Fundamenta Informaticae 191(3-4) : 197–229 (2024)
DOI 10.3233/FI-242179
IOS Press

On Three Domination-based Identification Problems in Block Graphs

Dipayan Chakraborty[*†]
Université Clermont-Auvergne, CNRS
Mines de Saint-Étienne, Clermont-Auvergne-INP
LIMOS, 63000 Clermont-Ferrand, France
dipayan.chakraborty@uca.fr

Florent Foucaud
Université Clermont-Auvergne, CNRS
Mines de Saint-Étienne, Clermont-Auvergne-INP
LIMOS, 63000 Clermont-Ferrand, France
florent.foucaud@uca.fr

Aline Parreau
Univ Lyon, CNRS, INSA Lyon, UCBL
Centrale Lyon, Univ Lyon 2, LIRIS, UMR5205
69622 Villeurbanne Cedex, France
aline.parreau@univ-lyon1.fr

Annegret Wagler
Université Clermont-Auvergne, CNRS
Mines de Saint-Étienne, Clermont-Auvergne-INP
LIMOS, 63000 Clermont-Ferrand, France
annegret.wagler@uca.fr

Abstract. The problems of determining the minimum-sized *identifying*, *locating-dominating* and *open locating-dominating codes* of an input graph are special search problems that are challenging from both theoretical and computational viewpoints. In these problems, one selects a dominating set C of a graph G such that the vertices of a chosen subset of $V(G)$ (i.e. either $V(G) \setminus C$ or $V(G)$ itself) are uniquely determined by their neighborhoods in C. A typical line of attack for these problems is to determine tight bounds for the minimum codes in various graph classes. In this work, we present tight lower and upper bounds for all three types of codes for *block graphs* (i.e. diamond-free chordal graphs). Our bounds are in terms of the number of maximal cliques (or *blocks*) of a block graph and the order of the graph. Two of our upper bounds verify conjectures from the literature with one of them being now proven for block graphs in this article. As for the lower bounds, we prove them to be linear in terms of both the number of blocks and the order of the block graph. We provide examples of families of block graphs whose minimum codes attain these bounds, thus showing each bound to be tight.

Keywords: identifying code, locating-dominating code, open locating-dominating code, block graph

[*]Also works: Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park, 2006, South Africa

[†]Address for correspondence: LIMOS, 1 rue de la Chebarde, Campus des Cézeaux, 63178 Aubière Cedex, France

1. Introduction

For a graph G that models a facility or a multiprocessor network, detection devices can be placed at its vertices to locate an intruder (like a faulty processor, a fire or a thief). Depending on the features of the detection devices, different types of dominating sets can be used to determine the optimum distributions of these devices across the vertices of G. In this article, we study three problems arising in this context, namely the three types of dominating sets called the *identifying codes*, *locating-dominating codes* and *open locating-dominating codes* of a given graph. Each of these problems has been extensively studied during the last decades (see the bibliography maintained by Lobstein [1]). These three types of codes are among the most prominent notions within the larger research area of identification problems in discrete structures pioneered by Rényi [2], with numerous applications, for example in fault-diagnosis [3], biological testing [4] or machine learning [5].

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ denote the set of vertices (also called the *vertex set*) and the set of edges (also called the *edge set*), respectively, of G. The *(open) neighborhood* of a vertex $u \in V(G)$ is the set $N_G(u)$ of all vertices of G adjacent to u; and the set $N_G[u] = \{u\} \cup N_G(u)$ is called the *closed neighborhood* of u. A subset $C \subseteq V(G)$ is called an *identifying code* [6] (or an *ID-code* for short) of G if

- $N_G[u] \cap C \neq \emptyset$ for all vertices $u \in V(G)$ (i.e. C is said to be a *dominating set* of G, or is said to possess the property of *domination* in G); and
- $N_G[u] \cap C \neq N_G[v] \cap C$ for all distinct vertices $u, v \in V(G)$ (i.e. C is called a *closed-separating set* of G, or is said to possess the property of *closed-separation* in G).

See Figure 1a for an example of an ID-code. A graph G admits an ID-code if and only if G has no *closed-twins* (i.e. a pair of distinct vertices $u, v \in V$ with $N_G[u] = N_G[v]$). Said differently, a graph G admits an ID-code if and only if G is *closed-twin-free*.

A subset $C \subseteq V(G)$ is called a *locating-dominating code* [7, 8] (or an *LD-code* for short) of G if

- $N_G[u] \cap C \neq \emptyset$ for all vertices $u \in V(G)$ (i.e. C is a dominating set of G); and
- $N_G(u) \cap C \neq N_G(v) \cap C$ for all distinct vertices $u, v \in V(G) \setminus C$ (i.e. C is called a *locating set* of G, or is said to possess the property of *location* in G).

See Figure 1b for an example of an LD-code. Note that every graph has an LD-code.

Finally, a subset $C \subseteq V(G)$ is called an *open locating-dominating code* [9] (or an *OLD-code* for short) of G if

- $N_G(u) \cap C \neq \emptyset$ for all vertices $u \in V(G)$ (i.e. C is called an *open-dominating set* of G, or is said to possess the property of *open-domination* in G)[1]; and

[1]The property of open-domination is often called *total-domination* in the literature. See for example [10]

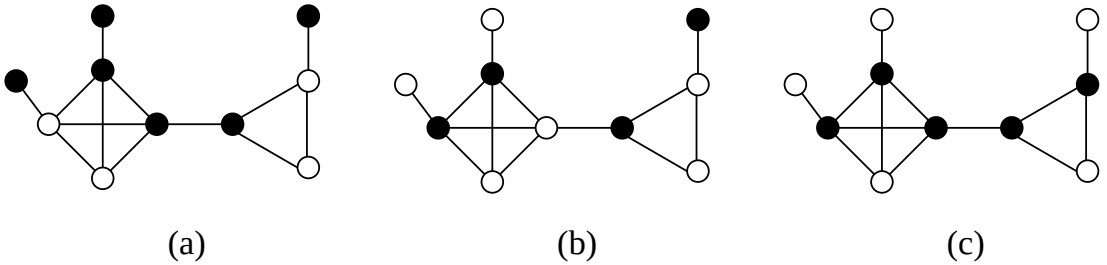


Figure 1: Examples of (a) an ID-code, (b) an LD-code and (c) an OLD-code. The set of black vertices in each of the three graphs constitute the respective code of the graph.

- $N_G(u) \cap C \neq N_G(v) \cap C$ for all distinct vertices $u, v \in V(G)$ (i.e. C is called an *open-separating set* of G, or is said to possess the property of *open-separation* in G).

See Figure 1c for an example of an OLD-code. A graph G admits an OLD-code if and only if G has neither isolated vertices nor *open-twins* (i.e. a pair of distinct vertices $u, v \in V(G)$ such that $N_G(u) = N_G(v)$). Again, said differently, a graph G admits an OLD-code if and only if G has no isolated vertices and is *open-twin-free*.

A graph with neither open- nor closed-twins is simply referred to as *twin-free*.

For the rest of this article, we often simply use the word *code* to mean any of the above three ID-, LD- or OLD-codes without distinction. Given a graph G, the *identifying code number* $\gamma^{ID}(G)$ (or *ID-number* for short), the *locating-dominating number* $\gamma^{LD}(G)$ (or *LD-number* for short) and the *open locating-dominating number* $\gamma^{OLD}(G)$ (or *OLD-number* for short) of a graph G are the minimum cardinalities among all ID-codes, LD-codes and OLD-codes, respectively, of G. In other words, for simplicity, for any symbol X $\in$ {ID, LD, OLD}, we have the X-number: $\gamma^X(G) = \min\{|C| : C \text{ is an X-code of } G\}$. In the case that all three codes are addressed together as one unit anywhere in the text, i.e. any specific symbol for X $\in$ {ID, LD, OLD} is irrelevant to the context, we then simply refer to the X-numbers as the *code numbers* of G.

Given two sets A and B, the set $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is called the *symmetric difference* of A and B. For a subset $C \subset V(G)$ and distinct vertices $u, v \in V(G)$, if there exists a vertex $w \in (N_G(u) \cap C) \triangle (N_G(v) \cap C)$ (resp. $(N_G[u] \cap C) \triangle (N_G[v] \cap C)$), then w and C are said to *open-separate* (resp. *closed-separate*) the vertices u and v in G.

1.1. Known results

Given a graph G, determining $\gamma^{ID}(G)$ or $\gamma^{LD}(G)$ is, in general, NP-hard [11] and remains so for several graph classes where other hard problems become easy to solve. These include bipartite graphs [11] and two subclasses of chordal graphs, namely split graphs and interval graphs [12]. In fact, for both bipartite and split graphs, it is NP-hard to even approximate the ID-number and LD-number within a factor of $\log |V(G)|$ [13]. Determining $\gamma^{OLD}(G)$ is also, in general, NP-hard [9] and remains so for perfect elimination bipartite graphs [14] and interval graphs [12]. On the other hand, determining $\gamma^{OLD}(G)$ becomes APX-complete for chordal graphs with maximum degree 4 [14].

As these problems are computationally very hard, a typical line of attack is to determine bounds on the code numbers for specific graph classes. Closed formulas for these parameters have so far been found only for restricted graph families (e.g. for paths and cycles [8, 9, 15], for stars [16], for complete multipartite graphs [17, 18] and for some subclasses of split graphs including thin headless spiders [19]). Lower bounds for all three code numbers for several graph classes like interval graphs, permutation graphs, cographs [20] and lower bounds for ID-numbers for trees [21], line graphs [22], planar graphs [23] and many others of bounded VC-dimension [24] have been determined. As far as upper bounds for the code numbers are concerned, for certain graph classes, upper bounds for ID-codes (see [25, 26, 27]), LD-codes (see [25, 28, 29]) and OLD-codes (see [30]) have been obtained.

1.2. Our work

In this paper, we consider the family of block graphs, defined by Harary in [31], see also [32] for equivalent characterizations. A *block graph* is a graph in which every maximal 2-connected subgraph is complete. In a block graph, every maximal complete subgraph is called a *block*. Equivalently, block graphs are diamond-free chordal graphs [33]. Linear-time algorithms to compute all three code numbers in block graphs have been presented in [34]. In this paper, we complement these results by determining tight lower and upper bounds for all three code numbers for block graphs. We give bounds using (i) the number of vertices, i.e. the order of a graph, as has been done for several other classes of graphs; and (ii) the number of blocks of a block graph, a quantity equally relevant to block graphs. In doing so, we also prove the following conjecture.

Conjecture 1.1. ([35, Conjecture 1])
The ID-number of a closed-twin-free block graph is bounded above by the number of blocks in the graph.

In addressing LD-codes for twin-free block graphs, we prove (for block graphs) the following conjecture posed by Garijo et al. [29] and reformulated in a slightly stronger form by Foucaud et al. [28].

Conjecture 1.2. ([28, Conjecture 2])
Every twin-free graph G with no isolated vertices satisfies $\gamma^{LD}(G) \leq \frac{|V(G)|}{2}$.

A short version of this article has also been published in the conference proceedings of CALDAM 2023 [36].

