\vec{b}_i, \quad L^T_{gh}\vec{b}_j = \sum_{i=1}^n (L^T_{gh})_{ij}\vec{a}_i, \quad [L]|_E |_G [h]|_E [\vec{b}_j]|_G = ([L^T_{gh}]|_F)^T |_E [h]|_E [\vec{b}_j]|_F, \\
\text{dual : } L\vec{a}_j = \sum_{i=1}^n L^T_{gh}\vec{b}_i, \quad L^T_{gh}\vec{b}_j = \sum_{i=1}^n (L^T_{gh})^T_{ij}\vec{a}_i, \quad [L^T_{gh}]|_F |_G [h]|_F [\vec{b}_j]|_F = ([L^T_{gh}]|_E)^T |_E [h]|_E [\vec{b}_j]|_F.
\end{aligned}
$$

(A.26)

gives, for (A.25)$_1$,

$$
\sum_{k=1}^n g_{ik}(L^T_{gh})_{kj} = \sum_{k=1}^n L_{ik} h_{kj} = \sum_{k=1}^n g_{ik}(L^T_{gh})^T_{kj} = \sum_{k=1}^n L^T_{ik} h_{kj},
$$

(A.27)

and, for (A.25)$_2$,

$$
(L^T_{gh})_{ij} = \sum_{k=1}^m \sum_{l=1}^m (g^{ij})_{kl} L_{kl} h_{lj} = (L^T_{gh})^T_{ij} = \sum_{k=1}^m \sum_{l=1}^m g^{ij} L^T_{kl} h_{lj} \quad \text{when } [g^{ij}] := ([g]|_E)^{-1}.
$$

(A.28)

**Exercise A.30** Prove: If $L$ is invertible then $(L^T_{gh})^{-1} = (L^{-1})^T_{gh}$.

**Answer.** $(L^T_{gh}, (L^{-1})^T_{gh})|_G = ((L^{-1})^T_{gh}, L|_G) = (\vec{x}, L^{-1}.L|_G|_G) = (\vec{x}, \vec{y})_F = (L^T_{gh}, (L^{-1})^T_{gh})|_G, \text{ true } \forall \vec{x}, \vec{y}$. 

A.7.3 Deformation gradient symmetric: Absurd

The symmetry of a linear map $L \in \mathcal{L}(E; F)$ is a nonsense if $F \neq E$.

Application: The gradient of deformation $F^0_t(p_{tu}) = d\Phi^0_t(p_{tu}) \in \mathcal{L}(T_{p_{tu}}(\Omega_t); T_{p_{tu}}(\Omega_t))$ cannot be symmetric since $F^0_t(p_{tu})^T = (F^0_t)^T(p_{tu}) \in \mathcal{L}(T_{p_{tu}}(\Omega_t); T_{p_{tu}}(\Omega_t))$. (Reason for the introduction of the second Piola–Kirchhoff tensor in $\mathcal{L}(T_{p_{tu}}(\Omega_t); T_{p_{tu}}(\Omega_t))$: Symmetric, see Madsen–Hughes [12] or § M.3.2.)
A.7.4 Isometry

**Definition A.31** A linear map \( L \in \mathcal{L}(E; F) \) is an isometry relative to \((\cdot, \cdot)_g\) and \((\cdot, \cdot)_h\) iff
\[
\forall \vec{x}, \vec{y} \in E, \quad (L \vec{x}, L \vec{y})_h = (\vec{x}, \vec{y})_g, \quad \text{i.e.} \quad L^T g \circ L = I_E \quad \text{(identity in } E\text{).} \quad (A.29)
\]

In particular, if \( L \) is an isometry and \((\vec{e}_i)\) is a \((\cdot, \cdot)_g\)-orthonormal basis, then \((L \vec{e}_i)\) is a \((\cdot, \cdot)_h\)-orthonormal basis, since \((L \vec{e}_i, L \vec{e}_j)_g = (\vec{e}_i, \vec{e}_j)_g = \delta_{ij}\) for all \( i, j \).

In particular, an endomorphism \( L \in \mathcal{L}(E; E) \) is a \((\cdot, \cdot)_g\)-isometry iff
\[
\forall \vec{x}, \vec{y} \in E, \quad (L \vec{x}, L \vec{y})_g = (\vec{x}, \vec{y})_g, \quad \text{i.e.} \quad L^T g \circ L = I. \quad (A.30)
\]

Thus, if \( L \) is an isometry and \((\vec{e}_i)\) is a \((\cdot, \cdot)_g\)-orthonormal basis, then \((L \vec{e}_i)\) is a \((\cdot, \cdot)_g\)-orthonormal basis.

**Exercise A.32** Let \( E = \mathbb{R}^3 \) be the usual affine space and \( E = \mathbb{R}^3 \) the usual associated vector space; Let \((\cdot, \cdot)_g\) be an inner dot product in \( \mathbb{R}^3 \). Definition: A distance-preserving function \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is a function s.t. \( ||f(p)f(q)||_g = ||pq||_g \) for all \( p, q \in \mathbb{R}^3 \). Prove: If \( f \) is a distance-preserving function, then \( f \) is affine, and \( df \) is an isometry.

**Answer.** Let \( O \in \mathbb{R}^3 \) (an origin), and consider the vectorial associated function \( \tilde{f} : \vec{x} = O \vec{p} \in \mathbb{R}^3 \rightarrow f(\vec{x}) := f(O)\vec{p} \). Let \( \vec{y} = O \vec{q} \). Then \( 2(f(\vec{x}), f(\vec{y}))(g) = ||f(\vec{x})||_g^2 + ||f(\vec{y})||_g^2 - ||f(\vec{x}) - f(\vec{y})||_g^2 = ||f(O)\vec{p}||_g^2 + ||f(O)\vec{q}||_g^2 - ||f(O)\vec{p} - f(O)\vec{q}||_g^2 = ||\vec{x}||_g^2 + ||\vec{y}||_g^2 - ||\vec{x} - \vec{y}||_g^2 = 2(\vec{x}, \vec{y})_g \) for all \( \vec{x}, \vec{y} \), thus \( \tilde{f} \) is an isometry, cf. (A.30). Thus, if \((\vec{e}_i)\) is a \((\cdot, \cdot)_g\)-orthonormal basis, then \((f(\vec{e}_i))\) is also a \((\cdot, \cdot)_g\)-orthonormal basis. Thus \( f(\vec{x}) = \sum_{i=1}^n (f(\vec{x}), f(\vec{e}_i)) f(\vec{e}_i) = \sum_{i=1}^n (\vec{x}, \vec{e}_i) f(\vec{e}_i) = \sum_{i=1}^n x_i f(\vec{e}_i) \) when \( \vec{x} = \sum_{i=1}^n x_i \vec{e}_i \), thus \( f(\vec{x} + \lambda \vec{y}) = \sum_{i=1}^n (x_i + \lambda y_i) f(\vec{e}_i) = f(\vec{x}) + \lambda f(\vec{y}) \), thus \( f(\vec{x}) \) is linear. Thus \( d(f(\vec{x})) = d(f(\vec{x})) \) is an isometry and \( f(\vec{x}) \) is affine. \( f(p) = f(O) + f(\vec{p}) f(O) = f(O) + f(\vec{p}) - f(\vec{O} \vec{p}) \).

A.8 Dual basis

A.8.1 Linear forms: “Covariant vectors”

With \( F = \mathbb{R} \) in the definition A.14, we get

**Definition A.33** The set \( E^* := \mathcal{L}(E; \mathbb{R}) \) of linear scalar valued functions is called the dual of \( E \):
\[
E^* := \mathcal{L}(E; \mathbb{R}) \quad \text{is the dual of } E. \quad (A.31)
\]

And a linear scalar valued function \( \ell \in E^* \) is called a linear form.

(More precisely, \( E^* \) is the algebraic dual, the topological dual being, if \( E \) is a Banach space, the subset of continuous linear forms; Recall: If \( E \) is finite dimensional, all norms are equivalent and any linear form is continuous.)

And (A.12) gives: If \( \ell \in E^* = \mathcal{L}(E; \mathbb{R}) \), then
\[
\forall \vec{u} \in E, \quad \ell(\vec{u}) := \ell \vec{u} = \ell, \vec{u} \in E^* \quad \text{is a linear form}. \quad (A.32)
\]

\( \ell(\vec{u}) \) is also written \( \langle \ell, \vec{u} \rangle_{E^*} \) where \( \langle \cdot, \cdot \rangle_{E^*} \) is the duality bracket: It is not an inner dot product (\( \ell \notin E \)), it is a duality product, the calculation \( \ell(\vec{u}) = \ell, \vec{u} = \langle \ell, \vec{u} \rangle_{E^*} \) being called a covariant calculation, the result being independent of an observer.

**Definition A.34** A linear form \( \ell \in E^* \) is also called a “covariant vector”; Covariant refers to:

1- The action of a function on a vector, cf. (A.32) (covariant calculation), and
2- The change of coordinate formula \( \ell_{\text{new}} = [\ell]_{\text{old}, P} \), see (A.68) (covariant formula).

NB: \( E^* \) being a vector space, an element \( \ell \in E^* \) is indeed a vector. But \( E^* \) has no existence if \( E \) has not been specified first, a “covariant vector” \( \ell \in E^* \) being a function on \( E \): It can’t be confused with a vector in \( E \) since there is no natural canonical isomorphism between \( E \) and \( E^* \).

**Remark A.35** Misner–Thorne–Wheeler [14], box 2.1, insist: “Without it [the distinction between covariance and contravariance], one cannot know whether a vector is meant or the very different object that is a linear form.”

**Interpretation of a linear form (a covariant vector):** It answers the question: What does a function \( E \rightarrow \mathbb{R} \) do? Answer: Like any function, it gives values to vectors: \( \ell(\vec{u}) \) is the value of \( \vec{u} \) through \( \ell \).
A.8.2 Covariant dual basis (the functions which give the components of a vector)  

Notation: If \( \vec{u}_1, ..., \vec{u}_k \) are \( k \) vectors in \( E \), then \( \text{Vect}\{\vec{u}_1, ..., \vec{u}_k\} \) := the vector space spanned by \( \vec{u}_1, ..., \vec{u}_k \).

**Definition A.36** Let \( (\vec{e}_i)_{i=1,...,n} \) be a basis in \( E \) (finite dimensional). Let \( i \in [1, n] \). The scalar projection on \( \text{Vect}\{\vec{e}_1, ..., \vec{e}_{i-1}, \vec{e}_{i+1}, ..., \vec{e}_n\} \) is the linear form named \( \pi_{ei} \in E^* \) with the classical notation, and named \( e^i \in E^* \) with the duality notation, defined by, for all \( i, j, \)

\[
\begin{align*}
\text{class.: } & \quad \pi_{ei}(\vec{e}_j) = \delta_{ij}, \quad \text{written } \pi_{ei}\vec{e}_j = \delta_{ij}, \\
\text{dual.: } & \quad e^i(\vec{e}_j) = \delta^i_j, \quad \text{written } e^i\vec{e}_j = \delta^i_j,
\end{align*}
\]

(A.33)

the dot notations since \( \pi_{ei} = e^i \) is linear. Drawing.

**Proposition A.37 and definition of the dual basis.** \((\pi_{ei})_{i=1,...,n} = (e^i)_{i=1,...,n} \) is a basis in \( E^* \), called the covariant dual basis of the basis \((\vec{e}_i)\), or, simply, the dual basis of the basis \((\vec{e}_i)\) (in this manuscript).

**Proof.** If \( \sum_{i=1}^n \lambda_i \pi_{ei} = 0 \), then \( (\sum_{i=1}^n \lambda_i \pi_{ei}) (\vec{e}_j) = 0 \) for all \( j \), thus \( \sum_{i=1}^n \lambda_i \pi_{ei}(\vec{e}_j) = 0 = \sum_{i=1}^n \lambda_i \delta_{ij} = \lambda_j \) for all \( j \), thus \((\pi_{ei})_{i=1,...,n} \) is a family of \( n \) independent vectors in \( E^* \); And dim \( E^* = n \), cf. prop. A.18, thus \((\pi_{ei})_{i=1,...,n} \) is a basis in \( E^* \). (You can use duality notations if you prefer.) \( \Box \)

Thus, \( \pi_{ei} = e^i \) being linear, if \( \vec{x} = \sum_{i=1}^n x_i \vec{e}_i = \sum_{i=1}^n x^i \vec{e}_i \) (classical or duality notations), then (A.33) gives

\[
\pi_{ei}(\vec{x}) = x_i = e^i(\vec{x}) = x^i
\]

(A.34)

So \( \pi_{ei} = e^i \) is the linear function \( E \to \mathbb{R} \) that gives the \( i \)-th component of a vector \( \vec{x} \) relative to the basis \((\vec{e}_i)\). Or, with the linearity dot notation, \( \pi_{ei} \vec{x} = x_i \) and \( \vec{e}^i \vec{x} = x^i \).

**Example A.38** Following example 1.1. The size of a child is represented on a wall by a bipoint vector \( \vec{u} \). And observer (e.g. René Descartes) chooses a unit of length through a bipoint vector which he names \( \vec{e} \). And then defines a linear form \( \pi_{e} : \mathbb{R} \to \mathbb{R} \) by \( \pi_e \vec{e} = 1 \). Thus \( \pi_e \) is a measuring instrument, built from \( \vec{e} \), which gives the size \( s \) of the child relative to \( \vec{e} \). It gives \( s = \pi_e(\vec{u}) = \pi_e \vec{u} \) so \( \vec{u} = (\pi_e \vec{u}) \vec{e} \).

**Exercise A.39** Let \((\vec{a}_i)\) and \((\vec{b}_i)\) be bases and let \( \pi_{ai} \) and \( \pi_{bi} \) be the dual bases. Let \( \lambda \neq 0 \). Prove:

\[
\text{If } \forall i = 1, ..., n, \quad \vec{b}_i = \lambda \vec{a}_i, \quad \text{then } \forall i = 1, ..., n, \quad \pi_{bi} = \frac{1}{\lambda} \pi_{ai}.
\]

(A.35)

With duality notations, \( b^i = \frac{1}{\lambda} a^i \).

**Answer.** \( \pi_{ai}\vec{b}_j = \delta_{ij} = \pi_{ai}\vec{a}_j = \pi_{ai}\vec{a}_i = \frac{1}{\lambda} \pi_{ai}\vec{a}_j \) for all \( j \) (since \( \pi_{ai} \) is linear), thus \( \pi_{ai} = \frac{1}{\lambda} \pi_{ai} \), true for all \( i \). \( \Box \)

NB: The dual basis is independent of any inner dot product (no inner dot product \((\cdot, \cdot)_g\) has been introduced). In particular, \( \pi_{ei}\vec{e}_j = e^i\vec{e}_j \) is not an inner dot product between two vectors, because \( \pi_{ei} = e^i \) (function) and \( \vec{e}_i \) (vector) do not belong to the same vector space. It means \( \pi_{ei}(\vec{e}_j) = e^i(\vec{e}_j) \), cf. (A.32).

A.8.3 Example: aeronautical units

**Example A.40** International aeronautical units: Horizontal length = nautical mile (NM), altitude = English foot (ft). Application: An air traffic controller chooses the point \( O \) = the position of its control tower, and a plane \( p \) is located thanks to the bipoint vector \( \vec{x} = \vec{O} \vec{p} \). And the traffic controller chooses \( \vec{e}_1 \) = the vector of length 1 NM indicating the south (first runway), \( \vec{e}_2 \) = the vector of length 1 NM indicating the southwest (second runway), \( \vec{e}_3 \) = the vector of length 1 ft indicating the vertical. Thus his referential is \( \mathcal{R} = (O, (\vec{e}_1, \vec{e}_2, \vec{e}_3)) \), and his dual basis \((\pi_{e1}, \pi_{e2}, \pi_{e3})\) is defined by \( \pi_{ei}(\vec{e}_j) = \delta_{ij} \) for all \( i, j, \) cf. (A.33). He writes \( \vec{x} = \sum_{i=1}^n x_i \vec{e}_i \in \mathbb{R}^n \), so that \( x_1 = \pi_{e1}(\vec{x}) \) = the distance to the south in NM, \( x_2 = \pi_{e2}(\vec{x}) \) = the distance to the southwest in NM, \( x_3 = \pi_{e3}(\vec{x}) \) = the altitude in ft. (If you prefer, with duality notations, \( \pi_{ei} = e^i \) and \( \vec{x} = \sum_{i=1}^n x_i^i \vec{e}_i \) and \( x^i = e^i(\vec{x}) \).)

Here the basis \((\vec{e}_i)\) is not a Euclidean basis. This non Euclidean basis \((\vec{e}_i)\) is however vital if you take a plane. (A Euclidean basis is not essential to life...). See next remark A.41. \( \Box \)
Remark A.41 The meter is the international unit for NASA that launched the Mars Climate Orbiter probe; The foot is the international vertical unit for aviation; And for the Mars Climate Orbiter landing procedure, NASA asked Lockheed Martin to do the computation. Result? The Mars Climate Orbiter space probe burned in the Martian atmosphere because of $\lambda \sim 3$ times too high a speed during the landing procedure: One meter is $\lambda \sim 3$ times one foot, and someone forgot it... Although everyone used a Euclidean dot product... But not the same (one based on a meter, and one based on the foot). (Objectivity and covariance can be useful.)

A.8.4 Matrix representation of a linear map
Let $\ell \in E^*$, let $(e_i)$ be a basis: The components of $\ell$ are the $n$ reals

$$\ell_i := \ell(e_i) = \ell, e_i,$$

and $[\ell]_e = (\ell_1 \ldots \ell_n)$ (row matrix)

is the matrix of $\ell$ relative to the basis $(e_i)$. In particular for the dual basis $(e_i)$, $[\pi]_e = (e^i)$,

$$[\pi_e]] = [e^j] = (0 \ldots 0 1 \ldots 0) = (e^i)^2 (\text{row matrix}),$$

And we get (usual representation of components with a basis), with classical and duality notations,

$$\ell : = \sum_{i=1}^n \ell_i \pi_{e_i} \Rightarrow \sum_{i=1}^n \ell_i e^i.$$

Thus with $\vec{x} = \sum_{i=1}^n x_i e_i = \sum_{i=1}^n x_i e_i \in E$, linearity gives, with classical and duality notations,

$$\ell \cdot \vec{x} = [\ell]_e [\vec{x}] = \sum_{i=1}^n \ell_i x_i = \sum_{i=1}^n \ell_i x_i.$$

Remark A.42 Relative to a basis, a vector is represented by a column matrix, cf. (A.2), and a linear form by a row matrix, cf. (A.36). This enables:

- The use of matrix calculation to compute $\ell \cdot \vec{x} = [\ell]_e [\vec{x}]$, cf. (A.39), not to be confused with an inner dot product calculation (depends on the choice of some inner dot product), and

- Not to confuse the “nature of objects”: Relative to a basis, a (contravariant) vector is a mathematical object represented by a column matrix, while a linear form (covariant) is a mathematical object represented by a row matrix. Cf. remark A.35.

A.8.5 Example: Thermodynamic
Consider the Cartesian space $\mathbb{R}^2 = \{(T,V) \in \mathbb{R} \times \mathbb{R} \} = \{(\text{temperature}, \text{volume})\}$. There is no meaningful inner dot product in this $\mathbb{R}^2$: What would $\sqrt{T^2 + V^2}$ mean (Pythagoras: Can you add Kelvin degrees and meters)? Thus, in thermodynamics, (contravariant) dual bases are the main ingredient.

E.g., consider the basis $(\vec{E}_1 = (1,0), \vec{E}_2 = (0,1))$ in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ (after a choice of temperature and volume units), let $\vec{X} \in \mathbb{R}^2$, $\vec{X} = (T,V) = T \vec{E}_1 + V \vec{E}_2$, and let $(\pi_{E_1}, \pi_{E_2}) = (E^1, E^2) = \text{ame}(dT,dV)$ be the (covariant) dual basis. The first principle of thermodynamics tells that there exists a density $\alpha$ of internal energy which is exact, that is, $\alpha$ is an exact differential form: $\exists U \in C^1(\mathbb{R}^2, \mathbb{R})$ s.t. $\alpha = dU$. So, at any $\vec{X}_0 = (T_0, V_0)$,

$$\alpha(\vec{X}_0) = dU(\vec{X}_0) = \frac{\partial U}{\partial T}(\vec{X}_0) dT + \frac{\partial U}{\partial V}(\vec{X}_0) dV \quad \text{and} \quad [dU(\vec{X}_0)]_e = \left( \frac{\partial U}{\partial T}(\vec{X}_0), \frac{\partial U}{\partial V}(\vec{X}_0) \right).$$

(Relative to a basis, a linear form is represented by a row matrix.) So, the variation rate at $\vec{X}_0 = (T_0, V_0)$ in the direction $\Delta \vec{X} = \Delta T \vec{E}_1 + \Delta V \vec{E}_2 = (\Delta T, \Delta V)$ in $\mathbb{R}^2$ is

$$dU(\vec{X}_0) \Delta \vec{X} = \frac{\partial U}{\partial T}(T_0, V_0) \Delta T + \frac{\partial U}{\partial V}(T_0, V_0) \Delta V.$$

And we have the first order Taylor expansion (in the vicinity of $\vec{X}_0$, i.e., with $\delta X = h \Delta \vec{X}$ and in the vicinity of $h=0$):

$$U(\vec{X}_0 + \delta \vec{X}) = U(\vec{X}_0) + \frac{\partial U}{\partial T}(T_0, V_0) \delta T + \frac{\partial U}{\partial V}(T_0, V_0) \delta V + o(\delta T, \delta V).$$

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Matrix computation: Column matrices for vectors, row matrices for linear forms:

\[ [E_1|E] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [E_2|E] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [X_0|E] = \begin{pmatrix} T_0 \\ V_0 \end{pmatrix}, \quad [\delta X|E] = \begin{pmatrix} \delta T \\ \delta V \end{pmatrix}, \]

(A.43)

\[ [dT|E] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [dV|E] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [dU|E] = \left( \frac{\partial U}{\partial T} \frac{\partial U}{\partial V} \right) \]

(A.44)

give

\[ dU(X_0)\delta X = \left( \frac{\partial U}{\partial T}(X_0) \frac{\partial U}{\partial V}(X_0) \right) \delta T + \left( \frac{\partial U}{\partial T}(X_0) \frac{\partial U}{\partial V}(X_0) \right) \delta V. \]

(A.45)

This is a “covariant calculation” (in particular there is no inner dot product).

A.9 Tensorial product and tensorial notations

A.9.1 Definition

Let \( E \) and \( F \) be vector spaces. Let \( \varphi \in \mathcal{F}(E; \mathbb{R}) \) and \( \psi \in \mathcal{F}(F; \mathbb{R}) \) be two scalar valued functions. Their tensorial product is the function \( \zeta = \varphi \otimes \psi \in \mathcal{F}(E \times F; \mathbb{R}) \) defined by, for all \( \vec{u}, \vec{w} \in E \times F \),

\[ \zeta(\vec{u}, \vec{w}) = \varphi(\vec{u})\psi(\vec{w}). \]

(A.46)

(A function with separate variables.) E.g., \( \cos \otimes \sin : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) \((x,y) \to (\cos \circ \sin)(x,y) = \cos(x)\sin(y)\).

A.9.2 Application to bilinear forms

With (A.46), if \( \varphi = \ell \in E^* \) and \( \psi = m \in F^* \) are two linear forms then

\[ (\ell \otimes m)(\vec{u}, \vec{w}) = \ell(\vec{u})m(\vec{w}) = (\ell, \vec{u})(m, \vec{w}) \]

(A.47)

(product of two real numbers, the dot notations since \( \ell \) and \( m \) are linear), and \( \ell \otimes m \) is trivially bilinear: \( \ell \otimes m \in \mathcal{L}(E,F; \mathbb{R}) \).

Quantification: If \((\vec{a}_i)\) and \((\vec{b}_j)\) are bases in \( E \) and \( F \) with dual bases \((\pi_{ai}) = (a^i)\) and \((\pi_{bj}) = (b^j)\) then (A.47) gives, with \( \vec{u} = \sum_{i=1}^n u_i \vec{a}_i = \sum_{i=1}^n u^i \vec{a}_i \) and \( \vec{w} = \sum_{j=1}^m w_j \vec{b}_j = \sum_{j=1}^m w^j \vec{b}_j \) (classical and duality notations),

\[ (\pi_{ai} \otimes \pi_{bj})(\vec{u}, \vec{w}) = u_i w_j = (a^i \otimes b^j)(\vec{u}, \vec{w}) = u^i w^j. \]

(A.48)

And \((\pi_{ai} \otimes \pi_{bj}) = (a^i \otimes b^j)\) is a basis of \( \mathcal{L}(E,F; \mathbb{R}) \), cf. prop. A.11. Thus a bilinear form \( g(\cdot, \cdot) \in \mathcal{L}(E,F; \mathbb{R}) \) is expressed in this basis thanks to its components \( g_{ij} \):

\[ g(\cdot, \cdot) = \sum_{i=1}^n \sum_{j=1}^m g_{ij} \pi_{ai} \otimes \pi_{bj} = \sum_{i=1}^n \sum_{j=1}^m g_{ij} a^i \otimes b^j, \quad \text{where} \quad g_{ij} = g(\vec{a}_i, \vec{b}_j). \]

(A.49)

Indeed, \( g(\vec{u}, \vec{w}) = \sum_{i=1}^n \sum_{j=1}^m g_{ij} \pi_{ai} \otimes \pi_{bj}(\vec{u}, \vec{w}) = \sum_{i=1}^n \sum_{j=1}^m g_{ij} (\pi_{ai} \otimes \pi_{bj})(\vec{u}, \vec{w}) = \sum_{i=1}^n \sum_{j=1}^m g_{ij} (\vec{a}_i, \vec{w})(\vec{b}_j, \vec{w}) = \sum_{i=1}^n \sum_{j=1}^m g_{ij} u_i w_j \]

(Use duality notations if you prefer.)

A.9.3 Bidual basis (and contravariance)

Definition A.43 The dual of \( E^* \) is \( E^{**} = (E^*)^* = \mathcal{L}(E^*; \mathbb{R}) \) and is named the bidual of \( E \). (In differential geometry, it is this space which is usually referred to as the space of contravariant vectors = the space or derivative covariants.)

Definition A.44 Let \((\vec{e}_i)\) be a basis in \( E \), let \((\pi_{ei}) = (e^i)\) be its dual basis (basis in \( E^* \)). The dual basis \((\delta_i)\) of \((\pi_{ei}) = (e^i)\) is called the bidual basis of \((\vec{e}_i)\). (The notation \( \delta_i \) is related to the derivation in the direction \( \vec{e}_i \), that is, \( \delta_i(df(\vec{f})) = df(\vec{f}) \cdot \vec{e}_i = \frac{\partial f}{\partial x_i} \), see § S.1.)
Thus, the $\partial_i \in E^{**}$ are characterized by, for all $j$,
\begin{equation}
\partial_i(\pi_{e_j}) = \delta_{ij} = \partial_i \pi_{e_j} = \pi_{e_j}(e^i) = \pi_{e_j}(\epsilon^i) = (\partial_i(e^j) = \delta^j_i = e^j(\epsilon^i)),
\end{equation}
with classical (and duality) notation (the dot notation $\partial_i \pi_{e_j}$ and $\pi_{e_j}(\epsilon^i)$ since $\partial_i$ and $\pi_{e_j}$ are linear). Thus, one $\ell \in E^* = \mathcal{L}(E; \mathbb{R})$ is expressed in a basis as
\begin{equation}
\ell = \sum_{i=1}^n \ell_i \pi_{e_i} = \sum_{i=1}^n \ell_i e^i \quad \text{where} \quad \ell_i = \partial_i \ell = (\ell_i \epsilon_i).
\end{equation}
Indeed, $\partial_i(\ell) = \partial_i(\sum_{j=1}^n \ell_j \pi_{e_j}) = \sum_{j=1}^n \ell_j \partial_i(\pi_{e_j}) = \sum_{j=1}^n \ell_j \delta_{ij} = \ell_i$. (Use duality notations if you prefer.)

**Notation.** Consider the natural canonical isomorphism (see (T.9))
\begin{equation}
\mathcal{J} : \left\{ \begin{array}{l}
E \to E^{**} = \mathcal{L}(E^*; \mathbb{R}) \\
\bar{u} \to \mathcal{J}(\bar{u}) \quad \text{where} \quad \mathcal{J}(\bar{u})(\ell) := \ell.\bar{u}, \quad \forall \ell \in E^*.
\end{array} \right.
\end{equation}
Thus we can identify $\bar{u}$ and $\mathcal{J}(\bar{u})$ (observer independent identification), thus $\partial_i = \mathcal{J}(\partial_i)$ is identified with $\bar{\epsilon}_i$, and $\partial_i$ = noted $\bar{\epsilon}_i$; Thus (A.50) reads
\begin{equation}
\partial_i \pi_{e_j} = \bar{\epsilon}_i \pi_{e_j} = \delta_{ij} \quad (= \bar{\epsilon}_i e^j = \delta^j_i),
\end{equation}
with classical (and duality) notations.

### A.9.4 Tensorial representation of a linear map

Consider the natural canonical isomorphism (between linear maps $E \to F$ and bilinear forms $F^* \times E \to \mathbb{R}$)
\begin{equation}
\tilde{\mathcal{J}} : \left\{ \begin{array}{l}
\mathcal{L}(E; F) \to \mathcal{L}(F^*; \mathbb{R}) \\
\mathcal{L}(E; F) \to \hat{L} = \mathcal{J}(L)
\end{array} \right.
\end{equation}
where $\tilde{L}(m, \bar{u}) := m.(\bar{u})$, $\forall (m, \bar{u}) \in F^* \times E$, see § T.4. And $\mathcal{J}(L)$ is also named $L$ for calculations purposes, see (A.57).

**Quantification:** Let $(\bar{a}_i)_{i=1, \ldots, n}$ be a basis in $E$ and $(\bar{b}_j)_{j=1, \ldots, m}$ be a basis in $F$. Let $L \in \mathcal{L}(E; F)$, and let
\begin{equation}
L\bar{a}_j = \sum_{i=1}^m L_{ij} \bar{b}_i = \sum_{i=1}^m \bar{L}^i_j \bar{b}_i,
\end{equation}
with classical (and duality) notation. And let $(\pi_{ai}) (= (a^i))$ be the dual basis of $(\bar{a}_i)$, cf. (A.33). Then we immediately get
\begin{equation}
\mathcal{J}(L) = \tilde{L} := \sum_{i=1}^m \sum_{j=1}^n L_{ij} \bar{b}_i \otimes \pi_{aj} \overset{\text{dual}}{=} \sum_{i=1}^m \sum_{j=1}^n \bar{L}^i_j \bar{b}_i \otimes a^j,
\end{equation}
with classical (and duality) notation. Indeed, on the one hand $\mathcal{J}(L)(\pi_{bt}, \bar{a}_j) = (\sum_{k=1}^n L_{kt} \pi_{bt} \otimes \pi_{at})(\pi_{bt}, \bar{a}_j) = \sum_{k=1}^n L_{kt} \delta_{tk} \delta_{bij} = L_{ij}$, and on the other hand $\pi_{bt}(L.\bar{a}_j) = \pi_{bt}(\sum_k L_{kj} \bar{b}_k) = \sum_k L_{kj} \pi_{bt}(\bar{b}_k) = \sum_k L_{kj} \delta_{bj} = L_{ij}$; Same result for all $i, j$.

**Definition A.45** (A.56) is the tensorial representation of a linear map $L$ which can be used for computation purposes: For any $\bar{u} \in E$, to get $L.\bar{u}$ you can use $\sum_{i=1}^m \sum_{j=1}^n L^i_j \bar{b}_i \otimes a^j$ together with the contraction rule:
\begin{equation}
\tilde{L}.\bar{u} := \sum_{i=1}^m \sum_{j=1}^n L^i_j \bar{b}_i \otimes a^j, \bar{u} = \sum_{i=1}^m \sum_{j=1}^n L^i_j \bar{b}_i (\pi_{aj}, \bar{u}) = \sum_{i=1}^m \sum_{j=1}^n L^i_j \bar{b}_i (A.15) = L.\bar{u}.
\end{equation}
(With duality notations: $\tilde{L}.\bar{u} := \sum_{i=1}^m \sum_{j=1}^n L^i_j \bar{b}_i \otimes a^j, \bar{u} = \sum_{i=1}^m \sum_{j=1}^n L^i_j \bar{b}_i (a^j, \bar{u}) = \sum_{i=1}^m \sum_{j=1}^n L^i_j \bar{u} a^j (A.15) = L.\bar{u}$.)

**Remark A.46** Warning: The bilinear form $\tilde{L}$ cannot be confused with the linear map $L$: The domain of definition of $\tilde{L}$ is $F^* \times E$, and $L$ acts on the two objects $\ell$ and $\bar{u}$ to get a **scalar** result; While the domain of definition of $L$ is $E$, and $L$ acts one object $\bar{u}$ to get a **vector** result. However, for calculation purposes (after choices of bases), you can use the computation rule (A.57) to get $L.\bar{u}$.}

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A.10 Einstein convention

A.10.1 Definition

When you work with components (quantification), after having chose some basis, the goal is to visually differentiate a linear function from a vector (to visually differentiate covariance from contravariance).

**Einstein Convention:** Requires a finite dimension vector space $E$, $\dim E = n$, and duality notations:

1. A basis in $E$ (contravariant) is written with bottom indices: e.g., $(e_i)$ is a basis in $E$;
   A (contravariant) vector $\vec{x} \in E$ (qualitative) has its components relative to $(e_i)$ (quantification) written with top indices: $\vec{x} = \sum_{i=1}^{n} x^i e_i$, and is represented by the column matrix $\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$.

2. A basis in $E^* = \mathcal{L}(E; \mathbb{R})$ (covariant) is written with top indices: e.g., $(e^i)$ is a basis in $E^*$;
   A linear form $\ell \in E^*$ (covariant) has its components relative to $(e^i)$ (quantification) written with bottom indices: $\ell = \sum_{i=1}^{n} \ell_i e^i$, and its matrix representation is the row matrix $[\ell]_{e^i} = (\ell_1 \ldots \ell_n)$.

3. You can also omit the sum sign $\sum$ when there are repeated indices at a different position, e.g. $\sum_{i=1}^{n} x^i e_i = \text{new}, e_i$, e.g. $L^i_j e_i = L_j^i e_i := L_j^1 e_1 + L_j^2 e_2 + \ldots + L_j^n e_n = \sum_{i=1}^{n} L_j^i e_i = \sum_{i=1}^{n} L^i_j e_i$. In fact, before computers and word processors, to print $\sum_{i=1}^{n}$ was not easy. But with \LaTeX this is no more a problem, so in this manuscript the sum sign $\sum$ is not omitted, and some confusions are removed.

**Remark A.47** Einstein’s convention is not mandatory. E.g. Arnold doesn’t use it when he doesn’t need it, or when it makes reading difficult, or when it induces misunderstandings. In fact, in classical mechanics Einstein’s convention often induces more confusion than understandings.

A.10.2 Do not mistake yourself

This convention is just a help:

1. This convention deals only with quantification of mathematical objects relative to a basis, not with the objects themselves (qualitative): The same mathematical object has multiple components representations and notations, see e.g. the classical or duality notations.

2. Classical notations are as good as duality notations: The main difference is that classical notation cannot detect obvious errors in component manipulations... But duality notations are often misused, and thus add confusion to the confusion...

3. Einstein’s convention can be satisfied while the results are meaningless. Trivial example: $\sum_{i=1}^{n} e_i e^i$ when $\ell = \sum_{i=1}^{n} \ell_i e^i \in E^*$ is a linear form on $E$ and $\bar{w} = \sum_{i=1}^{n} w_i e_i \in F$ is some vector with $F \neq E$. E.g., with the deformation gradient, the convention can be satisfied while the results are absurd. And see § A.12.

4. The convention fully used, e.g. with $g(\vec{u}, \vec{v}) = \sum_{i,j=1}^{n} g_{ij} u^i v^j$ shows the observer dependence thanks to the $g_{ij}$: Even if $g_{ij} = \delta_{ij}$ you cannot write $g(\vec{u}, \vec{v}) = \sum_{i=1}^{n} u^i v^j$ if you want to use the Einstein convention.

5. Golden rule: Return to classical notations if in doubt. (Einstein’s convention is not useful in learning continuum mechanics: It mostly adds confusions and untruths.)

A.11 Change of basis formulas

$E$ being a finite dimension vector space, $\dim E = n$, let $(e_{\text{old}, i})$ and $(e_{\text{new}, i})$ be two bases in $E$. Let $(\pi_{\text{old}, i}) = (e_{\text{old}, i}^t)$ and $(\pi_{\text{new}, i}) = (e_{\text{new}, i}^t)$ be the dual bases in $E^*$ (classical and duality notations).

A.11.1 Change of basis endomorphism and transition matrix

**Definition A.48** The change of basis endomorphism $\mathcal{P} \in \mathcal{L}(E; E)$ from $(e_{\text{old}, i})$ to $(e_{\text{new}, i})$ is the endomorphism defined by, for all $j \in [1, n]_{\mathbb{N}}$,

$$\mathcal{P} e_{\text{old}, j} = e_{\text{new}, j} \quad (A.58)$$

(Thanks to prop. A.18: A linear map is defined by its values on a basis.) And the transition matrix $P = [P_{ij}]$ from $(e_{\text{old}, i})$ to $(e_{\text{new}, i})$ is the matrix $P := [P]_{e_{\text{old}}} e_{\text{new}}$ of the endomorphism $\mathcal{P}$ relative to the
basis \((\vec{e}_{old,i,i})\), that is, with classical notations,
\[
(\vec{e}_{new,j}) = P \vec{e}_{old,i} = \sum_{i=1}^{n} P_{ij} \vec{e}_{old,i}, \quad \text{i.e.} \quad [P]_{\vec{e}_{old}} = P = [P_{ij}].
\] (A.59)

Duality notations: \((\vec{e}_{new,j}) = P \vec{e}_{old,i} = \sum_{i=1}^{n} P_{ji} \vec{e}_{old,i}, \quad \text{i.e.} \quad [P]_{\vec{e}_{old}} = P = [P_{ji}].
\]

Thus the components of \(\vec{e}_{new,j}\) relative to the basis \((\vec{e}_{old,i,i})\) are the \(P_{ij}\) (classical notations):
\[
\vec{e}_{new,j} = \sum_{i=1}^{n} P_{ij} \vec{e}_{old,i}, \quad \text{i.e.} \quad [\vec{e}_{new,j}]_{\vec{e}_{old}} = \begin{pmatrix} P_{1j} \\ \vdots \\ P_{nj} \end{pmatrix} \quad (j\text{-th column of } P)
\] (A.60)

(the matrix column containing the components of \(\vec{e}_{new,j}\)). Duality notations:
\[
\vec{e}_{new,j} = \sum_{i=1}^{n} P_{ji} \vec{e}_{old,i} = \left(\begin{array}{c} (P_{1j})^{i} \\
\vdots \\
(P_{nj})^{i} \end{array}\right) \quad \text{(A.61)}
\]

\((P_{j})^{i}\) being the \(i\)-th component of \(\vec{e}_{new,j}\) in the basis \((\vec{e}_{old,i,i})\).

### A.11.2 Inverse of the transition matrix

\(P\) being defined in (A.58), let \(Q = P^{-1}: E \rightarrow E\) (the inverse endomorphism). So \(Q\) is defined by, for all \(j \in [1,n]_{N}\),
\[
\vec{e}_{old,j} = Q \vec{e}_{new,j} \quad (= P^{-1} \vec{e}_{new,j}).
\] (A.62)

Define \(Q := [Q]_{\vec{e}_{old}}^{\text{clau}} = [Q]_{\vec{e}_{new}}^{\text{dual}}\); It is the transition matrix from \((\vec{e}_{new,i})\) to \((\vec{e}_{old,i})\): for all \(j \in [1,n]_{N}\),
\[
\vec{e}_{old,j} = \sum_{i=1}^{n} Q_{ij} \vec{e}_{new,i}, \quad [\vec{e}_{old,j}]_{\vec{e}_{new}} = \begin{pmatrix} Q_{1j} \\ \vdots \\ Q_{nj} \end{pmatrix} 
\] (A.63)

**Proposition A.49** We have
\[
Q = P^{-1}.
\] (A.64)

And we also have
\[
\begin{cases}
P \vec{e}_{new,j} = \sum_{i=1}^{n} P_{ij} \vec{e}_{old,i} = \sum_{i=1}^{n} P^{i}_{ji} \vec{e}_{new,i}, \quad \text{i.e.} \quad [P]_{\vec{e}_{old}} = [P]_{\vec{e}_{new}} = P, \\
Q \vec{e}_{old,j} = \sum_{i=1}^{n} Q_{ij} \vec{e}_{old,i} = \sum_{i=1}^{n} Q^{i}_{ji} \vec{e}_{old,i}, \quad \text{i.e.} \quad [Q]_{\vec{e}_{old}} = [Q]_{\vec{e}_{new}} = Q.
\end{cases}
\] (A.65)

**Proof.** \(\vec{e}_{new,j} = P \vec{e}_{old,j} = \sum_{i=1}^{n} P^{i} \vec{e}_{old,i} = \sum_{i=1}^{n} P^{i}_{ji} \vec{e}_{new,i} = \sum_{i=1}^{n} (\sum_{k} Q^{i}_{k} P^{k})_{ji} \vec{e}_{new,i} = \sum_{k} (Q P)^{i}_{ji} \vec{e}_{new,k}\), thus \((Q P)^{i}_{ji} = \delta^{i}_{k}\) for all \(j, k\). Hence \(Q P = I\), that is, \(Q = P^{-1}\), i.e. (A.64).

And \(Z = [Z]_{\vec{e}_{new}} = [P]_{\vec{e}_{old}}\) means \(P \vec{e}_{new,j} = \sum_{i=1}^{n} Z^{i}_{ji} \vec{e}_{new,i}\), i.e. \(\vec{e}_{new,j} = Q \vec{e}_{old,j} = Q \sum_{i=1}^{n} Z^{i}_{ji} \vec{e}_{old,i} = \sum_{i=1}^{n} \sum_{k} (Q^{i}_{k} Z^{k}_{ji}) \vec{e}_{new,k} = \sum_{k} (Q Z)^{i}_{ji} \vec{e}_{new,k}\), thus \((Q Z)^{i}_{ji} = \delta^{i}_{k}\), true for all \(j\), thus \(Q Z = I\), thus \(Z = P\). Idem for \(Q\), thus (A.65).

**Remark A.50** \(P^{T} \neq P^{-1}\) in general. E.g., \((\vec{e}_{old,i,i}) = (\vec{a}_{i})\) is a foot-built Euclidean basis, and \((\vec{e}_{new,i,i}) = (\vec{b}_{i})\) is a meter-built Euclidean basis, and \(\vec{b}_{i} = \lambda \vec{a}_{i}\) for all \(i\) (the basis are “aligned”); then \(P = \lambda I\), \(P^{T} = \lambda I\), and \(P^{-1} = \frac{1}{\lambda} I \neq P^{T}\), since \(\lambda = \frac{1}{0.3018} \neq 1\). Thus it is essential not to confuse \(P^{T}\) and \(P^{-1}\) (not to confuse covariance with contravariance), see e.g. the Mars Climate Orbiter crash (remark A.41).
A.11.3 Change of dual basis

Proposition A.51 For all $i \in [1, n]$, with classical and duality notations, we have

$$\begin{cases}
\text{clas. not.} : \pi_{\text{new},i} = \sum_{j=1}^{n} Q_{ij} \pi_{\text{old},i}, \\
\text{dual. not.} : e_{\text{new},i} = \sum_{j=1}^{n} Q_{ij}^* e_{\text{old},j},
\end{cases}$$

(A.66)

to compare with (A.59). So (matrices of linear forms are row matrices)

$$[\pi_{\text{new},i}]|e_{\text{add}} = (Q_{11} \ldots Q_{1n}) = e_{\text{new},i}|e_{\text{add}} = (Q_{i1} \ldots Q_{in}) \quad \text{(row matrix)},$$

(A.67)

and $[\pi_{\text{old},i}]|e_{\text{add}} = (P_{11} \ldots P_{mn}) = e_{\text{old},i}|e_{\text{new}} = (P_{i1} \ldots P_{in})$.

Proof. On the one hand $e_{\text{new},i}^\ell e_{\text{old},k} = \sum_{j} Q_{jk}^* e_{\text{new},j} = \sum_{j} Q_{jk} \delta_{ij} = Q_{ik}$, and on the other hand $(\sum_{j} Q_{ij} e_{old,j}) e_{new,k} = \sum_{j} Q_{ij}^* \delta_{ij} = Q_{ik}$, true for all $i, k$, hence (A.66). Similarly we get (A.67).

A.11.4 Change of coordinate system for vectors and linear forms

Proposition A.52 Let $\vec{x} \in E$ and $\ell \in E^\ast$. Then

$$\begin{align*}
\bullet [\vec{x}]|e_{\text{add}} &= P^{-1} \cdot [\vec{x}]|e_{\text{add}} \quad \text{(contravariance formula for vectors)}, \\
\bullet [\ell]|e_{\text{add}} &= [\ell]|e_{\text{add}} \cdot P \quad \text{( covariance formula for linear forms).}
\end{align*}$$

(A.68)

(The contravariance formula deals with column matrices, the covariance formula deals with row matrices.) And the scalar value $\ell \cdot \vec{x}$ (the measure of $\vec{x}$ by $\ell$) is computed incorrectly with one or the other basis:

$$\ell \cdot \vec{x} = [\ell]|e_{\text{add}} \cdot [\vec{x}]|e_{\text{add}} = [\ell]|e_{\text{new}} \cdot [\vec{x}]|e_{\text{new}}.$$  

(A.69)

(It is a dimensionless result, independent of the chosen basis by an observer.)

Proof. Let $\vec{x} = \sum_{i=1}^{n} x_i e_{\text{old},i}$, i.e. $[\vec{x}]|e_{\text{add}} = (x_1 \ldots x_n)$, and let

$\ell = \sum_{i=1}^{n} \ell_i e_{\text{old},i} \in E^\ast$, i.e. $[\ell]|e_{\text{add}} = (\ell_1 \ldots \ell_n)$, and $[\ell]|e_{\text{add}} = (m_1 \ldots m_n)$. (You can use classical notations if you prefer.) Thus

$$\sum_{i} y_i^* e_{\text{new},i} = \vec{x} = \sum_{j} x_j^* e_{\text{old},j} \quad \text{(A.63)}$$

gives $y_i = \sum_{j} Q_{ij} x_j$, for all $i$, i.e. (A.68) $1$. And

$$\sum_{j} m_j^* e_{\text{new},j} = \ell = \sum_{i} \ell_i e_{\text{old},i} = \sum_{i} \ell_i P_{ij}^* e_{\text{new},i} \quad \text{gives } m_j = \sum_i \ell_i P_{ij} \text{ for all } j,$$

i.e. (A.68) $2$. And $[\ell]|e_{\text{add}} \cdot [\vec{x}]|e_{\text{add}} = \ell \cdot \vec{x} = [\ell]|e_{\text{new}} \cdot [\vec{x}]|e_{\text{new}}$ thus (A.69) (or (A.68) gives $[\ell]|e_{\text{new}} \cdot [\vec{x}]|e_{\text{add}} = (P^{-1} \cdot [\vec{x}]|e_{\text{add}}) \cdot (P \cdot [\ell]|e_{\text{add}})$, hence (A.69).

Notation: (A.68) gives $x_{new}^i = \sum_{j=1}^{n} Q_{ij} x_{old}^j$ and $x_{old}^i = \sum_{j=1}^{n} P_{ij}^* x_{new}^j$, and thus the notation

$$Q_{ij} = \frac{\partial x_{new}^j}{\partial x_{old}^i} \quad \text{and } P_{ij}^* = \frac{\partial x_{old}^i}{\partial x_{new}^j},$$

(A.70)

Full notation with $x_{new}^{i_1, \ldots, i_n}(x_{old}^{1, \ldots, n}) = \sum_{j=1}^{n} Q_{ij} x_{old}^{j}$ and $x_{old}^{i_1, \ldots, i_n}(x_{new}^{1, \ldots, n}) = \sum_{j=1}^{n} P_{ij}^* x_{new}^{j}$.

A.11.5 Notations for transitions matrices for linear maps and bilinear forms

Let $E$ and $F$ be finite dimension vector spaces, dim $E = n$, dim $F = m$. Let $(\vec{u}_{\text{old},i})$ and $(\vec{u}_{\text{new},i})$ be two bases in $E$, and $(\vec{v}_{\text{old},i})$ and $(\vec{v}_{\text{new},i})$ be two bases in $F$. Let $\mathcal{P}_E$ and $\mathcal{P}_F$ be the change of basis endomorphisms from old to new bases, that is,

$$\begin{align*}
\vec{u}_{\text{new},j} &= \mathcal{P}_E \vec{u}_{\text{old},j} = \sum_{i,j=1}^{n} P_{ij} \vec{u}_{\text{old},j}, \\
\vec{v}_{\text{new},i} &= \mathcal{P}_F \vec{v}_{\text{old},i} = \sum_{i,j=1}^{m} P_{ij}^* \vec{v}_{\text{old},i},
\end{align*}$$

(A.71)

with classical and duality notations.
A.11.6 Change of coordinate system for bilinear forms

Let $g \in \mathcal{L}(E, F; \mathbb{R})$ be a bilinear form, let, for all $(i, j) \in [1, n] \times [1, m]_n$,

$$g(\vec{a}_{\text{old},i}, \vec{b}_{\text{old},j}) = M_{ij}, \quad g(\vec{a}_{\text{new},i}, \vec{b}_{\text{new},j}) = N_{ij}, \quad \text{i.e.} \quad \begin{cases} [g]_{\text{old}} = M = [M_{ij}]_{i=1,...,m, j=1,...,n} \\ [g]_{\text{new}} = N = [N_{ij}]_{i=1,...,m, j=1,...,n}. \end{cases} \quad (A.72)$$

**Proposition A.53** Change of basis formula for bilinear forms:

$$[g]_{\text{new}} = P^T_E [g]_{\text{old}} P_F, \quad \text{i.e.} \quad N = P^T_E M P_F. \quad (A.73)$$

In particular, if $E = F$ and $(\vec{a}_{\text{old}},i) = (\vec{b}_{\text{old}},i)$ and $(\vec{a}_{\text{new}},i) = (\vec{b}_{\text{new}},i)$, then $P_E = P_F = \text{named } P$, and

$$[g]_{\text{new}} = P^T_E [g]_{\text{old}} P_F, \quad \text{i.e.} \quad N = P^T E P_F. \quad (A.74)$$

**Proof.**

$$N_{ij} = g(\vec{a}_{\text{new},i}, \vec{b}_{\text{new},j}) = \sum_{k=1}^n \sum_{\ell=1}^m P_{F,k} P_{E,\ell} g(\vec{a}_{\text{old},k}, \vec{b}_{\text{old},\ell}) = \sum_{k=1}^n \sum_{\ell=1}^m P_{F,k} P_{E,\ell} \delta_{ik} g(\vec{a}_{\text{old},k}, \vec{b}_{\text{old},\ell}) = (P^T_E N P_F)_{ij}. \quad \blacksquare$$

**Exercise A.54**

Prove:

$$g(\vec{a}, \vec{u}, \vec{w}) = [\vec{u}]_{\text{new}}^T [g]_{\text{new}} [\vec{w}]_{\text{new}} = [\vec{u}]_{\text{old}}^T [g]_{\text{old}} [\vec{w}]_{\text{old}}. \quad (A.75)$$

(In particular, with (A.74) and a Euclidean dot product $(\cdot, \cdot)_g$, the length $||\vec{u}||_g$ depends only on the unit of measure chosen to build $(\cdot, \cdot)_g$. It is independent of the bases used to express vectors.)

**Answer.**

$$[\vec{u}]_{\text{new}}^T [g]_{\text{new}} [\vec{w}]_{\text{new}} = (P^T_E [\vec{u}]_{\text{old}})(P^T_E [g]_{\text{old}})(P^{-1}_E [\vec{w}]_{\text{old}}). \quad \blacksquare$$

### A.11.7 Change of coordinate system for linear maps

Notation of § A.11.5. Let $L \in \mathcal{L}(E; F)$ be a linear map, and let, for all $j = 1, \ldots, n$,

$$L.\vec{a}_{\text{old},j} = \sum_{i=1}^m M_{ij} \vec{b}_{\text{old},i} \quad \text{i.e.} \quad [L]_{\text{old}} = M = [M_{ij}]_{i=1,...,m, j=1,...,n},$$

$$L.\vec{a}_{\text{new},j} = \sum_{i=1}^m N_{ij} \vec{b}_{\text{new},i} \quad \text{i.e.} \quad [L]_{\text{new}} = N = [N_{ij}]_{i=1,...,m, j=1,...,n}, \quad (A.76)$$

with classical and duality notations. The change of basis formula for vectors $[L.\vec{u}]_{\text{new}} = P^{-1}_F [L.\vec{u}]_{\text{old}}$ gives:

**Proposition A.55** Change of bases formula for linear maps:

$$[L]_{\text{new}} = P^{-1}_F [L]_{\text{old}} P_F, \quad \text{i.e.} \quad N = P^{-1}_F M P_F. \quad (A.77)$$

In particular, if $E = F$, if $(\vec{a}_{\text{old},i}) = (\vec{b}_{\text{old},i})$, $(\vec{a}_{\text{new},i}) = (\vec{b}_{\text{new},i})$, then $P_E = P_F = \text{named } P$ and

$$[L]_{\text{new}} = P^{-1}_F [L]_{\text{old}} P_F, \quad \text{i.e.} \quad N = P^{-1}_F M P_F. \quad (A.78)$$

**Proof.**

$$[L.\vec{u}]_{\text{new}} = (A.68) P^{-1}_F [L.\vec{u}]_{\text{old}} \quad \text{and (A.15) give} \quad [L]_{\text{new}} [\vec{u}]_{\text{new}} = P^{-1}_F [L]_{\text{old}} [\vec{u}]_{\text{old}} = P^{-1}_F [L]_{\text{old}} P_F P^{-1}_F [\vec{u}]_{\text{old}} = P^{-1}_F [\vec{u}]_{\text{old}} P_F P^{-1}_F [\vec{u}]_{\text{old}} = P^{-1}_F [\vec{u}]_{\text{old}} [\vec{u}]_{\text{old}} = P^{-1}_F [\vec{u}]_{\text{old}} [\vec{u}]_{\text{old}} \quad \text{for all } \vec{u}. \quad \blacksquare$$

Or, if you prefer, on the one hand $L.\vec{a}_{\text{new},j} = \sum_{i=1}^m N_{ij} \vec{b}_{\text{new},i} = \sum_{i=1}^m N_{ij} P_{F,k} \vec{b}_{\text{old},k}$, and on the other hand $L.\vec{a}_{\text{new},j} = L.(\sum_{i=1}^m P_{E,\ell} \vec{a}_{\text{old},i}) = \sum_{i=1}^m P_{E,\ell} \sum_{k=1}^m M_{ik} \vec{b}_{\text{old},k} = \sum_{k=1}^m (M.P_F)_{kj} \vec{b}_{\text{old},k}$, for all $j$, thus $(P^{-1}_F N)_{kj} = (M.P_F)_{kj}$ for all $k, j$, thus $P^{-1}_F N = M.P_F$, i.e. (A.77). \hfill \blacksquare

**Remark A.56**

Bilinear forms in $\mathcal{L}(E, E; \mathbb{R})$ and endomorphisms in $\mathcal{L}(E; E)$ behave differently: The formulas (A.74) and (A.78) should not be confused since $P^{-1} \neq P^T$ in general. E.g., if an English observer uses a Euclidean (old) basis $(\vec{a}_i) = (\vec{a}_{\text{old},i})$ in foot, if a French observer uses a Euclidean (new) basis $(\vec{b}_i) = (\vec{b}_{\text{new},i})$ in meter, and if (simple case) $\vec{b}_i = \lambda \vec{a}_i$ for all $i$ (change of unit), then

$$[L]_{\text{new}} = [L]_{\text{old}}, \quad \text{while} \quad [g]_{\text{new}} = \lambda^2 [g]_{\text{old}} \quad \text{with} \quad \lambda > 10. \quad (A.79)$$

Quite different results, see the Mars Climate Orbiter crash (remark A.41). General case: Compare (A.73) and (A.77). \hfill \blacksquare
A.12 The “vectorial dual bases” of one basis

A vectorial dual basis is made of (contravariant) vectors. Not to be confused with the (covariant) dual basis (made of linear forms) which is unique, when there are as many vectorial dual bases as there are inner dot products.

Setting: $E$ is a vector space, $\dim E = n$. An observer chooses an inner dot product $(\cdot, \cdot)_g$ in $E$ (e.g., a foot-built Euclidean dot product, or a meter-built Euclidean dot product).

A.12.1 An inner dot product and the associated “vectorial dual basis”

**Definition A.57** Consider a basis $(\vec{e}_i)$ in $E$. Its $(\cdot, \cdot)_g$-dual vectorial basis (or $(\cdot, \cdot)_g$-dual basis) is the basis $(\vec{e}_{ig})$ in $E$ made of $n$ the vectors $\vec{e}_{ig} \in E$ defined by

$$\forall j = 1, \ldots, n, \quad (\vec{e}_{ig}, \vec{e}_j)_g = \delta_{ij}, \quad \text{also written} \quad \vec{e}_{ig} \cdot \vec{e}_i = \delta_{ij}. \quad (A.80)$$

NB: $(\vec{e}_{ig})$ is a basis, thus the index $i$ is a bottom index, with classical and dual notation, cf. § A.10.1.

**Example A.58** If $(\vec{e}_i)$ is a $(\cdot, \cdot)_g$-orthonormal basis we trivially get $\vec{e}_{ig} = \vec{e}_i$ for all $i$, i.e., $(\vec{e}_{ig}) = (\vec{e}_i)$. □

**Exercise A.59** Consider two inner dot products $(\cdot, \cdot)_a$ (e.g., a foot-built Euclidean dot product) and $(\cdot, \cdot)_b$ (e.g., a meter-built Euclidean dot product). Let $(\vec{e}_i)$ be a basis in $E$, and $(\vec{e}_{ia})$ and $(\vec{e}_{ib})$ be the $(\cdot, \cdot)_a$ and $(\cdot, \cdot)_b$-dual basis. Prove:

$$(\cdot, \cdot)_a = \lambda^2(\cdot, \cdot)_b \implies \vec{e}_{ib} = \lambda^2 \vec{e}_{ia}, \quad \forall i. \quad (A.81)$$

(E.g., $\lambda^2 > 10$ with foot and meter built Euclidean bases: $\vec{e}_{ib}$ is very different from $\vec{e}_{ia}$. Drawing.)

**Answer.** (A.80) gives $(\vec{e}_{ib}, \vec{e}_j)_b = \delta_{ij} = (\vec{e}_{ia}, \vec{e}_j)_a = \lambda^2(\vec{e}_{ia}, \vec{e}_j)_a$, thus $\vec{e}_{ib} - \lambda^2 \vec{e}_{ia} = \delta_{ij}$, for all $i, j$. □

NB: Here the name “$(\cdot, \cdot)_g$-dual” does not refer to the “covariance-contravariance duality” (the action of a linear form on a vector) since $\vec{e}_{ig} \in E$ is a (contravariant) vector, not a linear form.

NB: $\vec{e}_{ig}$ depends on the chosen inner dot product, e.g. (A.81).

A.12.2 An interpretation: $(\cdot, \cdot)_g$-Riesz representatives

Let $(\vec{e}_i)$ be a basis in $E$. Let $(\pi_{ei}) = (\vec{e}_i)$ be its (unique covariant) dual basis in $E^\ast$, i.e. the $\pi_{ei} = e^i$ are the linear forms defined by $\pi_{ei}(\vec{v}) = (\vec{e}_i, \vec{v})_g$. (A.33).

**Proposition A.60** The $(\cdot, \cdot)_g$-vectorial dual vector $\vec{e}_{ig}$ is the $(\cdot, \cdot)_g$-Riesz representation vector of the linear form $\pi_{ei} = e^i$, that is,

$$\forall i = 1, \ldots, n, \quad \forall \vec{v} \in E, \quad \pi_{ei}(\vec{v}) = (\vec{e}_{ig}, \vec{v})_g. \quad \text{(A.82)}$$

And $\pi_{ei} = e^i$ being linear, $\pi_{ei}(\vec{v}) = e^i(\vec{v})$ is a $\text{noted } \pi_{ei}(\vec{v}) = e^i, \vec{v}$, so, for all $i$,

$$\pi_{ei}(\vec{v}) = (\vec{e}_{ig}, \vec{v})_g = \vec{e}_{ig} \cdot \vec{v} = e^i, \vec{v}. \quad (A.83)$$

In other words, $\vec{e}_{ig} = R_g(\pi_{ei})$ where $R_g$ is the $(\cdot, \cdot)_g$-Riesz operator, see (C.6).

**Proof.** $\pi_{ei}(\vec{e}_j) = (A.33) \delta_{ij} = (A.80)(\vec{e}_{ig}, \vec{e}_j)_g$ for all $i, j$, and linearity of $\pi_{ei}$ and of $(\vec{e}_{ig}, \cdot)_g$. □

A.12.3 $\vec{e}_{ig}$ is a contravariant vector

$\vec{e}_{ig}$ is a vector in $E$: It is a “contravariant vector”. If you want to “see” that its components do satisfy the contravariant change of basis formula, i.e., if you want to check the formula

$$[\vec{e}_{ig}]_{\text{new}} = P^{-1} \cdot [\vec{e}_{ig}]_{\text{old}}, \quad \text{(A.84)}$$

for all $j$, so to exercise A.61, or apply § C.1.6 and (C.18).

**Exercise A.61** Check (A.84) directly from the definition (A.80).

**Answer.** Let $(\vec{a}_i)$ and $(\vec{b}_i)$ two bases (old and new), and let $P = [P_{ij}]$ be the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, i.e., $\vec{b}_j = \sum_{i=1}^n P_{ij} \vec{a}_i$ for all $j$. We have $[\vec{e}_{ig}]_g = P^{-1} \cdot [\vec{e}_{ig}]_a \text{ (change of basis formula for the vectors), cf. (A.68)}$, and $[\vec{e}_{ig}]_a = P^{T \cdot T}. [\vec{e}_{ig}]_g \text{ (change of basis formula for the bilinear forms, cf. (A.74)). Then (A.80) and (A.75)}$

$\text{gives } [\vec{e}_{ig}]_g^T \cdot [\vec{e}_{ig}]_g = (\vec{e}_{ig}, \vec{e}_{ig})_g = [\vec{e}_{ig}]_g^T \cdot [\vec{e}_{ig}]_g \cdot [\vec{e}_{ig}]_g^T \cdot [\vec{e}_{ig}]_g \cdot [\vec{e}_{ig}]_g^T \cdot [\vec{e}_{ig}]_g \cdot [\vec{e}_{ig}]_g^T \cdot [\vec{e}_{ig}]_g$ for all $i, j$, thus $[\vec{e}_{ig}]_g = P \cdot [\vec{e}_{ig}]_a$ for all $j$, i.e. $[\vec{e}_{ig}]_g = P^{-1} \cdot [\vec{e}_{ig}]_a$ for all $j$, i.e. (A.84). □
A.12.4 Components of $\vec{e}_{ij}$ relative to $(\vec{e}_i)$

(A.80) gives, for all $i, j,$

$$[e_{ij}]_c^T [g]_c[\vec{e}_{ij}]_c = \delta_{ij}, \quad \text{and} \quad \delta_{ij} = e^j_i \vec{e}_i = [e^j_i]_c^T [\vec{e}_i]_c = [e^j_i]_c^T [\vec{e}_i]_c,$$  \hspace{2cm} (A.85)

cf. (A.37), thus, for all $j$,

$$[g]_c[\vec{e}_{ij}]_c = [\vec{e}_j]_c$$  \hspace{2cm} (A.86)

(implicit equation to compute $\vec{e}_{ij}$). Thus (explicit formula to compute $\vec{e}_{ij}$), for all $j$,

$$[\vec{e}_{ij}]_c = ([g]_c^{-1})_c [\vec{e}_i]_c$$  \hspace{2cm} (A.87)

i.e., for all $j$,

$$\vec{e}_{ij} = \sum_{i=1}^n ([g]_c^{-1})_c [\vec{e}_i]_c : \text{The } i\text{-th component of } \vec{e}_{ij} \text{ is } ([g]_c^{-1})_c [\vec{e}_i]_c.$$  \hspace{2cm} (A.88)

Remark: (A.88) is also deduced from (C.15) (Riesz representation vector) with $\ell = \pi_e$ and $\vec{e}_{\ell} = \vec{e}_{ij}$.

(A.88) is also written (unfortunately an often confusing notation)

$$\text{when } ([g]_c^{-1})_c [\vec{e}_i]_c = g^{ij}, \text{ then } \vec{e}_{ij} = \sum_{i=1}^n g^{ij} \vec{e}_i,$$  \hspace{2cm} (A.89)

where the Einstein convention is not satisfied.

Example A.62 \(\mathbb{R}^2, [g]_c = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\), thus $[g]_c^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$ Thus $\vec{e}_{1g} = \vec{e}_1, \vec{e}_{2g} = \frac{1}{2} \vec{e}_2, g^{22} = \frac{1}{2}.$ \hspace{2cm} \(\blacksquare\)

Remark A.63 The Einstein convention is not satisfied in (A.89): Why?

- Answer 1: $M = [g]_c = [M_{ij}]$ is a matrix, and its inverse is the matrix $N = [N_{ij}] = M^{-1} = [M_{ij}]^{-1}$; A matrix is just a collection of scalars (has nothing to do with the Einstein convention), and its inverse is also a collection of scalars (has nothing to do with the Einstein convention), and you do not change this fact by calling $[N_{ij}] = \text{named}[M_{ij}]$.

- Answer 2: With duality notations (needed for the Einstein convention): $\vec{e}_{ij}$ being a vector in $E,$ let $(P_j)^i$ be its $i$-th component relative to the basis $(\vec{e}_i)$:

$$\vec{e}_{ij} = \sum_{\ell=1}^n (P_j)^i \vec{e}_\ell, \text{ also written } \vec{e}_{ij} = \sum_{\ell=1}^n P_{ij}^\ell \vec{e}_\ell,$$  \hspace{2cm} (A.90)

$P := [(P_j)^i]_{i=1}^n = \text{named}[P_{ij}]$ being the transition matrix from the basis $(\vec{e}_i)$ to the basis $(\vec{e}_{ij}).$ Thus (A.88)-(A.89) give

$$(\text{the } i\text{-th component of } \vec{e}_{ij} \text{ is } (P_j)^i = ([g]_c^{-1})_c [\vec{e}_i]_c \text{ noted } g^{ij}),$$  \hspace{2cm} (A.91)

which tells that the components of $\vec{e}_{ij}$ do depend on $[g]$; Of course, see (A.80) (definition) and (A.86).

But (A.91) has nothing to do with the Einstein convention: there is no summation; (A.91) is just the scalar result when solving (A.86) (system of $n$ equations with $n$ unknowns).

- Answer 3: If you want to see where the Einstein convention apply: With (A.90) we get

$$\text{(A.85) } \iff \sum_{k,\ell=1}^n \delta^j_k g_{\ell k} (P_j)^\ell = \delta_{ij}, \text{ thus } \text{(A.86) } \iff \sum_{\ell=1}^n g_{ij} (P_j)^\ell = \delta_{ij}$$  \hspace{2cm} (A.92)

for all $i, j$: The Einstein convention is satisfied here: Plain straight calculations with components. And you immediately get (A.87), i.e. (A.88). Now, if you want to rename the components of the matrix $N = ([g]_c^{-1})_c = [N_{ij}] = ([g]_c^{-1})_c$, it is up to you, e.g., you can rename $N_{ij} = \text{renamed} g^{ij}$... Why not... But it has nothing to do with the Einstein convention.

- Answer 4: See remark A.23.

- NB: Recall: If in trouble with notations which come as a surprise (the notation $g^{ij}$ here), go back to the classical notations: (A.80) gives (A.85) and (A.86), thus $\vec{e}_j = \sum_{k=1}^n \delta^j_k \vec{e}_k$ and $\vec{e}_{ij} = \sum_{\ell=1}^n (P_j)^\ell \vec{e}_\ell$ give $\sum_{k,\ell=1}^n \delta^j_k g_{\ell k} (P_j)^\ell = \delta_{ij}$ and $\sum_{k=1}^n g_{ij} (P_j)_k = \delta_{ij},$ for all $i, j$ which is (A.92) with classical notations. Thus $(P_j)_i = ([g]_c^{-1})_c [\vec{e}_i]_c$ which is (A.91) with classical notations: No misuse of Einstein’s convention. \(\blacksquare\)
Remark A.64 We have just seen multiple notations for the components of $\vec{e}_{ij}$ relative to the basis $(\vec{e}_i)$:

$$\vec{e}_{ij} = \sum_{j=1}^{n} (P_j)_{ij} \vec{e}_i = \sum_{j=1}^{n} P_{ij} \vec{e}_i = \sum_{j=1}^{n} \varepsilon = P \varepsilon,$$

(A.93)

where $P \in L(\mathbb{R}^n; \mathbb{R}^n)$ is the change of basis endomorphism from $(\vec{e}_i)$ to $(\vec{e}_{ij})$, which matrix $[P]_{ij} = [P]_{ij}$ is also written $[(P)_{ij}] = [P]_{ij}$. Thus $[\vec{e}_{ij}] = [(P)_{ij}] = [P]_{ij}$. If you want to see the Einstein convention, see (A.92).

A.12.5 Notation problem

Notation problem with duality notations: Because the components $(P)_{ij}$ of $\vec{e}_{ij}$ are equal to $(g^{-1})_{ij}$, cf. (A.88)-(A.89), some authors rename $\vec{e}_{ij}$ as $\vec{e}^i$... to get $\vec{e} = \sum_{i=1}^{n} g^{ij} \varepsilon_i$.

But doing so they despise Einstein’s convention:

- It is up to you to rename $\vec{e}_{ij}$ to $\vec{e}^i$, but it has nothing to do with Einstein’s convention which is meant to respect covariance and contravariance. If you do want to see the Einstein convention, see (A.92).

We insist: The contravariant vectors $\vec{e}_{ij}$ make a basis $(\vec{e}_{ij})$ in $E$, cf. (A.84): Named with low indices, cf. § A.10.1. And naming $\vec{e}_{ij}$ to $\vec{e}^i$ has nothing to do with the the covariance-contravariance duality notation (Einstein’s convention). This renaming just adds confusion to the confusion.

- With classical notations, there is no possible confusion regarding covariance or contravariance: If you know from the start that $\vec{e}_{ij} = \sum_{i=1}^{n} P_j \vec{e}_i$ is contravariant, then it is!

Recall: If you have the slightest doubt about duality notations (or with notations that appear as a surprise), go back to classical notations: Here $\vec{e}_{ij} = \sum_{i=1}^{n} P_j \vec{e}_i = \sum_{i=1}^{n} (g^{-1})_{ij} \varepsilon_i \in E$ (a contravariant vector), and no possible misinterpretation.

