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Bounds and extremal graphs for monitoring edge-geodetic sets in graphs

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ABSTRACT

A monitoring edge-geodetic set, or simply an MEG-set, of a graph G is a vertex subset $M \subseteq V(G)$ such that given any edge e of G, e lies on every shortest u-v path of G, for some $u, v \in M$. The monitoring edge-geodetic number of G, denoted by $meg(G)$, is the minimum cardinality of such an MEG-set. This notion provides a graph theoretic model of the network monitoring problem.

In this article, we compare $meg(G)$ with some other graph theoretic parameters stemming from the network monitoring problem and provide examples of graphs having prescribed values for each of these parameters. We also characterize graphs G that have $V(G)$ as their minimum MEG-set, which settles an open problem due to Foucaud et al. (CALDAM 2023), and prove that some classes of graphs fall within this characterization. We also provide a general upper bound for $meg(G)$ for sparse graphs in terms of their girth, and later refine the upper bound using the chromatic number of G. We examine the change in $meg(G)$ with respect to two fundamental graph operations: clique-sum and subdivisions. In both cases, we provide a lower and an upper bound of the possible amount of changes and provide (almost) tight examples.

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1. Introduction

In the field of network monitoring, the networking components are monitored for faults and evaluated to maintain and optimize their availability. In order to detect failures, one of the popular methods for such monitoring processes involves setting up distance probes [3–5,16]. At any given time, a distance probe can measure the distance to any other probe in the network. If there is any failure in the connection, then the probes should be able to detect it as there would be a change in the distances between the components. Such networks can be modeled by graphs whose vertices represent the components and the edges represent the connections between them. We select a subset of vertices of the graph and call them probes. This concept of probes that can measure distances in graphs has many real-life applications, for example, it is useful in the fundamental task of routing [12,21], or using path-oriented tools to monitor IP networks [5], or problems concerning network verification [3,4,6]. Based on the requirements of the networks, there have been various related parameters that were defined on graphs in order to study the problem and come up with an effective solution. To name a few, we may mention the geodetic number [10,11,15,22], the edge-geodetic number [2,31], the strong edge-geodetic number [24,28], and the distance-edge monitoring number [16]. The focus of this article is on studying the

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monitoring edge-geodetic number of a graph, a concept related to the above ones and introduced in [20] (see also [14,23]).

1.1. Main definition

We deal with simple graphs, unless otherwise stated, and will use standard graph terminology and notation according to West [32].

Definition 1.1. Let G be a graph. A pair of vertices $u, v \in V(G)$, or any vertex subset $M \subseteq V(G)$ containing them, *monitors* an edge $e \in E(G)$ if e belongs to all shortest paths between u and v. A *monitoring edge-geodetic set* or simply, an *MEG-set* of G is a vertex subset $M \subseteq V(G)$ that monitors every edge of G. The *monitoring edge-geodetic number*, denoted by $meg(G)$, is the cardinality of a minimum MEG-set of G.

The notion of MEG-sets were introduced in [20], motivated by the above network monitoring application: the vertices of the MEG-set are distance-probes, that can measure the distance between each other. By the definition of an MEG-set, if some edge of the network fails, then there is at least one pair of probes whose distance changes, and so, the system of probes is able to detect the failure.

1.2. Previous works

In the paper that introduced MEG-sets [20], the value of $meg(G)$, when G belongs to some basic graph families, were determined. These graph families include the family of trees, unicyclic graphs, complete graphs, complete multipartite graphs, rectangular grids, and hypercubes. The authors of [20] also showed a relation with the feedback edge set number f and the number ℓ of leaves of the graph: $meg(G) \leq 9f + \ell - 8$ (for $f \geq 2$). This was improved to $meg(G) \leq 3f + \ell + 1$ (which is tight up to an additive factor of 1) in [8]. In [23], $meg(G)$ when G is the result of certain types of graph products was studied, in particular, Cartesian products and strong products. The case when G is a corona product was studied in [14]. In [23], it was shown that determining $meg(G)$ is an NP-complete problem. This was refined to hold for graphs of maximum degree at most 9 in [14].

1.3. Related parameters

There are some network monitoring based graph parameters studied in the literature, whose definitions are relevant in our context since they are related to $meg(\cdot)$ (see Section 3). We list them below.

- A *geodetic set* of a graph G is a vertex subset $S \subseteq V(G)$ such that every vertex of G lies on some shortest path between two vertices $u, v \in S$. The *geodetic number*, denoted by $g(G)$, is the minimum $|S|$, where S is a geodetic set of G. The concept was introduced by Harary et al. in 1993 [22] and received considerable attention since then, both from the structural side [10,11,15] and from the algorithmic side [7,26].
- An *edge-geodetic set* of a graph G is a vertex subset $S \subseteq V(G)$ such that every edge of G lies on some shortest path between two vertices $u, v \in S$. The *edge-geodetic number*, denoted by $eg(G)$, is the minimum $|S|$, where S is an edge-geodetic set of G. This was introduced in 2003 by Atici et al. [2] and further studied from the structural angle [31] as well as algorithmic angle [9,13].
- A *strong edge-geodetic set* of a graph G is a vertex subset $S \subseteq V(G)$ and an assignment of a particular shortest u-v path P_{uv} to each pair of distinct vertices $u, v \in S$ such that every edge of G lies on P_{uv} for some $u, v \in S$. The *strong edge-geodetic number*, denoted by $seg(G)$, is the minimum $|S|$, where S is a strong edge-geodetic set of G. This concept was introduced in 2017 by Manuel et al. [28]. See [24] for some structural studies, and [13] for some algorithmic results.
- A *distance edge-monitoring set* of a graph G is a vertex subset $S \subseteq V(G)$ such that for every edge e of G, there are two vertices $x \in S$ and $y \in G$ for which e lies on all shortest paths between x and y. Let $dem(G)$ be the smallest size of a distance edge-monitoring set of G. This concept was introduced in 2020 by Foucaud et al. [16]. Refer to [16,17] for both structural and algorithmic results for this parameter.

Apart from the above, some well-known graph parameters too have relation with $meg(\cdot)$. See Fig. 1 (reproduced from [14]) for a diagram showing the relations between these parameters, and others.

1.4. Our results, and organization of the paper

- In Section 3, we recall that $g(G) \leq eg(G) \leq seg(G) \leq meg(G)$ for any connected non-trivial graph G [16]. To complement our findings, we construct examples of graphs $G_{a,b,c,d}$ having $g(G) = a, eg(G) = b, seg(G) = c$, and $meg(G) = d$, where $a \leq b \leq c \leq d$ (and a few additional constraints). We also show that $dem(G) < meg(G)$, and construct examples of graphs $G_{p,q}$ having $dem(G) = p$ and $meg(G) = q$, where $1 \leq p < q$.

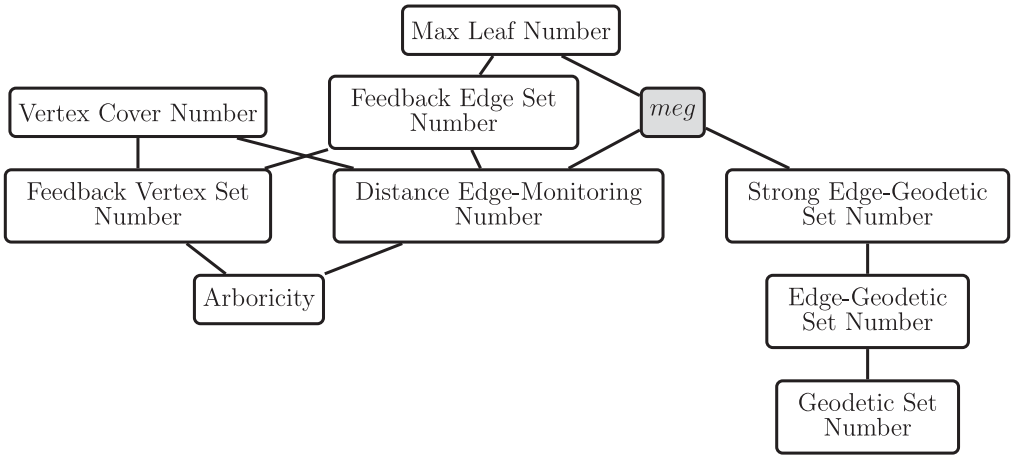


Fig. 1. Relations between the parameter *meg* and other structural parameters in graphs (with no isolated vertices). For the relationships of distance edge-monitoring sets, see [16]. Edges between parameters indicate that the value of the bottom parameter is upper-bounded by a function of the top parameter.
Source: Figure reproduced from [14].