1.3. Notations

For a block graph G, we let $\mathcal{K}(G)$ denote the set of all blocks of G, i.e. the set of all maximal cliques of G. Noting that any two distinct blocks K and K' of G intersect at at most a single vertex, any vertex $x \in V(G)$ such that $\{x\} = V(K) \cap V(K')$ is called an *articulation vertex* of both K and K'. We define $art(K)$ to be the set of all articulation vertices of a block $K \in \mathcal{K}(G)$. For a connected block graph, we fix a *root block* $K_0 \in \mathcal{K}(G)$ and define a system of assigning numbers to every block of G depending on "how far" the latter is from K_0. So, define a *layer function* $f : \mathcal{K}(G) \to \mathbb{Z}$ *on* G by:

$f(K_0) = 0$, and for any other $K(\neq K_0) \in \mathcal{K}(G)$ (also called a *non-root* block), define inductively $f(K) = i$ if $V(K) \cap V(K') \neq \emptyset$ for some block K' $(\neq K) \in \mathcal{K}(G)$ such that $f(K') = i - 1$. For a pair of blocks $K, K' \in \mathcal{K}(G)$ such that $f(K) = f(K') + 1$, define $art^-(K) = V(K) \cap V(K')$; and for the root block K_0, define $art^-(K_0) = \emptyset$. Note that for a block $K \in \mathcal{K}(G)$ such that $f(K) \geq 1$, we have $|art^-(K)| = 1$, and the only vertex in $art^-(K)$ is called the *negative articulation* vertex of the block K. In contrast to the negative articulation vertices of G, define $art^+(K) = art(K) \setminus art^-(K)$ to be the set of all *positive articulation* vertices of the block K and $\overline{art}(K) = V(K) \setminus art(K)$ to be the set of all *non-articulation* vertices of K. Any block K with $|art(K)| = 1$ is called a *leaf block* and all blocks that are not leaf blocks are called *non-leaf blocks*. Note that a block graph always has leaf-blocks. For simplicity, we also denote the set $f^{-1}(\{i\})$ by $f^{-1}(i)$. Then, for each $i \geq 0$, $f^{-1}(i)$ is called the *i-th layer* of G and each block $K \in f^{-1}(i)$ is said to be *in the* i-th layer of G. See Figure 2 for an illustration of the layers and the related concepts in a connected block graph.

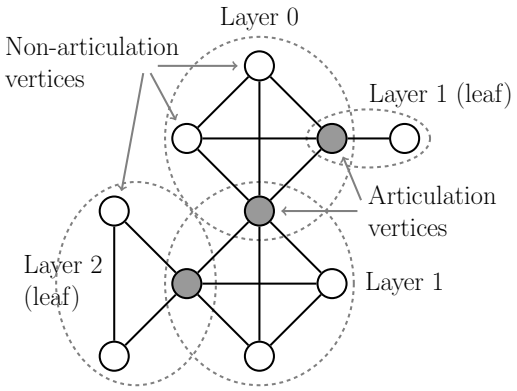


Figure 2: Example of different layer numbers, articulation vertices (grey) and non-articulation vertices (white) of a connected block graph.

1.4. Structure of the paper

Our results on the upper bounds for the code numbers are contained in Section 2 of this paper, whereas Section 3 is dedicated to the lower bounds for the code numbers. Section 2 is further subdivided into three subsections with each of the latter containing the results for a particular code. We conclude in Section 4.

2. Upper bounds

In this section, we establish upper bounds on the ID-, LD- and OLD-numbers for block graphs. Two of these upper bounds are in fact proving Conjectures 1.1 and 1.2. All our results in this section are for *connected* block graphs. However, applying the results to each connected component of a block graph, the results of Theorem 2.1, 2.2 and 2.4 hold equally well for all block graphs.

2.1. Identifying codes

The number of blocks is, structurally speaking, a quantity as relevant for block graphs as is the number of vertices for trees. In the next result, we prove Conjecture 1.1 to provide an upper bound for $\gamma^{ID}(G)$ for a block graph G in terms of its number of blocks.

Theorem 2.1. Let G be a connected closed-twin-free block graph and let $\mathcal{K}(G)$ be the set of all blocks of G. Then $\gamma^{ID}(G) \leq |\mathcal{K}(G)|$.

Proof:
Assume by contradiction that there is a closed-twin-free block graph G of minimum order such that $\gamma^{ID}(G) > |\mathcal{K}(G)|$. We also assume that G has at least four vertices since it can be easily checked that the theorem is true for closed-twin-free block graphs with at most three vertices (which are only P_1 and P_3). Suppose that $K \in \mathcal{K}(G)$ is a leaf-block of G. Due to the closed-twin-free property of G, one can assume that $V(K) = \{x, y\}$ and, without loss of generality, that x and y are the non-articulation and the negative articulation vertices, respectively, of K. Let $G' = G - x$ be the graph obtained by deleting the vertex $x \in V(G)$ (and the edge incident on x) from G. Then G' is a block graph with $|\mathcal{K}(G')| = |\mathcal{K}(G)| - 1$. We now consider the following two cases.

Case 1. (G' is closed-twin-free)
By the minimality of the order of G, there is an ID-code C' of G' such that $|C'| \leq |\mathcal{K}(G')| = |\mathcal{K}(G)| - 1$. First, assume that $y \notin C'$. Then by the property of domination of C', there exists a vertex $z \in V(G')$ such that $z \in N_{G'}(y) \cap C'$. We claim that $C = C' \cup \{x\}$ is an ID-code of G. First of all, that C is a dominating set of G is clear from the fact that C' is a dominating set of G'. To prove that C is a closed-separating set of G, we see that x is closed-separated in G from all vertices in $V(G') \setminus \{y\}$ by itself and is closed-separated in G from y by the vertex $z \in C'$. Moreover, all other pairs of distinct vertices are closed-separated by C' are also closed-separated by C. Thus, C, indeed, is an ID-code of G. This implies that $\gamma^{ID}(G) \leq |C| \leq |\mathcal{K}(G)|$, contrary to our assumption.

We therefore assume that $y \in C'$. If again, there exists a vertex $z \in N_{G'}(y) \cap C'$, then by the same reasoning as above, $C = C' \cup \{x\}$ is an ID-code of G. Otherwise, we have $N[y] \cap C' = \{y\}$. Now, since G is connected, we have $deg_G(y) > 1$ and therefore, there exists a vertex $w \in N_G(y) \setminus \{x\}$. Then $C = C' \cup \{w\}$ is an ID-code of G. This is because, first of all, C still closed-separates every pair of distinct vertices in $V(G')$. The vertex x is closed-separated from y by the vertex $w \in C$; and from w by w itself. Moreover, for any vertex v in $V(G') \setminus \{y, w\}$, v is closed-separated from y in G' by some vertex $u_v \in N_{G'}[v] \cap C'$. Then x is closed-separated from all such v in $V(G') \setminus \{y, w\}$ by the vertices $u_v \in C'$. Moreover, C is clearly also a dominating set of G. Hence, this leads to the same contradiction as before.

Case 2. (G' has closed-twins)
Assume that vertices u, $v \in V(G')$ are a pair of closed-twins of G'. Since u and v were not closed-twins in G, it means that x is adjacent to, say, u, without loss of generality. This implies that $u = y$. Note that v is then unique with respect to being a closed-twin with y in G'. This is because, if u and some vertex $v'(\neq v) \in V(G')$ were also closed-twins in G', then it would mean that v and v' were closed-twins in G, contrary to our assumption. Now, let $G'' = G' - v$. We claim the following.

Claim 2A. G'' is closed-twin-free.

Proof of Claim 2A. Toward a contradiction, if vertices $z, w \in V(G'')$ were a pair of closed-twins in G'', it would then mean that the vertex $z \in N_{G'}(v)$, without loss of generality. This would, in turn, imply that $z \in N_{G'}(y)$ (since the vertices y and v are closed-twins in G'). Or, in other words, $y \in N_{G''}(z)$. Now, since z and w are closed-twins in G'', we have $y \in N_{G''}(w)$, i.e. $w \in N_{G'}(y)$. Again, by virtue of y and v being closed-twins in G', we have $w \in N_{G'}(v)$. This implies that z and w are closed-twins in G which is a contradiction to our assumption. ♦

We also note here that the vertices y and v must be from the same block, as the two are adjacent on account of being closed-twins in G'. Thus, G'' is a connected closed-twin-free block graph. Therefore, by the minimality of the order of G, there is an ID-code C'' of G'' such that $|C''| \leq |\mathcal{K}(G'')| < |\mathcal{K}(G)|$. If $y \notin C''$, then we claim that $C = C'' \cup \{x\}$ is an identifying code of G. This is true because, firstly, C is a dominating set of G (note that, by the property of domination of C'' in G'', there exists a vertex $z \in N_{G''}(y) \cap C''$; and since y and v are closed-twins in G', we have $z \in N_G(v) \cap C$). Moreover, x is closed-separated in G from every other vertex in $V(G) \setminus \{y\}$ by x itself; and the vertices x and y are closed-separated in G by some vertex in $N_{G''}(y) \cap C''$ that dominates the vertex y. The vertex y is closed-separated from all the vertices in $V(G) \setminus \{y, x\}$ by x and; since y and v have the same closed neighborhood in G', v is closed-separated in G from all vertices in $V(G'') \setminus \{y\}$ because y is by C''. Finally, any two distinct vertices closed-separated by C'' still remain so by C. Thus, C, indeed, is an ID-code of G. This implies that $\gamma^{ID}(G) \leq |C| \leq |\mathcal{K}(G)|$; again a contradiction.

Let us, therefore, assume that $y \in C''$. This time, we claim that $C = (C'' \setminus \{y\}) \cup \{x, v\}$ is an ID-code of G. That C is a dominating set of G is clear. So, as for the closed-separating property of C is concerned, as before, x is closed-separated in G from every other vertex in $V(G) \setminus \{y\}$ by x itself; vertices x and y are closed-separated in G by v; the vertices y and v are closed-separated in G by x and the vertices v and x are closed-separated in G by v. Since y and v have the same closed neighbourhood in G' and since y is closed-separated in G'' by C'' from every other vertex in $V(G'')$, both v and y are also each closed-separated in G from every vertex in $V(G'') \setminus \{v, y\}$.

Finally, any two distinct vertices of G'' are closed-separated by C'' still remain so by C. This proves that C is an ID-code of G and hence, again, we are led to the contradiction that $\gamma^{ID}(G) \leq |C| \leq |\mathcal{K}(G)|$.

This proves the theorem. □

Note that, besides for stars [16], the upper bound in Theorem 2.1 is attained by the ID-numbers of thin headless spiders [19] which, therefore, serve as examples of cases where the bound in Theorem 2.1 is tight. Noting that blocks are simply the maximal complete subgraphs of a block graph, the statement of Theorem 2.1 does not hold in terms of the number of maximal complete subgraphs even for chordal graphs, let alone for graphs in general. A counterexample to the bound in Theorem 2.1 (replacing the number of blocks by the number of maximal cliques) for chordal graphs is the graph P_{2k}^{k-1} (the graph obtained from a path on $2k$ vertices with edges introduced between all pairs of vertices $u, v \in V(P_{2k})$ such that $d_{P_{2k}}(u, v) \leq k - 1$) which is closed-twin-free, has only two maximal complete subgraphs, but needs $2k - 1$ vertices in any identifying code [37].

2.2. Locating-dominating codes

In this subsection, we prove two results on upper bounds for the LD-numbers of block graphs. The first result is a more general one in which the upper bound is in terms of the number of blocks and other quantities arising out of the structural properties of a block graph. On the other hand, the second result is proving Conjecture 1.2 for block graphs. We begin with the more general result.

Theorem 2.2. Let G be a connected block graph and $\mathcal{K}(G)$ be the set of blocks in G. Then, we have

$$\gamma^{LD}(G) \leq |\mathcal{K}(G)| + \sum_{\substack{K \in \mathcal{K}(G), \\ |\overline{art}(K)| \geq 2}} (|\overline{art}(K)| - 2).$$

Proof:
We define a set $C \subset V(G)$ by the following rules.

Rule 1: For every block $K \in \mathcal{K}(G)$ which does not contain any closed-twins, i.e. with at most one non-articulation vertex, pick any one vertex from $V(K) \setminus art^{-}(K)$ in C.

Rule 2: For every block $K \in \mathcal{K}(G)$ which contains closed-twins, i.e. with at least two non-articulation vertices, pick any $|\overline{art}(K)| - 1$ vertices from $\overline{art}(K)$ in C.