A.12.6 (Huge) differences between the (covariant) dual basis and a “dual vectorial basis”

1. A basis $(\vec{e}_i)$ has an infinite number of vectorial dual bases $(\vec{e}_{ij})$, as many as the number of inner dot products $(\cdot, \cdot)_g$ (as many as observers), see (A.87).

2. While a basis $(\vec{e}_i)$ has a unique dual basis $(\pi_{ei})$ (e.g., intrinsic to the basis $(\vec{e}_i)$), cf. (A.33). In particular two observers who consider the same basis $(\vec{e}_i)$ do have the same dual basis $(\pi_{ei}) = (e^i)$ even if they use different Euclidean dot products.

3. $\pi_{ei} = e^i$ is covariant, while $\vec{e}_i$ and $\vec{e}_{ij}$ are contravariant. We insist: $(\vec{e}_{ij})$ is linked to $(\vec{e}_i)$ by an endomorphism $P \in L(E; E)$ and its transition matrix $[P]_{ij}$, cf. (A.93). When there can’t be any transition matrix between $(\vec{e}_{ij})$ and $(\pi_{ei}) = (e^i)$: The vector $\vec{e}_i \in E$ and the (linear) function $\pi_{ei} = e^i \in E^*$ don’t live in the same vector space.

4. If you fly, it is vital to use the dual basis $(\pi_{ei}) = (e^i)$: It is possibly fatal if you confuse foot and meter at takeoff (through the choice of a Euclidean dot product $(\cdot, \cdot)_g$ or $(\cdot, \cdot)_h$), and at landing (if you survived takeoff). See e.g. the Mars Climate Orbiter crash, remark A.41. Einstein’s convention can help... only if it is really followed...

5. The push-forward of a vector field $\vec{v}$, cf. (10.28), is different from the push-forward of a one-form $\alpha$, cf. (13.3). Thus vectors and linear forms (contravariance and covariance) should not be confused.

6. The Lie derivative of a vector field $\vec{v}$, cf. (15.17), is different from the Lie derivative of a one-form $\alpha$, cf. (15.49). Thus vectors and linear forms (contravariance and covariance) should not be confused.

A.12.7 The notation $g^{ij}$

The notation $[g] := [g_{ij}]^{-1} = [g]^{-1}_{ij}$ of the inverse of the matrix $[g_{ij}]$, cf. (A.88), is due to the definition of the Riesz associated inner dot product $g^2$ in $L(E^*, E^*; \mathbb{R})$: It is defined by, for all $\ell, m \in E^*$,

$$g^i(\ell, m) := g(\ell_g, m_g) \quad \forall \omega \in E, \quad \ell \omega = (\ell_g, \omega)_g \quad \text{and} \quad m \omega = (m_g, \omega)_g,$$

(A.94)
the $(\cdot, \cdot)_g$-Riesz representation having been applied twice, cf. (C.2). In particular, relative to a basis $(e_i)$ in $E$ and with its dual basis $(e^i)$ (duality notations),

$$
(g^i)^{ij} := g^i(e^i, e^j) = (\ell, [\ell, \ell]_g)(\ell, [\ell, \ell]_g) = (\ell, [\ell, \ell]_g)(\ell, [\ell, \ell]_g),
$$

(A.94)

$$
= [\ell, \ell]_{[\ell, \ell]_g}[\ell, \ell]_{[\ell, \ell]_g}[\ell, \ell]_{[\ell, \ell]_g} = (\ell, [\ell, \ell]_g)(\ell, [\ell, \ell]_g),
$$

(A.95)

thus

$$
[g^i]_{[\ell, \ell]_g} = (\ell, [\ell, \ell]_g).
$$

(A.96)

With the classical notations:

$$
(g^i)^{ij} := g^i(e_i, e_j) = [\pi_{e_i}, e_j]_{[\ell, \ell]_g}[\pi_{e_i}, e_j]_{[\ell, \ell]_g} = (\ell, [\ell, \ell]_g)(\ell, [\ell, \ell]_g).
$$

(A.97)

(NB: $g^i$ is rarely useful in continuum mechanics.)

**Exercise A.65** How do we compute $g^i(\ell, m)$ with matrix computations?

**Answer.** With a basis $(e^i)$ in $E^*$, $\ell = \sum_i \ell^i e^i$ and $m = \sum_i m_i e^i$, we get $g^i(\ell, m) = \sum_i \ell^i m_i g^i(e^i, e^i) = \sum_i \ell^i g^i(m_i)$. A linear form being represented by a row matrix, we get $g^i(\ell, m) = [\ell]_{[\ell, \ell]_g}[\ell, \ell]_g$.

**Exercise A.66** What is the $(\cdot, \cdot)_g$ tensor that you create from the $(\cdot, \cdot)_g$ tensor $(\cdot, \cdot)_g$ when using the $(\cdot, \cdot)_g$-Riesz representation theorem just once? (We have just seen that the $(\cdot, \cdot)_g$ tensor $g^i$ was created from the $(\cdot, \cdot)_g$ tensor $(\cdot, \cdot)_g$ using twice the $(\cdot, \cdot)_g$-Riesz representation theorem.) Why is the reverse question trivial?

**Answer.** Let $g^i \in \mathcal{L}(E^*, E; \mathbb{R})$ be defined by $g^i(\ell, \bar{u}) = (\ell, \bar{u})_g$ where $\ell^i_g$ is the $(\cdot, \cdot)_g$-Riesz representation vector of $\ell$, that is, $\ell^i_g$ is defined by $\ell^i_g = (\ell^i_g, \bar{u})_g$ for all $\bar{u} \in E$. We immediately get $g^i(\ell, \bar{u}) = \ell^i \bar{u}$ for all $(\ell, \bar{u}) \in E^* \times E$. We have $g^i(\ell, \bar{u}) = \ell^i \bar{u}$. Thus $g^i$ (bilinear form) is naturally associated with the identity (endomorphism).

The reverse is trivial: 1- Start from $\ell \in \mathcal{L}(E, E)$ the identity operator (observer independent); 2- Then identified $I$ with the $(\cdot, \cdot)_g$ tensor $\tilde{I} \in \mathcal{L}(E^*, E; \mathbb{R})$ defined by $I(\ell, \bar{u}) = \bar{u}$; 3- Then $(\cdot, \cdot)_g$-associate $\tilde{I}$ with the $(\cdot, \cdot)_g$ tensor $\tilde{g}^i$ created when using the $(\cdot, \cdot)_g$-Riesz representation theorem just once, that is, $\tilde{g}^i(\ell^i_g, \bar{u}) = \tilde{I}(\ell, \bar{u})$ when $\ell^i_g$ is the $(\cdot, \cdot)_g$-Riesz representation vector of $\ell$: you get $\tilde{\ell} = g^i$.

(So, $\tilde{I} = g^i$ and $\tilde{\ell} = g$ and $\tilde{g}^i = g^i$ for Riesz representations with $(\cdot, \cdot)_g$.)

**A.13 The adjoint of a linear map**

(Difference with the transposed: If $L$ is a linear map, then $L$ as only one adjoint linear map $L^*$, while $L$ has many transposed $L^T_g$ since a transposed depends on inner dot products $(\cdot, \cdot)_g$ and $(\cdot, \cdot)_h$.)

Here no inner dot product is introduced. And, as usual, we first give the definition (qualitative approach), and second, the quantification with bases and components.

Let $E$ and $F$ be two finite dimension vector spaces.

**Definition A.67** If $L \in \mathcal{L}(E; F)$, then its adjoint is the linear map $L^* \in \mathcal{L}(F^*; E^*)$ canonically defined by

$$
L^* : \begin{cases} F^* \to E^* \\
m \to L^*(m) := m \circ L,
\end{cases}
$$

(A.98)

And the linearity of $L$ and $L^*$ makes it possible to use the dot notation (linearity notation)

$$
L^*.m := m.L, \quad \text{and} \quad (L^*.m).\bar{u} := m.L.\bar{u}.
$$

(A.100)

(The dual $L^*$ cannot be confused with the transposed which requires inner dot products, cf. (A.24).)
**Quantification:** Let \((\vec{a}_i)\) be a basis in \(E\), \((\vec{b}_i)\) be a basis in \(F\), and \((\pi_a)\) and \((\pi_b)\) be the dual bases. Let

\[
[L]_{\vec{a},\vec{b}} = [L_{ij}]_{i=1,...,m} \quad \text{and} \quad [L^*]_{\vec{b},\vec{a}} = [(L^*)_{ij}]_{j=1,...,n},
\]

that is,

\[
L\vec{a}_j = \sum_{i=1}^m L_{ij}\vec{b}_i \quad \text{and} \quad L^*\vec{b}_j = \sum_{i=1}^n (L^*)_{ij}\pi_a_i.
\]

Then (A.100) gives \((L^*\circ \pi_b_j)\vec{a}_i = \pi_b_j (L\vec{a}_i)\), that is,

\[
(L^*)_{ij} = L_{ji}, \quad \text{i.e.} \quad [L^*]_{\vec{b},\vec{a}} = [(L)_{\vec{a},\vec{b}}]^T \quad \text{(transposed matrix)}.
\]

To compare with (A.25) (there is no inner dot product here).

(Duality notations: \(L^*\circ \vec{b} = \sum_{i=1}^n (L^*)_i^j\vec{a}^i\), and \((L^*)^j_i = L_j^i\).)

### B Euclidean Frameworks

Time and space are decoupled (classical mechanics). \(\mathbb{R}^n\) is the geometric affine space, \(n = 1, 2, 3\), and \(\mathbb{R}^n\) is the associated vector space (made of “bi-point vectors”) which will be equipped with measuring instruments: Euclidean dot products to get angles and lengths.

#### B.1 Euclidean basis

**Manufacturing of a Euclidean basis.**

An observer chooses a unit of measure (foot, meter, a unit of length used by Euclid, the diameter of pipe...) and makes a “unit rod” of length 1 in this unit. Postulate: The length of the rod does not depend on its direction in space.

- **Space dimension** \(n = 1\): This rod models a vector \(\vec{e}_1\) which makes a basis \((\vec{e}_1)\) called the Euclidean basis relative to the chosen unit of measure.

- **Space dimension** \(n \geq 2\):
  - The observers make three rods of length 3, 4 and 5, and makes a triangle \((A, B, C)\), where \(A, B\) and \(C\) are the vertices, and \(A\) not on the side on length 5.
  - Since \(3^2 + 4^2 = 5^2\), the triangle \((A, B, C)\) is said to have a right angle at \(A\) (Pythagoras).
  - Two vectors \(\vec{a}\) and \(\vec{w}\) in \(\mathbb{R}^n\) are orthogonal iff the triangle \((A, B, C)\) can be positioned such that \(\overrightarrow{AB}\) and \(\overrightarrow{AC}\) are parallel to \(\vec{a}\) and \(\vec{w}\).
  - A basis \((\vec{e}_i)_{i=1,...,n}\) is Euclidean relative to the chosen unit of measurement iff the \(\vec{e}_i\) are two to two orthogonal and their length is 1 (relative to the chosen unit).

**Example B.1** An English observer defines a Euclidean basis \((\vec{a}_i)\) using the foot. A French observer defines a Euclidean basis \((\vec{b}_i)\) using the meter. We have

\[
1 \text{ foot} = \mu \text{ meter}, \quad \mu = 0.3048, \quad \text{and} \quad 1 \text{ meter} = \lambda \text{ foot}, \quad \lambda = \frac{1}{\mu} \simeq 3.28.
\]

\((\mu = 0.3048\) is the official length in meter for the English foot.) E.g., the bases are “aligned” iff, for all \(i\),

\[
\vec{b}_i = \lambda \vec{a}_i \quad \text{(change of measurement unit),}
\]

and the transition matrix from \((\vec{a}_i)\) to \((\vec{b}_i)\) is then \(P = \lambda I\), thus \(P^T = P\), \(P^TP = \lambda^2 I\) and \(P^{-1} = \frac{1}{\lambda^2} I\).

**Remark B.2** The bases used in practice are not all Euclidean. See example A.40, especially if you fly.

#### B.2 Euclidean dot product

**Definition B.3** Consider an observer who has built (or chosen) his Euclidean basis \((\vec{e}_i)\), cf. § B.1. The associated Euclidean dot product is the bilinear form \(g(\cdot, \cdot) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})\) defined by

\[
g(\vec{e}_i, \vec{e}_j) = \delta_{ij} = g_{ij}, \quad \forall i, j, \quad \text{i.e.} \quad [g]_{\vec{e}} = [\delta_{ij}] = I,
\]

cf. prop. A.11.
That is, with tensorial and duality notations, cf. (A.49), $(\pi_{ei}) = (e^i)$ being the dual basis of $(e_i)$ (with classical and duality notations),

\[
(\cdot, \cdot)_g = \sum_{i=1}^{n} \pi_{ei} \otimes \pi_{ei} = \sum_{i=1}^{n} e^i \otimes e^i.
\]

(If you want to use the Einstein convention: $(\cdot, \cdot)_g := \sum_{i,j=1}^{n} \delta_{ij} e^i \otimes e^j$, cf. (B.3).) So, for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, with $\vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i = \sum_{i=1}^{n} x^i e_i$ and $\vec{y} = \sum_{i=1}^{n} y_i \vec{e}_i = \sum_{i=1}^{n} y^i e_i$,

\[
(\vec{x}, \vec{y})_g = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x^i y^i = [\vec{x}]_{\vec{e}}^T [\vec{y}]_{\vec{e}} = (\vec{x}, \vec{y}).
\]

(If you want to use the Einstein convention: $(\vec{x}, \vec{y})_g := \sum_{i,j=1}^{n} \delta_{ij} x^i y^j = (\vec{x}, \vec{y}).$)

**Definition B.4** The associated norm is $||.||_g := \sqrt{\langle \cdot, \cdot \rangle_g}$, and the length of a vector $\vec{x}$ relative to the chosen Euclidean unit of measurement is $||\vec{x}||_g := \sqrt{(\vec{x}, \vec{x})_g}$.

Thus with the Euclidean basis $(e_i)$ used to build $(\cdot, \cdot)_g$, if $\vec{x} = \sum_{i=1}^{n} x^i e_i = \sum_{i=1}^{n} x^i e^i$, then $||\vec{x}||_g = \sqrt{\sum_{i=1}^{n} x^i} = \sqrt{\sum_{i=1}^{n} (x^i)^2}$ is the length of $\vec{x}$ relative to the chosen Euclidean unit of measure (Pythagoras).

(If you want to use the Einstein convention: $||\vec{x}||_g := \sqrt{\sum_{i,j=1}^{n} \delta_{ij} x^i x^j}$.)

**Definition B.5** Two vectors $\vec{x}, \vec{y}$ are $(\cdot, \cdot)_g$-orthogonal iff $(\vec{x}, \vec{y})_g = 0$.

**Definition B.6** The angle $\theta(\vec{x}, \vec{y})$ between two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n - \{\vec{0}\}$ is defined by

\[
\cos(\theta(\vec{x}, \vec{y})) = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}||_g \cdot ||\vec{y}||_g} = (\cos(\theta(\vec{y}, \vec{x})).
\]

(With a calculator, this formula gives $\theta(\vec{x}, \vec{y}) = \arccos((\vec{x}, \vec{y})_g) \text{ a value in } [0, \pi].$)

**Definition B.7** A basis $(\hat{\vec{e}}_i)$ such that $(\vec{b}_i, \vec{b}_j)_{\hat{\vec{e}}} = 0$ for all $i, j = 1, \ldots, n$, is called a $(\cdot, \cdot)_{\hat{\vec{e}}}$-orthonormal basis.

**Proposition B.8** If $(\vec{a}_i)$ and $(\vec{b}_i)$ are two $(\cdot, \cdot)_g$-orthonormal bases, and if $P$ is the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, then $P^T P = I$, and we have $(\cdot, \cdot)_g = \sum_{i=1}^{n} a_i \otimes a_i = \sum_{i=1}^{n} b_i \otimes b_i$.

**Proof.**

\[
\vec{b}_i = \sum_{k=1}^{n} P_{ik} \vec{a}_k
\]

defines $\delta_{ij} = (\vec{b}_i, \vec{b}_j)_{\hat{\vec{e}}} = \sum_{k=1}^{n} P_{ik} P_{jk} (\vec{a}_k, \vec{a}_l)_{\hat{\vec{e}}} = \sum_{k=1}^{n} P_{ik} P_{jk} = \sum_{k=1}^{n} (P^T)^{jk} P_{jk} = (P^T P)_{ij} = [\delta_{ij}] = I.

And for the dual bases we have $a^i = \sum_{j=1}^{n} P^{ij} b^j$, cf. (A.67). Thus $\sum_{i,j=1}^{n} a_i \otimes a_i = \sum_{i,j=1}^{n} P_{ik} P_{jk} b^i \otimes b^j = \sum_{k=1}^{n} b_k \otimes b_k$, that is, $[\vec{b}]_{\hat{\vec{e}}} = [\vec{y}]_{\hat{\vec{e}}} = I$.

**B.3 Change of Euclidean basis**

Let $(\vec{a}_i)$ (e.g. built with the foot) and $(\vec{b}_i)$ (e.g. built with the meter) be two Euclidean bases in $\mathbb{R}^n$. Let $(\cdot, \cdot)_g$ and $(\cdot, \cdot)_h$ be the associated Euclidean dot products in $\mathbb{R}^n$.

**B.3.1 Two Euclidean dot products are proportional**

**Proposition B.9** If $\lambda = ||\vec{b}_i||_g$ then $||\vec{b}_i||_h = \lambda$ for all $i = 1, \ldots, n$ (change of unit). The Euclidean dot products $(\cdot, \cdot)_g$ and $(\cdot, \cdot)_h$ are necessarily proportional:

\[
\text{if } \lambda = ||\vec{b}_i||_g \text{ then } (\cdot, \cdot)_g = \lambda^2 (\cdot, \cdot)_h \text{ and } ||.||_g = \lambda ||.||_h. \tag{B.7}
\]

(Useful if e.g. you want to use results of Newton, Descartes... or work with another observer.)

**Proof.** By definition of a Euclidean basis, the length of the rod that enabled to define $(\vec{b}_i)$ is independent of $i$, cf. the postulate § B.1, thus $||\vec{b}_i||_g = ||\vec{b}_i||_g$ for all $i$.

Thus $(\vec{b}_i, \vec{b}_j)_g = ||\vec{b}_i||^2_g = \lambda^2 (\vec{b}_i, \vec{b}_i)_h = 1$ and if $i \neq j$ then two Euclidean basis vectors form a right angle (Pythagoras), thus $(\vec{b}_i, \vec{b}_j)_g = 0 = (\vec{b}_i, \vec{b}_j)_h$ when $i \neq j$, cf. (B.4). Hence $(\vec{b}_i, \vec{b}_j)_g = \lambda^2 (\vec{b}_i, \vec{b}_j)_h$ for all $i,j$, thus (B.7).
Example B.10 Continuation of example B.1. $(\cdot,\cdot)_a = \sum_{i=1}^n a^i \otimes a^i$ is the English Euclidean dot product (foot), and $(\cdot,\cdot)_b = \sum_{i=1}^n b^i \otimes b^i$ is the French Euclidean dot product (meter). (B.7) and (B.1) give:

$$(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b \quad \text{and} \quad ||\cdot||_a = \lambda||\cdot||_b,$$

with $\lambda \simeq 3.28$ and $\lambda^2 \simeq 10.76$. (B.8)

E.g., if $\vec{w}$ is s.t. $||\vec{w}||_a = 1$ (its length is 1 meter), then $||\vec{w}||_b = \lambda$ (its length is $\lambda \simeq 3.28$ foot).

B.3.2 Counterexample: non existence of a Euclidean dot product

1- Thermodynamic: Let $T$ be the temperature and $V$ the volume, and consider the Cartesian vector space $\mathbb{R}^2 = \{(T,V)\} = \mathbb{R} \times \mathbb{R}$. There is no associated Euclidean dot product: An associated norm would give $||(T,V)|| = \sqrt{T^2 + V^2} \in \mathbb{R}$ which is meaningless (you don’t add Kelvin and cubic meters, even squared...). See § A.8.5.

2- Polar coordinate system, see example 10.17: A parametric vector $\vec{q} = (r, \theta) \in \mathbb{R} \times \mathbb{R}$ has no norm that is physically meaningful: $\sqrt{r^2 + \theta^2}$ has no physical meaning.

B.4 Euclidean transposed of the deformation gradient

Let $n \in \{1,2,3\}$ and consider a linear map $L \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ (e.g., $L = F^n_t(P)$).

Let $(\cdot,\cdot)_G$ be an inner dot product in $\mathbb{R}^n$ (used in the past by someone), and let $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ be inner dot products in $\mathbb{R}^n$ (the actual space where the results are observed by two observers, e.g., English $(\cdot,\cdot)_a$ and French $(\cdot,\cdot)_b$). Let $L^T_{Ga}$ and $L^T_{Gb}$ be the transposed of $L$ relative to the dot products, that is, $L^T_{Ga}$ and $L^T_{Gb}$ in $\mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ are characterized by

$$(L^T_{Ga}, \vec{X}; \vec{Y})_a = (L, \vec{X}, \vec{Y})_a \quad \text{and} \quad (L^T_{Gb}, \vec{X}; \vec{Y})_G = (L, \vec{X}, \vec{Y})_G,$$

for all $\vec{X} \in \mathbb{R}^n$ and all $\vec{Y} \in \mathbb{R}^n$, cf. (A.24).

Proposition B.11 If $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ are Euclidean dot products, then

$$(\cdot,\cdot)_a = \lambda^2(\cdot,\cdot)_b \implies L^T_{Ga} = \lambda^2 L^T_{Gb} \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \quad (\text{B.10})$$

(a vector type relation, as in (5.34) or (C.10)). NB: Do not forget $\lambda^2$ if an French observer works with an English observer at $t$ (or NASA with Boeing, cf. remark A.41).

Proof. $(L^T_{Ga}, \vec{y}, \vec{X})_G \overset{(B.9)}{=} (L, \vec{X}, \vec{y})_a \overset{(B.7)}{=} \lambda^2 (L, \vec{X}, \vec{y})_b \overset{(B.9)}{=} (L^T_{Gb}, \vec{y}, \vec{X})_G$ for all $\vec{X} \in \mathbb{R}^n$ and all $\vec{y} \in \mathbb{R}^n$,

thus $L^T_{Ga} \vec{y} = \lambda^2 L^T_{Gb} \vec{y}$ for all $\vec{y} \in \mathbb{R}^n$.

B.5 The Euclidean transposed for endomorphisms

Let $n \in \{1,2,3\}$ and consider an endomorphism $L \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ (e.g. $L = d\vec{\tau}(p) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ the differential of the Eulerian velocity). Let $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ be dot products in $\mathbb{R}^n$. Let $L^T_a$ and $L^T_b$ be the transposed of $L$ relative to $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$, that is, $L^T_a$ and $L^T_b$ in $\mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ are the endomorphisms defined by, for all $\vec{x}, \vec{y} \in \mathbb{R}^n$, cf. (A.17),

$$(L^T_a, \vec{y}, \vec{x})_a = (L, \vec{x}, \vec{y})_a, \quad \text{and} \quad (L^T_b, \vec{y}, \vec{x})_b = (L, \vec{x}, \vec{y})_b \quad (\text{B.11})$$

Proposition B.12 If $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ are Euclidean dot products, then

$$(\text{Euclidean dot products}) \quad L^T_a = L^T_b \overset{\text{noted}}{=} L^T \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \quad (\text{B.12})$$

(an endomorphism type relation): Thus we can speak of “the Euclidean transposed of an endomorphism”.

Proof. $(L^T_a, \vec{y}, \vec{x})_a \overset{(B.11)}{=} (L, \vec{x}, \vec{y})_a \overset{(B.7)}{=} \lambda^2 (L, \vec{x}, \vec{y})_b \overset{(B.11)}{=} \lambda^2 (L^T_b, \vec{y}, \vec{x})_b \overset{(B.7)}{=} (L^T_b, \vec{y}, \vec{x})_b$ for all $\vec{x}, \vec{y} \in \mathbb{R}^n$,

thus $L^T_a \vec{y} = L^T_b \vec{y}$ for all $\vec{y} \in \mathbb{R}^n$.  

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C Riesz representation theorem

C.1 The Riesz representation theorem

The goal is, when we have an inner dot product $(\cdot, \cdot)_g$, to represent a linear function by a vector thanks to $(\cdot, \cdot)_g$. It is in no way an identification: It is only a representation, or an association, allowed by a choice of a measuring tool $(\cdot, \cdot)_g$ (observer dependent). In particular two observers with two different measuring tools get two different representation vectors (two different associations).

C.1.1 Framework

Framework: A Hilbert space $(E, (\cdot, \cdot)_g)$, that is, a vector space $E$ equipped with an inner dot product $(\cdot, \cdot)_g$ and $\|\cdot\|_g$ the associated norm such that $(E, \|\cdot\|_g)$ is a complete space (a Banach space). And $E^* = \mathcal{L}(E; \mathbb{R}) = \{\text{the space of the linear and continuous forms } E \to \mathbb{R}\}$. (C.1)

NB:
1- If $E$ is finite dimensional: All norms are equivalent, $(E, (\cdot, \cdot)_g)$ is a Hilbert space, and any linear form is continuous. This is the usual framework in continuum mechanics with $E = \mathbb{R}^3$.
2- If $E$ is infinite dimensional: Then two norms can be non equivalent, $(E, \|\cdot\|_g)$ can be non complete, a linear form can be non continuous; And the Riesz representation theorem can only be applied to continuous linear forms. This is the usual framework in continuum mechanics with $E = L^2(\Omega)$ the space of finite energy functions in $\Omega$, thus it is the framework of the finite element method.

C.1.2 Riesz representation theorem

We have the easy statement, $(\cdot, \cdot)_g$ being some inner dot product in $E$:

If $\vec{v} \in E$ (vector) then $\exists! v_g \in E^*$ (linear continuous form) s.t. $v_g(\vec{x}) = (\vec{v}, \vec{x})_g$, $\forall \vec{x} \in E$. (C.2)

Moreover $\|v_g\|_{E^*} = \|\vec{v}\|_g$.

Proof: Define $v_g \in E^*$ by $v_g(\vec{x}) = (\vec{v}, \vec{x})_g$ for all $\vec{x} \in E$, then use the Cauchy–Schwarz inequality to get $|v_g(\vec{x})| = \|\vec{v}\|_g \|\vec{x}\|_g$ for all $\vec{x} \in E$, thus $\|v_g\|_{E^*} \leq \|\vec{v}\|_g$, and with $\vec{x} = \vec{v}$ we get $|v_g(\vec{v})| = \|\vec{v}\|_g \|\vec{x}\|_g$, thus $\|v_g\|_{E^*} \geq \|\vec{v}\|_g$. Thus $\|v_g\|_{E^*} = \|\vec{v}\|_g < \infty$, thus $v_g$ is continuous.

The Riesz representation theorem concerns the converse:

Theorem C.1 (Riesz representation theorem and definiton) Let $(E, (\cdot, \cdot)_g)$ be a Hilbert space, and $E^*$ be the space of linear continuous form on $E$.

If $\ell \in E^*$ (linear continuous form) then $\exists \vec{\ell}_g \in E$ (vector) s.t. $\ell(\vec{x}) = (\vec{\ell}_g, \vec{x})_g$, $\forall \vec{x} \in E$. (C.3)

Moreover $\|\vec{\ell}_g\|_g = \|\ell\|_{E^*}$. The vector $\vec{\ell}_g = \vec{\ell}(g) \in E$ (which depends on $g$) is called the $(\cdot, \cdot)_g$-Riesz representation vector of $\ell$.

Proof. If $\ell = 0$ then $\vec{\ell}_g = \vec{0}$ (trivial). Suppose $\ell \neq 0$. Let $\text{Ker}\ell = \ell^{-1}\{0\}$ (the kernel).

If $\dim E = 1$, it is trivial (exercise). Suppose $\dim E \geq 2$. Then $\text{Ker}\ell \neq \{0\}$. Since $\ell$ is continuous, $\text{Ker}\ell = \ell^{-1}\{0\}$ is closed in the Hilbert space $(E, (\cdot, \cdot)_g)$. Thus, if $\vec{x} \in E$, then its $(\cdot, \cdot)_g$-projection $\vec{x}_0 \in \text{Ker}\ell$ on $\text{Ker}\ell$ exists, and is given by:

$$\forall \vec{y}_0 \in \text{Ker}\ell, \, (\vec{x} - \vec{x}_0, \vec{y}_0)_g = 0.$$ (C.4)

In particular, if $\vec{x} \notin \text{Ker}\ell$ (possible since $\ell \neq 0$), let $\vec{n} = \frac{\vec{x} - \vec{x}_0}{\|\vec{x} - \vec{x}_0\|_g}$. So $\vec{n}$ is a unitary $(\cdot, \cdot)_g$-normal vector to $\text{Ker}\ell$ and $(\text{Ker}\ell)^\perp = \text{Vec}\{\vec{n}\}$ since, $\ell$ being a linear form, $\dim(\text{Ker}\ell) = 1$ and $\dim(\text{Ker}\ell)^\perp = 1$. And $E = \text{Ker}\ell \oplus (\text{Ker}\ell)^\perp$ since both vector spaces are closed. Thus if $\vec{x} \in E$ then $\vec{x} = \vec{x}_0 + \lambda \vec{n} \in \text{Ker}\ell \times (\text{Ker}\ell)^\perp$, where $\vec{x}_0 \in \text{Ker}\ell$ and $\lambda \in \mathbb{R}$; Thus $(\vec{x}, \vec{n})_g = \lambda$ and $\ell(\vec{x}) = \lambda \ell(\vec{n})$; Thus $\ell(\vec{x}) = (\vec{x}, \vec{n})_g \ell(\vec{n}) = (\vec{x}, \ell(\vec{n}))_g = (\vec{x}, (\vec{n})_g)_g$ (bilinearity of $(\cdot, \cdot)_g$); Thus $\vec{\ell}_g := \ell(\vec{n})\vec{n}$ satisfies (C.3). And if vectors $\vec{\ell}_g$ and $\vec{\ell}_g$ satisfy (C.3) then $(\vec{\ell}_g - \vec{\ell}_g, \vec{x})_g = 0$ for all $\vec{x} \in E$, thus $\vec{\ell}_g - \vec{\ell}_g = 0$. Thus $\vec{\ell}_g$ is unique. And the Cauchy–Schwarz theorem gives $\|\ell\|_{E^*} = \sup_{\|\vec{x}\|_g = 1} |\ell(\vec{x})| = \sup_{\|\vec{x}\|_g = 1} (\vec{\ell}_g, \vec{x})_g = \|\vec{\ell}_g\|_g$. \hfill □
C.1.3 Riesz representation mapping

Let $SIDP(E)$ be the space of inner dot products in $E$. Then (C.3) gives the Riesz representation mapping:

$$R : \left\{ \begin{array}{l} SIDP \times E^* \to E \\ (g, \ell) \to R(g, \ell) := \bar{\ell}_g, \end{array} \right.$$  \hfill (C.5)

In particular, for a $g \in SIDP(E)$ (that is, for the observer who has chosen $g$),

$$R_g = R(g, \cdot) : \left\{ \begin{array}{l} (E^*, |||E^*||) \to (E, |||E||) \\ \ell \to R_g(\ell) = R(g, \cdot)(\ell) := R(g, \ell) \ (= \bar{\ell}_g) \end{array} \right.$$  \hfill (C.6)

is an isomorphism (linear, bijective and $|||\ell|||_{E^*} = |||\bar{\ell}_g|||_E$). And, for any $\ell \in E^*$,

$$R(\cdot, \ell) : \left\{ \begin{array}{l} SIDP \to E \\ g \to R(\cdot, \ell)(g) := R(g, \ell) \ (= \bar{\ell}_g) \end{array} \right.$$  \hfill (C.7)

associates an observer (the one who chooses $\cdot, \ell$) with $\bar{\ell}(g) = \bar{\ell}_g$.

**Remark C.2** You cannot identify a linear form $\ell$ with one of its Riesz representation vector: Which one? $\bar{\ell}_g = R(g, \ell)$? $\bar{\ell}_h = R(h, \ell)$? (See e.g. (C.11).)

**Notation.** A $\ell \in E^*$ being linear $\ell(\tilde{x})^{=\text{noted } \ell.\tilde{x}}$: The dot between $\ell$ and $x$ is the usual duality dot (covariance-contravariance dot = linear form acting on a vector): It is not the dot of a “inner dot product”. If you want to use a dot notation for the inner dot product $\langle \cdot, \cdot \rangle_g$, then (C.3) should be at least written

$$\ell.\tilde{x} = \bar{\ell}_g \cdot \tilde{x}, \text{ where } \bar{\ell}_g \cdot \tilde{x} := (\bar{\ell}_g, \tilde{x}).$$  \hfill (C.8)

At least, use a big dot notation like $\ell.\tilde{x} = \bar{\ell} \cdot \tilde{x}$.

C.1.4 Change of Riesz representation vector: Euclidean case

For one linear form $\ell$, two observers with two Euclidean dot products $\langle \cdot, \cdot \rangle_a$ and $\langle \cdot, \cdot \rangle_b$ (e.g., one in foot and one in meter) get two Riesz representation vectors $\bar{\ell}_a$ and $\bar{\ell}_b$ given by, cf. (C.3),

$$\forall \tilde{x} \in E, \ \ell.\tilde{x} = \langle \tilde{\ell}_a, \tilde{x} \rangle_a = \langle \tilde{\ell}_b, \tilde{x} \rangle_b.$$  \hfill (C.9)

**Proposition C.3** Change of Euclidean Riesz representation vector:

If $\exists \lambda > 0, \langle \cdot, \cdot \rangle_a = \lambda^2 \langle \cdot, \cdot \rangle_b$ then $\bar{\ell}_b = \lambda^2 \bar{\ell}_a.$  \hfill (C.10)

Conversely, if $\bar{\ell}_b = \lambda^2 \bar{\ell}_a$ for all linear forms $\ell$, then $\langle \cdot, \cdot \rangle_a = \lambda^2 \langle \cdot, \cdot \rangle_b$.

NB: (C.10) is a change of vector formula (one linear form, two vectors relative to two inner dot products), not a “change of basis formula for one vector” (concerns one vector and its representation in components relative to bases: In (C.10) no basis has been introduced).

**Proof.** $\langle \cdot, \cdot \rangle_a = \lambda^2 \langle \cdot, \cdot \rangle_b$ and $\ell.\tilde{x} = \langle \tilde{\ell}_a, \tilde{x} \rangle_a = \langle \bar{\ell}_a, \tilde{x} \rangle_b$ give $\lambda^2 \langle \tilde{\ell}_a, \tilde{x} \rangle_b = \langle \bar{\ell}_b, \tilde{x} \rangle_b$, thus $\lambda^2 \bar{\ell}_a - \bar{\ell}_b = 0$, for all $\tilde{x}$, hence $\lambda^2 \bar{\ell}_a - \bar{\ell}_b = 0$, i.e. (C.10).

$\bar{\ell}_b = \lambda^2 \bar{\ell}_a$ for all $\ell$ gives $\langle \tilde{\ell}_b, \tilde{x} \rangle_b = \lambda^2 \langle \tilde{\ell}_a, \tilde{x} \rangle_b = \lambda^2 \langle \tilde{\ell}_a, \tilde{x} \rangle_b$, for all $\tilde{x}$ and for all $\tilde{\ell}_a$, thanks to the isomorphism $R_{\bar{\ell}_b}$, cf. (C.6), thus $\langle \cdot, \cdot \rangle_a = (\lambda^2 \bar{\ell}_a, \tilde{x})_b = \lambda^2 \langle \cdot, \cdot \rangle_b.$  \hfill (C.11)

**Example C.4** $\langle \cdot, \cdot \rangle_a$ and $\langle \cdot, \cdot \rangle_b$ are the Euclidean dot products resp. made with the foot and the meter:

$$\bar{\ell}_b = \lambda^2 \bar{\ell}_a, \ \text{ with } \lambda^2 > 10,$$  \hfill (C.11)

so $\bar{\ell}_b$ is quite different from $\bar{\ell}_a$: A Riesz representation vector is not intrinsic to $\ell$: It depends on a $\langle \cdot, \cdot \rangle$.  \hfill (C.11)
Exercise C.5 Framework of example C.4: Prove: $||\vec{e}_b||_b = \lambda ||\ell||_a$. Does it contradict the Riesz representation theorem which also states that $||\ell||_{a^{**}} = ||\ell||_{a^*}$?

Answer. (C.11) and $||\ell||_a = \lambda ||\ell||_b$ give $||\vec{e}_b||_b = \lambda ||\ell||_a$. There is no contradiction with $||\ell||_{a^{**}} = ||\ell||_{a^*}$ (Riesz representation theorem), since $||\ell||_{a^{**}} = ||\ell||_{a^*}$ depends on the choice of the norm $||\cdot||_a$: Here $||\cdot||_a$ is either $||\cdot||_a$ or $||\cdot||_b$. And $||\ell||_b = \sup_{\vec{e} \in E} \frac{\ell (\vec{e})}{||\vec{e}||_b} = \sup_{\vec{e} \in E} \frac{\ell (\vec{e})}{||\vec{e}||_a} = \lambda \sup_{\vec{e} \in E} \frac{\ell (\vec{e})}{||\vec{e}||_a} = \lambda ||\ell||_a$.

Check: (C.11) and $||\ell||_a = \lambda ||\ell||_b$ give $||\vec{e}_b||_b = \lambda ||\ell||_a$.

Example C.6 If $f$ is $C^1$ and $p \in \mathbb{R}^n$, then $df(p)$ (the differential at $p$) is the linear form defined by, for all $\vec{w} \in \mathbb{R}^n$, see (S.3),

$$
    df(p).\vec{w} := \lim_{h \to 0} \frac{f(p + h\vec{w}) - f(p)}{h} \quad \text{(definition independent of any inner dot product).} \quad (C.12)
$$

And the gradient $\text{grad}_g f(p)$ relative to an inner dot product $(\cdot, \cdot)_g$ is defined by $df(p).\vec{w} = (\text{grad}_g f(p), \vec{w})_a$ for all $\vec{w}$, that is, $\text{grad}_g f(p)$ is the $(\cdot, \cdot)_g$-Riesz representation vector of $df(p)$ (when it exists: This is not the case in thermodynamic). Thus (C.11) gives

$$
    \text{grad}_g f(p) = \lambda^2 \text{grad}_a f(p) \quad \text{with} \quad \lambda^2 > 10 \quad \text{for English and French:} \quad (C.13)
$$

The gradient really depends on the observer (the gradient is subjective).

Example C.7 Aviation: If you do want to use a Riesz representation vector to represent $\ell$ thanks to an inner dot product, it is vital to know which inner dot product is in use (at least, at the time of landing...): Recall that the foot is the international unit for altitude for aviation. So do not forget $\lambda > 3$.

C.1.5 Quantification with a basis

Suppose dim $E$ is finite dimensional, dim $E = n \in \mathbb{N}^*$. Let $(\vec{e}_i)$ be a basis in $E$, and $(\pi_{e_i}) = (e^i)$ be its dual basis in $E^*$ (classical or duality notation), cf. (A.33).

Let $g(\cdot, \cdot) = (\cdot, \cdot)_g$ be an inner dot product and $g_{ij} := g(\vec{e}_i, \vec{e}_j)$ for all $i, j$, i.e., $[g]_{ij} = [g_{ij}]$.

Let $\ell \in E^*$, $\ell = \sum_{i=1}^n \ell_i \vec{e}^i$, i.e., $[\ell]_{e^i} = (\ell_1 \ldots \ell_n)$ (row matrix: $\ell$ is a linear form).

Let $\ell_g \in E$ be the $(\cdot, \cdot)_g$-Riesz representation vector of $\ell$, $\ell_g = \sum_{i=1}^n \ell_i \vec{e}^i$, i.e., $[\ell_g]_{e^i} = \begin{pmatrix} \ell_g^1 \\ \vdots \\ \ell_g^n \end{pmatrix}$ (column matrix: $\ell_g$ is a vector). Then (C.3) gives $[\ell]_{[e^i][\vec{v}]} = [\ell_g]_{[e^i][\vec{v}]} = [g]_{[\vec{v}][\vec{w}]} = [\ell_g]^T_{[\vec{v}]}$, i.e.,

$$
    [\ell_g]_{[\vec{v}]} = [g]_{[\vec{v}]}^{-1} [\ell]_{[\vec{v}]}^T. \quad (C.14)
$$

That is, for all $i$,

$$
    \ell_g^i = \sum_{j=1}^n ([g]^{-1}_{[\vec{v}][\vec{w}]} g_{ij}) \ell_j \quad \text{noted} \quad \sum_{j=1}^n g^{ij} \ell_j \quad \text{when} \quad [g]^{-1}_{[\vec{v}][\vec{w}]} = [g^{ij}]. \quad (C.15)
$$

Remark C.8 If a unique inner dot product $(\cdot, \cdot)_g$ is used (isometric framework), e.g., the Euclidean dot product made with the English foot, then a usual notation for $\ell_g$ is $\overline{\ell}_g$, because, if $\ell = \sum_{i=1}^n \ell_i \vec{e}^i$ then $\ell^2 = \sum_{i=1}^n \ell_i^2 \vec{e}^i$. The index $i$ in $\ell_i$ has been pulled up to get $i$ in $\ell^i$ cf. (C.15) (duality notations). And (C.3) and (C.15) read

$$
    \ell \cdot \vec{x} = \ell^i \cdot \vec{x} \quad \text{where} \quad [\ell^i]_{[\vec{v}]} = [g]^{-1}_{[\vec{v}][\vec{w}]} [\ell]_{[\vec{w}]}^T. \quad (C.16)
$$

(In particular, if $(\vec{e}_i)$ is a $(\cdot, \cdot)_g$-orthonormal basis, then $[g]_{[\vec{v}][\vec{w}]} = I$.)

C.1.6 A Riesz representation vector is contravariant

$\ell_g$ is by definition a vector in $E$, cf. (C.3), so it is contravariant. If you do need to check that the contravariance of basis formula applies to the vector $\ell_g$, consider two bases $(\vec{e}_{old,i})$ and $(\vec{e}_{new,i})$
in $E$. Let $\tilde{\ell}_g$ be the $(\cdot,\cdot)_g$-Riesz representation vector of $\ell$. Thus (C.3), with $[\tilde{x}]_{\text{new}} = P^{-1} [\tilde{x}]_{\text{old}}$ and $[g]_{\text{new}} = P^T [g]_{\text{old}} P$, gives

$$
[x]_{\text{new}} = P^{-1} [\tilde{x}]_{\text{old}} = \ell.\tilde{\ell}_g
$$

\hspace{1cm} (C.17)

True for all $\tilde{x}$, thus $[\tilde{\ell}_g]_{\text{new}} = P [\tilde{\ell}_g]_{\text{old}}$, i.e.,

$$
[\tilde{\ell}_g]_{\text{new}} = P^{-1} [\tilde{\ell}_g]_{\text{old}} \quad \text{(contravariance formula).}
$$

(C.18)

### C.1.7 Change of Riesz representation vector, general case

For a linear form $\ell$, two observers with two inner dot products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ get two Riesz representation vectors $\tilde{\ell}_a$ and $\tilde{\ell}_b$ given by (C.9): $\ell.\tilde{x} = (\tilde{\ell}_a, \tilde{x})_a = (\tilde{\ell}_b, \tilde{x})_b$ for all $\tilde{x} \in E$.

**Proposition C.9** If $(\cdot,\cdot)$ is one basis in $E$, then

$$
g_a[\tilde{\ell}_a]_\varepsilon \mid \varepsilon = [g_b]_\varepsilon [\tilde{\ell}_b]_\varepsilon \quad \text{i.e.,} \quad [\tilde{\ell}_b]_\varepsilon = [g_b]_\varepsilon^{-1} [g_a]_\varepsilon [\tilde{\ell}_a]_\varepsilon,
$$

(C.19)

is the change of Riesz representation vector formula (one basis, two vectors). NB: (C.19) is not a change of basis formula (two bases, one vector): It is a change of vector formula.

**Proof.** (C.9) gives $[x]_{\text{new}}^T [g_a]_\varepsilon [\tilde{x}]_\varepsilon = [x]_{\text{old}}^T [g_b]_\varepsilon [\tilde{x}]_\varepsilon$ for all $\tilde{x}$, hence (C.19).

### C.2 Question: What is a vector versus a $(\cdot,\cdot)_g$-vector?

1. Originally, a vector was a bipoint vector $\vec{v} = \vec{A}\vec{B}$ in $\mathbb{R}^3$ used to represent of a “real object”. See Maxwell [13]. E.g., to measure the size of a child, a “wooden stick representing a child” is used to make the vector $\vec{v} = \vec{A}\vec{B}$; And any observer can see the same bipoint vector (the same wooden stick); But the size of $\vec{v}$ (its norm) depends on the observer: depends on the chosen measuring unit (foot, meter...).

2. Then, the concept of vector space was introduced: It is a quadruplet $(E, +, K,.)$ where $+$ is an inner law, $(E, +)$ is a group, $K$ is a field, $.$ is an external law on $E$ (called a scalar multiplication) compatible with $(E, +)$ (see any math book). Then the general concept of inner dot product in a vector space was introduced.

Example: Consider the vector space $(\mathbb{R}^n, +, K,.)$ noted $\mathbb{R}^n$ the dual space of $\mathbb{R}^n$: Let $\ell \in \mathbb{R}^n$. Then an English observer, resp. a French observer, introduces an inner dot product $(\cdot,\cdot)_a$ in $\mathbb{R}^n$, resp. $(\cdot,\cdot)_b$; And he builds the mathematical vector $\tilde{\ell}_a \in \mathbb{R}^n$, resp. $\tilde{\ell}_b$, which is the $(\cdot,\cdot)_a$-Riesz representation vector associated to $\ell$, resp. “the $(\cdot,\cdot)_b$-Riesz representation vector associated to $\ell$”. These vectors $\tilde{\ell}_a$ and $\tilde{\ell}_b$ are two different vectors, cf. (C.11): They are abstract mathematically built vectors which can be called “$(\cdot,\cdot)_a$-vector” and “$(\cdot,\cdot)_b$-vector”. (Compare with 1-)

Here the Riesz mapping $R(\cdot,\cdot)$, cf. (C.5), gives the tool equipped with two buttons: $R(a,\cdot): \ell \rightarrow \tilde{\ell}_a$ for the English man, and $R(b,\cdot): \ell \rightarrow \tilde{\ell}_b$ for the French man.

3. With differential geometry: the definition of a vector in $\mathbb{R}^n$ is defined without ambiguity: “A vector $\vec{v}$ is a tangent vector”, which means that there exists a curve $c: s \in [a,b] \rightarrow c(s) \in \mathbb{R}^n$ and $\exists s \in [a,b]$ such that $\vec{v}$ is defined at $c(s)$ by $\vec{v} = \vec{c}'(s)$. And it is shown that $\vec{v}$ is equivalent $\frac{dc}{ds}$ is the directional derivative in the direction $\vec{v}$. For other equivalent definitions, see Abraham-Marsden [1].

### C.3 Problems due to a Euclidean framework

The Riesz isomorphism $R_{\ell} = R(g,\cdot): \ell \rightarrow \tilde{\ell}_g$, cf. (C.6), is neither canonical nor natural: We cannot identify a linear form $\ell$ which one of its representation vector (which one?), cf. C.10. (More generally, there is no natural canonical isomorphism between $E$ and $E^*$, see § T.2.)

Moreover, a Riesz representation vector may possibly make you loose track of the “nature” of the considered mathematical object: A linear form $\ell$ on $E$ is covariant while a vector in $E$ is contravariant (the change of basis formulas are different, cf. (A.68)).
Remark C.10 Other problems with the Riesz representation vector:

- Incompatibility with the use of push-forwards, cf. § 13.2 (transport by a motion).
- Incompatibility with the use of the Lie derivative, cf. (15.59).
- Problem with the usual divergence div of an order two tensor $\tau$ used by mechanical engineers (their divergence is not objective), see § S.10. About that, we can define the objective divergence div of a $\binom{1}{1}$ tensor $\vec{\tau}$ in $T^1_1(U)$, cf. (S.54). (And see example S.32.)

D Determinants

D.1 Alternating multilinear form

Let $E$ be a finite dimension vector space. Let $\mathcal{L}(E, \ldots, E; \mathbb{R}) = \text{named } \mathcal{L}(E^n; \mathbb{R})$ (with $E$ $n$-times) be the set of multilinear forms (the set $\mathcal{L}^n(E)$ of uniform $\binom{0}{n}$ tensors), that is, $m \in \mathcal{L}(E^n; \mathbb{R})$ iff

$$m(\ldots, \vec{x} + \lambda \vec{y}, \ldots) = m(\ldots, \vec{x}, \ldots) + \lambda m(\ldots, \vec{y}, \ldots)$$

(D.1)

for all $\vec{x}, \vec{y} \in E$ and all $\lambda \in \mathbb{R}$ and for all “slot”. In particular, $m(\lambda \vec{x}_1, \ldots, \lambda \vec{x}_n) = (\prod_{i=1}^{n} \lambda_i) m(\vec{x}_1, \ldots, \vec{x}_n)$ for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and all $\vec{x}_1, \ldots, \vec{x}_n \in E$.

Definition D.1 If $n = 1$ then a 1-alternating multilinear function is a linear form, also called a 1-form. If $n \geq 2$ then $\mathcal{A}: \mathbb{R}^n \rightarrow \mathbb{R}$ is a 1-alternating multilinear form iff, for all $\vec{u}, \vec{v} \in E$,

$$\mathcal{A}(\ldots, \vec{u}, \ldots, \vec{v}, \ldots) = -\mathcal{A}(\ldots, \vec{v}, \ldots, \vec{u}, \ldots),$$

(D.2)

the other elements being unchanged. If $n = 1$, the set of 1-forms is $\Omega^1(E) = E^*$. If $n \geq 2$, the set of $n$-alternating multilinear forms is

$$\Omega^n(E) = \{ m \in \mathcal{L}(E^n; \mathbb{R}) : m \text{ is alternating} \}.$$  

(D.3)

If $\mathcal{A}, \mathcal{B} \in \Omega^n(E)$ and $\lambda \in \mathbb{R}$ then $\mathcal{A} + \lambda \mathcal{B} \in \Omega^n(E)$ thanks to the linearity for each variable. Thus $\Omega^n(E)$ is a vector space, sub-space in $(\mathcal{F}(E^n; \mathbb{R}), +, \cdot)$.

D.2 Leibniz formula

Particular case $E = \mathbb{R}$. Let $\mathcal{A} \in \Omega^n(E)$ (a $n$-alternating multilinear form). Let $(\vec{e}_i)_{i=1}^{n} = \text{named } (\vec{e}_i)$ be a basis in $E$. Recall (see e.g. Cartan [5]):

1- A permutation $\sigma : [1, n] \rightarrow [1, n]$ is a bijective map (one-to-one and onto); Let $S_n$ be the set of permutations of $[1, n]$.

2- A transposition $\tau : [1, n] \rightarrow [1, n]$ is a permutation that exchanges two elements, that is, $\exists i, j$ s.t. $\tau(\ldots, i, \ldots, j, \ldots) = (\ldots, j, \ldots, i, \ldots)$, the other elements being unchanged.

3- A permutation is a composition of transpositions. And a permutation is even iff the number of transpositions is even, and a permutation is odd iff the number of transpositions is odd. Based on: The parity (even or odd) of a permutation is an invariant.

4- The signature $\varepsilon(\sigma) = \pm 1$ of a permutation $\sigma$ is +1 if $\sigma$ is even, and is -1 if $\sigma$ is odd.

Proposition D.2 (Leibniz formula) (dim $E = n$ and $\mathcal{A} \in \Omega^n(E)$.) Let $(\vec{e}_i)$ be a basis in $E$. Then, for all vectors $\vec{v}_1, \ldots, \vec{v}_n \in E$, with $\vec{v}_j = \sum_{i=1}^{n} v_{ij} \vec{e}_i$ for all $j$, we have,

$$c := \mathcal{A}(\vec{e}_1, \ldots, \vec{e}_n), \quad \mathcal{A}(\vec{v}_1, \ldots, \vec{v}_n) = c \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} v_{i\sigma(i)} = c \sum_{\tau \in S_n} \varepsilon(\tau) \prod_{i=1}^{n} v_{i\tau(i)}.$$  

(D.4)

Thus if $c = \mathcal{A}(\vec{e}_1, \ldots, \vec{e}_n)$ is known, then $\mathcal{A}$ is known. Thus dim$(\Omega^n(E)) = 1$.

Proof. Let $F := \mathcal{F}([1, n][1, n]) = \text{named } [1, n][1, n]$ be the set of functions $i : \{ [1, n][1, n] \}_{k \rightarrow i_k}^{k \rightarrow i_k}$.

$\mathcal{A}$ being multilinear, $\mathcal{A}(\vec{v}_1, \ldots, \vec{v}_n) = \sum_{j=1}^{n} v_{1j}^{1} \mathcal{A}(\vec{e}_1, \vec{v}_2, \ldots, \vec{v}_n)$ ("the first column" development). By recurrence we get $\mathcal{A}(\vec{e}_1, \ldots, \vec{e}_n) = \sum_{j=1}^{n} \sum_{i=1}^{n} v_{ij}^{1} \cdots v_{jn}^{n} \mathcal{A}(\vec{e}_j, \ldots, \vec{e}_n) = \sum_{j \in F} \prod_{k=1}^{n} v_{jk}^{k} \mathcal{A}(\vec{e}_{j(1)}, \ldots, \vec{e}_{j(n)})$.  

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And \( \mathcal{A}(\vec{e}_1, ..., \vec{e}_n) \neq 0 \) iff \( i : k \in \{1, ..., n\} \rightarrow i(k) = i_k \in \{1, ..., n\} \) is one-to-one (thus bijective). Thus \( \mathcal{A}(\vec{v}_1, ..., \vec{v}_n) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} v_{i}^{\sigma(i)} \mathcal{A}(\vec{e}_{\sigma(1)}, ..., \vec{e}_{\sigma(n)}) = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} v_{i}^{\sigma(i)} \mathcal{A}(\vec{e}_1, ..., \vec{e}_n) \), which is the first equality in (D.4). Then \( \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} v_{i}^{\sigma(i)} = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} v_{\sigma^{-1}(i)}^{\sigma^{-1}(i)} \) since \( \sigma \) is bijective, thus \( \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} v_{i}^{\sigma(i)} = \sum_{\tau \in S_n} \epsilon(\tau^{-1}) \prod_{i=1}^{n} v_{i}^{\tau(i)} \). The second equality in (D.4) since \( \epsilon(\tau^{-1}) = \epsilon(\tau) \). (See Cartan [5].)

D.3 Determinant of vectors

**Definition D.3** Let \((\vec{e}_i)_{i=1, ..., n}\) be a basis in \(E\). The alternating multilinear form \(\det|_{\vec{e}} \in \Omega^n(E)\) defined by

\[
\det(\vec{e}_1, ..., \vec{e}_n) = 1 \tag{D.5}
\]

is called the determinant relative to \((\vec{e}_i)\). That is, with prop. D.2, \(\det|_{\vec{e}}\) is defined by

\[
\det(\vec{v}_1, ..., \vec{v}_n) = \sum_{\sigma \in S_n} \epsilon(\sigma) \prod_{i=1}^{n} v_{i}^{\sigma(i)} = \sum_{\tau \in S_n} \epsilon(\tau) \prod_{i=1}^{n} v_{i}^{\tau(i)}, \tag{D.6}
\]

and \(\det|_{\vec{e}}(\vec{v}_1, ..., \vec{v}_n)\) is called the determinant of the \(n\) vectors \(\vec{v}_i\) relative to \((\vec{e}_i)\).

And prop. D.2 tells:

\[
\Omega^n(E) = \text{Vect}\{\det|_{\vec{e}}\} \quad (1\text{-D vector space spanned by } \det|_{\vec{e}}),
\]

and if \(\mathcal{A} \in \Omega^n(E)\) then

\[
\mathcal{A} = \mathcal{A}(\vec{e}_1, ..., \vec{e}_n) \det|_{\vec{e}}. \tag{D.8}
\]

In particular, if \((\vec{b}_i)\) is another basis then

\[
\forall \vec{c} \in \mathbb{R}, \quad \det|_{\vec{b}} = c \det|_{\vec{e}}, \quad \text{with} \quad c = \det|_{\vec{b}}(\vec{c}_1, ..., \vec{c}_n). \tag{D.9}
\]

**Exercise D.4** Change of measuring unit: If \((\vec{a}_i)\) is a basis and \(\vec{b}_j = \lambda \vec{a}_j\) for all \(j\), prove

\[
\forall j = 1, ..., n, \quad \vec{b}_j = \lambda \vec{a}_j \implies \det|_{\vec{b}} = \lambda^n \det|_{\vec{a}}. \tag{D.10}
\]

**Answer.** \(\det|_{\vec{b}}(\vec{b}_1, ..., \vec{b}_n) = \det(\lambda \vec{a}_1, ..., \lambda \vec{a}_n) = \lambda^n \det|_{\vec{a}}(\vec{a}_1, ..., \vec{a}_n) \) \((\text{D.5}) \equiv \lambda^n \det|_{\vec{b}}(\vec{b}_1, ..., \vec{b}_n) \) gives \(\det|_{\vec{b}} = \lambda^n \det|_{\vec{a}}\) (relation between volumes relative to a change of measuring unit).

**Proposition D.5** \(\det|_{\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0\) iff \((\vec{v}_1, ..., \vec{v}_n)\) is a basis, or equivalently, \(\det|_{\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0\) iff \((\vec{v}_1, ..., \vec{v}_n)\) are linearly dependent.

**Proof.** If one of the \(\vec{v}_i\) is \(\vec{0}\) then \(\det|_{\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0\) (multilinearity) and if \(\sum_{i=1}^{n} \epsilon_i \vec{v}_i = 0\) and one of the \(\epsilon_i \neq 0\) and then \(\vec{v}_i\) is a linear combination of the others thus \(\det|_{\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 0\) (since \(\det|_{\vec{e}}\) is alternate). Thus \(\det|_{\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0 \implies \vec{v}_i\) are independent. And if the \(\vec{v}_i\) are independent then \((\vec{v}_1, ..., \vec{v}_n)\) is a basis, thus \(\det|_{\vec{e}}(\vec{v}_1, ..., \vec{v}_n) = 1 \neq 0\), with \(\det|_{\vec{e}} = c \det|_{\vec{e}}\) then \(\det|_{\vec{e}}(\vec{v}_1, ..., \vec{v}_n) \neq 0\).

**Exercise D.6** In \(\mathbb{R}^2\). Let \(\vec{v}_1 = \sum_{i=1}^{2} v_{i}^{1} \vec{e}_i\) and \(\vec{v}_2 = \sum_{j=1}^{2} v_{j}^{2} \vec{e}_j\) (duality notations). Prove:

\[
\det|_{\vec{e}}(\vec{v}_1, \vec{v}_2) = v_{1}^{1} v_{2}^{2} - v_{1}^{2} v_{2}^{1}. \tag{D.11}
\]

**Answer.** Development relative to the first column (linearity used for the first vector \(\vec{v}_1 = v_{1}^{1} \vec{e}_1 + v_{2}^{1} \vec{e}_2\)): \(\det|_{\vec{e}}(\vec{v}_1, \vec{e}_2) = \det|_{\vec{e}}(v_{1}^{1} \vec{e}_1 + v_{2}^{1} \vec{e}_2, \vec{e}_2) = v_{1}^{1} \det|_{\vec{e}}(\vec{e}_1, \vec{e}_2) + v_{2}^{1} \det|_{\vec{e}}(\vec{e}_2, \vec{e}_1)\). Thus (linearity used for the second vector \(\vec{v}_2 = v_{2}^{2} \vec{e}_1 + v_{2}^{2} \vec{e}_2\)): \(\det|_{\vec{e}}(\vec{v}_1, \vec{v}_2) = 0 + v_{1}^{2} v_{2}^{2} \det|_{\vec{e}}(\vec{e}_1, \vec{e}_2) + v_{1}^{2} v_{2}^{1} \det|_{\vec{e}}(\vec{e}_2, \vec{e}_1) + 0 = v_{1}^{1} v_{2}^{2} - v_{1}^{2} v_{2}^{1} .\)
Exercise D.7 In $\mathbb{R}^3$, with $\vec{v}_j = \sum_{i=1}^{n} v^i_j \vec{e}_i$, prove:

$$\det(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} v^1_i v^2_j v^3_k,$$

(D.12)

where $\varepsilon_{ijk} = 1$ if $(i,j,k) = (1,2,3)$, $(3,1,2)$ or $(2,3,1)$ (even signature), $\varepsilon_{ijk} = -1$ if $(i,j,k) = (3,2,1)$, $(1,3,2)$ and $(2,1,3)$ (odd signature), and $\varepsilon_{ijk} = 0$ otherwise.

Answer. Development relative to the first column (as in exercise D.6).

Result: $v^1_1 v^2_2 v^3_3 + v^1_2 v^2_3 v^3_1 - v^1_3 v^2_1 v^3_2 - v^1_2 v^2_1 v^3_3 - v^1_3 v^2_2 v^3_1 - v^1_3 v^2_3 v^3_1$.

D.4 Determinant of a matrix

Let $M = [M_{ij}]_{i=1}^{n} \in \mathbb{R}^{n \times n}$ be a $n^2$ real matrix. Let $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ (Cartesian product $n$-times), and consider its canonical basis $(\vec{A}_i)$. Let $\vec{v}_j \in \mathbb{R}^n$, $\vec{v}_j = \sum_{i=1}^{n} M_{ij} \vec{A}_i$; So $([\vec{v}_1]_{\vec{A}}, ..., [\vec{v}_n]_{\vec{A}}) = [M_{ij}] = M$.

Definition D.8 The determinant of the matrix $M$ is

$$\det(M) := \det(\vec{v}_1, ..., \vec{v}_n).$$

(D.13)

Proposition D.9 Let $M^T$ be the transposed matrix, i.e., $(M^T)_{ij} = M_{ji}$ for all $i,j$. Then

$$\det(M^T) = \det(M).$$

(D.14)

Proof. $\det[M_{ij}] = \det(\vec{v}_1, ..., \vec{v}_n) = \sum_{\sigma \in S_n} \varepsilon(\sigma) \prod_{i=1}^{n} v^\sigma(i) = \sum_{\tau \in S_n} \varepsilon(\tau) \prod_{i=1}^{n} v^\tau(i) = \det[M_{ij}]$.

D.5 Volume

Definition D.10 Let $(\vec{e}_i)$ be a Euclidean basis. The algebraic volume, relative to $(\vec{e}_i)$, of a parallelepiped in $\mathbb{R}^n$ which sides are the vectors $\vec{v}_1, ..., \vec{v}_n$ is

$$\text{algebraic volume} = \det(\vec{v}_1, ..., \vec{v}_n).$$

(D.15)

And its volume, relative to $(\vec{e}_i)$, is (non negative)

$$\text{volume} = |\det(\vec{v}_1, ..., \vec{v}_n)|.$$

(D.16)

E.g., if $n = 1$ and $\vec{v} = v^1 \vec{e}_1$, then $\det_{\vec{e}}(\vec{v}) = |v^1|$ is the algebraic length of $\vec{v}$ (relative to the unit of measurement given by $\vec{e}_1$). And $|\det_{\vec{e}}(\vec{v})| = |v^1|$ is the length of $\vec{v}$ (the norm of $\vec{v}$).

E.g., if $n = 2$ or $3$, see exercises D.6-D.7.

Remark D.11 The volume function $(\vec{v}_1, ..., \vec{v}_n) \rightarrow |\det_{\vec{e}}(\vec{v}_1, ..., \vec{v}_n)|$ is not a multilinear form, because the absolute value function is not linear.

Remark D.12 (Notation.) Let $(\vec{e}_i)$ be a Cartesian basis and $(e^i) = (dx^i)$ be the dual basis. Then, cf. Cartan [6],

$$\det_{\vec{e}} \text{ noted } e^1 \wedge ... \wedge e^n = dx^1 \wedge ... \wedge dx^n.$$  

(D.17)

And for integration

$$d\Omega = |dx^1 \wedge ... \wedge dx^n| \text{ noted } dx^1 ... dx^n$$

(D.18)

is the volume element (non negative).
D.6 Determinant of an endomorphism

D.6.1 Definition and basic properties

**Definition D.13** Let \( L \in \mathcal{L}(E; E) \) be an endomorphism. Let \((\tilde{e}_i)\) be a basis. The determinant of \( L \) relative to the basis \((\tilde{e}_i)\) is

\[
\widetilde{\det}(L) := \det(\tilde{L}e_1, ..., \tilde{L}e_n). \tag{D.19}
\]

This define \( \det_{\tilde{e}} : \mathcal{L}(E; E) \to \mathbb{R} \). (If the context is not ambiguous, then \( \det_{\tilde{e}} = \det_{\tilde{e}} \).

**Proposition D.14** 1- If \( L = I \) the identity, then \( \widetilde{\det}(I) = 1 \) for all basis \((\tilde{e}_i)\).

2- For all \( L \in \mathcal{L}(E; E) \), for all \( \tilde{v}_1, ..., \tilde{v}_n \in E \),

\[
\widetilde{\det}(L, \tilde{v}_1, ..., \tilde{v}_n) = \det(L, \tilde{v}_1, ..., \tilde{v}_n). \tag{D.20}
\]

3- If \( L\tilde{e}_j = \sum_{i=1}^n L_{ij} \tilde{e}_i \), i.e. \([L]_{\tilde{e}} = [L_{ij}]\), then

\[
\widetilde{\det}(L) = \det([L]_{\tilde{e}}) = \det([L_{ij}]). \tag{D.21}
\]

4- For all \( L, M \in \mathcal{L}(E; E) \),

\[
\widetilde{\det}(M \circ L) = \widetilde{\det}(M) \widetilde{\det}(L) = \widetilde{\det}(L \circ M). \tag{D.22}
\]

(And the linearity gives the notation \( M \circ L \).)

5- \( L \) is invertible if \( \widetilde{\det}(L) \neq 0 \).

6- If \( L \) is invertible then

\[
\widetilde{\det}(L^{-1}) = \frac{1}{\widetilde{\det}(L)}. \tag{D.23}
\]

7- If \((\cdot, \cdot)_g\) is a dot product in \( E \) and \( L_g^T \) is the \((\cdot, \cdot)_g\) transposed of \( L \) i.e. \((L_g^T \tilde{w}, \tilde{v})_g = (\tilde{v}, L \tilde{u})_g \) for all \( \tilde{u}, \tilde{v} \in E \) then

\[
\widetilde{\det}(L_g^T) = \widetilde{\det}(L). \tag{D.24}
\]

8- If \((\tilde{e}_i)\) and \((\tilde{b}_i)\) are two \((\cdot, \cdot)_g\)-orthonormal bases in \( \mathbb{R}^n \) (e.g. two Euclidean bases for the same measuring unit), then \( \det_{\tilde{g}} = \pm \det_{\tilde{e}} \) with +. (And the bases are said to have the same orientation iff \( \det_{\tilde{g}} = + \det_{\tilde{e}} \).)

**Proof.** 1- \( \det_{\tilde{e}}(I) = \det_{\tilde{e}}(\tilde{L}e_1, ..., \tilde{L}e_n) = \det_{\tilde{e}}(\tilde{e}_1, ..., \tilde{e}_n) = 1 \), true for all basis.

2- Let \( m : (\tilde{v}_1, ..., \tilde{v}_n) \to m(\tilde{v}_1, ..., \tilde{v}_n) := \det_{\tilde{e}}(L \tilde{v}_1, ..., L \tilde{v}_n) \). It is a multilinear alternated form, since \( L \) is linear; And \( m(\tilde{e}_1, ..., \tilde{e}_n) = \det_{\tilde{e}}(L) \).

3- \( \det_{\tilde{e}}((M_1 \circ M_2) \tilde{e}_1, ..., M_1 \circ M_2 \tilde{e}_n) = \det_{\tilde{e}}(M_1 \tilde{e}_1, ..., M_1 \tilde{e}_n) \).

5- If \( L \) is invertible, then \( \det_{\tilde{e}}(L) = \det_{\tilde{e}}(L^{-1}) = \det_{\tilde{e}}(L) \det_{\tilde{e}}(L^{-1}) \), thus \( \det_{\tilde{e}}(L) \neq 0 \).

6- \( \det_{\tilde{e}}(L_{ij} \tilde{e}_j, ..., L_{ij} \tilde{e}_n) = \det_{\tilde{e}}(M) \).

7- \( \det_{\tilde{e}}(L_{ij}) = \det_{\tilde{e}}(L_{ij}) \).

8- Let \( P \) be the change of basis endomorphism from \((\tilde{e}_i)\) to \((\tilde{b}_i)\), and \( P \) be the transition matrix from \((\tilde{e}_i)\) to \((\tilde{b}_i)\). Both basis being \((\cdot, \cdot)_g\)-orthonormal, \( P^T \circ P = I \), thus \( \det(P) = \pm 1 = \det_{\tilde{g}}(P) \).

**Exercise D.15** Prove \( \widetilde{\det}(L) = \lambda^n \tilde{e} \).

**Answer.** \( \det(\lambda L) = \lambda^n \det_{\tilde{e}}(L) \).

\[
\det(L) = \det(\lambda \tilde{e}_1, ..., \lambda \tilde{e}_n) = \lambda^n \det_{\tilde{e}}(\tilde{e}_1, ..., \tilde{e}_n) = \lambda^n \det_{\tilde{e}}(L). \tag{D.25}
\]
D.6.2 The determinant of an endomorphism is objective

**Proposition D.16** Let \((\vec{a}_i)\) and \((\vec{b}_i)\) be bases in \(E\). The determinant of an endomorphism \(L \in \mathcal{L}(E; E)\) is objective (observer independent, here basis independent):

\[
\det([L]_{\vec{a}}) = \frac{\det(L)}{\det(L)} = \det([L]_{\vec{b}}) = \det([L]_{\vec{b}}).
\]  

(D.25)

NB: But the determinant of \(n\) vectors is not objective, cf. (D.9) (compare the change of basis formula for vectors \([\vec{w}]_{\vec{b}} = P^{-1}[\vec{w}]_{\vec{b}}\) with the change of basis formula for endomorphisms \([L]_{\vec{b}} = P^{-1}[L]_{\vec{b}} P\).