- In Section 4 we prove a necessary and sufficient condition for a vertex to be part of every MEG-set of G. As a corollary, we characterize graphs G having $meg(G) = |V(G)|$, which answers an open question posed by Foucaud, Krishna and Ramasubramony Sulochana [20]. It was worth recalling that this open question was addressed (not solved) by Haslegrave [23]. These results can act as fundamental tools in the study of MEG-set and related problems.
- In Section 5, we will explore the impact of the tools built in Section 4 to expand the list of known graphs whose minimum MEG-set is the entire vertex set (defined as *MEG-extremal* graphs in Section 5). To be precise, we completely characterize $meg(G)$ when G is a cograph, a block graph, a well-partitioned chordal graph, or a proper interval graph, and observe that the 2-connected graphs from these families are MEG-extremal. Moreover, if G is an MEG-extremal graph, then the Cartesian and the strong products of G with another graph is also MEG-extremal. The former was already known due to Haslegrave [23], we however provide a shorter proof. Furthermore, we also show that the tensor product of two MEG-extremal graphs is MEG-extremal.
- In Section 6, we show that $meg(G) \leq \frac{4|V(G)|}{g-3}$, where $g \geq 4$ denotes the girth of G. As a consequence, we show that any vertex cover is an MEG-set for a graph having girth at least 5. Later we also prove a refinement of the main result of this section using the chromatic number of G.
- In Section 7, we explore the effect of two fundamental graph operations, namely, the clique-sum and the subdivision on $meg(G)$. We show that $meg(G)$ is both lower and upper bounded by functions related to the operations and that the bounds are almost tight.
- In Section 8, we share our concluding remarks which also contain suggestions for future works in this direction.

Note: A preliminary version of this article was published in the proceedings of the CALDAM 2024 conference [18]. This version contains an extended introduction, proofs missing from the conference paper, extensions of the results, additional results, and a corrected version of Theorem 6.1. However, the algorithmic results from [18] are not contained in the current article, but presented in a separate extended version [19].

2. Preliminary results

Right before we proceed with our contributions, let us recall a few results from [20] since we think they will give a proper initial insight into the notion of MEG-set, and will also be used on several occasions throughout this work.

A vertex is *simplicial* if its neighborhood forms a clique. A vertex v of a graph G is *pendent vertex* if it is of degree 1.

Lemma 2.1 (*Lemma 2.1 of [20]*)**.** *In a connected graph G with at least one edge, any simplicial vertex belongs to any edge-geodetic set and thus, to any MEG-set of G.*

Lemma 2.1 implies that all pendent vertices of a graph G are part of any MEG-set of G.

Two distinct vertices u and v are said to be *open twins* if $N(u) = N(v)$ and *closed twins* if $N[u] = N[v]$. If u, v are either closed or open twins, then we simply call them *twins*.

Lemma 2.2 (*Lemma 2.2 of [20]*)**.** *If two vertices are twins of degree at least 1 in a graph G, then they must belong to any MEG-set of G.*

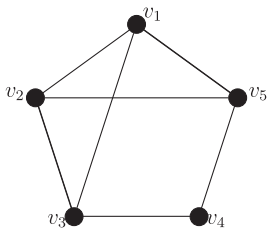


Fig. 2. A graph G with $2 = g(G) < 3 = eg(G) < 4 = seg(G) < 5 = meg(G)$. Note that, a minimum geodetic set of G is $\{v_3, v_5\}$, a minimum edge-geodetic set of G is $\{v_1, v_2, v_4\}$, a minimum strong edge-geodetic set of G is $\{v_1, v_2, v_3, v_4\}$ (the assigned shortest paths between a pair of adjacent vertices is the edge between them, and between a pair of non-adjacent vertices is the 2-path through v_5) and a minimum MEG-set of G is $\{v_1, v_2, v_3, v_4, v_5\}$.

3. Relation between network monitoring parameters

In this section, we find examples of graphs having prescribed values of network monitoring parameters.

3.1. Relation with geodetic parameters

We start with proving an if and only if condition for which a strong edge-geodetic set is also an MEG-set.

Proposition 3.1. *Let $M \subseteq V(G)$ be a vertex subset of a graph G and let f be an assignment of a shortest path to each pair of vertices of M. Then M is an MEG-set if and only if M, along with the assignment f, is a strong edge-geodetic set for any choice of f.*

Proof. Let M be an MEG-set of G. As all the edges of G are monitored by M, for each edge e there exists a pair of vertices $u, v \in M$ such that e lies on every shortest path between u and v. Therefore, if we arbitrarily assign any shortest path to each pair of vertices of M, the set M along with this so-obtained assignment of shortest paths is a strong edge-geodetic set.

On the other hand, let M, along with the assignment f, be a strong edge-geodetic set for any choice of f. Now if some edge e of G is not monitored, then it is possible to find an assignment of a path P_{uv} to each pair u, v of vertices of M such that e is not contained in P_{uv} for any u, v. This is a contradiction to our assumption. Thus, M must be an MEG-set. □

It is known [20] (see also Fig. 1) that the following relation holds among the geodetic parameters:

$$g(G) \leq eg(G) \leq seg(G) \leq meg(G).$$

Moreover, we know that for any connected graph $G \neq K_1$, $g(G) \geq 2$. Therefore, it is natural to ask the question, given four positive integers a, b, c, d satisfying $2 \leq a \leq b \leq c \leq d$, is there a graph $G_{a,b,c,d}$ such that we have $g(G_{a,b,c,d}) = a$, $eg(G_{a,b,c,d}) = b$, $seg(G_{a,b,c,d}) = c$, and $meg(G_{a,b,c,d}) = d$? The following remark captures some basic version of this answer. Later, in Theorem 3.3 we provide a positive answer to this question except for some specific cases.

Remark 3.2. Notice that, for any complete graph K_n on $n \geq 2$ vertices, the values of all the parameters are equal to n. That is, equality holds in all the inequalities of the above chain of inequations.

On the other hand, Fig. 2 gives an example of a graph where all the inequalities of the above chain of inequations are strict. To be specific, in this particular example, the values of the parameters increase exactly by one in each step.

Theorem 3.3. *For any positive integers $4 \leq a \leq b \leq c \leq d$, except for $d = c+1$ when $b = a+1$ and for $d = c$, $d = c+2$ when $b \neq a+1$, there exists a connected graph $G_{a,b,c,d}$ with $g(G_{a,b,c,d}) = a$, $eg(G_{a,b,c,d}) = b$, $seg(G_{a,b,c,d}) = c$, and $meg(G_{a,b,c,d}) = d$.*

Proof. We begin the proof by describing the construction of $G_{a,b,c,d}$.