Note that the vertices added in C by the above rules are all distinct. Therefore, the following is the size of C.

$$\begin{aligned} |C| &= \Big|\{K \in \mathcal{K}(G) : |\overline{art}(K)| \leq 1\}\Big| + \sum_{\substack{K \in \mathcal{K}(G), \\ |\overline{art}(K)| \geq 2}} (|\overline{art}(K)| - 1) \\ &= |\mathcal{K}(G)| + \sum_{\substack{K \in \mathcal{K}(G), \\ |\overline{art}(K)| \geq 2}} (|\overline{art}(K)| - 2). \end{aligned}$$

The result, therefore, follows from proving that C is an LD-code of G.

First of all, we notice that, by the construction of C, for every block $K \in \mathcal{K}(G)$, there exists a vertex $v_K \in V(K) \cap C$. Therefore, C is a dominating set of G. We now show that C is also a locating set of G. So assume that $u, v \in V(G) \setminus C$ are distinct vertices of G. Then, by the construction of C, we must have $u \in V(K)$ and $v \in V(K')$ for a distinct pair of blocks $K, K' \in \mathcal{K}(G)$.

Claim A. There exist vertices $v_K \in V(K) \cap C$ and $v_{K'} \in V(K') \cap C$ such that $v_K \neq v_{K'}$.

Proof of Claim A. On the contrary, if $V(K) \cap C = V(K') \cap C = \{v_K\}$, this would imply that v_K is the negative articulation vertex of either K or K'. Without loss of generality, let us assume that $art^{-}(K) = \{v_K\}$. If K does not contain any closed twins, then by Rule 1, there exists a vertex in K other than v_K which belongs to C, a contradiction to $V(K) \cap C = \{v_K\}$. Therefore, K contains closed twins. In other words, $|\overline{art}(K)| \geq 2$. Now, by Rule 2, at least one vertex w, say, of $\overline{art}(K)$

belongs to C. Since $\overline{art}(K) \cap art^{-}(K) = \emptyset$, we have $w \neq v_K$ and thus, we run into the same contradiction. This proves the claim. ♦

This implies that at least one of v_K and $v_{K'}$ must open-separate u and v in G and, hence, C is a locating set of G. □

There is an infinite number of arbitrarily large connected block graphs whose LD-numbers attain the upper bound in Theorem 2.2. One such subclass of block graphs is the following. For positive integers $t \geq 2$ and $m_1, m_2, \ldots, m_t$ with $m_i \geq 3$ for each i, we define a class of graphs $S_t(m_1, m_2, \ldots, m_t)$ by the following rule. Let X be a copy of the complete graph on t vertices and name its vertices $v_1, v_2, \ldots, v_t$. Also, for all $1 \leq i \leq t$, let Y_i be a copy of the complete graph on m_i vertices. Let $S_t(m_1, m_2, \ldots, m_t)$ be the block graph obtained by identifying a vertex of Y_i with v_i of X for every $1 \leq i \leq t$. For brevity, we continue to call the identified vertices resulting in $S_t(m_1, m_2, \ldots, m_t)$ by the same names of $v_1, v_2, \ldots, v_t$ as before. See Figure 3 for an example of the graph $S_t(m_1, m_2, \ldots, m_t)$ constructed with $t = 5$, and $m_1 = m_3 = 4$, $m_2 = m_5 = 3$ and $m_4 = 5$. We then show the following.

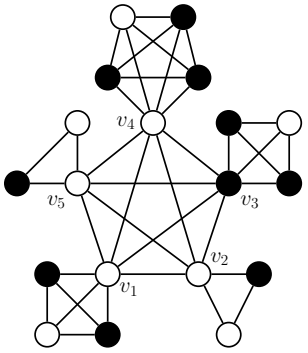


Figure 3: Graph $S_5(4, 3, 4, 5, 3)$ whose LD-number attains the upper bound in Theorem 2.2. The black vertices represent those included in the LD-code C of G as described in the proof of Theorem 2.2.

Proposition 2.3. For $t \geq 2$, and $m_1, m_2, \ldots, m_t$ such that $m_i \geq 3$ for all $1 \leq i \leq t$, we have

$$\gamma^{LD}(S_t(m_1, m_2, \ldots, m_t)) = |\mathcal{K}(S_t(m_1, m_2, \ldots, m_t))| + \sum_{\substack{K \in \mathcal{K}(S_t(m_1, m_2, \ldots, m_t)), \\ |\overline{art}(K)| \geq 2}} (|\overline{art}(K)| - 2).$$

Proof:
Let $G = S_t(m_1, m_2, \ldots, m_t)$. We note here that the number of blocks in G is $t + 1$; and the only blocks $K \in \mathcal{K}(G)$ with $|\overline{art}(K)| \geq 2$ are $Y_1, Y_2, \ldots, Y_t$ (as per the notations used in the preceeding discussion). More precisely, for each $1 \leq i \leq t$, we have $|\overline{art}(Y_i)| = m_i - 1$. So, the upper bound for the LD-number of G by Theorem 2.2 is $t + 1 + \sum_{i=1}^{t}(m_i - 3) = 1 - 2t + \sum_{i=1}^{t} m_i$.

Now, assume that C is a minimum LD-code of G. Let us first assume that $V(X) \cap C = \emptyset$. Then, since any two vertices in $V(Y_i)$ have the same neighborhood in C, it implies that we must

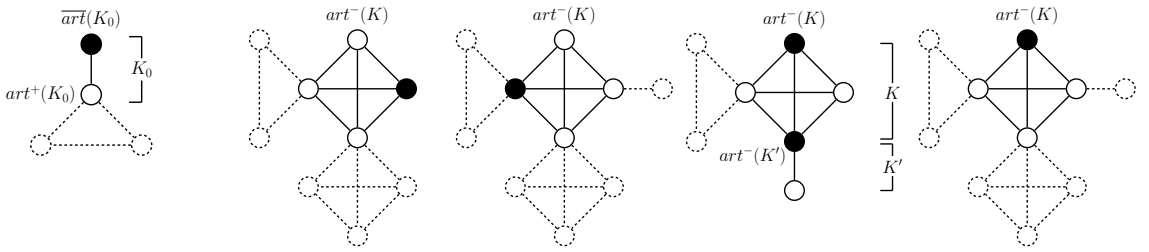
have $|V(Y_i) \cap C| = m_i - 1$. This further implies that $|C| = \sum_{i=1}^{t}(m_i - 1) = -t + \sum_{i=1}^{t} m_i > 1 - 2t + \sum_{i=1}^{t} m_i$ (since $t \geq 2$), the upper bound by Theorem 2.2 resulting in a contradiction. Therefore, we must have $V(X) \cap C \neq \emptyset$. Thus, let $v_i \in C$ for some $1 \leq i \leq t$. Note that $v_i \in V(Y_i)$. Since any two vertices of $V(Y_i) \setminus \{v_i\}$ are twins in G, we have $|V(Y_i) \cap C| \geq m_i - 1$. Moreover, for $1 \leq j \leq t$ such that $j \neq i$, again, since any two vertices of $V(Y_j) \setminus \{v_j\}$ are twins in G, we now have $|V(Y_j) \cap C| \geq m_j - 2$. Hence, we have $|C| \geq |V(Y_i) \cap C| + \sum_{j=1, j\neq i}^{t} |V(Y_j) \cap C| \geq 1 - 2t + \sum_{i=1}^{t} m_i$. □

Imposing additional structural constraints on a block graph, one could still limit the number of vertices one needs to choose from each of its blocks in order to form an LD-code of the graph. Our next result shows exactly that.

Theorem 2.4. Let G be a connected twin-free block graph. Then $\gamma^{LD}(G) \leq \frac{|V(G)|}{2}$.

Proof:
To prove the theorem, we partition the vertex set of G into two special subsets C^* and D^*.

Assume that $K_0 \in \mathcal{K}(G)$ is a leaf block of G. Then, $|V(K_0)| = 2$, as G is twin-free. Assign K_0 to be the root block of G, i.e. define a layer function $f : \mathcal{K}(G) \to \mathbb{Z}$ on G such that $f(K_0) = 0$. We then construct the sets C^* and D^* by the following rules applied inductively on $i \in f(\mathcal{K}(G))$. See Figure 4 for a demonstration of this construction.



(a) Rule 1 (b) Rule 2 (c) Rule 3 (d) Rule 4 (e) Rule 5

Figure 4: Example of each rule in the proof of Theorem 2.4 for the construction of the sets C^* or D^*. In each example, the black vertices represent those picked in C^* and the white vertices represent those picked in D^*. The blocks with solid edges represent those blocks (in i-th layer, say) from which vertices are chosen either in C^* or D^*. The blocks with dashed edges represent those blocks in the next layer (the $(i+1)$-th) which, inductively, are yet to be analysed for their choices of vertices in C^* and D^*; but whose presence in the figure is necessary to determine the positive, negative and the non-articulation vertices of the block in i-th layer.

Rule 1: Pick the (positive) articulation vertex of the root block K_0 in D^* (i.e. let $art^+(K_0) \subset D^*$) and pick the (other) non-articulation vertex of K_0 in C^* (i.e. let $\overline{art}(K_0) \subset C^*$). See Figure 4a for an example.

Rule 2: For every non-root block $K \in \mathcal{K}(G)$ with at least one non-articulation vertex (i.e. $\overline{art}(K) \neq \emptyset$) and whose negative articulation vertex is in D^* (i.e. $art^-(K) \subset D^*$), pick all non-articulation vertices of K in C^* (i.e. let $\overline{art}(K) \subset C^*$); and all positive articulation vertices of K in D^* (i.e. let $art^+(K) \subset D^*$). See Figure 4b for an example.

Rule 3: For every non-root block $K \in \mathcal{K}(G)$ with no non-articulation vertices (i.e. $\overline{art}(K) = \emptyset$) and whose negative articulation vertex is in D^* (i.e. $art^-(K) \subset D^*$), pick *one* positive articulation vertex, say, w of K in C^* and the rest of the positive articulation vertices in D^* (i.e. let $art^+(K) \setminus \{w\} \subset D^*$). See Figure 4c for one such case.

Rule 4: For every non-root block $K \in \mathcal{K}(G)$ with at least one non-articulation vertex (i.e. $\overline{art}(K) \neq \emptyset$) and whose negative articulation vertex is in C^* (i.e. $art^-(K) \subset C^*$), pick *one* positive articulation vertex (if available), say, w of K in C^*; and pick all other vertices in $V(K)$, except the vertex w and the negative articulation vertex of K, in D^* (i.e. let $V(K) \setminus (art^-(K) \cup \{w\}) \subset D^*$). See Figure 4d for both examples of when articulation vertices are available (block K) and when they are not, i.e. in the case of leaf blocks (block K').

Rule 5: For every non-root block $K \in \mathcal{K}(G)$ with no non-articulation vertices (i.e. $\overline{art}(K) = \emptyset$) and whose negative articulation vertex is in C^* (i.e. $art^-(K) \subset C^*$), pick all positive articulation vertices of K in D^* (i.e. $art^+(K) \subset D^*$). See Figure 4e for an illustration.

From the construction, C^* and D^* are complements of each other in $V(G)$. We claim that both C^* and D^* are LD-codes of G. We first show that both are dominating sets of G.

Claim A. Both C^* and D^* are dominating sets of G.

Proof of Claim A. To prove that both C^* and D^* are dominating sets of G, it is enough to show that, for every block $K \in \mathcal{K}(G)$, both $V(K) \cap C^* \neq \emptyset$ and $V(K) \cap D^* \neq \emptyset$. By Rule 1, the claim is true for the root block K_0. So, assume $K \in \mathcal{K}(G)$ to be a non-root block. First, suppose that the negative articulation vertex of K belongs to D^*. Then, by Rules 2 and 3, we have $V(K) \cap C^* \neq \emptyset$. Next, suppose that the negative articulation vertex of K belongs to C^*. Then, by Rules 4 and 5, we have $V(K) \cap D^* \neq \emptyset$. ♦

We now show that both C^* and D^* are also locating sets of G. We start with C^*.

