Broadcast domination and multipacking in graphs and digraphs

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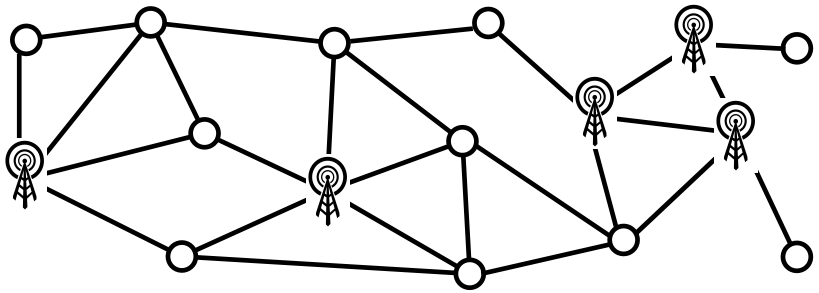
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Sk Samim Islam (Indian Statistical Institute, India)
Joydeep Mukherjee (Ramakrishna Mission Vivekananda Edu. and Res. Institute, India)

February 2026

Covering and packing: dual problems

Covering: cover the vertices of a graph using as few structures as possible

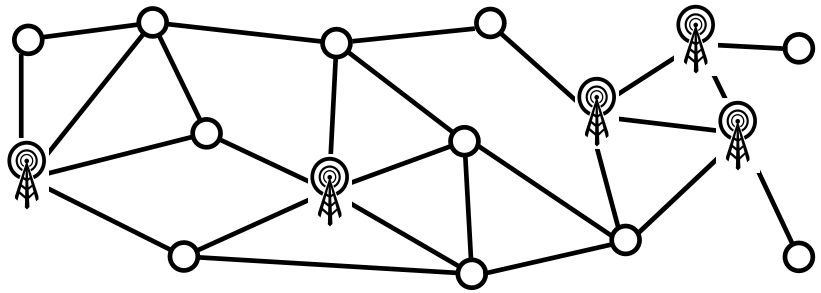
Example: dominating set: covering using 1-balls
$\rightarrow$ domination number $\gamma(G)$



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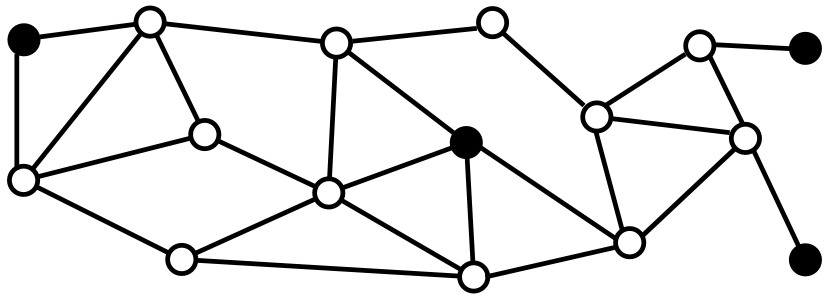
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Packing: pack as many structures as possible without interference

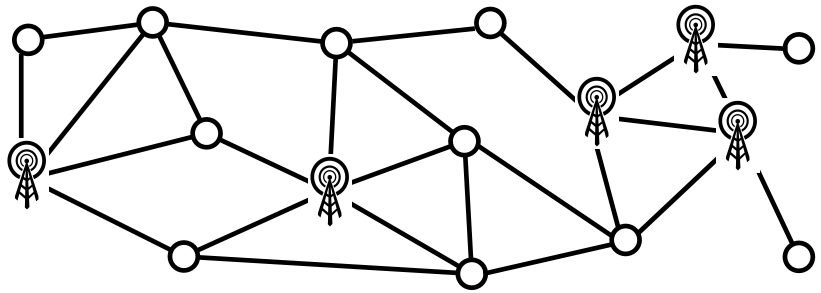
Example: dist. 3-independent set / 2-packing: packing 1-balls without overlap
$\rightarrow$ 2-packing number $\rho_2(G)$



Covering and packing: dual problems

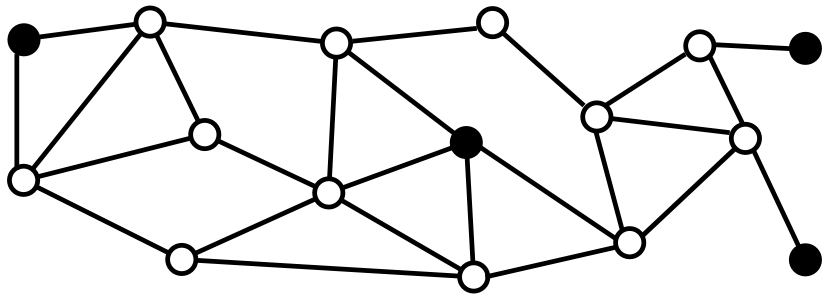
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These problems are dual (in the sense of LP) and $\rho_2(G) \leq \gamma(G)$.

Classic packing and covering problems are typically **NP-hard** (Karp, 1971):

$\Rightarrow$ Unless $P = NP$, there exists no efficient algorithm to solve them.

"Efficient algorithm": polynomial-time in terms of the size of the input graph (Cobham, Edmonds, 1965)



Richard C. Karp (1935-)



Jack Edmonds (1934-)



Alan B. Cobham (1927-2011)

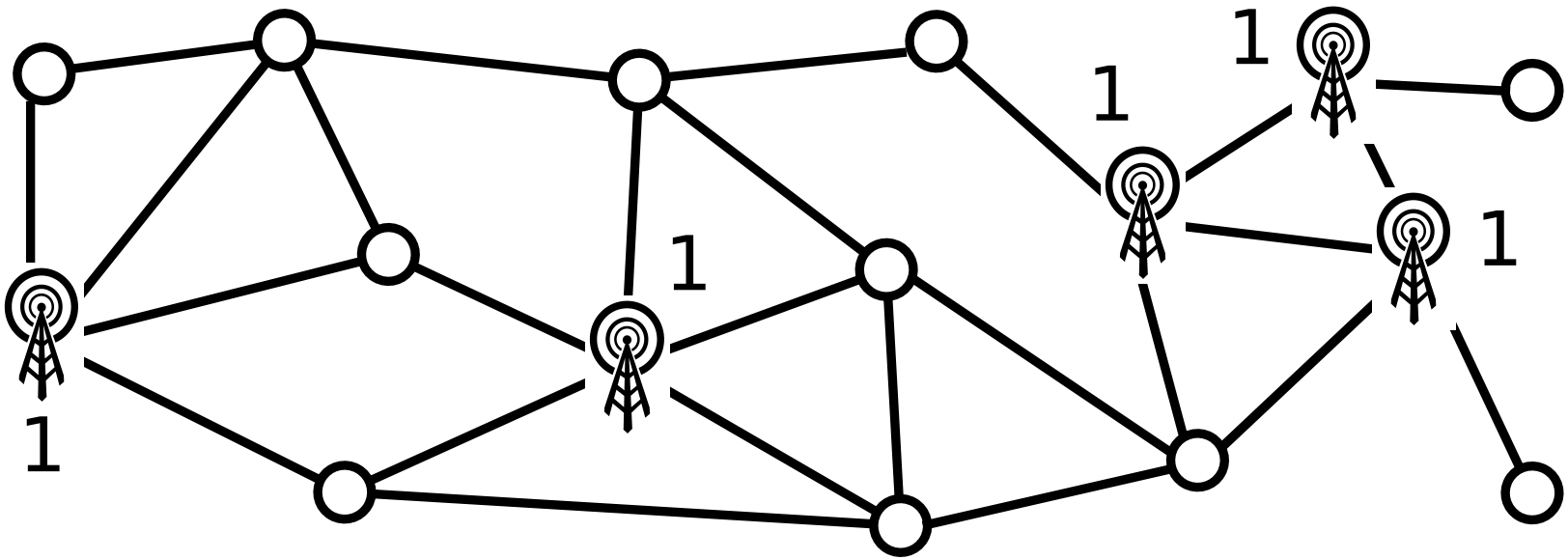
Definition - Dominating broadcast of graph G (Erwin, 2001)

A function $f : V(G) \to \mathbb{N}$ s.t. for every $v \in V(G)$, there exists $x \in V(G)$ with

- $f(x) > 0$ and
- $f(x) \geq d_G(x, v)$.

The cost of f is $\sum_{v \in V(G)} f(v)$.

Broadcast number $\gamma_b(G)$: smallest cost of a dominating broadcast of G.