Construction of $G_{a,b,c,d}$: In the first phase of the construction, we start with a $K_{2,2+b-a}$, where the partite set of size two has the vertices x_1 and y, and the partite set of size $(2 + b - a)$ has the vertices z_1, z_2 and $w_1, w_2, \ldots, w_{b-a}$. Moreover, we add some edges in such a way that the set

$$W = \{w_1, w_2, \ldots, w_{b-a}\}$$

becomes a clique. We also add the edge z_2w_1 only if $b = a + 1$.

In the second phase of the construction, we add $(c - b + 1)$ parallel edges between the vertices z_1 and z_2. After that we subdivide (once) each of the above-mentioned parallel edges and name the degree two vertices created due to the

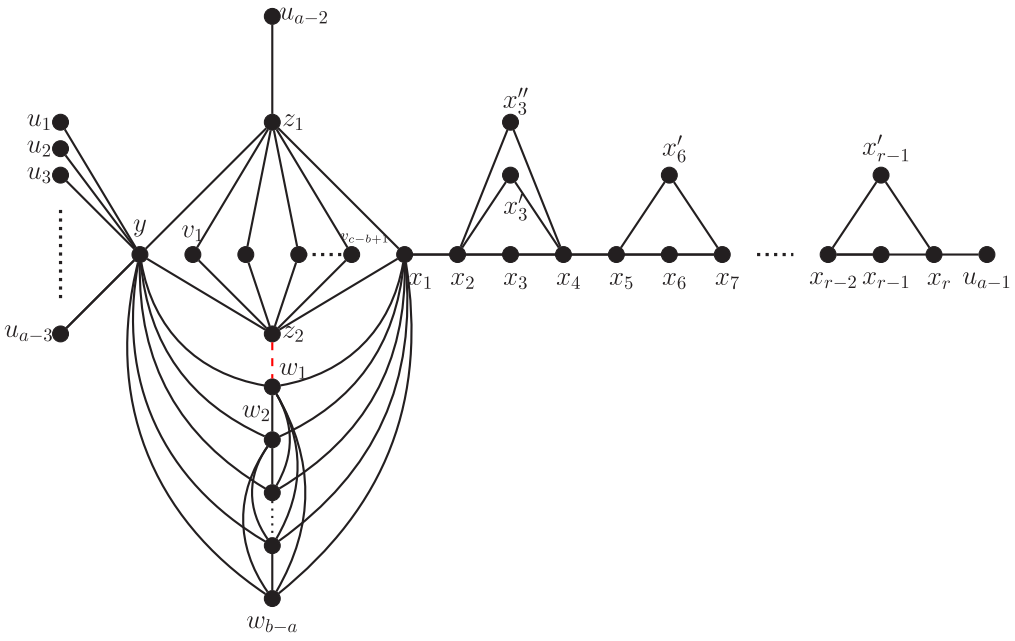


Fig. 3. The structure of $G_{a,b,c,d}$. The red dashed edge between z_2 and w_1 represents an edge/non-edge depending on the values of a and b.

subdivisions as $v_1, v_2, \ldots, v_{c-b+1}$. The set of vertices created by the subdivisions is given by

$$V = \{v_1, v_2, \ldots, v_{c-b+1}\}.$$

In the third phase of the construction, we add $(a-3)$ pendent neighbors $u_1, u_2, \ldots, u_{a-3}$ to y, and one pendent neighbor u_{a-2} to z_1. Moreover, we attach a long path $x_1x_2 \ldots x_r u_{a-1}$ with the vertex x_1, where u_{a-1} is a pendent vertex and $r = 3\left\lfloor \frac{d-c}{2} \right\rfloor + 1$. Next we will add a open twin x'_{3i} to the vertices of the form x_{3i} for all $i \in \left\{1, 2, \ldots, \left\lfloor \frac{d-c}{2} \right\rfloor\right\}$. Additionally, if $(d-c)$ is odd, then we will add another twin x''_3 to the vertex x_3. For convenience, let

$$U = \{u_1, u_2, \ldots, u_{a-1}\}$$

denote the set of all pendent vertices and

$$X = \left\{x_{3i}, x'_{3i} \mid i = 1, 2, \ldots, \left\lfloor \frac{d-c}{2} \right\rfloor\right\} \cup \{x''_3\}$$

denote the set of all degree 2 vertices on the paths connecting x_1 and u_{a-1}.

Note that, x''_3 exists in X if and only if $(d-c)$ is odd. Similarly, when $b \neq a+1$, we maintain the construction same as above except for a small change where we take $r = 3\lfloor \frac{d-c-1}{2} \rfloor + 1$, and x''_3 exists in X if and only if $(d-c)$ is even. This completes the description of the construction of the graph $G_{a,b,c,d}$ (see Fig. 3 for a pictorial reference).

As U is the set of all pendents, we know that it will be part of any geodetic set, edge-geodetic set, strong edge-geodetic set, and monitoring edge-geodetic set by Lemma 2.1. However, the vertices of U cannot cover the vertices of V using shortest paths between the vertices of U. Therefore, we need at least one more vertex to form a geodetic set. As $U \cup \{z_2\}$ is a geodetic set of G, we can infer that

$$g(G_{a,b,c,d}) = |U| + |\{z_2\}| = (a-1) + 1 = a.$$

Next, observe that the vertices of U are not able to cover any edge of the clique W using shortest paths between the vertices of U. Moreover, the only way to monitor those edges is by taking W in our edge-geodetic set. Similarly none of the edges of the form v_iz_j are monitored with only vertices from U and W. On the other hand, $U \cup W \cup \{z_2\}$ is an edge-geodetic set when $b \geq a+2$. Note that when $b = a+1$, z_2 and w_1 are adjacent. In this case too, we need the vertices from the set U and the vertex z_2 to monitor all the edges except z_2w_1. Thus, we take $W = \{w_1\}$ in our edge-geodetic set. Therefore,

$$eg(G_{a,b,c,d}) = |U| + |W| + |\{z_2\}| = (a-1) + (b-a) + 1 = b.$$

We know that the vertices of U are in any strong edge-geodetic set. Moreover, note that, the vertices of W must be in any strong edge-geodetic set to cover the edges of the clique W if $b \neq a+1$, or to monitor w_1z_2 if $b = a+1$. Now, let us

see how we can cover the edges of the $(c-b+1)$ 2-paths between z_1, z_2, having the vertices of V as their internal vertex. We can assign one shortest path involving one of v_i's between u_{a-2} and z_2, but for the remaining edges, we must take $c-b$ vertices in our strong edge-geodetic set to cover those incident edges. Moreover, the set $U \cup W \cup (V \setminus \{v_1\}) \cup \{z_2\}$ is indeed a strong edge-geodetic set, as every other edge can be monitored by assigning a shortest path between u_1 to u_{a-1} using all x_{3i}, a shortest path from u_{a-2} to u_{a-1} using all x'_{3i} and a shortest path from u_{a-2} to u_{a-1} using x''_{3i} (if it exists). Thus,

$$seg(G_{a,b,c,d}) = |U| + |W| + |V-1| + |\{z_2\}| = (a-1) + (b-a) + (c-b) + 1 = c.$$