**Proof.** Let \((\vec{a}_i)\) and \((\vec{b}_i)\) be bases in \(E\), and \(P\) be the transition matrix from \((\vec{a}_i)\) to \((\vec{b}_i)\). The change of basis formula \([L]_{\vec{b}} = P^{-1}[L]_{\vec{a}} P\) and (D.22) give \(\det([L]_{\vec{b}}) = \det(P^{-1}) \det([L]_{\vec{a}}) \det(P) = \det([L]_{\vec{a}})\), thus \(\det([L]_{\vec{a}}) = \det([L]_{\vec{b}})\), then (D.21) gives (D.25). \(\blacksquare\)

**Exercise D.17** Let \((\vec{a}_i)\) and \((\vec{b}_i)\) be bases in \(E\), and \(P \in \mathcal{L}(E; E)\) be the change of basis endomorphism from \((\vec{a}_i)\) to \((\vec{b}_i)\) (i.e., \(P \vec{a}_j = \vec{b}_j\) for all \(j\)). Prove

\[
\det([\vec{b}_1, ..., \vec{b}_n]_{\vec{a}}) = \det(P), \quad \text{thus} \quad \det([\vec{b}_1, ..., \vec{b}_n]_{\vec{b}}) = \det(P) \det([\vec{a}_1, ..., \vec{a}_n]_{\vec{b}}) = \det(P) \det([\vec{a}_1, ..., \vec{a}_n]_{\vec{a}}) = \det([\vec{a}_1, ..., \vec{a}_n]_{\vec{a}}),
\]  

(D.26)

**Answer.** \(\det([\vec{v}_1, ..., \vec{v}_n]_{\vec{a}}) = \det([\vec{v}_1, ..., \vec{v}_n]_{\vec{b}})\), thus (D.26), thus \(\det([\vec{v}_1, ..., \vec{v}_n]_{\vec{a}}) = \det([\vec{v}_1, ..., \vec{v}_n]_{\vec{b}})\).

\(\blacksquare\)

D.7 Determinant of a linear map

(Needed for the deformation gradient \(F_{t_0}^t(P) = d\Phi_{t_0}^t(P) : \mathbb{R}_n \to \mathbb{R}_n\).) Let \(A\) and \(B\) be vector spaces, \(\dim A = \dim B = n\). Let \((\vec{a}_i)\) and \((\vec{b}_i)\) be bases in \(A\) and \(B\).

**Definition D.18** The determinant of a linear map \(L \in \mathcal{L}(A; B)\) relative to the bases \((\vec{a}_i)\) and \((\vec{b}_i)\) is

\[
\det(L) := \det([L]_{\vec{a}})_{\vec{b}} := \det(L). \quad \text{(D.27)}
\]

(And \(\det([L]_{\vec{a}, \vec{b}}) = \text{det}(L)\) if the bases are implicit.) That is, if \(L \vec{a}_j = \sum_{i=1}^n L_{ij} \vec{b}_i\), i.e., \([L]_{\vec{a}, \vec{b}} = [L]_{ij}\), then

\[
\det(L) = \det([L]_{\vec{a}, \vec{b}}) = \det([L_{ij}]). \quad \text{(D.28)}
\]

**Proposition D.19** Let \(\vec{u}_1, ..., \vec{u}_n \in A\). Then

\[
\det([\vec{u}_1, ..., \vec{u}_n]_{\vec{b}}) = \det([\vec{u}_1, ..., \vec{u}_n]_{\vec{a}}) \det([\vec{u}_1, ..., \vec{u}_n]_{\vec{a}}). \quad \text{(D.29)}
\]

**Proof.** \(m : (\vec{u}_1, ..., \vec{u}_n) \in A^n \to m(\vec{u}_1, ..., \vec{u}_n) := \det([L]_{\vec{a}})_{\vec{b}} \in \mathbb{R}\) is a multilinear alternated form, since \(L\) is linear; And \(m(\vec{u}_1, ..., \vec{u}_n) = \det([\vec{u}_1, ..., \vec{u}_n]_{\vec{a}}) = \det([L]_{\vec{a}, \vec{b}}) = \det([L]_{\vec{a}})_{\vec{b}} \det([\vec{u}_1, ..., \vec{u}_n]_{\vec{a}}).\)

Thus \(m = \det([L]_{\vec{a}, \vec{b}}) \det([\vec{a}_i]_{\vec{a}})\), cf. (D.9), thus (D.29). \(\blacksquare\)

**Corollary D.20** Let \(A, B, C\) be vector spaces such that \(\dim A = \dim B = \dim C = n\). Let \((\vec{a}_i)\), \((\vec{b}_i)\), \((\vec{c}_i)\) be bases in \(A, B, C\). Let \(L : A \to B\) and \(M : B \to C\) be linear. Then

\[
\det(M \circ L) = \det(L) \det(M). \quad \text{(D.30)}
\]

**Proof.** \(\det(M \circ L) = \det([L]_{\vec{a}})_{\vec{b}} \det([M]_{\vec{b}})_{\vec{a}} = \det([L]_{\vec{a}})_{\vec{b}} \det([M]_{\vec{b}})_{\vec{a}} = \det(L) \det(M). \quad \text{(D.30)}\)
D.7.2 Jacobian of a motion, and dilatation

Let $\tilde{\Phi}$ be a motion, let $t_0, t \in \mathbb{R}$, let $\Phi_t^{t_0}$ be the associated motion, let $F_t^{t_0}(p_0) := d\Phi_t^{t_0}(p_0) : \mathbb{R}^n \to \mathbb{R}^n$ the deformation gradient at $p_0 \in \Omega_{t_0}$ relative to $t_0$ and $t$, cf. (5.2). Let $(\tilde{E}_i)$ be a Euclidean basis in $\mathbb{R}^n_{t_0}$ and $(\tilde{e}_i)$ be a Euclidean basis in $\mathbb{R}^n_{t}$ for all $t \geq t_0$, and $|F_t^{t_0}(p_0)|_{\tilde{E},\tilde{E}} = |F_{ij}(p_0)|$, i.e., $F_t^{t_0}(p_0) : \tilde{E}_j = \sum_i F_{ij}(p_0) \tilde{e}_i$ for all $J$ (we use Marsden–Hughes duality notations).

Definition D.21 The “volume dilatation” at $p_0$, relative to the Euclidean bases $(\tilde{E}_i)$ in $\mathbb{R}^n_{t_0}$ and $(\tilde{e}_i)$ in $\mathbb{R}^n_{t}$, is

$$J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) := \det(F_t^{t_0}(p_0)) = \det(F_{ij}(p_0)). \tilde{E}_1, ..., F_{ij}(p_0). \tilde{E}_n = \det(|F_{ij}|)). \quad (D.31)$$

More precisely, $(\tilde{E}_1, ..., \tilde{E}_n)$ being the unit parallelepiped at $p_0$ at $t_0$ for an observer $A$ relative to the unit of measurement he chose, $J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) = \det_{\tilde{E}}(F_t^{t_0}(p_0). \tilde{E}_1, ..., F_{ij}(p_0). \tilde{E}_n)$ is the volume of the parallelepiped $(F_t^{t_0}(p_0). \tilde{E}_1, ..., F_{ij}(p_0). \tilde{E}_n)$ at $p_t = \Phi_t^{t_0}(p_0)$ at $t$ for an observer $B$ relative to the unit of measurement he chose. The purpose is to see the evolution of this volume as seen by observer $B$:

- Dilatation if $J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) > 1$ (volume increase),
- contraction if $J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) < 1$ (volume decrease), and
- incompressibility if $J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) = 1$ for all $t$ (volume conservation).

In particular, if $(\tilde{e}_i) = (\tilde{E}_i)$ then $J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) = 1$, and then

- Dilatation if $J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) > 1$ (volume increase),
- contraction if $J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) < 1$ (volume decrease), and
- incompressibility if $J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) = 1$ for all $t$ (volume conservation).

$(J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) > 0$ since $\Phi_t^{t_0}$ is supposed regular and $F_t^{t_0}(p_0) = I_{t_0}$ gives $J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) = 1 > 0$.)

Notation: If $t_0$ and $t$ and $p_0$ are implicit then $F_t^{t_0}(p_0)$ is named $F$, and if $(\tilde{e}_i) = (\tilde{E}_i)$ and is implicit then,

$$J_{\tilde{E},\tilde{e}}(\Phi_t^{t_0})(p_0) := J = \det(F). \quad (D.32)$$

Exercise D.22 Let $(\tilde{E}_i)$ be a Euclidean basis in $\mathbb{R}^n_{t_0}$ and let $(\tilde{a}_i)$ and $(\tilde{b}_i)$ be two Euclidean bases in $\mathbb{R}^n_{t}$ for the same Euclidean dot product $(\cdot, \cdot)$. Prove: $J_{\tilde{E},\tilde{a}}(\Phi_t^{t_0}(P)) = \pm J_{\tilde{E},\tilde{b}}(\Phi_t^{t_0}(P)).$

Answer. The bases being both $(\cdot, \cdot)$-orthonormal and $P$ being the transition matrix from $(\tilde{a}_i)$ to $(\tilde{b}_i)$, det($P$) = ± 1. And (5.34) gives $|F|_{\tilde{E},\tilde{a}} = P.|F|_{\tilde{E},\tilde{b}}$, thus det($(|F|_{\tilde{E},\tilde{a}}) = \pm \det(|F|_{\tilde{E},\tilde{b}})$, thus det$_{\tilde{a}}(F.\tilde{E}_1, ..., F.\tilde{E}_n) = \pm \det_{\tilde{b}}(F.\tilde{E}_1, ..., F.\tilde{E}_n)$. \hfill \black三角

D.7.3 Determinant of the transposed

Let $(E, (\cdot, \cdot)_E)$ and $(F, (\cdot, \cdot)_F)$ be finite dimensional Hilbert spaces. Let $L \in \mathcal{L}(E; F)$ (a linear map). The transposed $L^T_{gh} \in \mathcal{L}(F; E)$ is defined by, for all $\tilde{u} \in E$ and all $\tilde{w} \in F$, cf. (A.24)

$$(L^T_{gh}, \tilde{w}, \tilde{u}) := (\tilde{w}, L.\tilde{u})_h. \quad (D.33)$$

Let $(\tilde{a}_i)$ be a basis in $E$ and $(\tilde{b}_i)$ be a basis in $F$. Then

$$\tilde{\text{det}}(|L^T_{gh}|_{\tilde{E},\tilde{E}}) = \det(|L|_{\tilde{E},\tilde{E}}) \frac{\det(|(\cdot, \cdot)_E|)}{\det(|(\cdot, \cdot)_F|)}. \quad (D.34)$$

Indeed, (D.33) gives $|(\cdot, \cdot)_g|_{\tilde{a}}. |L^T_{gh}|_{\tilde{E},\tilde{E}} = (|(\cdot, \cdot)_E|)^T. |(\cdot, \cdot)_h|_{\tilde{E}}$.

In particular, if $L$ is an endomorphism and $(\tilde{a}_i) = (\tilde{b}_i)$ and $(\cdot, \cdot)_g = (\cdot, \cdot)_h$, then we recover (D.24).

D.8 Dilatation rate

A unique Euclidean basis $(\tilde{e}_i)$ at all time is chosen.
D.8.1 \( \frac{\partial J^0}{\partial t}(t, P) = J^0(t, P) \text{div}\vec{v}(t, p_t) \)

A motion \( \tilde{\Phi} \) is considered, cf. (1.5); \( t_0 \) is given, the associated motion \( \Phi^t \) is given by (3.1), is supposed to be at least \( C^2 \), the Lagrangian velocity is \( \tilde{V}(t, P) = \frac{\partial \Phi^t}{\partial x}(t, P) \), the Eulerian velocity is \( \vec{v}(t, p_t) = \frac{\partial \Phi^t}{\partial P}(t, P) \) when \( p_t = \Phi^t(t, P) \), cf. (4.13). Let \( F^0(t, P) = d\Phi^t(t, P) = F^0_t(P) = \Phi^t_t(P) \), and consider the Jacobian

\[
J^0_t(P) = \det(F^0_t(P)) = J^0(t, P),
\]

(D.35)

Lemma D.23 \( \frac{\partial J^0}{\partial t}(t, P) \) satisfies, with \( p = \Phi^t_t(P) \),

\[
\frac{\partial J^0}{\partial t}(t, P) = J^0(t, P) \text{div}\vec{v}(t, p_t).
\]

(D.36)

In particular, \( \tilde{\Phi} \) is incompressible iff \( \text{div}\vec{v}(t, p_t) = 0 \).

Proof. Let \( \mathcal{O} \) be a origin in \( \mathbb{R}^n \). Let \( \Phi = \Phi^t \). Let \( \Phi = \sum^n_{i=1} \Phi^i \vec{e}_i, \vec{V} = \sum^n_{i=1} V^i \vec{e}_i, \vec{v} = \sum^n_{i=1} v^i \vec{e}_i, \)

\[
F = d\Phi = \sum^n_{i=1} \vec{e}_i \otimes d\Phi^i.
\]

Thus \( J = \det F = \det \left( \begin{array}{c}
\frac{\partial(d\Phi^1)}{\partial x_1} \\
\vdots \\
\frac{\partial(d\Phi^n)}{\partial x_n}
\end{array} \right) \), thus (a determinant is multilinear)

\[
\frac{\partial J}{\partial t} = \det \left( \begin{array}{c}
\frac{\partial(d\Phi^1)}{\partial x_1} \\
\vdots \\
\frac{\partial(d\Phi^n)}{\partial x_n}
\end{array} \right) + \ldots + \det \left( \begin{array}{c}
\frac{\partial(d\Phi^1)}{\partial x_1} \\
\vdots \\
\frac{\partial(d\Phi^n)}{\partial x_n}
\end{array} \right).
\]

And \( \Phi \) being \( C^2 \), Schwarz theorem gives \( \frac{\partial(d\Phi^i)}{\partial x}(t, P) = d\frac{\partial\Phi^i}{\partial x}(t, P) = dV^i(t, P) \) and \( p = \Phi(t, P) \) and \( V^1(t, P) = v^1(t, p) \) and \( dv^i(t, p) = \sum^n_{i=1} \frac{\partial v^i}{\partial x} \vec{e}_i \) give \( dV^1(t, P) = dv^1(t, p).F(t, P) = \sum^n_{i=1} \frac{\partial v^i}{\partial x} \Phi^i(t, P) \).

Thus \( \frac{\partial(d\Phi^i)}{\partial x}(t, P) = \sum^n_{i=1} \frac{\partial v^i}{\partial x} \Phi^i(t, P) \), written \( \sum^n_{i=1} \frac{\partial v^i}{\partial x} \Phi^i(t, P) \), thus

\[
\det \left( \begin{array}{c}
\frac{\partial(d\Phi^1)}{\partial x_1} \\
\vdots \\
\frac{\partial(d\Phi^n)}{\partial x_n}
\end{array} \right) = \det \left( \begin{array}{c}
\frac{\partial v^1}{\partial x^1} \Phi^1 \\
\vdots \\
\frac{\partial v^n}{\partial x^n} \Phi^n
\end{array} \right) = \det \left( \begin{array}{c}
\frac{\partial v^1}{\partial x^1} \\
\vdots \\
\frac{\partial v^n}{\partial x^n}
\end{array} \right) = \frac{\partial v^1}{\partial x^1} \frac{\partial v^n}{\partial x^n} \frac{\partial J}{\partial x^1}.
\]

Ident for the other terms, thus

\[
\frac{\partial J}{\partial t}(t, P) = \frac{\partial v^1}{\partial x^1}(t, P) J(t, P) + \ldots + \frac{\partial v^n}{\partial x^n}(t, P) J(t, P) = \text{div}\vec{v}(t, p_t) J(t, P),
\]

i.e. (D.36).

Definition D.24 \( \text{div}\vec{v}(t, p_t) \) is the dilatation rate.

D.8.2 Leibniz formula

Proposition D.25 (Leibniz formula) Under usual regularity assumptions (e.g. hypotheses of the Lebesgue theorem to be able to derive under \( \int \)) we have

\[
\frac{d}{dt} \left( \int_{t_0} f(t, p_t) \, d\Omega_t \right) = \int_{t_0} \left( \frac{Df}{dt} + df \text{div}\vec{v} \right)(t, p_t) \, d\Omega_t
\]

\[
= \int_{t_0} \left( \frac{\partial f}{\partial t} + df.\vec{x} + f \text{div}(\vec{v})(t, p_t) \right) \, d\Omega_t
\]

\[
= \int_{t_0} \left( \frac{\partial f}{\partial t} + \text{div}(f\vec{v})(t, p_t) \right) \, d\Omega_t.
\]

(D.37)
Proof. Let
\[ Z(t) := \int_{p \in \Omega_t} f(t, p) \, d\Omega_t = \int_{p \in \Omega_0} f(t, \Phi^0(t, P)) \, J^o(t, P) \, d\Omega_0. \]
(The Jacobian is positive for a regular motion.) Then (derivation under \(f\))
\[ Z'(t) \overset{\text{noted}}{=} \frac{d}{dt} \left( \int_{p \in \Omega_t} f(t, p) \, d\Omega_t \right) = \int_{p \in \Omega_0} \frac{Df}{Dt}(t, p_t) \, J^o(t, P) + f(t, p_t) \frac{\partial J^o}{\partial t}(t, P) \, d\Omega_0 \]
\[ = \int_{p \in \Omega_0} \left( \frac{Df}{Dt}(t, p_t) + f(t, p_t) \, \text{div}(\bar{v})(t, p_t) \right) \, J^o(t, P) \, d\Omega_0, \]
thanks to (D.36). And \(\text{div}(f \bar{v}) = df \bar{v} + f \, \text{div} \bar{v}\) gives (D.37).

Corollary D.26

\[ \frac{d}{dt} \int_{\Omega_t} f(t, p_t) \, d\Omega_t = \int_{\Omega_t} \frac{\partial f}{\partial t}(t, p_t) \, d\Omega_t + \int_{\partial \Omega_t} (f \bar{v} \cdot \bar{n})(t, p_t) \, d\Gamma_t, \quad (D.38) \]

sum of the temporal variation within \(\Omega_t\) and the flux through the surface \(\partial \Omega_t\).

Proof. Apply (D.37) \(\_3\).

D.9 \(\partial J/\partial F = J F^{-T}\)

D.9.1 Meaning of \(\frac{\partial}{\partial M_{ij}}\) for linear maps?

Setting: \(E\) and \(F\) are Banach spaces, consider a \(C^1\) function
\[ Z : \left\{ \begin{array}{l} \mathcal{L}(E; F) \rightarrow \mathbb{R} \\ L \rightarrow Z(L). \end{array} \right. \]
(e.g., \(E = F\) and \(Z(L) = (\text{Tr}(L))^2\)). The differential is given at \(L\) in a direction \(M\) by \(dZ(L)(M) = \lim_{h \rightarrow 0} \frac{Z(L + hM) - Z(L)}{h}\).

Let \(L \in \mathcal{L}(E; F)\), let \((\bar{E}_i)\) and \((\bar{e}_i)\) be bases in \(E\) and \(F\), and \((\pi_{ei})\) and \((\pi_{\bar{E}i})\) be the dual bases, name \(L_{ij} = e^j \cdot L \cdot \bar{E}_i\), i.e. \(L \cdot \bar{E}_j = \sum_{i=1}^n L_{ij} \bar{e}_i\) for all \(j\), i.e. \(L_{ij} = [L]_{\bar{E} \bar{E}} = [L_{ij}]\). Let \((\bar{e}_i \otimes \pi_{Ej})\) be the associated basis in \(\mathcal{L}(E; F)\) (defined by \((\bar{e}_i \otimes \pi_{Ej}) \cdot \bar{E}_k = \delta_{jk} \bar{e}_i\) for all \(k\)). Then, by (usual) definition,
\[ \frac{\partial Z}{\partial L_{ij}}(L) := \lim_{h \rightarrow 0} \frac{Z(L + h \bar{e}_i \otimes \pi_{Ej}) - Z(L)}{h}. \]

Matrix point of view: Let \(M_{nm}\) be the spaces of \(nm\) matrices, and define \(\bar{Z} : M_{nm} \rightarrow \mathbb{R}\) by
\[ \bar{Z}([L]_{\bar{E} \bar{E}}) = Z(L). \]
Let \([m_{ij}]_{i,j=1,2,\ldots,n}\) be the canonical basis of \(M_{nm}\), i.e. \([m_{ij}]\) is defined by: all its elements vanish but the element at intersection of row \(i\) and line \(j\) which equals 1. Thus, \([M] = [m_{ij}] \in M_{nm} \text{ iff } [M] = \sum_{i,j=1}^n M_{ij} [m_{ij}]\), and
\[ \frac{\partial \bar{Z}}{\partial M_{ij}}([M]) := d \bar{Z}(\{[M]\}, [m_{ij}]) = \lim_{h \rightarrow 0} \frac{\bar{Z}([M] + h [m_{ij}]) - \bar{Z}([M])}{h}, \quad (D.41) \]
and \(\frac{\partial Z}{\partial M_{ij}}(L) := \frac{\partial \bar{Z}}{\partial M_{ij}}([L]_{\bar{E} \bar{E}})\).

D.9.2 Meaning of \(\partial J/\partial F\)?

Let \(\Phi\) be a motion, let \(t_0, t \in \mathbb{R}\), let \(\Phi = \Phi^0\) be the association motion, let \(F := df\), let \(p_{t_0} \in \Omega_{t_0}\), let \(F_{p_{t_0}} := F(p_{t_0}) = df(p_{t_0}) \in \mathcal{L}([E^n_{t_0}; E^n_{t}]), \) let \(p_t = \Phi^0(t_0) \in \Omega_t\). Let \((\bar{E}_i)\) and \((\bar{e}_i)\) be bases in \(E^n_{t_0}\) and \(E^n_t\), let \(o \in \mathbb{R}^n\) be an origin, let \(\bar{\phi} = o \Phi(p_{t_0}) = \sum_{i=1}^n \Phi_i(p_{t_0}) \bar{e}_i\) (we use classical index notations to remove any
possible ambiguity; Thus \( F_{p_0}, \vec{E}_j = \sum_{i=1}^{n} F_{ij}(p_0)\vec{e}_i \) where \( F_{ij} = \frac{\partial x_i}{\partial X_j} \) for all \( i, j \). With (D.27), consider the Jacobian function

\[
J_{\Phi, E, \varepsilon} \overset{\text{noted}}{=} J_{\Phi} := \begin{cases} 
\Omega_{t_0} \to \mathbb{R} \\
p_0 \to J_{\Phi}(p_0) = -\det(F_{p_0}) = \det([F_{ij}(p_0)]) = \det([\frac{\partial \Phi}{\partial X_j}(p_0)]) 
\end{cases}
\]  

(D.42)

(the only derivations into play are along the directions \( \vec{E}_j \) in \( \mathbb{R}^n \); \( \frac{\partial \Phi}{\partial X_j}(p_0) = d\Phi_i(p_0) . \vec{E}_j \)). Then define the function

\[
\zeta_{E, \varepsilon} \overset{\text{noted}}{=} \zeta : C^1(\Omega_{t_0}; \Omega_{t_0}) \to C(\Omega_{t_0}; \mathbb{R}) \\
\Phi \to \zeta(\Phi) := \det([\frac{\partial \Phi}{\partial X_j}]) = \det([F_{ij}]), \text{ so } \zeta(\Phi)(p_0) := J_{\Phi}(p_0) \quad \forall p_0 \in \Omega_{t_0}.
\]

And let

\[
Z_{E, \varepsilon}(F) \overset{\text{noted}}{=} Z(F) := \det([F_{ij}]), \quad \text{so } Z(F)(p_0) := J_{\Phi}(p_0) \quad \forall p_0 \in \Omega_{t_0}.
\]

Proposition D.27

\[
\frac{\partial Z}{\partial F} = Z[F]^{-T}, \text{ usually written } \frac{\partial J}{\partial F} = J[F]^{-T}.
\]

(WARNING: \( Z \) is a function of \( F \), while \( J \) is a function of \( p_0 \), both depending on \((\vec{E}_j)\) and \((\vec{e}_i)\).)

Proof. \( F_{p_0} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \) is identified with the bi-point tensor \( \vec{F}_{p_0} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}) \); Let \((\pi_{E_i})\) be the dual basis of \((\vec{E}_j)\). Thus \( \vec{F} = \sum_{j=1}^{n} F_{ij}\vec{e}_i \otimes \pi_{E_j} \) and \( Z(F) = \det([F]) = \det([\vec{F}]) = \det([F_{ij}]) \). And \( \frac{\partial Z}{\partial F_j} := \lim_{h \to 0} \frac{\det([\vec{F} + h\vec{e}_j \otimes \pi_{E_j}]) - \det([F])}{h} \); The development of the determinant relative to the column \( j \) gives \( \det([\vec{F} + h\vec{e}_j \otimes \pi_{E_j}]) = \det([\vec{F}]) + h c_{ij} \) with \( c_{ij} \) the cofactor of \([F] := [F]_{i,j} \vec{e}_i \otimes \pi_{E_j} \). And \( [F]^{-1} = \frac{1}{\det([F])} [\vec{e}_i]^T \), thus \( \frac{\partial Z}{\partial F_j} = c_{ij} = \det([F]) ([F]^{-1})_{i,j} \), which is the meaning of (D.46). \( \blacklozenge \)

Remark D.28. The meaning of the derivation \( \frac{\partial J}{\partial X_j} \) is not obvious (in fact, is mysterious): It is a derivation in both directions \( \pi_{E_j} \) (in the past at \( t_0 \)) and \( \vec{e}_i \) (at \( t \)), cf. (D.40), when \( F \) is itself the result of a derivation in \( \mathbb{R}^n \). May be we should be content with \( J : p_0 \to J(p_0) = \det|_{E, \varepsilon}(F(p_0)) = \det|_{E}(F(p_0), \vec{E}_1, \ldots, F(p_0), \vec{E}_n) \), and the differential \( dJ(p_0) \) and its components

\[
\frac{\partial J}{\partial X_j}(p_0) = dJ(p_0).\vec{E}_j = \lim_{h \to 0} \frac{J(p_0 + h\vec{E}_j) - J(p_0)}{h} = \lim_{h \to 0} \frac{\det(F(p_0 + h\vec{E}_j), \vec{E}_1, \ldots, F(p_0 + h\vec{E}_j), \vec{E}_n) - \det(F(p_0), \vec{E}_1, \ldots, F(p_0), \vec{E}_n)}{h}.
\]

(Results of past deformations observed at \( t_0 \).) \( \blacklozenge \)

E Cauchy–Green deformation tensor \( C \)

The construction (definition) of the Cauchy–Green deformation tensor \( C := F^T \circ F \) is detailed, starting with the essential definition of the transposed \( F^T \).

E.1 Introduction and remarks

The “stress” may be measured using the Lie derivative of one Eulerian “force vector”. However, the Lie derivative did not exist during Cauchy’s lifetime, and Cauchy proposed to measure the stress starting with two vectors: 1. Consider two vectors \( \vec{W}_1 \) and \( \vec{W}_2 \) at \( t_0 \), 2. Consider \( F\vec{W}_1 \) and \( F\vec{W}_2 \) (the deformed by the motion at \( t \)), 3. Consider the inner dot product \( (\cdot, \cdot)_G \) used by the observer who made the measurements at \( t_0 \), 4. Choose an inner dot product \( (\cdot, \cdot)_G \) to make your measurements at \( t \), 5. You get \((F\vec{W}_1, F\vec{W}_2)_g = ((F^T \circ F)\vec{W}_1, \vec{W}_2)_G \) (by definition of \( F^T \)): The Cauchy–Green deformation tensor \( C := F^T \circ F \) is born (Cauchy strain tensor).
E.2 Transposed $F^T$

E.2.1 Framework

Consider a motion $\Phi: \mathbb{R} \times \text{Obj} \to \mathbb{R}^n$, cf. (1.5), and let $\Omega_t := \Phi(t, \text{Obj})$. Let $t_0, t \in \mathbb{R}$ be fixed, and let $\Phi := \Phi_t^{t_0}: \Omega_t \to \Omega_t$ be the associated motion, cf. (3.5), that is, $p_t = \Phi(p_{t_0}) = \Phi(t, \text{Obj})$ is the position at $t$ of the particle $\text{Obj}$ which was at $t_0$ at $p_{t_0} = \Phi(t_0, \text{Obj})$.

Suppose $\Phi$ is $C^1$, and for a fixed $p_{t_0} \in \Omega_{t_0}$, name $F := F^0_t(p_{t_0}) := \Phi^0_t(p_{t_0})$ be the deformation gradient between $t_0$ and $t$ at $p_{t_0}$, cf. (5.2).

E.2.2 Definition of $F^T$: Inner dot products required

Consider the inner dot product $(\cdot, \cdot)_G$ in $\mathbb{R}^n_{t_0}$ chosen by the observer who made the measurements at $t_0$ (in the past), and choose an inner dot product $(\cdot, \cdot)_g$ in $\mathbb{R}^n_t$ to make the measurements at $t$. Then the transposed of the linear map $F^0_t(p_{t_0}) \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_t)$ can be considered, cf. (A.24): It is the map $F^T_G(p_t)$ defined at $p_t = \Phi(p_{t_0}) \in \Omega_t$ by

$$F^T_G(p_t) := (F^0_t(p_{t_0}))^T \in \mathcal{L}(\mathbb{R}^n_t; \mathbb{R}^n_{t_0}).$$

That is, it is characterized by, for all $\tilde{u}_{t_0}(p_{t_0}) \in \mathbb{R}^n_{t_0}$ vector at $t_0$ at $p_{t_0}$ and all $\tilde{w}_t(p_t) \in \mathbb{R}^n_t$ vector at $t$ at $p_t = \Phi(p_{t_0})$:

$$(F^T_G(p_t) \cdot \tilde{u}_{t_0}(p_{t_0}), \tilde{w}_t(p_t))_{G} = (\tilde{w}_t(p_t), F^0_t(p_{t_0}) \cdot \tilde{u}_{t_0}(p_{t_0}))_g,$$

in short

$$F^T_G \cdot \tilde{u}_{t_0}(p_{t_0}), \tilde{w}_t(p_t))_G = (\tilde{w}_t, F \cdot \tilde{u}_{t_0})_g,$$

also written $(F^T \cdot \tilde{w})(p_t, \tilde{u}_{t_0}(p_{t_0}))_G = (\tilde{w}_t, F \cdot \tilde{u}_{t_0})_g$.

**Full definition** $T_{p_{t_0}}(\Omega_{t_0})$ being the tangent space at $p_{t_0} \in \Omega_{t_0}$ (called $\mathbb{R}^n_{t_0}$ above), and $T_{p_t}(\Omega_t)$ being the tangent space at $p_t = \Phi^{t_0}(p_{t_0})$ (called $\mathbb{R}^n_t$ above), and $(\cdot, \cdot)_G$ and $(\cdot, \cdot)_g$ being inner dot products in $T_{p_{t_0}}(\Omega_{t_0})$ and $T_{p_t}(\Omega_t)$, the transposed of $F^0_t(p_{t_0}) \in \mathcal{L}(T_{p_{t_0}}(\Omega_{t_0}); T_{p_t}(\Omega_t))$ relative to $(\cdot, \cdot)_G$ and $(\cdot, \cdot)_g$ is the linear map $F^0_t(p_{t_0})^T = \text{noted} (F^0_t(p_{t_0}))^T \in \mathcal{L}(T_{p_{t_0}}(\Omega_{t_0}); T_{p_t}(\Omega_t))$ characterized by (E.2) for all $\tilde{w}_t(p_t) \in T_{p_t}(\Omega_t)$ (vector at $t$ at $p_t$) and all $\tilde{u}_{t_0}(p_{t_0}) \in T_{p_{t_0}}(\Omega_{t_0})$ (vector at $t_0$ at $p_{t_0}$).

And, relative to $(\cdot, \cdot)_G$ and $(\cdot, \cdot)_g$, we have thus defined the field of linear maps

$$(F^0_t)^T_G: \begin{cases} 
\Omega_t &\to \mathcal{L}(T_{\Omega_{t_0}}; T_{\Omega_{t}}) \\
 p_t &\to (F^0_t)^T_G(p_t).
\end{cases}$$

**Remark E.1** Dangerous notation (source of confusions and errors): $F^T_{\tilde{z}} \tilde{W} = \tilde{z} \cdot F \cdot \tilde{W} = F \cdot \tilde{W} \cdot \tilde{z} = \tilde{W} \cdot F^T \cdot \tilde{z}$. Which dots are inner dot products? E.g.: What does $F \cdot \tilde{W}_1 \cdot F \cdot \tilde{W}_2 = \tilde{W}_1 \cdot F^T \cdot F \cdot \tilde{W}_2$ mean?

Answer: It means (there is no choice) $\tilde{W}_1, \tilde{W}_2 \in \mathbb{R}^n_{t_0}$ and $(F(\tilde{W}_1), F(\tilde{W}_2))_G = (\tilde{W}_1, F^T(F(\tilde{W}_2)))_G$, or if you prefer $(F(\tilde{W}_1))_g (F(\tilde{W}_2))_g = \tilde{W}_1 \cdot F^T_\cdot \tilde{W}_2$.

E.2.3 Quantification with bases (matrix representation)

We use Marsden–Hughes duality notations which are duality notations moreover with upper-case letters for the past, to emphasize that past and present should not be confused. And we recall the classical notations. Let $(\tilde{e}_i)$ be a basis in $\mathbb{R}^n_{t_0}$ chosen by an observer in the past, and $(\tilde{e}_i)$ be a basis in $\mathbb{R}^n_t$ chosen by an observer at $t$ (Marsden and Hughes notations). Classical notations: $(\tilde{a}_i) = (\tilde{e}_i)$ and $(\tilde{b}_i) = (\tilde{e}_i)$.

Let

$$\begin{align*}
G(\tilde{e}_i, \tilde{e}_j) &= G_{ij}, \quad \text{i.e.} \quad [G]_{\tilde{e}, \tilde{e}} = [G]_{ij} \quad \text{notated} [G],
\end{align*}$$

$$g_{ij} = g(\tilde{e}_i, \tilde{e}_j), \quad \text{i.e.} \quad [g]_{\tilde{e}, \tilde{e}} = [g]_{ij} \quad \text{notated} [g].$$

Classical notations $G(\tilde{a}_i, \tilde{a}_j) = G_{ij}$ and $g_{ij} = g(\tilde{a}_i, \tilde{a}_j)$. And, with shortened notations,

$$\begin{align*}
F.\tilde{e}_j &= \sum_{i=1}^n F^i_j \tilde{e}_i, \quad \text{i.e.} \quad [F]_{\tilde{e}, \tilde{e}} = [F^i_j] \quad \text{notated} [F],
\end{align*}$$

$$\begin{align*}
F^T \cdot \tilde{e}_j &= \sum_{i=1}^n (F^T)^i_j \tilde{e}_i, \quad \text{i.e.} \quad [F^T]_{\tilde{e}, \tilde{e}} = [(F^T)^i_j] \quad \text{notated} [F^T];
\end{align*}$$

Classical notations $F(\tilde{a}_j) = \sum_{i=1}^n F_{ij} \tilde{b}_i$ and $F^T(\tilde{b}_j) = \sum_{i=1}^n (F^T)^{ij} \tilde{a}_i$ and $[F] = [F_{ij}]$ and $[F^T] = [(F^T)^{ij}]$. 

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Proposition E.2 We have

\[ [F]_E := [(F^a)_E](p_i)_E^T \cdot \varepsilon, \quad i.e. \quad [(F^a)_E(p_i)]_E^T \cdot \varepsilon = [F]_E^{-1} \cdot [(F(p_i)]_E^T \cdot \varepsilon, \quad (E.6) \]

written in short

\[ [F]_E[F^T] = [F]^T[\varepsilon], \quad i.e. \quad [(F^T)]_E = [F]^{-1}_E \cdot [(F^T)]_E \cdot \varepsilon. \quad (E.7) \]

that is, for all \( I, j = 1, \ldots, n \),

\[ G_{ik}(F^T)_{k-j} = \sum_{k=1}^n F_{ik} g_{kj}, \quad i.e. \quad (F^T)_{ij} = \sum_{k,l=1}^n G_{ik} F_{kj} g_{lj} \quad \text{when} \quad [G]^{IJ} := [G]^{-1}. \quad (E.8) \]

Classical notations (to avoid confusions with the notation \( G^{IJ} \)): For all \( i, j = 1, \ldots, n \),

\[ \sum_{k=1}^n G_{ik}(F^T)_{k-j} = \sum_{k=1}^n F_{ik} g_{kj}, \quad i.e. \quad (F^T)_{ij} = \sum_{k,l=1}^n [(G^{-1})_{ik} F_{lj} g_{kj}. \quad (E.9) \]

In particular, if \( (\vec{E}_1) \) is \((\cdot, \cdot)_G \)-orthonormal and \( (\vec{E}_1) \) is \((\cdot, \cdot)_G \)-orthonormal, then \( [F^T] = [F]^T \), that is, \( (F^T)_{ij} = F_{ij} \) for all \( I, j = 1, n \) (classical notations: \( (F^T)_{ij} = F_{ji} \) for all \( i, j = 1, n \)).

Proof. (E.2) gives \( [\vec{W}]^T . [G] . [F^T] = [F \vec{W}]^T . [G] . [\vec{W}] \) for all \( \vec{W} \), \( \vec{Z} \), thus \( [G] . [F^T] = [F]^T . [\varepsilon] \), i.e. (E.7).

E.2.4 Remark: \( F^* \)

The adjoint of a linear map \( F \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m) \) (defined on vectors) is the linear map \( F^* \in \mathcal{L}(\mathbb{R}^m; \mathbb{R}^n) \) defined on linear forms at (A.98): For all \( m \in \mathbb{R}^m \) and all vectors \( \vec{W} \in \mathbb{R}^n \),

\[ F^*(m) := m \circ F \in \mathbb{R}^n, \quad (E.10) \]

also written \( F^* . m = m . F \) thanks to linearity. I.e., for all linear forms \( m \) on \( \mathbb{R}^n \) and all vectors \( \vec{W} \) in \( \mathbb{R}^n \),

\[ F^*(m)(\vec{W}) = m(F(\vec{W})), \quad (E.11) \]

also written \( (F^* . m)\vec{W} = m . F \vec{W} \) thanks to linearity.

NB: This definition does not need any inner dot product.

Matrix representation (quantification), relative to bases \( (\vec{a}_i) \) and \( (\vec{b}_i) \) in \( \mathbb{R}^n_{\vec{a}} \) and \( \mathbb{R}^m_{\vec{b}} \) and their dual bases \( (\pi_{ai}) \) and \( (\pi_{bi}) \) in \( \mathbb{R}^n^*_{\vec{a}} \) and \( \mathbb{R}^m^*_{\vec{b}} \): Let \( (F^*_i)_{ij} \) the components of \( F^* \):

\[ F^*_{ij} = \sum_{i=1}^n (F^*_{ij})_{pi} \pi_{ai}, \quad i.e. \quad [F^*_i]_{a} = [(F^*)_{ij}]_a. \quad (E.12) \]

We have, cf (A.103),

\[ [F^*_i]_{a} = [(F^*_i)]_{a} = [(F^*_{ij})_i]_a, \quad i.e. \quad [(F^*)_{ij}]_a = [F_{ji}], \quad (E.13) \]

Duality notations: \( F^* \cdot e^j = \sum_{i=1}^n F^*_{ij} e^j, \) and \( (F^*_i)_{ij} = [(F^*_i)]_e = [(F^*)_{ij}]_e, \) and \( [(F^*)_{ij}]_e = [F^*_j]. \)

E.3 Cauchy–Green deformation tensor

E.3.1 Definition

Let \( P \in \Omega_\varepsilon \) and \( F_{P} = F(P) := d\Phi^b_\varepsilon(P) \). Let \( i = 1, \ldots, n \), let \( \vec{W}_i = (\vec{W}_i)^T \) be vectors at \( P \in \Omega_\varepsilon \), and consider their pull-forward \( \vec{w}_i = \vec{w}_i(p) \) by \( \Phi = \Phi^b_\varepsilon \) at \( p = \Phi(P) \), i.e.,

\[ \vec{w}_i = F_{\vec{P}}(\vec{W}_i) = F_{\vec{P}}(\vec{W}_i), \quad \text{in short} \quad \vec{w}_i = F_{\vec{P}}(\vec{W}_i), \quad (E.14) \]

the dot notation \( F_{\vec{P}}.\vec{W}_i := F_{\vec{P}}(\vec{W}_i) \) thanks to the linearity of \( F_{\vec{P}} \).
If $(\cdot, \cdot)_G$ is a Euclidean dot product in $\mathbb{R}^n$ (used by an observer in the past), and if $(\cdot, \cdot)_g$ is a Euclidean dot product in $\mathbb{R}^n_t$ (used by an observer at $t$), then (E.14) and (E.2) (definition of the transposed) give
\[
\langle \vec{w}_{1p}, \vec{w}_{2p} \rangle_g = (F_P(\vec{W}_{1p}), F_P(\vec{W}_{2p}))_g = ((F^T_{Gg,p} \circ F_P)(\vec{W}_{1p}), \vec{W}_{2p})_G. \tag{E.15}
\]
And, by linearity, \( F^T_{Gg,p} \circ F_P \) noted \( F^T_{Gg,p} \) (this is not a matrix product: No bases have been introduced yet). In short,
\[
\langle \vec{w}_1, \vec{w}_2 \rangle_g = (F_{\vec{W}_1}, F_{\vec{W}_2})_g = (F^T.F_{\vec{W}_1}, \vec{W}_2)_G, \tag{E.16}
\]
or, \( \vec{w}_1 \cdot \vec{w}_2 = (F_{\vec{W}_1})_g (F_{\vec{W}_2}) = (F^T.F_{\vec{W}_1})_g \vec{W}_2. \)

**Definition E.3** The (right) Cauchy–Green deformation tensor at \( P \in \Omega_0 \) relative to \((\cdot, \cdot)_G\) and \((\cdot, \cdot)_g\), is the endomorphism \( C_{Gg,p} = \Phi^t_{Gg}(P) \in \mathcal{L}(\mathbb{R}^n_{t0}; \mathbb{R}^n_{ta}) \) defined by, with \( p = \Phi^t_{G0}(P) \),
\[
C_{Gg,p} := F^T_{Gg,p} \circ F_P, \quad \text{in short} \quad C = F^T \circ F = F^T.F, \tag{E.17}
\]
that is, \( C(\vec{W}) = F^T(\vec{W}) \) for all \( \vec{W} \in \mathbb{R}^n_t \) (full notation: \( C_{Gg,p} (\vec{W}) = F^T_{Gg,p} (F_P(\vec{W})) \)).

That is: The (right) Cauchy–Green deformation tensor at \( P \in \Omega_0 \) relative to \( t_0 \), \((\cdot, \cdot)_G\) and \((\cdot, \cdot)_g\), is the endomorphism \((C^t_{G0}(P) \in \mathcal{L}(\mathbb{R}^n_{t0}; \mathbb{R}^n_{ta})\) defined by \((C^t_{G0}(P) := (F^t_{G0})^T(\vec{p}) \circ F^t_{G0}(P)\) when \( p = \Phi^t_{G0}(P) \).

Thus (E.15) reads
\[
\langle \vec{w}_{1p}, \vec{w}_{2p} \rangle_g = ((C^t_{G0}(Gg,p), \vec{W}_{1p}), \vec{W}_{2p})_G, \quad \text{in short} \quad \langle \vec{w}_1, \vec{w}_2 \rangle_g = (C, \vec{W}_1, \vec{W}_2)_G. \tag{E.18}
\]
or, \( \vec{w}_1 \cdot \vec{w}_2 = (C, \vec{W}_1) (F_{\vec{W}_1})_g \vec{W}_2 \) with \( \vec{w}_i = F_{\vec{W}_i} \).

**Exercise E.4** Consider a curve \( \alpha : s \in [a, b] \rightarrow \alpha(s) \in \Omega_0 \) (drawn in \( \Omega_0 \)). Prove: The length of the transported curve \( \beta = \Phi^t_{G0} \circ \alpha : [a, b] \rightarrow \Omega_t \) is \( |\beta| = \int_a^b \sqrt{(C^t_{G0}(\alpha(s)), \vec{\alpha}'(s), \vec{\alpha}'(s))_G} \, ds \).

**Answer.** By definition of the length, the length of \( \beta \) is \( L_t := \int_a^b |\vec{\beta}'(s)|_g \, ds \). Here \( \vec{\beta}'(s) = F^t_{G0}(\alpha(s)), \vec{\alpha}'(s) \).

Thus \( |\vec{\beta}'(s)|^2 = (\vec{\beta}'(s), \vec{\beta}'(s))_g = (F^t_{G0}(\alpha(s)), \vec{\alpha}'(s), F^t_{G0}(\alpha(s)), \vec{\alpha}'(s))_g = (C^t_{G0}(\alpha(s)), \vec{\alpha}'(s), \vec{\alpha}'(s))_G. \)

**E.3.2 Quantification with bases**

With a basis \( (\vec{E}_i) \) in \( \mathbb{R}^n_{t0} \) and a basis \( (\vec{e}_i) \) in \( \mathbb{R}^n \), (E.17) gives \( [C] = [F^T], [F] \), thus (E.7) gives, in short,
\[
[C] = [G]^{-1} [F^T], [F], \quad (= [F^T], [F]). \tag{E.19}
\]
more precisely, \([C^t_{G0}(Gg,p)]_{\vec{e}} = [(G)_{\vec{E}}]^{-1} ([(F^t_{G0})_p]_{\vec{e}} [g]_{\vec{E}} [F^t_{G0}]_{\vec{e}} \vec{e} \vec{e}] \). That is, if \( C, \vec{E}_J = \sum_{i=1}^n C^j_{ij} \vec{E}_i \), then
\[
C^j_{ij} = \sum_{K, k, m=1}^n G^{KIK F^k_{KL} g_{LM} F^m_{jL}} \quad \text{when} \quad [G]^{-1} \quad \text{and} \quad [G^T] := [G]^{-1}. \tag{E.20}
\]
(Classical notations: \( C, \vec{E}_j = \sum_{i=1}^n C_{ij} \vec{E}_i \) and \( C_{ij} = \sum_{K, k, m=1}^n ([G]^{-1})_{ik} F^k_{Le} g_{Le} F^m_{jL} \).)

In particular, if \( (\vec{E}_i) \) and \( (\vec{e}_i) \) are \((\cdot, \cdot)_G\) and \((\cdot, \cdot)_g\)-orthonormal basis, then \([G]_{\vec{e}} = [g]_{\vec{E}} = I \), thus
\[
[C] = [F^T] [F], \quad \text{i.e.} \quad C^j_{ij} = \sum_{k=1}^n F^k_{Ij} F^k_{Jk}, \quad \forall I, J = 1, ..., n. \tag{E.21}
\]
(Classical notations: \( C_{ij} = \sum_{k=1}^n F_{ik} F_{kj} \) for all \( i, j = 1, ..., n \).)
E.4 Applications

E.4.1 Stretch

**Definition E.5** The stretch ratio for $\bar{W}_P \in \mathbb{R}^n_0$ between $t_0$ and $t$ is

$$\lambda(\bar{W}_P) := \frac{\|F_P(\bar{w}_P)\|_g}{\|\bar{w}_P\|_g} = \frac{\|\bar{w}_P\|_g}{\|\bar{w}_P\|_g},$$

with $\bar{w}_P = F_P \bar{W}_P$, cf. (E.14).

Thus

$$\forall \bar{W}_P \in \mathbb{R}^n_0 \text{ s.t. } \|\bar{W}_P\|_g = 1, \quad \lambda(\bar{W}_P) = \|F_P \bar{W}_P\|_g = \sqrt{(C_{\bar{W}_P}, \bar{W}_P)_g}. \quad (E.23)$$

You may also find the notation: $\lambda(d\bar{X}) = \|F_P (d\bar{X})\|_g = \sqrt{(C_{d\bar{X}}, d\bar{X})_g}$ with $d\bar{X}$ a unit vector (!). This notation should be avoided, see § 5.2.

E.4.2 Change of angle

Euclidean framework with the same Euclidean dot product $(\cdot, \cdot)_G = (\cdot, \cdot)_g$ at $t_0$ and $t$.

The angle $\theta(P) = (\bar{W}_{1P}, \bar{W}_{2P})$ formed by two non vanishing vectors $\bar{W}_{1P}$ and $\bar{W}_{2P}$ in $\mathbb{R}^n_0$ at $P \in \Omega_0$ is given by

$$\cos(\theta(P)) = \frac{(\bar{W}_{1P}, \bar{W}_{2P})_g}{\|\bar{W}_{1P}\|_g \|\bar{W}_{2P}\|_g}.$$

After deformation by the motion, $\bar{W}_{iP}$ has become $\bar{w}_{ip} = F^i_1(P).\bar{W}_{iP}$ when $p = \Phi^i_1(P)$ (push-forward). Thus, at $t$ (after deformation), the angle $\theta(t)(p) = (\bar{w}_{1p}, \bar{w}_{2p})$ is

$$\cos(\theta(t)(p)) = \frac{(\bar{w}_{1p}, \bar{w}_{2p})_g}{\|\bar{w}_{1p}\|_g \|\bar{w}_{2p}\|_g}. \quad (E.24)$$

E.4.3 Spherical and deviatoric tensors

**Definition E.6** The deformation spheric tensor is

$$C_{sph} = \frac{1}{n} \text{Tr}(C) I,$$

with $\text{Tr}(C) = \sum_{i=1}^n C_{ii}$ the trace of the endomorphism $C$.

**Definition E.7** The deviatoric tensor is

$$C_{dev} = C - C_{sph}. \quad (E.26)$$

So $\text{Tr}(C_{dev}) = 0$, and $C = C_{sph} + C_{dev}$.

E.4.4 Rigid motion

The deformation is rigid iff, for all $t_0, t$,

$$(F^t_{t_0})^T F^t_{t_0} = I,$$

i.e. $C^t_{t_0} = I$, written $C_1 = I = F^T F$.

(E.27)

Thus, after a rigid body motion, lengths and angles are left unchanged.

E.4.5 Diagonalization of $C$

**Proposition E.8** $C = F^T F$ being symmetric positive, $C$ is diagonalizable, its eigenvalues are positive, and $\mathbb{R}_0$ has an orthonormal basis made of eigenvectors of $C$.

**Proof.** $(C^t_{t_0}(P), \bar{W}_{1P}, \bar{W}_{2P})_G = (F^t_{t_0}(P), F^t_{t_0}(P), F^t_{t_0}(P))_G = (\bar{W}_{1P}, C^t_{t_0}(P), \bar{W}_{2P})_G$, thus $C$ is $(\cdot, \cdot)_G$-symmetric.

$$(C, \bar{W}_{1P}, \bar{W}_{1P})_G = (F \bar{W}_{1P}, F \bar{W}_{1P})_G = \|F \bar{W}_{1P}\|_g^2 \geq 0, \text{ since } F \text{ invertible (} \Phi^i_1 \text{ is supposed to be a diffeomorphism). Thus } C \text{ est } (\cdot, \cdot)_G \text{-symmetric positive.}$$

Therefore, $C$ being a real endomorphism, $C$ is diagonalizable with positive eigenvalues. ☑

**Definition E.9** Let $\lambda_i$ be the eigenvalues of $C$. Then the $\sqrt{\lambda_i}$ are called the principal stretches. And the associated eigenvectors give the principal directions.
E.4.6 Mohr circle

This § deals with general properties of 3 × 3 symmetric positive endomorphism, like $C^3_1(P)$. Consider $\mathbb{R}^3$ with a Euclidean dot product $(\cdot, \cdot)_{\mathbb{R}^3}$ and a $(\cdot, \cdot)_{\mathbb{R}^3}$-orthonormal basis $(\vec{e}_i)$. Let $\mathcal{M} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a symmetric positive endomorphism. Thus $\mathcal{M}$ is diagonalizable in a $(\cdot, \cdot)_{\mathbb{R}^3}$-orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, that is, $\exists \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, $\exists \vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathbb{R}^3$ s.t.

$$\mathcal{M} \vec{e}_i = \lambda_i \vec{e}_i \quad \text{and} \quad (\vec{e}_i, \vec{e}_j)_{\mathbb{R}^3} = \delta_{ij}.$$ (E.28)

So $[\mathcal{M}]_{\mathbb{R}} = \text{diag}(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$. (The matrix $[\mathcal{M}]_{\mathbb{R}}$ is not supposed to be diagonal.) And the orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is ordered s.t. $\lambda_1 \geq \lambda_2 \geq \lambda_3 (> 0)$.

Let $S$ be the unit sphere in $\mathbb{R}^3$, that is the set $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$. Its image $\mathcal{M}(S)$ by $\mathcal{M}$ is the ellipsoid $\{(x, y, z) : \frac{x^2}{\lambda_1^2} + \frac{y^2}{\lambda_2^2} + \frac{z^2}{\lambda_3^2} = 1\}$.

Then consider $\vec{n} = \sum_{i} n_i \vec{e}_i$ s.t. $||\vec{n}||_{\mathbb{R}^3} = 1$, that is,

$$[\vec{n}]_{\mathbb{R}} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \quad \text{with} \quad n_1^2 + n_2^2 + n_3^2 = 1.$$ (E.29)

Thus its image $\vec{A} = \mathcal{M} \vec{n} \in \mathcal{M}(S)$ satisfies

$$\vec{A} = \mathcal{M} \vec{n}, \quad [\vec{A}]_{\mathbb{R}} = \begin{pmatrix} \lambda_1 n_1 \\ \lambda_2 n_2 \\ \lambda_3 n_3 \end{pmatrix}.$$ (E.30)

Then define

$$A_\perp = (\vec{A}, \vec{n})_{\mathbb{R}^3}, \quad \vec{A}_\perp = \vec{A} - A_\perp \vec{n}, \quad A_\perp := ||\vec{A}_\perp||.$$ (E.31)

So $\vec{A} = A_\perp \vec{n} + \vec{A}_\perp \in \text{Vect}(\vec{n}) \otimes \text{Vect}(\vec{n})^\perp$. (Note that $\vec{A}_\perp$ is not orthonormal to the ellipsoid $\mathcal{M}(S)$, but is orthonormal to the initial sphere $S$.)

**Mohr Circle purpose:** To find a relation:

$$A_\perp = f(A_\perp),$$ (E.32)

relation between “the normal force $A_\perp$” (to the initial sphere) and the “tangent force $A_\perp$” (to the initial sphere).

(E.29), (E.30) and $A_\perp = (\mathcal{M} \vec{n}, \vec{n})_{\mathbb{R}^3}$ give

$$\begin{cases} n_1^2 + n_2^2 + n_3^2 = 1, \\ A_\perp = \lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2 \end{cases}$$

$$||\vec{A}_\perp||^2 = A_\perp^2 + A_\perp^\perp = \lambda_1^2 n_1^4 + \lambda_2^2 n_2^4 + \lambda_3^2 n_3^4.$$ (E.33)

That is,

$$\begin{cases} n_1^2 + n_2^2 + n_3^2 = 1, \\ \lambda_1 n_1^2 + \lambda_2 n_2^2 + \lambda_3 n_3^2 = A_\perp, \quad \lambda_1^2 n_1^4 + \lambda_2^2 n_2^4 + \lambda_3^2 n_3^4 = A_\perp^4 + A_\perp^{\perp 4}. \end{cases}$$

This is linear system with the unknowns $n_1^2, n_2^2, n_3^2$. The solution is

$$\begin{cases} n_1^2 = \frac{A_\perp^2 + (A_\perp - \lambda_2)(A_\perp - \lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \\ n_2^2 = \frac{A_\perp^2 + (A_\perp - \lambda_3)(A_\perp - \lambda_1)}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)}, \\ n_3^2 = \frac{A_\perp^2 + (A_\perp - \lambda_1)(A_\perp - \lambda_2)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}. \end{cases}$$ (E.34)

The $n_i^2$ being non negative, and with $\lambda_1 > \lambda_2 > \lambda_3 \geq 0$, we get

$$\begin{cases} A_\perp^2 + (A_\perp - \lambda_2)(A_\perp - \lambda_3) \geq 0, \\ A_\perp^2 + (A_\perp - \lambda_3)(A_\perp - \lambda_1) \leq 0, \\ A_\perp^2 + (A_\perp - \lambda_1)(A_\perp - \lambda_2) \geq 0. \end{cases}$$ (E.35)
Then let $x = A_n$ and $y = A_\perp$, and consider, for some $a, b \in \mathbb{R}$, the equation
\[ y^2 + (x - a)(x - b) = 0, \quad \text{so} \quad (x - \frac{a+b}{2})^2 + y^2 = \frac{(a-b)^2}{4}. \]
This is the equation of a circle centered at $\left( \frac{a+b}{2}, 0 \right)$ with radius $\frac{a-b}{2}$.

Thus (E.35) tells that $A_n$ and $A_\perp$ are inside the circle centered at $\left( \frac{\lambda_1 + \lambda_2}{2}, 0 \right)$ with radius $\frac{\lambda_3 - \lambda_1}{2}$, and (E.35), 1.3, tell that $A_n$ and $A_\perp$ are outside the other circles (adjacent and included in the first, drawing).

**Exercise E.10** What happens if $\lambda_1 = \lambda_2 = \lambda_3 > 0$?

**Answer.** Then
\[
\begin{align*}
n_1^2 + n_2^2 + n_3^2 &= 1, \\
n_1^2 + n_2^2 + n_3^2 &= \frac{A_n}{\lambda_i}, \\
n_1^2 + n_2^2 + n_3^2 &= \frac{A_n^2 + A_\perp^2}{\lambda_i^2}.
\end{align*}
\]
Thus $A_n = \lambda_1$ and $A_n^2 + A_\perp^2 = \lambda_i^2$, thus $A_\perp = 0$. Here $C = \lambda_1 I$, and we deal with a pure dilation: $A_{\perp} = 0$.

**Exercise E.11** What happens if $\lambda_1 = \lambda_2 > \lambda_3 > 0$?

**Answer.** Then
\[
\begin{align*}
n_1^2 + n_2^2 + n_3^2 &= 1, \\
(\lambda_1 - \lambda_3)n_3^2 &= A_n, \\
(\lambda_1^2 - \lambda_3^2)n_3^2 &= A_n^2 + A_\perp^2.
\end{align*}
\]
Thus $A_n = \lambda_1 - (\lambda_1 - \lambda_3)n_3^2 \in [\lambda_3, \lambda_1]$, and $A_\perp = \pm (\lambda_1^2 - (\lambda_1^2 - \lambda_3^2)n_3^2)^{1/2}$, with $A_n^2 + A_\perp^2$ a point on the circle with radius $\lambda_i^2(1 - n_3^2) + \lambda_i^2n_3^2$.

### E.5 $C^0$ and pull-back $g^*$

#### E.5.1 The flat $^b$ notation (endomorphism $L$ and its $(\cdot, \cdot)_g$-associated $0$ $2$ tensor $L^g$)

Let $(\cdot, \cdot)_g$ be a inner dot product in a vector space $E$. Let $L \in \mathcal{L}(E; E)$ (an endomorphism).

**Definition E.12** The bilinear map $L^g : \mathcal{L}(E, E; \mathbb{R})$ (the $^{(2)}$ uniform tensor) which is $(\cdot, \cdot)_g$-associated to $L$ is defined by: for all $\vec{u}, \vec{w} \in E$,
\[
L^g(\vec{u}, \vec{w}) := (L.\vec{u}, \vec{w})_g.
\]
(The bilinearity is trivial.) We have thus defined the operator
\[
(\cdot)_g = \mathcal{J}_g(\cdot) : \begin{cases} \mathcal{L}(E; E) \to \mathcal{L}(E, E; \mathbb{R}) \\ L \mapsto L^g = \mathcal{J}_g(L), \end{cases}
\]
which depends on $(\cdot, \cdot)_g$. (The linearity of $\mathcal{J}_g$ is trivial.)

The notation $^b$ refers to the change of position of the $i$-index from top $L^i_j$ to bottom $L_{ij}$, once a basis $(\vec{e}_i)$ has been chosen and the duality notation is used. $L.\vec{a}_j = \sum_{i=1}^n L^i_j \vec{a}_i$ and $L^g = \sum_{i,j=1}^n L^i_j a^i \otimes a^j$. And $L_{ij} = L^g(\vec{a}_i, \vec{a}_j) = (E.39)(L.\vec{a}_i, \vec{a}_j)_g = \sum_{k=1}^n L^k_i (\vec{a}_k, \vec{a}_j)_g = \sum_{k=1}^n L^k_i (\vec{a}_k, \vec{a}_j)_g$.

\[
[L^g]_{ij} = ([L]_i)^T[g]_{ij} \quad \text{written} \quad [L^g] = [L]^T[g].
\]

NB: A change of variance, from a $^{(1)}$ tensor to a $^{(2)}$ tensor, as with $\mathcal{J}_g$, is necessarily observer dependent: There is no natural canonical isomorphism between a vector space $E$ and its dual $E^*$, see § T.2.

#### E.5.2 Two inner dot products and $C^0$

Let $(\cdot, \cdot)_g$ be the inner dot product in $\mathbb{R}^n_0$ used by the observer who makes the measurements at $t$. Let $(\cdot, \cdot)_\tilde{g}$ be the inner dot product in $\mathbb{R}^n_0$ which was used by the observer who made the measurements at $t_0$.

Recall (E.18): $(C.\tilde{W}_1, \tilde{W}_2)_g = (\tilde{w}_1, \tilde{w}_2)_g$ when $\tilde{w}_i = F.\tilde{W}_i$ (push-forward). Thus, with (E.36),
\[
C^0_{\tilde{g}}(\tilde{W}_1, \tilde{W}_2) = (C.\tilde{W}_1, \tilde{W}_2)_G = (F.\tilde{W}_1, F.\tilde{W}_2)_g.
\]
Thus, with a basis $(\tilde{E}_i)$ in $\mathbb{R}^n_0$, (E.38) gives, with $C = \sum_{i,j=1}^n C^i_j \tilde{E}_i \otimes \tilde{E}_j$, $C^0_{\tilde{g}} = \sum_{i,j=1}^n C_{ij} E^i \otimes E^j$ and $[C]_{\tilde{g}}$ symmetric,
\[
[C^0]_{\tilde{g}} = [C]_{\tilde{g}}[C]_{\tilde{g}} \quad \text{written} \quad [C^0] = [C][C].
\]
E.5.3 The pulled-back metric $g^*$

Let $g$ be a $(0,2)$ tensor in $\Omega_t$ (e.g., a metric in $\Omega_t$). Its pull-back $g^*$ by $\Phi_t^{i_0}$ is the $(0,2)$ tensor in $\Omega_{t_0}$ defined by (14.13): In short,

$$g^*(\tilde{W}_1, \tilde{W}_2) := g(F\tilde{W}_1, F\tilde{W}_2), \quad \text{(E.41)}$$

which means, $g^*(P)(\tilde{W}_1(P), \tilde{W}_2(P)) := g(p)(F(P)\tilde{W}_1(P), F(P)\tilde{W}_2(P))$ when $p = \Phi_t^{i_0}(P)$ and $\tilde{W}_1, \tilde{W}_2$ are vector fields in $\Omega_{t_0}$. In other words, $g^*$ is defined thanks to the push-forward of vector fields, in short,

$$g^*(\tilde{W}_1, \tilde{W}_2) := g(\tilde{w}_1, \tilde{w}_2) \quad \text{when} \quad \tilde{w}_i = F\tilde{W}_i \text{ (push-forward)}, \quad \text{(E.42)}$$

i.e., $g^*(P)(\tilde{W}_1(P), \tilde{W}_2(P)) = g(p)(\tilde{w}_1(p), \tilde{w}_2(p))$ when $P = (\Phi_t^{i_0})^{-1}(p)$ and $\tilde{W}_i(P) = F(P)^{-1}\tilde{w}_i(p)$. In particular, if $g$ is a metric in $\Omega_t$ then $g^*$ is a metric in $\Omega_{t_0}$ (for a regular motion).

E.5.4 $C_{Gg}^0 = g^*$

Since $C := C_{Gg}$ also depends on $(\cdot, \cdot)_G$, cf. (E.17), $C_{G}^0 := (C_{Gg})_{G}^0$ also depends on $(\cdot, \cdot)_g$, and $C_{G}^0$ should be named e.g. $C_{Gg}^0$. Thus, by definition of $g^*$, cf. (E.41), we get, for all $\tilde{W}_1, \tilde{W}_2 \in \mathbb{R}^n_{t_0}$,

$$C_{Gg}^0(\tilde{W}_1, \tilde{W}_2) = g^*(\tilde{W}_1, \tilde{W}_2), \quad \text{i.e.} \quad C_{Gg}^0 = g^*. \quad \text{(E.43)}$$

NB: $C$ and $C^0$ both depend on $(\cdot, \cdot)_G$, while the pull-back $g^*$ does not depend on $(\cdot, \cdot)_G$.

E.6 Time Taylor expansion for $C$

E.6.1 First and second order

Euclidean dot products $(\cdot, \cdot)_G$ in $\mathbb{R}^n_{t_0}$ and $(\cdot, \cdot)_g$ in $\mathbb{R}^n_{t_0}$ are chosen, $C_{G}^{0n}(t) := C_{G}^{0n}(P) = F_{t_0}^{T}(P)_{G}^{T}, F_{t_0}^{lo}(P) := \text{named } F^T, F$ is the Cauchy–Green deformation tensor, $\tilde{V}_{t_0}^{lo}(P) := \frac{\partial g^{i_0}_t}{\partial t}(t, P)$ named $\tilde{V}$ and $\tilde{A}^{lo}_t(P) := \tilde{A}^{t_0}(t, P)$ named $\tilde{A}$ the Lagrangian velocity and acceleration, $p_t = \Phi_t^{i_0}(P)$, $\tilde{v}(t, p_t) = \tilde{V}^{lo}_t(t, P)$ and $\tilde{\gamma}(t, p_t) = \tilde{A}^{lo}_t(t, P)$ are the Eulerian velocity and acceleration, and $d\tilde{v}^T : = \text{named } d\tilde{v}^T$.

Then (5.41) gives (knowing $d\tilde{V} = d\tilde{v}^T$)

$$C_{G}^{0n}(t+h) = [F^T + h d\tilde{V}^T + \frac{h^2}{2} d\tilde{A}^T + o(h^2)][F + h d\tilde{V} + \frac{h^2}{2} d\tilde{A} + o(h^2)]$$

$$= F^T.F + h(F^T.d\tilde{V} + d\tilde{V}.F) + \frac{h^2}{2} (F^T.d\tilde{A} + 2d\tilde{V}.d\tilde{V} + d\tilde{A}^T.F) + o(h^2)$$

$$= F^T.[I + h(d\tilde{v}^T + \frac{h^2}{2} d\tilde{\gamma}^T + o(h^2))][I + h(d\tilde{v} + \frac{h^2}{2} d\tilde{\gamma} + o(h^2))]\cdot F$$

$$= F^T.(I + h(d\tilde{v} + d\tilde{v}^T) + \frac{h^2}{2} (d\tilde{\gamma} + d\tilde{\gamma}^T + 2d\tilde{\gamma}^T.d\tilde{v})(t, p(t)).F + o(h^2).$$

$$= F^T.(I + h(d\tilde{v} + d\tilde{v}^T) + \frac{h^2}{2} (d\tilde{\gamma} + d\tilde{\gamma}^T + 2d\tilde{\gamma}^T.d\tilde{v})(t, p(t)).F + o(h^2).$$

So:

$$C_{G}^{0n'}(t) = F^T.d\tilde{V} + d\tilde{V}.F$$

$$= F^T.(d\tilde{v} + d\tilde{v}^T).F, \quad \text{(E.45)}$$

and

$$C_{G}^{0n''}(t) = F^T.d\tilde{A} + 2d\tilde{V}.d\tilde{V} + d\tilde{A}^T.F$$

$$= F^T.(d\tilde{\gamma} + 2d\tilde{\gamma}^T + d\tilde{\gamma}^T).F, \quad \text{(E.46)}$$

In particular $C_{G}^{0n'}(t_0) = d\tilde{v} + d\tilde{v}^T = 2\mathcal{D}$ and $C_{G}^{0n''}(t_0) = d\tilde{\gamma} + d\tilde{\gamma}^T + 2d\tilde{\gamma}^T.d\tilde{v}$ (non linear in $\tilde{v}$).
E.6.2 Associated results and interpretation problems

\[ 2D = d\bar{u} + d\bar{v}^T \] gives \( 2 \frac{DD}{Dt} = \frac{D(d\bar{u})}{Dt} + \frac{D(d\bar{v})}{Dt} = d\bar{\gamma} + d\bar{v}.d\bar{v} - d\bar{v}^T.d\bar{v}^T, \]
thus \( E_{138} \) F.46 also reads

\[ C^a_{\alpha} (t) = F^T.(2 \frac{DD}{Dt} + d\bar{v}.d\bar{v} + d\bar{v}^T.d\bar{v}^T + 2d\bar{v}.d\bar{v}^T(t,p(t)).F) \]
\[ = 2F^T.(\frac{DD}{Dt} + D.d\bar{v} + d\bar{v}^T.D)(t,p(t)).F. \]

And the term in parentheses, \( Z = \frac{DD}{Dt} + D.d\bar{v} + d\bar{v}^T.D, \) resembles a lower-convected Maxwell derivative, but for \( d\bar{v}^T \) instead of \( d\bar{v} \), cf. (15.67), so you may find \( E_{138} \) written as \( C^a = 2F^T.L_cD_p.F. \)

- \( \frac{DD}{Dt} \) is not (covariant) objective unlike \( d\bar{v} \), and \( D \) is not (covariant) objective.
- \( d\bar{v} \) in \( D = \frac{dd\bar{v} + dd\bar{v}^T}{2} \) is an endomorphism naturally canonically identified to a mixed tensor (a \((1)_t\)-tensor), and the Lie derivative \( L_cD \) of a mixed tensor is different, cf. (15.62).
- To justify the notation \( L_cD_p \), you may try to consider \( d\bar{v}^T \) the \((1)_t\) tensor which is \((\cdot,\cdot)\)-associated with the endomorphism \( d\bar{v} \), and consider the \((0)_t\) tensor \( D_p :\frac{dd\bar{v} + dd\bar{v}^T}{2} \), and propose the product \( F^T.L_cD_p.F = \sum_{i,j}^n F_{i,j} (L_cD_p)^{ji} kF^T.E_i \otimes E_j? \)
- You get disappointing results when using the lower-convected Maxwell derivative, e.g. for viscoelastic fluids where the results are not the expected results (they really differ from experimental results and upper Maxwell derivatives are often considered even if the obtained results are disappointing), and for deformable solids where the Jaumann derivative is usually preferred (and the way the Jaumann derivative is obtained has nothing to do with the definition of a Lie derivative, see footnote page 26, although this Jaumann derivative is often said to be a kind of Lie derivative).

Remark: May be the use of the (covariant objective) Lie derivatives of vectors fields \( \bar{u} \) or differential forms \( \alpha \) could be preferred to characterize and build constitutive laws or materials, instead of the Lie derivatives of order two tensors \( T \) which are measuring tools used to give values to \( \bar{u} \)'s and \( \alpha \)'s (e.g. \( T(\alpha, \bar{u}) \in \mathbb{R} \) gives values to \( \alpha \) and \( \bar{u} \)): Illustration: see § F.4 or, more precisely, https://www.isima.fr/leborgne/IsimathMeca/PpvObj.pdf.

F Prospect: Elasticity and objectivity?

F.1 Introduction: Remarks

The introduction of the Cauchy strain tensor \( C = F^T \circ F \) raises questions:

Remark F.1 The linearization of \( E = \frac{C-L}{2} = \frac{E^T + E - I}{2} \) (the Green–Lagrange tensor) gives \( \varepsilon = \frac{E + E^T}{2} - I \)

(1) (the infinitesimal strain tensor). Steps:

- Start with the linear map \( F = d\Phi^{C}_{C} (P) \).
- Introduce Euclidean dot products to define \( F^T \).
- Create the quadratic type map \( C = F^T.F \), then the quadratic type map \( E = \frac{C-L}{2} \).
- Linearize \( E \) to get back to a linear result: the \( \varepsilon \) expression.