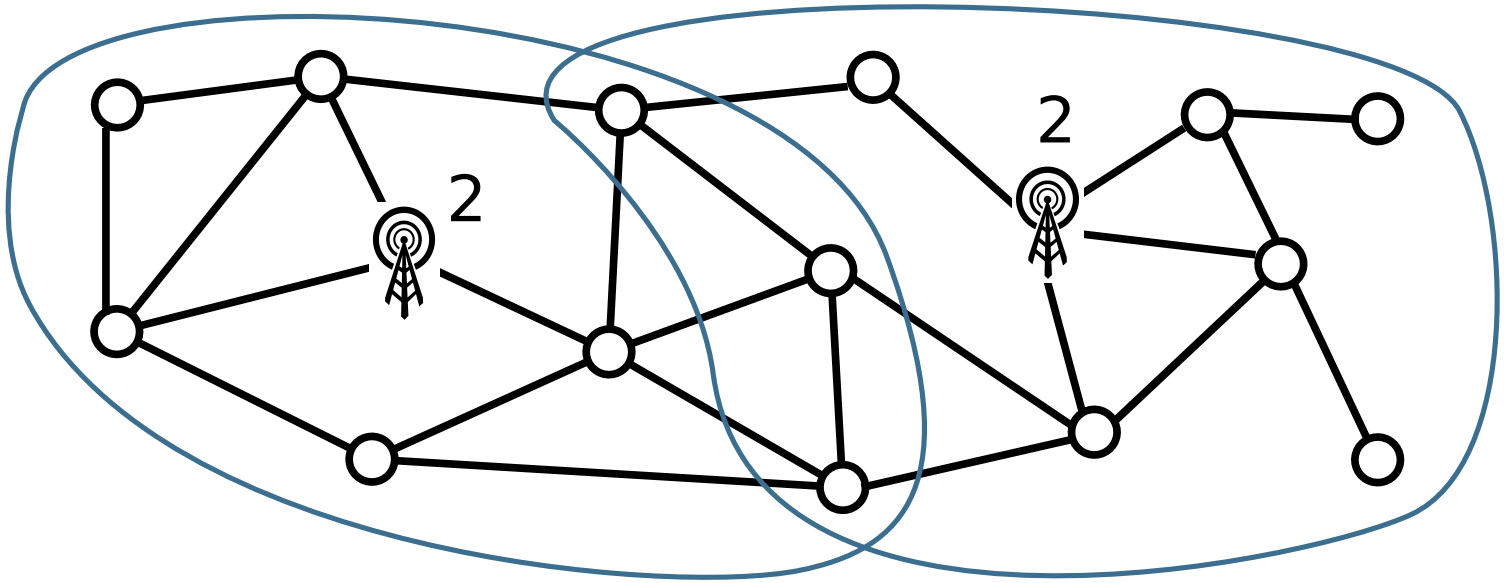
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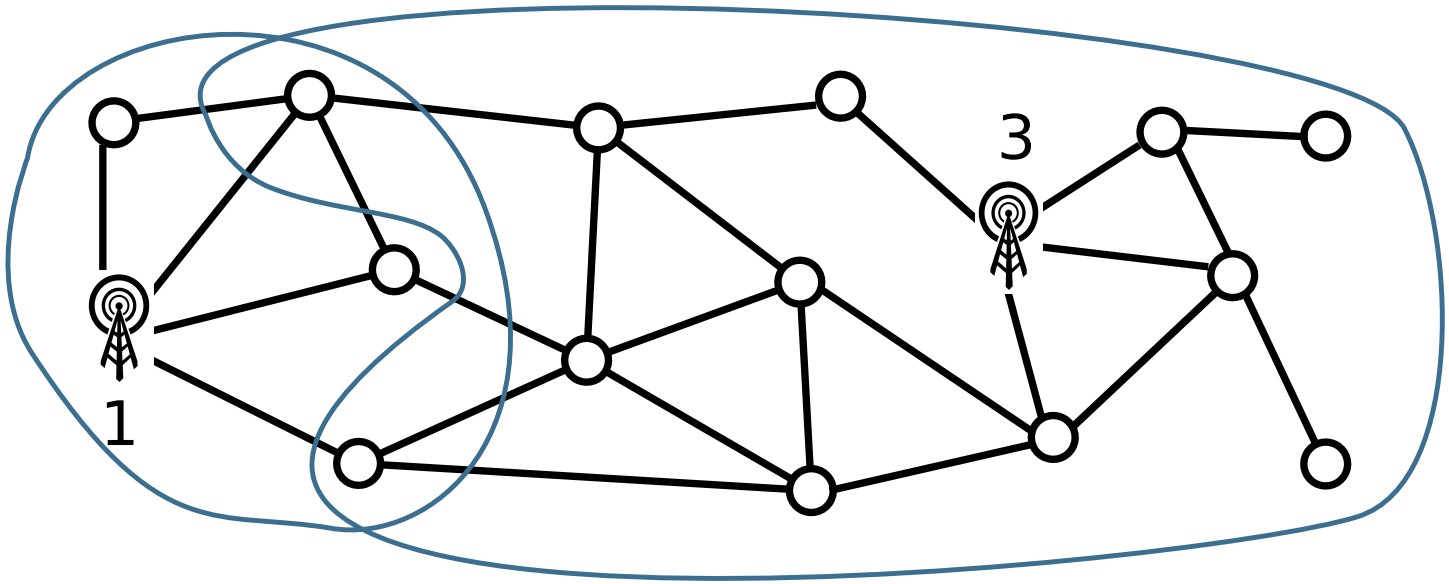
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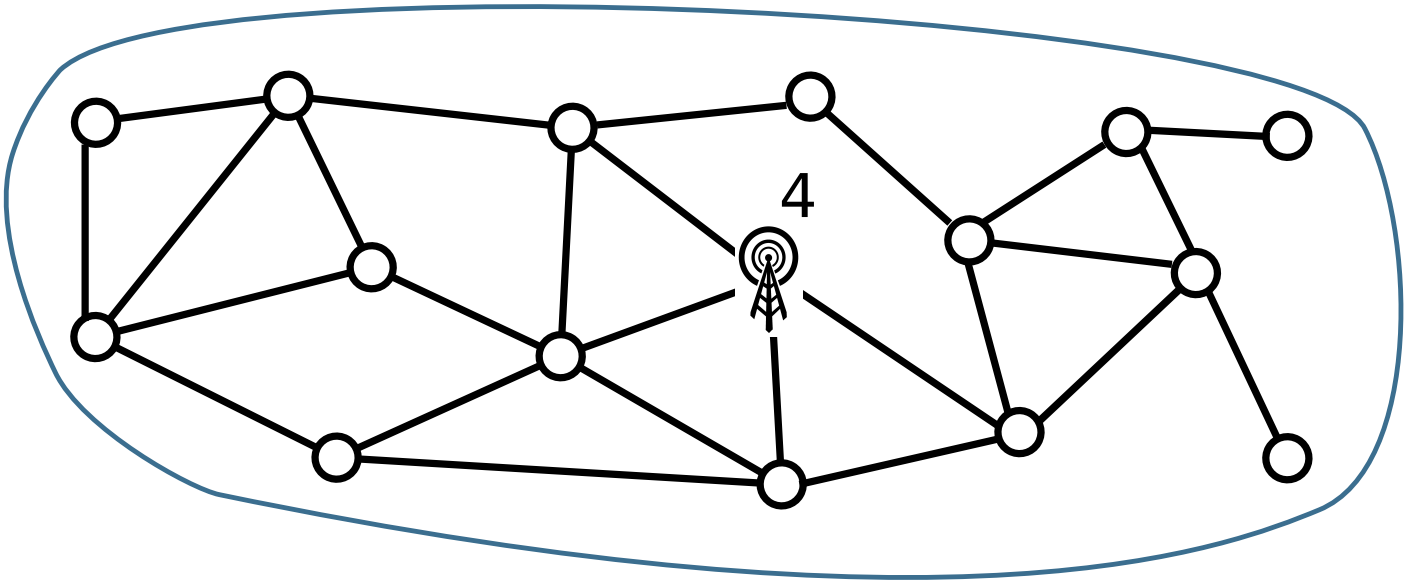
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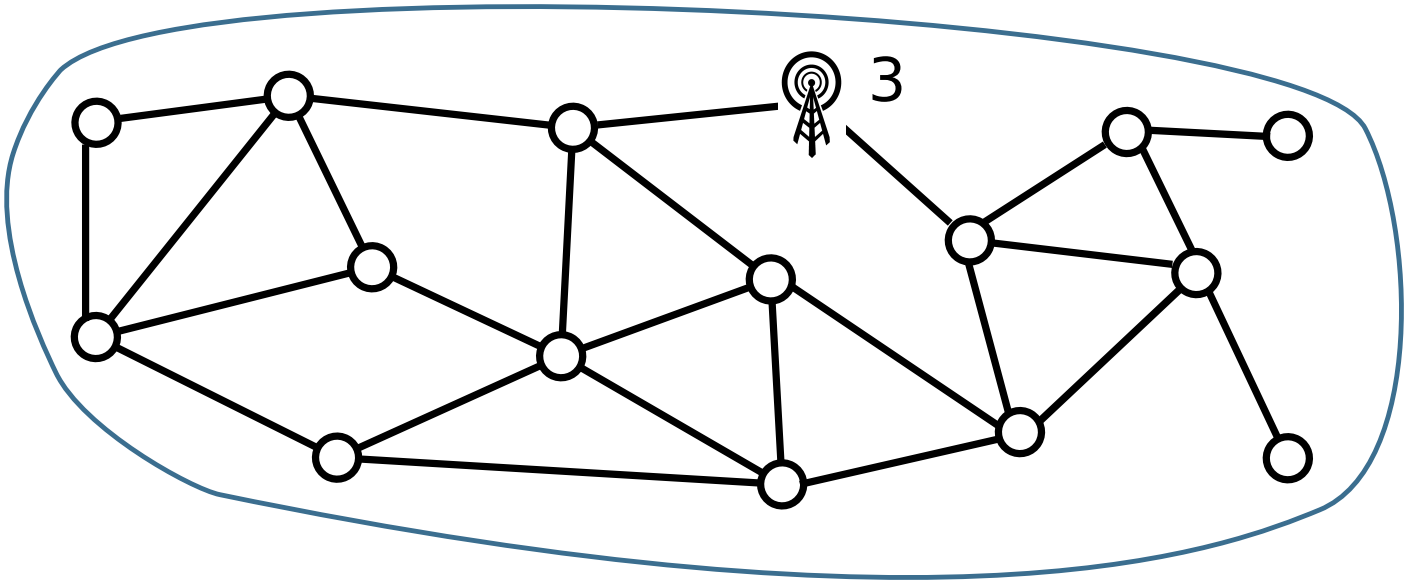
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Broadcast domination: an interesting fact