Finally, for any MEG-set, we have to take the vertices of U (as they are pendents), the vertices of W (to monitor the edges of the clique W), the vertices of X (as they are twins, see Lemma 2.2). However, even with these vertices, we cannot monitor the edges of the $(c-b+1)$ 2-paths between z_1, z_2, having the vertices of V as their internal vertex. To do so, we have to take all vertices of V in our MEG-set. These vertices also help monitor the edge z_2w_1. Hence, the set $U \cup W \cup V \cup X$ is an MEG-set when $b = a+1$, Hence,

$$meg(G_{a,b,c,d}) = |U| + |W| + |V| + |X| = (a-1) + 1 + (c-b+1) + 2\left\lfloor \frac{d-c}{2} \right\rfloor + \epsilon = d,$$

where $\epsilon = 0$ (resp., 1) if $(d-c)$ is (even (resp., odd), and $d \neq c+1$. Observe that, when we have $b \neq a+1$, that is, when $b = a$ or when $b \geq a+2$, we do not have the edge z_2w_1, hence, we have to take z_2 in our MEG-set to monitor the edges of the type v_iz_j. In this case, the set $U \cup V \cup W \cup X \cup \{z_2\}$ is an MEG-set. Hence,

$$\begin{aligned} meg(G_{a,b,c,d}) &= |U| + |W| + |V| + |X| + |\{z_2\}| \\ &= (a-1) + (b-a) + (c-b+1) + 2\left\lfloor \frac{d-c-1}{2} \right\rfloor + \epsilon + 1 \\ &= d, \end{aligned}$$

where $\epsilon = 0$ (resp., 1) if $(d-c)$ is odd (resp., even), and $d \neq c, c+2$. This completes the proof. □

3.2. Relation with distance-edge monitoring sets

Observe that the concept of MEG-set is closely related to that of distance-edge monitoring set. Notice that, MEG-sets are particular types of distance-edge monitoring sets (see the definition of distance-edge monitoring set in Section 1.1), and hence we have $dem(G) \leq meg(G)$. However, we show that this inequality is strict.

Lemma 3.4. *For any graph G having at least one edge, we have $dem(G) < meg(G)$.*

Proof. Let M be a minimum MEG-set of G. We claim that $M \setminus \{x\}$ for any $x \in M$ is a distance-edge monitoring set of G. This is correct as any edge of G is monitored by two vertices of M, and at least one of them must belong to $M \setminus \{x\}$. Thus $dem(G) \leq |M| - 1 < meg(G)$. □

In view of the above lemma, given any two positive integers $p < q$, we prove the existence of a graph $G_{p,q}$ satisfying $dem(G_{p,q}) = p$ and $meg(G_{p,q}) = q$. Note that it is possible to have $dem(G) = 1$ for a graph G, (for example paths [16]). Whereas $meg(G) \geq 2$ for all graphs G having at least one edge.

Theorem 3.5. *For any positive integers $1 \leq p < q$, there exists a connected graph $G_{p,q}$ with $dem(G_{p,q}) = p$ and $meg(G_{p,q}) = q$.*

Proof. We provide two separate constructions for the proof. One for the case when $p = 1$, and the other for the case when $p \geq 2$.

Case 1: When $1 = p < q$: Consider the star graph $K_{1,q}$ with partite sets $\{y\}$ and $\{u_1, u_2, \ldots, u_q\}$. Notice that $\{y\}$ is a minimum distance-edge monitoring set of G [16]. Thus $dem(G_{p,q}) = dem(K_{1,q}) = 1 = p$. On the other hand, the set of all leaves, that is, $\{u_1, u_2, \ldots, u_q\}$ is a minimal MEG-set [20]. Hence $meg(G_{p,q}) = meg(K_{1,q}) = q$.

Case 2: When $2 \leq p < q$: Let $u_1, u_2, \ldots, u_{p+1}$ be the vertices of a complete graph K_{p+1}. We add $(q-p)$ pendent vertices in the neighborhood of u_1. The so-obtained graph is $G_{p,q}$. We know that a minimum distance-edge monitoring set of $G_{p,q}$ is $\{u_2, u_3, \ldots, u_{p+1}\}$ [16], and a minimum MEG-set of this graph is $\{v_1, v_2, \ldots, v_{q-p}, u_2, u_3, \ldots, u_{p+1}\}$ [20]. Hence $dem(G_{p,q}) = p$ and $meg(G_{p,q}) = q$. □

4. Conditions for a vertex being in all or no optimal MEG-sets

In their introductory paper on monitoring edge-geodetic sets, Foucaud, Krishna and Ramasubramony Sulochana [20] asked to characterize the graphs G having $meg(G) = |V(G)|$. We provide a definitive answer to their question, and to this end, we give a necessary and sufficient condition for a vertex to be in any MEG-set of a graph. An *induced 2-path* of a graph G is an ordered set of three vertices u, v, x such that u, x are adjacent to v while u is not adjacent to x.

Theorem 4.1. *Let G be a graph. A vertex $v \in V(G)$ is in every MEG-set of G if and only if there exists $u \in N(v)$ such that for any vertex $x \in N(v)$, any induced 2-path uvx is part of a 4-cycle.*

Proof. For the necessary condition, let us assume that a vertex $v \in V(G)$ is in every MEG-set of G. We have to prove that there exists $u \in N(v)$, such that any induced 2-path uvx is part of a 4-cycle. We prove it by contradiction.

Suppose for every $u \in N(v)$, there exists an induced 2-path uvx such that uvx is not part of a 4-cycle. As uvx is an induced 2-path, observe that u and x are not adjacent, also as uvx is not part of a 4-cycle, we get, $d(u, x) = 2$ and the only shortest path between u and x is via v. This implies that, if we take u and x in our MEG-set S, then xv and uv are monitored. Hence, all the neighbors of v can monitor all the edges incident to v. Therefore, in particular, $V(G) \setminus \{v\}$ is an MEG-set of G. This is a contradiction to the fact that v is in every MEG-set of G. Thus, the necessary condition for a vertex v to be part of every MEG-set of G is proved according to the statement.

For the sufficient condition, let us assume that for some vertex v of G, there exists $u \in N(v)$ such that any induced 2-path uvx is part of a 4-cycle. We need to prove that v is in every MEG-set. Thus, it is enough to show that $S = V(G) \setminus \{v\}$ is not an MEG-set of G. Therefore, we would like to find an edge which is not monitored by the vertices of S.

We first observe that if there does not exist any induced 2-path of the form uvx, then v must be a simplicial vertex, and thus we know that v belongs to every MEG-set of G by Lemma 2.1. On the other hand, if there exists an induced 2-path of the form uvx, then according to our assumption, there exists a 4-cycle of the form $uvxwu$. Assume there exist a, b two vertices of G monitoring the edge uv (where a and u could be the same vertex). We consider a shortest path from a to b that we denote P, where $P = a \dots uvx \dots b$ (where x and b could be the same vertex). Note that $P' = a \dots uwx \dots b$ is another shortest path from a to b. Hence a, b do not monitor the edge uv, which implies that v has to be part of every MEG-set. This concludes the proof. □

An immediate corollary characterizes all graphs G with $meg(G) = |V(G)|$. This answers an open question asked in [20].

Corollary 4.2. *Let G be a graph of order n. Then, $meg(G) = n$ if and only if for every $v \in V(G)$, there exists $u \in N(v)$ such that any induced 2-path uvx is part of a 4-cycle.*

Proof. The proof directly follows from Theorem 4.1, since if there exists $u \in V(G)$ that does not fulfill the condition, then $V(G) \setminus \{u\}$ would be an MEG-set. □

A *universal* vertex u of a graph G is a vertex that is adjacent to every vertex except itself in G.

Corollary 4.3. *Let G be a graph with a universal vertex u. Then any vertex $v \neq u$ is in every MEG-set of G. In particular, if G has n vertices, then $meg(G) \geq n - 1$.*

Proof. Since u is a universal vertex, for any $v, x \in V(G)$, uvx is never an induced 2-path as u, x are adjacent. Therefore, every vertex $v \neq u$ satisfies the necessary condition for being part of every MEG-set of G according to Theorem 4.1. □

The *girth* of a graph is the length of its smallest cycle.