Claim B. C^* is a locating set of G.

Proof of Claim B. Assume that $u, v \in D^*$ are distinct vertices of G. Since G is twin-free, there exist distinct blocks $K, K' \in \mathcal{K}(G)$ such that $u \in V(K)$ and $v \in V(K')$. By the proof of Claim A, there exist vertices $v_K \in V(K) \cap C^*$ and $v_{K'} \in V(K') \cap C^*$. If $v_K \neq v_{K'}$, then either one of v_K and $v_{K'}$ must locate u and v in G. So, let us assume that no such pairs of distinct vertices $v_K \in V(K) \cap C^*$ and $v_{K'} \in V(K') \cap C^*$ exist, i.e. $V(K) \cap V(K') \subset C^*$ and that $V(K) \triangle V(K') \subset D^*$.

We now claim that either u is an articulation vertex of K or v is an articulation vertex of K' (or both). So, toward contradiction, assume that both u and v are non-articulation vertices of K and K', respectively. Then the following two cases arise.

Case 1. (K and K' belong to different layers)
Without loss of generality, assume that $f(K') = f(K) + 1$. Then, $K \neq K_0$, or else, by Rule 1, u, being a non-articulation vertex of K, must belong to C^*, contrary to our assumption. Therefore, K is a non-leaf block. Now, $V(K) \triangle V(K') \subset D^*$ implies that the negative articulation vertex of K belongs to D^*. Since u is a non-articulation vertex of K, by Rule 2, $u \in C^*$ which is a contradiction to our assumption.

Case 2. (K and K' belong to the same layer)
In this case, K and K' cannot both be leaf blocks, or else, G would have twins. So, without loss of generality, suppose that K is a non-leaf block. Now, the negative articulation vertex of K belongs to C^*. Since K is a non-leaf block, there exists a positive articulation vertex of K and, hence, by Rule 4, $art^+(K) \cap C^* \neq \emptyset$ which contradicts the fact that $V(K) \triangle V(K') \subset D^*$.

This proves our claim that either u is an articulation vertex of K or v is an articulation vertex of K' (or both). So if, without loss of generality, we assume that u is an articulation vertex of K, then $\{u\} = V(K) \cap V(K'')$ for some block K'' $(\neq K) \in \mathcal{K}(G)$. Moreover, $K'' \neq K'$, or else, $V(K) \cap V(K') = \{u\} \subset D^*$ which contradicts our assumption that $V(K) \cap V(K') \subset C^*$. Hence, some vertex in $V(K'') \cap C^*$ (which exists due to the proof of Claim A) must open-separate u and v in G. This proves our current claim. ♦

We now prove the same for D^*.

Claim C. D^* is a locating set of G.

Proof of Claim C. Assume that $u, v \in C^*$ are distinct vertices of G. Since G is twin-free, there exist distinct blocks $K, K' \in \mathcal{K}(G)$ such that $u \in V(K)$ and $v \in V(K')$. By the proof of Claim A, there exist vertices $v_K \in V(K) \cap D^*$ and $v_{K'} \in V(K') \cap D^*$. If $v_K \neq v_{K'}$, then either one of v_K and $v_{K'}$ must open-separate u and v in G. So, let us assume that no such pairs of distinct vertices $v_K \in V(K) \cap D^*$ and $v_{K'} \in V(K') \cap D^*$ exist, i.e. $V(K) \cap V(K') \subset D^*$ and that $V(K) \triangle V(K') \subset C^*$.

We now claim that either u is an articulation vertex of K or v is an articulation vertex of K' (or both). So, toward contradiction, assume that both u and v are non-articulation vertices of K and K', respectively. Then the following two cases arise.

Case 1. (K and K' belong to different layers)
Without loss of generality, assume that $f(K') = f(K) + 1$. If $|V(K')| \geq 3$, since G is twin-free and since v is a non-articulation vertex of K', then K' contains exactly one non-articulation vertex and thus, $art^+(K') \cap D^* \neq \emptyset$ by Rule 2. This, however, is a contradiction to the fact that $V(K) \triangle V(K') \subset C^*$. So, assume that $|V(K')| = 2$, in which case, K' is a leaf block (since, again, v is a non-articulation vertex of K'). This implies that K is a non-leaf block, or else, G would have twins. So, in particular, $K \neq K_0$, the root block of G. Moreover, $V(K) \triangle V(K') \subset C^*$ implies that the negative articulation vertex of K belongs to C^*. Therefore, since u is a non-articulation vertex of K, by Rule 4, $u \in D^*$ which is a contradiction to our assumption.

Case 2. (K and K' belong to the same layer)
In this case, K and K' cannot both be leaf blocks, or else, G would have twins. So, without loss of generality, assume K to be a non-leaf block. Therefore, $|V(K)| \geq 3$, or else, u would be an articulation vertex of K, contrary to our assumption. The negative articulation vertex of K belongs to D^*. Therefore, by Rule 2, $art^+(K) \cap D^* \neq \emptyset$ which contradicts $V(K) \triangle V(K') \subset C^*$.

This, therefore, proves our claim that either u is an articulation vertex of K or v is an articulation vertex of K' (or both). If, without loss of generality, u is an articulation vertex of K, then $\{u\} = V(K) \cap V(K'')$ for some block K'' $(\neq K) \in \mathcal{K}(G)$. Moreover, $K'' \neq K'$, or else, $V(K) \cap V(K') = \{u\} \subset C^*$ which contradicts our assumption that $V(K) \cap V(K') \subset D^*$. Hence, some vertex in $V(K'') \cap D^*$ (which exists due to the proof of Claim A) must open-separate u and v in G. This, again, proves our current claim. ♦

Combining Claims A, B and C, we find that C^* and D^* are both LD-codes of the twin-free block graph G with no isolated vertices. Moreover, since C^* and D^* are complements of each other in $V(G)$, at least one of them must have cardinality of at most half the order of G. This proves the theorem. □

Theorem 2.4 therefore proves Conjecture 1.2 for block graphs.

Corollary 2.5. Let G be a twin-free block graph without isolated vertices. Then $\gamma^{LD}(G) \leq \frac{|V(G)|}{2}$.

The trees attaining the bound of Theorem 2.4 were characterized in [28]. There are also arbitrarily large twin-free block graphs that are not trees and whose LD-numbers attain the bound given in Theorem 2.4. To demonstrate this attainment, we look at the following subclass of block graphs which we denote by H_t: For a fixed integer $t \geq 1$, let $T_1, T_2, \ldots, T_t$ be t copies of K_3, the complete graph on three vertices. Suppose that $V(T_i) = \{v_i, w_i, x_i\}$ for each $1 \leq i \leq t$. Also, let $R, R_1, R_2, \ldots, R_t, R'_1, R'_2, \ldots, R'_t$ be $2t + 1$ copies of P_2, the path on two vertices. Also, let $V(R) = \{u, v\}$ and for all $1 \leq i, i' \leq t$, let $V(R_i) = \{y'_i, y_i\}$ and $V(R'_{i'}) = \{z'_{i'}, z_{i'}\}$. We then identify the vertices $v, v_1, v_2, \ldots, v_t$ to a single vertex which we continue to call v; and, for each $1 \leq i \leq t$, we identify the vertices w_i and y'_i to a single vertex and the vertices x_i and z'_i to a single

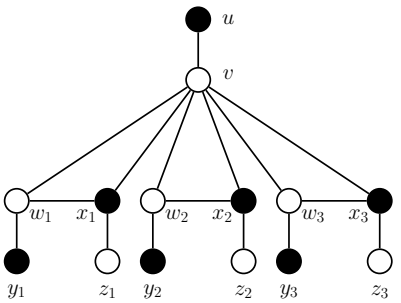


Figure 5: Graph H_3 whose LD-number attains the upper bound in Theorem 2.4. The black vertices represent those included in the LD-code C^* of G described in the proof of Theorem 2.4.

vertex. In the latter two cases, we continue to call the identified vertices w_i and x_i, respectively. The new resulting graph is what we call H_t. See Figure 5 for an example of H_t with $t = 3$. With that, we now prove the following.

Proposition 2.6. For each integer $t \geq 1$, $\gamma^{LD}(H_t) = \frac{|V(H_t)|}{2}$.

Proof:
Notice that the graph H_t is twin-free. Since $|V(H_t)| = 4t + 2$, we therefore have from Theorem 2.4 that $\gamma^{LD}(H_t) \leq 2t + 1$.

We now prove that $\gamma^{LD}(H_t) \geq 2t+1$. Since each of the $2t+1$ edges uv, w_iy_i, x_iz_i (for $1 \leq i \leq t$) of H_t contains a vertex of degree 1, therefore any LD-code of H_t, by its property of domination, must contain at least one endpoint of each of these edges. Since the above edges are all pairwise disjoint, any LD-code of H_t must contain at least $2t + 1$ vertices of H_t. □

2.3. Open locating-dominating codes

We now focus our attention on upper bounds for OLD-numbers for block graphs. Before we get to our results, we define the following two special graphs.

1. The 4-*path* (or P_4 in symbol) is a graph defined by its vertex set $V(P_4) = \{p_1, p_2, p_3, p_4\}$ and its edge set $E(P_4) = \{p_1p_2, p_2p_3, p_3p_4\}$.

2. The *bull graph* (or B_5 in symbol) is a graph defined by its vertex set $V(B_5) = \{b_1, b_2, b_3, b_4, b_5\}$ and its edge set $E(B_5) = \{b_1b_2, b_2b_3, b_3b_4, b_4b_5, b_2b_4\}$. See Figure 6 for a depiction of a bull graph.

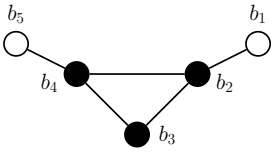


Figure 6: The Bull graph B_5. The set of black vertices constitute an OLD-code of B_5.



(a) $K_4 \rhd_q P_4$ (b) $K_4 \rhd_q B_5$

Figure 7: Examples of $G' \rhd_q X$, where $G' \cong K_4$ and $X \in \{P_4, B_5\}$. The vertex q (in grey) is obtained by identifying a vertex of G' and an articulation vertex of X.

We note here that both P_4 and B_5 are block graphs with articulation vertices p_2 and p_3 for P_4 and b_2 and b_4 for B_5. For P_4, the vertices p_1 and p_4 are called the *leaf vertices*; and for B_5, the vertices b_1 and b_5 are the leaf vertices. Assume G' to be any graph and X to be a graph which is either a copy of P_4 or B_5. For a fixed vertex $q \in V(G')$, we define a new graph $G' \rhd_q X$ to be the graph obtained by identifying the vertex q with an articulation vertex of X (see Figures 7a and 7b for examples of $K_4 \rhd_q P_4$ and $K_4 \rhd_q B_5$, respectively). As a matter of reference, we call the new vertex in $G' \rhd_q X$ obtained as a result of identifying two vertices as the *quotient vertex*; and continue to refer to the quotient vertex as q itself. We now turn to our results. Firstly, it is easy to establish the following.

Lemma 2.7. If P is a 4-path, then $\gamma^{OLD}(P) = 4$.

Lemma 2.8. If B is a bull graph, then $\gamma^{OLD}(B) = 3$.

Proof:
Let $V(B) = \{b_1, b_2, b_3, b_4, b_5\}$, where b_2 and b_4 are the articulation vertices; and b_1 and b_5 are the leaf vertices of B. Then it is easy to check that $\{b_2, b_3, b_4\}$ is an OLD-code of B and hence $\gamma^{OLD}(B) \leq 3$. See Figure 6 for the OLD-code demonstrated with black vertices in the figure.