n : number of vertices of the input graph

Theorem (Heggernes-Lokshtanov, 2006)

We can find a minimum-cost dominating broadcast in polynomial time $O(n^6)$.

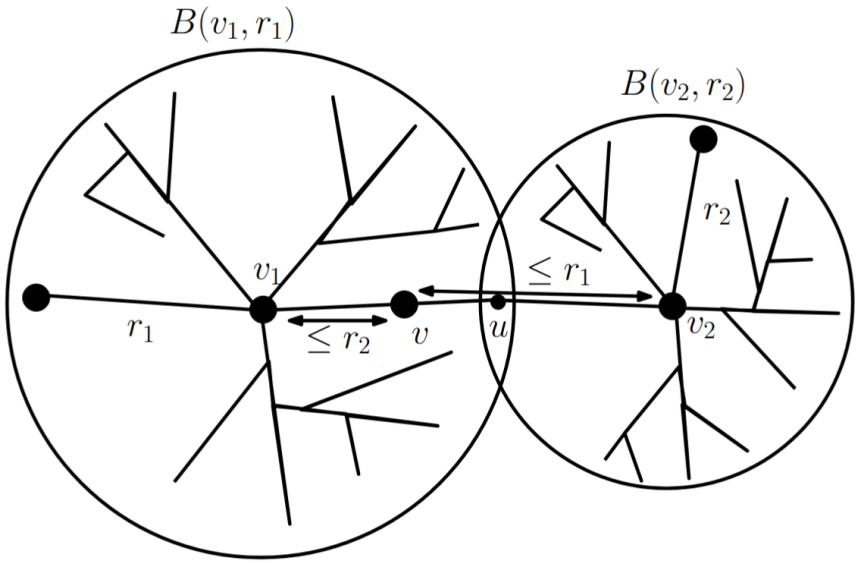
Proof idea:

- sufficient to find an efficient dominating broadcast (Erwin, 2001)
- The structure of broadcasting balls is a path or a cycle
- Dynamic programming on this structure

The broadcasting balls may be pairwise disjoint (efficient broadcast)

Lemma (Erwin, 2001)

In an optimum broadcast which minimizes the number of broadcasting balls, no two balls intersect.



Assume $r_1 \geq r_2$ and let u be in $B(v_1, r_1) \cap B(v_2, r_2)$.

Let u be the vertex on the shortest $v_1 - v$ path at distance r_2 from v_1.

$\rightarrow$ Replace $B(v_1, r_1)$ and $B(v_2, r_2)$ by $B(u, r_1 + r_2)$.

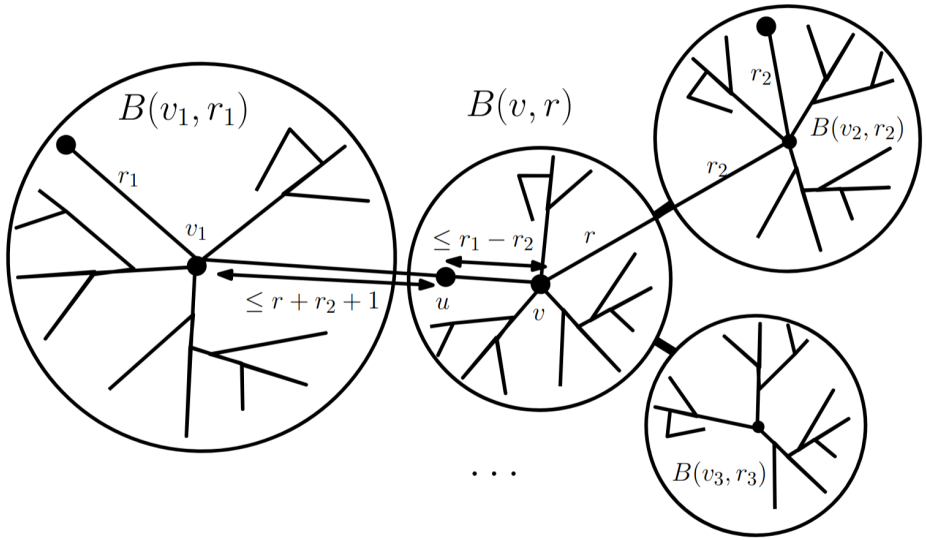
Image credit: Behsaz-Salavatipour, 2015

The structure of covering balls is a path or a cycle

Domination graph has broadcasting balls as vertex set, and two balls are adjacent IFF there is an edge joining some vertices of the balls in G.

Lemma (Heggernes-Lokshtanov, 2006)

In an optimum efficient broadcast which minimizes the number of broadcasting balls, every ball has maximum degree 2 in the domination graph.



Assume $r_1 \geq r_2 \geq r_3$.

u : vertex on shortest $v - v_1$ path at distance $\min\{r_1 + r + 1, r_1 - r_2\}$ from v.

$\rightarrow$ Replace the four balls by $B(u, r + r_1 + r_2 + 1)$.

Image credit: Behsaz-Salavatipour, 2015

Broadcast domination: ILP formulation

Vertices: $v_1, \ldots, v_n$.

$x_{i,k} \in \{0, 1\}$: whether vertex v_i broadcasts with radius k

We want to minimize:

$$\sum_{k=1}^{n} \sum_{i=1}^{n} k \cdot x_{i,k}$$

subject to:

$$\sum_{d(v_i, v_j) \leq k} x_{i,k} \geq 1 \text{ for each vertex } v_j.$$

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Dual ILP:

We want to maximize:

$$\sum_{i=1}^{n} y_i$$

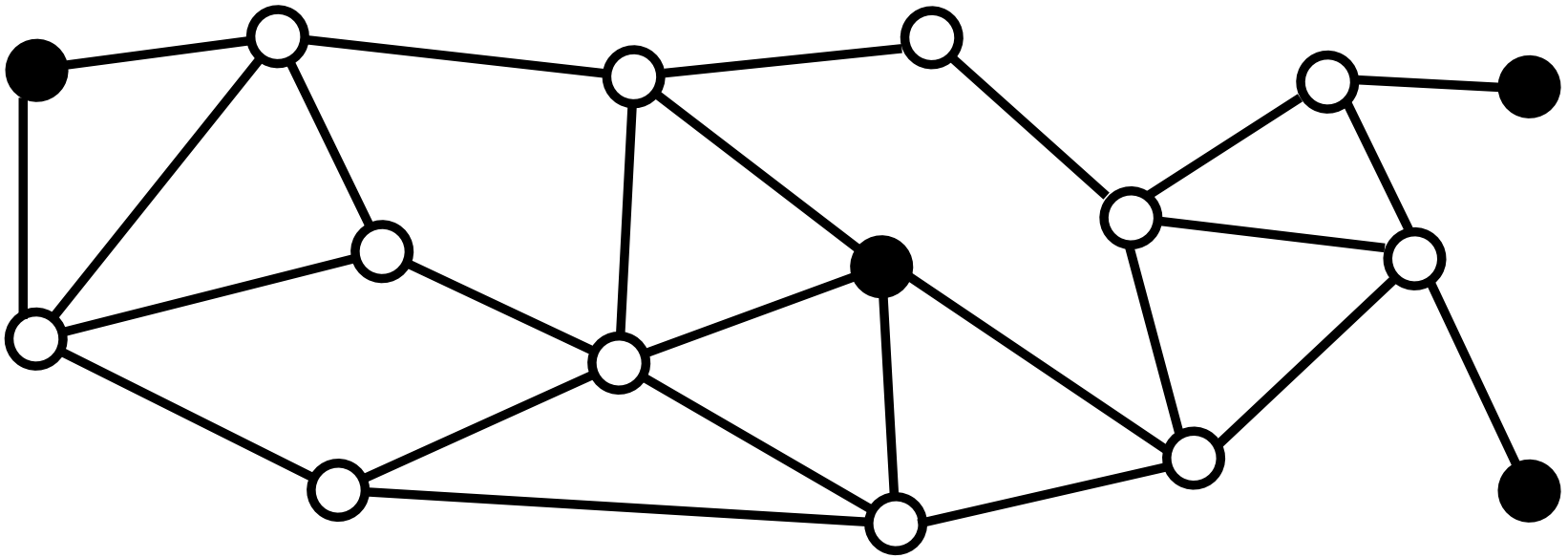
subject to:

$$\sum_{d(v_i, v_j) \leq k, y_j \geq 0} y_i \leq k \text{ for each vertex } v_j \text{ and integer } k \leq n.$$

Definition - Multipacking of graph G (Brewster-Mynhardt-Teshima, 2014)

A set S of vertices s.t. for every $v \in V(G)$ and every $d \in \mathbb{N}$, the d-ball $B_d(v)$ contains at most d vertices of S.

Multipacking number $mp(G)$: largest size of a multipacking of G.

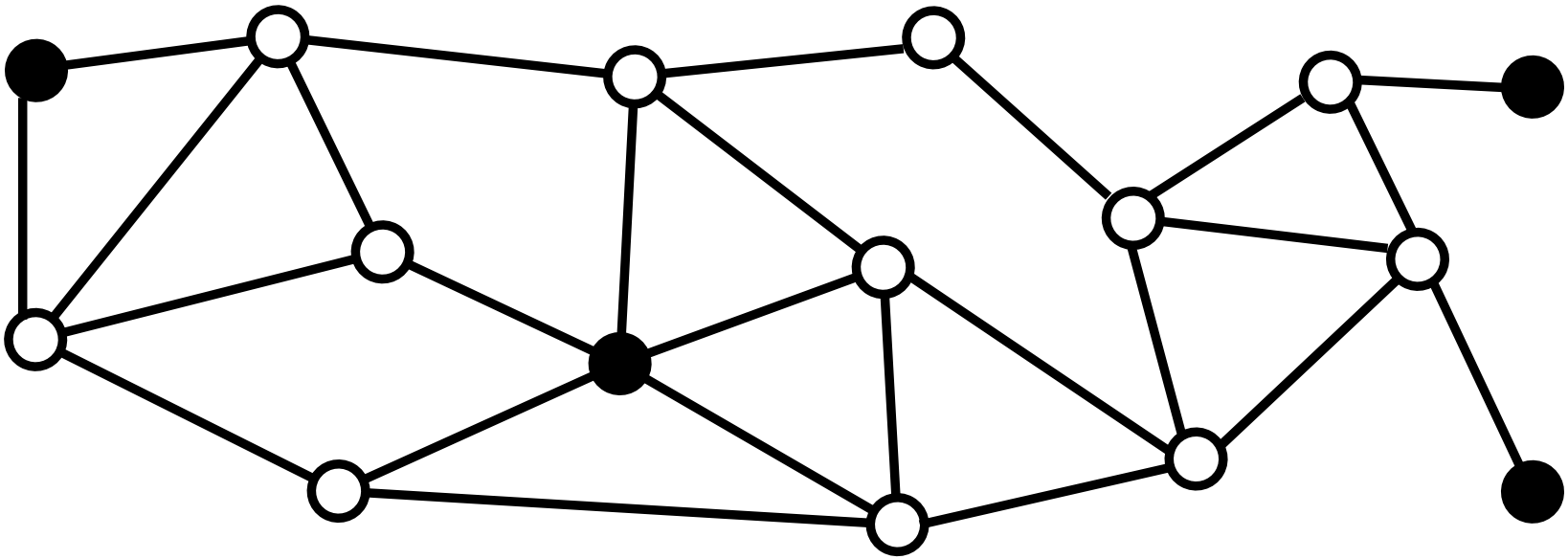


NOT a multipacking!

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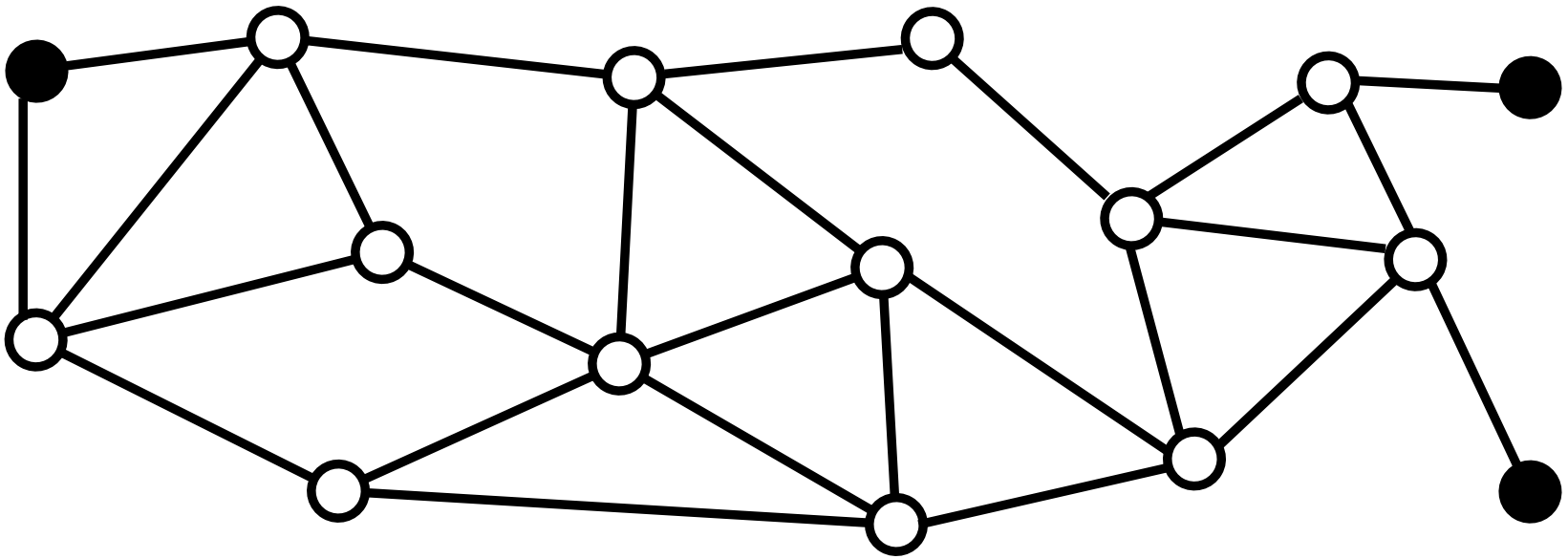


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Multipacking number $mp(G)$: largest size of a multipacking of G.

Theorem (Das, Islam, Lokshtanov, 2026+)

Determining whether G has a multipacking of size at least k is NP-complete.

Bounds for undirected graphs : general graphs

The two problems are dual (in the sense of LP).

Proposition

For every graph G, we have $mp(G) \leq \gamma_b(G)$.

Equality holds for:

- trees (Mynhardt-Teshima, 2017)
- more generally, strongly chordal graphs (Brewster-MacGillivray-Yang, 2019)
- rectangular grids (Beaudou-Brewster, 2019)

A chain of inequalities

diameter $diam(G)$: largest distance between two vertices in G

eccentricity of a vertex v: largest possible distance from v to another vertex

radius $rad(G)$: smallest eccentricity among all vertices

Proposition (Erwin, 2001 + Hartnell-Mynhardt, 2014)

For any graph G, $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq mp(G) \leq \gamma_b(G) \leq rad(G) \leq diam(G)$.