Corollary 4.4. *Let $G \neq K_1, K_2$ be a connected graph with girth at least 5. If G has n vertices, then $meg(G) \leq n - 1$.*

Proof. Notice that, if a vertex v of G satisfies the condition of Theorem 4.1, then either v has to be part of a 3-cycle, or a 4-cycle, or v is a pendent vertex. As G has girth at least 5, it cannot contain any 3-cycle or 4-cycle. Thus, if G has $meg(G) = n$, then all vertices of G are pendent vertices. This is only possible when $G = K_2$, which is not possible due to our assumption. Thus, not every vertex of G can satisfy the condition of Theorem 4.1. Hence, $meg(G) \leq n - 1$. □

Remark 4.5. Note that this bound is optimal for star as the girth of a star is infinite.

5. MEG-extremal graphs

An *MEG-extremal graph* G is a graph having $meg(G) = |V(G)|$. In [14,20], the following families of *MEG-extremal* graphs were presented: complete graphs, complete multipartite graphs (except stars), and hypercubes. In [23], the author showed that for any MEG-extremal graph G and any graph H, both the Cartesian product and the strong product of G and H are MEG-extremal graphs.

In this section we extend the same lines of work, primarily using the tools built in Section 4. We organize the section in two subsections. In the first subsection we provide complete characterization of $meg(G)$ when G belongs to the families of cographs, block graphs, well-partitioned chordal graphs, split graphs, and proper interval graphs. Using our characterization we then conclude that the 2-connected graphs in the above mentioned families are MEG-extremal. In the second subsection we present a short proof of a result from [23] where we show that the Cartesian product and the strong product of an MEG-extremal graph with any other graph is also MEG-extremal. Moreover, we show that the tensor product of two MEG-extremal graphs is always MEG-extremal.

5.1. Graph families

We start by recalling a useful lemma from [20] on cut-vertices.

Lemma 5.1 (*Lemma 2.3 of [20]*)**.** *Let G be a graph, and u be a cut-vertex of G. Then u is never part of any minimal MEG-set of G.*

Remark 5.2. Due to Lemma 5.1, for a connected graph $G \neq K_2$, every vertex of degree 1 must be part of any MEG-set, and its neighbor cannot be a part of any minimum MEG-set. Therefore, if $meg(G) = |V(G)|$, then G must have minimum degree at least 2.

A *cograph* G is a graph which does not contain any induced path on 4 vertices, that is, an induced P_4. A *complete join* of two graphs G_1 and G_2 is the graph G obtained by adding an edge between each vertex of G_1 and each vertex of G_2.

Theorem 5.3. *Let G be a connected cograph on n vertices. Then either G has a cut-vertex and $meg(G) = n - 1$, or else $meg(G) = n$.*

Proof. First assume that G has a cut vertex u. Note that, $G - u$ has at least two components. Observe that u must be a universal vertex as otherwise it is possible to find an induced P_4. Thus, u must be the only cut-vertex of G. Hence, by Corollary 4.3 and Lemma 5.1 $meg(G) = n - 1$.

A 2-connected cograph is of diameter at most 2. Take any vertex v and any of its neighbors u. If there is an induced 2-path of the form uvx in G, then by 2-connectedness there exists an induced path P connecting u and x which is internally vertex disjoint with uvx. However, P must have less than 4 vertices, and thus, either P is an edge, or an induced 2-path. Therefore, v must be part of every MEG-set of G according to Theorem 4.1. □

A *block graph* G is a graph all of whose 2-connected induced subgraphs are cliques.

Theorem 5.4. *Let G be a block graph with k cut-vertices. If G has n vertices, then $meg(G) = n - k$.*

Proof. In a block graph, a vertex is either simplicial or a cut-vertex. Hence, the result follows from Lemmas 2.1 and 5.1. □

A graph is *chordal* if it does not contain any induced cycle of length at least 4. A graph G is a *well-partitioned chordal graph* [1] if its vertex set can be partitioned into cliques $C_1, C_2, \ldots, C_\ell$ that satisfies the following properties:

(i) The cliques $C_1, C_2, \ldots, C_\ell$ are called bags.
(ii) Two bags C_i and C_j are non-adjacent if there is no edge with one end point in C_i and the other in C_j.
(iii) Two bags C_i and C_j are adjacent if the edges between C_i and C_j induce a complete bipartite graph. The complete bipartite graph does not need to span the two bags.
(iv) Two bags are either adjacent or non-adjacent and the graph obtained by considering bags as vertices with the above-mentioned adjacency rule is a forest.

Theorem 5.5. *Let G be a well-partitioned chordal graph with k cut-vertices. If G has n vertices, then $meg(G) = n - k$.*

Proof. Let v be a vertex of G which is neither a cut-vertex nor a simplicial vertex. It is enough to show that there exists a neighbor u of v such that any induced 2-path of the form uvx is part of a 4-cycle due to Lemmas 2.1 and 5.1. Assume that v belongs to the bag C_i.

Note that, as v is not a simplicial vertex, then v is adjacent to a vertex in a different bag C_k. We also assumed v was not a cut-vertex, hence there exists an edge incident to a vertex of C_k and a vertex of C_i distinct from v, hence C_i must contain at least one more vertex in it, say u. Suppose uvx is an induced 2-path. As u and x are non-adjacent, x must belong to a different bag C_j. Moreover, as v is not a cut vertex, there must be another vertex w in C_i which has an edge with a vertex of C_j. Thus, due to the definition of a well-partitioned chordal graph, w must be adjacent to both u and x. That is, $uvxwu$ is a 4-cycle. Hence, v must be part of all MEG-sets of G due to Theorem 4.1. □

A *split graph* G is graph whose vertices can be partitioned into a clique and an independent set.

Corollary 5.6. *Let G be a split graph with no component equal to K_1 or K_2 with k vertices having a pendent neighbor. If G has n vertices, then $meg(G) = n - k$.*

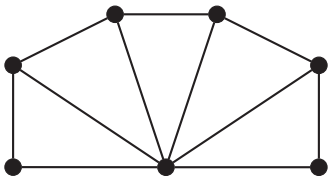


Fig. 4. A 2-connected interval graph with a non-extremal MEG-set. Note that the set of all vertices except the universal vertex is an MEG-set.

Proof. Let G be a split graph on n vertices with k vertices having a pendent neighbor. Suppose that G can be partitioned into a clique C and an independent set I. Now consider C as one bag, and each vertex of I as a bag. This shows that G is a well-partitioned graph as well. The proof follows from Theorem 5.5 observing that the only cut-vertices in a split graph are those having a pendent neighbor. □

An *interval graph* G is graph whose vertices correspond to intervals, and two vertices are adjacent if and only if their corresponding intervals intersect. Moreover, if it is possible to assign an interval to each vertex of G in such a way that none of the intervals contains another interval, then G is a *proper interval graph*.

Theorem 5.7. *Let G be a proper interval graph with k cut-vertices. If G has n vertices, then $meg(G) = n - k$.*

Proof. Let G be a proper interval graph on n vertices with k cut-vertices. For any vertex z in G, let us denote its corresponding interval by $[l_z, r_v]$ where l_z denotes the left endpoint of the interval and r_z denotes the right endpoint of the interval. As G is a proper interval graph, it is possible to provide a total ordering $\prec$ on the vertices of G by defining $z \prec z'$ if $l_z < l_{z'}$. Observe that, as G is a proper interval graph, $l_z < l_{z'}$ if and only if $r_z < r_{z'}$.

Let v be a vertex of G which is neither a cut-vertex nor a simplicial vertex. It is enough to show that there exists a neighbor u of v such that any induced 2-path of the form uvx is part of a 4-cycle due to Lemmas 2.1 and 5.1.

As v is not a simplicial vertex, v has neighbors which intersects l_v, and neighbors which intersects r_v. Let L_v denote the set of all neighbors y of v such that $y \prec v$. Let u be the maximum element of L_v with respect to $\prec$. We will show that, any induced 2-path of the form uvx must be part of a 4-cycle.