On the other hand, assume that C is an OLD-code of B. Since b_1 and b_5 are degree 1 vertices, their only neighbours, namely b_2 and b_4, respectively, must be in C for the latter to be an open-dominating set of B. Moreover, at least one of b_1 and b_3 must be in C for b_2 and b_5 to be open-separated in B. Hence, $|C| \geq 3$ and this establishes the result. □

This brings us to our result on the upper bound for OLD-numbers for block graphs.

Theorem 2.9. Let G be a connected open-twin-free block graph with no isolated vertices. Moreover, let G neither be a copy of P_2 nor of P_4. Moreover, let $m_Q(G)$ be the number of non-leaf blocks of G with at least one non-articulation vertex. Then $\gamma^{OLD}(G) \leq |V(G)| - m_Q(G) - 1$.

Proof:
To start with, if G is a copy of the bull graph, then $|V(G)| = 5$ and $m_Q(G) = 1$. Moreover, by Lemma 2.8, $\gamma^{OLD}(G) = 3 = |V(G)| - m_Q(G) - 1$ and so, we are done. So, let us assume that G is not a copy of the bull graph.

We first choose a root block $K_0 \in \mathcal{K}(G)$ according to the following two possibilities.

Possibility 1: $G \cong G' \rhd_q X$ for some block graph G' and some graph X that is a copy of either P_4 or B_5. In such a case, assume that x is the articulation vertex of X which is identified with the vertex q of G' to form G. Then, we choose K_0 to be the block of G isomorphic to a P_2 with vertices $\{x, z\}$, where z is the leaf vertex of X adjacent to x.

Possibility 2: $G \ncong G' \rhd_q X$ for all block graphs G' and X that is a copy of either P_4 or B_5. In this case, choose $K_0 \in \mathcal{K}(G)$ such that $|V(K_0)| = \min\{|V(K)| : K \text{ is a leaf block of } G\}$.

Next, we construct a set $C \subset V(G)$ by the following rules.

Rule 1: For every non-root leaf block $K \in \mathcal{K}(G)$, pick all vertices of K in C.

Rule 2: For every block $K \in \mathcal{K}(G)$ that is either the root block K_0 or is a non-leaf block in $\mathcal{K}(G) \setminus \{K_0\}$, (i) pick all articulation vertices of K in C; and (ii) pick all but one non-articulation vertices of K in C.

To compute the size of C, we note that, for the root block and every other non-leaf block K with at least one non-articulation vertex, exactly one vertex is left out from it in C. This gives $|C| = |V(G)| - m_Q(G) - 1$. Thus, the upper bound for $\gamma^{OLD}(G)$ in the theorem is established on showing that C, indeed, is an OLD-code of G; which is what we prove next.

To show that C is an OLD-code of G, we notice first of all that, if the root block K_0 is isomorphic to P_2, then the (only) non-articulation vertex of K_0 is open-dominated by the articulation vertex of K_0 which belongs to C. In every other case when the root block is not isomorphic to P_2, all blocks $K \in \mathcal{K}(G)$ have $|V(K) \cap C| \geq 2$. This makes C an open-dominating set of G. Now we show that C is also an open-separating set of $V(G)$. So, let us assume that $u, v \in V(G)$ are distinct vertices of G. We now consider the following three cases.

Case 1. ($u, v \in V(K)$ for some block $K \in \mathcal{K}(G)$)
We note here that, by the construction of C, for every block $K \in \mathcal{K}(G)$, at most one vertex of K is not in C. This implies that at least one of u and v, say u without loss of generality, must be in C. Then u open-separates u and v in G.

Case 2. ($u \in V(K)$, $v \in V(K')$ for distinct $K, K' \in \mathcal{K}(G)$ and both $u, v \notin V(K) \cap V(K')$)
In this case, first let us assume that $V(K) \cap V(K') = \emptyset$. Now, both K and K' cannot be root blocks of G. Without loss of generality, let us assume that K is not the root block of G. Since $|V(K) \cap C| \geq 2$ by the construction of C, the block K has a vertex x in C other than u. Then, x open-separates u and v in G. Therefore, assume that $V(K) \cap V(K') = \{w\}$ for some vertex $w \in V(G)$. Now, if either one of $V(K)$ and $V(K')$, say $V(K)$ without loss of generality, has size at least 4, then $|V(K) \cap C| \geq 3$ and so, at least one vertex in $V(K) \setminus \{u, w\}$ belongs to C which open-separates u and v in G. So, assume that $|V(K)| \leq 3$ and $|V(K')| \leq 3$. We next look at the following two subcases.

Subcase 2.1. ($|V(K)| = |V(K')| = 2$)
Then, at least one of K and K' must be a non-leaf block, or else, G would have open-twins. So, without loss of generality, suppose that K is a non-leaf block. Then $\{u\} = V(K) \cap V(K'')$ for some block $K'' (\neq K, K') \in \mathcal{K}(G)$. If however, K' too is a non-leaf-block, then $\{v\} = V(K') \cap V(K''')$ for some block $K''' (\neq K', K, K'') \in \mathcal{K}(G)$. Now, at least one of K'' and K''' is not the root block. Without loss of generality, therefore, assume that K'' is not the root block. Then there is at least one vertex in $V(K'') \setminus \{u\}$ which is in C and, hence, open-separates u and v in G. So, let us assume that K' is a leaf block, i.e. v is a non-articulation vertex of K'. Now again, if K'' is not the root block, then there is at least one vertex in $V(K'') \setminus \{u\}$ which is in C and, hence, open-separates u and v in G. So, now let us assume that K'' *is* the root block. If G satisfies the conditions of Possibility 1, then $|V(K'')| = 2$. However, if G satisfies the conditions of Possibility 2, then again, since K' is a leaf block and $|V(K')| = 2$, by the minimality in size of the root block, we must have $|V(K'')| = 2$. So, assume z to be the non-articulation vertex of K_0. If $P = G[z, u, w, v]$, we have $P \cong P_4$ with u and w being the articulation vertices and z and v being the leaf-vertices of P. Since $G \ncong P_4$, we

must have $G \cong G' \rhd_q P$ for some block graph G' and some vertex $q \in \{u, w\}$ (note that both z and v are non-articulation vertices of G). However, by the way we have chosen the root block K_0, we must have $q = u$. This implies that u is the negative articulation vertex of some $K^* \in \mathcal{K}(G)$ such that $K^* \notin \{K, K_0\}$. This, in turn, implies that u and v are open-separated in G by some vertex in $(V(K^*) \cap C) \setminus \{u\}$.

Subcase 2.2. (At least one of $V(K)$ and $V(K')$ has size 3)
Without loss of generality, let us assume that $|V(K)| = 3$. So, assume that $K = G[w, u, y]$ for some vertex $y \in V(G)$. We must have $y \notin C$ (otherwise y would open-separate u and v). We first assume that K is the root block. If v is an articulation vertex of K', then $\{v\} = V(K') \cap V(K''')$ for some block $K''' (\neq K') \in \mathcal{K}(G)$. This implies that there exists a vertex in $V(K''') \setminus \{v\}$ which open-separates u and v in G. Moreover, if v is a non-articulation vertex of K', then we must have $|V(K')| = 3$ (or else, K' is a leaf block of size smaller than the root block which is a contradiction). Assume that $V(K') = \{w, v, a\}$ for some vertex $a \in V(G)$. If a is also a non-articulation vertex of K', then K' is a leaf block that is not a root block and, hence, $a \in C$. If however, a is an articulation vertex of K', then also, $a \in C$. Thus, either way, a open-separates u from v in G. Thus, we are done in the case that K is the root block of G.

So, let us now assume that K is *not* the root block of G. Now, if $y \in C$, then y open-separates u and v in G. So, let us assume that $y \notin C$, which implies that y is a non-articulation vertex of K. This, in turn, implies that u is an articulation vertex of K, or else, K would be a leaf block of G that is not the root block and so, $y \in C$, contrary to our assumption. So, let $\{u\} = V(K) \cap V(K'')$ for some block $K'' (\neq K) \in \mathcal{K}(G)$. If however, v is also an articulation vertex of K', then $\{v\} = V(K') \cap V(K''')$ for some block $K''' (\neq K') \in \mathcal{K}(G)$. Now, at least one of K'' and K''' is not the root block. So, without loss of generality, assume that K'' is not the root block. Then, there is at least one vertex in $V(K'') \setminus \{u\}$ which is in C and, hence, open-separates u and v in G. So, let us assume that v is a non-articulation vertex of K'. Again, if K'' is not the root block, then there exists at least one vertex in $V(K'') \setminus \{u\}$ which is in C and, hence, open-separates u and v in G. So, now assume K'' to *be* the root block. If $|V(K'')| \geq 3$, then there exists a vertex of $V(K'') \setminus \{u\}$ in C and, hence, open-separates u and v in G. So, let us assume that $|V(K'')| = 2$ and that z is the non-articulation of K''. If $|V(K')| = 3$, then suppose that $V(K') = \{w, v, a\}$ for some vertex $a \in V(G)$. If a is also a non-articulation vertex of K', then K' is a leaf block that is not a root block and hence, $a \in C$. If however, a is an articulation vertex of K', then also, $a \in C$. Thus, either way, a open-separates u from v in G and we are done in the case that $|V(K')| = 3$. So, let us finally assume that $|V(K')| = 2$ and that $v \in V(K')$ is a non-articulation vertex of K'.

If $B = G[z, u, y, w, v]$, we have $B \cong B_5$ with u and w being the articulation vertices and z and v being the leaf-vertices of B. Since by our assumption, $G \ncong B_5$, we have $G \cong G' \rhd_q B$ for some block graph G' and some vertex $q \in \{u, w\}$ (note that z, y and v are non-articulation vertices of G). Now, by the way we have chosen the root block, this implies that we must have $q = u$. This further implies that u is the negative articulation vertex of K^* for some $K^* \in \mathcal{K}(G)$ such that $K^* \notin \{K, K_0\}$. Hence, u and v are open-separated in G by some vertex in $V(K^*) \cap C \setminus \{u\}$.

This, therefore, proves that C is an OLD-code of G and with that, we prove the theorem. □

Applying Theorem 2.9 to each connected component of a block graph, one has the following general result.

Corollary 2.10. Let G be an open-twin-free block graph with k connected components and no isolated vertices. Moreover, let no component of G be either a copy of P_2 or of P_4. Also, let $m_Q(G)$ be the number of non-leaf blocks of G with at least one non-articulation vertex. Then $\gamma^{OLD}(G) \leq |V(G)| - m_Q(G) - k$.

Foucaud et al. [38] have shown that, for any open-twin-free graph G, $\gamma^{OLD}(G) \leq |V(G)| - 1$ unless G is a special kind of bipartite graph called half-graph (a *half-graph* is a bipartite graph with both parts of the same size, where each part can be ordered so that the open neighbourhoods of consecutive vertices differ by exactly one vertex [39]). Noting that P_2 and P_4 are the only block graphs that are half-graphs, Theorem 2.9 can be seen as a refinement of this result for block graphs.

We now show that the upper bound given in Theorem 2.9 is tight and is attained by arbitrarily large connected block graphs. To prove so, for two non-negative integers k and l such that $k + l \geq 2$, let us define a subclass $G_{k,l}$ of block graphs by the following rule: Let $T_1, T_2, \ldots, T_k$ be k copies of K_3 with $V(T_i) = \{u_i, v_i, w_i\}$ for each $1 \leq i \leq k$. Further, let $A_1, A_2, \ldots, A_k$ are k copies of P_2 with $V(A_i) = \{a_i, b_i\}$ for each $1 \leq i \leq k$, and $L_1, L_2, \ldots, L_l$ be l copies of P_3 with $V(L_j) = \{x_j, y_j, z_j\}$ for each $1 \leq j \leq l$. Let $G_{k,l}$ be the graph obtained by identifying the vertices v_i with b_i for each $1 \leq i \leq k$ and identifying the vertices u_i and z_j for all $1 \leq i \leq k$ and $1 \leq j \leq l$ into a single vertex u, say. See Figure 8 for an example of a construction of $G_{k,l}$ with $k = 2$ and $l = 3$. As a matter of reference, we continue to call the vertices of $G_{k,l}$ obtained by identifying v_i with b_i, for all $1 \leq i \leq k$, as v_i itself.