So: 1- From the linear map \( F = d\Phi \) you get back to a linear “\( \varepsilon \)”, after having “squared” \( F \), to get \( C \) and \( E = \frac{1}{2}(C - I) \) which you linearize (the \( \frac{1}{2} \) introduced since your “squared” to begin with)...

And you end up with a kind of first order Taylor expansion of the motion \( \Phi \) (expected), but with an unwanted \( F^T \)

(\( \varepsilon \)...) And 2- \( \varepsilon \) is not a function (is not a tensor), see § F.3.1.

Remark F.2 It is simple to compose the differentials along a trajectory: \( F_{t_1}^{t_0} \circ F_{t_0}^{t_1} = F_{t_1}^{t_0} \), cf. (6.20). It is more difficult for the Cauchy–Green deformation tensors: \( C_{t_2}^{t_1} \circ C_{t_1}^{t_0} = (F_{t_2}^{t_1})^T \circ F_{t_1}^{t_0} \circ (F_{t_1}^{t_0})^T \circ F_{t_0}^{t_1} \)

is not of the type \( C_{t_2}^{t_0} \) (the composition \( C_{t_2} \circ C_{t_1}^{t_0} \) does not seem to be very used, and many descriptions requires starting from a “rest state”).

The above remarks end with the question: Can we start a linear theory without \( \varepsilon \)? Yes (proposals):

F.2 Polar decompositions of \( F \)

The motion is supposed regular. Let \( t_0, t \in \mathbb{R}, p_{t_0} \in \Omega_{t_0}, F := F_{t_1}^{t_0}(p_{t_0}) := d\Phi^{C}_{C}(p_{t_0}), \)
and let \( (\cdot, \cdot)_G \) and \( (\cdot, \cdot)_g \) be Euclidean dot products in \( \mathbb{R}^n_0 \) and \( \mathbb{R}^n_t \).
F.2.1 \( F = R . U \) (right polar decomposition)

Let \( C = F^T \circ F \overset{\text{noted}}{=} F^T . F \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \), the right Cauchy Green deformation tensor at \( p_{t_0} \) relative to \((\cdot, \cdot)_G\) and \((\cdot, \cdot)_g\). The endomorphism \( C \) being symmetric definite positive, \( \exists \alpha_1, \ldots, \alpha_n \in \mathbb{R}_*^+ \) (the eigenvalues), \( \exists \vec{e}_1, \ldots, \vec{e}_n \in \mathbb{R}_0^n \) (associated eigenvectors), such that, for all \( i = 1, \ldots, n \),

\[
C \vec{e}_i = \alpha_i \vec{e}_i, \quad \text{and} \quad (\vec{e}_i) \text{ is a } (\cdot, \cdot)_G \text{-orthonormal basis in } \mathbb{R}_0^n. \quad (F.1)
\]

Then, define the endomorphism \( U \in \mathcal{L}(\mathbb{R}^n_0; \mathbb{R}^n_0) \), called the right stretch tensor, by, for all \( i = 1, \ldots, n \),

\[
U \vec{e}_i = \sqrt{\alpha_i} \vec{e}_i, \quad \text{and} \quad U \overset{\text{noted}}{=} \sqrt{C}, \quad (F.2)
\]

the \( \sqrt{\alpha_i} \) being called the principal stretches. Then, define the linear map \( R \in \mathcal{L}(\mathbb{R}^n_0; \mathbb{R}^n_0) \), called the rotation map, by

\[
R := F \circ U^{-1} \overset{\text{noted}}{=} F.U^{-1}, \quad (F.3)
\]

so that

\[
F = R \circ U \overset{\text{noted}}{=} R.U, \quad \text{called the right polar decomposition of } F. \quad (F.4)
\]

**Proposition F.3** We have:

\[
C = U \circ U^{-1} \overset{\text{noted}}{=} U^2, \quad U \text{ is symmetric definite positive, } \quad R^{-1} = R^T. \quad (F.5)
\]

And

the right polar decomposition \( F = R \circ U \) is unique: \( (F.6) \)

If \( F = \hat{R} \circ \hat{U} \) where \( \hat{U} \in \mathcal{L}(\mathbb{R}^n_0; \mathbb{R}^n_0) \) is symmetric definite positive and \( \hat{R} \in \mathcal{L}(\mathbb{R}^n_0; \mathbb{R}^n_0) \) satisfies \( \hat{R}^{-1} = \hat{R}^T \), then \( \hat{U} = U \) and \( \hat{R} = R \).

**Proof.** \( (F.2) \) yields \((U \circ U)\vec{e}_j = \lambda \vec{e}_j = C \vec{e}_j \) for all \( j \), cf. \((F.1)\), thus \( U \circ U = C \); Then \((U^T \vec{e}_i, \vec{e}_j)_G = (\vec{e}_i, U^T \vec{e}_j)_G = (\vec{e}_i, \sqrt{\alpha_j} \vec{e}_j)_G = \sqrt{\alpha_j} \delta_{ij} = \sqrt{\alpha_i} \delta_{ij} = (\sqrt{\alpha_i} \vec{e}_i, \vec{e}_j)_G = (\sqrt{\alpha_i} \vec{e}_i, \vec{e}_j)_G \) for all \( i, j \), thus \( U^T = U \) (symmetry).

Then \( R^T \circ R = U^{-1} \circ F^T \circ F \circ U^{-1} = U^{-1} \circ C \circ U^{-1} = U^{-1} \circ (U \circ U) \circ U^{-1} = I_0 \) identity in \( \mathbb{R}_0^n \). (Details: \( R^T . R \vec{W}, \vec{Z}_G = (R . \vec{W}, R . \vec{Z})_g = (U^T . F \circ U - 1 . \vec{W}, U^T . F \circ U - 1 . \vec{Z})_g = (U^T \circ U^{-1} . \vec{W}, U^{-1} \vec{Z})_g \) Therefore \( R^{-1} = R \in \mathcal{L}(\mathbb{R}^n_0, \mathbb{R}^n_0) \). Thus \( R \circ R^T = R \circ R^{-1} = I_0 \) identity in \( \mathbb{R}_0^n \).

And \( F = \hat{R} \circ \hat{U} = R \circ U \) gives \( F^T \circ F \circ U^{-1} = (R^T \circ R \circ U^{-1} = (\hat{R} \circ \hat{U})^T \circ \hat{R} \circ \hat{U} = U \circ U^{-1} \circ \hat{R} \circ \hat{U} = U \circ U^{-1} \circ \hat{R} \circ \hat{U} \), with \( F^T \circ F = U \circ U \), thus \( \hat{U} = U \) and \( \hat{R} = U \) (uniqueness of the positive square root eigenvalues). Hence \( \hat{R} = R \).

**NB:** We can be more precise. Since we work with the affine space \( \mathbb{R}^n \), consider the Marsden’s shifter

\[
S := S^n_0(p_{t_0}) : \quad \left\{ \begin{array}{lr}
T_{p_{t_0}}(\Omega_{t_0}) \overset{\text{noted}}{=} \mathbb{R}^n_{t_0} \rightarrow T_{p_t}(\Omega_t) \overset{\text{noted}}{=} \mathbb{R}^n_t \\
\vec{w}_{t_0,p_{t_0}} \rightarrow (S.\vec{w}_{t_0,p_{t_0}})(t,p_t) = \vec{w}_{t_0,p_{t_0}} \quad \text{where} \quad p_t = \Phi_t^p(p_{t_0}).
\end{array} \right. \quad (F.7)
\]

The shifter \( S \) will be considered between the Hilbert spaces \((T_{p_{t_0}}(\Omega_{t_0}), (\cdot, \cdot)_G)\) and \((T_{p_t}(\Omega_t), (\cdot, \cdot)_g)\); So \( S \) is the algebraic identity, but \( S \) is not a topological identity: It changes the norms: You consider \( \|S.\vec{w}_{t_0,p_{t_0}}\|_G \) at \( t_0 \) (in the past) and \( \|S.\vec{w}_{t_0,p_{t_0}}\|_g = \|\vec{w}_{t_0,p_{t_0}}\|_g \) at \( t \) (at present time).

Then, let \( R_0 \in \mathcal{L}(\mathbb{R}^n_0; \mathbb{R}^n_0) \) be the endomorphism defined by, in short,

\[
R_0 := S^{-1} \circ R \overset{\text{noted}}{=} S^{-1}.R, \quad \text{so} \quad R = S \circ R_0 \overset{\text{noted}}{=} S.R_0. \quad (F.8)
\]

Full notations: \( (R_0)^{t_0}_{1,G,G_0}(p_{t_0}) := (S^{t_0})^{-1}(R^{t_0}_{1,G_0}(p_{t_0})). \) Thus

\[
R_0 : \quad \left\{ \begin{array}{lr}
\mathbb{R}^n_{t_0} \rightarrow \mathbb{R}^n_0 \\
\vec{w}_{p_{t_0}} \rightarrow R_0.\vec{w}_{p_{t_0}} := S^{-1}.R.\vec{w}_{p_{t_0}} \\
\end{array} \right. \quad \text{endomorphism in } \mathbb{R}^n_0. \quad (F.9)
\]
Proposition F.4 The endomorphism $R_0 = S^{-1} \circ R \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ is a rotation operator in $(\mathbb{R}^n, (\cdot, \cdot)_G)$:

$$R_0^{-1} = R_T^T \quad \text{in} \quad (\mathbb{R}^n, (\cdot, \cdot)_G).$$

(F.10)

And

$$F = S \circ R_0 \circ U.$$  

(F.11)

Interpretation: $F$ is composed of: The pure deformation $U$ (endomorphism in $\mathbb{R}^n_0$), the rotation $R_0$ (endomorphism in $\mathbb{R}^n_0$), and the shift operator $S: \mathbb{R}^n_0 \to \mathbb{R}^n_T$ (from past to present time).

Proof.

$$\left(R_0^T \tilde{W}_2, \tilde{W}_1\right)_G = (R_0, \tilde{W}_1, \tilde{W}_2)_G \quad \text{(definition of the transposed)}$$

$$= (S^{-1} \cdot R \cdot \tilde{W}_1, \tilde{W}_2)_G \quad \text{(definition of $R_0$)}$$

$$= (R \cdot \tilde{W}_1, \tilde{W}_2)_G \quad \text{($S$ is the algebraic identity)}$$

$$= (R^T \cdot \tilde{W}_2, \tilde{W}_1)_g \quad \text{(definition of $R^T$)}$$

$$= (R^{-1} \cdot \tilde{W}_2, \tilde{W}_1)_g \quad \text{(cf. (F.5))}$$

$$= (R_0^{-1} \cdot \tilde{W}_2, \tilde{W}_1)_G \quad \text{($S$ is the algebraic identity),}$$

true for all $\tilde{W}_1, \tilde{W}_2 \in \mathbb{R}^n_0$, thus $R_0^T = R_0^{-1}$ in $(\mathbb{R}^n_0, (\cdot, \cdot)_G)$. And (F.8) and (F.4) give (F.11).

Exercise F.5 Let $D = \text{diag}(\alpha_i)$, let $(\tilde{a}_i)$ be a basis in $\mathbb{R}^n_0$, let $P$ be the transition matrix from $(\tilde{a}_i)$ to $(\tilde{c}_i)$, so $[C]_{\tilde{a} \tilde{a}} = P.D.P^{-1}$; Prove $[U]_{\tilde{a} \tilde{a}} = P.\sqrt{D}.P^{-1}$. Case $(\tilde{a}_i)$ is a $(\cdot, \cdot)_g$-orthonormal basis?

Answer. The $n$ equations (F.1) (for $j = 1, \ldots, n$), read as the matrix equation $[C]_{\tilde{a} \tilde{a}} \cdot P = P.\sqrt{D}.P^{-1}$. And he $n$ equations (F.2) (for $j = 1, \ldots, n$), read as the matrix equation $[U]_{\tilde{a} \tilde{a}} \cdot P = P.\sqrt{D}$ since $[C]_{\tilde{a} \tilde{a}}$ is the $j$-th column of $P$.

If $(\tilde{a}_i)$ is a $(\cdot, \cdot)_g$-orthonormal basis, then $P^{-1} = P^T$ (costless computation of $P^{-1}$).

Remark F.6 Instead of $R_0 \in \mathcal{L}(\mathbb{R}^n_0; \mathbb{R}^n_0)$, cf. (F.8), you may prefer to consider $\tilde{R}_0 \in \mathcal{L}(\mathbb{R}^n_T; \mathbb{R}^n_T)$ defined by $\tilde{R}_0 = \tilde{R}_0 \circ S$, i.e., $\tilde{R}_0 = R \circ S^{-1}$.

F.2.2 $F = V R$ (left polar decomposition)

Same steps as for the right polar decomposition, but with pull-backs (with $F^{-1}$ instead of $F$).

Let $p_t = \Phi^T_t(p_0) \in \Omega_t$, let $b^T_0(p_t) := F^T_0(p_0) \circ (F^T_0)^T(p_t) \in \mathcal{L}(\mathbb{R}^n_T; \mathbb{R}^n_T)$, written $b = F \circ F^T$ (the left Cauchy–Green deformation tensor also called the Finger tensor). The endomorphism $b$ being symmetric definite positive: $\exists \beta_1, \ldots, \beta_n \in \mathbb{R}_+^n \text{ (the eigenvalues)}, \exists \tilde{d}_1, \ldots, \tilde{d}_n \in \mathbb{R}^n_0 \text{ (associated eigenvectors)}$, such that, for all $i = 1, \ldots, n$,

$$b \cdot \tilde{d}_i = \beta_i \tilde{d}_i, \quad \text{and} \quad (\tilde{d}_i) \text{ is a } (\cdot, \cdot)_g \text{-orthonormal basis in } \mathbb{R}^n_T.$$  

(F.13)

Then, define the unique endomorphism $V \in \mathcal{L}(\mathbb{R}^n_T; \mathbb{R}^n_T)$, called the left stretch tensor, by, for all $i = 1, \ldots, n$,

$$V \cdot \tilde{d}_i = b \cdot \tilde{d}_i, \quad \text{and} \quad V_{\text{noted}} := \sqrt{b}.$$  

(F.14)

(Full notation: $V_{\text{noted}}(p_t) = \sqrt{b^T_0(p_t)_{G_0}}$). Then define the linear map $R_t \in \mathcal{L}(\mathbb{R}^n_0; \mathbb{R}^n_T)$ by

$$R_t := V^{-1} \circ F_{\text{noted}} \equiv V^{-1} F,$$

so that

$$F = V \circ R_{\text{noted}} \equiv V.R_t, \quad \text{called the left polar decomposition of } F.$$  

(F.16)

Proposition F.7 We have: 1-

$$b = V \circ V_{\text{noted}} \equiv V^2, \quad \text{V is symmetric definite positive}, \quad R_t^{-1} = R_T^T.$$  

(F.17)

And the left polar decomposition $F = V \circ R$ is unique.

2- $R_0 = R$ and $V = R.U.R^{-1}$ (so $U$ and $V$ are similar), thus $U$ and $V$ have the same eigenvalues, i.e., $\alpha_i = \beta_i$ for all $i$, and $\tilde{d}_i = R.\tilde{c}_i$ for all $i$ gives a relation between eigenvectors.
1. Use $F^{-1}$ and $h^{-1} = (F^{-1})^T(F^{-1})$, instead of $F$ and $C = FT.F$, to get $F^{-1} = R_t^{-1}.U_t^{-1}$, cf. (F.3); Thus $F = U_t.R_t$. Then name $U_t = V$ to get (F.16) and (F.17).

2. $V.R_t = F = R.U = (R.U).R_t$, thus, by uniqueness of the right polar decomposition, $V = R.U.R_t^{-1}$ (so $U$ and $V$ are similar) and $R_t = R$. Thus, with (F.13), $\beta_i.d_i = V.d_i = R.U.R_t^{-1}.d_i$, thus with $\bar{c}_i = R^{-1}.d_i$, then $(\bar{c}_i)$ is an orthonormal basis in $\mathbb{R}^n_{t_0}$ and $\beta_i.R.\bar{c}_i = R.U.\bar{c}_i = \alpha_i.R.\bar{c}_i$ gives $\beta_i = \alpha_i$ and the $\bar{c}_i$ are eigenvectors of $U$, for all $i$.

**F.3 Elasticity: A Classical tensorial approach**

**F.3.1 Classical approach and issue**

With the infinitesimal strain “tensor”

$$\varepsilon = \frac{F + F^T}{2} - I,$$  \hfill (F.18)

the homogeneous isotropic elasticity constitutive law reads

$$\sigma(\Phi) = \sigma = \lambda \text{Tr}(\varepsilon)I + 2\mu\varepsilon, \hfill (F.19)$$

where $\lambda, \mu$ are the Lamé coefficients and $\sigma$ is the Cauchy stress “tensor”.

**Issue:** Adding $F$ and $F^T$ to make $\varepsilon$ is functionally a mathematical nonsense: $F = F_t^{\prime\prime}(p_t) : \mathbb{R}^n_{t_0} \rightarrow \mathbb{R}^n_{p_t}$ while $F^T = F^T(p_t) : \mathbb{R}^n_{p_t} \rightarrow \mathbb{R}^n_{t_0}$ and $I$ is some identity operator (in $\mathbb{R}^n_{t_0}$, $\mathbb{R}^n_{p_t}$). So what could be the set of definition for $\varepsilon$? What could be the meaning of $\varepsilon.\bar{n} = \frac{1}{2}(F(\bar{n}) + F^T.\bar{n})$, of $\text{Tr}(\bar{\varepsilon})$, and of

$$\sigma.\bar{n} = \lambda \text{Tr}(\varepsilon)\bar{n} + 2\mu\varepsilon.\bar{n} \hfill (F.20)$$

since $\bar{n}$ as to be defined at $(t_0, p_{t_0})$ for $F$ and at $(t, p_t)$ for $F^T$?

So, despite the claims, $\varepsilon$ in (F.18) is not a tensor. Neither is $\sigma$ (as defined in (F.19)).

So (F.18)-(F.19)-(F.20) don’t have a functional meaning (are not tensorial): They only have a matrix meaning (observer dependent), i.e. they mean, relative to some chosen bases,

$$[\varepsilon] := \frac{[F] + [F]^T}{2} - [I], \quad [\sigma] = \lambda \text{Tr}([\varepsilon])[I] + 2\mu[\varepsilon], \quad [\varepsilon].[\bar{n}] = \lambda \text{Tr}([\varepsilon])[\bar{n}] + 2\mu[\varepsilon].[\bar{n}]. \hfill (F.21)$$

**Remark F.8** Another pitfall: To justify the name “tensor” applied to $\varepsilon$ you may read: “For small displacements the Eulerian variable $p_t$ and the Lagrangian variable $p_{t_0}$ can be confused: $p_t \simeq p_{t_0}$. This means that, in the vicinity of $p_{t_0}$ along its trajectory, you use the zero-th order time Taylor expansion $p(t) = \Phi_t^{\prime\prime}(t, p_{t_0}) = p_{t_0} + o(1)$... But at the same time you also need to use $dV_{t_0}$ the differential of the (Lagrangian) velocity (e.g. for a finite element computation), so you first need to consider the first order time Taylor expansion $p(t) = \Phi_t^{\prime\prime}(t, p_{t_0}) = p_{t_0} + (t-t_0) \frac{d\Phi_t^{\prime\prime}}{dt}(t, p_{t_0}) + o(t-t_0) = p_{t_0} + hV_{t_0}(t, p_{t_0}) + o(t-t_0)$ (cf. (4.12)). No mathematics allows this (the zero-th order wins: $p(t) = p_{t_0} + o(1)$).”

**F.3.2 A functional (tensorial) formulation?**

**Question:** Can the constitutive law (F.19) be modified into a tensorial expression (a functional expression)? **Answer:** Yes: A proposal:

1. Consider the “right polar decomposition” $F = R.U$, cf. (F.3). The Green Lagrange tensor $E = \frac{\varepsilon - I}{2}$ (endomorphism in $\mathbb{R}^n_{t_0}$) then reads, with (F.5),

$$E = \frac{U^2 - I_{t_0}}{2} = \frac{(U - I_{t_0})^2 + 2(U - I_{t_0})}{2} \hfill (F.22)$$

(the Green–Lagrange tensor is independent of the rotation $R$), thus, with $U - I_{t_0} = O(h)$ (small deformation approximation), we get the modified infinitesimal strain tensor at $p_{t_0} \in \Omega_{t_0}$

$$\bar{\varepsilon} = U - I_{t_0} \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_{t_0}) \hfill (F.23)$$

endomorphism in $\mathbb{R}^n_{t_0}$ (to compare with $\varepsilon$ which is not a function, cf. the previous §). And an endomorphism is naturally canonically identified with a tensor: (F.23) is seen as a tensorial expression. (Full
notation $\tilde{\mathbb{E}}_{G}(p_{t_{0}}) = U_{t_{0}}^{G}(p_{t_{0}}) - I_{t_{0}}(p_{t_{0}})$ in $L(\tilde{\mathbb{E}}_{t_{0}}^{n}, \tilde{\mathbb{E}}_{t_{0}}^{n})$. And, for all $\tilde{W} \in \tilde{\mathbb{E}}_{t_{0}}^{n}$ we get

$$\tilde{\mathbb{E}}_{G}(\tilde{W}) = U_{t_{0}}^{G}(\tilde{W}) - R^{-1} \tilde{w} - \tilde{W} \in \tilde{\mathbb{E}}_{t_{0}}^{n}, \quad \text{when} \quad \tilde{w} = F.\tilde{W} = R.U.\tilde{W} \quad \text{(push-forward).} \quad (F.24)$$

**Interpretation:** From $\tilde{w} = F.\tilde{W} \in \tilde{\mathbb{E}}_{t_{0}}^{n}$ (the deformed by the motion), remove the “rigid body rotation” $R$, in $||\tilde{\mathbb{E}}_{G}(\tilde{W})||_{G} = ||(U-I_{t_{0}}).\tilde{W}||_{G}$ measures the relative elongation undergone by $\tilde{W}$. And

$$R.(\tilde{\mathbb{E}}_{G}(\tilde{W})) = F.\tilde{W} - R.\tilde{W} = \tilde{w} - R.\tilde{W} \in \tilde{\mathbb{E}}_{t_{0}}^{n}, \quad \text{when} \quad \tilde{w} = F.\tilde{W} = (\text{push-forward}), \quad (F.25)$$

enables to compare the motion deformed vector $\tilde{w} = F.\tilde{W}$ with $R.\tilde{W}$ the rotated (unstretched) initial $\tilde{W}$ to get $R.(\tilde{\mathbb{E}}_{G}(\tilde{W}))$, both vectors $\tilde{w}$ and $R.\tilde{W}$ being in $\tilde{\mathbb{E}}_{t_{0}}^{n}$.

2. Then, at $p_{t_{0}} \in \Omega_{t_{0}}$, consider the stress tensor $\tilde{\Sigma}(\Phi)$ noted $\tilde{\Sigma} \in L(\tilde{\mathbb{E}}_{t_{0}}^{n}, \tilde{\mathbb{E}}_{t_{0}}^{n})$ (functionally well defined) defined by

$$\tilde{\Sigma} = \lambda \text{Tr}(\tilde{\mathbb{E}})I_{t_{0}} + 2\mu \tilde{\mathbb{E}} = \lambda \text{Tr}(U(I_{t_{0}}))I_{t_{0}} + 2\mu(U(I_{t_{0}})). \quad (F.26)$$

In particular the trace $\text{Tr}(\tilde{\mathbb{E}})$ is well defined since $\tilde{\mathbb{E}}$ is an endomorphism. Then for any $\tilde{W} \in \tilde{\mathbb{E}}_{t_{0}}^{n}$ you get in $\tilde{\mathbb{E}}_{t_{0}}^{n}$, at $p_{t_{0}} \in \Omega_{t_{0}}$,

$$\tilde{\Sigma}.\tilde{W} = \lambda \text{Tr}(\tilde{\mathbb{E}})\tilde{W} + 2\mu \tilde{\mathbb{E}}\tilde{W} = \lambda \text{Tr}(U(I_{t_{0}}))\tilde{W} + 2\mu(U(I_{t_{0}}))\tilde{W} \quad (F.27)$$

(functionally well defined in $\tilde{\mathbb{E}}_{t_{0}}^{n}$), vectorial stress which is independent of any rotation $R$.

3. Then rotate and shift with $R$ to get into $\tilde{\mathbb{E}}_{t_{0}}^{n}$, at $p_{t_{1}}$,

$$R.(\tilde{\Sigma}) = \lambda \text{Tr}(\tilde{\mathbb{E}})R.\tilde{W} + 2\mu R.(\tilde{\mathbb{E}})R.\tilde{W} = \lambda \text{Tr}(U(I_{t_{0}}))R.\tilde{W} + 2\mu R.(U(I_{t_{0}}))R.\tilde{W} \quad (F.28)$$

$$R.\tilde{W} = \lambda \text{Tr}(\tilde{\mathbb{E}})R.\tilde{W} + 2\mu R.\tilde{\mathbb{E}}R.\tilde{W} = \lambda \text{Tr}(U(I_{t_{0}}))R.\tilde{W} + 2\mu(U(I_{t_{0}}))R.\tilde{W}, \quad \text{where} \quad \tilde{w} = F.\tilde{W} = R.U.\tilde{W}. \quad (F.29)$$

You have defined the two point tensor

$$R.\tilde{\Sigma} = \lambda \text{Tr}(\tilde{\mathbb{E}})R + 2\mu R.(\tilde{\mathbb{E}})R \in L(\tilde{\mathbb{E}}_{t_{0}}^{n}, \tilde{\mathbb{E}}_{t_{0}}^{n}). \quad (F.30)$$

4. Then you can propose the constitutive law with the stress tensor (more precisely the symmetric endomorphism) in $\tilde{\mathbb{E}}_{t_{0}}^{n}$ given by

$$(\tilde{\mathbb{E}}(\Phi)) = (\tilde{\mathbb{E}}) = R \circ \tilde{\Sigma} \circ R^{-1} \text{ noted } \tilde{\mathbb{E}}_{t_{0}}^{n} \in L(\tilde{\mathbb{E}}_{t_{0}}^{n}, \tilde{\mathbb{E}}_{t_{0}}^{n}). \quad (F.31)$$

(Functionally well defined.) And, for all vector fields $\tilde{w}$ defined in $\Omega_{t_{0}}$, you get the (functionally well defined) vector field

$$\tilde{\mathbb{E}}_{t_{0}}^{n} \tilde{w} = R.\tilde{\Sigma}R^{-1}.\tilde{w} \in \tilde{\mathbb{E}}_{t_{0}}^{n}. \quad (F.31)$$

**Interpretation** of (F.30)-(F.31): Shift and rigid rotate backward by applying $R^{-1}$, apply the elastic stress law with $\tilde{\Sigma}$ which corresponds to a rotation free motion (Noll’s frame indifference principle), then shift and rigid rotate forward by applying $R$.

**Detailed expression for (F.30)-(F.31):** With $\text{Tr}(R.R^{-1}) = \text{Tr}(\tilde{\mathbb{E}})$ (see exercise F.10), we get, at $(t, p_{t_{1}})$,

$$\tilde{\mathbb{E}} = \lambda \text{Tr}(\tilde{\mathbb{E}})I_{t} + 2\mu R.(\tilde{\mathbb{E}})R^{-1} = \lambda \text{Tr}(U(I_{t_{0}}))I_{t} + 2\mu R.(U(I_{t_{0}}))R^{-1} \quad (F.32)$$

And for any $\tilde{w} \in \tilde{\mathbb{E}}_{t_{0}}^{n}$, and with $\tilde{w} = R.\tilde{W}$, you get the tensorial expression

$$\tilde{\mathbb{E}}_{t_{0}}^{n} \tilde{w} = \lambda \text{Tr}(\tilde{\mathbb{E}})\tilde{w} + 2\mu R.(\tilde{\mathbb{E}})R.\tilde{W} = \lambda \text{Tr}(U(I_{t_{0}}))\tilde{w} + 2\mu R.(U(I_{t_{0}}))\tilde{W} \quad (F.33)$$

To compare with the classical functionally meaningless (F.20).
Remark F.9 Doing so, you avoid the use of the Piola–Kirchhoff tensors.

Exercise F.10 Prove: $\text{Tr}(R_{\mathbb{R}^p} R^{-1}) = \text{Tr}(\bar{\mathbb{R}^n}) = \sum_i (\alpha_i - 1)$. (NB: $\bar{\mathbb{R}^n}$ is an endomorphism in $\mathbb{R}^n$ while $R_{\mathbb{R}^p} R^{-1}$ is an endomorphism in $\bar{\mathbb{R}^n}$.)

Answer. $\det(R_{\mathbb{R}^p} R^{-1} - \lambda I) = \det(R_{\mathbb{R}^p} R^{-1} - \lambda I_{\mathbb{R}^n}) = \det(\bar{R}, \bar{R}^{-1} - \lambda I) = \det(\bar{R}, \bar{R}^{-1} - \lambda I)$ for all Euclidean bases $(\bar{E}_i)$ and $(\bar{e}_i)$ in $\bar{\mathbb{R}^n}$ and $\mathbb{R}^n$. (With $L = \bar{\mathbb{R}^n}$ and components, $\text{Tr}(R_{\mathbb{R}^p} R^{-1}) = \sum_{ij} (R L)^i_j (R L)^{-1}) = \sum_{ij} (R L)^i_j L_k^i = \sum_{ij} L_j^i = \text{Tr}(L)$.)

Exercise F.11 Elongation in $\mathbb{R}^2$ along the first axis: origin $O$, same Euclidean basis $(\bar{E}_1, \bar{E}_2)$ and Euclidean dot product at all time, $t > 0$, $L, H > 0$, $P \in [0, L] \times [0, H]$, $|\bar{OP}|_2 = \left(\frac{X_0}{Y_0}\right)$, and $\left[\bar{OP}^T_2(P)\right]_2 = \left(\frac{X_0 + \xi (t-t_0)L}{Y_0}\right)$, where $\xi = \theta (t-t_0)$ for $t > t_0$.

1. Give $F$, $C$, $U = \sqrt{C}$ and $R = F^{-1}U$. Relation with the classical expression?

2. Spring $\bar{OP} = \bar{OC}_0(s) = X_0 \bar{E}_1 + Y_0 \bar{E}_2 + s \bar{W}$, i.e. $|\bar{OP}|_2 = |\bar{OC}_0|_2 = \left(\frac{X_0 + s W_1}{Y_0 + s W_2}\right)$ with $s \in [0, L]$ and $\bar{W} = W_1 \bar{E}_1 + W_2 \bar{E}_2$. Give the deformed spring, i.e. give $p = \bar{C}_0(s) = \Phi^\alpha_1(c_0(s))$, and $\bar{C}_0$, and the stretch ratio.

Answer. 1. $[F] = [d\Phi] = \left(\begin{array}{cc} \frac{\alpha + 1}{\alpha} & 0 \\ 0 & 1 \end{array}\right)$, same Euclidean dot product and basis at all time, thus $[F^T] = [F] = [F]$, then $[C] = [F]^T[F] = [F] = \left(\begin{array}{cc} 0 & \frac{\alpha + 1}{\alpha} \\ \frac{\alpha + 1}{\alpha} & 0 \end{array}\right)$, thus $[U] = [F] = \left(\begin{array}{cc} 0 & \frac{\alpha + 1}{\alpha} \\ \frac{\alpha + 1}{\alpha} & 0 \end{array}\right)$, thus $[R] = [I]$. All the matrices are given relative to the basis $(\bar{E}_i)$, since $F, C, U, R$ (e.g., $C \bar{E}_1 = (\alpha + 1) \bar{E}_1$ and $C \bar{E}_2 = \bar{E}_2$).

2. $\bar{OC}_0(s) = \bar{OP} \left(\begin{array}{c} c \left(\frac{\alpha + 1}{\alpha}\right) \\ \frac{\alpha + 1}{\alpha} \end{array}\right)$, thus $\bar{C}_0(s) = \left(\begin{array}{c} W_1 \left(\frac{\alpha + 1}{\alpha}\right) \\ W_2 \frac{\alpha + 1}{\alpha} \end{array}\right)$, stretch ratio $\frac{W_1^2(\alpha + 1)^2 + W_2^2}{W_1^2 + W_2^2}$ at $(t, p, u)$.

Exercise F.12 Simple shear in $\mathbb{R}^2$: $\left[\bar{OP}^T_2(P)\right]_2 = \left(\begin{array}{cc} X + \xi (t-t_0)Y \\ Y \end{array}\right)$ or $\left(\begin{array}{c} X + \xi (t-t_0)Y \\ Y \end{array}\right)$.

Answer. 1. $[F] = \left(\begin{array}{cc} 1 & \frac{\alpha}{1} \\ 0 & 1 \end{array}\right)$, $[C] = \left(\begin{array}{cc} 1 & 0 \\ \frac{\alpha}{1} & 1 \end{array}\right)$, $[F] = \frac{1}{\frac{\alpha}{1} + 1}$. Eigenvalues: $\det(C - \lambda I) = \lambda^2 - (2 + \alpha^2) \lambda + 1$. Discriminant $\Delta = (2 + \alpha^2)^2 - 4 = \alpha^2 (\alpha^2 + 4)$.

2. $\bar{OC}_0(s) = \bar{OP} \left(\begin{array}{c} c \left(\frac{\alpha + 1}{\alpha}\right) \\ \frac{\alpha + 1}{\alpha} \end{array}\right)$, thus $\bar{C}_0(s) = \left(\begin{array}{c} W_1 \left(\frac{\alpha + 1}{\alpha}\right) \\ W_2 \frac{\alpha + 1}{\alpha} \end{array}\right)$, stretch ratio $\frac{W_1^2(\alpha + 1)^2 + W_2^2}{W_1^2 + W_2^2}$ at $(t, p, u)$.

F.3.3 Second functional formulation: With the the Finger tensor

The above approach uses the push-forward: It uses $F$, i.e. you arrive with your memory. You may prefer to use the pull-back, i.e. use $F^{-1}$ (you remember the past): Then you use $F^{-1} = R^{-1} V^{-1}$ the right polar decomposition of $F^{-1}$, and you consider the tensor

$$\bar{\mathbb{R}^n} = V^{-1} - L_1 \in \mathcal{L}(\bar{\mathbb{R}^n}; \bar{\mathbb{R}^n}),$$

and

$$\sigma_i (\Phi) = \gamma \text{Tr}(\bar{\mathbb{R}^n}) L_1 + 2 \mu \bar{\mathbb{R}^n} \text{ and } \bar{\sigma}_i \bar{\n}_i = \gamma \text{Tr}(\bar{\mathbb{R}^n}) \bar{\n}_i + 2 \mu \bar{\mathbb{R}^n} \bar{\n}_i.$$

(Quantities well functionally defined: Gives a tensorial approach).
F.4 Elasticity: An objective approach?

In § F.3 you need to start with Euclidean dot products: From the start the result cannot be objective. Can you start without Euclidean dot products? Answer: Yes, a proposal:

Consider an initial curve \( c_t : \left\{ \begin{array}{l} -\varepsilon, \varepsilon \to \Omega_t \\ s \to c_t(s) \end{array} \right\} \) (a fiber) in \( \Omega_t \) with tangent vector \( \vec{w}_{t_0}(p_{t_0}) := c_{t_0}'(s) \) at \( p_{t_0} = c_{t_0}(s) \).

The motion transforms this curve into the curve \( c_t = \Phi_t \circ c_t : \left\{ \begin{array}{l} -\varepsilon, \varepsilon \to \Omega_t \\ s \to c_t(s) = \Phi_t(c_t(s)) \end{array} \right\} \) with tangent vector \( \vec{w}_{t_0}(p_{t_0}) := \Phi_{t_0}(c_{t_0}(s)) \) (push-forward) at \( p_{t_0} = \Phi_{t_0}(c_{t_0}(s)) \) (at \( s = c_{t_0}(s) \)). See (5.4) and figure 5.1

You can expect that, along the curve \( c_t \), the stress \( \tau(t, p_t) \) is a function of the strain \( \Phi_t(\vec{w}_{t_0}(p_{t_0})) = \vec{w}_{t_0}(p_{t_0}) \) (the push-forward).

And the Lie derivative enables to characterize the rate of stress, cf. § 15.5 and 15.7. Recall: With \( \vec{w}(t, p_t) = \frac{\partial \vec{w}}{\partial t}(t, P_{O_0}) \) the Eulerian velocity at \( t \) at \( p_t = \Phi(t, P_{O_0}) \) of a particle \( P_{O_0} \), the Lie derivative of a Eulerian vector field \( \vec{w} \) along \( \vec{v} \) is, at \( (t, p_t) \),

\[
\mathcal{L}_\vec{v}\vec{w}(t, p_t) = \lim_{\tau \to t} \frac{\vec{w}(\tau, p_\tau) - \vec{w}(t, p_t)}{\tau - t} = \frac{\partial \vec{w}}{\partial t} + d\vec{w}.\vec{v} - d\vec{v}.\vec{w}(t, p_t).
\]

(Moreover \( \mathcal{L}_\vec{v}\vec{w} \) is covariant objective = observer independent.) Hence the proposisiton in \( \mathbb{R}^n \), with the virtual power principle to measure the rate of stress (following Germain: “To know the weight of a suitcase you have to move it”).

1- Hypothesis: Suppose that \( n \) Eulerian vector fields \( \vec{w}_j \) (“force fields”), \( j = 1, ..., n \), enable to characterize a material.

2- Measure: Hypothesis: With a basis \( (\vec{e}_i) \) chosen in \( \mathbb{R}^n \), with \( (e^i) \) its (covariant) dual basis in \( \mathbb{R}^n \), assume that the internal power density at \( (t, p_t) \) is given by (at first order):

\[
p_{int}(\vec{v}) = \sum_{j=1}^{n} e^j \mathcal{L}_\vec{e}_j \vec{w}_j = \sum_{j=1}^{n} e^j (\frac{\partial \vec{w}_j}{\partial t} + d\vec{w}_j.\vec{v} - d\vec{v}.\vec{w}_j).
\]  

(F.37)

Up to now the approach is qualitative (objective, observer independent).

3- Then, in a Galilean referential, choose a Euclidean dot product \( \langle \cdot, \cdot \rangle \) in \( \mathbb{R}^n \): As usual (frame invariance hypothesis) the internal power vanishes if \( d\vec{v} = 0 \) (translation), thus we are left with

\[
p_{int}(\vec{v}) = -\sum_{j=1}^{n} e^j d\vec{v}.\vec{w}_j = -\tau \otimes d\vec{v}, \quad \text{where} \quad \tau = \sum_{j=1}^{n} \vec{w}_j \otimes e^j
\]  

(F.38)

is identified with an endomorphism in \( \mathbb{R}^n \). (The \( j \)-th column of \( [\tau]_{\vec{v}} \) is \( [\vec{w}_j]_{\vec{v}} \).) And (frame invariance hypothesis) the internal power vanishes if \( d\vec{v} - d\vec{v}^T = 0 \) (pure rotation), thus we are left with

\[
p_{int}(\vec{v}) = -\tau \otimes d\vec{v} + \tau \otimes d\vec{v}^T = -\sigma \otimes d\vec{v} \quad \text{where} \quad \sigma = \tau + \tau^T.
\]

(F.39)

Example F.13 E.g., 2-D case for simplicity. May be applied to orthotropic elasticity which fibers at rest are, e.g., along \( \vec{e}_1 \). With an elongation type motion \( \Phi_{x,t} \) given by \([F_{x,t}(p_{t_0})] = [d(\Phi_{x,t})(p_{t_0})] = \begin{pmatrix} 1+\alpha_{11}(p_{t_0}) & 0 \\ 0 & 1-\alpha_{22}(p_{t_0}) \end{pmatrix} \) you measure the Young moduli in the directions \( \vec{e}_1 \) and \( \vec{e}_2 \); And with a shear type motion given by \([F_{x,t}(p_{t_0})] = [d(\Phi_{x,t})(p_{t_0})] = \begin{pmatrix} 1 & \gamma_{12} \\ 0 & 1 \end{pmatrix} \) you measure the shear modulus.

For more complex material, you may need more vectors \( \vec{w}_j \) to describe the constitutive law, that is, (F.37) may be considered with \( \sum_{i=1}^{m} e^j \mathcal{L}_\vec{e}_j \vec{w}_j \) with \( m > n \).

NB: The Lie approach is different from the classic approach: 1- The classic approach looks for an order two stress tensor \( \sigma \) as a function of the deformation gradient \( F \), cf. (F.19). 2- The Lie approach begins with the measures of force vectors, which then enable to build \( \tau \) and the \( \sigma \) (the stress tensor), cf. (F.38)-(F.39).
E.g., application to visco-elasticity: You use Lie derivative of vector fields, instead of Lie derivative of tensor fields (which does not seem to give good result, see e.g. the Maxwell visco-elastic type laws, as well as footnote\(^1\) page 26).

And for second order approximation orders for \(p_{\text{data}}\) you can use second order Lie derivative \(\mathcal{L}_v(\mathcal{L}_v\bar{v}_i)\) of vector fields \(\bar{v}_i\); See e.g. https://www.isima.fr/leborgne/isimathMeca/PpvObj.pdf.

\section{G Finger tensor (left Cauchy–Green tensor)}

There is a lot of misunderstandings, as was the case for the Cauchy–Green deformation tensor \(C\), due to the lack of precision in the notation: Points at stake \((p \text{ or } P)\)? Definition domain and value domain? Euclidean dot product (English? French?)? Covariance? Contravariance?...

Finger’s approach is consistent with the foundations of relativity (Galileo classical relativity or Einstein general relativity), see remark 1.3: We can only do measurements at the current time \(t\), and we can refer to the past (memory), but we cannot refer to the future.

\subsection{G.1 Definition}

Let \(\Phi\) be motion, \(t_0 \in \mathbb{R}\), \(\Phi_{t_0}\) the associated motion, \(P \in \Omega_{t_0}, t \in \mathbb{R}\), and \(F_t^{t_0}(P) := \frac{d}{dt}\Phi_t^{t_0}(P) \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_t)\).

And let \((\cdot, \cdot)_G\) and \((\cdot, \cdot)_g\) be Euclidean dot products in \(\mathbb{R}^n_{t_0}\) and \(\mathbb{R}^n_t\).

**Definition G.1** The Finger tensor \(b_t^{t_0}(p_t)\), or left Cauchy–Green deformation tensor, at \(t\) at \(p_t\) relative to \(t_0\) is the endomorphism \(K(t, g) \in \mathcal{L}(\mathbb{R}^n_{t_0}, \mathbb{R}^n_t)\) defined with by

\[ b_t^{t_0}(p_t) := F_t^{t_0}(P).F_t^{t_0}(P)^T \quad \text{when } P = \Phi_t^{t_0}^{-1}(p_t), \quad \text{abusively written } b = F.F^T, \quad (G.1) \]

the last notation being confusing (domain, codomain, inner dot products?). (And \(b_t^{t_0}(p_t) := F_t^{t_0}(P).F_t^{t_0}(P)^T(t, p_t)\))

Thus, for all \(\bar{w}_{11}(p_t), \bar{w}_{22}(p_t) \in \Omega_t,

\[ b_t^{t_0}(p_t) = \begin{pmatrix} \bar{w}_{11}(p_t) & \bar{w}_{12}(p_t) \\ \bar{w}_{21}(p_t) & \bar{w}_{22}(p_t) \end{pmatrix} \] \quad (G.2)

for all \(\bar{w}_{11}(p_t), \bar{w}_{22}(p_t) \in \Omega_t\). Shortened abusive notation:

\[ b_t^{t_0} := F(F^T), \quad \text{and } \begin{pmatrix} \bar{w}_{11} & \bar{w}_{12} \\ \bar{w}_{21} & \bar{w}_{22} \end{pmatrix} = (F^T.F, F^T.F)(G.3) \]

To compare with \(C = F^T.F\) and \((C.W_1, W_2) = (F.W_1, F.W_2)(G.4)\).

So, the Finger tensor relative at \(t_0\) is defined by

\[ b_t^{t_0} : = \begin{cases} \bigcup_{t_0} \{t \times \Omega_t\} \rightarrow \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_t) \\ (t, p_t) \rightarrow b_t^{t_0}(t, p_t) := b_t^{t_0}(p_t) \end{cases} \quad (G.4) \]

**NB:** \(b_t^{t_0}\) looks like an Eulerian function, but isn’t, since it depends on a \(t_0\).

**Other definition:**

\[ B_t^{t_0}(P) := F_t^{t_0}(P).F_t^{t_0}(P)^T, \quad \text{written } B = F.F^T. \quad (G.5) \]

In other words \(B_t^{t_0} := b_t^{t_0} \circ (\Phi_t^{t_0})^{-1}\).

Pay attention: \(B_t^{t_0}(P) \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_t)\) is an endomorphism at \(t = p_t\), not at \(t_0\) at \(P\); E.g., \(B_t^{t_0}(p_t).\bar{w}_i(p_t)\) is meaningful, while \(B_t^{t_0}(p_t).\bar{W}_{i_j}(P)\) is absurd.

**Remark G.2** The push-forward by \(\Phi_t^{t_0}\) of the Cauchy–Green deformation tensor \(C = F^T.F\) is \(\Phi_*(C) = F.C.F^{-1} = F.F^T = b_t^{t_0}\) cf. (14.20): It is the Finger tensor. So the endomorphism \(C\) in \(\mathbb{R}^n_{t_0}\) is the pull-back of the endomorphism \(b_t^{t_0}\) in \(\mathbb{R}^n_t\); But this is not really interesting, since a push-forward (and a pull-back) doesn’t depend on any inner dot product, whereas the transposed \(F^T\) does. The Finger tensor, or more precisely its flat representation \(b_t^{t_0}\), can also be interpreted as being the metric \((\cdot, \cdot)_g\) at \(t\) (with the help of the Riesz representation theorem), since \(C^*\) is the pull-back of the metric \((\cdot, \cdot)_g\).
G.2 \( b^{-1}_s \)

With pull-backs (adapted to the virtual power principle at \( t \)), let \( \vec{w}_i(p_t) \in \Omega_t \) and let \( \vec{W}_i(P) = (F_t^{s_0}(P))^{-1}.\vec{w}_i(p_t) \). Then, with \( F := F_t^{s_0}(P), \vec{W}_i := \vec{W}_i(P) \) and \( \vec{w}_i := \vec{w}_i(p_t) \),

\[
(\vec{W}_1, \vec{W}_2)_G = (F^{-1}.\vec{w}_1, F^{-1}.\vec{w}_2)_G = (F^{-T}.F^{-1}.\vec{w}_1, \vec{w}_2)_g = (b^{-1}_s.\vec{w}_1, \vec{w}_2)_g. \tag{G.6}
\]

So \( (b^{s_0}_e)^{-1} \) is useful: with \( p(t) = \Phi^s_t(P) \),

\[
(b^{s_0}_e)^{-1} : \begin{cases} 
\Omega_t & \to \mathcal{L}(\overline{\mathbb{R}}^n_t; \overline{\mathbb{R}}^n_t) \\
\quad p_t & \to (b^{s_0}_e)^{-1}(p_t) = F_t^{s_0}(P)^{-T}.F_t^{s_0}(P)^{-1} = H_t^{s_0}(p_t)^T.H_t^{s_0}(p_t) 
\end{cases} \tag{G.7}
\]

where \( H_t^{s_0}(p_t) = (F_t^{s_0}(P))^{-1} \) cf. (12.3). Thus we can define

\[
(b^{s_0}_e)^{-1} : \begin{cases} 
\cup \{\{t\} \times \Omega_t\} & \to \mathcal{L}(\overline{\mathbb{R}}^n_t; \overline{\mathbb{R}}^n_t) \\
\quad (t, p_t) & \to (b^{s_0}_e)^{-1}(t, p_t) := (b^{s_0}_e)^{-1}(p_t) 
\end{cases} \tag{G.8}
\]

**Remark:** \( (b^{s_0}_e)^{-1} \) looks like a Eulerian function, but isn’t, since it depends on \( t_0. \)

Simplified notation:

\[
\begin{align*}
\frac{b^{-1}_s}{\frac{D}{D_t}} &= H^T.H, & \text{to compare with } C = F^T.F. \tag{G.9}
\end{align*}
\]

With \( \vec{w} = F.\vec{W} \) we have

\[
\frac{b^{-1}_s}{\frac{D}{D_t}}.\vec{w} = H^T.\vec{W}, & \text{ to compare with } C.\vec{W} = F^T.\vec{w}. \tag{G.10}
\]

With \( \vec{W}_i = F^{-1}.\vec{w}_i \), so \( \vec{w}_i = F.\vec{W}_i \), we have:

\[
(b^{-1}_s.\vec{w}_1, \vec{w}_2)_g = (\vec{W}_1, \vec{W}_2)_G, \quad \text{to compare with } (C.\vec{W}_1, \vec{W}_2)_G = (\vec{w}_1, \vec{w}_2)_g. \tag{G.11}
\]

**Remark G.3** \( b(p_t) = F(P).F(P)^T \) and \( C(P) = F(P)^T.F(P) \) give

\[
\frac{b(p_t)}{\frac{D}{D_t}}.F(P) = F(P).C(P), \quad \text{ when } p_t = \Phi_t^{s_0}(P) \tag{G.12}
\]

written \( b = F.C.F^{-1}. \). Thus \( b^{-1}_s = F.C^{-1}.F^{-1} \), so

\[
\Phi_t^{s_0}(b^{-1}_s) = F^{-1}.b^{-1}_s.F = F^{-1}.F^{-T} = (F^T.F)^{-1} = C^{-1}, \tag{G.13}
\]

i.e. the pull-back of \( b^{-1}_s \) is \( C^{-1} \), i.e. \( b^{-1}_s \) is the push-forward of \( C^{-1} \). However see remark G.2.

G.3 **Time derivatives of \( b^{-1}_s \)**

With (G.8) let \( (b^{s_0}_e)^{-1} \) - noted \( b^{-1}_s = H^T.H \). Thus, along a trajectory, and with (12.7), we get

\[
\frac{D}{D_t}b^{-1}_s = \frac{D}{D_t}H^T.H + H^T.\frac{DH}{D_t} = -d\vec{v}^T.H^T.H - H^T.H.d\vec{v} = -b^{-1}_s.d\vec{v} - d\vec{v}^T.b^{-1}_s. \tag{G.14}
\]

So,

\[
\frac{D^2b^{-1}_s}{D^2t} = \frac{D}{D_t}b^{-1}_s.d\vec{v} - b^{-1}_s \frac{D}{D_t}(d\vec{v}) + (idem)^T
\]

\[
= (b^{-1}_s.d\vec{v} + d\vec{v}^T.b^{-1}_s).d\vec{v} - b^{-1}_s.(d\vec{v} - d\vec{v}^T.d\vec{v}) + d\vec{v}^T.(b^{-1}_s.d\vec{v} + d\vec{v}^T.b^{-1}_s) - (d\vec{v}^T - d\vec{v}^T.d\vec{v}^T.b^{-1}_s)
\]

\[
= b^{-1}_s.(2d\vec{v}^T.d\vec{v} - d\vec{v}) + 2d\vec{v}^T.b^{-1}_s.d\vec{v} + (2d\vec{v}^T.d\vec{v} - d\vec{v})b^{-1}_s
\]

\[
= b^{-1}_s.(d\vec{v}^T.d\vec{v} - \frac{D}{D_t}(d\vec{v}) + 2d\vec{v}^T.b^{-1}_s.d\vec{v} + (d\vec{v}^T.d\vec{v} - \frac{D}{D_t}(d\vec{v})^T)b^{-1}_s. \tag{G.15}
\]
Exercise G.4 Prove (G.14) with (G.11).

Answer: (G.11) gives \( \frac{D}{D t} \left( \frac{\delta^{-1}}{\delta} (\bar{w}_1, \bar{w}_2) \right)_g = 0 = \left( \frac{D}{D t} - \frac{\delta}{\delta} \right) \left( \bar{w}_1, \bar{w}_2 \right)_g + \left( \frac{D}{D t} - \frac{\delta}{\delta} \right) \left( \bar{w}_1, \bar{w}_2 \right)_g \), and \( \bar{w}_1(t, p(t)) = F_t^o(t, P, W_{t0}(P)) \) gives \( \frac{DA}{D t} \bar{w}_1, \bar{w}_2 = \bar{d} \bar{w}_1, \bar{w}_2, \) thus \( \left( \frac{D}{D t} \bar{w}_1, \bar{w}_2 \right)_g + \left( \frac{D}{D t} \bar{d} \bar{w}_1, \bar{w}_2 \right)_g + \left( \frac{D}{D t} \bar{d} \bar{w}_1, \bar{w}_2 \right)_g = 0, \) thus (G.14).

Exercise G.5 Prove (G.14) with \( FT \cdot \delta^{-1} F = I_{t0} \).

Answer: \( \delta^{-1} = (F, F')^{-1} = F^{-1} F'^{-1} \) gives \( FT \cdot \delta^{-1} F = I_{t0} \), thus \( (F, F')' \delta^{-1} F + F' \cdot \frac{D}{D t} \delta^{-1} F + F' \cdot \delta^{-1} F' = 0, \) thus \( F' \cdot \delta^{-1} F + F' \cdot \frac{D}{D t} \delta^{-1} F + F'T \cdot \delta^{-1} F' = 0 \), thus (G.14).

H Green–Lagrange deformation tensor

H.1 Definition

(E.15) gives

\( (\bar{w}_1, \bar{w}_2)_g = (\bar{W}_1, \bar{W}_2)_G = ((C_{t0}^o(P) - I_{t0}), \bar{W}_1, \bar{W}_2)_G. \) (H.1)

Definition H.1 The Green–Lagrange tensor (or Green–Saint Venant tensor) at \( P \) relative to \( t_0 \) and \( t \) is the endomorphism \( E_{t0}^o(P) \in \mathbb{L}(\mathbb{R}^n; \mathbb{R}^n) \) defined by

\[ E_{t0}^o(P) := \frac{1}{2} (C_{t0}^o(P) - I_{t0}), \text{ written } E = \frac{1}{2} (C - I) = \frac{F^T F - I}{2}. \] (H.2)

(See \( E_{t0}^o = 0 \) for solid body motions.) And \( E_{t0}^o : \Omega \rightarrow \mathbb{L}(\mathbb{R}^n; \mathbb{R}^n) \) is the Green–Lagrange tensor relative to \( t_0 \) and \( t \).

Thus

\( (\bar{w}_1, \bar{w}_2)_g = (\bar{W}_1, \bar{W}_2)_G = 2(E.\bar{W}_1, \bar{W}_2)_G. \) (H.3)

Example H.2 If \( \bar{W}_1 \perp \bar{W}_2 \), then \( (\bar{w}_1, \bar{w}_2)_g = 2(E.\bar{W}_1, \bar{W}_2)_G, \) cf. (H.1), therefore \( 2E \) quantifies the evolution of vectors initially at right angle (at \( t_0 \)) that then let themselves be transported (deformed) by the flow.

Example H.3 Comparison of lengths: If \( \bar{W}_1 = \bar{W}_2 = \text{note} \bar{W} \) and \( \bar{w} = F.\bar{W} \), then, cf. (H.3),

\[ \left\| \bar{w} \right\|_g = \left\| \bar{W} \right\|_G \] (H.4)

For small deformations, i.e. \( \left\| \bar{w} \right\| - \left\| \bar{W} \right\| = o(\left\| \bar{W} \right\|) \), we get

\[ \left\| \bar{w} \right\|_g - \left\| \bar{W} \right\|_G = (||\bar{w}||_g - ||\bar{W}||_G)(||\bar{w}||_g + ||\bar{W}||_G) = 2(E, \bar{W}, \bar{W})_G. \] (H.5)

so \( 2 \frac{||\bar{w}||_g - ||\bar{W}||_G}{||W||_G}(1 + o(1)) = 2(E, \frac{\bar{W}}{\|\bar{W}\|_G}, \frac{\bar{W}}{\|\bar{W}\|_G})_G. \) Thus, if \( ||\bar{w}|| - ||\bar{W}|| = o(\|\bar{W}\|) \) (small deformations) then

\[ \frac{||F, \bar{W}||_g - ||\bar{W}||\bar{W}||_G}{||\bar{W}||_G} = (E, \frac{\bar{W}}{\|\bar{W}\|_G}, \frac{\bar{W}}{\|\bar{W}\|_G})_G + o(1). \] (H.6)

Therefore, for small deformations, and for a vector \( \bar{W} \) at \( t_0 \) such that \( ||\bar{W}||_G = 1 \) the Green–Lagrange tensor \( E \) gives the rate of variations of its length when it let itself be transported (deformed) by the flow (when \( \bar{W} \) becomes \( \bar{w} = F.\bar{W} \)). This is meaningful when \( (\cdot, \cdot)_G = (\cdot, \cdot)_g \) (the same observer, using the same tools, at \( t_0 \) and \( t \).

H.2 Time Taylor expansion of \( E \)

\( E_{t0}^o(t) = \frac{1}{2} (C_{t0}^o(t) - I_{t0}) \), cf. (H.2), \( p(t) = \Phi_{t0}^o(t) \) and (E.46) give

\[ E_{t0}^o(t + h) = F_{t0}^o(t)^T \left( \frac{D}{D t} \bar{d} \bar{w} + \bar{d} \bar{w}^T + \frac{h^2}{2} \left( \frac{D^2}{D t^2} + D \cdot \bar{d} \bar{w} + D \cdot \bar{d} \bar{w}^T \cdot D \right)(t, p(t)) \right) F_{t0}^o(t + o(h^2)) \] (H.7)

See § E.6.2.
I  Euler–Almansi tensor

Euler–Almansi approach is consistent with the foundations of relativity (Galileo relativity or Einstein general relativity), see remark 1.3: We can only do measurements at the current time \( t \), and we can refer to the past (memory), but we cannot refer to the future.

I.1  Definition

We are at \( t \) in \( \Omega_0 \). Consider the Finger tensor \( \bar{b} = F.F^T \) and its inverse \( \bar{b}^{-1} = F^{-T}.F^T = H^T.H \) cf. (G.9).

**Definition I.1** Euler–Almansi tensor at \( p_t \in \Omega_0 \) is the endomorphism \( a_{\bar{b}}^{t_0}(p_t) \in \mathcal{L}(\mathbb{R}^n_0; \mathbb{R}^n_0) \) defined by

\[
a_{\bar{b}}^{t_0}(p_t) = \frac{1}{2} (I_t - \bar{b}^{t_0}(p_t)^{-1}) = \frac{1}{2} (I_t - H(p_t)^T.H(p_t)),
\]

written

\[
a_{\bar{b}} = \frac{1}{2} (I - \bar{b}^{-1}) = \frac{1}{2} (I - H^T.H),
\]

to compare with the Green–Lagrange tensor \( E = \frac{1}{2}(C - I) = \frac{1}{2}(F^T.F - I) \in \mathcal{L}(\mathbb{R}^n_0, \mathbb{R}^n_0) \).

**Remark:** \( a_{\bar{b}}^{t_0} \) looks like a Eulerian function, but isn’t, since it depends on \( t_0 \).

(G.11) gives \( \bar{w}_i = F.\bar{W}_i \)

\[
(\bar{w}_1, \bar{w}_2)_g - (\bar{W}_1, \bar{W}_2)_G = 2(\bar{a}, \bar{w}_1, \bar{w}_2)_g,
\]

to compare with \( (\bar{w}_1, \bar{w}_2)_g - (\bar{W}_1, \bar{W}_2)_G = 2(E.\bar{W}_1, \bar{W}_2)_G \). This also gives \( (\bar{a}, \bar{w}_1, \bar{w}_2)_g = (E.\bar{W}_1, \bar{W}_2)_G \).

And (I.2) gives

\[
F^T.a.F = E, \quad \text{i.e.} \quad a_{\bar{b}} = F^{-T}.E.F^{-1},
\]

standing for \( F_{\bar{t}}^{t_0}(P)^T.a_{\bar{b}}^{t_0}(p_t), F_{\bar{t}}^{t_0}(P) = E_{\bar{t}}^{t_0}(P) \) when \( p_t = \Phi_{t_0}^{\bar{t}}(P) \).

**Remark I.2** \( a_{\bar{b}}^{t_0} \) is not the push-forward of \( E_{\bar{t}}^{t_0} \) by \( \Phi_{t_0}^{\bar{t}} \) (the push-forward is \( F.E.F^{-1} \)), which is not a surprise: A push-forward is independent of any inner dot product, whereas the transposed \( F^T \) does depend on an inner dot product. Once again, an imposed Euclidean structure does not allow to go "objectively" from a configuration at \( t_0 \) to a configuration at \( t \).

I.2  Time Taylor expansion for \( a_{\bar{b}} \)

(G.14) and (G.15) give

\[
\frac{D a_{\bar{b}}}{D t} = \frac{b^{-1}.d\bar{v} + d\bar{v}^T.b^{-1}}{2}.
\]

Then

\[
\frac{D^2 a_{\bar{b}}}{D t^2} = \frac{b^{-1}.(d\bar{v}_T - 2d\bar{v}.d\bar{v}) + (d\bar{v}_T - 2d\bar{v}.d\bar{v}_T).b^{-1} - 2d\bar{v}.b^{-1}.d\bar{v})}{2}.
\]

J  Infinitesimal strain tensor \( \varepsilon_{\bar{b}} \)

J.1  Small displacement hypothesis

The small displacement displacement hypothesis for a regular motion (a diffeomorphism) reads: we work at \( t \) close to \( t_0 \).

Then the Taylor expansion at zeroth order gives

\[
F_{\bar{t}}^{t_0}(t_0 + h) = I_{t_0} + O(h) \quad (= F_{\bar{t}}^{t_0}(t_0) + O(h)).
\]

(Reminder: \( O(h) \) is the notation for a function satisfying \( |O(h)| \leq Ch \) with \( C \) independent of \( h \).)

Thus, since \( F_{\bar{t}}^{t_0}(t_0 + h) := F_{t_0 + h}^{t_0}(P) \), for any \( \bar{W}_P \in \mathbb{R}^n_0 \), we have

\[
F_{t_0 + h}^{t_0}(P).\bar{W}_P - \bar{W}_P = O(h) \quad \text{small displacement hypothesis when } h \ll 1.
\]

This impose the use of a unique inner dot product \( (\cdot, \cdot)_G = (\cdot, \cdot)_g \). (J.2) means \( ||F_{\bar{t}}^{t_0}(P).\bar{W}_P - \bar{W}_P||_g = O(t - t_0) \).
J.2 Definition of $\varepsilon$

A unique Euclidean dot product $(\cdot, \cdot)_G = (\cdot, \cdot)_G^I$ is given in $\mathbb{R}^n_{t_0}$ and in $\mathbb{R}^n_I$.

**Definition J.1** The infinitesimal strain tensor $\varepsilon$ is the matrix defined by

$$\varepsilon := \frac{F + F^T}{2} - I \quad \text{(matrix meaning)},$$

that is, if some common basis $(\tilde{e}_i)$ is chosen in $\mathbb{R}^n_{t_0}$ and in $\mathbb{R}^n_I$,

$$[\varepsilon^{\alpha}(P)]_{ij} = \frac{[F^\alpha_t(P)]_{ij} + [F^\alpha_t(P)^T]_{ij}}{2} - I.$$

NB: (J.3) is not a tensorial definition, it is a matrix definition, see next remark J.2.

**Remark J.2** (J.3) has to be a matrix definition since $F^\alpha_t(P) \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_I)$ and $I_{t_0} \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_{t_0})$, and thus $F^\alpha_t(P), F^\alpha_t(P)^T$ and $I_{t_0}$ do not have the same domain of definition, therefore they cannot be added. So $\varepsilon$ is not a tensor... But it is called an infinitesimal strain tensor... (some more confusion).

**Proposition J.3** With the small displacement hypothesis and the Green–Lagrange tensor $E = \frac{F^T F - I}{2}$, cf. (H.2), then (J.3) gives (matrix meaning)

$$E^\alpha_t = \varepsilon^\alpha + O(t-t_0),$$

that is, if some common basis $(\tilde{e}_i)$ is chosen in $\mathbb{R}^n_{t_0}$ and in $\mathbb{R}^n_I$,

$$[E^\alpha_t(P)]_{ij} = \frac{[F^\alpha_t(P)]_{ij} + [F^\alpha_t(P)^T]_{ij}}{2} - I + O(t-t_0).$$

**Proof.** (H.2) gives (matrix meaning)

$$2E = C - I = F^T F - I = (F^T - I) (F - I) + F^T F - 2I,$$

and (J.1) gives $(F^T - I) (F - I) = O(h)O(h) = O(h^2)$ which is negligible compared with $F^T F - 2I = O(h)$.

J.3 A second mathematical definition (Euler–Almansi)

We may prefer a definition from $\mathbb{R}^n_I$ (and looking at the past), that is, we may prefer a definition with $\tilde{g}$ the Euler–Almansi tensor, cf. §1 and (1.1).

We have $I - \tilde{b}^{-1} = I - H^T H = -(I - H^T) (I - H) + 2I - H^T H$ where $H$ stands for $H^\alpha_t(p_t)$. Thus, for small displacement we get $I - \tilde{b}^{-1} = 2I - H^T H + O(h)$, so

$$g(t, p(t)) = \tilde{g}(t, p(t)) + O(h) \quad \text{where} \quad \tilde{g} := I - \frac{H + H^T}{2}.$$

And, with $t = t_0 + h$ we have $F^\alpha_t(t, P) = I + (t-t_0) \tilde{d}(t, P) + o(t-t_0)$, cf. (5.42), and we have $H^\alpha_t(t, p(t)) = I - (t-t_0) \tilde{d}(t, P) + o(t-t_0)$ when $p(t) = \Phi^\alpha_t(t, P)$, cf. (12.10). Thus

$$F^\alpha_t(t, P) - I = I - H^\alpha_t(t, p(t)) + o(t-t_0).$$

Therefore, for small displacements:

$$g(t, p(t)) \simeq \tilde{g}(t, p(t)) \simeq \tilde{g}^\alpha_t(t, P) \simeq F^\alpha_t(t, P).$$

(Which has to be understood as a matrix relation, see remark J.2.)

**Example J.4** An elastic solid could thus be modeled as:

$$\tilde{g}^\alpha_t(p_t) = \lambda \text{Tr}(\tilde{g}(t, p_t)) I_t + 2\mu \tilde{g}(t, p_t),$$

with $\lambda, \mu \in \mathbb{R}$, to compare with $g^\alpha_t(P) = \lambda \text{Tr}(E^\alpha_t(P)) I_t + 2\mu E^\alpha_t(P)$ with $\lambda, \mu \in \mathbb{R}$. And for small deformations and matrix computations, $[a]$ can be estimated with $[\varepsilon]$, and $[E]$ can be estimated with $[\varepsilon]$.  

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Remark J.5 With $\gamma^{tn}$ tensor defined at $t$, cf. (J.11), we can use the objective double contraction $\gamma^{tn}(p_t) \cdot d\nu^t(p_t)$ in $\mathbb{R}^n$ as an operation on functions, see (Q.33)-(Q.34), and the application of the virtual power principle with $\gamma$ is meaningful.

To compare with $\gamma^{tn} = \sigma^{tn}(e)$ and $\gamma^{tn}(P) : d\nu^t(p_t)$ which is functionally meaningless since $\gamma^{tn}(P) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $d\nu^t(p_t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$: It is at most a matrix computation.

And $\text{div}(\gamma^{tn}(p_t))$ is meaningful, as well as the fundamental law of dynamics $\text{div}(\gamma^{tn}(p_t) + \gamma_t(p_t) = \rho \gamma(t, p_t)$ in $\mathbb{R}^n$. And doing this, we don’t need to use the Marsden and Hughes shifter, cf. (K.14)).

But the use of a Euclidean dot product, to define $F^T$, prevents this approach from being objective. 

K Displacement

K.1 The displacement vector $\vec{U}$

In $\mathbb{R}^n$, let $p_t = \Phi_t^{tn}(p_{t_0})$. Then the bi-point vector

$$\vec{U}^{tn}(p_{t_0}) = \Phi_t^{tn}(p_{t_0}) - I_{t_0}(p_{t_0}) = p_t - p_{t_0} = \overrightarrow{p_{t_0}}$$

(K.1)

is called the displacement vector at $p_{t_0}$, relative to $t_0$ and $t$. This defines the map

$$\vec{U}^{tn}_t : \begin{cases}
\Omega_{t_0} \to \mathbb{R}^n \\
p_{t_0} \to \vec{U}^{tn}_t(p_{t_0}) := p_t - p_{t_0} = \overrightarrow{p_{t_0}} \quad \text{when} \quad p_t = \Phi_t^{tn}(p_{t_0}).
\end{cases}$$

(K.2)

Remark K.1 $\vec{U}^{tn}(p_{t_0})$ doesn’t define a vector field (it is not tensorial), because $\vec{U}^{tn}(p_{t_0}) = p_t - p_{t_0}$ requires the time and space ubiquity gift: At $t$, $p_t \in \Omega_t$, and at $t_0$, $p_{t_0} \in \Omega_{t_0}$. So $\vec{U}^{tn}(p_{t_0})$ is just a bipoint vector in the affine space $\mathbb{R}^3$ which is straddling two configurations (time and space ubiquity required). In particular, it makes no sense on a non-plane surface (manifold). See §K.5 for example.

Remark K.2 For elastic solids in $\mathbb{R}^n$, the function $\vec{U}^{tn}$ is often considered to be the unknown (to be computed): But the “real” unknown is the motion $\Phi^{tn}$ (determined using fundamental equations, the law of behavior of the material, initial conditions and boundary conditions). And it is sometimes confused with the extension of a spring 1-D case. But see figure 5.1: $||\vec{w}_{t_0}(p_{t_0})||$ is the initial length and $||\vec{w}_{t_0}(p_{t_0})||$ is the current length of the spring, while the length of the displacement vector $\vec{U}^{tn}_t = p_t - p_{t_0}$ can be very long for a small elongation of the spring.