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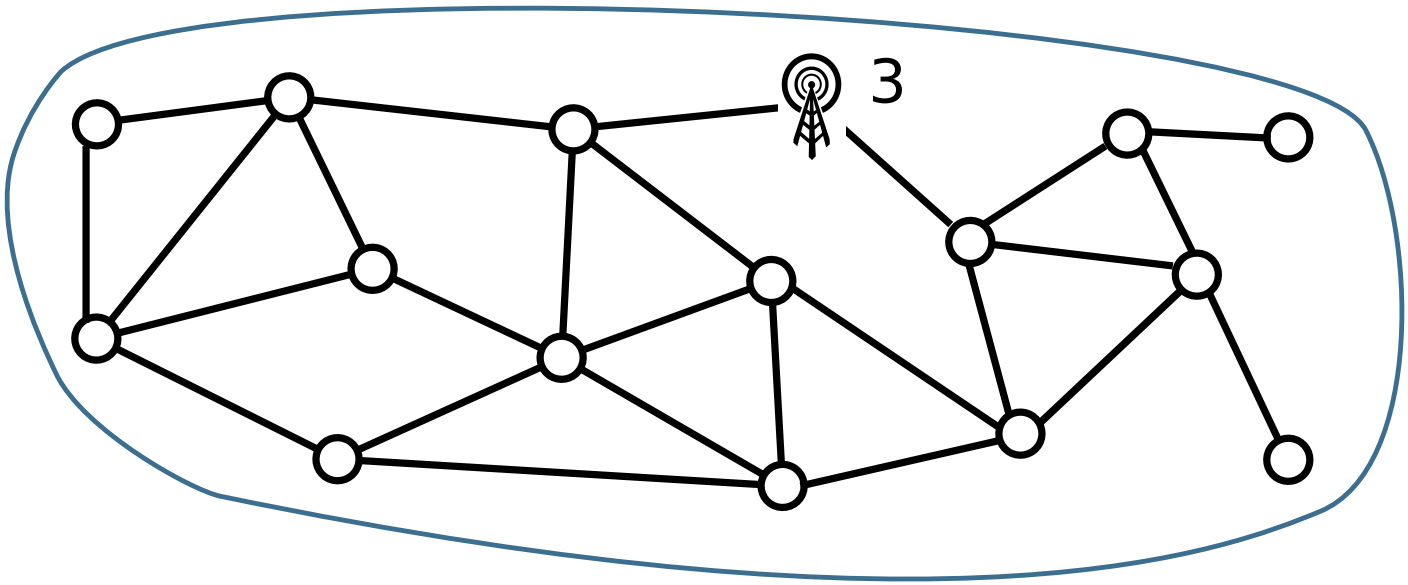
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$\gamma_b(G) \leq rad(G)$: consider a radial vertex v. Set $f(v) = rad(G)$.



A chain of inequalities

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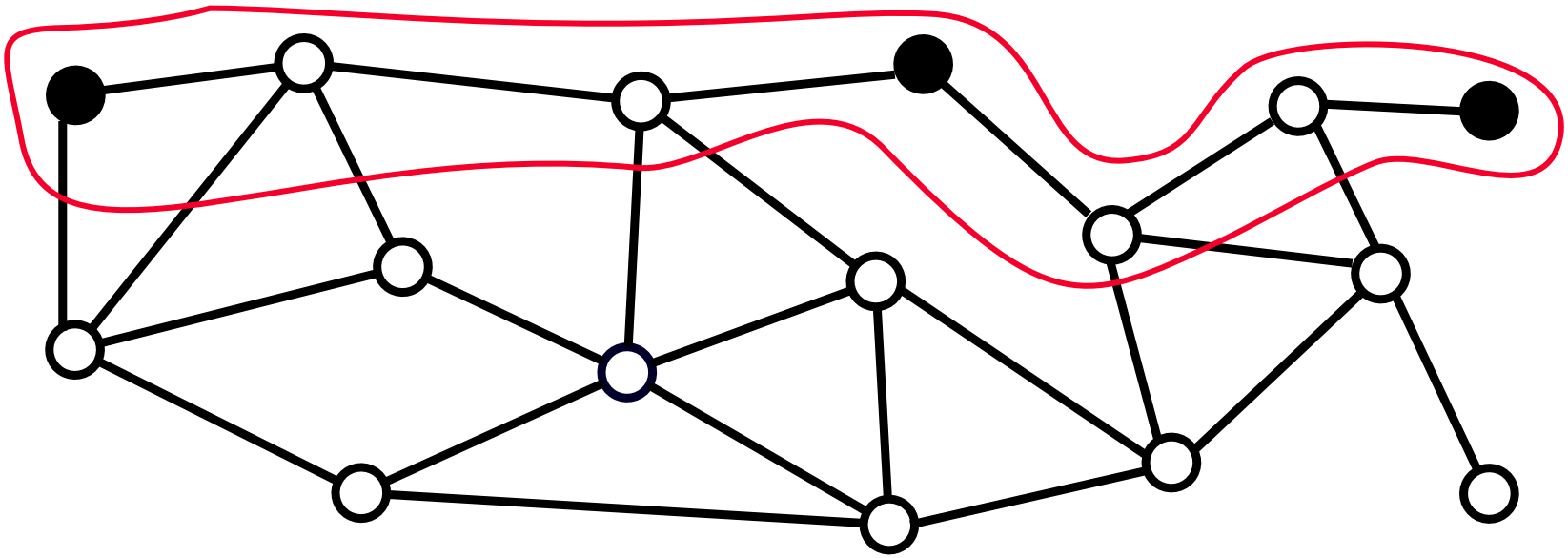
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$\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq mp(G)$: consider a diametral path P, select every third vertex.



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Proposition (Erwin, 2001 + Hartnell-Mynhardt, 2014)

For any graph G, $\left\lceil \frac{diam(G)+1}{3} \right\rceil \leq mp(G) \leq \gamma_b(G) \leq rad(G) \leq diam(G)$.

Corollary

For any graph G, we have $\gamma_b(G) < 3mp(G)$, hence $\frac{\gamma_b(G)}{mp(G)} < 3$.

Question (Hartnell-Mynhardt, 2014)

What is the largest possible ratio $\frac{\gamma_b(G)}{mp(G)}$?

Our theorem

Theorem (Beaudou, Brewster, F., 2019)

For any graph G, we have $\gamma_b(G) \leq 2mp(G) + 3$, hence $\frac{\gamma_b(G)}{mp(G)} \leq 2 + \epsilon$.

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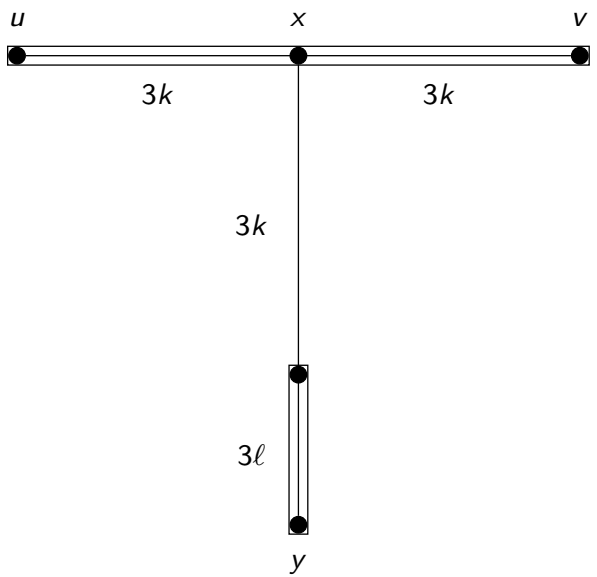
Lemma

Proof sketch

Let u, v, x, y be 4 vertices with:

- $d(u, v) = 6k$
- $d(x, u) = d(x, v) = 3k$
- $d(x, y) = 3k + 3\ell$.

Then, $mp(G) \geq 2k + \ell$.



Theorem (Beaudou, Brewster, F., 2019)

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Then, $mp(G) \geq 2k + \ell$.

Let $diam(G) = 6k + i$ and $rad(G) = 3k + 3\ell + j$ $(0 \leq i < 6$ and $0 \leq j < 3)$

Apply the lemma with x, a vertex of eccentricity $rad(G)$.

$$\begin{aligned} mp(G) &\geq 2k + \ell \\ &\geq \frac{diam(G)}{3} + \frac{rad(G)}{3} - \frac{diam(G)}{6} - c \\ &\geq \frac{rad(G)}{2} - c \\ &\geq \frac{\gamma_b(G)}{2} - c \end{aligned}$$

□

Conjecture

Theorem (Beaudou, Brewster, F., 2019)

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The conjecture is true when $mp(G) \leq 4$.