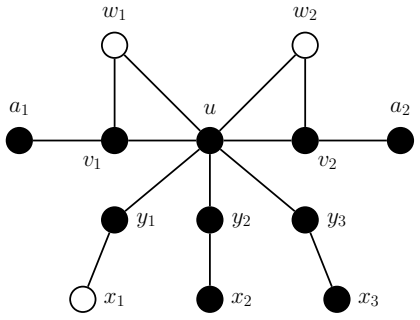


Figure 8: Graph $G_{2,3}$ whose OLD-number attains the upper bound in Theorem 2.9. The black vertices represent those included in the OLD-code C of G as described in the proof of the Theorem 2.9.

Proposition 2.11. For all positive integers k and l with $l + k \geq 2$, $\gamma^{OLD}(G_{k,l}) = |V(G_{k,l})| - m_Q(G_{k,l}) - 1$.

Proof:
First of all, we have $|V(G_{k,l})| = 3k + 2l + 1$, and $m_Q(G_{k,l}) = k$. Therefore, by Theorem 2.9, we have $\gamma^{OLD}(G_{k,l}) \leq 2(k + l)$.

Now, let C be an OLD-code of $G_{k,l}$. For C to be an open-dominating set of $G_{k,l}$, the only neighbours v_i and y_j of the degree 1 vertices a_i and x_j, respectively, of $G_{k,l}$ must be in C. Moreover, $u \in C$ for each pair of w_i and a_i to be open-separated in $G_{k,l}$ by C. Similarly, at least all but one of the x_i's must belong to C for each pair of vertices y_i and y_j, for $1 \leq i < j \leq l$, to be open-separated in $G_{k,l}$ by C. Let us first assume that $x_2, x_3, \ldots, l_l$ belong to C, without loss of generality. Then, for all $1 \leq i \leq k$, at least one of w_i and a_i must be in C for the pair y_1 and v_i, for $1 \leq i < j \leq k$, to be open-separated in $G_{k,l}$ by C. Adding up, therefore, we have $|C| \geq 2(k+l)$. On the other hand, if $x_1, x_2, \ldots, x_l \in C$, then, to open-separate the vertices v_i and v_j (for $1 \leq i < j \leq k$), we need at least one of w_i and a_i (for $k-1$ such i) to belong to C. Then again, the count adds up to $|C| \geq 2(k+l)$. This proves the proposition. □

3. Lower bounds

The general lower bound for the size of an identifying code using the number of vertices is $\gamma^{ID}(G) \geq \lceil \log_2(|V(G)|+1) \rceil$ [6]. However, to reach this bound, a graph needs to have a large VC-dimension [24] (the *VC-dimension* of a graph G is the size of a largest *shattered set*, that is, a set S of vertices such that for every subset S' of S, some closed neighbourhood in G intersects S exactly at S'). Indeed, if a graph has VC-dimension c, then any identifying code has size at least $O(|V(G)|^{1/c})$ [24]. The value $1/c$ is not always tight for specific subclasses of graphs of VC-dimension c, see for example the case of line graphs, which have VC-dimension at most 4 but for which the tight order for the lower bound is $\Omega(|V(G)|^{1/2})$ [22]. Similar results hold for LD- and OLD-codes, by using the same techniques as in [24].

Block graphs have VC-dimension at most 2 (one can check that a shattered set of size 3 would imply the existence of an induced 4-cycle or diamond), and thus, using the result from [24], their ID-number is lower bounded by $\Omega(|V(G)|^{1/2})$. In this section, we improve this lower bound to a linear one, and give a tight result.

For the rest of this article, given a block graph G, by $\mathcal{K}_{leaf}(G)$ we shall mean the set of all leaf blocks of G with at least one edge in the block. Moreover, by the symbol $n_i(G)$, we shall mean the number of vertices of degree i in the graph G.

Lemma 3.1. Let G be a connected block graph with at least one edge and with blocks $B_1, B_2, \ldots B_h$, say. Then, there exist distinct vertices $v_0, v_1, v_2, \ldots, v_h$ of G such that $v_0, v_1 \in V(B_1)$ and $v_i \in V(B_i)$ for all $2 \leq i \leq h$.

Proof:
Since G has at least one edge, note that every block of G has at least one edge. The proof is by induction on h with the base case being $h = 1$. In the base case, we have $G = B_1$. Since G has at least one edge v_0v_1, say, the two distinct vertices are v_0 and v_1 and we are done. So, let us assume the induction hypothesis for any connected block graph G' with at least one edge and with h' blocks, where $1 \leq h' \leq h-1$. Therefore, let $h \geq 2$. Without loss of generality, let us assume that B_h is a leaf block of G. Let $G' = G - (V(B_h) \setminus art^-(B_h))$. Then G' is also a connected block graph on $h-1$ blocks. Since G' contains the block B_1, the former has at least one edge. Then by the

induction hypothesis, there exist distinct vertices $v_0, v_1, v_2, \ldots, v_{h-1}$ of G such that $v_0, v_1 \in V(B_1)$ and $v_i \in V(B_i)$ for all $2 \leq i \leq h-1$. Since B_h has at least one edge, it implies that there exists a vertex $v_h \in V(B_h) \setminus art^-(B_h)$. This implies that $v_h \neq v_i$ for all $1 \leq i \leq h-1$ and hence, the result holds. □

Lemma 3.2. Let G be a connected block graph with at least one edge. Then we have

$$|\mathcal{K}(G)| \leq |V(G)| - 1 - |\mathcal{K}_{leaf}(G)| + n_1(G).$$

Proof:
Let $\mathcal{L}(G) = \{L \in \mathcal{K}_{leaf}(G) : L \cong K_2\}$ and G^* be a graph obtained from G by, for each $L \in \mathcal{L}(G)$, introducing a new vertex and making it adjacent to both elements of $V(L)$. Thus, G^* is a block graph in which every leaf block has at least 3 vertices. We also note here that

(1) $|\mathcal{L}(G)| = n_1(G)$,

(2) $|V(G^*)| = |V(G)| + |\mathcal{L}(G)| = |V(G)| + n_1(G)$,

(3) $|\mathcal{K}(G)| = |\mathcal{K}(G^*)|$ and that

(4) $|\mathcal{K}_{leaf}(G)| = |\mathcal{K}_{leaf}(G^*)|$.

Now, let $|\mathcal{K}(G^*)| = h$ and $\mathcal{K}(G^*) = \{B_1, B_2, \ldots, B_h\}$. Then, by Lemma 3.1, there exist distinct vertices $v_0, v_1, v_2, \ldots, v_h$ of G such that $v_0, v_1 \in V(B_1)$ and $v_i \in V(B_i)$ for all $2 \leq i \leq h$. Moreover, since at least one vertex in each leaf block of G^* is not any of the vertices v_i, we have

$$\begin{aligned} |\mathcal{K}(G)| = |\mathcal{K}(G^*)| = h = |\{v_0, v_1, v_3, \ldots, v_h\}| - 1 &\leq |V(G^*)| - |\mathcal{K}_{leaf}(G^*)| - 1 \\ &= |V(G)| + n_1(G) - |\mathcal{K}_{leaf}(G)| - 1. \end{aligned}$$

□

Corollary 3.3. Let G be a block graph with k connected components. Then, we have

$$|\mathcal{K}(G)| - n_0(G) \leq |V(G)| - k - |\mathcal{K}_{leaf}(G)| + n_1(G).$$

Proof:
Assume that $k = p + q$ such that $G_1, G_2, \ldots, G_p$ are the connected components of G, each with at least one edge; and that $S_1, S_2, \ldots, S_q$ are the components of G, each with a single vertex. Then, we have

$$\begin{aligned} |\mathcal{K}(G)| &= \sum_{1 \leq i \leq p} |\mathcal{K}(G_i)| + \sum_{1 \leq j \leq q} |\mathcal{K}(S_j)| \\ &\leq q - p + \sum_{1 \leq i \leq p} \Big(|V(G_i)| - |\mathcal{K}_{leaf}(G_i)| + n_1(G_i) \Big) && \text{[using Lemma 3.2]} \\ &= |V(G)| - k - |\mathcal{K}_{leaf}(G)| + n_1(G) + n_0(G) && \text{[since } q = n_0(G)\text{]} \end{aligned}$$

□

Corollary 3.4. Let G be a block graph with k connected components. Then we have

$$|\mathcal{K}(G)| - n_0(G) \leq |V(G)| - k.$$

Proof:
The result follows from Corollary 3.3 and the fact that $n_1(G) \leq |\mathcal{K}_{leaf}(G)|$. □

Before we come to our results, we define the following notations. For a given code C of a connected block graph G, let us assume that the sets $C_1, C_2, ..., C_k$ partition the code C such that the induced subgraphs $G[C_1], G[C_2], \ldots, G[C_k]$ of G are the k connected components of the subgraph $G[C]$ of G induced by C. Note that each C_i is a block graph (since every induced subgraph of a block graph is also a block graph).

Definition 3.5. Given a block graph G, its vertex set $V(G)$ is partitioned into the four following parts.

(1) $V_1 = C$,

(2) $V_2 = \{v \in V(G) \setminus V_1 : |N_G(v) \cap C| = 1\}$,

(3) $V_3 = \{v \in V(G) \setminus V_1 :$ there exist distinct $i, j \leq k$ such that $N_G(v) \cap C_i \neq \emptyset$ and $N_G(v) \cap C_j \neq \emptyset\}$,

(4) $V_4 = V(G) \setminus (V_1 \cup V_2 \cup V_3)$. Note that, for all $v \in V_4$, we have $N_G(v) \cap C \subset C_i$ for some i and that $|N_G(v) \cap C_i| \geq 2$.

We now prove a series of lemmas establishing upper bounds on the sizes of each of the vertex subsets V_1, V_2, V_3 and V_4 of a connected block graph G.

Lemma 3.6. Let G be a connected block graph and C be a code of G. Then the following are upper bounds on the size of the vertex subset V_2 of G.

(1) $|V_2| \leq |C| - n_0(G[C])$ if C is an ID-code.

(2) $|V_2| \leq |C|$ if C is an LD-code.

(3) $|V_2| \leq |C| - n_1(G[C])$ if C is an OLD-code.

Proof:
By definition of V_2, each vertex $v \in V_2$ has a unique neighbor u in C, i.e. $N_G(v) \cap C = \{u\}$. Hence, there can be at most $|C|$ vertices in V_2 and this proves (2).

If C is an ID-code, u cannot be isolated in $G[C]$ (or else, u and v will not be closed-separated in G by C). Thus, there are at most $|C| - n_0(G[C])$ vertices in V_2 and this proves (1).

Finally, if C is an OLD-code and u has a neighbor $w \in N_G(u) \cap C$ such that $deg_{G[C]}(w) = 1$, then v and w are not open-separated. Thus, there are at most $|C| - n_1(G[C])$ vertices in V_2 and this proves (3). □

Definition 3.7. Given a connected block graph G and a code C of G, let $G[C_1], G[C_2], \ldots, G[C_k]$ be all the connected components of $G[C]$, where $C_1, C_2, \ldots, C_k$ are subsets of C. Moreover, let the vertex set $V(G)$ be partitioned into the subsets V_1, V_2, V_3, V_4 as in Definition 3.5. Then, consider the bipartite graph $F_C(G)$, where $A = \{a_j : v_j \in V_3\}$ and $B = \{u_i : G[C_i]$ is a connected component of $G[C]\}$ are the two *parts* of $V(F_C(G))$. As for the edge set $E(F_C(G))$, for each vertex v_j in V_3, we add an edge between a_j and u_i if v_j is adjacent to a vertex in C_i. The graph $F_C(G)$ is called the *auxiliary graph* of G.