K.2 The differential of the displacement vector

The differential of $\vec{U}^{tn}_t$ at $p_{t_0}$ is the linear map (matrix meaning cf. remark K.1)

$$d\vec{U}^{tn}_t(p_{t_0}) = d\Phi_t^{tn}(p_{t_0}) - I_{t_0} = \Phi_t^{tn}(p_{t_0}) - I_{t_0}, \text{ written } d\vec{U} = F - I.$$ 

(K.3)

That is, with a unique basis at any time, (K.3) means

$$[d\vec{U}^{tn}_t(p_{t_0})]_{\xi} = [d\Phi_t^{tn}(p_{t_0})]_{\xi} - I.$$

(K.4)

Thus, with $\vec{w}_{t_0}(p_{t_0}) \in \mathbb{R}^n$ and its push-forward $\vec{w}_{t_0^*}(p_t) = F_{t_0^*}(t).\vec{w}_{t_0}(p_{t_0}) \in \mathbb{R}^t$, we get (matrix meaning)

$$d\vec{U}^{tn}_t(p_{t_0}).\vec{w}_{t_0}(p_{t_0}) = \vec{w}_{t_0^*}(p_t) - \vec{w}_{t_0}(p_{t_0}).$$

(K.5)

see figure 5.1. Thus we have defined (matrix meaning)

$$\vec{U}^{tn}_t : \begin{cases}
[t_0, T] \times \Omega_{t_0} \to \mathbb{R}^n \\
(t, p_{t_0}) \to \vec{U}^{tn}_t(t, p_{t_0}) := \vec{U}^{tn}_t(p_{t_0}),
\end{cases}$$

and $\vec{U}^{tn}_{\overrightarrow{p_{t_0}}} : \begin{cases}
[t_0, T] \to \mathbb{R}^n \\
t \to \vec{U}^{tn}_{\overrightarrow{p_{t_0}}}(t) := \vec{U}^{tn}_t(p_{t_0}).
\end{cases}$

(K.6)
K.3 Deformation "tensor" (matrix) $\varepsilon$, bis

(K.3) gives (matrix meaning)

$$F_\varepsilon^{t_o}(p_{t_0}) = I_{t_0} + d\tilde{U}_n(p_{t_0}), \quad \text{written } F = I + d\tilde{U}. \tag{K.7}$$

Therefore, Cauchy–Green deformation tensor $C = F^T.F$ reads (matrix meaning)

$$C_\varepsilon^{t_o}(p_{t_0}) = I_{t_0} + d\tilde{U}_n(p_{t_0}) + d\tilde{U}_n(p_{t_0})^T + d\tilde{U}_n(p_{t_0})^T d\tilde{U}_n(p_{t_0}), \quad \text{written } C = I_{t_0} + d\tilde{U} + d\tilde{U}^T + d\tilde{U}^T .d\tilde{U}. \tag{K.8}$$

Thus the Green–Lagrange deformation tensor $E = \frac{C - I}{2}$, cf. (H.2), reads (matrix meaning)

$$E_\varepsilon^{t_o}(p_{t_0}) = \frac{d\tilde{U}_n(p_{t_0}) + d\tilde{U}_n(p_{t_0})^T + d\tilde{U}_n(p_{t_0})^T .d\tilde{U}_n(p_{t_0})}{2}, \quad \text{written } E = \frac{d\tilde{U} + d\tilde{U}^T}{2} + \frac{1}{2} d\tilde{U}^T .d\tilde{U}. \tag{K.9}$$

Thus the deformation tensor $\varepsilon$ cf. (J.3), reads (matrix meaning)

$$\varepsilon = E - \frac{1}{2}(d\tilde{U})^T .d\tilde{U}, \tag{K.10}$$

with $\varepsilon$ the “linear part” of $E$ (small displacements).

Remark K.3 $C = F^T.F : \Omega_{t_0} \rightarrow \Omega_{t_0}$ is meaningful, cf. (E.15), but $I_{t_0} + d\tilde{U}_n(p_{t_0}) + d\tilde{U}_n(p_{t_0})^T + d\tilde{U}_n(p_{t_0})^T d\tilde{U}_n(p_{t_0})$ is meaningless since $d\tilde{U}_n(p_{t_0}) + d\tilde{U}_n(p_{t_0})^T$ is meaningless: How do you define $d\tilde{U}_n(p_{t_0})^T$? Is it $(F_\varepsilon^{t_o}(p_{t_0}) - I_{t_0})^T = F_\varepsilon^{t_o}(p_{t_0})^T - I_{t_0}$ while $F_\varepsilon^{t_o}(p_{t_0}) : \tilde{R}_n^n \rightarrow \tilde{R}_n^n$ and $I_{t_0} : \tilde{R}_n^n \rightarrow \tilde{R}_n^n$ do not have the same definition domain (if no ubiquity)? See § K.5 to deal with the problem.

K.4 Small displacement hypothesis, bis

(Usual introduction.) Let $p_t = \Phi_{t_0}^t(p_{t_0}), \tilde{W}_t \in \tilde{R}^n_{t_0}, \tilde{w}_t(p_t) = F_{t_0}^{t_o}(p_{t_0}) \tilde{W}_t(p_{t_0}) \in \tilde{R}^n_{t_0}$ (the push-forwards), written $\tilde{w}_t = F.\tilde{W}_t$. Then define (matrix meaning)

$$\Delta_t := \tilde{w}_t - \tilde{W}_t = d\tilde{u}_t.\tilde{W}_t, \quad \text{and } \|\Delta_t\|_{\infty} = \max(||\Delta_1||_{\tilde{R}^n_{t_0}}, ||\Delta_2||_{\tilde{R}^n_{t_0}}). \tag{K.11}$$

Then the small displacement hypothesis reads (matrix meaning):

$$\|\Delta_t\|_{\infty} = o(\|\tilde{W}_t\|_{\infty}). \tag{K.12}$$

Thus $\tilde{w}_t = \tilde{W}_t + \Delta_t$ (with $\Delta_t$ “small”) and the hypothesis $(\cdot, \cdot)_{G} = (\cdot, \cdot)_{G}$ (same inner dot product at $t_0$ and $t$) give

$$(\tilde{w}_1, \tilde{w}_2)_{G} = (\tilde{W}_1, \tilde{W}_2)_{G} + (\Delta_1, \tilde{W}_2)_{G} + (\tilde{W}_1, \Delta_2)_{G} + (\Delta_1, \Delta_2)_{G}.$$  \tag{K.12}

So (K.10) gives $2(E.\tilde{W}_1, \tilde{W}_2)_{G} = 2(\varepsilon \tilde{W}_1, \tilde{W}_2)_{G} + (d\tilde{U}^T .d\tilde{U} \tilde{W}_1, \tilde{W}_2)_{G}$, And (K.12) gives

$$(E.\tilde{W}_1, \tilde{W}_2)_{G} = (\varepsilon \tilde{W}_1, \tilde{W}_2)_{G} + O(||\Delta_t||_{\infty}^2), \tag{K.13}$$

so $E_\varepsilon^{t_o}$ is approximated by $\varepsilon_\varepsilon^{t_o}$, that is, $E_\varepsilon^{t_o} \simeq \varepsilon_\varepsilon^{t_o}$ (matrix meaning).

K.5 Displacement vector with differential geometry

K.5.1 The shifter

We give the steps, see Marsden–Hughes [12]. The complexity introduced is due to the small displacement hypothesis applied to the Green–Lagrange tensor $E = E_\varepsilon^{t_o}$: In fact, to get a linear result (a result as a function of $F = F_\varepsilon^{t_o}$), cf. (J.5), you take “the square of $F$” (more precisely you take $F^T.F$) that you linearize to get back to $F$...

(...When it is simpler just to consider $F$ from the start, isn’t it? That’s what the push-forward and the Lie derivative do. But here, classic approach, so we “linearize the square of $F$”... to get back to $F$...)

Let $P \in \Omega_{t_0}, \tilde{W}_P \in \tilde{R}_n^{t_0}, p_t = \Phi_{t_0}^t(P) \in \Omega_t$, and $\tilde{w}_{p_t} = F_{t_0}^{t_o}(P) \tilde{W}_P \in \tilde{R}_n^{t_0}$ (push-forward).
• Affine case $\mathbb{R}^n$ (continuum mechanics): With $p_t = \Phi^t_0(P)$, the shifter is:

$$\widetilde{S}^t_0 : \begin{cases} 
\Omega_0 \times \tilde{\mathbb{R}}^n_0 \to \Omega_0 \times \tilde{\mathbb{R}}^n_0 \\
(P, \tilde{Z}^t_0) \to \tilde{S}^t_0(P, \tilde{Z}^t_0) = (p_t, S^t_0(\tilde{Z}^t_0)) \quad \text{with} \quad S^t_0(\tilde{Z}^t_0) = \tilde{Z}^t_0.
\end{cases} \quad (K.14)$$

(The vector is unchanged but the time and the application point have changed: A real observer has no ubiquity gift.) So:

$$S^t_0 \in \mathcal{L}(\tilde{\mathbb{R}}^n_0; \tilde{\mathbb{R}}^n_0) \quad \text{and} \quad [S^t_0]_{\Omega} = I \text{ identity matrix}, \quad (K.15)$$

the matrix equality being possible after the choice of a unique basis at $t_0$ and at $t$. And (simplified notation) $\tilde{S}^t_0(P, \tilde{Z}^t_0) = \text{noted} S^t_0(\tilde{Z}^t_0)$. Then the deformation tensor $\varepsilon^t_0$ at $P$ can be defined by

$$\varepsilon^t_0(P, \tilde{Z}^t_0(P)) = \frac{(S^t_0)^{-1}(F^t_0(P, \tilde{Z}^t_0(P)) + F^t_0(P)(S^t_0(\tilde{Z}^t_0(P)), \tilde{Z}^t_0(P)) - \tilde{Z}^t_0(P).} {2} \quad (K.16)$$

• In a manifold: $\Omega$ is a manifold (like a surface in $\mathbb{R}^3$ from which we cannot take off). Let $T_P\Omega_{t_0}$ be the tangent space at $P$ (the fiber at $P$), and $T_{\Omega_{t_0}}$ be the tangent space at $p_t$ (the fiber at $p_t$). In general $T_P\Omega_{t_0} \neq T_{\Omega_{t_0}}$ (e.g. on a sphere “the Earth”). The bundle (the union of fibers) at $t_0$ is $T\Omega_{t_0} = \bigcup_{P \in \Omega_{t_0}} (P) \times T_P\Omega_{t_0}$, and the bundle at $t$ is $T\Omega_t = \bigcup_{P \in \Omega_t} \{p_t\} \times T_{p_t}\Omega_t$. Then the shifter

$$\widetilde{S}^t_0 : \begin{cases} 
T\Omega_{t_0} \to T\Omega_t \\
(P, \tilde{Z}^t_0) \to \tilde{S}^t_0(P, \tilde{Z}^t_0) = (p_t, S^t_0(\tilde{Z}^t_0)),
\end{cases} \quad (K.17)$$

is defined such that $\tilde{Z}^t_0 \in T_P\Omega_{t_0}$ “as little distorted as possible” along a path. E.g., on a sphere, if the path is a geodesic, if $\theta_t$ is the angle between $\tilde{Z}_P$ and the tangent vector to the geodesic at $P$, then $\theta_t$ is also the angle between $S^t_0(\tilde{Z}_P)$ and the tangent vector to the geodesic at $p_t$, and $S^t_0(\tilde{Z}_P)$ has the same length than $\tilde{Z}_P$ (at constant speed in a car you think the geodesic is a straight line, although $S^t_0(\tilde{Z}_P) \neq \tilde{Z}_P$: the Earth is not flat).

K.5.2 The displacement vector

(Affine space framework, $\Omega_{t_0}$ open set in $\mathbb{R}^n$.) Let $P \in \Omega_{t_0}$, $\tilde{W}_P \in \tilde{\mathbb{R}}^n_{t_0}$, $p_t = \Phi^t_0(P) \in \Omega_t$, and $d\Phi^t_0 = F^t_0 = \mathcal{L}(\tilde{\mathbb{R}}^n_0; \tilde{\mathbb{R}}^n_0)$. Define

$$\widetilde{\delta U}^t_0 : \begin{cases} 
\Omega_0 \times \tilde{\mathbb{R}}^n_0 \to \Omega_0 \times \mathcal{L}(\tilde{\mathbb{R}}^n_0; \tilde{\mathbb{R}}^n_0) \\
(P, \tilde{Z}_P) \to \tilde{\delta U}^t_0(P, \tilde{Z}_P) = (p_t, \tilde{\delta U}^t_0(\tilde{Z}_P)) \quad \text{with} \quad \tilde{\delta U}^t_0(\tilde{Z}_P) = (F^t_0 - S^t_0),\tilde{Z}_P.
\end{cases} \quad (K.18)$$

Then $\tilde{\delta U}^t_0 = F^t_0 - S^t_0$ is a two-point tensor. And

$$C^t_0 = (F^t_0)^T.F^t_0 = (\tilde{\delta U}^t_0 + S^t_0)^T.(\tilde{\delta U}^t_0 + S^t_0)$$

$$= 1 + (\tilde{S}^t_0)^T.\tilde{\delta U}^t_0 + (\tilde{\delta U}^t_0)^T.S^t_0 + (\tilde{\delta U}^t_0)^T.\tilde{\delta U}^t_0,$$  

$$\text{since} \quad (S^t_0)^T.S^t_0 = I \text{ identity in } T\Omega_{t_0}. \text{ Indeed,} \quad ((S^t_0)^T.S^t_0)_{\tilde{\mathbb{R}}^n_{t_0} \tilde{\mathbb{R}}^n_{t_0}} = (S^t_0 \tilde{A}, S^t_0 \tilde{B})_{\tilde{\mathbb{R}}^n_{t_0}} \cong (\tilde{A}, \tilde{B})_{\tilde{\mathbb{R}}^n_{t_0}},$$

cf. (K.14), for all $\tilde{A}, \tilde{B}$. Then the Green–Lagrange tensor is defined on $\Omega_{t_0}$ by

$$E^t_0 = \frac{1}{2}(C^t_0 - I_{t_0}) = (S^t_0)^T.\tilde{\delta U}^t_0 + (\tilde{\delta U}^t_0)^T.S^t_0 \quad + \frac{1}{2}(\tilde{\delta U}^t_0)^T.\tilde{\delta U}^t_0, \quad (K.20)$$

to compare with (H.2).

L Transport of volumes and areas

Here $\mathbb{R}^n = \mathbb{R}^3$ the usual affine space. Let $t_0, t \in \mathbb{R}$, and $\Phi^t_0 : \mathbb{R} \times \Omega_{t_0} \to \Omega_t$, see (3.1). Let $F_P = d\Phi^t_0(P)$. Let $(\cdot, \cdot)_g$ be a Euclidean dot product in $\mathbb{R}^3$ (English, French,...), with $||\cdot||_g$ the associated norm.
L.1 Transformed parallelepiped

The Jacobian of $\Phi^t_p$ at $P$ relative to a $(\cdot,\cdot)_g$ Euclidean basis is defined in (D.31), used here with $(\vec{e}_i) = (\vec{E}_i)$ a $(\cdot,\cdot)_g$-Euclidean basis: With $F_P = F_{10}^p$ and $\det \vec{e} = \det \vec{e}$,

$$J_P := \det(F_P) = \det(F_{10}^p(\vec{e}_1, ..., F_{10}^p(\vec{e}_n)), \quad \text{and} \quad J_P > 0 \quad \text{(L.1)}$$

the motion being supposed regular. Thus, if $(\vec{U}_1P, ..., \vec{U}_nP)$ is a parallelepiped at $P$ at $t_0$, if $\vec{u}_{ip} = F_P \vec{U}_i P$, then $(\vec{u}_{1p}, ..., \vec{u}_{np})$ is a parallelepiped at $p$ at $t$ which volume is

$$\det(\vec{u}_{1p}, ..., \vec{u}_{np}) = J_P \det(\vec{U}_1P, ..., \vec{U}_nP). \quad \text{(L.2)}$$

L.2 Transformed volumes

Riemann integrals and (L.2) give the change of variable formula: For any regular function $f : \Omega_t \rightarrow \mathbb{R}$,

$$\int_{p_i \in \Omega_t} f(p_i) \, d\Omega_t = \int_{P \in \Omega_0} f(\Phi^t_0(P)) \, |J(P)| \, d\Omega_0. \quad \text{(L.3)}$$

(See (D.18): $d\Omega_t$ is a positive measure. It is not a multilinear form.) In particular,

$$|\Omega_t| = \int_{p_i \in \Omega_t} d\Omega_t(p_i) = \int_{P \in \Omega_0} |J(P)| \, d\Omega_0(P). \quad \text{(L.4)}$$

(With $J(P) > 0$ for regular motions.)

L.3 Transformed parallelogram

Consider two independent vectors $\vec{U}_1P, \vec{U}_2P \in \mathbb{R}^n_{t_0}$ at $t_0$ at $P$, and the vectors $\vec{u}_{1p} = F_P \vec{U}_1P$ and $\vec{u}_{2p} = F_P \vec{U}_2P$ at $t$ at $p = \Phi^t_0 (P)$. Since $\Phi^t_0$ is a diffeomorphism, $\vec{u}_{1p}$ and $\vec{u}_{2p}$ are independent.

Then choose a Euclidean dot product $(\cdot,\cdot)_g$ (English, French...) to be able to use the vectorial product, cf. (9.15), the same at all time $t$. Then the areas of the parallelograms are

$$||\vec{U}_1P \wedge \vec{U}_2P||_g \quad \text{and} \quad ||\vec{u}_{1p} \wedge \vec{u}_{2p}||_g, \quad \text{(L.5)}$$

and unit normal vectors to the quadrilaterals are

$$\vec{N}_P = \frac{\vec{U}_1P \wedge \vec{U}_2P}{||\vec{U}_1P \wedge \vec{U}_2P||_g} \in \mathbb{R}^n_{t_0}, \quad \text{and} \quad \vec{n}_P = \frac{\vec{u}_{1p} \wedge \vec{u}_{2p}}{||\vec{u}_{1p} \wedge \vec{u}_{2p}||_g} \in \mathbb{R}^n_t. \quad \text{(L.6)}$$

**Proposition L.1** If $\vec{u}_{1p} = F_P \vec{U}_1P$ and $\vec{u}_{2p} = F_P \vec{U}_2P$, then

$$\vec{u}_{1p} \wedge \vec{u}_{2p} = J_P F^{-T}_P \vec{U}_1P \wedge \vec{U}_2P, \quad \text{and} \quad ||\vec{u}_{1p} \wedge \vec{u}_{2p}||_g = J_P ||F^{-T}_P \vec{U}_1P \wedge \vec{U}_2P||_g, \quad \text{(L.7)}$$

since $J_P > 0$ (for regular motions), and

$$\vec{n}_P = \frac{F_P^T \vec{N}_P}{||F_P^T \vec{N}_P||_g} \neq F_P \vec{N}_P \text{ in general}. \quad \text{(L.8)}$$

**Proof.** Let $\vec{W}_P \in \mathbb{R}^n_{t_0}$ and $\vec{w}_P = F_P \vec{W}_P$. Then the volume of the parallelepiped $(\vec{u}_{1p}, \vec{u}_{2p}, \vec{w}_P)$ is

$$((\vec{u}_{1p}, \vec{u}_{2p}, \vec{w}_P)_g = \det(\vec{u}_{1p}, \vec{u}_{2p}, \vec{w}_P) = \det(F_P \vec{U}_1P, F_P \vec{U}_2P, \vec{W}_P) = \det(F_P) \det(\vec{U}_1P, \vec{U}_2P, \vec{W}_P) = J_P (\vec{U}_1P \wedge \vec{U}_2P, \vec{W}_P)_g = J_P (\vec{U}_1P \wedge \vec{U}_2P, F^{-1}_P \vec{w}_P)_g = J_P (F^{-T}_P \vec{U}_1P \wedge \vec{U}_2P, \vec{w}_P)_g, \quad \text{(L.9)}$$

for all $\vec{w}_P$, thus (L.7), thus

$$\vec{u}_{1p} \wedge \vec{u}_{2p} = \frac{J_P F^{-T}_P \vec{U}_1P \wedge \vec{U}_2P}{J_P ||F^{-T}_P \vec{U}_1P \wedge \vec{U}_2P||_g}, \quad \text{thus} \quad \text{(L.8).}$$
L.4 Transformed surface

L.4.1 Deformation of a surface

A parametrized surface $\Psi_{t_0}$ in $\Omega_{t_0}$ and the associated geometric surface $S_{t_0}$ are defined by

$$\Psi_{t_0} : \begin{cases} (a, b] \times [c, d] \to \Omega_{t_0} \\ (u, v) \to P = \Psi_{t_0}(u, v) \end{cases} \quad \text{and} \quad S_{t_0} = \text{Im}(\Psi_{t_0}) \subset \Omega_{t_0}. \quad (L.10)$$

(It is also represented after a choice of an origin $O$ by the vector valued parametrized surface $\tilde{r}_{t_0} = \overrightarrow{OP}$.)

The transformed parametric surface is $\Psi_t := \Phi^t_0 \circ \Psi_{t_0}$ and the associated geometric surface is $S_t$:

$$\Psi_t := \Phi^t_0 \circ \Psi_{t_0} : \begin{cases} (a, b] \times [c, d] \to \Omega_{t_0} \\ (u, v) \to P = \Psi_t(u, v) = \Phi^t_0(\Psi_{t_0}(u, v)) = \Phi^t_0(P) \end{cases} \quad \text{and} \quad S_t = \Phi^t_0(S_{t_0}). \quad (L.11)$$

(After a choice of an origin $O$, the associated vector valued parametrized surface $\tilde{r}_t = \overrightarrow{OP}$.)

Let $(\tilde{E}_1, \tilde{E}_2)$ be the canonical basis in the space $\mathbb{R} \times \mathbb{R} \supset [a, b] \times [c, d] = \{(u, v)\}$ of parameters. The surface $\Psi_{t_0}$ is supposed to be regular, that is, $\Psi_{t_0}$ is $C^1$ and, for all $P = \Psi_{t_0}(u, v) \in S_{t_0}$, the tangents vectors $\tilde{T}_{1,P}$ and $\tilde{T}_{2,P}$ at $P$ are independent, that is,

$$\begin{align*}
\tilde{T}_{1,P} &:= d\Psi_{t_0}(u, v).\tilde{E}_1 = \frac{\partial \Psi_{t_0}}{\partial u}(u, v), \quad \text{so} \quad \tilde{r}_1 = F_P.\tilde{T}_{1,P} \quad (= d\Phi^0_0(P) \frac{\partial \Psi_{t_0}}{\partial u}(u, v)), \\
\tilde{T}_{2,P} &:= d\Psi_{t_0}(u, v).\tilde{E}_2 = \frac{\partial \Psi_{t_0}}{\partial v}(u, v), \quad \text{so} \quad \tilde{r}_2 = F_P.\tilde{T}_{2,P} \quad (= d\Phi^0_0(P) \frac{\partial \Psi_{t_0}}{\partial v}(u, v)).
\end{align*} \quad (L.12)$$

And the tangent vectors at $S_t$ at $p = \Phi^t_0(P)$ at $t$ are

$$\begin{align*}
\tilde{r}_1 &:= d\Psi_t(u, v).\tilde{E}_1 = \frac{\partial \Psi_t}{\partial u}(u, v), \quad \text{so} \quad \tilde{r}_1 = F_P.\tilde{T}_{1,P} \\
\tilde{r}_2 &:= d\Psi_t(u, v).\tilde{E}_2 = \frac{\partial \Psi_t}{\partial v}(u, v), \quad \text{so} \quad \tilde{r}_2 = F_P.\tilde{T}_{2,P}
\end{align*} \quad (L.13)$$

These vectors are independent since $\Phi^t_0$ is a diffeomorphism and $\Psi_{t_0}$ is regular. In fact, we used tangent vectors to curves and their push-forwards, cf. figure 5.1 and § 10.5.2.

L.4.2 Euclidean dot product and unit normal vectors

Then choose a Euclidean dot product $(\cdot, \cdot)_g$ (English, French...), to be able to use the vectorial product, cf. (9.15), the same at all time $t$. Then the scalar area elements $d\Sigma_P$ at $P$ at $S_{t_0}$ relative to $\Psi_{t_0}$, and $d\sigma_P$ at $P$ at $S_t$ relative to $\Psi_t$, are

$$\begin{align*}
d\Sigma_P &= ||\frac{\partial \Psi_t}{\partial u}(u, v) \wedge \frac{\partial \Psi_t}{\partial v}(u, v)||_g \, du \, dv \quad (= ||\tilde{T}_{1,P} \wedge \tilde{T}_{2,P}||_g \, du \, dv), \\
d\sigma_P &= ||\frac{\partial \Psi_{t_0}}{\partial u}(u, v) \wedge \frac{\partial \Psi_{t_0}}{\partial v}(u, v)||_g \, du \, dv \quad (= ||\tilde{r}_1 \wedge \tilde{r}_2||_g \, du \, dv).
\end{align*} \quad (L.14)$$

And the areas of $S_{t_0}$ and $S_t$ are

$$\begin{align*}
|S_{t_0}| &= \int_{P \in S_{t_0}} d\Sigma_P := \int_{u=a}^{b} \int_{v=e}^{d} ||\frac{\partial \Psi_{t_0}}{\partial u}(u, v) \wedge \frac{\partial \Psi_{t_0}}{\partial v}(u, v)||_g \, du \, dv, \\
|S_t| &= \int_{P \in S_{t}} d\sigma_P := \int_{u=a}^{b} \int_{v=e}^{d} ||\frac{\partial \Psi_t}{\partial u}(u, v) \wedge \frac{\partial \Psi_t}{\partial v}(u, v)||_g \, du \, dv. 
\end{align*} \quad (L.15)$$

(See (D.18): $d\Sigma_P$ and $d\sigma_P$ are positive measures: They are not multilinear forms.)

And the unit normal vectors $\vec{N}_P$ at $S_{t_0}$ at $P$ at $t_0$ and $\vec{n}_P$ at $S_t$ at $P$ at $t$ are

$$\begin{align*}
\vec{N}_P = \frac{\frac{\partial \Psi_{t_0}}{\partial u}(u, v) \wedge \frac{\partial \Psi_{t_0}}{\partial v}(u, v)}{||\frac{\partial \Psi_{t_0}}{\partial u}(u, v) \wedge \frac{\partial \Psi_{t_0}}{\partial v}(u, v)||_g} \quad (= \frac{\tilde{T}_{1,P} \wedge \tilde{T}_{2,P}}{||\tilde{T}_{1,P} \wedge \tilde{T}_{2,P}||_g}), \\
\vec{n}_P = \frac{\frac{\partial \Psi_t}{\partial u}(u, v) \wedge \frac{\partial \Psi_t}{\partial v}(u, v)}{||\frac{\partial \Psi_t}{\partial u}(u, v) \wedge \frac{\partial \Psi_t}{\partial v}(u, v)||_g} \quad (= \frac{\tilde{r}_1 \wedge \tilde{r}_2}{||\tilde{r}_1 \wedge \tilde{r}_2||_g}).
\end{align*} \quad (L.16)$$

Then the vectorial area elements $d^3\Sigma_P$ at $P$ at $S_{t_0} = \text{Im}(\Psi_{t_0})$ relative to $\tilde{r}_{t_0}$ and $d\vec{n}_P$ at $P$ at $S_t = \text{Im}(\Psi_t)$
relative to $\Psi$, are

$$
\begin{align*}
\|d\Sigma_p\| = \vec{N}_p \, d\Sigma_p &= \frac{\partial \Psi_{t_0}}{\partial u}(u,v) \wedge \frac{\partial \Psi_{t_0}}{\partial v}(u,v) \, du \, dv \quad (= \vec{T}_1 P \wedge \vec{T}_2 P \, du \, dv) \\
\|d\sigma_p\| = \vec{n}_p \, d\sigma_p &= \frac{\partial \Psi_{t_1}}{\partial u}(u,v) \wedge \frac{\partial \Psi_{t_1}}{\partial v}(u,v) \, du \, dv \quad (= \vec{l}_1 P \wedge \vec{l}_2 P \, du \, dv).
\end{align*}
$$

(Useful to get the flux through a surface: $\int\vec{F} \cdot \vec{n} \, d\sigma = \int\vec{F} \cdot \vec{d}\sigma$.)

( NB: $d\Sigma_p$ and $d\sigma_p$ are not multilinear since $d\Sigma_p$ and $d\sigma_p$ are not.)

### L.5 Piola identity

Reminder: Let $M = [M_{ij}]$ be a 3×3 matrix function. We use the usual divergence in continuum mechanics

(non objective) given by $\text{div}M := \left(\frac{\partial M_{11}}{\partial x^1} + \frac{\partial M_{21}}{\partial x^2} + \frac{\partial M_{31}}{\partial x^3}\right) = \left(\sum_{j=1}^{n} \frac{\partial M_{ij}}{\partial X^j}\right)$, cf. (S.66). And if $\text{Cof}(M)$ is the matrix of cofactors (in $\mathbb{R}^3$: $\text{Cof}(M)_{ij} = M_{ij+1}M_{j+2} - M_{i+1,j+2}M_{j+1}$), then $M^{-1} = \frac{1}{\det M} \text{Cof}(M)^T$, i.e.,

$$
(\det M)M^{-1} = \text{Cof}(M)^T.
$$

The framework being Euclidean, we use a Euclidean basis and the associated matrix, and thus (matrix meaning)

$$(JF^{-T})(P) = J(P)F(P)^{-T} = \text{Cof}(F(P))^\text{noted} \text{Cof}(F)(P).$$

### Proposition L.2 (Piola identity) In $\mathbb{R}^3$, we have

$$
\text{div}(JF^{-T})(P) = 0, \quad \text{i.e.} \quad \sum_{j=1}^{n} \frac{\partial \text{Cof}(F)(P)}{\partial X^j} = 0.
$$

Also written $\sum_{j=1}^{n} \frac{\partial (JX^{-T})}{\partial X^j} = 0$... NB: (L.22) is just a matrix computation since we used the divergence of a matrix (we used components relative to a given basis).

**Proof.** We are in $\mathbb{R}^3$, thus $\text{Cof}(F)_{ij} = F_{i+1,j+2}^{j+1} - F_{i+2,j+1}^{j+2}$, and $F = |d\Psi| = [\frac{\partial \varphi^i}{\partial X^j}]$, that is, $F_i = \frac{\partial \varphi^i}{\partial X^j}$. Thus

$$
\frac{\partial \text{Cof}(F)(P)}{\partial X^j} = \frac{\partial^2 \varphi^{i+1}}{\partial X^j \partial X^{i+1}} \frac{\partial \varphi^{i+2}}{\partial X^{j+2}} + \frac{\partial^2 \varphi^{i+1}}{\partial X^j \partial X^{i+1}} \frac{\partial \varphi^{i+2}}{\partial X^{j+2}} + \frac{\partial^2 \varphi^{i+1}}{\partial X^j \partial X^{i+1}} \frac{\partial \varphi^{i+2}}{\partial X^{j+2}} + \frac{\partial \varphi^{i+1}}{\partial X^j} \frac{\partial^2 \varphi^{i+2}}{\partial X^{j+2}} \frac{\partial \varphi^{i+2}}{\partial X^{j+2}}.
$$

Thus, for all $i = 1, 2, 3$, we get $\sum_{j=1}^{n} \frac{\partial \text{Cof}(F)(P)}{\partial X^j} = 0$ (the terms cancel each other out two by two). \hfill \Box
L.6 Piola transformation

Let \( \vec{u} \) be a vector field in \( \Omega_t \). The goal is to find a vector field \( \vec{U}_{\text{Piola}} \) in \( \Omega_{t_0} \) s.t. for all open subset \( \omega_t \subset \Omega_t \) with \( \omega_{t_0} = \Phi_t^{-1}(\omega_t) \subset \Omega_t \),

\[
\int_{\partial \omega_{t_0}} \vec{U}_{\text{Piola}} \cdot \vec{N} \ d\Sigma = \int_{\partial \omega_t} \vec{u} \cdot \vec{n} \ d\sigma \quad (L.23)
\]

(which means \( \int_{P \in \partial \omega_{t_0}} (\vec{U}_{\text{Piola}}(P), \vec{N}(P))_g \ d\Sigma(P) = \int_{P \in \partial \omega_t} (\vec{u}(p), \vec{n}(p))_g \ d\sigma(p) \)). So that

\[
\int_{\omega_{t_0}} \text{div}(\vec{U}_{\text{Piola}}) \ d\Omega_t = \int_{\omega_t} \text{div}(\vec{u}) \ d\Omega_t \quad (L.24)
\]

(which means \( \int_{P \in \omega_{t_0}} \text{div}(\vec{U}_{\text{Piola}})(P) \ d\Omega_t = \int_{P \in \omega_t} \text{div}(\vec{u})(p) \ d\Omega_t \)), and thus

\[
\int_{P \in \omega_{t_0}} \text{div}(\vec{U}_{\text{Piola}})(P) \ d\Omega_t = \int_{P \in \omega_t} \text{div}(\vec{u})(\Phi_t(P)) \ J(P) \ d\Omega_t. \quad (L.25)
\]

(The motion is supposed to be regular, so \( J(P) > 0 \).

Thus we want \( \text{div}(\vec{U}_{\text{Piola}})(P) = J(P) \text{div}\vec{u}(p) \) when \( p = \Phi_t(P) \).

**Definition L.3** The Piola transform is the map

\[
\begin{cases}
C^\infty(\Omega_t; \mathbb{R}^n) \rightarrow C^\infty(\Omega_{t_0}; \mathbb{R}^n) \\
\vec{u} \rightarrow \vec{U}_{\text{Piola}}, \quad \vec{U}_{\text{Piola}}(P) := J(P)F(P)^{-1}\vec{u}(P) \quad \text{when} \quad p = \Phi_t(P).
\end{cases}
\]

(\( \vec{U}_{\text{Piola}}(P) = J(P)(\Phi_t(P))^{-1}(\vec{u}(P)) \) where \( (\Phi_t(P))^{-1}(\vec{u}) = F(P)^{-1}\vec{u}(p) \) the pull-back.)

**Proposition L.4** With \( p = \Phi_t(P) \), \( \vec{U}_{\text{Piola}} = \sum_{i=1}^n U_{\text{Piola}}^i \vec{c}_i \) and \( \vec{u} = \sum_{i=1}^n u^i \vec{c}_i \), we get

\[
\text{div}\vec{U}_{\text{Piola}} = J(P) \text{div}\vec{u}, \quad \text{i.e.} \quad \sum_{i=1}^n \partial U_{\text{Piola}}^i / \partial X^i = J(P) \sum_{i=1}^n \partial u^i / \partial x^i(p). \quad (L.27)
\]

**Proof.** (S.71) gives \( \text{div}(|JF|^{-1})(\vec{u} \circ \Phi_t(P))(P) = \text{div}(|JF|^{-1})(\vec{u}(P))d\Omega_t + (J(P)F(P)^{-1}) \otimes (\text{div}(\vec{u}(p)).F(P)) \). Thus \( (L.22) \) and \( (L.26) \) gives \( \text{div}\vec{U}_{\text{Piola}} = 0 + (J(P)F(P)^{-1}) \otimes (\text{div}(\vec{u}(p)).F(P)) \). And \( F^{-1}(P) \otimes (\text{div}(\vec{u}(p)).F(P)) = (F(P), F(P)^{-1}) \otimes \vec{d} = I \otimes \vec{d} = \text{div}\vec{u} \), which gives \( (L.27) \). \hfill \Box

M Work and power

M.1 Introduction

M.1.1 Work for a 1-D material

(Thermodynamic like approach.) The elementary work is a differential form \( \alpha \), e.g. \( \alpha = \delta W \) in thermodynamics. Let \( t_0 < T \) and consider a regular curve \( c : t \in [t_0, T] \rightarrow c(t) \in \mathbb{R}^n \). And let \( \vec{c}(t) := c'(t) \).

The work of \( \alpha \) along the curve is

\[
\int_c \alpha := \int_{t_0}^{T} \alpha(t, c(t)).\vec{c}'(t) \ dt = \int_{t_0}^{T} \alpha(t, c(t)).d\vec{c}' \quad (M.1)
\]

\[
\int_c \alpha := \int_{t_0}^{T} \alpha(t, c(t)).\vec{c}'(t) \ dt = \int_{t_0}^{T} \alpha(t, c(t)).d\vec{c}' \quad (M.1)
\]

\[
\int_c \alpha := \int_{t_0}^{T} \alpha(t, c(t)).\vec{c}'(t) \ dt = \int_{t_0}^{T} \alpha(t, c(t)).\vec{c}'(t) \ dt = \int_{t_0}^{T} \alpha(t, c(t)).d\vec{c}' \quad (M.1)
\]

Special case: If \( \alpha \) is a stationary and exact differential form, e.g. \( \alpha = du \) in thermodynamics, then \( \int_c du = \int_{t_0}^{T} dU(c(t)).\vec{c}'(t) \ dt = \int_{t_0}^{T} \frac{dU(c(t))}{dt} \ dt = U(c(T)) - U(c(t_0))) = \text{noted} \Delta U \) only depends on the extremities \( c(t_0) \) and \( c(T) \) of the curve \( c \).

**NB (continuum mechanics):** An observer chooses a Euclidean dot product \( (\cdot, \cdot)_g \), and writes \( (\vec{a}, \vec{w})_g = \text{noted} \vec{a} \cdot \vec{w} = \text{noted} \vec{a} \cdot \vec{w} \) if \((\cdot, \cdot)_g \) is implicit. And if he chooses to represent a linear form \( \alpha_i(p_i) \) with its \((\cdot, \cdot)_g\)-Riesz representation vector \( f_i(p_i) \), cf. (B.8), thus

\[
\int_c \alpha = \int_{t_0}^{T} \alpha(t, c(t)).\vec{c}'(t) \ dt = \int_{t_0}^{T} f_i(t) d\vec{c}' = \int_{t_0}^{T} f_i \cdot \vec{c}' \ dt. \quad (M.2)
\]

The two last equalities depend on the choice of the inner dot product \((\cdot, \cdot)_g\) (English? French?).
M.1.2 Power density for a 1-D material

(M.1) gives the power density along the curve $c$: for all $t \in [t_0, T]$ and $p = c(t)$,

$$\psi(t, p) = \alpha(t, p) \bar{v}(t, p), \quad \text{i.e.} \quad \psi := \alpha \bar{v},$$

(M.3)

And if $\Omega_t$ is a set at $t_0$, if $\Phi^{0o}(t, p_{t_0}) \in [t_0, T] \times \Omega_t$, then with the curves $c_{p_{t_0}}(t) = \Phi^{0o}(t, p_{t_0})$, we have defined the power density $\psi : \cup_{t \in [t_0, T]}(t \times \Omega_t) \to \mathbb{R}$, cf. (M.3), which is a Eulerian function.

And with a Euclidean dot product $(\cdot, \cdot)_g$ and the $(\cdot, \cdot)_g$-Riesz representation vector $\vec{f}$ of $\alpha$, we get

$$\psi(t, p) = (\vec{f}(t, p), \bar{v}(t, p)) \quad \text{namely} \quad \vec{f}(t, p) \cdot \bar{v}(t, p) \quad \text{(observer dependent).}$$

(M.4)

M.2 Definitions for a n-D material

M.2.1 Power to work

Consider a motion $\tilde{\Phi}$, cf. (1.5), and let $\Omega_t := \tilde{\Phi}(t, \partial \Phi)$. Hypothesis: The power at $t$ is:

$$\mathcal{P}(t, \tilde{\Phi}) := \sum_{p_t \in \Omega_t} \psi(t, p_t) = \int_{p_t \in \Omega_t} \psi(t, p_t) \, d\Omega_t,$$

(M.5)

where $\psi : \cup_{t \in [t_0, T]}(t \times \Omega_t) \to \mathbb{R}$ is the power density (a Eulerian function). Here $\sum$ is meant for a finite number of points, while $\sum = \text{inted} \int$ (Riemann notation) is meant for a continuous material.

Example M.1 For external forces

$$\psi_{ext}(t, p_t) = \alpha(t, p_t) \bar{v}(t, p_t),$$

(M.6)

cf. (M.3) (objective), or (M.4) (non objective). For internal forces, the classical homogeneous isotropic elasticity formulation reads, relative to a Euclidean basis $(\vec{e}_i)$ in a Galilean referential,

$$\psi_{int}(t, p_t) = \tau(t, p_t) \otimes d\bar{v}(t, p_t) = \sum_{i,j=1}^n \tau^i_j(t, p_t) \frac{\partial \bar{v}^j}{\partial x^i}(t, p_t),$$

(M.7)

where $\tau$ is the stress tensor and $d\bar{v}$ the differential of the velocity. Then let

$$\tau = \sum_{i,j=1}^n \tau^i_j \bar{e}_i \otimes \bar{e}^j = \sum_{i=1}^n \bar{e}_i \otimes \alpha^i \quad \text{where} \quad \alpha^i = \sum_{j=1}^n \tau^i_j e^j,$$

(M.8)

together with $\bar{v} = \sum_{j=1}^n v^j \bar{e}_j$, so that $d\bar{v} = \sum_{j=1}^n \bar{e}_j \otimes dv^j = \sum_{j,k=1}^n \frac{\partial v^j}{\partial x^k} \bar{e}_j \otimes e^k$.

And

$$d\bar{v} \tau = \left( \sum_{j=1}^n \bar{e}_j \otimes dv^j \right) \otimes \left( \sum_{i=1}^n \bar{e}_i \otimes \alpha^i \right) = \sum_{i,j=1}^n \frac{\partial v^j}{\partial x^i} \bar{e}_j \otimes \alpha^i = \sum_{i=1}^n \frac{\partial d\bar{v}}{\partial x^i} \otimes \alpha^i$$

(M.9)

or,

$$\psi_{int}(t, p_t) = \left( \sum_{i=1}^n \bar{e}_i \otimes \alpha^i \right) \otimes \left( \sum_{j=1}^n \bar{e}_j \otimes dv^j \right) \quad \text{where} \quad \alpha^i = \sum_{j=1}^n \tau^i_j e^j,$$

(M.10)

when $\tau = \sum_{i,j=1}^n \tau^i_j \bar{e}_i \otimes e^j = \sum_{i=1}^n \bar{e}_i \otimes \left( \sum_{j=1}^n \tau^i_j e^j \right) = \sum_{i=1}^n \bar{e}_i \otimes \alpha^i \quad \text{where} \quad \alpha^i = \sum_{j=1}^n \tau^i_j e^j$, and

Indeed $\tau(t, p_t) d\bar{v}(t, p_t) = \sum_{j=1}^n (\alpha^i \bar{e}_j) \bar{e}_i \otimes dv^j = \sum_{i,j=1}^n \tau^i_j \bar{e}_i \otimes dv^j$ gives $\sum_{i,j=1}^n (\alpha^i \bar{e}_j) \bar{e}_i \otimes dv^j$;

And $d\bar{v} \tau = \sum_{i,j=1}^n \frac{\partial v^j}{\partial x^i} \bar{e}_j \otimes e^k$, thus $d\bar{v} \tau = \sum_{i,j=1}^n (\tau^i_j e^j) \bar{e}_j \otimes dv^j = \sum_{i=1}^n (\text{div} \bar{w}) \bar{e}_i$, And $\tau \bar{w} = \sum_{i,j=1}^n \tau^i_j n^i \bar{e}_j = \sum_{i=1}^n n^i \bar{w}_i$; Thus by integration by parts,

$$\mathcal{P}_{int}(t, \tilde{\Phi}) = - \int_{\Omega_t} \text{div} \tau \bar{v} \, d\Omega_t + \int_{\Omega_t} (\tau \bar{w}) \, d\Omega_t = \sum_{j=1}^n \left( - \int_{\Omega_t} (\text{div} \bar{w}^j) e^j + \int_{\Omega_t} n^j \bar{w}_j \cdot \bar{v} \, d\Omega_t \right).$$

(M.11)

We deduce the work between $t_0$ and $T$:

$$W^{\text{int}}_{1D}(\tilde{\Phi}) := \sum_{j=1}^n \left( - \int_{t_0}^T \int_{\Omega_t} \psi(t, p_t) \, d\Omega_t \right) \, dt.$$
M.2.2 Work to power

For one particle (one trajectory), the work is
\[ W_{\Omega}^{\tau} (\tilde{\Phi}_{P_{0}}) := \int_{t=t_{0}}^{T} \psi(t, \tilde{\Phi}_{P_{0}}(t)) \, dt. \quad (M.13) \]

And for a finite number of particles (a finite number of trajectories)
\[ W_{\Omega}^{\tau} (\tilde{\Phi}) := \sum_{P_{0i}=1, \ldots, n} W_{\Omega}^{\tau} (\tilde{\Phi}_{P_{0i}}) = \sum_{P_{0i}=1, \ldots, n} \left( \int_{t=t_{0}}^{T} \psi(t, \tilde{\Phi}_{P_{0i}}(t)) \, dt \right). \quad (M.14) \]

Then we also have
\[ W_{\Omega}^{\tau} (\tilde{\Phi}) = \int_{t=t_{0}}^{T} \left( \sum_{P_{0j}=1, \ldots, n} \psi(t, \tilde{\Phi}_{P_{0j}}(t)) \right) \, dt. \quad (M.15) \]

And the power at \( t \) is defined by
\[ \mathcal{P}(\Omega_t) := \sum_{P_{0j}=1, \ldots, n} \psi(t, \tilde{\Phi}_{P_{0j}}(t)). \quad (M.16) \]

And, for an infinite number of particles (e.g. continuum mechanics), the work is defined by
\[ W_{\Omega}^{\tau} (\tilde{\Phi}) := \int_{t=t_{0}}^{T} \left( \int_{P_{t} \in \Omega_t} \psi_1(p_t) \, d\Omega_t \right) \, dt. \quad (M.17) \]

Hypothesis: the \( \int \) signs can be inverted (\( \psi \) is supposed regular enough). Thus
\[ W_{\Omega}^{\tau} (\tilde{\Phi}) = \int_{t=t_{0}}^{T} \left( \int_{P_{t} \in \Omega_t} \psi_1(p_t) \, d\Omega_t \right) \, dt. \quad (M.18) \]

And the power at \( t \) is defined by
\[ \mathcal{P}(\Omega_t) := \int_{P_{t} \in \Omega_t} \psi_1(p_t) \, d\Omega_t. \quad (M.19) \]

In what follows we only look at the continuum mechanics case (if finite number of particles: Exercise).

M.2.3 Objective internal power

We will use an “objective tensorial product” (independent of the unit of measurement chosen by an observer). Consider a Eulerian velocity field \( \tilde{\Phi} \). Thus \( d\tilde{v} \) is a \( (1,1) \) tensor. Thus, to consider a objective tensorial product with \( d\tilde{v} \), we need a \( (1,1) \) tensor \( \mathfrak{T} \) to contract with \( d\tilde{v} \). The objective contraction
\[ \mathfrak{T}(t, p) \odot d\tilde{v}(t, p) \]

is then a real value independent of the observer. Its expression with a basis \( (\tilde{e}_i) \) at \( t \) is
\[ \mathfrak{T} \odot d\tilde{v} = \sum_{i,j=1}^{n} \tau_{ij}^j v_i^j. \quad (M.20) \]

when \( d\tilde{v} = \sum_{i,j} v_i^j \tilde{e}_i \otimes e^j \) and \( \mathfrak{T} = \sum_{i,j} \tau_{ij}^j \tilde{e}_i \otimes e^j \), cf. (Q.34). The scalar value \( \mathfrak{T} \odot d\tilde{v} \) is the same for all observer, whatever is the measuring units, foot, meter... (The Einstein convention is satisfied).

To compare with the double matrix contraction \( \mathfrak{T} \odot d\tilde{v} = \sum_{i,j} \tau_{ij}^j v_i^j \) which is not objective, cf. (Q.41)-(Q.42): the result depends on the observer (depends on the measuring units).

Hypothesis: At first order, at \( t \) at \( p \in \Omega_t \), the power density \( \psi(t, p) \) is
\[ \psi(t, p) = \mathfrak{T}(t, p) \odot d\tilde{v}(t, p), \quad (M.21) \]

where \( \mathfrak{T} \) is some \( (1,1) \) tensor. (A posteriori, if we have a Euclidean dot product, then if \( \mathfrak{T} = \mathfrak{T}^T \) we get \( (M.7) \).) And the power at \( t \) is
\[ \mathcal{P}_t(\tilde{v}_t) = \int_{P_{t} \in \Omega_t} \mathfrak{T}(p_t) \odot d\tilde{v}(p_t) \, d\Omega_t. \quad (M.22) \]

And we deduce the work (with Eulerian functions):
\[ W_{\Omega}^{\tau} (\tilde{\Phi}) = \int_{t=t_{0}}^{T} \mathcal{P}_t(\tilde{v}_t) \, dt = \int_{t=t_{0}}^{T} \int_{P_{t} \in \Omega_t} \mathfrak{T}(p_t) \odot d\tilde{v}(p_t) \, d\Omega_t \, dt. \quad (M.23) \]
**Exercise M.2** $\mathbb{R}^n$ Euclidean. If $S, T \in \mathcal{T}^1(\Omega)$, prove:

$$\frac{S + S^T}{2} \otimes T = \frac{S + S^T}{2} \otimes \frac{T + T^T}{2}. \quad (M.24)$$

(If $S = S^T$ then $\frac{S + S^T}{2} \otimes T = S \otimes T$.)

**Answer.** $S \otimes T = \sum_{ik} S^k_i T^j_k = \sum_{ik} S^k_i T^i_k = S \otimes T$. With $\frac{S + S^T}{2}$ symmetric, we get $\frac{S + S^T}{2} \otimes T = \frac{S + S^T}{2} \otimes \frac{T + T^T}{2}$, hence (M.24).

**M.2.4 Power and initial configuration**

To get back to $\Omega_{t_0}$ (pull-back) we use the change of variable formula to get

$$\mathcal{P}(\Omega_{t_0}) = \int_{P \in \Omega_{t_0}} \psi_t(\Phi_t^{t_0}(P)) |J_t^{t_0}(P)| d\Omega_{t_0}, \quad (M.25)$$

where $J_t^{t_0}(P) = \det(F_t^{t_0}(P))$ (the Jacobian of $\Phi_t^{t_0}$ at $P$).

The motion is supposed to be regular, so the Jacobian is positive, thus

$$W_{t_0}^{t_0}(\tilde{\Phi}) = \int_{t=t_0}^{T} \int_{P \in \Omega_{t_0}} \psi(t, \Phi_t^{t_0}(P)) J_t^{t_0}(t, P) d\Omega_{t_0} dt. \quad (M.26)$$

**Remark M.3** The power $\mathcal{P}$ is an Eulerian concept, cf. (M.22). To get back to the initial configuration imposes the use of some initial time and Lagrangian variables. And it can make things complicated. E.g., the introduction of the Lie derivatives with the intermediary of an initial configuration makes the understanding of the Lie derivative quite impossible. See footnote page 26.

**Remark M.4** With the pull-backs, (M.25) reads ($J_t^{t_0}(P)$ being positive)

$$\mathcal{P}(t, \tilde{\Phi}) = \int_{P \in \Omega_{t_0}} ((\Phi_t^{t_0})^* \psi_t)(P) ((\Phi_t^{t_0})^* d\Omega_t) \quad (M.27)$$

since $((\Phi_t^{t_0})^* d\Omega_t) = J_t^{t_0}(P) d\Omega_{t_0}$ and $((\Phi_t^{t_0})^* \psi_t)(P) = \psi_t(p_t)$ (scalar function). It gives the Piola–Kirchhoff tensor (pull-back to the initial configuration), see next § M.3. E.g., with $\psi = \alpha \vec{v}$, cf. (M.6): Then $((\Phi_t^{t_0})^* \alpha)(P) = \alpha_t(\Phi_t^{t_0}(P)), F_t^{t_0}(P)$, and $((\Phi_t^{t_0})^* \vec{v})(P) = F_t^{t_0}(P)^{-1} \vec{v}_t(\Phi_t^{t_0}(P))$, and $((\Phi_t^{t_0})^* \alpha_t, \vec{v}_t) = (\Phi_t^{t_0})^* \alpha_t(\Phi_t^{t_0}(P), \vec{v}_t(\Phi_t^{t_0}(P))).$ (M.28)

**Remark M.5** Reminder: The pull-back $((\Phi_t^{t_0})^* \vec{v})(P) = F_t^{t_0}(P)^{-1} \vec{v}_t(\Phi_t^{t_0}(P))$ of the Eulerian velocity isn’t the Lagrangian velocity $\vec{V}_t^{t_0}(P) = \vec{v}_t(\Phi_t^{t_0}(P))$ (unless $F_t^{t_0}(P) = I$). Besides, a Lagrangian velocity doesn’t define a vector field, cf. (4.10) and § 4.1.2, when the pull-back of a Eulerian velocity (at $t$) is vector field (at $t_0$).

**M.3 Piola–Kirchhoff tensors**

**M.3.1 The first Piola–Kirchhoff tensor**

The Piola–Kirchhoff tensor is not that easy to master for a simple reason: If everything is quite simple in a Eulerian setting (the configuration at $t$), then everything is made complicated when expressed in a past configuration (at $t_0$), for, in the end (one more complication), get back to the current configuration (at $t$). The Piola–Kirchhoff tensor is also used to introduce the Lie derivative of tensors (if you want the Lie derivative to remain mysterious), see footnote page 26.

The Piola–Kirchhoff approach consists in transforming Eulerian quantities into Lagrangian quantities to refer to the initial configuration: (M.22) becomes

$$\mathcal{P}_t(\vec{v}_t) = \int_{P \in \Omega_{t_0}} \bar{\varepsilon}(\Phi_t^{t_0}(P)) \otimes d\vec{v}_t(\Phi_t^{t_0}(P)) J_t^{t_0}(P) d\Omega_{t_0}. \quad (M.28)$$

With $p_t = \Phi_t^{t_0}(t, P)$ and with the Lagrangian velocity $\vec{V}_t^{t_0}(P) = \vec{v}_t(\Phi_t^{t_0}(P))$, cf. (4.13), we have $d\vec{v}_t(p_t, F_t^{t_0}(P)) = d\vec{V}_t^{t_0}(P)$. Thus $\bar{\varepsilon}(p_t) \otimes d\vec{v}_t(p_t) = \varepsilon(p_t) \otimes (d\vec{V}_t^{t_0}(P), F_t^{t_0}(P)^{-1}) = \varepsilon(p_t) \otimes (d\vec{v}_t(p_t), F_t^{t_0}(P)), \quad (M.29)$
Remark M.8 It is the time derivative of $\mathbf{SK}(t) = J(t) F(t)^{-1} \dot{\varphi}(t) F(t)^{-T}$ that leads to some kind of Lie derivative as explain in books in continuum mechanics, which then are expressed in the current configuration. See footnote page 26.
M.4 Classical hyper-elasticity: $\partial W/\partial F$

We use Marsden and Hughes notations. (One of the difficulties is in the notations.)

M.4.1 Framework: A scalar function acting on linear maps

Consider a function

$$\tilde{W} : \left\{ \begin{array}{c} \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R} \\ L \to \tilde{W}(L) \end{array} \right. \tag{M.39}$$

**Example M.9** $m = n$, endomorphisms $L \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$, and $\tilde{W}(L) = \text{Tr}(L)$ (the trace). For the classical hyper-elasticity, $L \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$.

The differential $d\tilde{W}$:

$$d\tilde{W} : \left\{ \begin{array}{c} \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m) \to \mathcal{L}(\mathbb{L}(\mathbb{R}^n; \mathbb{R}^m); \mathbb{R}) \\ L \to d\tilde{W}(L) \end{array} \right. \tag{M.40}$$

is defined at $L$ by, in a direction $M$, cf. (S.2),

$$d\tilde{W}(L)(M) = \lim_{h \to 0} \frac{\tilde{W}(L + hM) - \tilde{W}(L)}{h} = \frac{\partial \tilde{W}(L)(M)}{\partial L} \tag{M.41}$$

Also written $d\tilde{W}(L)(M) = d\tilde{W}(L,M)$ where the dot is the linearity dot. Warning: This dot is not a priori related to “product matrix calculations”, so we stick to the notation $d\tilde{W}(L)(M)$.

**Example M.10** Continuing example M.9: $\tilde{W}(L) = \text{Tr}(L)$ give $d\tilde{W}(L)(M) = \lim_{h \to 0} \frac{\text{Tr}(L + hM) - \text{Tr}(L)}{h} = \text{Tr}(M)$ (the trace is linear), thus $d\tilde{W}(L) = \text{Tr} = \tilde{W}$ is the trace operator: $d\tilde{W}(L)(M) = \text{Tr}(M)$, expected since $\text{Tr}$ is linear.

M.4.2 Expression with bases: The $\partial W/\partial L^i_j$

Quantification: Let $(\hat{E}_i)$ and $(\hat{e}_i)$ be bases in $\mathbb{R}^n$ and $\mathbb{R}^m$. Let $(E^i_j)$ be the dual basis of $(\hat{E}_i)$. We use tensorial notations, cf. § A.9.4 (here we compute): $\hat{e}_i \otimes E^j \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ is defined by $(\hat{e}_i \otimes E^j)\tilde{W} = (E^j \tilde{W})\hat{e}_i$ for all $\tilde{W} \in \mathbb{R}^n$ and $(\hat{e}_i \otimes E^j)_{j=1,\ldots,m}$ is a basis of $\mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$. And, as in (D.40), if $L \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ and $L = \sum_{i,j=1}^n L^i_j \hat{e}_i \otimes E^j$ we have

$$d\tilde{W}(L)(\hat{e}_i \otimes E^j) \text{ named } \frac{\partial \tilde{W}(L)(E^j)}{\partial L^i_j} \tag{M.42}$$

and (associated matrix relative to the bases $(\hat{E}_i)$ and $(\hat{e}_i)$)

$$[d\tilde{W}(L)]_{\hat{e},\hat{E}} := [d\tilde{W}(L)^i_j]_{i=1,\ldots,n} := \left[ \frac{\partial \tilde{W}(L)}{\partial L^i_j} \right]_{i=1,\ldots,n} \tag{M.43}$$

Thus $M = \sum_{i,j} M^i_j \hat{e}_i \otimes E^j \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ gives $d\tilde{W}(L)(M) = \sum_{i,j} M^i_j d\tilde{W}(L)(\hat{e}_i \otimes E^j)$ since $d\tilde{W}(L)$ is linear, that is,

$$d\tilde{W}(L)(M) = \sum_{i,j} M^i_j \frac{\partial \tilde{W}(L)(E^j)}{\partial L^i_j} = [M]_{\hat{e},\hat{E}} : [d\tilde{W}(L)]_{\hat{e},\hat{E}} \tag{M.44}$$

(double matrix contraction).

**Remark M.11** The notation $[M]_{\hat{e},\hat{E}} : [d\tilde{W}(L)]_{\hat{e},\hat{E}}$ is a pure matrix computation, since $M = \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ and $d\tilde{W}(L) \in \mathcal{L}(\mathcal{L}(\mathbb{R}^n; \mathbb{R}^m); \mathbb{R})$ are different kinds of mathematical objects: $M$ acts on vectors while $\tilde{W}(L)$ acts on maps.

**Example M.12** Continuing example M.10 with $(\hat{e}_i) = (\hat{E}_i)$: Then $\tilde{W}(L) = \text{Tr}(L)$ gives $d\tilde{W}(L)(M) = \text{Tr}(M) = \sum_i M^i_i$, thus $d\tilde{W}(L)(E^j) = \delta^j_i$ for all $i,j$, thus $d\tilde{W}(L)(\hat{e}_i) = [I]$ (identity matrix), and we recover $d\tilde{W}(L)(M) = [I] : [M] = \sum_{i=1}^n M^i_i = \text{Tr}(M) = \tilde{W}(M)$.

**Remark M.13** With $F = d\Phi^t_i(P) \in \mathcal{L}(\mathbb{R}^n_t; \mathbb{R}^m_t)$ (deformation gradient), the real meaning of the derivation $d\tilde{W}/dF$ is intriguing: It is a derivation in the direction $\hat{e}_i \otimes E^j$, cf. (M.41), that is a derivation both in $\mathbb{R}^n_t$ and in $\mathbb{R}^m_t$.
M.4.3 Motions and ω-lemma
Let \( \bar{\Phi} \) be a \( C^2 \) motion, let \( t_0, t \in \mathbb{R} \), let \( \Phi := \Phi_{t_0}^t : p_{t_0} \in \overline{\Omega_{t_0}} \rightarrow p \in \overline{\Omega_t} \) be the association motion, let \( F(p_{t_0}) := F_{t_0}^t(p_{t_0}) = d\bar{\Phi}_{t_0}^t(p_{t_0}) \in \mathcal{L}(\mathbb{R}^n_{t_0}, \mathbb{R}^n_t) \) be the deformation gradient between \( t_0 \) and \( t \) at \( p_{t_0} \).

Generalization of (M.39) (to non-homogenous functions): Consider a \( C^1 \) function
\[
\bar{W} : \left\{ \begin{array}{l}
\Omega_{t_0} \times \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_t) \rightarrow \mathbb{R} \\
(p_{t_0}, L) \rightarrow \bar{W}(p_{t_0}, L).
\end{array} \right.
\]

And at \( p_{t_0} \in \overline{\Omega_{t_0}} \) let \( \bar{W}_{p_{t_0}}(L) := \bar{W}(p_{t_0}, L) \) for all \( L \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_t) \). In particular the derivative of \( \bar{W} \) at \( (p_{t_0}, L) \) relative to the second variable in a direction \( M \in \mathcal{L}(\mathbb{R}^n_{t_0}; \mathbb{R}^n_t) \) is the real \( D_2 \bar{W}(p_{t_0}, L)(M) := d(\bar{W}_{p_{t_0}})(L)(M) = \lim_{h \rightarrow 0} \frac{\bar{W}(p_{t_0}, L + hM) - \bar{W}(p_{t_0}, L)}{h} \) named \( \frac{\partial \bar{W}}{\partial E}(p_{t_0}, L) \).

Define
\[
f : \begin{cases} 
C^1(\overline{\Omega_{t_0}}; \mathbb{R}) & \rightarrow C^0(\Omega_{t_0}; \mathbb{R}) \\
\Phi & \rightarrow f(\Phi) := \bar{W}(\cdot, d\Phi(\cdot))
\end{cases}
\]
(a function of \( \Phi \) which only depends on its “first gradient”), that is, \( f(\Phi)(p_{t_0}) = \bar{W}(p_{t_0}, d\Phi(p_{t_0})) \in \mathbb{R} \) for all \( p_{t_0} \in \Omega_{t_0} \). (This kind of relation is generally deduced after application of the frame invariance principle, and the hypothesis of dependence on only the first order derivative \( F = d\Phi \).)

Lemma M.14 (ω-lemma) For all \( \Phi, \Psi \in C^1(\overline{\Omega_{t_0}}; \mathbb{R}) \),
\[
df(\Phi)(\Psi) = D_2\bar{W}(\cdot, d\Phi)(d\Psi) \text{ noted } \frac{\partial \bar{W}}{\partial F}(\cdot, d\Phi(d\Psi)),
\]
that is, for all \( p_{t_0} \in \Omega_{t_0} \),
\[
df(\Phi)(\Psi)(p_{t_0}) := \frac{\partial \bar{W}}{\partial F}(p_{t_0}, d\Phi(p_{t_0}))(d\Phi(p_{t_0})).
\]

With basis (\( \vec{E}_i \)) and (\( \vec{e}_i \)) in \( \mathbb{R}^n_{t_0} \) and \( \mathbb{R}^n_t \) and an origin \( o \in \mathbb{R}^n \) at \( t \), if \( o\Phi(p_{t_0}) = \sum_{i=1}^n \Phi_i(p_{t_0}) \vec{e}_i, \)
\[
D_2\bar{W}( \vec{e}_i ) \text{ and } D_2\bar{W}( \vec{e}_i ),
\]
and let \( \vec{\alpha} \Psi(p_{t_0}) = \sum_{i=1}^n \Psi_i(p_{t_0}) \vec{e}_i, F_\Psi = d\Psi \) and \( F_\Phi \) = \( d\Phi \), with \( F_\Psi.\vec{E}_j = \sum_{i=1}^n (F_\Psi)_i \vec{e}_i \) and \( F_\Phi.\vec{E}_J = \sum_{i=1}^n (F_\Phi)_i \vec{e}_i \), then
\[
df(\Phi)(\Psi) = \sum_{i,j=1}^n \frac{\partial \bar{W}}{\partial (F_i)}(F_\Psi)_i (F_\Phi)_j \text{ noted } \sum_{i,j=1}^n \frac{\partial \bar{W}}{\partial X_i}(F_\Psi)_i (F_\Phi)_j.
\]

(The derivatives considered are along the basis vectors \( \vec{E}_i \) in \( \mathbb{R}^n_{t_0} \).)

Proof. \( C^1(\overline{\Omega_{t_0}}; \mathbb{R}) \) is a vector space, and \( df(\Phi) \in \mathcal{L}(C^1(\overline{\Omega_{t_0}}; \mathbb{R}); C^0(\Omega_{t_0}; \mathbb{R})) \) for any \( \Phi \in C^1(\overline{\Omega_{t_0}}; \mathbb{R}) \) is given by, cf. (S.4), \( df(\Phi)(\Psi) = \lim_{h \rightarrow 0} \frac{(\Phi + h\Psi)(p_{t_0}) - \Phi(p_{t_0})}{h} \in C^0(\Omega_{t_0}; \mathbb{R}) \). Thus, \( p_{t_0} \) being fixed in \( \Omega_{t_0} \),
\[
df(\Phi)(\Psi)(p_{t_0}) = \lim_{h \rightarrow 0} \frac{(\Phi + h\Psi)(p_{t_0}) - \Phi(p_{t_0})}{h} \in \mathbb{R},
\]
thus, with \( L = d\Phi(p_{t_0}) \) and \( M = d\Psi(p_{t_0}) \),
\[
df(\Phi)(\Psi)(p_{t_0}) = \lim_{h \rightarrow 0} \frac{\bar{W}(p_{t_0}, \frac{\Phi(p_{t_0}) + h\Psi(p_{t_0}) - \Phi(p_{t_0})}{h}) - \bar{W}(p_{t_0}, d\Phi(p_{t_0}))}{h} = \lim_{h \rightarrow 0} \frac{\bar{W}(p_{t_0}, L + hM) - \bar{W}(p_{t_0}, L)}{h}.
\]

M.4.4 Application to classical hyper-elasticity: \( IK = \partial W/\partial F \)

Let \( (\cdot, \cdot) \) be a unique Euclidean dot product in \( \mathbb{R}^n \) at all times \( t \), and let \( (\vec{E}_i) \) and \( (\vec{e}_i) \) be Euclidean bases at \( t_0 \) and at \( t \). Thus we can consider the transposed \( F^T \) and the Jacobian \( J(p_{t_0}) = \det_{\vec{E}_i, \vec{e}_i} (F(p_{t_0})) \).

Let \( p_{t_0} = F(p_{t_0}) = \det_{\vec{E}_i, \vec{e}_i} (F(p_{t_0}))^{-T} \) be the first Piola–Kirchhoff (two point) tensor at \( p_{t_0} \), cf. (M.31). Since \( IK \) depends on \( \Phi \), the full notation is \( IK = IK(\Phi) \) given by
\[
IK(\Phi)(p_{t_0}) = J(p_{t_0}) \sigma_{\vec{E}_i}(\Phi(p_{t_0}))) d\Phi(p_{t_0})^{-T}.
\]

Definition M.15 If there exists a function \( IK \) such that \( IK \) reads
\[
IK(\Phi)(p_{t_0}) = IK(p_{t_0}, F(p_{t_0})),
\]
then \( IK \) is called a constitutive function. (First order hypothesis: \( IK \) only depends on the first order derivative of \( \Phi \).)
The material is hyper-elastic iff there exists a function $\hat{W} : \{ \Omega_{t_0} \times C(\mathbb{R}^{n_x}; \mathbb{R}^n) \rightarrow \mathbb{R} \}
abla (p_{t_0}, L) \rightarrow \hat{W}(p_{t_0}, L)\}$ such that
\[
(\mathcal{K}(\Phi) = \hat{\mathcal{K}}(\cdot, d\Phi) = \frac{\partial \hat{W}}{\partial F}(\cdot, d\Phi), \text{ written } \hat{\mathcal{K}} = \frac{\partial \hat{W}}{\partial F}, \] (M.51)\]
that is, $\hat{\mathcal{K}}(p_{t_0}, F(p_{t_0})) = \frac{\partial \hat{W}}{\partial F}(p_{t_0}, F(p_{t_0}))$ for all $p_{t_0} \in \Omega_{t_0}$, where $F = d\Phi$. (A full notation is $\hat{W}^{t_0}$ with $\hat{\mathcal{K}}^{t_0}(p_{t_0}, d\Phi(p_{t_0})) = \frac{\partial \hat{W}^{t_0}}{\partial F^{t_0}}(p_{t_0}, d\Phi^{t_0}(p_{t_0})).$)

Thus, with bases $(\hat{E}_i)$ and $(\hat{e}_i)$ in $\hat{\mathbb{R}}^n_{t_0}$ and $\mathbb{R}^n_{t}$ and $\mathcal{K} = \sum_{i,j=1}^n \hat{K}_{ij}^{t} \hat{e}_i \otimes \hat{E}_j$,
\[
[A(\mathcal{K}(\Phi))|_{\hat{E}, \hat{c}} = \frac{\partial \hat{W}^{t_0}}{\partial F^{t_0}}(\cdot, F)|_{\hat{E}, \hat{c}}, \text{ i.e. } [\mathcal{K}^t] = \frac{\partial \hat{W}^{t_0}}{\partial F^{t_0}}(\cdot, F)]. \tag{M.52}
\]
Thus, for any (virtual) motion $\Psi : \Omega_{t_0} \rightarrow \Omega_t$, with (M.46) and (M.43),
\[
\hat{\mathcal{K}}(d\Phi)(d\Psi) = \frac{\partial \hat{W}}{\partial F}(d\Phi)(d\Psi) = \left( \sum_{i,j} \frac{\partial \hat{W}}{\partial F_i^{t_0}}(F)^i \frac{\partial \hat{W}}{\partial F_j^{t_0}}(F)^j \right) = \left[ \hat{\mathcal{K}} : [d\Psi] \right], \tag{M.53}
\]
that is, $\hat{\mathcal{K}}(d\Phi)(d\Psi)(p_{t_0}) = \sum_{i,j} \frac{\partial \hat{W}}{\partial F_i^{t_0}}(p_{t_0}, F^{t_0}(p_{t_0})) \frac{\partial \hat{W}}{\partial F_j^{t_0}}(p_{t_0})$ for all $p_{t_0} \in \Omega_{t_0}$.