Let uvx be an induced 2-path. Notice that we must have $u \prec v \prec x$. As v is not a cut-vertex, there exists an induced path $P = uw_iw_2 \dots w_\ell x$ connecting u and x which does not contain the vertex v. Notice that as P is an induced path, we must have

$$u \prec w_1 \prec w_2 \cdots \prec w_\ell \prec x.$$

That means, every w_i is a neighbor of v. Moreover, due to the maximality of u, we must have $v \prec w_1$. That means w_1 intersects r_v, and hence intersects $[l_x, r_x]$. Hence, $uvxw_1u$ is a 4-cycle. □

The results proved in this section together imply the following theorem:

Theorem 5.8. *If G is a 2-connected cograph, block-graph, well-partitioned chordal graph, split graph, or proper interval graph, then G is MEG-extremal.*

Remark 5.9. As split graphs are a subclass of well-partitioned chordal graphs, it was not necessary to mention them separately in Theorem 5.8. However, we still do so since split graphs are a well-known class of graphs.

Remark 5.10. A natural question to ask is whether we can include superclasses of the graph families mentioned in Theorem 5.8. In Fig. 4 we provide an example of a 2-connected interval graph which is not MEG-extremal. As interval graphs are also chordal graphs, we cannot hope to extend Theorem 5.8 to the family of interval graphs or chordal graphs.

5.2. Graph products

Let G and H be two graphs. Now we are going to define three (product) graphs, each on the set of vertices

$$V(G) \times V(H) = \{(a, b) : a \in V(G) \text{ and } b \in V(H)\}.$$

The *strong product* of G and H, denoted by $G \boxtimes H$, has the following adjacency rule: two vertices (a, b) and (a', b') are adjacent if and only if one of the following conditions are satisfied: (i) $a = a'$ and $bb' \in E(H)$, (ii) $aa' \in E(G)$ and $b = b'$, (iii) $aa' \in E(G)$ and $bb' \in E(H)$.

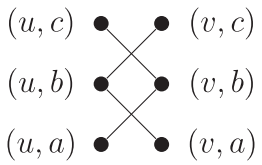


Fig. 5. The tensor product of a K_2 graph uv and the P_3 graph abc. Here, the set of degree 1 vertices is an MEG-set.

The *Cartesian product* of G and H, denoted by $G \square H$, has the following adjacency rule: two vertices (a, b) and (a', b') are adjacent if and only if either $a = a'$ and $bb' \in E(H)$, or $aa' \in E(G)$ and $b = b'$.

The *tensor product* of G and H, denoted by $G \times H$, has the following adjacency rule: two vertices (a, b) and (a', b') are adjacent if and only if $aa' \in E(G)$ and $bb' \in E(H)$.

Our next result provides a shorter proof of Corollary 2 of [23].

Theorem 5.11 ([23])**.** *Let G be an MEG-extremal graph and let H be any graph. Then their Cartesian product $G \square H$ and strong product $G \boxtimes H$ are both MEG-extremal.*

Proof. We prove this equality for the Cartesian product $G \square H$ first. Let (a, b) be a vertex of $G \square H$. Since G is MEG-extremal, by Theorem 4.1, for the vertex $a \in V(G)$, there exists $u \in V(G)$ such that every induced 2-path uax is part of a 4-cycle. We will show that, all induced 2-path of the form $(u, b)(a, b)(x, y)$ is part of a 4-cycle in $G \square H$. Since (a, b) is adjacent to (x, y), we have the following cases:

- Let $ax \in E(G)$ and $b = y$. First notice that if $(u, b)(a, b)(x, y)$ is an induced 2-path, (u, b) and (x, y) are not adjacent. Since $b = y$, the vertices x, u are not adjacent in G. Thus, uax is an induced 2-path in G. This implies that there exists a $w \in V(G)$ such that $uaxwu$ is a 4-cycle in G. Hence, observe that (w, b) is adjacent to both (u, b) and (x, y).
- Let $a = x$ and $by \in E(H)$. In this case note that the vertex (u, y) is adjacent to both (u, b) and (x, y).

Thus, by Theorem 4.1, (a, b) is in every MEG-set of $G \square H$, and hence $G \square H$ is MEG-extremal.

For proving that the strong product $G \boxtimes H$ is also MEG-extremal, we may consider the above cases here. But the second case is redundant for strong product of graphs as (u, b) is adjacent to (a, y) and the path is no longer induced. That apart, we need to consider a third case, which is as follows.

- Let $ax \in E(G)$ and $by \in E(G)$. We know that there exists a vertex $w \in V(G)$ such that $uaxw$ is a 4-cycle in G. Then observe that (w, y) is adjacent to both (u, b) and (x, y).

Thus, by Theorem 4.1, (a, b) is in every MEG-set of $G \boxtimes H$, and hence $G \boxtimes H$ is MEG-extremal. □

Theorem 5.12. *Let G, H be two MEG-extremal graphs. Then their tensor product $G \times H$ is also MEG-extremal.*

Proof. Let (a, b) be a vertex of $G \times H$. As G is MEG-extremal, there exists a $u \in V(G)$ such that any induced 2-path of the form uax is part of a 4-cycle $uaxwu$ in G. Similarly, as H is MEG-extremal, there exists a $v \in V(H)$ such that any induced 2-path of the form vby is part of a 4-cycle $vbyzv$ in H.

Let us consider an induced 2-path of the form $(u, v)(a, b)(x, y)$ in $G \times H$. Notice that at least one of uax and vby is an induced 2-path in G or H, respectively. If both uax and vby are induced 2-paths, then due to the observations noted in the last paragraph, there exists w that is adjacent to both u, x in G and there exists z that is adjacent to both v, y in H. Hence, (w, z) is adjacent to both (u, v) and (x, y). Otherwise, without loss of generality, we assume that uax is an induced two path and consider w adjacent to both u, x in G, and observe that (w, b) is adjacent to both (u, v) and (x, y), which concludes the proof. □

Remark 5.13. One can check that the tensor product of an MEG-extremal graph with any graph does not always yield an MEG-extremal graph. For example, the tensor product of the MEG-extremal graph K_2 and the path P_3 on 3 vertices (see Fig. 5) is not MEG-extremal.

6. Graphs of given girth

Due to Remark 5.2, it makes sense to study connected graphs with minimum degree 2. In Corollary 4.4, we noted that, if G has girth 5 or more, then G cannot be MEG-extremal. Therefore, it is natural to wonder whether $meg(G)$ will become even smaller (with respect to the order of G) if G becomes sparser. One way to consider sparse graphs is to study graphs having high girth. The following bound is meaningful for graphs of girth at least 8. Note that, as mentioned in the introduction, an incorrect version of this bound appeared in the conference version of this paper [18].

Theorem 6.1. *Let G be a 2-connected graph with girth g at least 4. If G has n vertices, then $meg(G) \leq \frac{4n}{g-3}$.*

Proof. Let G be a 2-connected graph on n vertices with girth $g \geq 4$. We construct a vertex subset M of G recursively, and claim that M is an MEG-set of G. To begin with, we initialize M by picking an arbitrary vertex of G. Next, we add an arbitrary vertex to M that is at distance at least $\frac{g-3}{4}$ from any vertex of M. We repeat this process until every vertex of $V(G) \setminus M$ are at a distance strictly less than $\frac{g-3}{4}$ from some vertex of M. We want to note that M can also be obtained by applying a greedy algorithm to find a maximal independent set in the $\lceil(\frac{g-3}{4}) - 1\rceil$-th power of G when the girth is large enough.