Lemma 3.8. For a connected block graph G and a code C of G, the auxiliary graph $F_C(G)$ is a forest.

Proof:
If there is a cycle in $F_C(G)$, there would be a cycle in G involving two vertices of different connected components $G[C_i]$ and $G[C_j]$, say. By the definition of a block graph, the latter cycle in G has to induce a complete subgraph in G. However, that would imply that $G[C_i]$ and $G[C_j]$ must be the same component of $G[C]$ which is a contradiction. Thus, $F_C(G)$ is cycle-free and, hence, is a forest. □

Lemma 3.9. Let G be a connected block graph, C be a code of G and $G[C_1], G[C_2], \ldots, G[C_k]$ be all the connected components of $G[C]$, where $C_1, C_2, \ldots, C_k$ are subsets of C. Then, we have $|V_3| \leq k - 1$.

Proof:
By Lemma 3.8, $F_C(G)$ is a forest. Let $V(F_C(G)) = A \sqcup B$ be as defined above. Then we have $|B| = k$. Therefore, $F_C(G)$, on account of being a forest, has at most $|A| + k - 1$ edges. Also, since $F_C(G)$ is bipartite, its number of edges is $\sum_{a \in A} deg_{F_C(G)}(a) \geq 2|A|$ (the last inequality holds since any vertex in the part A of $V(F_C(G))$ is adjacent to at least two distinct vertices of B). Thus, the result follows from the fact that $|V_3| = |A|$. □

Lemma 3.10. Let G be a connected block graph, C be a code of G and $G[C_1], G[C_2], \ldots, G[C_k]$ be all the connected components of $G[C]$, where $C_1, C_2, \ldots, C_k$ are subsets of C. Then, we have $|V_4| \leq |C| - k$. In particular,

(1) $|V_4| \leq |\mathcal{K}(G[C])| - |\mathcal{K}_{leaf}(G[C])| \leq |C| - 3k$ if C is an ID-code;

(2) $|V_4| \leq |\mathcal{K}(G[C])| \leq |C| - 2k + n_1(G[C])$ if C is an OLD-code.

Proof:
Let v be any vertex in V_4 and let $G[C_i]$ be the component of $G[C]$ such that $N_G(v) \cap C \subseteq C_i$. Moreover, $|N_G(v) \cap C_i| \geq 2$. Then, notice that $N_G(v) \cap C$ must be a subset of exactly one block of $G[C_i]$, or else, $G[C_i]$ would be disconnected, as $v \notin C$. This implies that $|V_4| \leq |\mathcal{K}(G[C])| - n_0(G[C]) \leq |C| - k$, by Corollary 3.4. We now prove the more specific bounds for ID- and OLD-codes.

(1) First consider the case where C is an ID-code. Let $G[C_i]$ be a connected component of $G[C]$ such that at least one vertex of V_4 is adjacent to some vertices in C_i. In particular, $|C_i| \geq 2$ and since C is a ID-code, $G[C_i]$ is closed-twin-free. We first show the following.

Claim A. No element of V_4 is adjacent to the vertices of the leaf blocks of $G[C_i]$.

Proof of Claim A. Suppose that L is a leaf block of $G[C_i]$. Then we must have $L \cong K_2$, or else, at least two vertices in $V(L)$ are not closed-separated in G by C. So, assume that $V(L) = \{x, y\}$. Then at least one of x and y must be a non-articulation vertex of $G[C_i]$. Without loss of generality, suppose that y is a non-articulation vertex of $G[C_i]$. If there exists a vertex v of V_4 such that $N_G(v) \cap C = V(L)$, then v and y would not be closed-separated in G by C which is a contradiction. Hence, no element of V_4 is adjacent to the vertices of the leaf blocks of $G[C_i]$. ♦

This implies that the number of vertices of V_4 that can be adjacent to the vertices of $G[C_i]$ are at most $|\mathcal{K}(G[C_i])| - |\mathcal{K}_{leaf}(G[C_i])|$. Now, we must have the following.

Claim B. $|\mathcal{K}_{leaf}(G[C_i])| \geq 2$

Proof of Claim B. If, on the contrary, $|\mathcal{K}_{leaf}(G[C_i])| = 1$, then $|C_i| = 1$; or else, all pairs of vertices of $G[C_i]$ are not closed-separated in G by C. This contradicts the fact that $|C_i| \geq 2$. ♦

Therefore, by the above two claims and the fact that any vertex of V_4 having its neighbors in a component $G[C_i]$ of $G[C]$ is adjacent to the vertices of exactly one block of $G[C_i]$, the number of vertices of V_4 adjacent to the vertices of $G[C_i]$ is at most $|\mathcal{K}(G[C_i])| - |\mathcal{K}_{leaf}(G[C_i])| \leq |\mathcal{K}(G[C_i])| - 2 \leq |C_i| - 3$ (the last inequality is by the fact that $G[C_i]$ is connected, that is, $k = 1$ and $n_0(C_i) = 0$ in Corollary 3.4). Hence,

$$|V_4| \leq \sum_{1 \leq i \leq k} \Big(|\mathcal{K}(G[C_i])| - |\mathcal{K}_{leaf}(G[C_i])| \Big) \leq \sum_{1 \leq i \leq k} \Big(|C_i| - 3 \Big) = |C| - 3k.$$

(2) In the case that C is an OLD-code of G, we have $n_0(G[C]) = 0$. So, assume that $G[C_i]$ is a connected component of $G[C]$ with at least one edge. Then, by Lemma 3.2, we have $|\mathcal{K}(G[C_i])| \leq |C_i| - 1 - |\mathcal{K}_{leaf}(G[C_i])| + n_1(G[C_i]) \leq |C_i| - 2 + n_1(G[C_i])$. This implies that

$$|V_4| \leq |\mathcal{K}(G[C])| = \sum_{1 \leq i \leq k} |\mathcal{K}(G[C_i])| \leq \sum_{1 \leq i \leq k} \Big(|C_i| - 2 + n_1(G[C_i]) \Big) = |C| - 2k + n_1(G[C]).$$

This proves the lemma. □

Theorem 3.11. Let G be a connected block graph. Then we have

- $\gamma^{ID}(G) \geq \frac{|V(G)|}{3} + 1$;
- $\gamma^{LD}(G) \geq \frac{|V(G)|+1}{3}$; and
- $\gamma^{OLD}(G) \geq \frac{|V(G)|+2}{3}$.

Proof:
Let $|V(G)| = n$. Assume C to be a code of G and $G[C_1], G[C_2], \ldots, G[C_k]$ be all the connected components of $G[C]$, where $C_1, C_2, \ldots, C_k$ are subsets of C. Recalling from Definition 3.5 the sets V_1, V_2, V_3, V_4 that partition $V(G)$, we prove the theorem using the relation $|V(G)| = |C| + |V_2| + |V_3| + |V_4|$ and the upper bounds for $|V_2|$, $|V_3|$ and $|V_4|$ in Lemmas 3.6, 3.9 and 3.10, respectively.

If C is an ID-code, then we have

$$\begin{aligned} n &= |C| + |V_2| + |V_3| + |V_4| \\ &\leq |C| + |C| - n_0(G[C]) + k - 1 + |C| - 3k \\ &\leq 3|C| - 2k - 1 \leq 3|C| - 3, \quad \text{using } k \geq 1. \end{aligned}$$

Hence, the result holds.

If C is an LD-code, then

$$\begin{aligned} n &= |C| + |V_2| + |V_3| + |V_4| \\ &\leq |C| + |C| + k - 1 + |C| - k \\ &= 3|C| - 1 \end{aligned}$$

and, hence, the result holds.

Finally, if C is an OLD-code, then we have

$$\begin{aligned} n &= |C| + |V_2| + |V_3| + |V_4| \\ &\leq |C| + |C| - n_1(G[C]) + k - 1 + |C| - 2k + n_1(G[C]) \\ &= 3|C| - k - 1 \leq 3|C| - 2, \quad \text{using } k \geq 1. \end{aligned}$$

Hence, the result holds. □

We now look at examples of which connected block graphs are extremal with respect to the bounds of the previous theorem. To that end, we show that there are infinitely many connected block graphs whose ID- and LD-numbers reach the first and the second lower bounds, respectively, of Theorem 3.11. In contrast however, there is exactly one connected block graph whose OLD-number reaches the last lower bound in Theorem 3.11. In order to calculate the code numbers in the extremal cases attaining the bounds of Theorem 3.11, the inequalities in the previous calculations in some of the earlier results leading to the bounds will have to be changed to equalities.

Proposition 3.12. For $k \geq 3$, consider a path on vertices $u_1, u_2, \ldots, u_\ell$. For all $1 \leq i \leq \ell$, attach a vertex v_i by the edge v_iu_i and, for each pair u_i, u_{i+1} for $\ell \geq 4$ and $2 \leq i \leq \ell - 2$, attach a vertex w_i by edges w_iu_i and w_iu_{i+1}. See Figure 9a with $\ell = 6$. Then, such graphs are the only ones whose ID-numbers attain the lower bound in Theorem 3.11.

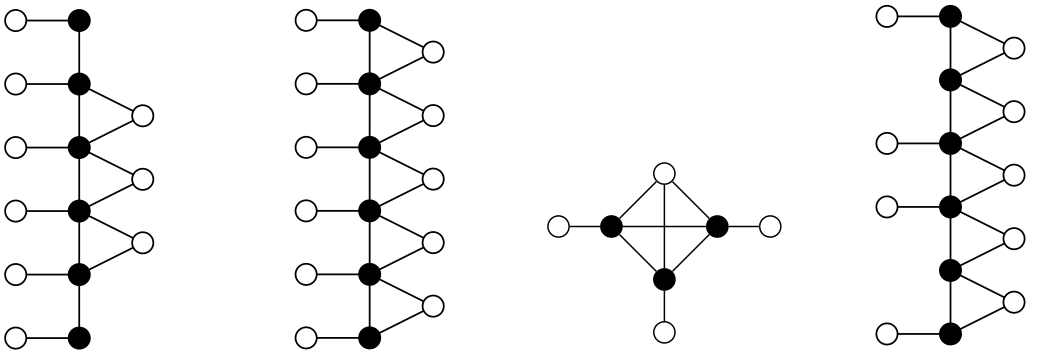


Figure 9: Extremal cases where the lower bounds in Theorem 3.11 are attained. The black vertices form a minimum (a) ID-code, (b) LD-code, (c) and (d) OLD-code.

Proof:
For extremal graphs whose ID-numbers attain the lower bound in Theorem 3.11, we have equalities in the equations in the proof of Theorem 3.11 when C is an ID-code. We therefore have in this case $k = 1$ (that is, $G[C]$ is connected) which implies that $|V_2| = |C|$, $|V_3| = 0$ and $|V_4| = |C| - 3$. Tracing back the equality for $|V_4|$, it stems from equality in the statement of Lemma 3.10(1), that is,

$$|V_4| = |\mathcal{K}(G[C])| - |\mathcal{K}_{leaf}(G[C])| = |C| - 3k.$$

This further implies $|\mathcal{K}_{leaf}(G[C])| = 2$ in Claim B of the proof of Lemma 3.10. Recall that $k = 1$. Hence, putting these values in the preceding equation, we have $|V_4| = |\mathcal{K}(G[C])| - 2 = |C| - 3$ or $|\mathcal{K}(G[C])| = |C| - 1$. This implies that $G[C]$ must be a tree and more particularly, a path, since $|\mathcal{K}_{leaf}(G[C])| = 2$. Moreover, with the values of $|V_2| = |C|$ and $|V_4| = |C| - 3$ coupled with the fact that no vertex of V_4 can have neighbors in leaf blocks of C (Claim A in proof of Lemma 3.10), any extremal graph with respect to the lower bound for ID-numbers in Theorem 3.11 must be as described in the statement of the proposition. □

Proposition 3.13. There exist arbitrarily large connected block graphs whose LD-number attains the lower bound in Theorem 3.11.