**Exercise M.17** With a unique Euclidean dot product $(\cdot, \cdot)_g$ both in $\hat{\mathbb{R}}^n_{t_0}$ and $\mathbb{R}^n_{t}$, let $C = F^{T} \cdot F$. With Euclidean bases $(\hat{E}_i) \in \hat{\mathbb{R}}^n_{t_0}$ and $(\hat{e}_i) \in \mathbb{R}^n_{t}$, prove (derivation in the direction $\hat{e}_i \otimes E^j$):

\[
\frac{\partial C}{\partial F^i_j} (F) = \sum_K F^i_K \hat{E}_K \otimes E^K + \sum_K F^j_K \hat{E}_K \otimes E^K = (\begin{array}{c} C(F)(\hat{e}_i \otimes E^j) \end{array}) = \lim_{h \rightarrow 0} \frac{C(F + h\hat{e}_i \otimes E^j) - C(F)}{h}, \tag{M.54}
\]

\[
\frac{\partial \sqrt{C}}{\partial F^i_j} (F) = \frac{1}{2} \left( \sqrt{C(F)} \right)^{-1} \frac{\partial C}{\partial F^i_j} (F). \tag{M.55}
\]

\[
\frac{\partial \sqrt{C}}{\partial C} = \frac{1}{2} \left( \sqrt{C} \right)^{-1}. \tag{M.56}
\]

**Answer.** Let $F = \sum_{i,j} F^i_j \hat{E}_i \otimes E^j$, so $F^T = \sum_{i,j} (F^T)^i_j \hat{E}_i \otimes e^j = \sum_{i,j} F^j_i \hat{E}_i \otimes e^j$, and $C = \sum_{i,j} C^i_j \hat{E}_i \otimes E^j = F^{T} \cdot F = \sum_{i,j} \sum_{k} (F^T)^i_k F^j_k \hat{E}_i \otimes E^j = \sum_{i,j,k} F^j_k F^i_k \hat{E}_i \otimes E^j = C(F)$, so $C^i_j = \sum_k F^j_k F^i_k = C^{i_j}(F)$. And
\[
C(F + h\hat{e}_i \otimes E^j) = (F + h\hat{e}_i \otimes E^j)^T \cdot (F + h\hat{e}_i \otimes E^j) = (F^{T} + h\hat{E}_j \otimes e^j) \cdot (F + h\hat{e}_i \otimes E^j)
\]

\[
= C(F) + h(\hat{E}_j \otimes e^j), + hF^{T} \cdot (\hat{e}_i \otimes E^j) + h^2 \hat{E}_j \otimes E^j = C(F) + h \sum_{k} F^j_k \hat{E}_k \otimes E^k + \sum_K (F^T)^i_k \hat{E}_K \otimes E^K + h^2 \hat{E}_j \otimes E^j \tag{M.57}
\]

Thus (M.54) and $C(F + h\hat{e}_i \otimes E^j) - C(F) = \left( \begin{array}{c} \sqrt{C(F + h\hat{e}_i \otimes E^j)} - \sqrt{C(F)} \end{array} \right)$, which gives $\frac{dC(F)(\hat{e}_i \otimes E^j) = 2\sqrt{C(F)}dC(F)(\hat{e}_i \otimes E^j)$ and $\frac{\partial \sqrt{C}}{\partial F^i_j} = \frac{1}{2} \left( \sqrt{C(F)} \right)^{-1} \frac{\partial C}{\partial F^i_j}$ gives
\[
(C + h\hat{e}_i \otimes e^j) - C = (\sqrt{C + h\hat{e}_i \otimes e^j} - \sqrt{C}) \cdot (\sqrt{C + h\hat{e}_i \otimes e^j} - \sqrt{C}), \text{ divided by } h, \text{ gives } \hat{e}_i \otimes e^j = 2\sqrt{C} \cdot \lim_{h \rightarrow 0} \frac{\sqrt{C + h\hat{e}_i \otimes e^j} - \sqrt{C}}{h} = 2\sqrt{C} \cdot \frac{dC}{dC.L} \text{ for all } L \text{ (linearity of } d\sqrt{C}) \text{, thus } d\sqrt{C}.L = \frac{1}{2} \left( \sqrt{C} \right)^{-1} \cdot L. \]

**M.4.5 Corollary (hyper-elasticity):** $\mathcal{K} = \partial W/\partial C$

With the symmetry of the second Piola–Kirchhoff tensor $\mathcal{K} = F^{-1} \cdot \hat{\mathcal{K}}$, we deduce $\mathcal{K}^{t_0}(\Phi^{t_0})(P) = \hat{\mathcal{K}}^{t_0}(P, F^{t_0}(P))$ (constitutive function). And we deduce the existence of a function $\hat{W} : \{ \Omega_{t_0} \times C(\hat{\mathbb{R}}^{n_x}, \mathbb{R}^n_{t_0}) \rightarrow \mathbb{R} \}
abla (p_{t_0}, L) \rightarrow \hat{W}(p_{t_0}, L)\}$ such that
\[
\hat{W}(p_{t_0}, L) = \frac{\partial \hat{W}}{\partial C}(\cdot, C). \tag{M.58}
\]
(See Marsden and Hughes for details and the thermodynamical hypotheses required.)

**Remark M.18** The previous issues remain: Derivation at $t_0$ (the tensor $C$ is defined in $\hat{\mathbb{R}}^n_{t_0}$ and the derivations $\frac{\partial \hat{W}}{\partial C^i_j}(\cdot, C) := \hat{\mathcal{W}}(\hat{E}_i \otimes \hat{E}_j)$ are considered) while a derivation at $t$ could be more connected to Cauchy’s approach which starts with considerations at $t$, see theorem O.3.
M.5 Hyper-elasticity and Lie derivative

We look for a “stored energy function” in the actual state. Thus we apply a motion and measure the work or the power. With \( \vec{w} \) be the Eulerian velocity, the Lie Derivative \( L_{\vec{w}} \) (which means the rates of evolution along the motion) seems to be an adequate tool, approach proposed in www.isima.fr/leborge/IsimathMeca/PtrvObj.pdf.

Hypothesis, first order (linear) approximation: general H is a Euclidean differential forms \( \alpha_i, \ i = 1, \ldots, n \), characterize the elastic material (instead of a Cauchy stress tensor as a starting point), and at \( t \) in a referential \( (O, (\vec{e}_i)) \) in \( \mathbb{R}^n \), along a virtual motion which Eulerian velocity field is \( \vec{w} \), the density of virtual power due to \( \vec{w} \) is \( \text{pow}(\vec{w})(t, p_0) = \sum_{i=1}^n \mathcal{L}_{\vec{w}}\alpha_i(t, p_0) \delta t \) where \( \mathcal{L}_{\vec{w}}\alpha = \frac{\partial}{\partial t} \alpha + \alpha . \vec{w} \) is the Lie derivative of a differential form \( \alpha \) and in a Galilean setting, with a Cartesian basis \( (\vec{e}_i) \), its dual basis \( (\vec{e}^i) \), and the usual hypothesis (isometric objectivity = independence of any rigid body motion), the power reduces to

\[
pow(\vec{w}) = \sum_{i=1}^n \alpha_i . \vec{w} \cdot \vec{e}_i = -\vec{r} \cdot \vec{d} \vec{w}, \quad \text{where} \quad \vec{r} = -\sum_{i=1}^n \vec{e}_i \otimes \alpha_i, \quad (M.59)
\]

i.e., \( \varepsilon^i = -\alpha_i \). That is, for all \( p_0 \in \Omega_t \), \( \text{pow}(\vec{w})(t, p_0) = \sum_{i=1}^n \alpha_i(t, p_0) . \vec{w} \cdot \vec{e}_i = -\vec{r}(t, p_0) \cdot \vec{d} \vec{w}(t, p_0) \)

where \( \vec{r}(t, p_0) = -\sum_{i=1}^n \vec{e}_i \otimes \alpha_i(t, p_0) \). With components, \( \vec{w} = \sum_{k=1}^n w_k \vec{e}_k \) gives \( \vec{d} \vec{w} = \sum_{i,k=1}^n \frac{\partial w_k}{\partial x_i} \vec{e}_k \otimes \vec{e}_i \) and \( \vec{d} \vec{w} \cdot \vec{e}_i = \sum_{k=1}^n \frac{\partial w_k}{\partial x_i} \vec{e}_k \) and \( \vec{d} \vec{w} \cdot \vec{e} = \sum_{k=1}^n \frac{\partial w_k}{\partial x} \vec{e}_k \otimes \vec{e}_i \), thus \( \vec{r} = -\sum_{i,j=1}^n \alpha_i^j \vec{e}_i \otimes \vec{e}_j \) (the \( i \)-th row of \( \vec{r} \))

Example M.19 Isotropic homogeneous elastic material: With \( (\vec{E}_i) \) a Cartesian basis in \( (\mathbb{R}^n_0) \),

\[
\alpha_i(t, p_0) = \alpha_i(t_0, p_{00}) . (F^t_0)^{-1}(p_0), \quad \text{where} \quad \alpha_i(t_0, p_{00}) = 2\mu \epsilon^i
\]

(push-forward), and \( (F^t_0)^{-1}(p_0) = \sum_{i=1}^n \epsilon^i \vec{E}_i \otimes \vec{e}^i \) gives \( \vec{r}(t, p_0) |_{\vec{e}^i} = -2\mu |(F^t_0)^{-1}(p_0)|_{\vec{e}^i}. \)

Example M.20 2-D transversely isotropic linear elasticity: With \( (\vec{E}_1, \vec{E}_2) \) a Cartesian basis in \( (\mathbb{R}^n_0) \), and \( \vec{F} \) the direction of the fibers,

\[
\alpha_i(t, p_0) = \alpha_i(t_0, p_{00}) . (F^t_0)^{-1}(p_0), \quad \text{where} \quad \alpha_i(t_0, p_{00}) = 2\mu \epsilon^i
\]

(push-forward), and \( (F^t_0)^{-1}(p_0) = \sum_{i=1}^n \epsilon^i \vec{E}_i \otimes \vec{e}^i \) gives \( \vec{r}(t, p_0) |_{\vec{e}^i} = -2\mu |(F^t_0)^{-1}(p_0)|_{\vec{e}^i}. \)

Example M.21 Isotropic homogeneous elastic material: Let \( (\vec{e}_i) \) be a Euclidean basis, the same at all \( t \), and \( \beta^i_0(p_0) = \epsilon^i \) for all \( i \).

1- Classic approach: Let \( \varepsilon^i = \lambda \text{Tr}[(F^t_0)[I] + 2\mu(F^t_0)[I]] \) (usual law with \( \varepsilon^i = \frac{[\vec{e}^i] + [\vec{e}_i]}{2} \), and \( [\alpha^i] = [\epsilon^i] \) the \( i \)-th-row. Here \( \alpha^i(t_0, p_{00}) = [\varepsilon^i(\beta^0_0(p_0), \ldots, \beta^0_0(p_0), F(p_0))] := \lambda \text{Tr}[(F^t_0)[I][\beta^0] + 2\mu |(I)|]^{-1} \).

2- Functional approach with \( F^t_0(p_0) = R^t_0(p_0) \circ U^t_0(p_0) \), written \( F = R.U \), cf. (F.3), and with \( \Sigma_t = \lambda \text{Tr}(U_t-U^t_0)I + 2\mu |(U_t-U^t_0)|^{-1}. \) (stress in \( \mathbb{R}^n_0 \))

Let \( \Sigma = \lambda \text{Tr}(U_t-U^t_0)I + 2\mu |(U_t-U^t_0)|^{-1}. \) (isotropic homogeneous elasticity). And classical type law: \( \sigma := \frac{\varepsilon^i + \varepsilon^j}{2} \) (Cauchy stress).

Definition M.22 Classical type definition:

E.g., if \( \alpha^i_0 \) is the result of deformation,

\[
\alpha^i_0(p_0) = \alpha^i_0(p_{00}), \quad (\text{push-forward: deformation}), \quad (M.62)
\]

where \( H^t_0(p_0) = (F^t_0(p_{00}))^{-1} \). cf. (12.3), i.e., \( \vec{r} = -\sum_{i=1}^n \vec{e}_i \otimes (\alpha^i_0, H^t_0) \), i.e.,

\[
\vec{r}(p_0) = \vec{r}_0(p_{00}), \quad H^t_0(p_{00}) \quad \text{where} \quad \vec{r}_0(p_{00}) = -\sum_{i=1}^n \vec{e}_i \otimes (\alpha^i_0(p_{00}), (M.63)
\]

(two different motions can start from \( \Omega_t \) and end at \( \Omega_t \) with the same \( F^t_0 \).) With components: \( \alpha^i_0 = \sum_{k=1}^n (\alpha^i_0)_k \vec{e}^k \) and \( H = \sum_{k=1}^n (\alpha^i_0)_k H^k_0 \). giv e \( \alpha^i = \sum_{j,k=1}^n (\alpha^i_0)_k H^k_0 \vec{e}^j \) and \( \vec{r}_0 = -\sum_{i,k=1}^n (\alpha^i_0)_k \vec{e}_i \otimes \vec{e}^k \)

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\[
\tau_i = -\sum_{i,j,k=1} \left( \alpha^{i}_{t_{0}} \right)_{k} H^{i}_{t} \tilde{e}_i \otimes e^j.
\]
And
\[
pow_{t}(\tilde{\omega}_{t})(p_{t}) = \sum_{i=1}^{n} \alpha^{i}_{t_{0}}(p_{t_{0}}) H^{i}_{t} \tilde{e}_i = \sum_{i=1}^{n} \left( \tilde{e}_i \otimes \alpha^{i}_{t_{0}}(p_{t_{0}}) \right) (H^{i}_{t_{0}}(p_{t}) \tilde{d}\tilde{u}_{t}).
\]
\[
= -\tau_{\tilde{u}_{t_{0}}}(p_{t}) (H^{i}_{t_{0}}(p_{t}) \tilde{d}\tilde{u}_{t}).
\]

And the Cauchy stress vector \( \tau_{\tilde{u}_{t_{0}}}(p_{t}) \tilde{n}_{t}(p_{t}) \) in a direction \( \tilde{n}_{t}(p_{t}) \) is
\[
\tau_{\tilde{u}_{t_{0}}} \tilde{n}_{t} = \sum_{i=1}^{n} \left( \alpha^{i}_{t_{0}} \tilde{n}_{t} \right) \tilde{e}_i = \sum_{i=1}^{n} \left( \alpha^{i}_{t_{0}} H^{i}_{t} \tilde{n}_{t} \right) \tilde{e}_i = \sum_{i=1}^{n} \left( \alpha^{i}_{t_{0}} \tilde{N}_{t_{0}} \right) \tilde{e}_i, \text{ where } \tilde{N}_{t_{0}} := H^{i}_{t_{0}} \tilde{n}_{t}.
\]

**Remark M.23** Special case of a isotropic material: With \( \alpha^{i}_{t_{0}} = \mu E i \) for all \( i \) we get \( \alpha^{i}_{t_{0}} = \mu E i H^{i}_{t} = \mu \sum_{j=1}^{n} H^{j}_{t} e^j, \) and \( \tau_{\tilde{u}_{t_{0}}} = -\mu \sum_{j=1}^{n} H^{j}_{t} \tilde{e}_i \otimes e^j = -\mu H^{i}_{t} \tilde{n}_{t}, \) and \( \tilde{n}_{t} = -\mu H^{i}_{t_{0}} \tilde{n}_{t}. \)

Consider a trajectory \( c = \Phi_{t_{0}}^{t_{0}} : u \in [t_{0}, t] \to \Phi_{t_{0}}^{t_{0}}(u) \in \mathbb{R}^{3}, \) and its tangent vector \( \frac{\partial c}{\partial u}(u) = \tilde{n}(u, p_{u}) \) at \( p_{u} = c(u) \). The work of the Cauchy stress vector along this trajectory is
\[
W(\Phi_{t_{0}}^{t_{0}}) = \int_{u_{t_{0}}}^{t} \tilde{T}(u, \Phi_{t_{0}}^{t_{0}}(u)) \frac{\partial \Phi_{t_{0}}^{t_{0}}}{\partial u}(u) dx = \int_{u_{t_{0}}}^{t} \tilde{T}(u, \Phi_{t_{0}}^{t_{0}}(u)) \frac{\partial \tilde{W}_{t_{0}}^{t_{0}}}{\partial u}(u) dx.
\]

The work along a trajectory \( \Phi_{t_{0}}^{t_{0}} : u \to p_{u} = \Phi_{t_{0}}^{t_{0}}(u) \) is, here with \( \tilde{n}(u, p_{u}) = \frac{\partial \Phi_{t_{0}}^{t_{0}}}{\partial u}(u) = \frac{\partial \tilde{W}_{t_{0}}^{t_{0}}}{\partial u}(u) \) for all \( u, \nabla \)
\[
W(\Phi_{t_{0}}^{t_{0}}) = -\sum_{j=1}^{n} \int_{u_{t_{0}}}^{t} \sum_{j=1}^{n} \sum_{j=1}^{n} \tilde{a}_{j}(u, p_{u}) \tilde{n}_{j}(u, p_{u}) \tilde{W}_{t_{0}}^{t_{0}}(u) dx
\]
\[
= -\sum_{j=1}^{n} \int_{u_{t_{0}}}^{t} \left( \tilde{W}_{t_{0}}^{t_{0}}(u, p_{u}) \tilde{a}_{j}(u, p_{u}) \right) \tilde{W}_{t_{0}}^{t_{0}}(u, p_{u}) dx
\]
\[
= -\sum_{j=1}^{n} \tilde{n}_{j}(p_{u}) \int_{u_{t_{0}}}^{t} \left( \tilde{W}_{t_{0}}^{t_{0}}(u, p_{u}) \tilde{a}_{j}(u, p_{u}) \right) \tilde{W}_{t_{0}}^{t_{0}}(u, p_{u}) dx
\]

**Definition M.24** The material is hyper-elastic iff \( \exists \tilde{w} \in C^{1} \) s.t. \( \tilde{T}(u, \Phi_{t_{0}}^{t_{0}}(u)) \frac{\partial \tilde{w}}{\partial u}(u) dx = dw(u). \)

A possible hypothesis: \( n \) vector fields \( (\tilde{a}_{i,j}) \) characterize the elastic material at \( t \) (instead of \( \sigma \) the Cauchy stress tensor), and, given a virtual velocity field \( \tilde{w} \) at \( t \) relative to a virtual motion \( \tilde{\Psi} \), the density of virtual power due to \( \tilde{w} \) is \( p_{t}(\tilde{w}) = \sum_{j=1}^{n} \tilde{a}_{j}(\tilde{w}) \tilde{w}_{j} \) where the \( \tilde{L}_{j} \tilde{w}_{j} \) are the Lie derivatives. In a Galilean Euclidean setting with a Euclidean basis \( (\tilde{e}_i) \) and dual basis \( (e^j) \), we get
\[
pow_{t}(\tilde{\omega}_{t})(p_{t}) = \sum_{i=1}^{n} \tilde{a}_{i}(p_{t}) \otimes e^j,
\]
\[
= -\tau_{\tilde{u}_{t_{0}}}(p_{t}) \otimes \tilde{d}\tilde{u}_{t}, \text{ where } \tau_{\tilde{u}_{t_{0}}} = -\sum_{j=1}^{n} \tilde{a}_{j} \otimes e^j.
\]
is the deduced Eulerian tensor characterizing the elastic material at \( t \). So, for all \( p_{t} \in \Omega_{t} \), \( \text{pow}_{t}(\tilde{\omega}_{t})(p_{t}) = \sum_{j=1}^{n} \tilde{a}_{j}(p_{t}) \otimes e^j = -\tau_{\tilde{u}_{t_{0}}}(p_{t}) \otimes \tilde{d}\tilde{u}_{t} \) where \( \tau_{\tilde{u}_{t_{0}}}(p_{t}) = -\sum_{j=1}^{n} \tilde{a}_{j}(p_{t}) \otimes e^j. \) With components, \( \tilde{a}_{i,j} = \sum_{i=1}^{n} \tilde{a}_{i,j} \tilde{e}_i \) and \( \tilde{d}\tilde{u}_{t} = \sum_{i,k=1}^{n} \frac{\partial \tilde{u}_{t}}{\partial x^i} \tilde{e}_k \otimes e^j, \) then \( e^j \tilde{d}\tilde{u}_{t} = \sum_{i=1}^{n} \frac{\partial \tilde{u}_{t}}{\partial x^i} \tilde{e}_k \otimes e^j \) and \( \text{pow}_{t}(\tilde{\omega}_{t}) = \sum_{i=1}^{n} \frac{\partial \tilde{u}_{t}}{\partial x^i} (\tilde{a}_{i,j})^j \).

Suppose that the material is such that, for all \( t_{0}, t \) and all motions \( \Phi_{t}^{t_{0}} : \Omega_{t_{0}} \to \Omega_{t}, \)
\[
\tilde{a}_{j}(p_{t}) = F_{t}^{t_{0}}(p_{t}) \tilde{a}_{j}(p_{t_{0}}), \text{ i.e. } \tau_{\tilde{u}_{t_{0}}}(p_{t}) = -F_{t}^{t_{0}}(p_{t}) \sum_{j=1}^{n} \tilde{a}_{j}(p_{t_{0}}) \otimes e^j.
\]
when \( p_{t} = \Phi_{t}^{t_{0}}(p_{t_{0}}) \) (two different motions can start from \( \Omega_{t_{0}} \) and end at \( \Omega_{t} \) with the same \( F_{t}^{t_{0}} \)).

(With components: \( F = \sum_{i,k=1}^{n} F_{k}^{i} \tilde{e}_i \otimes \tilde{e}_k \) and \( \tilde{a}_{i,j} = \sum_{k=1}^{n} (\tilde{a}_{i,j})^{j,k} \tilde{E}_{k} \) give \( \tilde{a}_{i,j} = \sum_{k=1}^{n} (\tilde{a}_{i,j})^{j,k} \tilde{E}_{k} \))

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M.5. Hyper-elasticity and Lie derivative
and

\[ \tau_j = -\sum_{i,j,k=1}^{n} F^i_k (\tilde{a}_{i,j})^k \tilde{e}_i \otimes e^j. \]

And the Cauchy stress vector is

\[ \tau(p_t, \tilde{n}_t(p_t)) = -\sum_{i,j,k=1}^{n} F^i_k (\tilde{a}_{i,j}(p_t))^k \tilde{n}_t(p_t) \tilde{e}_i. \]

Thus

\[ \text{pow}_x(\tilde{n}_t)(p_t) = \sum_{j=1}^{n} e_j \cdot d\tilde{w}_t(p_t). F^{t_0}_t(p_t, \tilde{a}_{jt_0}(p_t)) = \sum_{j=1}^{n} e_j \cdot d\tilde{W}^{t_0}_t(p_t, \tilde{a}_{jt_0}(p_t)) \]

\[ = \tau_x(p_t) \otimes \tilde{n}_t(p_t), \quad \text{where} \quad \tau_x(p_t) = -\sum_{j=1}^{n} \tilde{a}_{jt_0}(p_t) \otimes e^j, \]

where \( \tilde{W}^{t_0}_t(p_t) \) is the Lagrangian velocity, i.e., \( \tilde{W}^{t_0}_t(p_t) = \tilde{w}_t(p_0) \) when \( p_0 = (\Phi^{t_0}_t)^{-1}(p_t) \).

In a direction \( \tilde{n}_t \) (unit normal to a surface) we get the Cauchy stress vector

\[ \tilde{T}(t, p_t) = \tilde{x}(t, p_t) \tilde{n}(t, p_t) = -\sum_{j=1}^{n} \tilde{a}_{jt_0}(t, p_t) n^j(t, p_t) = -F^{t_0}_t(t, p_t), \sum_{j=1}^{n} (e_j \cdot \tilde{n}(t, p_t)) \tilde{a}_{jt_0}(t, p_t). \]

(M.71)

With components, \( \tilde{T} = -\sum_{i,j=1}^{n} (\tilde{a}_{ij}(t, p_t) n^j \tilde{e}_i = -\sum_{i,j,k=1}^{n} F^i_k (\tilde{a}_{ij}(t, p_t))^k n^j \tilde{e}_i. \)

Consider a trajectory \( c = \Phi^{t_0}_{p_0} : u \in [0, t] \rightarrow c(u) = \Phi^{t_0}_{p_0}(u) \in \mathbb{R}^3 \), and its tangent vector \( \frac{dc}{du} \) named \( \tilde{n}(u, p_0) \) at \( p_0 = c(u) \). The work of the Cauchy stress vector along this trajectory is

\[ W(\Phi^{t_0}_{p_0}) = \int_{u=t_0}^{t} \tilde{T}(u, \Phi^{t_0}_{p_0}(u)) \cdot \frac{d\Phi^{t_0}_{p_0}}{du}(u) du = \int_{u=t_0}^{t} \tilde{T}(u, \Phi^{t_0}_{p_0}(u)) \cdot \tilde{W}^{t_0}_{p_0}(u) du. \]

(M.72)

The work along a trajectory \( \Phi^{t_0}_{p_0} : u \rightarrow p_u = \Phi^{t_0}_{p_0}(u) \) is, here with \( \tilde{n}(u, p_0) = \frac{d\Phi^{t_0}_{p_0}}{du}(u) = \tilde{W}^{t_0}_{p_0}(u) \) for all \( u \),

\[ W(\Phi^{t_0}_{p_0}) = -\sum_{j=1}^{n} \int_{u=t_0}^{t} \sum_{i,j=1}^{n} n^i(j, p_u) \tilde{a}_{ij}(u, p_u) \tilde{W}^{t_0}_{p_0}(u) du \]

\[ = -\sum_{j=1}^{n} \int_{u=t_0}^{t} (\tilde{W}^{t_0}_{p_0}j(u, p_u)(F^{t_0}_{p_0}(u, p_u), \tilde{a}_{jt_0}(p_u))) \tilde{W}^{t_0}_{p_0}(u, p_u) du \]

\[ = -\sum_{j=1}^{n} \tilde{a}_{jt_0}(p_u) \tilde{W}^{t_0}_{p_0}(u, p_u) \left( \int_{u=t_0}^{t} (\tilde{W}^{t_0}_{p_0}j(u)(F^{t_0}_{p_0}(u, p_u)) \right. \tilde{W}^{t_0}_{p_0}(u) du) \]

(M.73)

**Definition M.25** The material is hyper-elastic ifff \( \exists w \in C^1 \) s.t. \( \tilde{T}(u, \Phi^{t_0}_{p_0}(u)) \cdot \frac{d\Phi^{t_0}_{p_0}}{du}(u) du = dw(u) \).

**N Conservation of mass**

Let \( \rho(t, p) = \rho_t(p) \) be the (Eulerian) mass density for all \( t \) at \( p \in \Omega_t \), supposed to be > 0, and \( m(\omega_t) \) be the mass of a subset \( \omega_t \subset \Omega_t = \Phi(t, Obj) \), that is,

\[ m(\omega_t) = \int_{p \in \omega_t} \rho_t(p) d\omega_t. \]

(N.1)

**Conservation of mass principle** (no loss nor production of particles): For all \( \omega_t \subset \Omega_t \) and all \( t \),

\[ m(\omega_t) = m(\omega_{t_0}), \quad \text{i.e.} \quad \int_{p \in \omega_t} \rho_t(p) d\omega_t = \int_{p \in \omega_{t_0}} \rho_{t_0}(P) d\omega_{t_0}. \]

(N.2)

**Proposition N.1** If (N.2) and \( p = \Phi^{t_0}_{1}(P) \), then, with \( J^{t_0}_{1}(P) = \det(d\Phi^{t_0}_{1}(P)) \) (positive Jacobian),

\[ \rho_t(p) = \frac{\rho_{t_0}(P)}{J^{t_0}_{1}(P)}. \]

(N.3)

**Proof.** Green Formula gives

\[ \int_{p \in \omega_t} \rho_t(p) d\omega_t = \int_{p \in \omega_{t_0}} \rho_t(\Phi^{t_0}_{1}(P)) J^{t_0}_{1}(P) d\omega_{t_0}, \]

for all (measurable) \( \omega_t \), thus (N.2) gives \( \rho_t(p)J^{t_0}_{1}(P) = \rho_{t_0}(P) \).
Proposition N.2 \( \vec{v} = \vec{v}(t, p_t) \) being the Euclidian velocity at \( (t, p_t) \in \mathbb{R} \times \Omega_t \), (N.2) gives
\[
\frac{\partial \rho}{\partial t} + \rho \text{div} \vec{v} = 0, \quad \text{i.e.} \quad \frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) = 0. \tag{N.4}
\]

Then, for any open regular sub domain \( \omega_t \subset \Omega_t \),
\[
\int_{\omega_t} \frac{\partial \rho}{\partial t} \, d\omega_t = -\int_{\partial \omega_t} \rho \vec{v} \cdot \vec{n} \, d\sigma_t. \tag{N.5}
\]

Proof. (N.2) gives \( \frac{\partial}{\partial t}(\int_{\Omega(t)} \rho(t, p(t)) \, d\omega_t) = 0 \), and Leibniz formula (D.37) gives (N.4). Then the Green formula \( \int_{\Omega(t)} \text{div}(\rho \vec{v}) \, d\Omega_t = \int_{\partial \Omega_t} \rho \vec{v} \cdot \vec{n} \, d\sigma_t \) gives (N.5).

\[\Box\]

Exercise N.3 Use (N.3) to prove (N.4).

Answer. \( J(t, P) \rho(t, \Phi(t, P)) = \rho_{\partial}(P) \) give, with \( p_t = \Phi(t, P) \),
\[
\frac{\partial J}{\partial t}(t, P) \rho(t, p_t) + J(t, P) \left( \frac{\partial \rho}{\partial t}(t, p_t) + d\rho(t, p_t) \cdot d\Phi(t, P) \right) = 0.
\]
Thus \( \frac{\partial J}{\partial t}(t, P) = J(t, P) \text{div} \vec{v}(t, p) \), cf. (D.36), gives (N.4).

\[\Box\]

O Balance of momentum

O.1 Framework

Let \( \Phi : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R}^n \) be a regular motion, cf. (1.5), let \( \Omega_t = \Phi(t, \mathbb{R}) \) and \( \Gamma_t = \partial \Omega_t \) (the boundary), and let \( \vec{v} \) be the Eulerian velocity field, cf. (2.5). Let \( \omega_t \) be a regular sub domain in \( \Omega_t \) and \( \partial \omega_t \) be its boundary.

An observer chooses a Euclidean basis \( (\vec{e}_i) \) (e.g. made from the foot or made from the meter). Let \( (\cdot, \cdot) \) be the associated Euclidean dot product.

Let \( \vec{n}_t(p_t) = \vec{n}_t(p) \) be the outer unit normal at \( t \) at \( p_t \in \partial \omega_t \), and \( \vec{n}_t(p) = \sum_{i=1}^n n^i_1(p) \vec{e}_i \).

All the functions are assumed to be regular (regular enough to validate the following calculations).

Let \( \rho : \bigcup_{t \in [t_0, T]} \{ t \} \times \Omega_t \rightarrow \mathbb{R} \) (a mass density), let \( \vec{f} : \bigcup_{t \in [t_0, T]} \{ t \} \times \Omega_t \rightarrow \mathbb{R}^n \) (a body force density), and let \( \vec{T} : \bigcup_{t \in [t_0, T]} \{ t \} \times \partial \omega_t \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) (a surface force density) defined for any regular subset \( \omega_t \subset \Omega_t \).

O.2 Master balance law

Definition O.1 The balance of momentum is satisfied by \( \rho, \vec{f} \) and \( \vec{T} \) iff, for all regular open subset \( \omega_t \) in \( \Omega_t \),
\[
\frac{d}{dt} \left( \int_{\omega_t} \rho \vec{v} \, d\Omega_t \right) = \int_{\omega_t} \vec{f} \, d\Omega_t + \int_{\partial \omega_t} \vec{T} \, d\Gamma_t, \tag{O.1}
\]
called a master balance law. ((O.1) is in fact a linearity hypothesis, see next lemma O.3.)

Thus, with (D.37),
\[
\int_{\omega_t} \frac{D(\rho \vec{v})}{Dt} + \rho \text{div} \vec{v} \, d\Omega_t = \int_{\omega_t} \vec{f} \, d\Omega_t + \int_{\partial \omega_t} \vec{T} \, d\Gamma_t. \tag{O.2}
\]

Thus, with the conservation of mass hypothesis, cf. (N.4), we get
\[
\int_{\omega_t} \rho \frac{D\vec{v}}{Dt} \, d\omega_t = \int_{\omega_t} \vec{f} \, d\Omega_t + \int_{\partial \omega_t} \vec{T} \, d\Gamma_t, \tag{O.3}
\]
with \( \frac{D\vec{v}}{Dt} = \ddot{\vec{v}} \) = the Eulerian acceleration.
O.3 Cauchy theorem $\bar{T} = \sigma \bar{n}$ (stress tensor $\sigma$)

Theorem O.2 (Cauchy first law, and Cauchy stress tensor) If the master balance law (O.1) is satisfied, then $\bar{T}$ is linear in $\bar{n}$, that is, there exists a Eulerian tensor $\sigma \in T^1_1(\Omega)$, called the Cauchy stress tensor, s.t. for all $\partial \omega_i$,

$$\bar{T} = \sigma \bar{n}. \quad (O.4)$$

The proof is based on:

Lemma O.3 Let $\varphi : \{ \Omega \rightarrow \mathbb{R} \}$ and $\psi : \{(p, \bar{n}) \rightarrow \mathbb{R} \}$ s.t.

$$\int_{\partial \omega_i} \varphi(p) \, d\Omega = \int_{\partial \omega_i} \psi(p, \bar{n})(p) \, d\Gamma, \quad (O.5)$$

that is, at $p \in \partial \omega$, if $\psi$ only depends on $\bar{n}$ (no dependence on the curvature or on higher derivatives), then $\psi$ depends linearly on $\bar{n}$, that is,

$$\exists k \in \mathbb{R} \, s.t. \psi = k \bar{n}, \text{ and therefore } \varphi = \text{div } k \text{ (i.e., } \varphi \text{ is a divergence)}, \quad (O.6)$$

the last equality with (S.50) (the Gauss–Green–Ostrogradsky formula).

Proof. (Lemma O.3) (This proof is standard: We recall it.) Let $p \in \Omega \subset \mathbb{R}^3$. Consider the tetrahedral defined by its vertices $p, p + (h_1, 0, 0), p + (h_2, 0, 0)$ and $p + (0, h_3, 0)$, with $h_i > 0$ for all $i$. (On each face of a tetrahedron, the unit normal vector is uniform.) Let $\Sigma_i$ the side which outer unit normal is $-\bar{E}_i$: It is area $\sigma_1 = \frac{1}{2}h_1h_2\psi$ (square triangle). Idem for $\Sigma_2$ and $\Sigma_3$. Let $\Sigma$ be the fourth side: its area is $\sigma = \frac{1}{2}\sqrt{h_1^2h_2^2 + h_2^2h_3^2 + h_3^2h_1^2}$ and its outer unit normal is $\bar{n} = \frac{1}{2\psi}(h_2h_3, h_3h_1, h_1h_2)$ (see exercise O.4). that is $\bar{n} = (n_1, n_2, n_3)$ with $n_i = \frac{\sigma}{\psi}$ for $i = 1, 2, 3$. The volume of the tetrahedral is $1/6h_1h_2h_3 = \text{noted } \ell^3$. Let $M := \sup p \in \Omega | \varphi(p) |$; We have $M < \infty$, since $\varphi$ is continuous in $\Omega$. Then (O.6) give

$$\mathcal{M} \ell^3 \geq \int_{\partial \omega_i} \psi(p, \bar{n})(p) \, d\Gamma, \quad (O.7)$$

And $\psi$ being continuous, the mean value theorem applied on $\Sigma_i$ gives: There exists $p_i \in \Sigma_i$ s.t.

$$\int_{\Sigma_i} \psi(p, \bar{n})(p) \, d\Gamma = \sigma_i \psi(p_i, \bar{n}_i).$$

Thus

$$\int_{\partial \omega_i} \psi(p, \bar{n})(p) \, d\Gamma = \left( \sigma_1 \psi(p_1, -\bar{E}_1) + \sigma_2 \psi(p_2, -\bar{E}_2) + \sigma_3 \psi(p_3, -\bar{E}_3) + \psi(p_4, \bar{n}) \right).$$

Then, $\Psi$ being continuous, (O.7) gives

$$\sigma_1 \psi(p_1, -\bar{E}_1) + \sigma_2 \psi(p_2, -\bar{E}_2) + \sigma_3 \psi(p_3, -\bar{E}_3) + \psi(p_4, \bar{n}) = O(\ell^3). \quad (O.8)$$

We flatten the tetrahedron on the $yz$ face by taking $h_2 = h_3 = \text{noted } h$ and $h_1 = h^2$; Thus $\sigma_1 = \frac{1}{2}h^2$, $\sigma_2 = o(h^2), \sigma_3 = o(h^2), \sigma \sim \sigma_1, \ell^3 = \frac{1}{6}h^3$, with $\bar{n} \sim \bar{n}_1 = \bar{E}_1$ and $p_i \sim p_i$. Then

$$\psi(p, -\bar{E}_1) + \psi(p, +\bar{E}_1) = 0. \quad (O.9)$$

Idem with $xz$ and $xy$. And for a fixed tetrahedron with $h_1, h_2, h_3$ given, consider the smaller tetrahedron with $\varepsilon h_1, \varepsilon h_2, \varepsilon h_3$. Then as $\varepsilon \to 0$ (O.8) with (O.9) give

$$\psi(p, \bar{n}) = -\frac{\sigma_1}{\sigma} \psi(p, -\bar{E}_1) - \frac{\sigma_2}{\sigma} \psi(p, +\bar{E}_2) + \frac{\sigma_3}{\sigma} \psi(p, -\bar{E}_3) = \sum_{i=1}^3 n_i \psi(p, \bar{E}_i),$$

since $u_i = \frac{\sigma_i}{\sigma} \varepsilon$ for $i = 1, 2, 3$. The same steps can be done for any (inclined) tetrahedron (or apply a change of variable to get back to the above tetrahedron). Thus $\psi_p$ is a linear map in $\bar{n}_p$, that is, there exists a linear form $\alpha_p$ s.t. $\psi_p(\bar{n}_p) = \alpha_p \bar{n}_p$ for any $p \in \partial \omega$. And the Riesz representation theorem gives:

$$\exists\mathcal{R} \text{ s.t. } \alpha_p \bar{n}_p = (\mathcal{R}_p, \bar{n}_p) = \text{noted } \mathcal{R}_p \cdot \bar{n}_p.$$
Exercise O.4 Consider a triangle $T$ in $\mathbb{R}^3$ which vertices are $A = (h_1, 0, 0), B = (0, h_2, 0), C = (0, 0, h_3)$. Prove that $\vec{n} = (h_2h_3, h_3h_1, h_1h_2)$ is orthogonal to $T$ and that $\sigma = \frac{1}{2}\sqrt{h_2^2h_3^2 + h_3^2h_1^2 + h_1^2h_2^2}$ is its area.

Answer. Consider the parametric surface $\vec{r}(t, u) = A + t\vec{AB} + u\vec{AC}$ for $t, u \in [0, 1]$ describing the triangle. Thus

$$\vec{n} = \frac{\vec{AB} \wedge \vec{AC}}{||\vec{AB} \wedge \vec{AC}||} = \left(-\frac{h_1}{h_2}, 0, \frac{h_1}{h_3}\right) \wedge \left(0, -\frac{h_1}{h_2}, 1\right) = \left(\frac{h_2h_3}{h_1h_2}, \frac{h_3}{h_1h_2}, 0\right)$$

is symmetric. Idem for the other components: $\sigma$ is symmetric.

This excludes e.g. Cosserat continua.

Corollary O.5 With $\text{div}_{\varrho} := \sum_{i=1}^n (\sum_{j=1}^n \partial_{\varrho_{ij}}) \epsilon_i$ (definition of “the matrix divergence” see (S.66)),

$$f + \text{div}_{\varrho} = \rho \frac{D\vec{v}}{Dt} \text{ in } \Omega_t, \quad \varrho \vec{n} = \vec{T} \text{ on } \Gamma_t. \quad (O.10)$$

Proof. Apply Gauss Formula.

O.4 Toward an objective formulation

We cannot know $\vec{T}(t, p_t)$ by simply looking at the material: “to know the weight of a suitcase you have to lift it”, says Germain [8]. And the virtual power principle is used at all $t$ to deduce the forces acting on a body. In particular the virtual internal power states the existence of a tensor $\varrho$ s.t., for all virtual power in $\vec{w}$,

$$P_t(\vec{w}) = - \int_{p_t \in \Omega_t} \sigma(t, p_t) : d\vec{w}(t, p_t) \, dt. \quad (O.11)$$

Unfortunately, with the non-objective double dot used in $\sigma : d\vec{w}$, the value $P_t(\vec{w})$ depends on the observer (English? French?), cf. example Q.16: $P_t$ is quantitative not qualitative. Indeed the computation $\sigma : d\vec{w}$ requires a Euclidean dot product (in foot? in meter?)

With the objective Lie derivatives, an objective principle of virtual work can be stated, as well as a non-linear virtual power in $\vec{w}$ (while (O.11) is linear), see http://www.isima.fr/leborgne/IsimathMeca/PpvObj.pdf.

P Balance of momentum of motion

Definition P.1 The balance of momentum of motion is satisfied by $\rho$, $\vec{f}$ and $\vec{T}$ iff for all regular sub-open set $\in \Omega_t$

$$\frac{d}{dt} \int_{\omega_t} \rho \overrightarrow{OM} \wedge \vec{v} \, d\Omega_t = \int_{\omega_t} \rho \overrightarrow{OM} \wedge \vec{f} \, d\Omega_t + \int_{\partial\omega_t} \overrightarrow{OM} \wedge \vec{T} \, d\Gamma_t. \quad (P.1)$$

(This excludes e.g. Cosserat continua materials.)

Theorem P.2 (Cauchy second law.) The master balance law is supposed, so $\vec{T} = \varrho \vec{n}$, see (O.4).

Then, if (P.1) is satisfied, then $\varrho$ is symmetric.

Proof. (Standard proof.) Let $\vec{x} = \overrightarrow{OM} = \sum_i x_i \vec{E}_i$ and $\vec{T} = \sum_i T_i \vec{E}_i = \varrho \vec{n} = \sum_i \sigma_{ij} n_j \vec{E}_i$. Then (first component) $\vec{x} \wedge \vec{T} = x_2 T_3 - x_3 T_2 = x_2 (\sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3) - x_3 (\sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3) = (x_2 \sigma_{21} - x_3 \sigma_{31}) n_1 + (x_2 \sigma_{22} - x_3 \sigma_{32}) n_2 + (x_2 \sigma_{23} - x_3 \sigma_{33}) n_3$. Thus

$$\int_{\omega_t} \vec{x} \wedge \vec{T} \, d\Gamma_t = \int_{\omega_t} \frac{\partial (x_2 \sigma_{21} - x_3 \sigma_{31})}{\partial x_1} + \frac{\partial (x_2 \sigma_{22} - x_3 \sigma_{32})}{\partial x_2} + \frac{\partial (x_2 \sigma_{23} - x_3 \sigma_{33})}{\partial x_3} \, d\omega_t = \int_{\omega_t} x_2 (\text{div}_{\varrho})_1 + x_3 (\text{div}_{\varrho})_2 + \sigma_{32} - \sigma_{31} \, d\omega_t. \quad (O.10)$$

Idem for the other components: $\varrho$ is symmetric.
Q Uniform tensors in $L^*_n(E)$

The introduction of tensors in classical mechanics is often limited to a matrix presentation (after choices of bases) despite the claim of a tensorial presentation. E.g., with classical notations, the “double tensorial contraction” cannot be $M : N = \sum_{ij} M_{ij} N_{ij} = M N^T$ which is observer independent (use of an inner dot product to define $N^T$); But it can be $M \circ N = \text{Tr}(M \circ N) = \sum_{ij} M_{ij} N_{ij}^T$, iff the tensors are “compatible tensors”.

In this section we describe uniform tensors: It enables in particular to define without ambiguity the “contractions rules”. These rules are very simple and not “random”. And objective results are obtained without ambiguity.

Q.1 Tensorial product and multilinear form

Let $A_1, \ldots, A_n$ be $n$ finite dimension vector spaces, with $\dim(A_i) = d_i \in \mathbb{N}^*$.

Q.1.1 Tensorial product of functions

Let $f_1 : A_1 \to \mathbb{R}, \ldots, f_n : A_n \to \mathbb{R}$ be $n$ functions. Their tensorial product is the function $f_1 \otimes \cdots \otimes f_n : A_1 \times \cdots \times A_n \to \mathbb{R}$ defined by (separate variable function)

\[
(f_1 \otimes \cdots \otimes f_n)(\vec{x}_1, \ldots, \vec{x}_n) = f_1(\vec{x}_1) \cdots f_n(\vec{x}_n). \tag{Q.1}
\]

Q.1.2 Tensorial product of linear forms: multilinear forms

Let $\mathcal{L}(A_1, \ldots, A_n; \mathbb{R})$ be the set of $\mathbb{R}$-multilinear forms on the Cartesian product $A_1 \times \cdots \times A_n$, that is, the set of the functions $M : A_1 \times \cdots \times A_n \to \mathbb{R}$ s.t., for all $i = 1, \ldots, n$, all $\vec{x}_i, \vec{y}_i \in A_i$ and all $\lambda \in \mathbb{R}$,

\[
M(\ldots, \vec{x}_i + \lambda \vec{y}_i, \ldots) = M(\ldots, \vec{x}_i, \ldots) + \lambda M(\ldots, \vec{y}_i, \ldots), \tag{Q.2}
\]

the other variables being unchanged.

E.g., with (Q.1), if the $f_i$ are all linear, then their tensor product

\[
f_1 \otimes \cdots \otimes f_n \text{ is called an elementary } n\text{-multilinear form} \tag{Q.3}
\]

And we recall that $f_i(\vec{x}_i) \overset{\text{noted}}{=} f_i \vec{x}_i$ (notation for linear maps, cf. (A.12)), so that

\[
(f_1 \otimes \cdots \otimes f_n)(\vec{x}_1, \ldots, \vec{x}_n) = f_1(\vec{x}_1) \cdots f_n(\vec{x}_n) \overset{\text{noted}}{=} (f_1 \vec{x}_1) \cdots (f_n \vec{x}_n). \tag{Q.4}
\]

(The dot in $f_i \vec{x}_i$ is not the dot of an inner dot product: It is the duality dot defined to be $f_i(\vec{x}_i)$, cf. (A.12).)

Q.2 Uniform tensors in $L^0_n(E)$

Let $E$ be a real vector space, with $\dim(E) = n \in \mathbb{N}^*$. We consider a first overlay on $E$: the multilinear forms $M$ on $E$, called the uniform tensors of type $0$ s. (E.g., $M \in L^0_1(E)$ a linear form, $M \in L^0_2(E)$ an inner dot product, and $M \in L^0_n(E)$ a determinant).

Q.2.1 Definition of type $0$ uniform tensors

Let $L^0_0(E) = \mathbb{R}$. Then let $s \in \mathbb{N}^*$. The set

\[
L^0_s(E) := \mathcal{L}(E \times \cdots \times E; \mathbb{R}) \overset{s \text{ times}}{\text{}} \tag{Q.5}
\]

is called the set is called the set of uniform tensors of type $0$ on $E$. 

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Q.2.2 Example: Type \(^{(0)}\_1\) uniform tensor

A type \(^{(0)}\_1\) uniform tensor is an element of \(\mathcal{L}^1(E) = \mathcal{L}(E; \mathbb{R}) = E^*\): It is a linear form on \(E\).

Representations with a basis: Let \((\hat{e}_i)\) be a basis in \(E\), and \((e^i)\) its dual basis (basis in \(E^*\)). Let \(\ell \in \mathcal{L}^1(E)\). Define \(\ell_i := \ell(\hat{e}_i)\). Then:

\[
\ell = \sum_{i=1}^n \ell_ie^i, \quad \text{and} \quad [\ell]_{\hat{e}} = (\ell_1 \ldots \ell_n). \tag{Q.6}
\]

(The matrix representing \(\ell\) is a row matrix: Matrix of a linear form.)

Thus, if \(\vec{v} \in E\), \(\vec{v} = \sum_{i=1}^n v^i\hat{e}_i\), and \(\vec{v}\) is represented by \([\vec{v}]_{\hat{e}} = \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}\) (column matrix for a vector), then the matrix calculation gives

\[
\ell(\vec{v}) = [\ell]_{\hat{e}}[\vec{v}]_{\hat{e}} = (\ell_1 \ldots \ell_n) \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} = \sum_{i=1}^n \ell_iv^i \text{ noted } \ell.\vec{v}, \tag{Q.7}
\]

the resulting being objective (Einstein convention is satisfied).

Q.2.3 Example: Type \(^{(0)}\_2\) uniform tensor

A type \(^{(0)}\_2\) uniform tensor is an element of \(\mathcal{L}^2(E) = \mathcal{L}(E, E; \mathbb{R})\): It is a bilinear form on \(E\).

Representations with a basis: Let \((\hat{e}_i)\) be a basis in \(E\), and \((e^i)\) its dual basis (basis in \(E^*\)). Let \(T \in \mathcal{L}^2(E)\). Let \(T_{ij} := T(\hat{e}_i, \hat{e}_j)\). Then, with \(\vec{v} = \sum_{i=1}^n v^i\hat{e}_i\) and \(\vec{w} = \sum_{i=1}^n w^i\hat{e}_i\),

\[
T = \sum_{i,j=1}^n T_{ij}e^i \otimes e^j, \quad \text{and} \quad T(\vec{v}, \vec{w}) = \sum_{i,j=1}^n T_{ij}v^i w^j = [\vec{v}]_{\hat{e}}[T]_{\hat{e}}[\vec{w}]_{\hat{e}}. \tag{Q.8}
\]

(Einstein convention is satisfied.)

E.g., an inner dot product is a \(^{(0)}\_2\) uniform tensor.

An elementary uniform tensor in \(\mathcal{L}^2(E)\) is a tensor \(T = \ell \otimes m\), where \(\ell, m \in E^*\). And so, for all \(\vec{v}, \vec{w} \in E\),

\[
(\ell \otimes m)(\vec{v}, \vec{w}) = \ell(\vec{v})m(\vec{w}) \text{ noted } \ell.\vec{v}(m.\vec{w}), \tag{Q.9}
\]

the last equality with the notation for linear forms.

Q.2.4 Example: Determinant

The determinant is a alternating \(^{(0)}\_n\) uniform tensor, cf. (D.2).

Q.3 Uniform tensors in \(\mathcal{L}^r_s(E)\)

We consider an over-oklyn on \(E\): The multilinear forms acting on vectors \((\in E)\) and on functions \((\in E^*)\).

Q.3.1 Definition of type \(r\ s\) uniform tensors

Let \(r, s \in \mathbb{N}\) s.t. \(r + s \geq 1\). The set of multilinear forms

\[
\mathcal{L}^r_s(E) := \mathcal{L}(E^* \times \ldots \times E^*; \underbrace{E \times \ldots \times E}_{r \text{ times}}; \mathbb{R}) \tag{Q.10}
\]

is called the set of uniform tensors of type \(^{r\ s}\) on \(E\).

The case \(r = 0\) has been considered at § Q.2.

When \(r \geq 1\), a tensor \(T \in \mathcal{L}^r_s(E)\) is a functional: Its domain of definition contains a set of functions (the set \(E^* = \mathcal{L}(E; \mathbb{R})\)).
Q.3.2 Example: Type \((1,0)\) uniform tensor: Identified with a vector

A uniform \((1,0)\) tensor is a element of \(\mathcal{L}_0^1(E) = \mathcal{L}(E^*, \mathbb{R}) = E^{**}\).

We have the natural (i.e. independent of an observer) canonical isomorphism

\[
\mathcal{J} : \begin{cases}
E \to E^{**} \\
\vec{w} \to \mathcal{J}(\vec{w}) = w, \quad w(\ell) := \ell(\vec{w}) \quad \forall \ell \in E^*,
\end{cases}
\]  

\text{cf. (T.9) and Proposition T.5. (Q.3)}  

And \(w = \mathcal{J}(\vec{w}) \in E^{**}\) being linear, \(w(\ell) = \vec{w}(\ell)\), so \(w(\ell) = \ell(\vec{w})\). Then thanks to \(\mathcal{J}\) (natural canonical isomorphism), \(w\) is identified to \(\vec{w}\):

\[
w = \vec{w}, \quad \text{and} \quad w.\ell = \ell.\vec{w}.
\]  

So a \((1,0)\) type uniform tensor \(w\) is identified to the vector \(\vec{w} = \mathcal{J}^{-1}(w)\).

**Interpretation:** \(E^{**}\) is the set of directional derivatives. Indeed, if \(E\) is an affine space, if \(p \in E\), and if \(f\) is a differentiable function at \(p\), then \(w.\text{df}(p) = \text{df}(p).\vec{w}\), \text{cf. (Q.11)}, that is \(w\) is the directional derivative along \(\vec{w}\). In differential geometry, \(w.\text{df}\) is written \(\vec{w}(f)\), so \(\vec{w}(f)(p) := \text{df}(p).\vec{w}\).

**Representation with a basis** (quantification). Let \((\vec{e}_i)\) be a basis in \(E\), let \((e^i)\) be its dual basis (basis in \(E^*\)), and let \((\partial_i)\) be its bidual basis: Thus \(\partial_i \in (E^*)^* = E^{**}\) and

\[
\partial_i(e^j) = \delta^j_i = e_i(\vec{e}_j), \quad \text{thus} \quad \partial_i = \mathcal{J}(\vec{e}_i), \quad \text{and} \quad \partial_i(e^j)_{\text{noted}} = \vec{e}_i(e^j),
\]

for all \(i, j\), and \(\partial_i = \mathcal{J}(\vec{e}_i)\) is identified with \(\vec{e}_i\). So, if \(f\) is a \(C^1\) function and \(\text{df}(p) = \sum_{i=1}^n f_i(p) e^i\), then

\[
\partial_i(\text{df}(p)) = \text{df}(p).\vec{e}_i = f_i(p)_{\text{noted}} \equiv \partial_i(f)(p)_{\text{noted}} = \vec{e}_i(f)(p).
\]

Q.3.3 Example: Type \((1,1)\) uniform tensor

An elementary uniform tensor in \(\mathcal{L}_1^1(E)\) is a tensor \(T = u \otimes \beta\), where \(u \in E^{**}\) and \(\beta \in E^*\). And, with \(\vec{u} = \mathcal{J}^{-1}(u) \in E\), \text{cf. (Q.11)}, we also write \(T = \vec{u} \otimes \beta\). Thus,

\[
(u \otimes \beta)(\ell, \vec{w}) = u(\ell)\beta(\vec{w}) = \ell(\vec{u})\beta(\vec{w})_{\text{noted}} = \vec{u}(\ell)\beta(\vec{w})_{\text{noted}} = (\vec{u} \otimes \beta)(\ell, \vec{w}),
\]

for all \(\ell \in E^*\) and \(\vec{w} \in E\).

And any tensor is a sum of elementary tensors.

**Representation with a basis** (quantification). Let \(T \in \mathcal{L}_1^1(E) = \mathcal{L}(E^*, E; \mathbb{R})\). Let \((\vec{e}_i)\) be a basis in \(E\) and \((e^i)\) be its dual basis (basis in \(E^*\)). Then \(\vec{e}_i \otimes e^j\) is an elementary \((1,1)\) tensor. And \(T \in \mathcal{L}_1^1(E)\) being multilinear, \(T\) is determined as soon as the values \(T^i_j := T(e^i, \vec{e}_j)\) are known. And then

\[
T = \sum_{i,j=1}^n T^i_j \vec{e}_i \otimes e^j, \quad \text{and} \quad [T]_{\vec{e}} = [T]_{e}^i_j,
\]

\([T]_{\vec{e}} = [T]_{e}^i_j\) being the matrix of \(T\) relative to the basis \((\vec{e}_i)\). (Einstein convention is satisfied.)

Thus with \(\ell \in E^*\) and \(\vec{w} \in E\), then \(\ell = \sum_{i=1}^n \ell_i e^i \in E^*\) and \(\vec{w} = \sum_{i=1}^n \vec{e}_i \in E\) and (Q.16) give

\[
T(\ell, \vec{w}) = \sum_{i,j=1}^n T^i_j \vec{e}_i(e^j(\vec{w})) = \sum_{i,j=1}^n T^i_j \ell_i w^j = [\ell]_{\vec{e}} [T]_{\vec{e}} [\vec{w}]_{\vec{e}}.
\]

(Einstein convention is satisfied.)

Q.3.4 Example: Type \((1,2)\) uniform tensor

The same steps are applied to any tensor. E.g., if \(T \in \mathcal{L}_2^1(E)\), then, with a basis and \(T^i_{jk} = T(e^i, \vec{e}_j, \vec{e}_k)\),

\[
T = \sum_{i,j,k=1}^n T^i_{jk} e^i \otimes e^j \otimes e^k, \quad \text{and} \quad T(\ell, \vec{u}, \vec{w}) = \sum_{i,j,k=1}^n T^i_{jk} \ell_i u^j w^k.
\]

This example will be applied to \(d^2\vec{u}\) to get \(d^2\vec{u} = \sum_{i,j,k=1}^n T^i_{jk} \vec{e}_i \otimes e^j \otimes e^k\).
Q.4 Exterior tensorial products

Let $T_1 \in \mathcal{L}^1(E)$ and $T_2 \in \mathcal{L}^2(E)$. Their tensorial product is the tensor $T_1 \otimes T_2 \in \mathcal{L}^{1+2}(E)$ defined by

$$ (T_1 \otimes T_2)(\ell_1, \ldots, \ell_{1+2}, \ldots, \ell_{2+2}, \ldots) := T_1(\ell_1, \ldots, \ell_{1+1}, \ldots)T_2(\ell_{2+1}, \ldots, \ell_{2+2}, \ldots). \quad (Q.19) $$

Particular case: with $\lambda \in \mathcal{L}^0(E) = \mathbb{R}$ and $T \in \mathcal{L}^1(E)$,

$$ \lambda \otimes T = T \otimes \lambda := \lambda X \in \mathcal{L}^1(E). \quad (Q.20) $$

Example Q.1 Let $T_1, T_2 \in \mathcal{L}^1(E)$. Quantification: Let $(\tilde{e}_i)$ is a basis, then $T_1 = \sum_{i,j=1}^n (T_1)_{ij}\tilde{e}_i \otimes e^j$ and let $T_2 = \sum_{i,j,k,m=1}^n (T_2)_{ijkm}\tilde{e}_i \otimes \tilde{e}_j \otimes e^k \otimes e^m$; Then $T_1 \otimes T_2 = \sum_{i,j,k,m=1}^n (T_1)^{ij} (T_2)_{ijkm}\tilde{e}_i \otimes \tilde{e}_j \otimes e^k \otimes e^m \in \mathcal{L}^{2+2}(E)$. \(\blacklozenge\)

Remark Q.2 Alternative definition: $T_1 \otimes T_2 := \sum_{i,j,k,m=1}^n (T_1)^{ij} (T_2)_{ijkm}\tilde{e}_i \otimes e^j \otimes \tilde{e}_k \otimes e^m \in \mathcal{L}(E^*, E^*, E; \mathbb{R})$. And we get back to the previous definition thanks to the natural canonical isomorphism $\tilde{J} : \mathcal{L}(E^*, E^*, E; \mathbb{R}) \rightarrow \mathcal{L}^{2+2}(E) = \mathcal{L}(E^*, E^*, E; \mathbb{R})$ defined by $\tilde{J}(T) = T$ where $T(\ell, \tilde{v}, \tilde{w}) = \tilde{T}(\ell, \tilde{v}, m, \tilde{w})$. \(\blacklozenge\)

Q.5 Contractions

Q.5.1 Objective contraction of a linear form with a vector

Let $\ell \in \mathcal{L}^0(E) = E^*$ and $\tilde{w} \in E$. Their objective contraction is the value:

$$ \ell(\tilde{w}) \xrightarrow{\text{noted}} \ell.\tilde{w} \xrightarrow{\text{noted}} \tilde{w}.\ell. \quad (Q.21) $$

Thus, if $(\tilde{e}_i)$ is a basis and $(e^i)$ the dual basis, then

$$ \ell.\tilde{w} = \sum_{i=1}^n \ell_i \tilde{w}^i = \sum_{i=1}^n \tilde{w}^i \ell_i = \tilde{w}.\ell = \text{Tr}(\tilde{w} \otimes \ell), \quad (Q.22) $$

the result being independent of the choice of the basis (and Einstein convention is satisfied). The notation Tr in (Q.22) means the (linear) trace operator $\text{Tr} : \mathcal{L}^1(E) \rightarrow \mathbb{R}$ defined by $\text{Tr}(\tilde{e}_i \otimes e^j) = \delta_j^i$. And here $\tilde{w} \otimes \ell = \sum_{i,j=1}^n w^j \ell_j \tilde{e}_i \otimes e^j$ gives $\text{Tr}(\tilde{w} \otimes \ell) = \sum_{i,j=1}^n w^j \ell_j \text{Tr}(\tilde{e}_i \otimes e^j) = \sum_{i,j=1}^n w^j \ell_j \delta_j^i = \sum_{i=1}^n \ell_i \tilde{w}^i$.

Exercise Q.3 Use the change of coordinate formulas to prove that the computation $\ell.\tilde{w}$ in (Q.22) gives a result independent of the basis.

Answer. Let $P$ be the change of basis matrix. So $[\tilde{w}]_{\text{new}} = P^{-1}.[\tilde{w}]_{\text{old}}$ and $[\ell]_{\text{new}} = [\ell]_{\text{old}}.P$, cf. (A.68), thus $[\ell]_{\text{new}}.\tilde{w}]_{\text{new}} = ([\ell]_{\text{old}}.P).([\tilde{w}]_{\text{old}}) = ([\ell]_{\text{old}}.P^{-1}).[\tilde{w}]_{\text{old}}$ ($[\ell]_{\text{old}}[\tilde{w}]_{\text{old}} = [\ell.\tilde{w}]_{\text{old}}$). \(\blacklozenge\)

Q.5.2 Objective contraction of an endomorphism and a vector

Let $\ell \in E^*$ and $\tilde{w}, \tilde{u} \in E$. Define the contraction of the elementary tensor $\tilde{w} \otimes \ell$ with $\tilde{u}$ by:

$$ (\tilde{w} \otimes \ell).\tilde{u} = (\ell.\tilde{u})\tilde{w}. \quad (Q.23) $$

And more generally, by definition of the summation of functions, if $\ell^i$ and $\tilde{w}_i$ are $m$ linear forms and vectors, then the contraction of the tensor $T = \sum_{i=1}^m \tilde{w}_i \otimes \ell^i$ with a vector $\tilde{u}$ is

$$ T.\tilde{u} = \sum_{i=1}^m (\tilde{w}_i \otimes \ell^i).\tilde{u} = \sum_{i=1}^m (\ell^i.\tilde{u})\tilde{w}_i. \quad (Q.24) $$

(And any tensor is a sum of elementary tensors.) In particular, if $(\tilde{e}_i)$ is a basis in $E$ and $(e^j)$ is the dual basis, then, for any $\tilde{u} = \sum_{i=1}^n u^i \tilde{e}_i \in E$:

$$ T = \sum_{i,j=1}^n T_{ij} \tilde{e}_i \otimes e^j \Rightarrow T.\tilde{u} \overset{(Q.24)}{=} \sum_{i,j=1}^n T_{ij} u^i \tilde{e}_i, \quad (Q.25) $$

because $e^j(\tilde{w}) = u^j$. And since the set $\mathcal{L}(E) = \mathcal{L}(E; E)$ of endomorphisms is naturally canonically isomorphic to the set $\mathcal{L}(E, E^*; \mathbb{R}) = \mathcal{L}^1(E)$ of uniform tensors, see (T.7), any endomorphism $L \in \mathcal{L}(E)$
can be written
\[ L = \sum_{i,j=1}^{n} T_{ij}^2 e_i \otimes e_j \quad \text{where} \quad L e_j = T e_j, \quad \text{i.e.} \quad L e_j = \sum_{i=1}^{n} T_{ij} e_i, \quad \forall j. \quad (Q.26) \]

Thus (Q.23) or (Q.24) or (Q.25) is also called the contraction of an endomorphism \( L \) with the vector \( \vec{u} \), or simply the contraction of \( L \) and \( \vec{u} \) (as in (Q.21)).

### Q.5.3 Objective contractions of uniform tensors

More generally: The contraction of two tensors, if meaningful, is defined thanks to (Q.21). So:

Let \( T_1 \in \mathcal{L}_{r_1}^{l_1}(E), T_2 \in \mathcal{L}_{r_2}^{l_2}(E), \ell \in E^* \) and \( \vec{u} \in E \).

Consider \( T_1 \otimes \ell \in \mathcal{L}_{r_1+l_1}^{l_1}(E) \) and \( \vec{u} \otimes T_2 \in \mathcal{L}_{r_2+l_2}^{l_2}(E) \).

**Definition Q.4** The objective contraction of \( T_1 \otimes \ell \in \mathcal{L}_{r_1+l_1}^{l_1}(E) \) and \( \vec{u} \otimes T_2 \in \mathcal{L}_{r_2+l_2}^{l_2}(E) \) is the tensor 

\[ (T_1 \otimes \ell) \otimes (\vec{u} \otimes T_2) = (\ell.\vec{u}) T_1 \otimes T_2, \quad (Q.27) \]

In particular \((T_1 \otimes \ell).\vec{u} = (\ell.\vec{u}) T_1 \) (as in (Q.23)), and \((\vec{u} \otimes T_2) \cdot \ell = (\ell.\vec{u}) T_2 \) (as in (Q.20)).

**Remark Q.5** With natural canonical isomorphisms, as in remark Q.2, we also define the objective contraction of \( T_1 \otimes \vec{u} \in \mathcal{L}_{r_1+l_1}^{l_1}(E) \) and \( \ell \otimes T_2 \in \mathcal{L}_{r_2+l_2}^{l_2}(E) \) as being the tensor \((T_1 \otimes \vec{u}).(\ell \otimes T_2) \in \mathcal{L}_{r_1+l_2}^{l_1+l_2}(E) \) given by \((T_1 \otimes \vec{u}).(\ell \otimes T_2) = (\vec{u}.\ell) T_1 \otimes T_2 \). *

Representation with a basis \( (e_i) \):

**Example Q.6** Let \( T \in \mathcal{L}_{r_1}^{l_1}(E) \), and \( T = \sum_{i,j=1}^{n} T_{ij} e_i \otimes e_j \in \mathcal{L}_{r_1}^{l_1}(E) = \mathcal{L}_{0}^{0}(E) \). Let \( \vec{v} \in E \sim \mathcal{L}_{r_1}^{l_1}(E) = \mathcal{L}_{0}^{0}(E) = \mathcal{L}_{0}^{0+1}(E) \), and \( \vec{w} = \sum_{i=1}^{n} w_i e_i \). Then (Q.27) gives \( T.\vec{v} = \sum_{i=1}^{n} T_{ij} w_i e_i \) is a column matrix.

(Q.28)

(The Einstein convention is satisfied.) Indeed, \( T.\vec{v} = \sum_{i,j=1}^{n} T_{ij} w_i e_i \otimes e_j = \sum_{i,j,k=1}^{n} T_{ij} w_i e_i \delta_{jk} \).

Let \( \ell \in E^* \) and \( \ell = \sum_{i=1}^{n} \ell_i e_i \). (Q.27) and \( T \in \mathcal{L}_{r_1}^{l_1}(E) \) gives \( \ell. T \in \mathcal{L}_{r_2}^{l_2}(E) \) and \( \ell. T = \sum_{i,j=1}^{n} \ell_i T_{ij} e_j \).

(Q.29)

(The Einstein convention is satisfied.) Indeed, \( \ell. T = \sum_{i=1}^{n} \ell_i e_i \). (Q.27) gives \( \ell. T = \sum_{i,j,k=1}^{n} \ell_i T_{ij} e_{jk} \).

**Example Q.7** Let \( S, T \in \mathcal{L}_{r_1}^{l_1}(E) \), and \( S = \sum_{i,j=1}^{n} S_{ij} e_i \otimes e_j \) and \( T = \sum_{i,j=1}^{n} T_{ij} e_i \otimes e_j \).

(Q.30)

(The Einstein convention is satisfied.) Indeed, \( S. T = \sum_{i,j,k=1}^{n} S_{ij} T_{kj} e_i \otimes e_k \).

**Example Q.8** Let \( T \in \mathcal{L}_{r_1}^{l_1}(E) \), e.g. \( T = \delta^2 \vec{v} \), and \( \vec{w} \in E \sim \mathcal{L}_{r_1}^{l_1}(E) \). Then \( T = \sum_{i,j,k=1}^{n} T_{ijk} e_i \otimes e_j \) and \( \vec{w} = \sum_{i=1}^{n} w_i e_i \) gives \( (T.\vec{v}) . \vec{u} = \sum_{i,j,k=1}^{n} T_{ijk} w_i e_i \).

(Q.31)

(The Einstein convention is satisfied.) So, \( T.\vec{w} = \sum_{i,j,k=1}^{n} T_{ijk} w_i e_i \).

Q.1
Q.5.4 Objective double contractions of uniform tensors

**Definition Q.9** Let $S,T \in \mathcal{L}_1^1(E)$. The double objective contraction $S \odot T$ of $S$ and $T$ is defined by

$$S \odot T = \text{Tr}(S_T).$$  

(Q.33)

Thus, with a basis, (Q.30) gives

$$S \odot T = \sum_{i,j=1}^n S^i_j T^j_i. \quad \text{(Q.34)}$$

(Einstein convention is satisfied.)

**Proposition Q.10** $S \odot T$ defined in (Q.33) is an invariant: the real value $\sum_{i,j=1}^n S^i_j T^j_i$ in (Q.34) is independent of the chosen basis (it has the same value in all basis).

**Proof.** Indeed it is a Trace.

Proof with basis: Let $(\vec{a}_i)$ and $(\vec{b}_j)$ be two bases. Let $P = [P^i_j]$ be the transition matrix, i.e., $\vec{a}_j = \sum_{i=1}^n P^i_j \vec{a}_i$ for all $j$. Let $Q = [Q^i_j] = [P^i_j]^{-1}$. Then $b'_i = \sum_{i=1}^n Q^i_j a^j$ with $[Q^i_j] = [P^i_j]^{-1}$.

Then let $S = \sum_{ij}(S_{ij})^i_j \odot e^i \otimes e^j = \sum_{ij}(S_{ij})^i_j \otimes b^i$. So $([S_{ij}])^i_j = Q_i^j ([S_{ij}])^j_i P^i_j$. Then $\sum_{i,j=1}^n S^i_j T^j_i = \sum_{i,j=1}^n Q^i_k (S_{ij})^k_j P^k_i Q_i^j (T_{ij})^j_i = \sum_{i,j,k,m,\gamma,\epsilon}(S_{ij})^k_j (T_{ij})^j_i 2^j_i 2^j_i 2^j_i 2^j_i $.

**Definition Q.11** More generally the objective double contractions $S \odot T$ of uniform tensors, is obtained by applying the objective simple contraction twice consecutively, when applicable.

**Example Q.12** With $T_1 \otimes \ell_{11} \otimes \ell_{12}$ and $\vec{u}_{21} \otimes \vec{u}_{22} \otimes T_2$, then the first contraction gives

$$(T_1 \otimes \ell_{11} \otimes \ell_{12}) \odot (\vec{u}_{21} \otimes \vec{u}_{22} \otimes T_2) = (\ell_{12}, \vec{u}_{21}) \odot (T_1 \otimes \ell_{11} \otimes \vec{u}_{22} \otimes T_2), \quad \text{(Q.35)}$$

and the second contraction gives (concerns $T_1 \otimes \ell_{11} \otimes \vec{u}_{22} \otimes T_2$)

$$(T_1 \otimes \ell_{11} \otimes \ell_{12}) \odot (\vec{u}_{21} \otimes \vec{u}_{22} \otimes T_2) = (\ell_{12}, \vec{u}_{21}) \odot (T_1 \otimes \ell_{11} \otimes \vec{u}_{22} \otimes T_2). \quad \text{(Q.36)}$$

**Example Q.13** See (Q.34).

**Example Q.14** Let $S \in \mathcal{L}_1^1(E)$ and $T \in \mathcal{L}_1^1(E)$, $S = \sum_{i,j,k=1}^n S^i_j k^k_j \hat{e}^i \otimes e^j \otimes \hat{e}^k \otimes e^i$ and $T = \sum_{\alpha,\beta,\gamma=1}^n T^{\beta \gamma} \hat{e}_\alpha \otimes \hat{e}_\beta \otimes e^\gamma$. Then

$$S_T = \sum_{i,j,k,\alpha,\beta,\gamma=1}^n S^i_j T^{\beta \gamma} \hat{e}^i \otimes e^j \otimes \hat{e}_\beta \otimes e^\gamma, \quad \text{and} \quad S \odot T = \sum_{i,j,k,\alpha,\beta,\gamma=1}^n S^i_j T^{\beta \gamma} \hat{e}^i \otimes e^j \otimes \hat{e}_\beta \otimes e^\gamma. \quad \text{(Q.37)}$$

(The Einstein convention is satisfied.)