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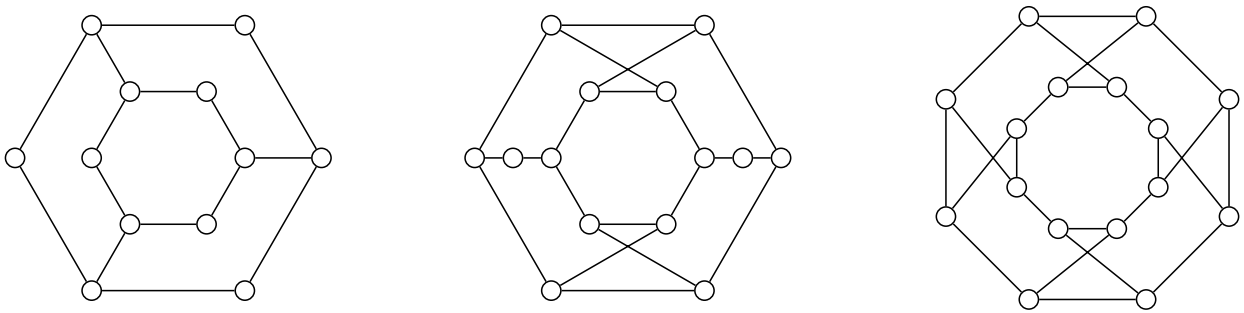
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Conjecture would be tight — infinitely many graphs G s.t. $\gamma_b(G) = 2mp(G)$:



$mp(G) = 2$ and $\gamma_b(G) = 4$

Connected graphs

Question

What happens for connected graphs?

Question

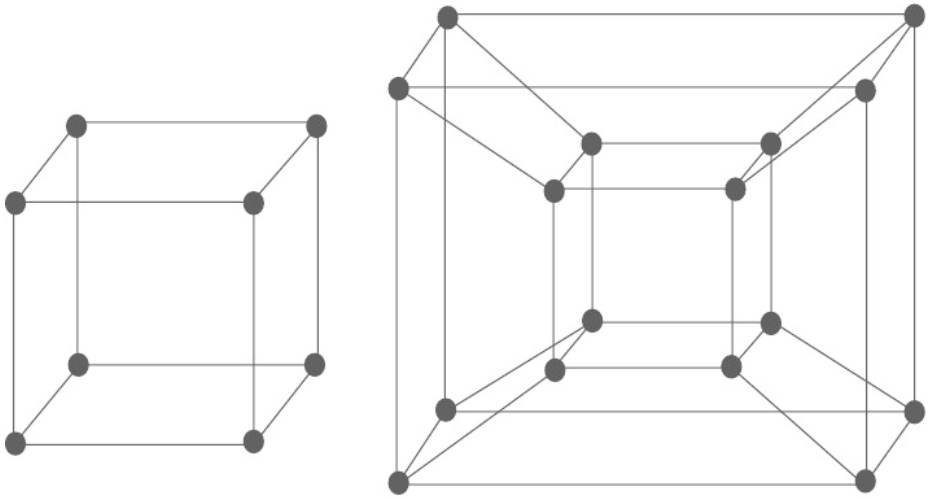
What happens for connected graphs?

Theorem (Brešar, Špacapan, 2019)

For the hypercube H_d: $\gamma_b(H_d) = d - 1$.

Theorem (Rajendraprasad, Sani, Sasidharan, Sen, 2025+)

For the hypercube H_d: $\lfloor \frac{d}{2} \rfloor \leq mp(H_d) \leq \frac{d}{2} + 6\sqrt{2d}$.



Q_3 Q_4

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For the hypercube H_d: $\lfloor \frac{d}{2} \rfloor \leq mp(H_d) \leq \frac{d}{2} + 6\sqrt{2d}$.

Corollary

For connected graphs G, $\lim_{mp(G)\to\infty} \sup \left\{ \frac{\gamma_b(G)}{mp(G)} \right\} = 2$.

Bounds for undirected graphs : chordal graphs

Chordal graph: graph where every cycle of length 4 or more has a *chord*

Proposition

If G is a chordal graph, $\gamma_b(G) \leq \left\lceil \frac{3}{2} mp(G) \right\rceil$.

Chordal graph: graph where every cycle of length 4 or more has a *chord*

Proposition

If G is a chordal graph, $\gamma_b(G) \leq \left\lceil \frac{3}{2} mp(G) \right\rceil$.

This can be proved using the following two theorems:

Theorem (Laskar, Shier, 1983)

If G is a chordal graph with radius r and diameter d, then $2r \leq d + 2$.

Theorem (Erwin 2001 & Hartnell-Mynhardt 2014)

If G is a connected graph of order at least 2 having radius r, diameter d, multipacking number $mp(G)$, broadcast domination number $\gamma_b(G)$ and domination number $\gamma(G)$, then $\left\lceil \frac{d+1}{3} \right\rceil \leq mp(G) \leq \gamma_b(G) \leq min\{\gamma(G), r\}$.

Bound for chordal graphs

Theorem (Das, F., Islam, Mukherjee, 2023)

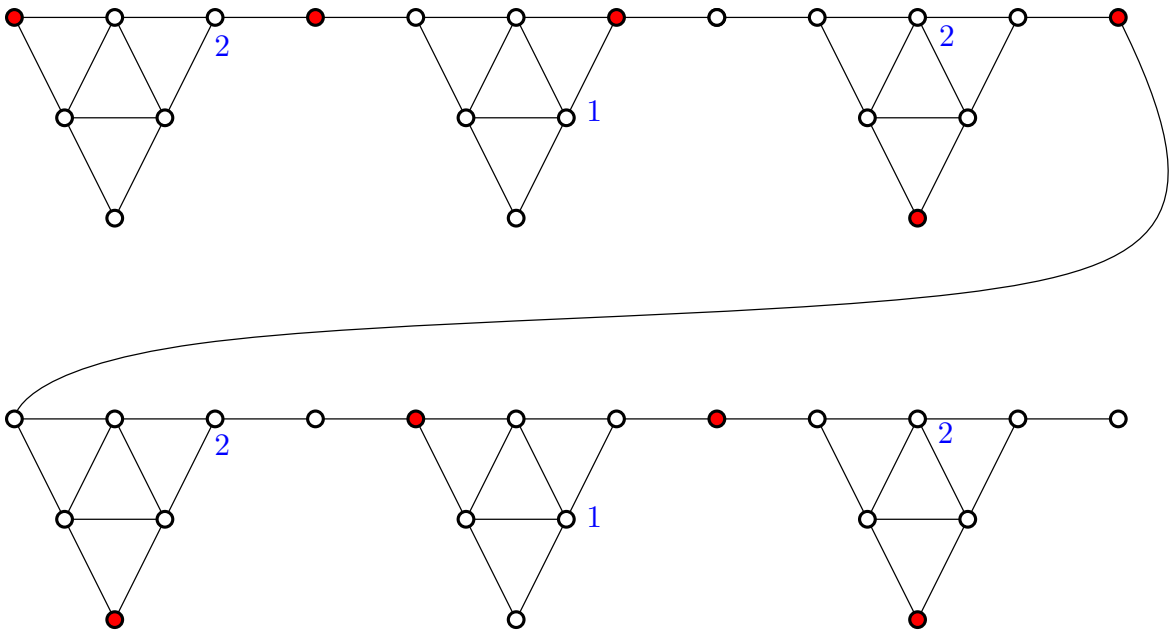
For connected chordal graphs G, $\frac{10}{9} \leq \lim_{mp(G)\to\infty} \sup \left\{ \frac{\gamma_b(G)}{mp(G)} \right\} \leq \frac{3}{2}$.

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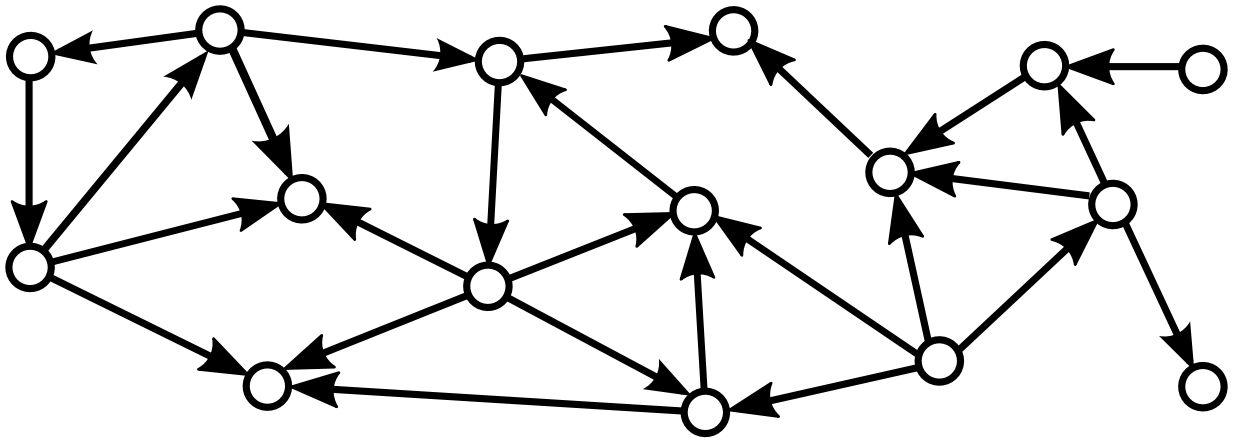
Lemma

$mp(G_{2k}) \leq 9k$ and $\gamma_b(G_{2k}) = 10k$.