Next, we will show that M is indeed an MEG-set of G. Let uv be an arbitrary edge of G. Note that, the distance between u (resp., v) and the set M is strictly less than $\frac{g-3}{4}$ due to the way we constructed M. Without loss of generality, we consider u' the nearest vertex to u in M, such that $d_G(u, u') \leq \frac{g-3}{4}$ and $d_G(v, u') > d_G(u, u')$. If the vertex of M nearest to u is closer to v, we simply swap the place of u and v. We consider the vertex of M closest to v in $G - uv$. Since G is 2-connected, $G - uv$ is connected, hence v' is always defined. Also observe that $d_G(v, v') \leq 2\left(\frac{g-3}{4}\right) - d_G(u, u')$ otherwise there would be a shortest path from u' to v' on more than $\frac{g-3}{4}$ vertices where at least one vertex is at distance more than $\frac{g-3}{4}$ from any vertex of M.

Let P_1 be a shortest path connecting u' and u, and P_2 be a shortest path connecting v and v' in $G - uv$. Let P be the path obtained by concatenating the path P_1, the edge uv, and the path P_2. Since neither of P_1 nor P_2 contains the edge uv, P is non-backtracking. Note that the length of P is at most

$$d(u', u) + 1 + 2\left(\frac{g-3}{4}\right) - d(u', u) = \frac{g-1}{2}.$$

Moreover, no vertex can appear twice in P otherwise there is a cycle of length strictly less than g in G. Therefore, P is the unique shortest path connecting u' and v', as otherwise it will imply that there is a cycle of length strictly less than g, contradicting the girth condition on G. Hence, $u'v'$ monitors the edge uv.

Now we are left with counting the cardinality of M. For any vertex $u \in M$, let S_u be the set of all vertices that are at distance at most ℓ from u, where ℓ is the biggest integer such that a vertex of G is at distance ℓ or less from at most one vertex of M. Notice that, for any two vertices $u, v \in M$, $S_u \cap S_v = \emptyset$, hence a path connecting u and v will be of length at least $2\ell + 1$. As any two vertices of M are at distance at least $\frac{g-3}{4}$ from one another, then

$$2\ell + 1 \geq \frac{g-3}{4} \implies \ell \geq \frac{g-7}{8}.$$

As G is a 2-connected graph, each vertex v of M is part of a cycle which implies that $|S_v| \geq 2\ell + 1$ (counting v itself too). Therefore, we must have

$$n = |V(G)| \geq |M|(2\ell + 1) \geq |M|\left(\frac{2(g-7)}{8} + 1\right) \implies |M| \leq \frac{4n}{g-3}.$$

This completes the proof. □

Remark 6.2. As the girth can be considered as a measure of sparseness of a graph, the above result shows that $meg(G)$ has a stricter upper bound when the sparseness (in terms of the girth) of G increases. However, the idea used in the proof is quite general and it may be possible to provide a better bound using the same idea for specific families of graphs, having more structural information.

Theorem 6.3. *For G a connected graph of minimum degree 2 and girth at least 5, any vertex cover of G is an MEG-set.*

Proof. Let G be a connected graph having minimum degree at least 2 and n vertices. Let I be an independent set. Since any vertex cover of G is the complement of an independent set, it is enough to show that $M = V(G) \setminus I$ is an MEG-set of G.

Let uv be any edge of G. If $u, v \notin I$, then that implies $u, v \in M$, and thus, uv is monitored. If $u \in I$ and $v \notin I$, then $v \in M$. As the minimum degree of G is at least 2, u must have a neighbor w (say) other than v. Note that as I is an independent set, w cannot belong to I. This means, in particular, $w \in M$. Notice that wuv must be the unique shortest path joining w and v as the girth of G is at least 5, and thus uv is monitored. □

Corollary 6.4. *Let G be a connected graph with girth at least 5 having minimum degree at least 2. If G has n vertices, and chromatic number $\chi(G)$, then $meg(G) \leq n\left(\frac{\chi(G)-1}{\chi(G)}\right)$.*

Proof. If G is a connected graph with $\chi(G) = k$ and n vertices, a maximum independent set of G is of size at least $\frac{n}{k}$, which immediately yields the bound using Theorem 6.3. □

We prove a more general version of Corollary 6.4 for graphs that have pendent vertices.

Corollary 6.5. *Let G be a graph with girth at least 5, and ℓ pendent vertices. If G has n vertices, and chromatic number $\chi(G)$, then $meg(G) \leq n\left(\frac{\chi(G)-1}{\chi(G)}\right) + \frac{\ell}{\chi(G)}$.*

Proof. We know that the pendent vertices are part of any MEG-set of G by Lemma 2.1. Now, let us remove the minimum number of vertices from G to obtain a subgraph G' which is a connected graph having minimum degree 2 (if possible). Notice that, to obtain G', we need to remove at least the ℓ pendents. Therefore, G' will have at most $(n-\ell)$ vertices. Therefore, using Corollary 6.4 we can find an MEG-set of G' having cardinality $(n-\ell)\left(\frac{\chi(G)-1}{\chi(G)}\right)$ as $\chi(G') \leq \chi(G)$. Observe that the MEG-set of G' together with the pendent vertices of G can monitor all the edges of G. Therefore,

$$meg(G) \leq (n-\ell)\left(\frac{\chi(G)-1}{\chi(G)}\right) + \ell = n\left(\frac{\chi(G)-1}{\chi(G)}\right) + \frac{\ell}{\chi(G)}.$$

If, in case, it is not possible to obtain such a G', then we can infer that G is a forest. In that case, the set of all pendents is an MEG-set of G. That means, $meg(G) \leq \ell$, which satisfies the statement trivially. □

7. Effects of clique-sum and subdivisions

Let G_1 and G_2 be two graphs having cliques C_1 and C_2 of size k, respectively. A *k-clique-sum* of G_1 and G_2, denoted by $G_1 \oplus_k G_2$, is a graph obtained by identifying the vertices of C_1 with the vertices of C_2 (each vertex of C_1 is identified with exactly one vertex of C_2).

This particular operation between two graphs is a fundamental operation in graph theory, and is used for characterizing chordal graphs, maximal planar graphs, K_5-minor-free graphs, etc. Some variants of the definition requires deletion of all or some edges of the clique which may be important in the context of the problem solved. For example, in the context of the illustrious graph structure theorem [25,29], it is allowed to delete some edges of the clique obtained by identification. In our context, as MEG-sets do not interact well with edge deletion, we investigate the changes in $meg(G)$ with respect to the clique-sum operation without edge deletion.

Theorem 7.1. *Let $G_1 \oplus_k G_2$ be a k-clique-sum of the graphs G_1 and G_2 for some $k \geq 2$. Then we have,*

$$meg(G_1) + meg(G_2) - 2k \leq meg(G_1 \oplus_k G_2) \leq meg(G_1) + meg(G_2).$$

Moreover, both the lower and the upper bounds are tight.

Proof. Let M_1 and M_2 be MEG-sets of G_1 and G_2. Observe that, the union $M_1 \cup M_2$ is an MEG-set of $G_1 \oplus_k G_2$. This implies the upper bound.

For the lower bound, first let $C = V(G_1) \cap V(G_2)$ be the clique of size k, common to G_1 and G_2. Let M be a minimum MEG-set of $G_1 \oplus_k G_2$. Note that, $M_1 = (M \setminus V(G_2)) \cup C$ is an MEG-set of G_1, and $M_2 = (M \setminus V(G_1)) \cup C$ is an MEG-set of G_2. Observe that,

$$\begin{aligned} meg(G_1) + meg(G_2) &\leq |M_1| + |M_2| \\ &= |(M \setminus V(G_2)) \cup C| + |(M \setminus V(G_1)) \cup C| \leq |M| + 2|C| \\ &= meg(G_1 \oplus_k G_2) + 2k \end{aligned}$$

This proves the lower bound.