**Exercise Q.15** If $S \in \mathcal{L}(E,F;\mathbb{R})$, $T \in \mathcal{L}(F,G;\mathbb{R})$ and $U \in \mathcal{L}(G,E;\mathbb{R})$ then prove

$$S \odot (T,U) = (S,T) \odot U = (U_S) \odot T. \quad \text{(Q.38)}$$

**Answer.** If $S = \sum S^i_j a^i \otimes b^j$, $T = \sum T^i_j \hat{b}^i \otimes c^j$ and $U = \sum U^i_j \hat{e}^i \otimes a^j$, then $T \odot U = \sum T^i_j U^i_j \hat{b}^i \otimes a^j$, thus $S \odot (T,U) = \sum S^i_j T^i_j \hat{b}^i \otimes c^j$, and $S \odot T = \sum S^i_j \hat{b}^i \otimes c^j$, so $(S,T) \odot U = \sum S^i_j T^i_j U^i_j$. And the second equality thanks to the symmetry of $\odot$, that is, $(S,T) \odot U = U \odot (S,T) = (U_S) \odot T$ with the previous calculation.

We define in the same way the triple objective contraction (apply the simple contraction three times consecutively). E.g., with (Q.37) we get

$$S \odot T = \sum_{i,j,k=1}^n S^i_j T^k_j. \quad \text{(Q.39)}$$

(The Einstein convention is satisfied.)
Q.5.5 Non objective double contraction: Double matrix contraction

The engineers often use the double matrix contraction of second order tensors they defined by

\[ S : T := \sum_{i,j=1}^{n} S_{ij}^i T_j^i, \quad \text{or} \quad S : T = \sum_{i,j=1}^{n} S_{ij} T_{ij}, \quad \text{(Q.40)} \]

when \( S = [S^i_i] \) and \( T = [T^i_i] \), or \( S = [S_{ij}] \) and \( T = [T_{ij}] \). NB: The Einstein convention is not satisfied.

And they also write \( S : T := \text{tr}(S T^T) \): It is then obvious that their double (matrix) contraction is not objective since a transposed requires an inner dot product to be defined. See example Q.16 to be convinced.

It is also written \( S : T = (S, T)_g = \sum_{i,j=1}^{n} S_{ij} T_{ij} \) the vectorial usual inner dot product in \( \mathbb{R}^{n^2} \) when \( S \) and \( T \) is written like a long a column vector \( (S_{11}, S_{12}, \ldots, S_{1n}, S_{21}, \ldots, S_{2n}, \ldots, S_{nn})^T \in \mathbb{R}^{n^2} \).

Example Q.16 Issue: Let \( (\vec{e}_i) \) be a basis, let \( S \in \mathcal{L}(E; E) \) given by \( [S]_{\vec{e}} = \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix} \) (non symmetric matrix as the matrix \([d\vec{v}]\) of the differential of a Eulerian velocity field). Then the double matrix contraction (Q.40) gives

\[ S : S = [S]_{\vec{e}} : [S]_{\vec{e}} = 4 \times 4 + 2 \times 2 = 20. \quad \text{(Q.41)} \]

Change of basis: let \( \vec{b}_1 = \vec{e}_1 \) and \( \vec{b}_2 = 2\vec{e}_2 \) (similar to an aviation problem). Then the transition matrix from \( (\vec{e}_i) \) to \( (\vec{b}_i) \) is \( P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \). Then \( [S]_{\vec{b}} = P^{-1} \cdot [S]_{\vec{e}} \cdot P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 8 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 8 \\ 0 & 4 \end{pmatrix} \). Thus

\[ S : S = [S]_{\vec{b}} : [S]_{\vec{b}} = 8 \times 8 + 8 \times 1 = 65. \quad \text{(Q.42)} \]

So: Is \( S : S = 20 \) better than \( S : S = 65 \), cf. (Q.41)-(Q.42)? (Observer dependent results.)

To compare with the double objective contraction: \([S]_{\vec{e}} \cdot [S]_{\vec{e}} = 4 \times 2 + 2 \times 4 = 16 = 8 \times 1 + 8 = [S]_{\vec{b}} \cdot [S]_{\vec{b}} \) named \( S \oplus S \) (observer independent result = objective result).

Corollary Q.17 Let \( S, T \in \mathcal{L}_1^1(E) \). The double objective contraction \( S \oplus T \) is invariant, cf. prop. Q.10, while the double matrix contraction (Q.40) is not invariant: It depends on the choice of the basis (quite annoying), see (Q.41)-(Q.42).

(Thus the double objective contraction \( S \oplus T \) is suitable to get an objective virtual power principle.)

Remark Q.18 E.g., application to aviation where the altitude unit (the foot) is different from the horizontal unit (the Nautical Mile): The use of the double objective contraction is vital, while the use of the double matrix contraction is fatal.

Exercice Q.19 Let \( S \in \mathcal{L}_2^1(E) \) (e.g. a metric), let \( (\vec{a}_i) \) be a Euclidean basis in foot, and let \( (\vec{b}_i) = (\lambda \vec{a}_i) \) be the related euclidean basis in meter (change of unit). Give \([S]_{\vec{a}} : [S]_{\vec{a}} \) and \([S]_{\vec{b}} : [S]_{\vec{b}} \) and compare. (The double objective contraction is impossible here since \( S \) and \( T \) are not compatible even for a simple objective contraction.)

Answer: Let \( S = \sum_{i,j=1}^{n} S_{a,ij} a^i \otimes a^j = \sum_{i,j=1}^{n} S_{b,ij} b^i \otimes b^j \). Since \( (\vec{b}_i) = (\lambda \vec{a}_i) \) we have \( b^i = \frac{1}{\lambda} a^i \). Thus \( \sum_{i,j=1}^{n} S_{a,ij} a^i \otimes a^j = \sum_{i,j=1}^{n} S_{b,ij} \lambda^2 b^i \otimes b^j \), thus \( \lambda^2 S_{a,ij} = S_{b,ij} \). Thus

\[
[S]_{\vec{b}} : [S]_{\vec{b}} = \sum_{i,j=1}^{n} (S_{b,ij})^2 = \lambda^4 \sum_{i,j=1}^{n} (S_{a,ij})^2 = \lambda^4 [S]_{\vec{a}} : [S]_{\vec{a}}. \quad \text{(Q.43)}
\]

with \( \lambda^4 \geq 100 \): Quite a difference isn’t it?

Q.6 Endomorphism and tensorial notation

Q.6.1 Endomorphism identified to a 1 1 uniform tensor

We have the natural canonical isomorphism, cf. (T.7),

\[
\mathcal{J}_2 \colon \begin{cases} \mathcal{L}(E; E) \to \mathcal{L}(E^*, E; \mathbb{R}) \\ L \to T_L = \mathcal{J}_2(L) \end{cases}, \quad \text{with} \quad \forall (\vec{v}, m) \in E \times E^*, \quad T_L(m, \vec{v}) := m. L. \vec{v}. \quad \text{(Q.44)}
\]

Thus we note

\[
L^\text{noted} = \mathcal{J}_2(L) \quad (= T_L). \quad \text{(Q.45)}
\]

Let \( (\vec{e}_i) \) be a basis. Let \( L \in \mathcal{L}(E; E) \) (an endomorphism). Let \([L]_{\vec{e}} = [L]_{\vec{e}} \) be its matrix relative to
the basis \((\vec{e}_i)\), that is, the \(L^i_j\) are the components of \(L.\vec{e}_j\) relative to the basis \((\vec{e}_i)\), i.e.

\[ L.\vec{e}_j = \sum_{i,j=1}^{n} L^i_j \vec{e}_i, \quad \text{i.e.} \quad \vec{e}_i.L.\vec{e}_j = L^i_j \quad (= T_L(e^i, \vec{e}_j)), \quad (Q.46) \]

thanks to \((Q.21)\) and \((Q.44)\). And \(J_2\) allows to note

\[ L^n \equiv \sum_{i,j=1}^{n} L^i_j \vec{e}_i \otimes e^j \quad (= J_2(L) = T_L). \quad (Q.47) \]

So, if \(\vec{w} \in E\) and \(\vec{w} = \sum_{j=1}^{n} w^j \vec{e}_j\), then \((Q.47)\) gives (as expected)

\[ L.\vec{w} = \sum_{i,j=1}^{n} L^i_j w^j \vec{e}_i, \quad \text{and} \quad [L.\vec{w}]_\vec{e} = [L]_\vec{e} [\vec{w}]_\vec{e}. \quad (Q.48) \]

Indeed, \((\sum_{i,j=1}^{n} L^i_j \vec{e}_i \otimes e^j)(\sum_{k=1}^{n} w^k \vec{e}_k) = \sum_{i,j=1}^{n} L^i_j \vec{e}_i (e^j, \vec{e}_k) = \sum_{i,j=1}^{n} L^i_j w^k \delta^k_i = \sum_{i,j=1}^{n} L^i_j w^j \vec{e}_i\).

### Q.6.2 Simple and double objective contractions of endomorphisms

The simple contraction of \(L \in \mathcal{L}(E; E)\) et \(M \in \mathcal{L}(E; E)\) is the composed endomorphism \(L \circ M =: L.M\), the notation \(L.M\) being both the usual notation for linear maps as well as the notation of the contraction \(T_L.T_M\), cf. \((Q.44)\). So, if \(L = \sum_{i,j=1}^{n} L^i_j \vec{e}_i \otimes e^j\) and \(M = \sum_{k,m=1}^{n} M^k_m \vec{e}_k \otimes e^m\), then

\[ L \circ M =: L.M = \sum_{i,j,k=1}^{n} L^k_i M^k_j \vec{e}_i \otimes e^j, \quad \text{and} \quad [L.M]_\vec{e} = [L]_\vec{e} [M]_\vec{e}. \quad (Q.49) \]

also obtained with the contractions, cf. \((Q.30)\).

And the double objective contraction is, cf. \((Q.34)\),

\[ L \otimes M =: \text{Tr}(L \circ M) = \text{Tr}(L.M) = \sum_{i,j=1}^{n} L^i_j M^j_i \quad (Q.50) \]

(The trace of an endomorphism is objective, and the Einstein convention is satisfied.)

### Q.6.3 Double matrix contraction (not objective)

We refer to § Q.5.5: Let \([L^i_j]\) and \([M^j_i]\) be \(n \times n\) matrices. The double matrix contraction is defined by

\[ [L^i_j]: [M^j_i] := \sum_{i,j=1}^{n} L^i_j M^j_i. \quad (Q.51) \]

The Einstein convention is not satisfied.

### Q.7 Kronecker contraction tensor, trace

**Definition Q.20** The \(\binom{1}{1}\) uniform tensor \(\delta \in \mathcal{L}^1(E)\) defined

\[ \forall (\ell, \vec{u}) \in E^* \times E, \quad \delta(\ell, \vec{u}) := \ell.\vec{u} \quad (Q.52) \]

is named the Kronecker tensor.

Thus, if \((\vec{e}_i)\) is a basis then

\[ \delta^j_i = \delta(e^i, \vec{e}_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad \text{named the Kronecker symbols.} \quad (Q.53) \]

Thus

\[ \delta := \sum_{i=1}^{n} e^i \otimes e^i, \quad [\delta] = [\delta^j_i] = I \quad \text{(for any basis)}. \quad (Q.54) \]

With the canonical natural isomorphism \(\mathcal{L}(E; E) \cong \mathcal{L}(E^*; E; \mathbb{R})\), see \((T.16)\), \(\delta\) is identified to the identity
isomorphism: we then have \( I = \text{noted } \delta \):

\[
\delta \text{ named } = I, \text{ meaning (thanks to contractions) } \delta \bar{v} := \bar{v}, \forall \bar{v} \in E. \tag{Q.55}
\]

Indeed, the contractions rules give \((\sum_{i=1}^{n} e_i \otimes e^i) \bar{v} = \sum_{i=1}^{n} (e^i \bar{v}) e_i = \bar{v} \).

\[\text{R Tensors in } T^r_s(U)\]

First we need the definition of a (Eulerian) vector field. Then a first degree of complexity is introduced with the definition of functions acting on the vector fields, that is, the differential forms (or one-forms). Then a second degree of complexity is introduced with the tensors that are functionals acting on vector fields and on differential forms.

\[\text{R.1 Introduction, module, derivation}\]

Let \(A\) and \(B\) be any sets, and let \(F(A; B)\) be the set of scalar valued functions. The “plus” interior operation and the “dot” exterior operation are defined by: for all \(f, g \in F(A; B)\), all \(\lambda \in \mathbb{R}\) and all \(p \in A\),

\[
\begin{cases}
(f + g)(p) := f(p) + g(p), & \text{and} \\
(\lambda f)(p) := \lambda f(p), & \lambda f \text{ noted } \lambda f.
\end{cases}
\tag{R.1}
\]

Thus \((F(A; B), +, .., \mathbb{R})\) is a vector space on the field \(\mathbb{R}\) (easy proof). This is introduced in any elementary course.

But the field \(\mathbb{R}\) is “too small” to define a tensor which can be seen as “an linear object that satisfies the change of coordinate system rules”:

\[\text{Remark R.1 Fundamental counterexample} \text{ (the } \mathbb{R}\text{-linearity is not sufficient). Consider the derivation } d : \bar{w} \in C^\infty(\Omega) \to d\bar{w} \in C^\infty(\Omega). \text{ It is } \mathbb{R}\text{-linear (trivial), but its component on a basis of a coordinate system (the Christoffel symbols) do not satisfy the change of coordinate system rules. In fact, “the change of coordinate system rules” deal with algebra (where no derivation is used), while a derivation is not an algebraic concept but a functional analysis concept (a derivation is not a tensor: It is a “spray”, see Abraham-Marsden [1]). In fact the problem comes from the derivation rule, with } f \in C^\infty(\Omega; \mathbb{R}) \text{ a regular scalar valued function, } d(f\bar{w}) = f d\bar{w} + df.\bar{w}, \text{ so } d(f\bar{w}) \neq f d(\bar{w}). \tag{R.2}\]

Thus the \(\mathbb{R}\)-linearity of an operator \(T : E \to F\) (like a derivation) is not sufficient to characterize a tensor (corresponds to the case \(f\) is a constant function), and the requirement for some \(T\) to be a tensor will look like:

\[
T(f\bar{w}) = f T(\bar{w}), \tag{R.3}
\]

that is, \(T\) has to be \(C^\infty(\Omega; \mathbb{R})\)-linear, and not only \(\mathbb{R}\)-linear (the field \(\mathbb{R}\) is to small: It doesn’t introduce enough constraints; It has to be enlarged to \(C^\infty(\Omega; \mathbb{R})\)).

\[\text{Solution: To replace a “vector space” build from a field (like } \mathbb{R}\text{) by a “module” build from a ring (like } C^\infty(\Omega; \mathbb{R})). \text{ Reminder: A ring is almost like a field, except that in a ring some elements don’t have an inverse. E.g., a function } \varphi \in C^\infty(\Omega; \mathbb{R}) \text{ that vanishes at one point doesn’t have an inverse } \in C^\infty(\Omega; \mathbb{R}). \text{ And the definition of the module is very similar to the definition of vector space, but for the external product where a real is replaced by an element of a ring. }\]

Here for the group \((F(A; B), +)\), and compared to (R.1), the external dot product on \(\mathbb{R}\) is generalized to the external dot product on \(F(A; B)\) defined by: For all \(f \in F(A; B)\), all \(\varphi \in F(A; \mathbb{R})\) and all \(p \in A\),

\[
(\varphi f)(p) := \varphi(p)f(p), \quad \varphi f \text{ noted } = \varphi f \tag{R.4}
\]

(and \(f, \varphi := \varphi f\). And \((F(A; B); +, .., F(A; \mathbb{R}))\) is a module (over the ring \(F(A; \mathbb{R}))\).
R.2 Functions and vector fields

R.2.1 Framework

Let $U$ be an open set in an affine space $E$, and let $E$ be an associated vector. (More generally $U$ is an open set in a differentiable manifold.)

Classical mechanics: The definition of tensors is done at any (fixed) time $t$.

As before, the approach is first qualitative, then quantitative, and at $p \in E$ a basis will then be noted $(\vec{e}_i(p))$, and its dual basis $(e^i(p))$.

R.2.2 Field of functions

Let $f \in \mathcal{F}(U; \mathbb{R})$ be a function. The associated function field $\widetilde{f}$ is

$$\widetilde{f} : \begin{cases} U \to U \times \mathbb{R} \\ p \to \widetilde{f}(p) := (p; f(p)) \end{cases},$$

and $p$ is called the base point. So $\text{Im} \widetilde{f} = \{(p; f(p)) : p \in U\}$ is the graph of $f$. (A field of functions is of Eulerian type, not Lagrangian.)

Let $T^0_0(U)$ be the set of function fields on $U$, called the set of $(0,0)$ type tensor on $U$, or tensors of order 0 on $U$. In $T^0_0(U)$ (and for any type tensor), the internal sum and the external multiplication on the ring $\mathcal{F}(U; \mathbb{R})$ are defined by, for $\widetilde{f}, \widetilde{g} \in T^0_0(U)$ with $\widetilde{f}(p) = (p; f(p))$ and $\widetilde{g}(p) = (p; g(p))$, and for $\varphi \in \mathcal{F}(U; \mathbb{R})$:

$$\begin{cases} (\widetilde{f} + \widetilde{g})(p) := (p; (f + g)(p)) & (= (p; f(p) + g(p))), \\ (\varphi \widetilde{f})(p) := (p; (\varphi f)(p)) & (= (p; \varphi f(p))). \end{cases}$$

(R.6)

So the base point $p$ is kept, and (R.1) and (R.4) are applied: (R.6) models the actual computation made by an observer located at $p$ (no gift of ubiquity). Abusive notations:

$$\begin{cases} \widetilde{f}(p) \text{ noted } f(p) \text{ instead of } \widetilde{f}(p) = (p; f(p)), \\ T^0_0(U) \text{ noted } \mathcal{F}(U; \mathbb{R}). \end{cases}$$

(R.7)

It lightens the notations, but keep the base point in mind (Eulerian functions, no ubiquity gift).

R.2.3 Vector fields

Classical mechanics: Let $\vec{w} \in \mathcal{F}(U, E)$ be a vector valued function (sufficiently regular, at least Lipschitzian, to get integral curves, cf. Cauchy–Lipschitz theorem). The associated vector field $\vec{\widetilde{w}}$ is

$$\vec{\widetilde{w}} : \begin{cases} U \to U \times E \\ p \to \vec{\widetilde{w}}(p) = (p; \vec{w}(p)) \end{cases}.$$

(R.8)

So $\text{Im} \vec{\widetilde{w}} = \{(p; \vec{w}(p)) : p \in U\}$ is the graph of $\vec{w}$: the vector $\vec{w}(p)$ is drawn at $p$ (the base point). (A vector field is of Eulerian type, not Lagrangian.)

Let

$$\Gamma(U) := \text{the set of vector fields on } U.$$

(R.9)

Abusive notation:

$$\vec{\widetilde{w}}(p) \text{ noted } \vec{w}(p) \text{ instead of } \vec{\widetilde{w}}(p) = (p; \vec{w}(p)).$$

(R.10)

It lightens the notations, but keep the base point in mind (Eulerian functions, no ubiquity gift).

More precisely, we will use the following full definition of vector fields (see e.g. Abraham–Marsden [1]): A vector field is built from tangent vectors to curves. It makes sense on non planar surfaces, and more generally on differential manifolds.

E.g., see § 10.5.2 for a fundamental example in mechanics.
R.3 Differential forms, covariance and contravariance

Reminder: If \( f : U \to \mathbb{R} \) be \( C^1 \), then its differential \( df : U \to E^* \) is called an “exact differential form". Recall: If \( p \in U \) then \( df(p) \in E^* = \mathcal{L}(E; \mathbb{R}) \) is the linear form defined on \( E \) by, for all \( \vec{u} \in E \),
\[
df(p)\vec{u} := \lim_{h \to 0} \frac{f(c_p(h)) - f(c_p(0))}{h}.
\]
And if \( U \) is a non planar surface, then \( df(p)\vec{u} := \lim_{h \to 0} \frac{f(c_p(h)) - f(c_p(0))}{h} \) where \( c_p : h \to c_p(h) \in U \) is a regular curve s.t. \( c_p(0) = p \) and \( c_p'(h) = \vec{u} \). (If \( U \) were planar: \( c_p(h) = p + hi + o(h) \)).

An “exact differential form" is a particular case of a “differential form":

R.3.1 Differential forms

The basic concept is that of vector fields. A first degree of complexity (a first overlay) is introduced with the differential forms with are “functions defined on vector fields”:

**Definition R.2** Let \( \alpha \in \mathcal{F}(U; E^*) \) (so, if \( p \in U \) then \( \alpha(p) \in E^* = \mathcal{L}(E; \mathbb{R}) \), i.e., \( \alpha(p) \) is a linear form at each \( p \). The associated differential form (also called a 1-form) \( \tilde{\alpha} \) is

\[
\tilde{\alpha} : \begin{cases} U \to U \times E^* \\ p \to \tilde{\alpha}(p) = (p, \alpha(p)) \end{cases} \tag{R.11}
\]

So \( \text{Im} \tilde{\alpha} = \{(p, \alpha(p)) : p \in U \} \) is the graph of \( \alpha \): \( \alpha(p) \) is drawn at point \( p \) (base point). (A differential form is of Eulerian type, not Lagrangian.)

Let
\[
\Omega^1(U) := \text{the set of differential forms } U. \tag{R.12}
\]

Thus, if \( \tilde{\alpha} \in \Omega^1(U) \) (differential form) and \( \vec{w} \in \Gamma(U) \) (vector field), then \( \tilde{\alpha} \cdot \vec{w} \in \mathcal{T}_0^0(U) \) (scalar valued), and

\[
\tilde{\alpha} \cdot \vec{w} : \begin{cases} U \to U \times \mathbb{R} \\ p \to (\tilde{\alpha} \cdot \vec{w})(p) = (p, (\alpha \cdot \vec{w})(p)) = (p, \alpha(p), \vec{w}(p)) \in U \times \mathbb{R} \end{cases} \tag{R.13}
\]

Abuse notation:
\[
\tilde{\alpha}(p) \text{ noted } \alpha(p) \text{ instead of } \tilde{\alpha}(p) = (p, \alpha(p)). \tag{R.14}
\]

It lightens the notations, but keep the base point in mind (Eulerian functions, no ubiquity gift).

**Remark R.3** Thermodynamic: Let \( U \) be the internal energy. Then its differential \( dU \) is a (exact) differential form (first principle of thermodynamics). The elementary work \( w = \delta W \) is a differential form which is not exact in general (it doesn’t derive from a potential in general, e.g. because of frictions losses). And (thus) the elementary heat \( q = \delta Q := dU - \delta W \) is a non exact differential form in general.

R.3.2 Covariance and contravariance

**Definition R.4** A vector field \( \vec{w} \) is said to be covariant.

A differential form \( \alpha \) (a 1-form), which is a function acting on the vector fields \( \vec{w} \), is said to be covariant.

(See Misner, Thorne, Wheeler [14] box 2.1: “Without it [the distinction between covariance and contravariance], one cannot know whether a vector is meant or the very different () object that is a 1-form.”)

R.4 Definition of tensors

The basic concept is that of vector fields and of differential forms (first degree of complexity). A second degree of complexity (a second overlay) is introduced with the tensors with are “functions defined on vector fields and on differential forms”.

We need uniform tensors \( T \in \mathcal{L}_0^0(E) \), cf. § Q.3. Framework of § R.2.1.

The following definition of tensors (or tensor fields) enables to exclude the derivation operators which are \( \mathbb{R} \)-linear but are not tensors, cf. remark R.1.
**Definition R.5** (See e.g. Abraham–Marsden [1].) Let \( r, s \in \mathbb{N}, r+s \geq 1 \). Let \( T : \begin{cases} U \to \mathcal{L}^r_s(E) \\ p \to T(p) \end{cases} \) (so \( T(p) \) is a uniform \( \binom{r}{s} \) tensor for each \( p \), cf. (Q.3.1)). The associated function \( \tilde{T} \)

\[
\tilde{T} : \begin{cases} U \to U \times \mathcal{L}^r_s(E) \\ p \to \tilde{T}(p) = (p; T(p)) \end{cases}
\]

is a tensor of type \( \binom{r}{s} \) iff \( T \) is \( C^\infty(U; \mathbb{R}) \)-multilinear (not only \( \mathbb{R} \)-multilinear), that is, for all \( f \in C^\infty(U; \mathbb{R}) \), for all \( z_1, z_2 \) where applicable, for all \( p \in U \),

\[
T(p)(..., f(p)z_1(p) + z_2(p), ...) = f(p) T(p)(..., z_1(p)), ...
\]

written \( T(..., f z_1 + z_2, ...) = f T(..., z_1, ...) + T(..., z_2, ...) \), or

\[
\tilde{T}(..., f z_1 + z_2, ...) = f \tilde{T}(..., z_1, ...) + \tilde{T}(..., z_2, ...).
\]

Remark: If \( T = \{(p; T(p)) : p \in U\} \) is the graph of \( T \), and \( T(p) \) is drawn at point \( p \) (base point). (A tensor is of Eulerian type, not Lagrangian.)

It lightens the notations, but keep the base point in mind (Eulerian functions, no ubiquity gift).

**Example R.6 Fundamental counterexample.** See remark R.1. ☂

**R.5 Example: Type \( \binom{0}{1} \) tensor = differential forms**

Let \( T \in \mathcal{T}^1_0(U) \), so \( T(p) \in \mathcal{L}^1_0(E) = \mathcal{L}(E; \mathbb{R}) = E^* \). Thus \( T \) is a differential form: \( \mathcal{T}^0_1(U) \subset \Omega^1(U) \).

Converse: Does a differential form \( \alpha \in \Omega^1(U) \) defines a \( \binom{0}{1} \) type tensor on \( U \)? That is, is \( \alpha \) verified by \( \alpha \)? That is, is \( \alpha(f \bar{w}) = f \alpha(w) \) for all \( f \in \mathcal{F}(U; \mathbb{R}) \) and \( \bar{w} \in \Gamma(U) \)? That is, is \( \alpha \) verified by \( \alpha \)? That is, is \( \alpha(p)(f(p)\bar{w}(p)) = f(p) \alpha(p)(\bar{w}(p)) \) for all \( f \in \mathcal{F}(U; \mathbb{R}) \), all \( \bar{w} \in \Gamma(U) \) and all \( p \in U \)? The answer is yes since \( f(p) \in \mathbb{R} \), \( \bar{w}(p) \in E \) and \( \alpha(p) \) is \( \mathbb{R} \)-linear on \( E \) (there is no differentiation involved).

Therefore

\[
\mathcal{T}^1_0(U) = \Omega^1(U). \tag{R.20}
\]

**R.6 Example: Type \( \binom{1}{0} \) tensor = identified to a vector field**

Let \( T \in \mathcal{T}^1_0(U) \), so \( T(p) \in \mathcal{L}^1_0(E) = \mathcal{L}(E^*; \mathbb{R}) = E^{**} \) for all \( p \in U \). Thus, thanks to \( J \), cf. (Q.11), \( T(p) \) can be identified to a vector, thus \( \mathcal{T}^1_0(U) \) can be identified to a subset of \( \Gamma(U) \).

Converse: Does a vector field \( \bar{w} \in \Gamma(U) \) defines a \( \binom{1}{0} \) type tensor on \( U \)? That is, is \( \bar{w} \) verified by \( \bar{w} \)? That is, with \( w = J(\bar{w}) \), \( w(f \alpha) = f w(\alpha) \) for all \( f \in \mathcal{F}(U; \mathbb{R}) \) and \( \alpha \in \Omega^1(U) \)? That is, is \( \bar{w} \) verified by \( \bar{w} \)? That is, is \( \bar{w}(p)(f(p)\alpha(p)) = f(p) \bar{w}(p)(\alpha(p)) \) for all \( f \in \mathcal{F}(U; \mathbb{R}) \), all \( \alpha \in \Omega^1(U) \) and all \( p \in U \)? That is, is \( f(p)(\alpha(p)w(p)) = f(p) \alpha(p)w(p) \)? Yes!

Therefore (identification)

\[
\mathcal{T}^1_0(U) \simeq \Gamma(U). \tag{R.21}
\]

**R.7 Example: A metric is a type \( \binom{0}{2} \) tensor**

Let \( T \in \mathcal{T}^2_0(U) \), so \( T(p) \in \mathcal{L}^2_0(E) = \mathcal{L}(E^{**}; \mathbb{R}) = E^{***} \).

**Definition R.7** A metrics \( g \) on \( U \) is a \( \binom{0}{2} \) type tensor on \( U \) such that, for all \( p \in E \), \( g(p) = \text{noted } g_p(\cdot, \cdot) \) is an inner dot product on \( E \).

**R.8 Example: Type \( \binom{1}{1} \) tensor...**

Let \( T \in \mathcal{T}^1_1(U) \), so \( T(p) \in \mathcal{L}^1_1(E) \) and \( T(p)(\alpha(p), \bar{w}(p)) \in \mathbb{R} \) for all \( \alpha \in \Omega^1(U) \) and all \( \bar{w} \in \Gamma(U) \).
R.9 ... and identification with fields of endomorphisms

Let $L : p \in U \rightarrow L(p) \in \mathcal{L}(E; E)$, so $L(p)$ is an endomorphism. The associated field of endomorphisms on $U$ is

$$\mathcal{L} : \begin{cases} U \rightarrow U \times \mathcal{L}(E; E) \\ p \rightarrow (p, L(p)) \end{cases}$$

(So $L(p)$ is an endomorphism in $E$ for any $p \in E$.)

Abusive notation: $\mathcal{L}(p) = (p; L(p))$, instead of $\mathcal{L}(p) = (p, L(p))$, to lighten the notations.

And we have the natural (i.e. independent of an observer) canonical isomorphism $f_2$, cf. (Q.44). So we can identify a field of endomorphisms $L$ and the $(\mathcal{L})$ tensor $T_L = J_2(L)$: for all $p \in U$,

$$\forall q \in E^*, \forall \hat{u}_p \in E, \quad T_L(q, \hat{u}_p) = \ell_p(L_p \hat{u}_p), \quad \text{and} \quad T_L = J_2(L_p).$$

Example R.8 Let $\hat{w}$ be a vector field on $U$. Then $L = \hat{d} \hat{w}$ is a field of endomorphisms on $U$. Reminder: if $p \in U$ and $\hat{w} \subset C^1$, then $L(p) = \hat{d} \hat{w}(p) \subset \mathcal{L}(E; E)$ is defined by $\hat{d} \hat{w}(p, \hat{u}) = \lim_{h \rightarrow 0} \frac{\hat{u}(p+h \hat{w}) - \hat{u}(p)}{h} \in E$, for all $\hat{u} \in E$. And $L = \hat{d} \hat{w}$ is identified to the $(\mathcal{L})$ tensor $LT = J_2(L)$.

And $\hat{d} \hat{w}$ is a tensor in $T^1_1(U)$. Indeed $\hat{d} \hat{w}(p)(f(p), \hat{u}_1(p) + \hat{u}_2(p)) = f(p) \hat{d} \hat{w}(p, \hat{u}_1(p) + \hat{d} \hat{w}(p, \hat{u}_2(p))$ since $L(p) = \hat{d} \hat{w}(p) \in \mathbb{R}$-linear by definition of a differential. (We don’t deal with the derivation $d$, but with the result $\hat{d} \hat{w}$ of the derivation.)

So, if $(\hat{e}_i)$ is a basis, if $w^i_{j} := e^i \hat{d} \hat{w} \hat{e}_j$, then

$$d \hat{w} = \sum_{i,j=1}^{n} w^i_{j} \hat{e}_i \otimes e^j, \quad \text{i.e.} \quad d \hat{w} \hat{e}_j = \sum_{i=1}^{n} w^i_{j} \hat{e}_i \quad \forall j$$

thanks to the contraction rule (Q.23). (Einstein convention is satisfied.)

R.10 Example: Type $(\mathcal{L})$ tensor...

Same steps to define $(\mathcal{L})$ tensors, or more generally $(\mathcal{L})$ tensors.

R.11 Unstationary tensor

Let $t \in [t_1, t_2] \subset \mathbb{R}$. Let $(\hat{T}_t)_{t \in [t_1, t_2]}$ be a family of $(\mathcal{L})$ tensors, cf. (R.15). Then $\hat{T} : t \rightarrow \hat{T}_t := T_t$ is called an unstationary tensor. And the set of instationary tensors is also noted $T^1_1(U)$.

Example R.9 A Eulerian velocity field is a $(\mathcal{L})$ unstationary vector field.

S A differential, its eventual gradients, divergence

Qualitative to start, then quantitative (introduction of a basis or of an inner dot product if applicable).

S.1 Definitions

Let $\mathcal{E}$ and $\mathcal{F}$ be affine spaces associated with Banach spaces (normed complete vector spaces) $(E, \|\|_E)$ and $(F, \|\|_F)$. Let $\Omega_\mathcal{E}$ and $\Omega_\mathcal{F}$ be open sets in $\mathcal{E}$ and $\mathcal{F}$, and consider a function $\Psi : \begin{cases} \Omega_\mathcal{E} \rightarrow \Omega_\mathcal{F} \\ p_\mathcal{E} \rightarrow p_\mathcal{F} = \Psi(p_\mathcal{E}) \end{cases}$ (i.e., $\Psi \in \mathcal{F}(E; F)$). If applicable, $\mathcal{E}$ and/or $\mathcal{F}$ can be replaced by $\mathcal{E}$ and/or $F$.

Definition S.1 Let $p_\mathcal{E} \in \Omega_\mathcal{E}$. Then $\Psi$ is continuous at $p_\mathcal{E}$ iff $\Psi(q_{\mathcal{E}}) \rightarrow \Psi(p_\mathcal{E})$, that is, $\forall \varepsilon > 0$, $\exists \eta > 0, \forall q_{\mathcal{E}} \in \mathcal{E}$ s.t. $\|q_{\mathcal{E}} - p_\mathcal{E}\|_E < \eta$ we have $\|\Psi(q_{\mathcal{E}}) - \Psi(p_\mathcal{E})\|_F < \varepsilon$. And $\Psi \in C^0(E; F)$ iff $\Psi$ is continuous at all $p_\mathcal{E} \in \mathcal{E}$.

Definition S.2 Let $p_\mathcal{E} \in \Omega_\mathcal{E}$, let $\hat{u} \in E$, and let $f : h \in \mathbb{R} \rightarrow f(h) = \Psi(p_\mathcal{E} + h \hat{u}) \in F$. Then $\Psi$ is differentiable at $p_\mathcal{E}$ in the direction $\hat{u}$ iff $f$ is derivable at 0, that is, iff the following limit exists in $F$:

$$f'(0) = \lim_{h \rightarrow 0} \frac{\Psi(p_\mathcal{E} + h \hat{u}) - \Psi(p_\mathcal{E})}{h} = d\Psi(p_\mathcal{E})(\hat{u}) \in F.$$  

And $d\Psi(p_\mathcal{E})(\hat{u})$ is then called the directional derivative of $\Psi$ at $p_\mathcal{E}$ in the direction $\hat{u}$. 

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Definition S.3 Let \( p_E \in \Omega_E \). If \( d\Psi(p_E)(\vec{u}) \) exists (in \( F \)) for all \( \vec{u} \in E \) then \( \Psi \) is called Gâteaux differentiable at \( p_E \).

(And \( d\Psi(p_E) \) is homogeneous: \( d\Psi(p_E)(\lambda \vec{u}) = \lim_{h \to 0} \frac{\Psi(p_E + h\lambda \vec{u}) - \Psi(p_E)}{h} = \lambda \lim_{h \to 0} \frac{\Psi(p_E + h\vec{u}) - \Psi(p_E)}{h} \) for all \( \lambda \neq 0 \).)

Definition S.4 Let \( \Psi : \Omega_E \to \Omega_F \) and \( p_E \in \Omega_E \). If there exists a bounded (= continuous) linear map \( L_{p_E} \in L(E; F) \) such that for all \( \vec{u} \in E \),

\[
\Psi(p_E + h\vec{u}) = \Psi(p_E) + h L_{p_E} \vec{u} + o(h),
\]

then \( L_{p_E} \) is said to be (Fréchet) differentiable at \( p_E \). And (S.2) is called the first order Taylor expansion of \( \Psi \) at \( p_E \). (And the graph of \( \Psi \) admits a tangent plane at \( p_E \).)

Definition S.5 \( \Psi : \Omega_E \to \Omega_F \) is differentiable in \( \Omega_E \) iff it is differentiable at all \( p_E \in \Omega_E \). Then its differential is the map

\[
d\Psi : \Omega_E \to \mathcal{L}(E; F)
\]

where, \( \forall \vec{u} \in E \), \( d\Psi(p_E).\vec{u} = \lim_{h \to 0} \frac{\Psi(p_E + h\vec{u}) - \Psi(p_E)}{h} \). (S.3)

And if \( d\Psi \) is continuous in \( \Omega_E \) then \( \Psi \in C^1(\Omega_E; \Omega_F) \) (the set of \( C^1 \) maps \( \Omega_E \to \Omega_F \)).

Exercise S.6 Prove: \( \lim_{h \to 0} \frac{\Psi(p_E + h\vec{u}) - \Psi(p_E)}{h} \) does not depend on the measuring unit in \( \mathbb{R} \ni h \).

Answer. Let \( \lambda \neq 0 \). Then \( \lim_{h \to 0} \frac{\Psi(p_E + \lambda h\vec{u}) - \Psi(p_E)}{\lambda h} = \lim_{h \to 0} \frac{\Psi(p_E + h\vec{u}) - \Psi(p_E)}{h} = \lim_{h \to 0} \frac{\Psi(p_E + h\vec{u}) - \Psi(p_E)}{h} = \lim_{h \to 0} \frac{\Psi(p_E + h\vec{u}) - \Psi(p_E)}{h} = \lim_{h \to 0} \frac{\Psi(p_E + k\vec{u}) - \Psi(p_E)}{k} \).

Remark S.7 The definition of a tangent map in differential geometry is: If \( \Psi \in C^1(\Omega_E; \Omega_F) \), then its tangent map is defined with (S.2) by

\[
T\Psi : \Omega_E \times E \to \Omega_F \times F
\]

where \( (p_E, \vec{u}) \to T\Psi(p_E, \vec{u}) = (\Psi(p_E), d\Psi(p_E).\vec{u}) \). (S.4)

And the two points \( p_E \) (input) and \( \Psi(p_E) \) (output) are the base points, and the point \( (p_E, \Psi(p_E)) \) in \( \Omega_E \times \Omega_F \) is on the graph of \( \Psi \).

S.2 Quantification and the \( j \)-th partial derivative

Let \( E \) be of finite dimension, and let \( (\vec{e}_i(p_E)) \) be a basis in \( E \) at \( p_E \in \Omega_E \). If \( \Psi \) is differentiable at \( p_E \) then the \( j \)-th partial derivative \( \partial_j(p_E)(\Psi) \) of \( \Psi \) at \( p_E \) is its derivative along \( \vec{e}_j(p_E) \), that is,

\[
\partial_j(p_E)(\Psi) := d\Psi(p_E).\vec{e}_j(p_E) \quad \text{noted} \quad \partial_j \Psi(p_E) \quad (= \lim_{h \to 0} \frac{\Psi(p_E + h\vec{e}_j(p_E)) - \Psi(p_E)}{h}). \quad \text{(S.5)}
\]

If the \( \vec{e}_i \) are vector fields in \( \Omega_E \) such that \( (\vec{e}_i(p_E)) \) is a basis at all \( p_E \in E \), Then (S.5) defines the directional derivative operators: For \( j = 1, \ldots, n \),

\[
\partial_j : C^1(\Omega_E; E) \to C^0(\Omega_E; F)
\]

\[
\Psi \to \partial_j \Psi := d\Psi.\vec{e}_j. \quad \text{(S.6)}
\]

S.3 Example: Quantification for the differential of a scalar valued function

Let \( f : \{ \Omega_E \to \mathbb{R} \} \) be \( C^1 \). Let \( \dim(E) = n \). Let \( (\vec{e}_i(p_E)) \) be a basis at \( p_E \) in \( E \) (e.g. a polar basis at \( p_E \), or simply a Cartesian basis). The \( j \)-th partial derivative of \( f \) at \( p_E \) is the scalar value \( f_{ij}(p_E) \) defined by

\[
f_{ij}(p_E) := \frac{df(p_E).\vec{e}_j(p_E)}{h} \quad (= \lim_{h \to 0} \frac{f(p_E + h\vec{e}_j(p_E)) - f(p_E)}{h}). \quad \text{(S.7)}
\]

Thus, with \( (e^i(p_E)) \) the dual basis (in \( E^* \)) of the basis \( (\vec{e}_i) \), \( f_{ij}(p_E) \) is the \( j \)-th component of \( df(p_E) \) relative to the basis \( (e^i(p_E)) \):

\[
df(p_E) = \sum_{j=1}^n f_{ij}(p_E) e^j(p_E), \quad \text{i.e.} \quad df = \sum_{j=1}^n f_{ij} e^j. \quad \text{(S.8)}
\]
Definition S.8 The Jacobian matrix of $f$ at $p \in \mathbb{E}$ relative to $(e_i(p))$ is the row matrix

$$[df(p)|_{\hat{e}}] = (f_{i1}(p) \ldots f_{in}(p)).$$  \hfill (S.9)

(A row matrix since $df(p)$ is a linear form.)

So if $\vec{u}(p) = \sum_{j=1}^{n} u_j(p)e_j(p) \in E$, then, $df(p).\vec{u}(p) = \sum_{j=1}^{n} f_{ij}(p)u_j(p) = [df(p)|_{\hat{e}}][\vec{u}(p)]_{\hat{e}}$ that is,

$$df.\vec{u} = \sum_{j=1}^{n} f_{ij}u_j = [df]_{\hat{e}}[\vec{u}]_{\hat{e}}. \quad \hfill (S.10)$$

And if $f, g \in C^1(U; \mathbb{R})$ then (derivative of a product)

$$d(fg) = (df)g + f(dg), \quad \text{i.e.} \quad (fg)_h = f_{ij}g + f_{ji}, \quad \forall i, \ldots, n, \quad \hfill (S.11)$$

i.e.,

$$d(fg).\vec{a} = (df.\vec{a})g + f(dg.\vec{a})$$

for all $\vec{a}$. (Proof: $\lim_{h \to 0} \frac{f(pe + he \vec{a}) - f(pe)}{h} = \lim_{h \to 0} \frac{f(pe + he \vec{a}) - f(pe + he \vec{b})}{h} + \lim_{h \to 0} \frac{f(pe + he \vec{b}) - f(pe)}{h}$)

Proposition S.9 Let $(\vec{a}_i(p))$ and $(\vec{b}_i(p))$ be two bases at $p \in \mathbb{E}$, let $P(p) = [P_i^j(p)]$ be the transition matrix from $(\vec{a}_i(p))$ to $(\vec{b}_i(p))$, that is, $\vec{b}_j(p) = \sum_{i=1}^{n} P_i^j(p)\vec{a}_i(p)$ for all $j$. Then

$$[df]_{\vec{b}} = [df]_{\vec{a}}P, \quad \text{i.e.} \quad \forall j = 1, \ldots, n, \quad df(p).\vec{b}_j = \sum_{i=1}^{n} P_i^j(p)df(p).\vec{a}_i(p). \quad \hfill (S.12)$$

Proof. Apply (A.68) to the linear form $\ell = df(p) \in \mathcal{L}(E; \mathbb{R}) = E^*$. Or: $\vec{b}_j(p) = \sum_{i=1}^{n} P_i^j(p)\vec{a}_i(p)$ and $df(p)$ is linear, thus $df(p).\vec{b}_j = \sum_{i=1}^{n} P_i^j(p)df(p).\vec{a}_i(p)$, thus (S.12).

If $(e_i)$ is a Cartesian basis (independent of $p \in \mathbb{E}$), and if $p \in \Omega_x$, let $\vec{x} = \vec{e}_x(p) = \sum_{i} x^i e_i$, and let $(e^j)^{\text{named}}(dx^j)$ be its dual basis. Then, if $p \in E$,

$$df(p).e_j = \frac{\partial f}{\partial x^j}(pe) = \left( \lim_{h \to 0} \frac{f(pe + he_j) - f(pe)}{h} \right), \quad \hfill (S.13)$$

thus, cf. (S.8),

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x^j} dx^j, \quad \text{and} \quad [df]_{\hat{e}} = \left( \frac{\partial f}{\partial x^1} \ldots \frac{\partial f}{\partial x^n} \right). \quad \hfill (S.14)$$

NB: A notation that depends on the name of a variable (e.g. $x$ here) is often ambiguous since it depends on the user: $\frac{\partial f}{\partial x^a} \neq \frac{\partial f}{\partial x^b}$. When ambiguous, go back to the notation $\partial_i f = df.\hat{e}_i = f_{ij}$, cf. (S.8).

And if $df(p).\vec{a}_j = \frac{\partial f}{\partial x^a}(pe)$ and $df(p).\vec{b}_j = \frac{\partial f}{\partial x^b}(pe)$, then (S.12) reads

$$\forall j = 1, \ldots, n, \quad \frac{\partial f}{\partial x^j}(pe) = \sum_{i=1}^{n} P_i^j(p)\frac{\partial f}{\partial x^a}(pe). \quad \hfill (S.15)$$

Exercise S.10 (S.15) is also noted

$$\frac{\partial f}{\partial x^b} \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial x^a} \frac{\partial x^a}{\partial x^b}, \quad \hfill (S.16)$$

What does this notation mean?

Answer. Quick answer. We have $[\vec{x}]_{\hat{a}} = P.[\vec{x}]_{\hat{e}}$, cf. (A.68), that is,

$$\begin{pmatrix} x^1_a(x_1^b, \ldots, x_n^b) \\
\vdots \\
x^1_a(x_1^b, \ldots, x_n^b) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} P^1_j x^j_a \\
\vdots \\
\sum_{j=1}^{n} P^n_j x^j_a \end{pmatrix}, \quad \text{i.e.} \quad \frac{\partial x^i_a}{\partial x^b}(x_1^b, \ldots, x^n_b) = P^i_j, \quad \forall i. \quad \hfill (S.17)$$

Thus (S.16) means

$$\frac{\partial f}{\partial x^b}(pe) \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial x^a}(pe) \frac{\partial x^i_a}{\partial x^b}(x_1^b, \ldots, x^n_b). \quad \hfill (S.18)$$

Detailed answer. (S.18) is not satisfactory since $p \in \mathbb{E}$ and the $x^i_a$ and $x^j_b$ should be related: What are the relations?
Let $O$ be a point (origin) in $U_{\mathbb{E}}$. If $p_{\mathbb{E}} \in U_{\mathbb{E}}$, let $\bar{x} = O \bar{x}_{\mathbb{E}} = \sum_{i=1}^{n} x_{i} \bar{e}_{i} = \sum_{i=1}^{n} x_{i} \bar{b}_{i}$.

This define the function $[\bar{x}]_{a} : [\bar{x}]_{a} : [\bar{x}]_{a}([\bar{x}]_{a})$ where $[\bar{x}]_{a}([\bar{x}]_{a}) = P,[\bar{x}]_{a}$ (change of basis formula).

Then let $f_{a},f_{b} : \mathbb{R}^{n} \to \mathbb{R}$ be defined by $f_{a}(x_{1},...,x_{n}) := f(p_{\mathbb{E}})$ and $f_{b}(x_{1},...,x_{n}) := f(p_{\mathbb{E}})$.

That is, $f_{a}([\bar{x}]_{a}) = f_{a}([\bar{x}]_{a})$, thus $f_{a}$ is a function acting on a point $p_{\mathbb{E}}$ (independent of a referential), while $f_{a}$ and $f_{b}$ are functions acting on a matrix (dependent on the choice of a referential): The domain of definitions are different, so the functions $f_{a}$ and $f_{b}$ are different.

**Example S.11** Consider an English observer and its Euclidean basis ($\bar{a}_{i}$) in foot and a French observer and its Euclidean basis ($\bar{b}_{i}$) in meter. Suppose $\bar{b}_{i} = \lambda \bar{a}_{i}$ for all $i$ (change of unit cf. (B.2)). Then, $df(p)$ being linear,

$$df(p)\bar{b}_{i} = \lambda df(p)\bar{a}_{i},$$

written $\frac{df}{dx_{i}} = \lambda \frac{df}{dx_{i}}$ or $\frac{df}{dx_{i}} = \frac{df}{dx_{i}}$, (S.20)

where $x_{i}^{\lambda} = \lambda x_{i}^{\lambda}$ (contravariance formula): E.g., $x_{i}^{\lambda} = 1$ (meter) gives $x_{i}^{\lambda} = \lambda$ (foot). This is also obtained by applying (S.19): $\frac{df}{dx_{i}}([\bar{x}]_{a}) = \sum_{i=1}^{n} \frac{df}{dx_{i}}([\bar{x}]_{a})\frac{dx_{i}}{dx_{i}}([\bar{x}]_{a})$, here with $\frac{dx_{i}}{dx_{i}}([\bar{x}]_{a}) = \lambda^{\lambda}$.

**S.4 Possible gradient associated with a differential**

Let $E = \mathbb{R}^{n}$, let $f \in C^{1}(U_{\mathbb{E}}: \mathbb{R})$ (a $C^{1}$ scalar valued function), cf. § S.3, let $p_{\mathbb{E}} \in U_{\mathbb{E}}$.

Suppose that there exists a Euclidean dot product $(\cdot,\cdot)_{\mathbb{E}}$ in $E$.

**Definition S.12** The gradient $\nabla_{g}f(p_{\mathbb{E}})$ of $f$ at $p_{\mathbb{E}}$ relative to $(\cdot,\cdot)_{g}$, also called the $(\cdot,\cdot)_{g}$-gradient of $f$ at $p_{\mathbb{E}}$, is the $(\cdot,\cdot)_{g}$-Riesz representation vector of the linear form $df(p_{\mathbb{E}})$, that is, the vector characterized by, cf. (C.3),

$$\forall \bar{u} \in \mathbb{R}^{n}, \quad df(p)\bar{u} = \lambda df(p)\bar{u}_{g} \quad \text{(also written: } \nabla_{g}f(p_{\mathbb{E}}) = \lambda \nabla_{g}f(p_{\mathbb{E}})), \quad \text{(S.21)}$$

(Also written $df.\bar{u} = \nabla_{g}f(\bar{u})$ if only one inner dot product is imposed to all observers: Subjective approach.)

**Fundamental:**
- An inner dot product does not always exist (as a meaningful tool), see § B.3.2 (thermodynamics).
- $df(p_{\mathbb{E}})$ is a linear functions (covariant) while $\nabla_{g}f(p_{\mathbb{E}})$ is a vector (contravariant). Thus the change of basis formula differs: $[df(p_{\mathbb{E}})]_{old} = [df(p_{\mathbb{E}})]_{old} \lambda$, and $\nabla_{g}f(p_{\mathbb{E}}(p_{\mathbb{E}}))_{old} = \lambda^{-1} \nabla_{g}f(p_{\mathbb{E}}(p_{\mathbb{E}}))_{old}$.
- The (linear) function $df(p_{\mathbb{E}})$ has an infinity of gradient vectors: As many as inner dot products. E.g., see (C.13) where the gradient $\nabla_{g}f(p_{\mathbb{E}})$ of an English observer and his foot is much smaller than the that the gradient $\nabla_{g}f(p_{\mathbb{E}})$ of a French observer and his meter. Also see § C.2.

If you have an inner dot product, then the first degree Taylor expansion (S.2) gives

$$f(p + h\bar{u}) = f(p) + h(\nabla_{g}f(p_{\mathbb{E}}),\bar{u})_{g} + o(h). \quad \text{(S.22)}$$

**Remark S.13** If a unique inner dot product is used by all the observer (dictatorial management, quite subjective!) then $\nabla_{g}f(p_{\mathbb{E}})$ is noted $\vec{\nabla}f(p_{\mathbb{E}})$ (isometric framework). In such a framework, an English observer and a French observer cannot work together... and if they do, it can lead to an accident (e.g., the crash of the Mars Climate Orbiter, cf. remark A.41).

**Remark S.14** The differential $df$ is also called the “covariant gradient”, and the gradient vector is also called the “contravariant gradient” relative to an inner dot product.

**Remark S.15** In a general space $(E,(\cdot,\cdot)_{g})$ (e.g. in $\mathbb{R}^{n}$ with a non-Euclidean dot product $(\cdot,\cdot)_{g}$), then $\nabla_{g}f$ is called the conjugate gradient relative to $(\cdot,\cdot)_{g}$ (widely used in optimization).
S.5 Example: Quantization for the differential of a vector valued function

Let \( \dim(E) = n \) and \( \dim(F) = m \). Let \( \Psi \in C^1(\Omega_E, \Omega_F) \).

Quantification: Let \((\vec{e}_i(p_E))\) be a basis at \( p_E \) in \( E \). Let \((\vec{b}_i(p_F))\) be a basis at \( p_F = \Psi(p_E) \) in \( F \). Let \[ d\Psi(p_E)|_{\vec{e}} = (\vec{e}_j^i(p_F))|_{\vec{e}} \] be the Jacobian matrix of \( \Psi \) at \( p_E \) relative to the bases \((\vec{e}_i)\) and \((\vec{b}_i)\), that is,
\[
d\Psi(p_E)\vec{e}_j(p_E) = \sum_{i=1}^m \Psi_{ij}(p_E) \vec{b}_i(p_F), \quad \text{written } \quad d\Psi\vec{e}_j = \sum_{i=1}^m \Psi_{ij}\vec{b}_i, \quad \quad (S.23)
\]
the last notation being abusive (at what points? \( p_E? \), \( p_F? \)), but in a Cartesian coordinate system. Thus, if \( \vec{u}(p_E) \in E \) is a vector at \( p_E \), if \( \vec{u}(p_E) = \sum_{j=1}^n u_j(p_E)\vec{e}_j(p_E) \), then, by linearity of \( d\Psi(p_E) \),
\[
d\Psi(p_E)\vec{u}(p_E) = \sum_{i=1}^m \sum_{j=1}^n \Psi_{ij}(p_E)u_j(p_E)\vec{b}_i(p_F), \quad \text{and } \quad [d\Psi\vec{u}]_{\vec{b}} = [d\Psi]_{\vec{e}\vec{b}}[\vec{u}]_{\vec{e}}. \quad \quad (S.24)
\]
Cartesian setting: \((\vec{e}_i)\) and \((\vec{b}_i)\) are Cartesian bases, and \[ d\Psi(p_E)|_{\vec{e}} = \left[ \frac{\partial \Psi^i}{\partial p^j}(p_E) \right]. \]

S.6 Trace of an endomorphism

S.6.1 Definition

Let \( E \) be a vector space, \( \dim(E) = n \). Let \( L \in \mathcal{L}(E; E) \) (an endomorphism) Let \((\vec{e}_i)\) be a basis. Let \( L_{ij} \) be the components of \( L \), that is, \( L_{ij} = \sum_j L_{ij}\delta_j^i \) for all \( j \), and and \( [L]_{\vec{e}} = [L_{ij}] \) is its matrix relative to \((\vec{e}_i)\). (If you prefer to use duality notations, see § A: \( L.e = \sum_{i=1}^n L_{ij}^i \)).

Definition S.16 The trace of \( L \) is
\[
\text{Tr}(L) = \sum_{i=1}^n L_{ii} \in \mathbb{R}. \quad \quad (S.25)
\]
(With the duality notation \( \text{Tr}(L) = \sum_{i=1}^n L^{i}_i \).

Proposition S.17 The trace of an endomorphism is independent of the basis.

Proof. Let \((\vec{a}_i)\) and \((\vec{b}_i)\) be two bases. Let \([L]_{\vec{a}} = [(L_a)_{ij}]\) and \([L]_{\vec{b}} = [(L_b)_{ij}]\). Let \( P = [P_{ij}] \) be the transition matrix from \((\vec{a}_i)\) to \((\vec{b}_i)\). Let \( Q = P^{-1} \). Thus \([L]_{\vec{b}} = Q[L]_{\vec{a}} P\), cf. (A.77). Thus \( \sum_{i=1}^n (L_b)_{ii} = \sum_{i,j,k} Q_{ij}(L_a)_{jk} P_{ki} = \sum_{i,j,k} P_{ki} Q_{ij}(L_a)_{jk} = \sum_{i,j,k} \delta_{kj}(L_a)_{jk} = \sum_j (L_a)_{jj} \).

S.6.2 Alternative definition: With one-one tensors

Consider the natural canonical isomorphism \( \tilde{\mathcal{F}} : \mathcal{L}(E; E) \to \mathcal{L}(E^*, E; \mathbb{R}) \) given by \( L \to \tilde{T}_L \), \( \tilde{T}_L(\ell, \vec{u}) := \ell \cdot (L \vec{u}) \), cf. (T.7).

E.g., the elementary \((1,1)\) uniform tensor \( \psi \otimes \ell \) is naturally canonically associated to the endomorphism \( \tilde{\mathcal{F}}^{-1}(\psi \otimes \ell)(\vec{u}) = \ell(\vec{u}) \psi \).

Definition S.18 The trace operator is also the name given to the linear map \( \overline{\text{Tr}} : \mathcal{L}_1(E) \to \mathbb{R} \) by
\[
\forall (\vec{e}, \ell) \in E \times E^*, \quad \overline{\text{Tr}}(\vec{e} \otimes \ell) = \ell.\vec{e} = \delta(\ell, \vec{e}), \quad \quad (S.26)
\]
where \( \delta \) is the Kronecker \((1,1)\) tensor (in a basis, \( \delta = \sum_{i=1}^n \delta_i^i \otimes e^i \)).

Quantification: If \((\vec{e}_i)\) is a basis, if \( T_L \in \mathcal{L}_1(E) \), if \( T_L = \sum_{i,j=1}^n T_{ij}^i e_i \otimes e^j \), then \( \overline{\text{Tr}}(T_L) = \sum_{i,j=1}^n T_{ij}^i \text{Tr}(e_i \otimes e^j) = \sum_{i,j=1}^n T_{ij}^i \delta_i^j \), so
\[
\overline{\text{Tr}}(T_L) = \sum_{i=1}^n T_{ii}^i = \text{Tr}(L). \quad \quad (S.27)
\]
And the trace of a \((0,1)\) tensor is trivially invariant (observer independent): If \((\vec{a}_i)\) and \((\vec{b}_i)\) are two bases, then
\[
T_L = \sum_{i,j=1}^n A_{ij}^i \vec{a}_i \otimes a^j = \sum_{i,j=1}^n B_{ij}^j \vec{b}_i \otimes b^j \implies \overline{\text{Tr}}(T_L) = \sum_{i=1}^n A_{ii}^i = \sum_{i=1}^n B_{ii}^i, \quad \quad (S.28)
\]
since \( a^i(\vec{a}_j) = b^i(\vec{b}_j) = \delta_j^i \) for all \( i, j \).

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\[ \vec{v} = \sum_i v^i \vec{e}_i \text{ and } \ell = \sum_j \ell_j e^j \text{ give } \overline{\text{Tr}}(\vec{v} \otimes \ell) = \sum_i v^i \ell_i = \ell(\vec{v}). \]

**Example S.20** For the Kronecker tensor \( \delta = \sum_{i=1}^n e_i \otimes e^i \), cf. (Q.54),

\[ \overline{\text{Tr}}(\delta) = \sum_i \text{Tr}(e_i \otimes e^i) = \sum_{i=1}^n \delta_i^i = \sum_{i=1}^n 1 = n = \text{Tr}(I), \quad (S.29) \]

trace of the identity (endomorphism) \( I : E \to E \). And with the objective double contraction, we get:

\[ \overline{\text{Tr}}(T_L) = \delta \circ T_L, \quad (S.30) \]

since \((\sum_{i,j=1} \delta_i^j e_i \otimes e^j) \otimes \sum_{j,k=1}^n T_k^j e_k \otimes e^j) = \sum_{i=1}^n T_i^i. \]

**Exercise S.21** Let \( L \in \mathcal{L}(E; E) \), let \((\vec{a}_i)\) and \((\vec{b}_i)\) be bases in \( E \), and let \( L \vec{a}_j = \sum_{i=1}^n L_{ij}^k \vec{b}_k \). (NB: Here \( L \) is an endomorphism that is not described with only one basis.) Prove: if \( P \) is the change of basis from \((\vec{a}_i)\) to \((\vec{b}_i)\), then \( \text{Tr}L = \sum_{i=1}^n (PL)^{ij} \).

**Answer.** \( L = \sum_{i} \sum_k L_{ij}^k (\sum_l P_l^k \vec{a}_k) \otimes a^l = \sum_{i,j} (\sum_k L_{ij}^k \vec{a}_k) \otimes a^l = \sum_{i,j} (PL)^{ij} \vec{a}_k \otimes a^l. \)

### S.7 Divergence of a vector field: invariant

Let \( \Omega \) be an open set in \( \mathbb{R}^n \). Here \( \Gamma(\Omega) \) stands for the set of \( C^1 \) vector fields.

**Definition S.22** The divergence operator is

\[ \text{div} := \text{Tr} \circ d : \{ \Gamma(\Omega) \to C^0(\Omega; \mathbb{R}) \}, \quad \vec{w} \to \text{div}\vec{w} = \text{Tr}(d\vec{w}). \quad (S.31) \]

\[ \text{div} = \text{Tr} \circ d \] is \( \mathbb{R} \)-linear (it is composed of two \( \mathbb{R} \)-linear maps).

Let \((\vec{e}_i(p))\) be a basis in \( \mathbb{R}^n \) at \( p \). Let \( \vec{w} \in \Gamma(\Omega) \) (a \( C^1 \) vector field) and let \( w^i_j(p) \) be the components of \( d\vec{w} \), that is, \( \text{div}\vec{w} = \sum_{i=1}^n w^i_j(p) \vec{e}_i \) for all \( j \). Then

\[ \text{div}\vec{w} = \sum_{i=1}^n w^i_j, \quad (S.32) \]

with \( \text{div}\vec{w} \) independent of the chosen basis, proposition S.17: The divergence of a vector field is an invariant.

If \((\vec{e}_i)\) is a Cartesian basis, then \( w^i_j = dw^i.\vec{e}_j \) (noted \( \partial w^i / \partial x^j \) (usual), and then \( \text{div}\vec{w} = \sum_{i=1}^n \partial w^i / \partial x^i \). If \( \vec{w} \) is unstationary, \( \vec{w} : (t, p) \to \vec{w}(t, p) \), then, for \( t \) fixed, \( \vec{w}_t(p) := \vec{w}(t, p) \), and

\[ \text{div}\vec{w}(t, p) := \text{div}\vec{w}_t(p). \quad (S.33) \]

**Exercise S.23** Let \( U \) be an open set in a Cartesian product \( \mathbb{R}^n \), e.g., \( \mathbb{R}^n \) considered as a space of parameters (e.g., \( \mathbb{R}^2 \) and the polar coordinates \((\rho, \theta)\)). Let \( \Omega \) be an open set in the geometric affine space \( \mathbb{R}^n \). Let \( \Psi : \vec{q} \in U \to p = \Psi(\vec{q}) \in \Omega \) be a coordinate system in \( \Omega \) (a diffeomorphism \( U \to \Omega \)). Let \((\vec{A}_i)\) be the canonical basis in \( \mathbb{R}^n \) (parameters), and let \( \vec{c}_i(p) = d\Psi(\vec{q}).\vec{A}_i \) (noted \( \partial \Psi / \partial \vec{A}_i \) at \( p = \psi(\vec{q}) \), so \( \vec{c}_i(p) \)) is the coordinate basis at \( p \) of the coordinate system \( \Psi \) (e.g., see the polar coordinate system (10.43)). The maps \( \vec{c}_i : \Omega \to \mathbb{R}^n \) define vector fields, and, for all \( p \in \Omega \), the \( \vec{c}_i(p) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) are endomorphisms. Let \( \gamma^i_{jk} := e^i.d\vec{c}_k.\vec{e}_j \) (the Christoffel symbols), that is, \( \text{div}\vec{w} = \sum_{i,j=1}^n \gamma^i_{jk} \vec{e}_j \) (the \( \gamma^i_{jk} \) are the components of \( d\vec{w} \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \)) relative to the basis \((\vec{e}_i(p))\). Use the \( \gamma^i_{jk} \) to express \( d\vec{w} \) and \( \text{div}\vec{w} \).

**Answer.** \( d\vec{w} = \sum_{i=1}^n w^i_j \vec{e}_j \) gives \( d\vec{w} = \sum_{i=1}^n \vec{e}_i \otimes dw^i + \sum_{i=1}^n w^i_j \vec{e}_j = \sum_{i=1}^n \vec{e}_i \otimes dw^i + \sum_{k=1}^n w^k d\vec{e}_k. \) And \( dw^i = \sum_{j=1}^n (d\vec{w}^i.\vec{e}_j)e^j. \) So, with \( d\vec{w}_t(p) = \sum_{j=1}^n \gamma^i_{jk}(p)(\vec{e}_i(p) \otimes e^j(p)), \) we get

\[ d\vec{w} = \sum_{i,j=1}^n w^i_j \vec{e}_j \otimes e^j \quad \text{where} \quad w^i_j = dw^i.\vec{e}_j + \sum_{k=1}^n w^k \gamma^i_{jk} \quad \text{(noted} \quad \partial w^i / \partial x^k) + \sum_{k=1}^n w^k \gamma^i_{jk}, \quad (S.34) \]

the last notation since \( \partial (w^i.\vec{e}_j) / \partial (q^j(p)) = dw^i(p).\vec{e}_j(p) \) and \( \partial (w^i.\vec{e}_j) / \partial (q^j(p)) = \text{named} \partial w^i / \partial q^j(p) \) (be careful with notations...). Therefore:

\[ \text{div}\vec{w} = \sum_{i=1}^n w^i_j = \sum_{i=1}^n dw^i.\vec{e}_j + \sum_{k=1}^n w^k \gamma^i_{jk} \quad \text{noted} \quad \sum_{i=1}^n \partial w^i / \partial q^j + \sum_{k=1}^n w^k \gamma^i_{jk}. \quad (S.35) \]

If \((\vec{e}_i)\) is Cartesian, then the \( \gamma^i_{jk} \) vanish, then \( d\vec{w} = \sum_{i,j=1}^n \partial w^i / \partial x^j \vec{e}_j \otimes e^j \) and \( \text{div}\vec{w} = \sum_{i=1}^n \partial w^i / \partial x^i (\vec{e}_i(p) \otimes e^i). \)
Remark S.24 If $\alpha$ is a differential form, if $(e_i)$ is a basis and $(e^i)$ its dual basis, and if $\alpha = \sum_{i=1}^{n} \alpha_i e^i$, then $da = \sum_{i=1}^{n} \alpha_{ij} e^i \otimes e^j$ where $\alpha_{ij} := e^i da \cdot e^j$. Here it is impossible to define an objective trace $Tr(da) = \sum_{i=1}^{n} \alpha_{ii}$ of $da$ since the Einstein convention is not satisfied: The result depends on the choice of the basis (e.g., Euclidean made with the foot? With the meter?).

S.8 Unit normal vector, unit normal form, integration

S.8.1 Framework

The results in this § are not objective: We need a unit normal to use the Green integration formula, so we need a Euclidean dot product (need of orthogonality and length) (which one: English? French?).

Let $(e_i)$ be a Euclidean basis in $\mathbb{R}^n$, $(e^i) = \text{noted}(dx^i)$ be its dual basis, and $(\cdot, \cdot)_g$ be the associated Euclidean dot product, so $g_{ij} := g(e_i, e_j) = \delta_{ij}$.

Let $\Omega$ be a regular open set in $\mathbb{R}^n$, and let $\Gamma := \partial \Omega$. If $p \in \Gamma$ then $T_p \Gamma$ is the tangent hyperplane to $\Gamma$ at $p$. And consider a basis $(\vec{b}_i(p))_{i=1,\ldots,n-1}$ in $T_p \Gamma$ (e.g., a coordinate system basis, e.g., the polar coordinate basis cf. (10.46)).

S.8.2 Unit normal vector

Let

$$\vec{\beta}_n(p) := \vec{n}_g(p) = \sum_{i=1}^{n} n^i_g(p) \vec{e}_i \quad \text{(S.36)}$$

be the unit outward normal vector at $T_p \Gamma$ relative to $(\cdot, \cdot)_g$, that is, the vector given by

$$\forall i = 1, \ldots, n-1, \quad (\vec{n}_g(p), \vec{\beta}_i(p))_g = 0, \quad \text{and} \quad ||\vec{n}_g(p)||_g = 1. \quad \text{(S.37)}$$

Remark S.25 $\vec{n}_g(p)$ depends on the observer who has chosen the unit of measurement: So this vector is not objective. See next exercise S.26.

Exercise S.26 Let $(\vec{a}_i)$ be a Euclidean basis in foot, and let $(\vec{b}_i)$ be a Euclidean basis in meter. Let

$$(\cdot, \cdot)_a = \sum_{i=1}^{n} a^i \otimes a^j$$

and

$$(\cdot, \cdot)_b = \sum_{i=1}^{n} b^i \otimes b^j$$

(these are the associated inner dot products), and let $\lambda \in \mathbb{R}$ s.t. $(\cdot, \cdot)_a = \lambda^2 (\cdot, \cdot)_b$, cf. (B.7). Let $\vec{n}_a(p)$ and $\vec{n}_b(p)$ be the corresponding unit outward normal vectors, cf. (S.37). Prove:

$$\vec{n}_b = \lambda \vec{n}_a, \quad \text{and} \quad (\vec{w}, \vec{n}_a)_a = \lambda (\vec{w}, \vec{n}_b)_b \quad \forall \vec{w} \in \mathbb{R}^n \quad \text{(S.38)}$$

($\vec{n}_b$ is $\lambda$ times larger than $\vec{n}_a$). Let $\vec{n}_a = \sum_{i=1}^{m} n^i_a \vec{a}_i$ and $\vec{n}_b = \sum_{i=1}^{m} n^i_b \vec{b}_i$. Prove:

If

$$\forall i = 1, \ldots, n, \quad \vec{b}_i = \lambda \vec{a}_i \quad \text{then} \quad \forall i = 1, \ldots, n, \quad n^i_a = n^i_b. \quad \text{(S.39)}$$

(And of course $1 = ||\vec{n}_a||^2_a = \sum_{i=1}^{n} (n^i_a)^2 = \sum_{i=1}^{n} (n^i_b)^2 = ||\vec{n}_b||^2_b = 1$).