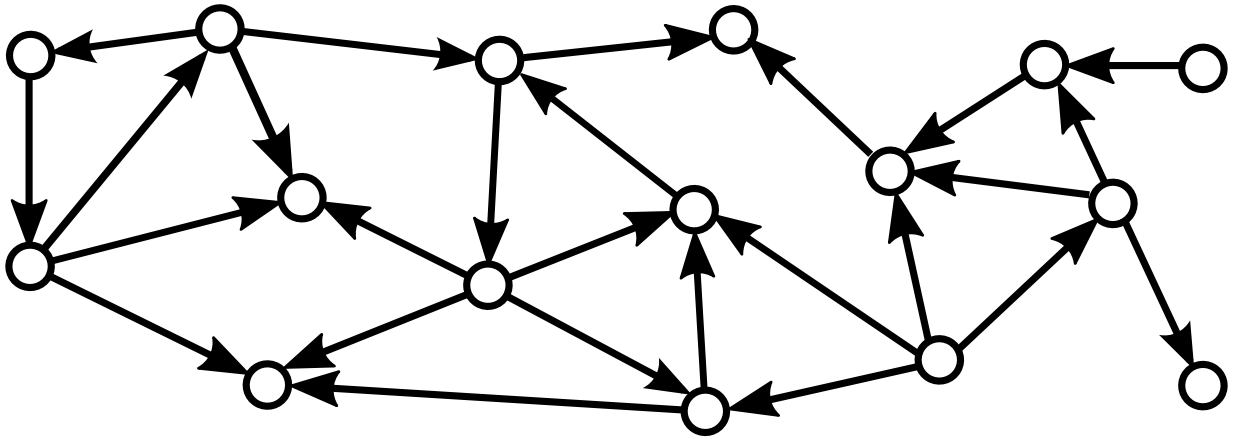


Complexity & algorithms for directed graphs

Broadcast domination in directed graphs

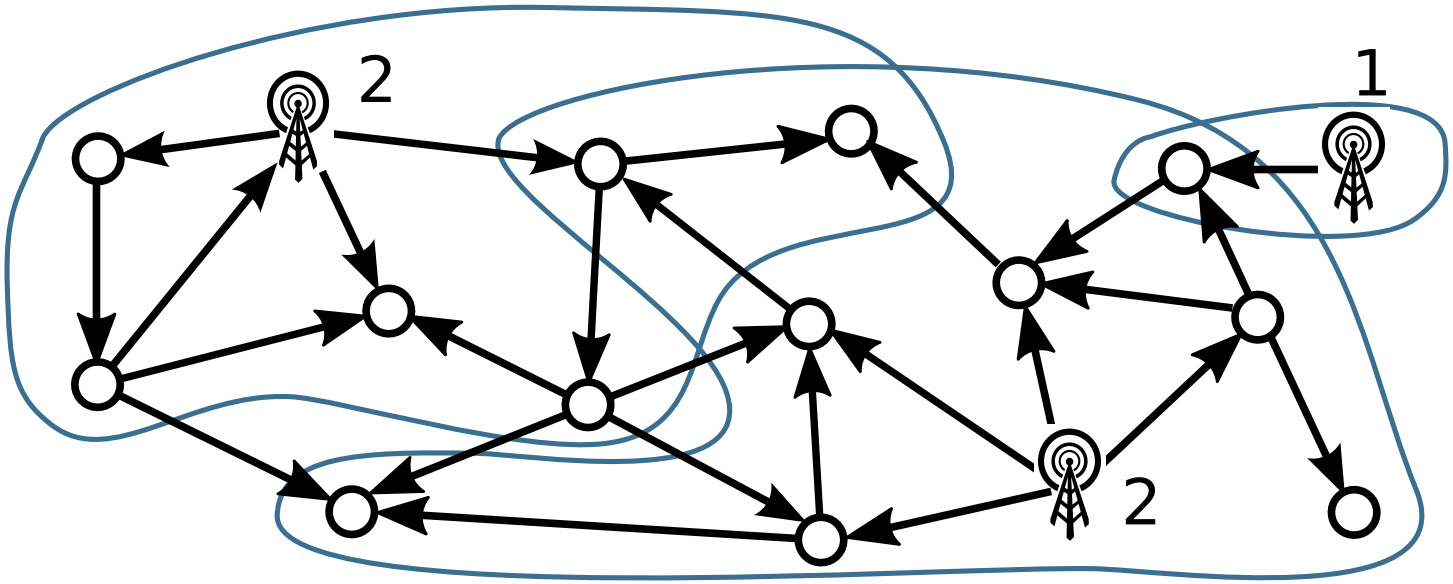


Broadcast domination in directed graphs



Note: an undirected graph can be seen as a symmetric directed graph!

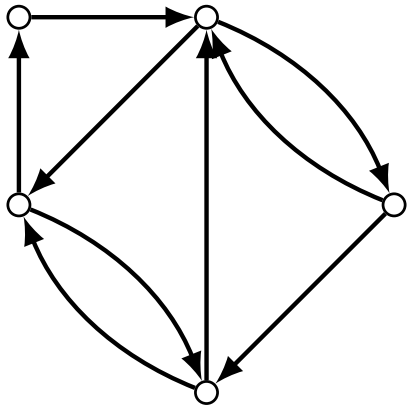
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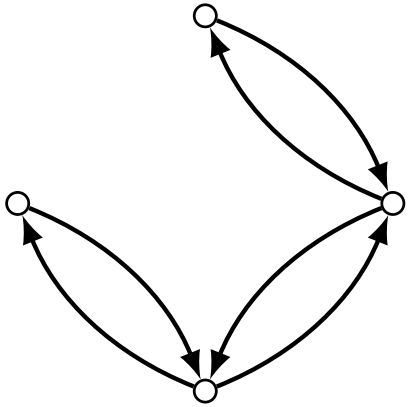
Broadcast domination for directed graphs:

A vertex v with $f(v) = r$ broadcasts to all vertices at directed distance up to r.

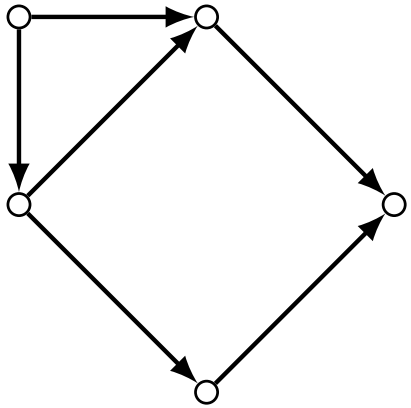
Some classes of directed graphs



General directed graphs (digraphs)



Symmetric digraphs = undirected graphs



Directed acyclic graphs (DAGs)

Algorithmic complexity of Broadcast domination

BROADCAST DOM
Input: A (directed) graph G, an integer k.
Question: Does G have a dominating broadcast of cost at most k?

Theorem (Heggernes-Lokshtanov, 2006)

BROADCAST DOM can be solved in polynomial time $O(n^6)$ for **undirected graphs**.

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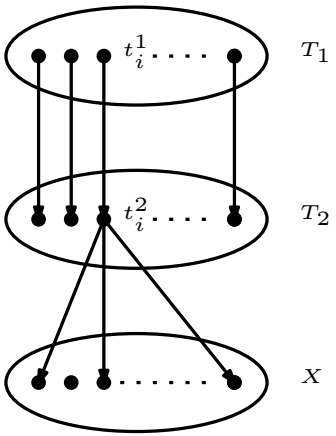
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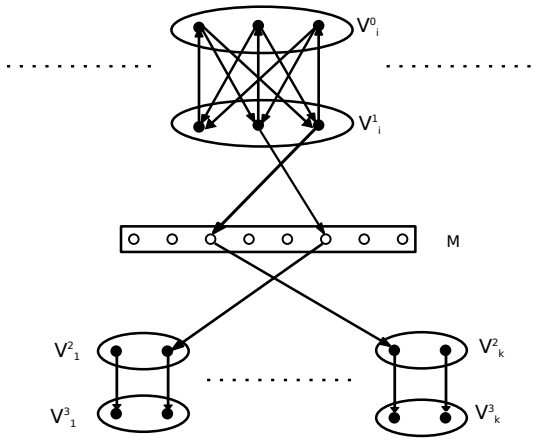
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Theorem (F., Gras, Perez, Sikora, 2020)

BROADCAST DOM is NP-hard and $W[2]$-hard: likely no algorithm of the form $f(k)poly(n)$, for any computable function f.

Proof: Reductions from SET COVER.





Algorithmic complexity for BROADCAST DOM (2)

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An **algorithmic problem** with input I and **parameter** k is **FPT** parameterized by k if it can be solved in time $f(k) \cdot |I|^{O(1)}$, where f is a computable function.