Tightness of the upper bound: Take $G_1 = G_2 = K_k^*$, where K_k^* denotes the graph obtained by adding a pendent neighbor to each vertex of the complete graph K_k. Due to Lemma 5.1, $meg(G_1) = meg(G_2) = k$. Note that, there is essentially a unique way to obtain a k-clique-sum of G_1 and G_2, and let $G = G_1 \oplus_k G_2$ be the graph obtained after performing the k-clique-sum. Observe that G is the graph obtained by adding two pendent neighbors to each vertex of the complete graph K_k. Therefore, by Lemma 5.1,

$$meg(G_1 \oplus_k G_2) = 2k = k + k = meg(G_1) + meg(G_2).$$

Tightness of the lower bound: First we will describe the construction of a graph H_k. To construct this graph, we start with a complete graph K_{k+1} on $(k+1)$ vertices named $v_0, v_1, v_2, \ldots, v_k$. Next, we add a 3-path of the form $v_{i,1}v_{i,2}v_{i,3}v_i$ attached to v_i for each $i \in \{0, 1, \ldots, k\}$. Moreover, we add edges between the vertices $v_{0,2}, v_{1,2}, \ldots, v_{k,2}$ to form a clique. Finally we subdivide each edge of the above mentioned clique exactly once. The so obtained graph is H_k. We take $G_1 = G_2 = H_k$. However, for convenience, the vertices of G_1 are denoted by the original names given to the vertices of H_k, while the

vertices of G_2 are denoted by placing a "bar" over the original names given to the vertices of H_k. That is, instead of using the names $v_0, v_1, v_{3,2}$, we will use the names $\bar{v}_0, \bar{v}_1, \bar{v}_{3,2}$ when we want to refer to the vertices of G_2. Let $G = G_1 \oplus_k G_2$ be the graph obtained by taking a k-clique-sum of G_1 and G_2 on the cliques $\{v_1, v_2, \ldots, v_k\}$ and $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k\}$.

Observe that, H_k has $(k+1)$ pendents, and thus they are part of any MEG-set of H_k. Next suppose that $M^* = V(H_k) \setminus \{v_0, v_1, v_{0,3}, v_{1,3}\}$. Notice that, this set is not able to monitor the edge v_0v_1. That means, we need to take at least k vertices, other than the $(k+1)$ pendent vertices, in any MEG-set of H_k. On the other hand, the set obtained by the pendent vertices and all but one vertex from the K_{k+1} clique gives us an MEG-set of H_k. Hence, $meg(H_k) = 2k+1$.

Furthermore, G has exactly $(2k+2)$ pendents, and the set of all pendents of G is an MEG-set. Therefore,

$$meg(G) = 2k+2 = (2k+1) + (2k+1) - 2k = meg(G_1) + meg(G_2) - 2k.$$

This concludes the proof. □

Let G be a graph. We obtain the graph S_G^ℓ by subdividing each edge of G exactly ℓ times. The graph operation subdivision is also a fundamental graph operation, integral in the theory of topological minors, which can be used for sparsification of a graph. Moreover, subdivision can be considered as the inverse operation to edge contraction, which is another fundamental notion that plays an instrumental role in the famous graph minor theorem [27,30]. The following result proves a relation between $meg(G)$ and $meg(S_G^\ell)$.

Theorem 7.2. *For any graph G and for all $\ell \geq 2$, we have*

$$1 \leq \frac{meg(G)}{meg(S_G^\ell)} \leq 2.$$

Moreover, the lower bound is tight, and the upper bound is asymptotically tight.

Proof. Let M be an MEG-set of G. Then observe that, M is also an MEG-set of S_G^ℓ. This proves $meg(S_G^\ell) \leq meg(G)$.

On the other hand, let M' be an MEG-set of S_G^ℓ. Now we construct an MEG-set M of G using M'. Let v be a vertex of M'. If v is also a vertex of G, then we put v in M. If v is not a vertex of G, then v must be one of the vertices which was used for subdividing an edge e of G to obtain S_G^ℓ. In such a case, we put both the end points of e in M. Note that, the so-obtained M is an MEG-set of G. As $|M| \leq 2|M'|$, we have $meg(G) \leq 2meg(S_G^\ell)$. This completes the proof of the inequality.

Tightness of the lower bound: Given any positive integer $n \geq 2$, take G to be a tree with n leaves, then S_G^ℓ is also a tree with n leaves, we know that $meg(G) = meg(S_G^\ell) = n$ [20]. Thus, we have infinitely many examples of G where,

$$\frac{meg(G)}{meg(S_G^\ell)} = 1.$$

Asymptotic tightness of the upper bound: Let $G = P_k \,\square\, P_2$, that is the Cartesian product of the paths P_k (on k vertices) and P_2 (on 2 vertices). For convenience, let us assume that G is embedded on the plane with its vertices placed on the points (i, j), where $i \in \{0, 1, \ldots, k-1\}$ and $j \in \{0, 1\}$. Moreover, let the vertex placed on (i, j) be denoted by $v_{i,j}$. Two vertices of this graph are adjacent if they are at a Euclidean distance exactly 1. We know that $meg(G) = 2k$ [20]. We will construct an MEG-set M of size $(k+1)$ for S_G^ℓ. First of all, put the vertices $v_{0,0}$ and $v_{0,1}$ in M. Let w_i be a vertex (choose any option) on the path obtained by subdividing the edge $v_{i,0}v_{i,1}$. Put w_i in M for all $i \in \{1, 2, \ldots, k-1\}$. Observe that, the so-obtained M monitors S_G^ℓ. Therefore, $meg(S_G^\ell) \leq k+1$. Hence,

$$\frac{meg(G)}{meg(S_G^\ell)} \leq \frac{2k}{k+1} = 2 - \frac{2}{k+1}.$$

As k tends to infinity, the ratio $\frac{meg(G)}{meg(S_G^\ell)}$ tends to the upper bound 2. Hence the upper bound is asymptotically tight. □

8. Concluding remarks

(1) In Section 3, we gave examples of graphs $G_{a,b,c,d}$ which attains $g(G_{a,b,c,d}) = a$, $eg(G_{a,b,c,d}) = b$, $seg(G_{a,b,c,d}) = c$, and $meg(G_{a,b,c,d}) = d$ for "almost" all $2 \leq a \leq b \leq c \leq d$. However, for some of the combinations of a, b, c, d we still do not know if an example exists or not. One problem to consider is to decide exactly for which prescribed values of a, b, c, d, such a graph $G_{a,b,c,d}$ exists, along with finding an explicit example.

(2) In Section 4, we have proved a necessary and sufficient condition for a vertex to be part of any minimum MEG-set. A question in this direction is to find a necessary and sufficient condition for a vertex not to be in any minimum MEG-set.

(3) In Section 5, we used the result of Theorem 4.1, as well as a more comprehensive definition of MEG-extremal, to prove that several well-known graph classes, restricted to 2-connected graphs, are MEG-extremal. We provide an example of a 2-connected interval graph that is not MEG-extremal. Thus, it will be interesting to characterize or devise efficient algorithms to find $meg(G)$ when G is an interval graph, or a chordal graph. One can also try to extend the results of Theorems 5.3 and 5.7 to perfect graphs.

(4) In Section 7, we deal with the effects on $meg(G)$ with respect to some fundamental graph operations like clique-sums, and subdivisions. It will be interesting to perform similar studies with respect to other fundamental graph operations such as vertex deletion, edge deletion, edge contraction, graph powers, etc. One could also wonder how the parameter interacts with bounded-diameter subgraph identification, as a direct generalization of our results.

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Data availability

No data was used for the research described in the article.

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