Answer. $\vec{n}_a(p) = \lambda \vec{n}_b(p)$, since the vectors are Euclidean and orthogonal to $T_p \Gamma$ cf. (S.37). And $||\vec{n}_a||_a = \lambda ||\vec{n}_b||_b$ cf. (B.8), thus $||\vec{n}_b||_b = 1 = ||\vec{n}_a||_a = \lambda ||\vec{n}_a||_a = ||\lambda \vec{n}_a||_b$, so $\vec{n}_b = \pm \lambda \vec{n}_a$. And they are outward vectors, so $\vec{n}_b = + \lambda \vec{n}_a$. Thus

$$\vec{w}, \vec{n}_a)_a = \lambda (\vec{w}, \vec{n}_b)_b = \lambda^2 (\vec{w}, \vec{n}_a)_a = (\vec{w}, \vec{n}_a)_a. \quad \text{And if} \quad \vec{b}_i = \lambda \vec{a}_i \quad \text{(S.38)} \quad \text{gives} \quad \sum_{i=1}^{m} n^i_b \vec{b}_i = \lambda \sum_{i=1}^{m} n^i_a \vec{a}_i = \sum_{i=1}^{m} n^i_a (\lambda \vec{a}_i) = \sum_{i=1}^{m} n^i_a \vec{b}_i, \quad \text{then} \quad n^i_a = n^i_b.$$

S.8.3 Unit normal form

Let $(\vec{\beta}_i(p))_{i=1,\ldots,n}$ be $(\vec{\beta}_i(p))_{i=1,\ldots,n}$ dual basis, cf. (A.33), that is, the linear forms $\beta^i(p) \in \mathbb{R}^{n^*}$ are given by

$$\forall i, j = 1, \ldots, n, \quad \beta^i(p).\vec{\beta}_j(p) = \delta^i_j. \quad \text{(S.40)}$$

Let

$$\beta_n^p(p) := n^p_g(p) = \sum_{i=1}^{n} n^i_g(p) e^i \quad \text{(S.41)}$$

(The linear form $\beta_n^p(p) = n^p_g(p)$ does not exist without a Euclidean dot product $(\cdot, \cdot)_g$. Thus

$$\vec{w}(p) = \sum_{i=1}^{n} w^i(p) \vec{\beta}_i(p) \quad \implies \quad w^p_n(p) = \sum_{i=1}^{n} n^i_g(p) \vec{w}(p), \vec{n}_g(p)_g, \quad \text{(S.42)}$$

and $\vec{n}_g(p)$ is then the $(\cdot, \cdot)_g$-Riesz representation vector of the linear form $n^p_g(p)$.)
In particular, \((\bar{e}_i)\) being Euclidean, \(n'_i(p)\bar{e}_i = (\bar{n}_g(p), \bar{e}_i)_g\) for all \(i\), cf. (S.42), thus

\[
\begin{align*}
\bar{n}_g(p) = \sum_{i=1}^n n'_i(p)\bar{e}_i, \\
n'_g(p) = \sum_{i=1}^n n_i(p)e^i,
\end{align*}
\] (S.43)

Indeed, on the one hand \(n'_i(p)\bar{e}_j = \sum_{i=1}^n n_i(p)\delta^i_j = n_i(p)\), and on the other hand \(n'_i\bar{e}_j = (\bar{n}_g, \bar{e}_j)_g = \sum_{i=1}^n n_i(p)\delta^i_j = n'_i\) for all \(j\).

NB: Einstein convention isn’t satisfied in the right hand side of (S.43); this is not a surprise because of the subjective choice of the unit of measurement (foot? meter?). We should have written \(n_i = \sum_{j=1}^n g_{ij}n'_j\), although \(g_{ij} = \delta_{ij}\) here, to view the dependence on the chosen Euclidean dot product.

### 8.8.4 Integration by parts

Let \(\varphi \in C^1(\Omega; \mathbb{R})\). Let \(\frac{\partial \varphi}{\partial x^i}(p) := d\varphi(p)\bar{e}_i\), and let \(\bar{n}_g(p) = \sum_{i=1}^n n'_i(p)\bar{e}_i\). The green formula reads, for \(i = 1, ..., n\):

\[
\int_{\Gamma} \frac{\partial \varphi}{\partial x^i}(p) d\Gamma = \int_{\Gamma} \varphi(p) n'_i(p) d\Gamma.
\] (S.44)

(The Einstein convention is not satisfied: \(i\) is at the bottom on the left hand side and at the top in the right hand side, which is not a surprise because of the use of a Euclidean dot product.)

Let \(\bar{v}(p) \in \mathbb{R}^n\), \(\bar{v}(p) = \sum_{i=1}^n v_i(p)\bar{e}_i\). Then \(d\varphi, \bar{v} = \sum_{i=1}^n d\varphi, \bar{e}_i = \sum_{i=1}^n v^i\frac{\partial \varphi}{\partial x^i}\), so

\[
\sum_{i=1}^n \int_{\partial \Omega} \frac{\partial \varphi}{\partial x^i}(p) v_i(p) d\Gamma = \sum_{i=1}^n \int_{\partial \Omega} \varphi(p) v'_i(p) n'_i(p) d\Gamma,
\] (S.45)

that is,

\[
\int_{\partial \Omega} d\varphi, \bar{v}(p) d\Omega = \int_{\partial \Omega} \varphi(p) (\bar{v}(p), \bar{n}_g(p))_g d\Gamma^{\text{noted}} = \int_{\partial \Omega} \varphi(p) \bar{v}(p) \cdot \bar{n}_g(p) d\Gamma,
\] (S.46)

the last notation with the classical notation \(\bar{v} \cdot \bar{w} := (\bar{v}, \bar{w})_g\) when all users use the same Euclidean dot product. Shortened notation:

\[
\int_{\partial \Omega} d\varphi, \bar{v} d\Omega = \int_{\partial \Omega} \varphi \bar{v} \cdot \bar{n}_g d\Gamma = \int_{\partial \Omega} \varphi \bar{v} d\Gamma.
\] (S.47)

Since a Euclidean dot product has been chosen, we can also use the gradient vector, thus

\[
\int_{\partial \Omega} \nabla \varphi, \bar{v} d\Omega = \int_{\partial \Omega} \varphi \nabla \cdot \bar{n}_g d\Gamma.
\] (S.48)

(S.44) and \(\varphi \psi\) instead of \(\varphi\) give the integration by parts formula: For \(i = 1, ..., n\),

\[
\int_{\partial \Omega} \frac{\partial \varphi}{\partial x^i} \psi d\Omega = - \int_{\partial \Omega} \varphi \frac{\partial \psi}{\partial x^i} d\Omega + \int_{\partial \Omega} \varphi \psi n_i d\Gamma.
\] (S.49)

Then \(\bar{w} = \sum_{i=1}^n w^i \bar{e}_i \in C^1(\overline{\Omega}; \mathbb{R}^n)\) and (S.44) give

\[
\int_{\Omega} \frac{\partial w^i}{\partial x^j} d\Omega = \int_{\Gamma} w^i n_j d\Gamma.
\]
(S.50)

the real \((\bar{w}(p), \bar{n}_g(p))_g = \sum_{i=1}^n n_i(p) w^i(p) = \bar{w}(p) \cdot \bar{n}_g(p)\) being the normal component of \(\bar{w}(p)\) at \(\Gamma\) at \(p\). And replacing \(\bar{w}\) with \(\bar{w}, \varphi\), we get

\[
\int_{\Omega} \left( \int_{\Gamma}(\text{div}\bar{w}) \varphi d\Gamma = - \int_{\partial \Omega} d\varphi, \bar{w} d\Omega + \int_{\partial \Omega} (\bar{n}_g, \bar{w})_g \varphi d\Gamma = - \int_{\partial \Omega} \bar{w} \cdot \nabla \varphi d\Omega + \int_{\partial \Omega} \bar{w} \cdot \bar{n}_g \varphi d\Gamma,
\] (S.51)

and differential operator is said to be the dual of the divergence operator, and the gradient operator relative to \((\cdot, \cdot)_g\) is said to be the dual of the divergence operator relative to \((\cdot, \cdot)_g\).
**Exercise S.27** Continuation of exercise S.26. Prove that the volumes satisfy \( \det \tau = \lambda^3 \det \tilde\tau \), and that the area satisfy \( \nu_a = \lambda^2 \nu_b \). Check that (S.46) is compatible with the change of unit of measurement.

**Answer.** In \( \mathbb{R}^3 \) (similar calculations in \( \mathbb{R}^2 \)), the observer \( A \) computes \( (\int_0^1 d\varphi \tilde{\varphi} \tilde{d}\Omega) A = (\int_0^1 \varphi \tilde{\varphi} \tilde{d}\Omega_a) A \) (unit given by \( \tilde{a}_i \)). And the observer \( B \) computes \( (\int_0^1 d\varphi \tilde{\varphi} \tilde{d}\Omega) B = (\int_0^1 \varphi \tilde{\varphi} \tilde{d}\Omega_b) B \). Let us prove that \( (\int_0^1 d\varphi \tilde{\varphi} \tilde{d}\Omega) A = \lambda^3 (\int_0^1 d\varphi \tilde{\varphi} \tilde{d}\Omega) B \), as well as \( (\int_0^1 \varphi \tilde{\varphi} \tilde{d}\Omega_a) A = \lambda^2 (\int_0^1 \varphi \tilde{\varphi} \tilde{d}\Omega_b) B \), which is the desired result.

**Volumes:** \( \det \tau (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = \det \tau (\tilde{P}, \tilde{a}_1, \tilde{P}, \tilde{a}_2, \tilde{P}, \tilde{a}_3) = \det \tau (\tilde{P}) \det \tau (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) = \det \tau (\tilde{P}) = \lambda^3 \), cf. (D.20), thus \( \det \tau (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) = \lambda^3 \det \tau (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3) \), thus \( \det \tau = \lambda^3 \det \tilde\tau \).

**Areas:** \( \nu_a (\tilde{a}_i, \tilde{a}_j) = \det \tau (\tilde{a}_i, \tilde{a}_j, \tilde{n}_a) = \lambda^3 \det \tau (\tilde{a}_i, \tilde{a}_j, \tilde{n}_b) = \lambda^2 \nu_b (\tilde{a}_i, \tilde{a}_j) \), thus \( \nu_a = \lambda^2 \nu_b \).

And \( d\varphi \tilde{\varphi} = (\sum_i \frac{\partial \varphi}{\partial \tilde{\varphi}_i}) (\sum_j \tilde{\varphi}_j e^j) \) is independent of the Euclidean basis, cf. (A.69). Thus \( (\int_0^1 d\varphi \tilde{\varphi} \tilde{d}\Omega) A = \lambda^3 (\int_0^1 d\varphi \tilde{\varphi} \tilde{d}\Omega) B \). And \( (\int_0^1 \varphi \tilde{\varphi} \tilde{d}\Omega_a) A = \lambda^2 (\int_0^1 \varphi \tilde{\varphi} \tilde{d}\Omega_b) B \). (S.38) \( ^{(2)} \lambda^3 (\int_0^1 \varphi \tilde{\varphi} \tilde{d}\Omega) B = \lambda^3 (\int_0^1 \varphi \tilde{\varphi} \tilde{d}\Omega) B \). ♦

### S.9 Object divergence for 1 1 tensors or endomorphisms

Thanks to the natural canonical isomorphism \( \tilde{J} : \left\{ \begin{array}{c} \mathcal{L}(E; E) \\ \mathcal{L}_1(E) \\ L \rightarrow T_L \end{array} \right. \) (cf. (T.16), we consider indifferently a \( \tau \mid_i \) tensor or an endomorphism.

#### S.9.1 Differential of a 1 1 tensor or of an endomorphism

Let \( \tau \in T_1^1 (U) \) be \( C^1 \). Its differential \( d\tau : \left\{ \begin{array}{c} U \rightarrow \mathcal{L}(E; E^*; E; \mathbb{R}) \\ p \rightarrow d\tau(p) \end{array} \right. \) is given by \( d\tau(p) \tilde{u} = \lim_{h \rightarrow 0} \frac{\tau(p + h) \tilde{u} - \tau(p) \tilde{u}}{h} \) for all \( \tilde{u} \in E \). Thanks to the natural canonical isomorphism

\[
J_3 : \left\{ \begin{array}{c} \mathcal{L}(E; E^*; E; \mathbb{R}) \\ \mathcal{L}_2(E) \\ T \rightarrow J_3(T), J_3(T)(\ell, \tilde{g}, \tilde{x}) = \ell ((T \tilde{x}) \tilde{g}), \end{array} \right. \quad (S.52)
\]

\( d\tau \) is identified to a \( \tau \) tensor, and written \( d\tau \in T_2^1 (U) \).

Let \( (e^j) \) be a basis in \( E \), let \( (e^k) \) be its dual basis, let \( \tau = \sum_{i,j=1}^n (\tau_{ij}^k) e^i \otimes e^j \), and thanks to \( J_3 \) let

\[
\forall k, \quad d\tau \tilde{e}_k = \sum_{i,j=1}^n (\tau_{ij}^k) \tilde{e}_i \otimes e^j \in T_1^1 (U), \quad \text{and } d\tau \text{ noted } \sum_{i,j,k=1}^n (\tau_{ij}^k) \tilde{e}_i \otimes e^j \otimes e^k \in T_2^1 (U). \quad (S.53)
\]

#### S.9.2 Definition: Objective divergence

To create an objective divergence for a second order tensor \( \tau \in T_1^1 (U) \), and satisfy Einstein convention with (S.53) and for the derived terms, we have no choice but to contract \( k \) (the derivation index) with \( i \):

**Definition S.28** Let \( \tau \in T_1^1 (U) \) be a \( C^1 \) tensor. With (S.53), its objective divergence \( \text{div}_\tau (\tau) \in T_1^1 (U) \) relative to \( (e^j) \) is the \( \tau \) tensor (the differential form in \( \Omega^1 (U) \)) defined by

\[
\text{div}_\tau (\tau) := \sum_{i,j=1}^n (\tau_{ij}^j) e^j \in T_1^1 (U), \quad [\text{div}_\tau (\tau)]_{ij} = \sum_{i=1}^n (\tau_{ij}^i) \quad (\text{row matrix}). \quad (S.54)
\]

(The Einstein convention is satisfied.) (The matrix \( [\text{div}_\tau (\tau)]_{ij} \) is a row matrix since it represents a differential form.) (So we take the divergences of the “column vectors” of \( \tau \) to make \( \text{div}_\tau (\tau) \) to make the row matrix \( [\text{div}_\tau (\tau)]_{ij} \)

**Proposition S.29** Let \( \tau \in T_1^1 (U) \). Let \( (\tilde{a}_i) \) and \( (\tilde{b}_i) \) be bases. Then

\[
\text{div}_\tau (\tau) \tilde{a}_i = \text{div}_\tau (\tau) \tilde{b}_i \quad \text{name } \Rightarrow \text{div}_\tau (\tau) \in T_1^1 (U), \quad (S.55)
\]

that is, for all \( \tilde{w} \in \Gamma (U) \) (vector field in \( U \)),

\[
\text{div}_\tau (\tau) \tilde{a}_i \tilde{w} = \text{div}_\tau (\tau) \tilde{b}_i \tilde{w} \in \mathbb{R}. \quad (S.56)
\]

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Let $\mathbf{d} \mathbf{i} = \mathbf{w}$. Observe that $\mathbf{d} \mathbf{i} = \mathbf{w}$ is a tensor for $\mathbf{v} \mathbf{f}$, because 

$$
\mathbf{d} \mathbf{i} = \mathbf{w} \quad \text{(covariance formula)} .
$$

Thus, the “objective divergence” $\mathbf{d} \mathbf{i} = \mathbf{w}$ of a (1, 0) tensor is objective.

\textbf{Proof.} Let $\mathbf{d} \mathbf{i} = \mathbf{w}^n_{i,j} a_{i,j} a_i = \mathbf{w}^n_{i,j} b_{i} \otimes b_j$. Let

$$
\ell := \mathbf{d} \mathbf{i} = \sum_{i,j} (\mathbf{d} \mathbf{i})_{i,j} a_i = \sum_{i,j} (\mathbf{d} \mathbf{i})_{i,j} b_j = \sum_{j} (\mathbf{d} \mathbf{i})_{j} b_j ,
$$

and

$$
m := \mathbf{d} \mathbf{i} = \sum_{i,j} (\mathbf{d} \mathbf{i})_{i,j} b_j = \sum_{j} (\mathbf{d} \mathbf{i})_{j} b_j .
$$

Let us prove (S.57), that is

$$
[m]_{\mathbf{i}} = [\ell]_{\mathbf{i}} .
$$

With $b_j = \sum_{\beta} \mathbf{P}_j^{\beta} a_{i,j}$ (definition of $P$), with $Q := q^{\mathbf{i}}$, and with $b^i = \sum_{\beta} e_{i}^{\mathbf{P}} a_{i,j}$, cf. (A.66), and with $(\mathbf{d} \mathbf{i})_{i,j} = b_j (d \mathbf{r})_{i,j} b_j$, cf. (S.53), we get

$$
(\mathbf{d} \mathbf{i})_{i,j} = \sum_{\alpha} P_{i}^{\alpha} \mathbf{P}_{j}^{\beta} (Q_{\alpha})_{i,j} = \sum_{\alpha} P_{i}^{\alpha} \mathbf{P}_{j}^{\beta} (Q_{\alpha})_{i,j} = \sum_{\alpha} \mathbf{P}_{i}^{\alpha} \mathbf{P}_{j}^{\beta} \delta_{\alpha\beta} = \sum_{\alpha} \mathbf{P}_{i}^{\alpha} \mathbf{P}_{j}^{\beta} ,
$$

which is (S.59).

Thus, with $\mathbf{d} \mathbf{i} = \sum_{i} w_i \mathbf{d} \mathbf{i} = \sum_{i} w_i \mathbf{d} \mathbf{i}$, since $|w_i| = q$, $|\mathbf{d} \mathbf{i}| = q$, that is $w_i = \sum_{k} Q_{i}^{k} w_k$ for all $i$, we get

$$
\mathbf{d} \mathbf{i} = \sum_{i} w_i \mathbf{d} \mathbf{i} = \sum_{i} w_i \mathbf{d} \mathbf{i} = \sum_{i} w_i \mathbf{d} \mathbf{i} = \sum_{i} w_i \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} ,
$$

i.e. (S.56), thus $m = \ell$, i.e. $\mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i}$. 

Proposition S.30 Let $\mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i}$ be a (1, 0) tensor and $\mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i}$ be a vector field. Then, we have the objective result

$$
\mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} .
$$

\textbf{Proof.} $\mathbf{d} \mathbf{i} = \sum_{i} \tau_i^{\mathbf{i}} \mathbf{e}_i$ and $\mathbf{d} \mathbf{i} = \sum_{i} \tau_i^{\mathbf{i}} \mathbf{e}_i$, give $\mathbf{d} \mathbf{i} = \sum_{i} \tau_i^{\mathbf{i}} \mathbf{e}_i$, thus $\mathbf{d} \mathbf{i} = \sum_{i} \tau_i^{\mathbf{i}} \mathbf{e}_i$. 

Example S.31 Let $f \in C^1(\mathbf{f} \mathbf{f}; \mathbf{f})$ and $\mathbf{d} \mathbf{i} = \sum_{i} \tau_i^{\mathbf{i}} \mathbf{e}_i$ be a tensor. Then

$$
\mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} .
$$

(“The only possibility for a derivation of a product.”) Indeed: $df = \sum_{i} f_i \mathbf{e}_i$, $\mathbf{d} \mathbf{i} = \sum_{i} \tau_i^{\mathbf{i}} \mathbf{e}_i$,

$$
\mathbf{d} \mathbf{i} = \sum_{i,j} (f_{i,j}^{\mathbf{i}} + f_{i,j}^{\mathbf{i}}) \mathbf{e}_i \mathbf{e}_i = \sum_{i,j} (f_{i,j}^{\mathbf{i}} + f_{i,j}^{\mathbf{i}}) \mathbf{e}_i \mathbf{e}_i ,
$$

and on the other hand

$$
\mathbf{d} \mathbf{i} = \sum_{i,j} (f_{i,j}^{\mathbf{i}} + f_{i,j}^{\mathbf{i}}) \mathbf{e}_i \mathbf{e}_i .
$$

Example S.32 Let $\mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i}$ be a basis, and $\mathbf{d} \mathbf{i} = \sum_{i} \tau_i^{\mathbf{i}} \mathbf{e}_i$, $\alpha = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i}$, so $\mathbf{d} \mathbf{i} = \sum_{i} \tau_i^{\mathbf{i}} \mathbf{e}_i \mathbf{e}_i \mathbf{e}_i$, and $\alpha = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i}$.

$$
\mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} = \mathbf{d} \mathbf{i} .
$$

(\text{Example S.32}) 

So $\mathbf{d} \mathbf{i}$ (contravariant) and $\alpha$ (covariant) do not play the same role: the whole of $\alpha$ is used, whereas only the trace $\text{Tr}(\mathbf{d} \mathbf{i}) = \mathbf{d} \mathbf{i}$ of $\mathbf{d} \mathbf{i}$ is used.
NB: the trace of a \(^{(n)}\) tensor \( T \in T^0_0(U) \) has no existence, as far as objectivity is concerned, since \( T = \sum_{i,j=1}^n T_{ij} e^i \otimes e^j \) and there are no index to contract to get an objective quantity (there is no upper index in \( T_{ij} \)). And since \( da = \sum_{i,j=1}^n a_{ij} e^i \otimes e^j \in T^2_0(U) \) there is no objective trace for \( da \).

Thus (S.62) gives, for all \( \vec{w} \in \Gamma(U) \),

\[
\text{div}(\vec{u} \otimes \alpha) \vec{w} = (\text{div} \vec{u}) (\alpha, \vec{w}) + (da, \vec{w}) \vec{w}. \tag{S.63}
\]

Example S.33 Let \( \vec{w} \in \Gamma(U) \cong T^0_0(U) \) be \( C^2 \). So \( d\vec{w} \in T^1_0(U) \) and \( d^2 \vec{w} \in T^2_0(U) \). Let \( (\vec{e}_i) \) be a basis, \( \vec{w} = \sum_{i=1}^n w^i \vec{e}_i \), \( d\vec{w} = \sum_{i,j=1}^n w^i_j \vec{e}_i \otimes e^j \) and \( d^2 \vec{w} = \sum_{i,j,k=1}^n w^i_{jk} \vec{e}_i \otimes e^j \otimes e^k \).

Then

\[
\text{div}(d\vec{w}) = \sum_{i,j=1}^n w^i_{j} \vec{e}^j = \sum_{i,j=1}^n w^i_{j} \vec{e}^j = \sum_{j=1}^n (\text{div} \vec{u})_j e^j = d(\text{div} \vec{u}). \tag{S.64}
\]

This is not the Laplacian \( \Delta \vec{w} = \sum_{i,j=1}^n w^i_{jj} \vec{e}_i \), where \( w^i_{jj} = \frac{\partial^2 w^i}{\partial x^j \partial x^j} \). The Laplacian is not objective: one divides by a length to the square: in which unit? And in \( \Delta \vec{w} = \sum_{i,j=1}^n w^i_{jj} \vec{e}_i \) the Einstein convention is not satisfied.

S.9.3 Objective divergences of a 2 0 tensor

Let \( \tau \in T^2_0(U) \) and \( \tau = \sum_{i,j=1}^n \tau_{ij} e^i \otimes e^j \), thus \( d\tau = \sum_{i,j,k=1}^n \tau_{ijk} e^i \otimes e^j \otimes e^k \); Then two objective divergences can be defined: by contracting \( k \) with \( i \), or \( k \) with \( j \). (The Einstein convention is then satisfied.)

S.9.4 Non existence of an objective divergence of a 0 2 tensor

Let \( \tau = \sum_{i,j=1}^n \tau_{ij} e^i \otimes e^j \in T^2_0(U) \). Thus \( d\tau = \sum_{i,j,k=1}^n \tau_{ijk} e^i \otimes e^j \otimes e^k \), and there are no indices to contract to satisfy Einstein convention. Thus no objective divergence of 0 2 tensors are defined.

S.10 Euclidean framework and “classic divergence” of a tensor (subjective)

S.10.1 “Classic divergence” of a 1 1 tensor or of an endomorphism

An observer chooses a unit of measurement, builds a related Euclidean basis \( (\vec{e}_i) \) and the associated Euclidean dot product \( (\cdot, \cdot)_g \). So \( (\cdot, \cdot)_g = \sum_{i=1}^n e^i \otimes e^i \). Let \( \sigma \in T^1_1(U) \) be \( C^1 \), and

\[
\sigma = \sum_{i,j=1}^n \sigma^i_{j} \vec{e}_i \otimes e^j, \quad \text{so} \quad d\sigma = \sum_{i,j,k=1}^n \sigma^i_{jk} \vec{e}_i \otimes e^j \otimes e^k, \quad \text{and} \quad \sigma^i_{j} = \frac{\partial \sigma^i}{\partial x^j}. \tag{S.65}
\]

Definition S.34 (Usual divergence in classical mechanics.) The divergence \( \text{div}_g \sigma \) of \( \sigma \), relative to the \( (\cdot, \cdot)_g \)-Euclidean basis \( (\vec{e}_i) \), is the column matrix (it is not a vector)

\[
\text{div}_g \sigma \equiv \begin{pmatrix} \sum_{j=1}^n \sigma^1_{j} \\ \vdots \\ \sum_{j=1}^n \sigma^n_{j} \end{pmatrix} \equiv \begin{pmatrix} \sum_{j=1}^n \frac{\partial \sigma^1}{\partial x^j} \\ \vdots \\ \sum_{j=1}^n \frac{\partial \sigma^n}{\partial x^j} \end{pmatrix} = \text{div}_g \sigma, \tag{S.66}
\]

the last notation to recall the name of the selected base (this divergence depends on the observer). So: Take the divergences of the rows (the row matrices as if they were vectors) of \( \sigma \) \( \sigma \) to make the “column vector” \( \text{div}_g \sigma \) (column matrix).

NB: Einstein convention is not satisfied in (S.66), which is expected since we use a Euclidean dot product (we use a Euclidean basis).

NB: The matrix \( \text{div}_g \sigma \) does not behave like a vector, see (S.68), since nothing is objective here, so nothing is “vectorial” or “tensorial”. However: Notation: The column matrix \( \text{div}_g \sigma \) (column vector) in (S.66) is noted as if it was a vector, that is,

\[
\text{div}_g \sigma \text{ noted} = \sum_{i,j=1}^n \sigma^i_{j} \vec{e}_i = \sum_{i,j=1}^n \frac{\partial \sigma^i}{\partial x^j} \vec{e}_i, \quad \text{with} \quad \sigma^i_{j} = \frac{\partial \sigma^i}{\partial x^j}. \tag{S.67}
\]

But it is not a vector.
Proposition S.35 The "column vector" \( \text{div}_\sigma \) cf. (S.66)-(S.67), does not satisfy the change of coordinate system rule: If \((\tilde{a}_i)\) and \((\tilde{b}_i)\) are bases and if \(P\) is the transition matrix from \((\tilde{a}_i)\) to \((\tilde{b}_i)\), then
\[
[\text{div}_\sigma]_{\tilde{b}} \neq P^{-1} [\text{div}_\sigma]_{\tilde{a}} \quad \text{in general}
\] (compare with (S.57)). (The introduction of an inner dot product produces "non natural" results: The results depend on an observer.) (It doesn’t either satisfy \( \text{div}_\sigma = \text{div}_\sigma \circ P \) if \( \text{div}_\sigma \) considered to be a row matrix.)

Proof. Consider two observers and their Euclidean basis \((\tilde{a}_i)\) and \((\tilde{b}_i)\) and the simple case \( \tilde{b}_i = \lambda \tilde{a}_i \) for all \( i, \lambda > 1 \), cf. (B.2). The transition matrix is then \( P = \lambda I \).

Let \( \sigma \in T^1_1(U) \), and \( \sigma = \sum_{i,j=1}^n (\sigma_{ij}) \tilde{b}_i \otimes \tilde{b}_j = \sum_{i,j=1}^n (\sigma_{ij}) \tilde{a}_i \otimes \tilde{a}_j \). And, \( \sigma \) being a \((1,1)\) tensor, \( [\sigma]_{\tilde{b}} = P^{-1} [\sigma]_{\tilde{a}} \). \( \lambda \) gives \( [\sigma]_{\tilde{a}} = [\sigma]_{\tilde{b}} \), that is, \( (\sigma_{ij}) = (\sigma_{ij}) \) for all \( i,j \).

Then (S.67) reads
\[
\begin{align*}
\text{div}_\sigma = \sum_{i,j=1}^n (d(\sigma_{ij})_{\tilde{b}}) \tilde{b}_i, \\
\text{div} \circ \sigma = \sum_{i,j=1}^n (d(\sigma_{ij})_{\tilde{a}}) \tilde{a}_i.
\end{align*}
\]
Since \( (\sigma_{ij}) = (\sigma_{ij}) \) we get
\[
\text{div}_\sigma = \sum_{i,j=1}^n (d(\sigma_{ij})_{\tilde{b}}) \tilde{b}_i = \sum_{i,j=1}^n (d(\sigma_{ij})_{\tilde{a}})(\lambda \tilde{a}_i) = \lambda^2 \sum_{i,j=1}^n (d(\sigma_{ij})_{\tilde{a}}) \tilde{a}_i = \lambda^2 \text{div}_\sigma = P^2 \text{div}_\sigma,
\]

(S.69) since \( P = \lambda I \). So, with \( \lambda \neq 1 \), \( \text{div}_\sigma \neq P^{-1} \text{div}_\sigma \).

(Alternative proof: Apply proposition S.36 with \((C.11)\).)

Proposition S.36 Let \( \tau \in T^1_1(U) \), and consider its \((\cdot, \cdot)_g\)-transposed \( \tau^T \in T^1_1(U) \) (defined by \( (\tau^T \tilde{a}, \tilde{w}) = g(\tilde{u}, \tau \tilde{w}) \)) for all \( \tilde{a}, \tilde{w} \)). Recall that \( \text{div}(\tau) \) defined in (S.54) is objective, cf. proposition S.29.

Then, for all \( \tilde{w} \in \Gamma(U) \),
\[
\text{div}(\tau, \tilde{w}) = (\text{div}_e(\tau^T), \tilde{w})_g + \tau^T : d\tilde{w}.
\]

(S.70) so, \( \text{div}_g(\tau^T) \) is the \((\cdot, \cdot)_g\)-Riesz representation vector of the objective differential form \( \text{div}(\tau) \). And (S.60) gives
\[
\text{div}(\tau, \tilde{w}) = (\text{div}(\tau^T), \tilde{w})_e + \tau^T : d\tilde{w}.
\]

(S.71)

Proof. Let \( \tau = \sum_{i,j=1}^n \tau_{ij}^e \tilde{e}_i \otimes \tilde{e}_j \). Thus \( \tau^T = \sum_{i,j=1}^n \tau_{ij}^e \tilde{e}_j \otimes \tilde{e}_i \). Thus \( \text{div}_e(\tau^T), \tilde{w}) = \sum_{i,j=1}^n \tau_{ij}^e \tilde{w}_i \tilde{e}_j \).

Therefore \( (\text{div}_e(\tau^T), \tilde{w})_e = \sum_{i,j=1}^n \tau_{ij}^e \tilde{w}_i \tilde{e}_j \).

(S.67)

S.10.2 Classic divergence for 2 0 and 0 2 tensors

With \( \sigma \in T^0_2(U) \), or \( \sigma \in T^0_2(U) \) or \( \sigma \in T^1_1(U) \), relative to a Euclidean basis, we have \([\sigma]_{\tilde{a}} = [\sigma]_{\tilde{b}} \). Here we do not care about covariance and contravariance, a linear form being represented by a vector thanks to a Riesz representation vector. The classic divergence is the column matrix
\[
\text{div}_\sigma = \begin{pmatrix}
\sum_{j=1}^n \frac{\partial \sigma_{ij}}{\partial x_j} \\
\vdots \\
\sum_{j=1}^n \frac{\partial \sigma_{nj}}{\partial x_j}
\end{pmatrix}
\]

(S.72)

The objectivity is of no concern here.

T Natural canonical isomorphisms

T.1 The adjoint of a linear map

Setting of § A.13: \( E \) and \( F \) are vector spaces, \( E^* = \mathcal{L}(E; \mathbb{R}) \) and \( F^* = \mathcal{L}(F; \mathbb{R}) \) are their duals, and the adjoint of a linear map \( \mathcal{P} \in \mathcal{L}(E; F) \) is the linear map \( \mathcal{P}^* \in \mathcal{L}(F^*; E^*) \) canonically defined by
\[
\forall \ell \in F^*, \quad \mathcal{P}^*(\ell) := \ell \circ \mathcal{P}, \quad \text{written} \quad \mathcal{P}^* \ell = \ell \mathcal{P}
\]

(T.1)
T.2 An isomorphism \( E \simeq E^* \) is never natural

**T.2.1 Definition**

Two observers \( A \) and \( B \) consider a linear map \( L \in \mathcal{L}(E; E^*) \); Let \( \mathcal{P} \in \mathcal{L}(E; E) \) be the change of observer endomorphism. Willing to work together, \( A \) and \( B \) ("naturally") consider the diagram

\[
\begin{array}{ccc}
\mathcal{P}^* & \quad \overset{L}{\longrightarrow} \quad & E^* \\
\downarrow & & \downarrow \mathcal{P}^* \\
E & \quad \overset{L}{\longrightarrow} \quad & E
\end{array}
\]

\( \left( \text{considered by observer } A \right) \quad \left( \text{considered by observer } B \right) \quad (T.3)

**Definition T.1** (Spivak [17].) A linear map \( L \in \mathcal{L}(E; E^*) \) is natural iff the diagram (T.3) commutes for all \( \mathcal{P} \in \mathcal{L}(E; E) \):

\[
L \in \mathcal{L}(E; E^*) \text{ is natural } \iff \forall \mathcal{P} \in \mathcal{L}(E; E), \quad \mathcal{P}^* \circ L \circ \mathcal{P} = L. \quad (T.4)
\]

(In that case, if \( A \) asks \( B \) to make the experiment then \( (\mathcal{P}^* \circ L \circ \mathcal{P}) \) gives the result \( L \vec{u} \) expected by \( A \).

**T.2.2 Question**

Does there exist an endomorphism \( L \) such that the diagram (T.3) commutes for all change of observers? That is, do we have

\[
\exists ? L \in \mathcal{L}(E; E), \forall \mathcal{P} \in \mathcal{L}_i(E; E), \quad \mathcal{P}^* \circ L \circ \mathcal{P} = L? \quad (T.5)
\]

**T.2.3 Theorem**

The answer to the question (T.5) is always **NO**;

**Theorem T.2** A (non-zero) linear map \( L \in \mathcal{L}(E; E^*) \) is not natural: If \( L \in \mathcal{L}(E; E^*) \setminus \{0\} \), then

\[
\exists \mathcal{P} \in \mathcal{L}_i(E; E) \quad \text{s.t.} \quad L \neq \mathcal{P}^* \circ L \circ \mathcal{P}. \quad (T.6)
\]

**Proof.** (Spivak [17].) It suffices to prove this proposition for \( E = \mathbb{R}^n \). Let \( L \in \mathcal{L}(\mathbb{R}^n; (\mathbb{R}^n)^*) \), \( L \neq 0 \).

Let \( (\vec{a}_i) \) be a basis in \( \mathbb{R}^n \) (chosen by \( A \)). Let \( (\vec{b}_i) \) be a basis in \( \mathbb{R}^n \) (chosen by \( B \)).

Consider \( \mathcal{P} \in \mathcal{L}_i(\mathbb{R}^n; \mathbb{R}^n) \) defined by \( \mathcal{P}(\vec{a}_i) = \vec{b}_i \) (change of observer), and let \( \lambda \in \mathbb{R} \) s.t. \( \vec{b}_1 = \lambda \vec{a}_1 \). Then (T.1) gives \( \mathcal{P}^*(\ell)(\vec{a}_i) := \ell(\mathcal{P}(\vec{a}_i)) = \ell(\vec{b}_i) = \ell(\lambda \vec{a}_1) = \lambda \ell(\vec{a}_1) \), thus \( \mathcal{P}^*(\ell) = \lambda \ell \) for all \( \ell \in \mathbb{R}^n \).

Thus \( \mathcal{P}^*(L(\mathcal{P}(\vec{a}_i))) = \mathcal{P}^*(L(\lambda \vec{a}_i)) = \lambda \mathcal{P}^*(L(\vec{a}_i)) = \lambda^2 L(\vec{a}_1) \neq L(\vec{a}_1) \) when \( \lambda^2 \neq 1 \). E.g., \( \mathcal{P} = 2I \) gives \( L \neq \mathcal{P}^* \circ L \circ \mathcal{P} \) (\( = 4L \)), thus (T.6): A (non-zero) linear map \( E \to E^* \) cannot be natural.

**T.2.4 Illustrations (two fundamental examples)**

**Example T.3** Consider \( E \) s.t. \( \dim E = 1 \), and consider the linear map \( L \in \mathcal{L}(E; E^*) \) which sends a basis \( (\vec{a}_1) \) onto its dual basis \( (\pi_{a_1}) \), that is, \( L_{\vec{a}_1} := \pi_{a_1} \).

Question: If \( (\vec{b}_1) \) is another basis, \( \vec{b}_1 = \lambda \vec{a}_1 \) with \( \lambda \neq \pm 1 \) (change of unit of measurement), does \( L \) also send \( (\vec{b}_1) \) onto its dual basis? I.e., does \( L_{\vec{b}_1} = \pi_{b_1} \)?

Answer: No. Indeed, \( b_1 = \lambda a_1 \) gives \( \pi_{b_1} = \frac{1}{\lambda} \pi_{a_1} \), thus \( L_{\vec{b}_1} = \lambda L_{\vec{a}_1} = \lambda \pi_{a_1} = \lambda^2 \pi_{b_1} \neq \pi_{b_1} \) since \( \lambda^2 \neq 1 \). In words: \( L \) is not natural, cf. (T.6).

A different presentation: Let \( L_A \) and \( L_B \) be defined by \( L_A \vec{a}_j = \pi_{a_j} \) and \( L_B \vec{b}_j = \pi_{b_j} \) for all \( j \). And suppose that \( \vec{b}_j = \lambda \vec{a}_j \) for all \( j \). Then, \( L_A \vec{b}_j = \lambda L_A \vec{a}_j = \lambda \pi_{a_j} = \lambda^2 \pi_{b_j} = \lambda^2 L_B \vec{b}_j \neq L_B \vec{b}_j \) when \( \lambda^2 \neq 1 \), that is, \( L_A \neq L_B \) when \( \lambda^2 \neq 1 \): An operator that sends a basis onto its dual basis is not natural.
Example T.4 Let $(\cdot, \cdot)_g$ be an inner dot product in $E = \mathbb{R}^n$. Let $R_g \in \mathcal{L}(E^*; E)$ be the Riesz representation map, that is, defined by $R_g(\ell) = \ell g$ where $\ell g$ is defined by $(\ell g, \overline{v}) = \ell \overline{v}$ for all $\overline{v} \in \mathbb{R}^n$, cf. (C.6).

Question: Is $R_g$ natural? Answer: No: Consider the diagram (T.3) with $R_g \circ \mathcal{P}$ where $\mathcal{P} = \lambda I$, $\lambda \neq \pm 1$. Then $\mathcal{P}^* = \lambda I$, and $\mathcal{P} R_g \mathcal{P}^* \ell = \lambda^2 R_g \ell \neq R_g \ell$ gives $\mathcal{P} R_g \mathcal{P}^* \neq R_g$. So $R_g$ is not natural, cf. (T.6).

A different presentation: Consider two distinct Euclidean dot products $(\cdot, \cdot)_g$ and $(\cdot, \cdot)_h$ (e.g., built with a foot and built with a meter). So $(\cdot, \cdot)_h = \lambda^2 (\cdot, \cdot)_g$ with $\lambda^2 \neq 1$. Let $R_g, R_h \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ be the Riesz operators relative to $(\cdot, \cdot)_g$ and $(\cdot, \cdot)_h$, that is $R_g \ell = \ell g$ and $R_h \ell = \ell h$, are given by $\ell \overline{v} = (\ell g, \overline{v}) = (\ell h, \overline{v})$ for all $\overline{v} \in \mathbb{R}^n$. We have $\ell g = \lambda^2 \ell h$, cf. (C.10), thus $R_h = \lambda^2 R_g$ since $\lambda^2 \neq 1$: A Riesz representation operator is not natural (it is observer dependent).

T.3 Natural canonical isomorphism $E \simeq E^{**}$

T.3.1 Framework and definition

Two observers $A$ and $B$ consider the same linear map $L \in \mathcal{L}(E; E^*)$ (where $E^* = (E^*)^* = \mathcal{L}(E^*; \mathbb{R})$).

Willing to work together, they ("naturally") consider the diagram

$$
\begin{array}{ccc}
\mathcal{P} & \downarrow E & E^{**} \\
\downarrow & & \downarrow \mathcal{P}^{**} \\
E & \rightarrow & E^{**}
\end{array}
$$

\hspace{1cm} \left\langle \text{considered by observer } A \right. \hspace{1cm}

\hspace{1cm} \left. \text{considered by observer } B \right. \hspace{1cm}

where $\mathcal{P}^{**} \in \mathcal{L}_{\ast}(E^*; E^{**})$ is given by $\mathcal{P}^{**}(u) = u \circ \mathcal{P}^\ast$ for all $u \in E^{**}$, cf. (T.1), that is, $\mathcal{P}^{**}$ is given by, for all $(\ell, u) \in E^* \times E^{**}$, cf. (T.1),

$$
(\mathcal{P}^{**}(u))(\ell) = u(\ell \circ \mathcal{P}), \quad \text{i.e.} \quad (\mathcal{P}^{**}, u) = u.(\ell, \mathcal{P}). 
$$

(T.8)

T.3.2 The Theorem

**Question:** Does there exist a linear map $L \in \mathcal{L}(E; E^{**})$ that is natural?

**Answer:** Yes (particular case of the next proposition):

**Proposition T.5** The canonical isomorphism

$$
\mathcal{J}_E : \begin{cases}
E \rightarrow E^{**} \\
\overline{u} \rightarrow u = \mathcal{J}_E(\overline{u})
\end{cases}
$$

is natural, that is, $F$ being another finite dimensional vector space, the diagram

$$
\begin{array}{ccc}
\mathcal{P} & \downarrow \mathcal{J}_E & E^{**} \\
\downarrow & & \downarrow \mathcal{J}_{F}^{**} \\
\mathcal{F} & \rightarrow & F^{**}
\end{array}
$$

(T.10)

commutes for all $\mathcal{P} \in \mathcal{L}(E; F)$:

$$
\forall \mathcal{P} \in \mathcal{L}(E; F), \quad \mathcal{P}^{**} \circ \mathcal{J}_E = \mathcal{J}_F \circ \mathcal{P}, \quad \text{and we write } E \overset{\text{natural}}{\simeq} E^{**}. 
$$

(T.11)

Thus we can use the unambiguous notation (observer independent)

$$
\mathcal{J}(\overline{u}) \overset{\text{natural}}{=} \overline{u}, \quad \text{and} \quad \mathcal{J}(\overline{u}).\ell \overset{\text{natural}}{=} \overline{u}.\ell \quad (= \ell, \overline{u}). 
$$

(T.12)

(And $u = \mathcal{J}(\overline{u})$ is the derivation operator in the direction $\overline{u}$.)

**Proof.** (Spivak [17].) It is trivial that $\mathcal{J}_E$ is linear and bijective ($E$ is finite dimensional): It is an isomorphism. Then $(\mathcal{P}^{**} \circ \mathcal{J}_E(\overline{u}))(\ell) = \mathcal{J}_E(\overline{u})(\ell, \mathcal{P}) \overset{\text{(T.8)}}{=} (\ell \circ \mathcal{P})(\overline{u}) = \ell(\mathcal{P}(\overline{u})) \overset{\text{(T.9)}}{=} \mathcal{J}_F(\mathcal{P}(\overline{u}))(\ell)$, for all $\ell \in F^*$ and all $\overline{u} \in E$, thus $\mathcal{P}^{**} \circ \mathcal{J}_E(\overline{u}) = \mathcal{J}_F(\mathcal{P}(\overline{u}))$, for all $\overline{u} \in E$, thus $\mathcal{P}^{**} \circ \mathcal{J}_E = \mathcal{J}_F \circ \mathcal{P}$. 

\hspace{1cm} □
**Proposition T.6** (Quantification.) \( J_E \) sends any basis \((\vec{a}_i)\) onto its bidual basis. (Expected, since \( J_E(\vec{u}) \) is the directional derivative in the direction \( \vec{u} \), whatever \( \vec{u} \).)

**Proof.** Let \((\vec{a}_i)\) be a basis and \((\pi_{ai})\) be its dual basis (defined by \( \pi_{ai}\vec{a}_j = \delta_{ij} \) for all \( i, j \)). Then (T.9) gives \( J_E(\vec{a}_j)\pi_{ai} = \pi_{ai}\vec{a}_j = \delta_{ij} \) for all \( i, j \), thus \((J_E(\vec{a}_j))\) is the dual basis of \((\vec{a}_i)\), i.e., is the bidual basis of \((\vec{a}_i)\); True for all basis: \( J_E(\vec{b}_j)\pi_{bi} = \pi_{bi}\vec{b}_j = \delta_{ij} \) for all \( i, j \). 

> T.4 Natural canonical isomorphisms \( \mathcal{L}(E; F) \simeq \mathcal{L}(F^*, E; \mathbb{R}) \simeq \mathcal{L}(E^*; F^*) \)

Consider the canonical isomorphism (\( E \) and \( F \) are finite dimensional)

\[
\mathcal{J}_{EF} : \begin{cases} \mathcal{L}(E; F) \to \mathcal{L}(F^*, E; \mathbb{R}) \\ L \to \tilde{L} = \mathcal{J}_{EF}(L) \end{cases}, \quad \tilde{L}(\ell, \vec{u}) := \ell.\tilde{L}.\vec{u}, \quad \forall(\ell, \vec{u}) \in F^* \times E. \tag{T.13}
\]

Let \( V \) and \( W \) be two more vector spaces, let \( \mathcal{P}_1 \in \mathcal{L}_1(E; V) \) and \( \mathcal{P}_2 \in \mathcal{L}(F; W) \), and consider the diagram

\[
\begin{array}{c}
\mathcal{L}(E; F) & \xrightarrow{\mathcal{J}_{EF}} & \mathcal{L}(F^*, E; \mathbb{R}) \\
\downarrow \mathcal{I}_P & & \downarrow \mathcal{\tilde{I}_P} \\
\mathcal{L}(V; W) & \xrightarrow{\mathcal{J}_{VW}} & \mathcal{L}(W^*, V; \mathbb{R})
\end{array} \tag{T.14}
\]

where

\[ \mathcal{I}_P(L).\vec{u} = \mathcal{P}_2.L.\mathcal{P}_1^{-1}.\vec{u} \quad \text{and} \quad \mathcal{\tilde{I}_P}(\tilde{L})(\ell, \vec{u}) = \tilde{L}(\ell.\mathcal{P}_2, \mathcal{P}_1^{-1}\vec{u}), \quad \forall(\ell, \vec{u}) \in W^* \times V. \tag{T.15} \]

(\( \mathcal{I}_P \) and \( \mathcal{\tilde{I}_P} \) are the push-forwards for linear maps \( L \in \mathcal{L}(E; F) \) and for bilinear forms \( \tilde{L} \in \mathcal{L}(F^*, E; \mathbb{R}) \).)

**Proposition T.7** The canonical isomorphism \( \mathcal{J}_{EF} \) is natural, that is, the diagram (T.14) commutes for all \( \mathcal{P}_1 \in \mathcal{L}_1(E, V) \) and all \( \mathcal{P}_2 \in \mathcal{L}(F, W) \):

\[
\forall(\mathcal{P}_1, \mathcal{P}_2) \in \mathcal{L}_1(E; V) \times \mathcal{L}(F; W), \quad \mathcal{\tilde{I}_P} \circ \mathcal{J}_{EF} = \mathcal{J}_{VW} \circ \mathcal{I}_P, \quad \text{and we write} \quad \mathcal{L}(E; F) \overset{\text{natural}}{\simeq} \mathcal{L}(F^*, E; \mathbb{R}). \tag{T.16}
\]

Thus \( \mathcal{L}(E^*; F^*) \overset{\text{natural}}{\simeq} \mathcal{L}(E; F) \).

**Proof.** \( \mathcal{J}_{VW}(\mathcal{I}_P(L))(\ell, \vec{u}) \overset{(T.13)}{=} \ell.\mathcal{I}_P(L).\vec{u} \overset{(T.15)}{=} \ell.(\mathcal{P}_2.L.\mathcal{P}_1^{-1}.\vec{u}) = (\ell.\mathcal{P}_2).L.(\mathcal{P}_1^{-1}\vec{u}) \overset{(T.13)}{=} \mathcal{J}_{EF}(L)(\ell.\mathcal{P}_2, \mathcal{P}_1^{-1}\vec{u}) \overset{(T.15)}{=} \mathcal{\tilde{I}_P}(\mathcal{J}_{EF}(L))(\ell, \vec{u}), \text{true for all } L \in \mathcal{L}(E; F), \ell \in W^*, \vec{u} \in V, \text{i.e. (T.16)}.

Thus \( \mathcal{L}(E^*; F^*) \simeq \mathcal{L}((F^*)^*, E^*; \mathbb{R}) \overset{T.11}{\simeq} \mathcal{L}(F, E^*; \mathbb{R}) \overset{T.10}{\simeq} \mathcal{L}(E^*, F^*) \overset{\text{natural}}{\simeq} \mathcal{L}(E; F). \tag{T.16} \]

**Proposition T.8** The linear map (canonical definition of the transposed of a bilinear map)

\[
\mathcal{J}_b : \begin{cases} \mathcal{L}(E, F; \mathbb{R}) \to \mathcal{L}(F, E; \mathbb{R}) \\ T \to \mathcal{J}_b(T) \end{cases}, \quad \mathcal{J}_b(T)(\vec{u}, \vec{w}) := T(\vec{u}, \vec{w}), \quad \forall(\vec{u}, \vec{w}) \in E \times F. \tag{T.17}
\]

is natural: For all \( (\mathcal{P}_1, \mathcal{P}_2) \in \mathcal{L}_1(E; V) \times \mathcal{L}(F; W) \), and with \( \mathcal{\tilde{I}_P} \mathcal{E}_F(T)(\vec{u}, \vec{w}) := T(\mathcal{P}_1^{-1}.\vec{u}, \mathcal{P}_2^{-1}.\vec{w}) \) for all

\[
(\vec{u}, \vec{w}) \in V \times W, \text{the diagram} \begin{array}{c}
\mathcal{I}_P \mathcal{E}_F \downarrow \\
\mathcal{I}_P \mathcal{E}_F \downarrow
\end{array} \begin{array}{c}
\mathcal{J}_b \downarrow \\
\mathcal{J}_b \downarrow
\end{array} \mathcal{L}(E, F; \mathbb{R}) \overset{\text{natural}}{\simeq} \mathcal{L}(E, F; \mathbb{R}).
\]

**Proof.** \( \mathcal{\tilde{I}_P} \mathcal{E}_F(\mathcal{J}_b(T))(\vec{u}, \vec{w}) = \mathcal{J}_b(T)(\mathcal{P}_1^{-1}.\vec{u}, \mathcal{P}_2^{-1}.\vec{w}) = T(\mathcal{P}_2^{-1}.\vec{u}, \mathcal{P}_1^{-1}.\vec{w}), \text{and } \mathcal{J}_b(\mathcal{\tilde{I}_P} \mathcal{E}_F(T))(\vec{u}, \vec{w}) = \mathcal{\tilde{I}_P} \mathcal{E}_F(T)(\vec{u}, \vec{w}) = T(\mathcal{P}_2^{-1}.\vec{u}, \mathcal{P}_1^{-1}.\vec{w}) \) gives \( \mathcal{\tilde{I}_P} \mathcal{E}_F \circ \mathcal{J}_b = \mathcal{J}_b \circ \mathcal{\tilde{I}_P} \mathcal{E}_F \).

**U Distribution in brief: A covariant concept**

(We refer to the books of Laurent Schwartz for a full description.) Let \( \Omega \) be an open set in \( \mathbb{R}^n \). In continuum mechanics, for the infinite dimensional space of the finite energy functions \( \mathcal{L}^2(\Omega) \) and its sub-spaces, a distribution gives a covariant formulation for the virtual power, as used by Germain.
U.1 Definitions

Usual notations: Let $p \in [1, \infty]$ (e.g. $p = 2$ for finite energy functions), and let
\begin{equation}
L^p(\Omega) := \{ f : \Omega \to \mathbb{R} : \int_{\Omega} |f(x)|^p \, dx < \infty \} \quad \text{and} \quad \|f\|_p = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}},
\end{equation}
the space of functions such that $|f|^p$ is Lebesgue integrable and its usual norm. Then $(L^p(\Omega), \|\cdot\|_{L^p})$ is a Banach space (a complete normed space). And let
\begin{equation}
L^\infty(\Omega) := \{ f : \Omega \to \mathbb{R} : \sup_{x \in \Omega} |f(x)| < \infty \}, \quad \text{and} \quad \|f\|_\infty = \sup_{x \in \Omega} |f(x)|,
\end{equation}
the space of Lebesgue measurable bounded functions and its usual norm. Then $(L^\infty(\Omega), \|\cdot\|_{L^\infty})$ is a Banach space (a complete normed space).

Definition U.1 If $f \in \mathcal{F}(\Omega; \mathbb{R})$, then its support is the set
\begin{equation}
\text{supp}(f) := \{ x \in \Omega : f(x) \neq 0 \} = \{ x \in \Omega : f(x) \neq 0 \}
\end{equation}
\begin{equation}
= \{ x \in \Omega : f(x) \neq 0 \}
\end{equation}
the set where it is interesting to study $f$.

The closure is required: E.g., if $\Omega = \mathbb{R}$ and $f(x) = 1_{[0,2\pi]}(x)\sin x$, then $\{ x \in \Omega : f(x) \neq 0 \} = [0, \pi]\cup\{2\pi\}$ and the point $\pi$ is a point of interest since $\sin$ varies in its vicinity: $\cos(\pi) = -1 \neq 0$.

E.g., if $\Omega = \mathbb{R}$ and $f(x) := e^{-\frac{1}{x^2}}$ if $x \in [-1, 1]$ and $f(x) := 0$ elsewhere: $\varphi \in \mathcal{D}(\mathbb{R})$ with supp($\varphi$) $= [-1, 1]$.

And $\mathcal{D}(\Omega)$ is a vector space which is dense in $(L^p(\Omega), \|\cdot\|_{L^p})$ for $p \in [1, \infty]$.

Definition U.2 (Schwartz notation, $D$ being the letter after $C$:)
\begin{equation}
\mathcal{D}(\Omega) := C_0^\infty(\Omega; \mathbb{R}) = \{ \varphi \in C^\infty(\Omega; \mathbb{R}) \text{ s.t. supp}(\varphi) \text{ is compact in } \Omega \}.
\end{equation}

The notation $\langle T, \varphi \rangle_{\mathcal{D}(\Omega), \mathcal{D}(\Omega)} = \langle T, \varphi \rangle = \langle T, \varphi \rangle$ is the “duality bracket” = the “covariance–contravariance bracket” between a linear function $T \in \mathcal{D}'(\Omega)$ and a vector $\varphi \in \mathcal{D}(\Omega)$.

Definition U.3 A distribution in $\Omega$ is a linear $\mathcal{D}(\Omega)$-continuous function
\begin{equation}
T : \quad \mathcal{D}(\Omega) \to \mathbb{R}
\end{equation}
\begin{equation}
\varphi \to T(\varphi) \quad \text{named } (T, \varphi)
\end{equation}
(see remark U.7). The space of distribution in $\Omega$ is named $\mathcal{D}'(\Omega)$ (the dual of $\mathcal{D}(\Omega)$).

The notation $\langle T, \varphi \rangle_{\mathcal{D}(\Omega), \mathcal{D}(\Omega)} = \langle T, \varphi \rangle = \langle T, \varphi \rangle$ is the “duality bracket” = the “covariance–contravariance bracket” between a linear function $T \in \mathcal{D}'(\Omega)$ and a vector $\varphi \in \mathcal{D}(\Omega)$.

Definition U.4 Let $f \in L^p(\Omega)$. The regular distribution $T_f \in \mathcal{D}'(\Omega)$ associated to $f$ is defined by
\begin{equation}
T_f(\varphi) := \int_{\Omega} f(x) \varphi(x) \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).
\end{equation}

Interpretation: $T_f$ is a measuring instrument associated with the density measure $dm_f(x) = f(x) \, dx$, since $T_f(\varphi) := \int_{\Omega} \varphi(x) \, dm_f(x)$.

Definition U.5 Let $x_0 \in \mathbb{R}^n$. The Dirac measure $\delta_{x_0}$ is the distribution $T^{\text{named}} := \delta_{x_0} \in \mathcal{D}'(\mathbb{R}^n)$ defined by, for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,
\begin{equation}
\langle \delta_{x_0}, \varphi \rangle = \langle \delta_{x_0}, \varphi \rangle = \varphi(x_0).
\end{equation}
And $\delta_{x_0}$ is not a regular distribution ($\delta_{x_0}$ is not a density measure): There is no integrable function $f$ such that $T_f = \delta_{x_0}$. Interpretation: $\delta_{x_0}$ corresponds to an ideal measuring device: The precision is perfect at $x_0$ (gives the exact value $\varphi(x_0)$ at $x_0$). In real life $\delta_{x_0}$ is the ideal approximation of $T_{f_n}$ where $f_n$ is e.g. given by $f_n(x) = nI_{[x_0-x_0+\frac{1}{n}, x_0+\frac{1}{n}]}$ (drawing): For all $\varphi \in \mathcal{D}(\Omega)$, $T_{f_n}(\varphi) \to_{n \to \infty} \delta_{x_0}(\varphi) = \varphi(x_0)$.

Generalization of the definition: In (U.5) $\mathcal{D}(\Omega) = C_c^\infty(\Omega; \mathbb{R})$ is replaced by $C_c^\infty(\Omega; \mathbb{R}^n)$. So if you consider a basis $\{e_i\}$ then $\varphi \in C_c^\infty(\Omega; \mathbb{R}^n)$ reads $\varphi = \sum_{i=1}^n \varphi^i e_i$ with $\varphi^i \in C_c^\infty(\Omega)$ for all $i$.

Example U.6 Power: Let $\alpha : \Omega \to \mathcal{D}_0^1(\Omega)$ be a differential form. Then $P = T_\alpha$ defined by $P(\varphi) = \int_{\Omega} \alpha.\vec{v} \, dx$ gives the virtual power associated to $\alpha$ relative to the vector field $\vec{v}$ (mechanics and thermodynamics).

Remark U.7 In the definition U.3, the $\mathcal{D}(\Omega)$-continuity of $T$ is defined by: 1. A sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ towards a function $\varphi \in \mathcal{D}(\Omega)$ iff there exists a compact $K \subset \Omega$ s.t. $\text{supp}(\varphi_n) \subset K$ for all $n$, and \( \| \varphi - \varphi_n \|_\infty \to_{n \to \infty} 0 \), and $\| \frac{\partial x}{\partial \alpha} - \frac{\partial x}{\partial \alpha} \|_{\infty} \to_{n \to \infty} 0$ for all $i$, and for all derivations $\delta_{x_k}$, $k \in \mathbb{N}$.

2. $T$ is continuous at $\varphi \in \mathcal{D}(\Omega)$ iff $T(\varphi_n) \to_{n \to \infty} T(\varphi)$ for any sequence $(\varphi_n)_{n \in \mathcal{N}} \to \varphi \in \mathcal{D}(\Omega)$.
U.2 Derivation of a distribution

Let $O$ be a point in $\mathbb{R}^n$ (an origin). If $x \in \mathbb{R}^n$ and if $(e_i)$ is a basis in $\mathbb{R}^n$, let $\bar{x} = \sum_{i=1}^n x^i e_i$.

**Definition U.8** The derivative $\frac{\partial T}{\partial x^i}$ of a distribution $T \in D'(\Omega)$ is the distribution in $D'(\Omega)$ defined by, for all $\varphi \in D(\Omega)$,

$$\frac{\partial T}{\partial x^i}(\varphi) := -T_i(\varphi), \quad \text{i.e., } \langle \frac{\partial T}{\partial x^i}, \varphi \rangle := - \langle T, \frac{\partial \varphi}{\partial x^i} \rangle. \quad \text{(U.8)}$$

($\frac{\partial T}{\partial x^i}$ is indeed a distribution: Easy check.)

**Example U.9** If $T = T_f$ is a regular distribution with $f \in C^1(\Omega)$, then $\frac{\partial (T_f)}{\partial x^i} = T_{\frac{\partial f}{\partial x^i}}$. Indeed, for all $\varphi \in D(\Omega)$, $\frac{\partial (T_f)}{\partial x^i}(\varphi) = -T_f(\frac{\partial \varphi}{\partial x^i}) = -\int_{\Omega} f(x) \frac{\partial \varphi}{\partial x^i} \, d\Omega = + \int_{\Omega} \frac{\partial f}{\partial x^i} \varphi(x) \, d\Omega + \int_{\Gamma} 0 \, d\Gamma$, since $\varphi$ vanishes on $\Gamma = \partial \Omega$ (the support of $\varphi$ is compact in $\Omega$), thus $\frac{\partial (T_f)}{\partial x^i}(\varphi) = T_{\frac{\partial f}{\partial x^i}}(\varphi)$ for all $\varphi \in D(\Omega)$.

**Example U.10** Consider the Heaviside function (the unit step function) $H_0 := 1_{\mathbb{R}_+}$ and the associated distribution $T = T_{H_0}$. Then $\langle (T_{H_0})', \varphi \rangle := -\langle T_{H_0}, \varphi' \rangle = -\int_{\Omega} H_0(x) \varphi'(x) \, dx = -\int_{0}^{\infty} \varphi'(x) \, dx = \varphi(0) = \langle \delta_0, \varphi \rangle$ for any $\varphi \in D(\mathbb{R})$, thus $(T_{H_0})' = \delta_0$. Written $H_0' = \delta_0$ in $D'(\Omega)$, which is not in a equality between distributions, because $H_0$ is not derivable at $0$ as a function, and $\delta_0$ is not a function; It is equality between distributions: The notation $H_0'$ can only be used to compute $H_0'(\varphi) = \langle H_0', \varphi \rangle := -\langle H_0, \varphi' \rangle$.

U.3 Hilbert space $H^1(\Omega)$

**U.3.1 Motivation**

Consider the hat function $\Lambda(x) \begin{cases} = x + 1 & \text{if } x \in [-1,0], \\ = 1 - x & \text{if } x \in [0,1], \\ = 0 & \text{otherwise} \end{cases}$ (drawing). When applying the finite element method, it is well-known that, if you use integrals (if you use the virtual power principle which makes you compute average values), then you can consider the derivative of the hat function $\Lambda$ as if it was the usual derivative, that is, at the points where the usual computation of $\Lambda'$ is meaningful, that is, $\Lambda'(x) = \begin{cases} = 1 & \text{if } x \in [-1,0], \\ = 1 - x & \text{if } x \in [0,1], \\ = 0 & \text{if } x \in \mathbb{R} \setminus \{-1,0,1\} \end{cases}$ (drawing).

**Problem:** $\Lambda'$ is not defined at $-1, 0, 1$ (the function $\Lambda$ is not derivable at $-1, 0, 1$).

**Question:** Does however the “usual” computation $I = \int_{\mathbb{R}} \Lambda'(x) \varphi(x) \, dx$ with (U.9) gives the good result? (This is not a trivial question: E.g., with $H_0 = 1_{\mathbb{R}_+}$ instead of $\Lambda$, we would get the absurd result $H_0' = 0$, absurd since $H_0' = \delta_0$.)

**Answer:** Yes. Justification:

1. Consider $T_{\Lambda}$ the regular distribution associated to $\Lambda$, cf. (U.6);

2. Then consider $(T_{\Lambda})'$, cf. (U.8): We get $\langle (T_{\Lambda})', \varphi \rangle = \int_{\mathbb{R}} \Lambda(x) \varphi'(x) \, dx = \int_{-1}^{0} \Lambda(x) \varphi'(x) \, dx + \int_{0}^{1} \Lambda(x) \varphi'(x) \, dx = + \int_{-1}^{0} \varphi(x) \, dx + \int_{0}^{1} \varphi(x) \, dx$, for any $\varphi \in D(\mathbb{R})$;

3. Thus $(T_{\Lambda})' = T_f$ where $f = \chi_{[-1,0]} + \chi_{[0,1]}$; that is $(T_{\Lambda})'$ is a regular distribution. And its is named $f = \Lambda'$ within the distribution setting, i.e., for computations $\langle \Lambda', \varphi \rangle = \langle (T_{\Lambda})', \varphi \rangle$ with $\varphi \in D(\mathbb{R})$ (value $= \int_{\mathbb{R}} f(x) \varphi(x) \, dx$).

**U.3.2 Definition of $H^1(\Omega)$**

The space $C^1(\Omega; \mathbb{R})$ is too small in many applications (e.g., for the $\Lambda$ function above); We need a bigger space where the functions are “derivable is a weaker sense”. Consider a basis in $\mathbb{R}^n$:

**Definition U.11** The Sobolev space $H^1(\Omega)$ is the subspace of $L^2(\Omega)$ restricted to functions whose generalized derivatives are in $L^2(\Omega)$:

$$H^1(\Omega) = \{ v \in L^2(\Omega) : \frac{\partial v}{\partial x^i} \in L^2(\Omega), \forall i = 1,...,n \}. \quad \text{(U.10)}$$

Usual shortened notation: $H^1(\Omega) = \{ v \in L^2(\Omega) : \text{grad } v \in L^2(\Omega)^n \}$. 

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Soto check that \( v \in H^1(\Omega) \), even if \( \frac{\partial v}{\partial n} \) does not exist in the classic way (see the above hat function \( \Lambda \)), you have to:

1. Consider its associated regular distribution \( T_v \).
2. Compute \( \frac{\partial T_v}{\partial x} \) in \( \mathcal{D}(\Omega) \).
3. And if, for all \( i \), there exists \( f_i \in L^2(\Omega) \) s.t. \( \frac{\partial T_v}{\partial x_i} = T_{f_i} \), then \( v \in H^1(\Omega) \).
4. Then \( f_i \) is named \( \frac{\partial v}{\partial x} \) only if it is used within the Lebesgue integrals \( \int_\Omega \frac{\partial v}{\partial x}(x) \varphi(x) \, dx \) with \( \varphi \in \mathcal{D}(\Omega) \).

E.g., \( \Lambda \in H^1(\mathbb{R}) \) since \( (T_\Lambda)' = T_f \) with \( f = 1_{[-1,0]} + 1_{[0,1]} \in L^2(\mathbb{R}) \). And \( (T_\Lambda)' = \Lambda' = f \) in the distribution context (integral computations).

Let \((\cdot,\cdot)_{L^2} \) and \( ||\cdot||_{L^2} \) be the usual inner dot product and norm in \( L^2(\Omega) \), that is,

\[
(u,v)_{L^2} = \int_\Omega u(x)v(x) \, dx, \quad \text{and} \quad ||v||_{L^2} = \sqrt{(u,v)_{L^2}} = (\int_\Omega |v(x)|^2 \, dx)^{\frac{1}{2}}. \tag{U.11}
\]

\((L^2(\Omega), (\cdot,\cdot)_{L^2}) \) is a Hilbert space. Then define, for all \( u,v \in H^1(\Omega) \),

\[
(u,v)_{H^1} = (u,v)_{L^2} + \sum_{i=1}^n \left( \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2}, \quad \text{and} \quad ||v||_{H^1} = (u,v)_{H^1}^{\frac{1}{2}}. \tag{U.12}
\]

Then \((H^1(\Omega), (\cdot,\cdot)_{H^1}) \) is a Hilbert space (Riesz-Fisher theorem). (With a Euclidean dot product \((\cdot,\cdot)_g \) in \( \mathbb{R}^n \) and a \((\cdot,\cdot)_g\)-orthonormal basis, \((u,v)_{H^1} = (u,v)_{L^2} + (\text{grad} u, \text{grad} v)_{L^2} \).)

### U.3.3 Subspace \( H^1_0(\Omega) \) and its dual space \( H^{-1}(\Omega) \)

The boundary \( \Gamma = \partial \Omega \) of \( \Omega \) is supposed to be regular. Let

\[
H^1_0(\Omega) := \{ v \in H^1(\Omega) : v_{|\Gamma} = 0 \}. \tag{U.13}
\]

Then \((H^1_0(\Omega), (\cdot,\cdot)_{H^1}) \) is a Hilbert space.

More generally (without any regularity assumption on \( \Gamma \)), \( H^1_0(\Omega) := \overline{\mathcal{D}(\Omega)}^{H^1} \) is the closure of \( \mathcal{D}(\Omega) \) in \((H^1(\Omega), ||\cdot||_{H^1}) \): This closure of \( \mathcal{D}(\Omega) \) in \( H^1(\Omega) \) enables the use of the distribution setting.

**Notation:** The dual space of \( H^1_0(\Omega) \) is the space

\[
H^{-1}(\Omega) = (H^1_0(\Omega))^\prime = \mathcal{L}(H^1_0(\Omega); \mathbb{R}) \tag{U.14}
\]

equipped with the (usual) norm \( ||T||_{H^{-1}} := \sup_{||v||_{H^1_0} = 1} |T(v)| \). And (duality bracket), if \( v \in H^1_0(\Omega) \) and \( T \in H^{-1}(\Omega) \) then

\[
T(v) \overset{\text{named}}{=} \langle T, v \rangle_{H^{-1}, H^1_0} \overset{\text{named}}{=} (T,v). \tag{U.15}
\]

**Theorem U.12** (Characterization of \( H^{-1}(\Omega) = (H^1_0(\Omega))^\prime \)) A distribution \( T \) is in \( H^{-1}(\Omega) \) iff

\[
\exists (f,g) \in L^2(\Omega) \times L^2(\Omega)^n \quad \text{s.t.} \quad T = f - \text{div} \, g \quad (\in \mathcal{D}(\Omega)'), \tag{U.16}
\]

that is, for all \( v \in H^1_0(\Omega) \),

\[
\langle T, v \rangle_{H^{-1}, H^1_0} = \int_\Omega f(v) \, dx + \int_\Omega \text{div} \, g \, dv. \tag{U.17}
\]

And if \( \Omega \) is bounded then we can choose \( f = 0 \). If moreover \( g \in H^1(\Omega)^n \) then

\[
\langle T, v \rangle_{H^{-1}, H^1_0} = \int_\Omega f(x)v(x) \, dx - \int_\Omega \text{div} \, g(x)v(x) \, dx. \tag{U.18}
\]

(In fact we only need \( g \in H_{\text{div}}(\Omega) = \{ g \in L^2(\Omega)^n : \text{div} \, g \in L^2(\Omega) \} \).)

**Proof.** E.g., see Brezis [4].

For boundary value problems with Neumann boundary conditions, we then need \((H^1(\Omega))^\prime \) the dual space of \( H^1(\Omega) \).

Characterization of \((H^1(\Omega))^\prime \): We still have (U.17), but we have to replace (U.16) or (U.18) by, with a Euclidean dot product in \( \mathbb{R}^n \),

\[
\langle T, v \rangle_{(H^1(\Omega))^\prime, H^1} = \int_\Omega f(x)v(x) \, dx - \int_\Omega \text{div} \, g(x)v(x) \, dx + \int_\Gamma g(x) \cdot \n(x) \, v(x) \, dx. \tag{U.19}
\]

See Brezis [4].
References