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Theorem (F., Gras, Perez, Sikora, 2020)

FPT $k^{O(k)}n$-time algorithm for BROADCAST DOM for directed acyclic graphs.

Proof: *Lemma: There always exists an optimal broadcast where each broadcasting vertex is covered only by itself.*
$\rightarrow$ iterative branching, starting from the sources: a previously uncovered vertex is either covered by itself, or by one of its broadcasting predecessors. □

Algorithmic complexity of Multipacking

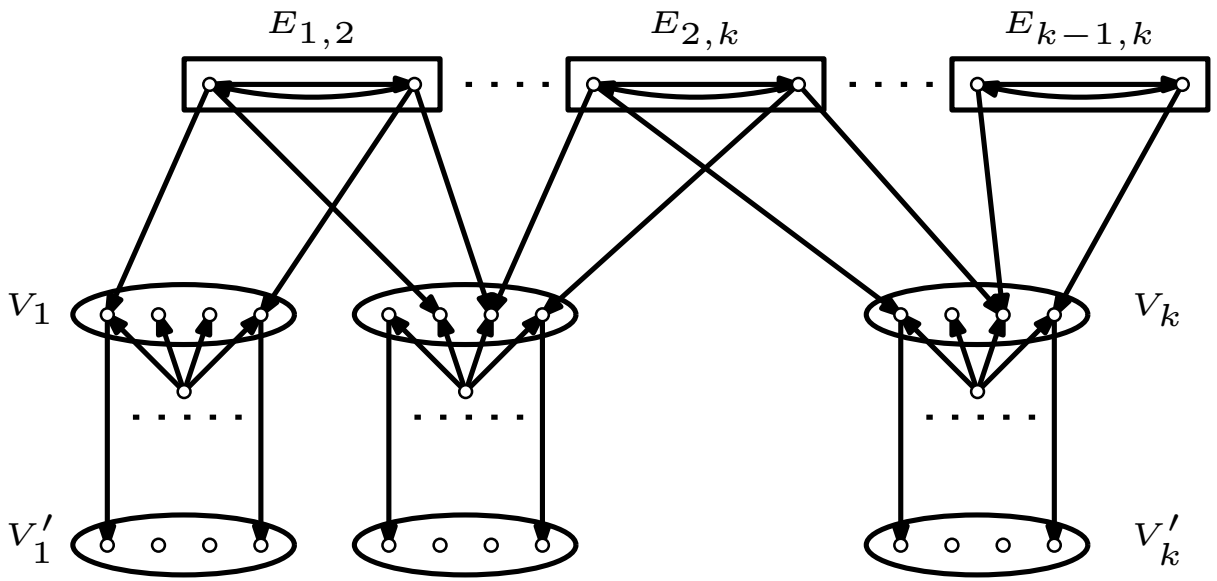
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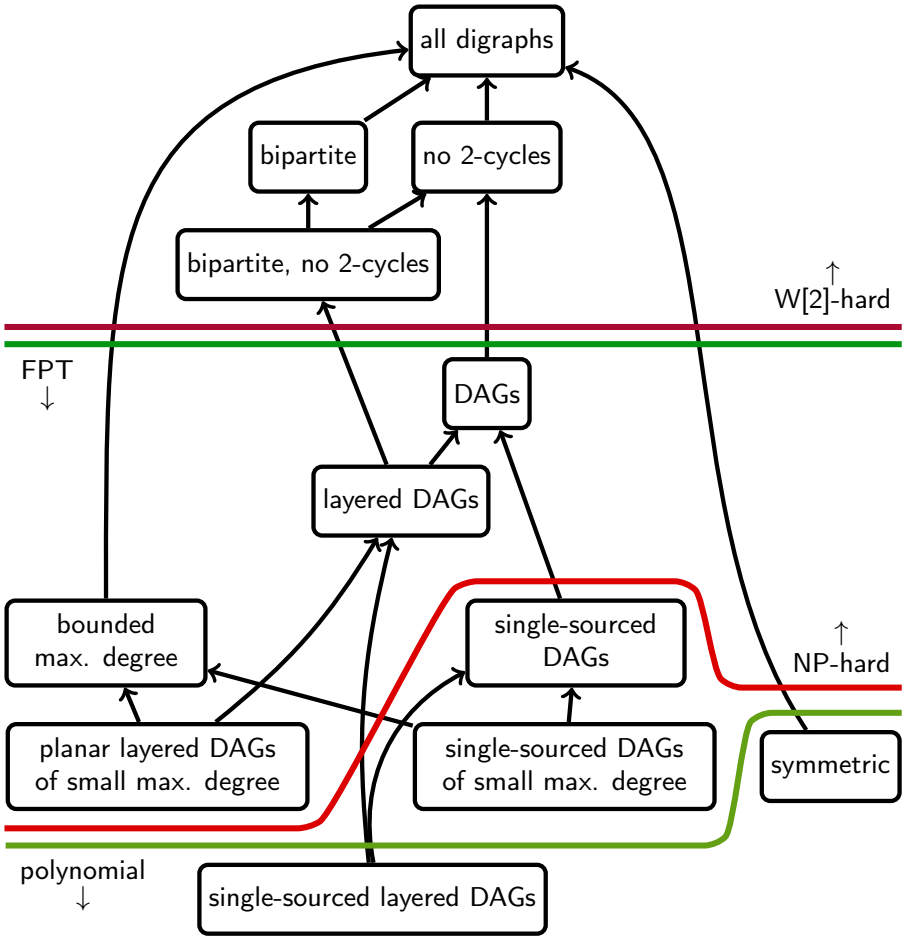
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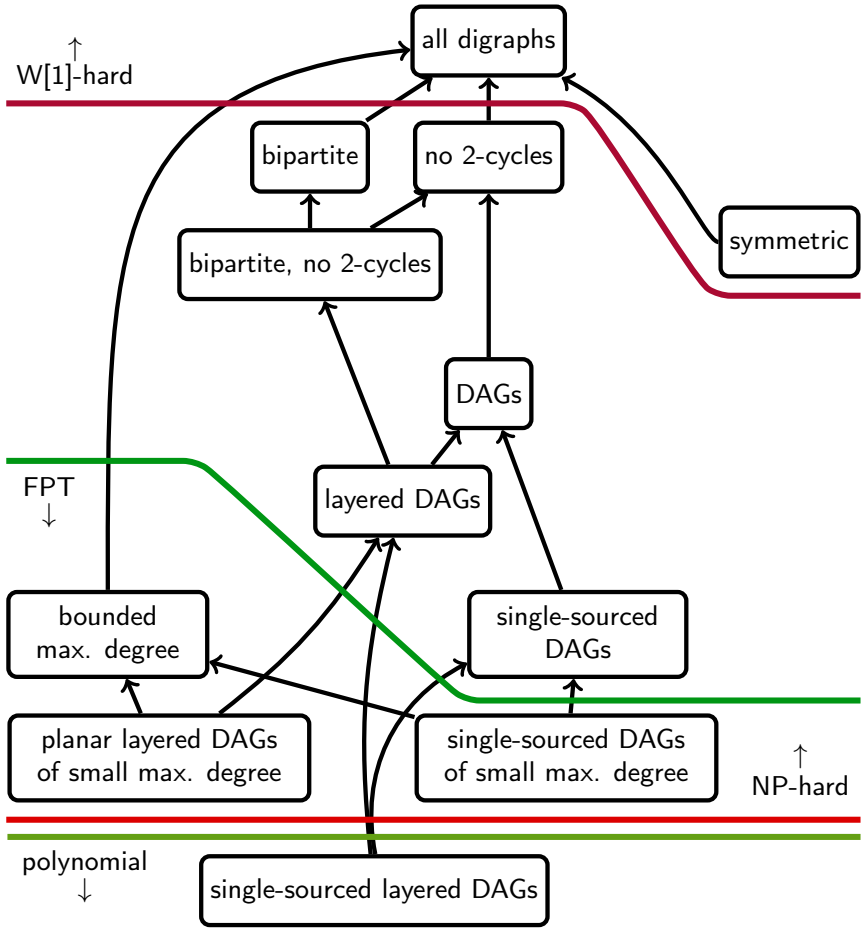
Proof: Reduction from INDEPENDENT SET.



Algorithmic complexity landscapes



BROADCAST DOM



MULTIPACKING

Takeaway: Problems tend to become more difficult on digraphs (even DAGs)

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Open problems:

Bounds:

- Conjecture: $\gamma_b(G) \leq 2mp(G)$ for any undirected graph G
- Better bounds for the multipacking number of the hypercube H_d?
- What is a tight bound for connected undirected chordal graphs? $\gamma_b(G) \leq \frac{10}{9}mp(G)$?

Algorithmic complexity:

- Is MULTIPACKING FPT (parameterized by solution size) for DAGs?
- Complexity of both problems on oriented trees?

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Thanks!