Proof:
For any $\ell \geq 1$, consider a path on vertices $u_1, u_2, \ldots, u_\ell$. For all $1 \leq i \leq \ell$, attach a vertex v_i by the edge $v_i u_i$ and, for each pair u_i, u_{i+1} for $\ell \geq 2$ and $1 \leq i \leq \ell - 1$, attach a vertex w_i by edges $w_i u_i$ and $w_i u_{i+1}$. We call the graph G. See Figure 9b for an example with $k = 6$. Then it can be verified that $C = \{u_1, u_2, \ldots, u_\ell\}$ is a minimum LD-code of G. Since $|V(G)| = 3\ell - 1$, the LD-number of G attains the lower bound. □

Proposition 3.14. (1) Consider the block graph Z consisting of a clique on four vertices, of which three are support vertices to one leaf each. See Figure 9c. Then Z is the only block graph whose OLD-number attains the bound in Theorem 3.11.

(2) For any $G \ncong Z$, the bound for the OLD-number in Theorem 3.11 becomes

$$\gamma^{OLD}(G) \geq \frac{|V(G)|}{3} + 1.$$

In this case, there are arbitrarily large connected block graphs whose OLD-number attains this lower bound.

Proof:
For any block graph G whose OLD-number attains the bound in Theorem 3.11, we must have equalities in the equations in the proof of Theorem 3.11 when C is an OLD-code. This implies that we must have $k = 1$. Moreover, we must have $|V_2| = |C| - n_1(G[C])$, $|V_3| = 0$ and

$$|V_4| = |C| - 2 + n_1(G[C]) = |\mathcal{K}(G[C])| \qquad \text{(using Lemma 3.10 and } k = 1)$$

(1) The last equation stems from an equality in the statement of Lemma 3.2, which implies that $|\mathcal{K}_{leaf}(G[C])| = 1$. This implies that C is a clique. Since C is an OLD-code, we must have $|C| \geq 2$. If $|C| = 2$, then we have $|V_4| = n_1(G[C]) = 1$. This is a contradiction, as $|C| = 2$ implies that $G[C] \cong P_2$ and hence, $n_1(G[C]) = 2$. Thus, we have $|C| \geq 3$. This implies that $n_1(G[C]) = 0$ and so, we get $|V_4| = |C| - 2 = 1 \implies |C| = 3$. Moreover, $|V_2| = 3$ determines the graph to be Z.

(2) We now assume that $G \ncong Z$. If $k \geq 2$, then putting so in the proof of Theorem 3.11, we get $|V(G)| \leq 3|C|-3$ and hence, the result follows. Therefore, for the rest of this proof, let us assume that $k = 1$. In other words, $G[C]$ is a connected subgraph of G. Now, we must have $|\mathcal{K}_{leaf}(G[C])| \geq 2$, or else, if $|\mathcal{K}_{leaf}(G[C])| = 1$, the same analysis as in (1) would follow and we would have $G \cong Z$, a contradiction. Thus, putting $|\mathcal{K}_{leaf}(G[C])| \geq 2$ in the proof of Lemma 3.10(2), we get $|V_4| \leq |\mathcal{K}(G[C])| \leq |C| - 3k + n_1(G[C])$. Feeding this further in the proof of Theorem 3.11 when C is an OLD-code, we get $|V(G)| \leq 3|C| - 3$ and hence, the lower bound holds.

To show that there exist arbitrarily large and infinitely many block graphs whose OLD-numbers attain this bound, for $\ell \geq 6$, we consider a path on vertices $u_1, u_2, \ldots, u_\ell$. For $i = 1, \ell$ and $3 \leq i \leq \ell - 2$, attach a vertex v_i by the edge v_iu_i. Moreover, for each pair u_i, u_{i+1} for $1 \leq i \leq \ell - 1$, attach a vertex w_i by edges w_iu_i and w_iu_{i+1}. We call the graph G. See Figure 9d for an example with $\ell = 6$. Then it can be verified that $C = \{u_1, u_2, \ldots, u_\ell\}$ is a minimum OLD-code of G. Since $|V(G)| = 3\ell - 3$, the OLD-number of G attains the lower bound. □

If we now consider the parameter $|\mathcal{K}(G)|$, we can use the relation $|V(G)| \geq |\mathcal{K}(G)| + 1$ to obtain a similar lower bound. But this lower bound can be improved as the next result shows.

Theorem 3.15. Let G be a connected block graph and $\mathcal{K}(G)$ be the set of all blocks of G. Then we have

- $\gamma^{ID}(G) \geq \frac{3(|\mathcal{K}(G)|+2)}{7}$;
- $\gamma^{LD}(G) \geq \frac{|\mathcal{K}(G)|+2}{3}$; and
- $\gamma^{OLD}(G) \geq \frac{|\mathcal{K}(G)|}{2} + 1$.

Proof:
Assume C to be a code of G and $G[C_1], G[C_2], \ldots, G[C_k]$ be all the connected components of $G[C]$, where $C_1, C_2, \ldots, C_k$ are subsets of C. First, we define $\mathcal{I}_G(C) = \{K \in \mathcal{K}(G) : V(L) \subset V(K)$ for some $L \in \mathcal{K}(G[C])\}$. Moreover, for each $1 \leq i \leq k$, let $\mathcal{I}_G(C_i) = \{K \in \mathcal{K}(G) : V(L) \subset V(K)$ for some $L \in \mathcal{K}(G[C_i])\}$. Next, we define the following types of blocks of G.

1. Let $\mathcal{K}_C(G) = \{K \in \mathcal{I}_G(C) : V(K) \subset C\}$, i.e. all blocks of G which are also blocks of the subgraph $G[C]$ (also a block graph) of G.
2. Let $\mathcal{K}_{\overline{C}}(G) = \mathcal{K}(G) \setminus \mathcal{I}_G(C)$. In other words, the set $\mathcal{K}_{\overline{C}}(G)$ includes all blocks of G which do not contain any vertices of the code C.
3. For $i = 2, 3, 4$, let $\mathcal{K}_i(G) = \{K \in \mathcal{I}_G(C) : V(K) \cap V_i \neq \emptyset\}$ (recall the sets V_1, V_2, V_3, V_4 from Definition 3.5).

We note here that, $\mathcal{K}(G) = \mathcal{K}_C(G) \cup \mathcal{K}_{\overline{C}}(G) \cup \mathcal{K}_2(G) \cup \mathcal{K}_3(G) \cup \mathcal{K}_4(G)$. We now have the following bounds.

Claim A. $|\mathcal{K}_2(G)| \leq |V_2|$.

Proof of Claim A. Since each vertex in the set V_2 belongs to a unique block $K \in \mathcal{K}_2(G)$, this claim is true. ♦

We now invoke the auxiliary graph $F_C(G)$ of G from Definition 3.7 and assume that there are l connected components of $F_C(G)$. Then we have the following claim.

Claim B. $|\mathcal{K}_3(G)| \leq 2(k-l)$.

Proof of Claim B. Since each vertex of $F_C(G)$ in the part A is of degree at least 2, we have $|E(F_C(G))| \geq 2|A| = 2|V_3|$. Combining this with the fact that $|E(F_C(G))| = |V_3| + k - l$ (since $F_C(G)$ is a forest by Lemma 3.8), we have $|V_3| \leq k - l$. Hence, we have $|\mathcal{K}_3(G)| \leq |E(F_C(G))| \leq 2(k-l)$. ♦

Claim C. $|\mathcal{K}_{\overline{C}}(G)| \leq l - 1$.

Proof of Claim C. Let $F_1, F_2, \ldots, F_l$ be the l connected components of the auxiliary (bipartite and forest) graph $F_C(G)$ of G. To count $|\mathcal{K}_{\overline{C}}(G)|$, we first observe that any $K \in \mathcal{K}_{\overline{C}}(G)$, does not contain any non-articulation vertex of K. This is because, as $V(K) \cap C = \emptyset$, any non-articulation vertex of K will remain non-dominated by the code C, a contradiction. This therefore implies that K cannot be a leaf block of G (and, in particular, the root-block of G as well). Hence, every block $K \in \mathcal{K}_{\overline{C}}(G)$ has a positive articulation vertex v_K, say. Then we have a block $K' \in \mathcal{K}(G)$ with its layer $f(K') = f(K) + 1$ such that $V(K) \cap V(K') = \{v_K\}$. Moreover, since $V(K) \cap C = \emptyset$, for the code C to dominate v_K, we may assume, without loss of generality, the block K' to be such that there exists a vertex $v_{K'}$ of G in $V(K') \cap C$. For every $K \in \mathcal{K}_{\overline{C}}(G)$, therefore, we fix such a triple $(v_K, K', v_{K'})$ for the rest of this proof. Assume $G[C_i]$ to be the component of $G[C]$ such that $v_{K'} \in C_i$. Moreover, let u_i (the vertex of the part B of $F_C(G)$ corresponding to $G[C_i]$) belong to the component F_j, for some $1 \leq j \leq l$. Then, we associate F_j with the block $K \in \mathcal{K}_{\overline{C}}(G)$. More precisely, we define the following mapping.

$$g : \mathcal{K}_{\overline{C}}(G) \to \{F_1, F_2, \ldots, F_l\}$$
$$K \mapsto F_j, \text{ where } (v_K, K', v_{K'}) \text{ is fixed, } v_{K'} \in C_i \text{ and } u_i \in F_j$$

Since the vertex $v_{K'} \in C$ can belong to exactly one component $G[C_i]$ of $G[C]$ and, similarly, the vertex u_i of $F_C(G)$ can belong to exactly one of its components F_j, the mapping g is therefore well-defined. We now claim that g is one-to-one. Indeed, consider any $K \in \mathcal{K}_{\overline{C}}(G)$ with $(v_K, K', v_{K'})$ associated to it, and such that $g(K) = F_j$. Assume by contradiction that there is $L \in \mathcal{K}_{\overline{C}}(G)$, $K \neq L$, with $g(L) = F_j$, and $(v_L, L', v_{L'})$ is associated with L. Since $V(K) \cap C = \emptyset$, for every vertex $u_r \in V(F_j)$ and any block $J \in \mathcal{I}_G(C_r)$, we have $f(J) \geq f(K) + 1$. Thus, $f(K') = f(K) + 1$ is minimum among all such blocks. By the same reasoning applied to L, we also have $f(L') = f(L) + 1$ minimum among all blocks $J \in \mathcal{I}_G(C_r)$, where $u_r \in V(F_j)$. This implies that $f(L') = f(K')$, since both values are minimim among all $f(J)$, where $J \in \mathcal{I}_G(C_r)$ and $u_r \in V(F_j)$. Therefore,

we also have $f(K) = f(L)$. Now, since $K \neq L$, we must have $V(K) \cap V(L) = \emptyset$, or else, we would have $|f(K) - f(L)| = 1$, a contradiction. Now, let G_K represent the *descendant block graph* of K, that is, the connected subgraph (also a block graph) of G rooted at K. Similarly define G_L to be the descendant block graph of L. Then, by the structure of the block graph G, its subgraphs G_K and G_L are vertex-disjoint. Since $V(K) \cap C = V(L) \cap C = \emptyset$, all components $G[C_r]$ of $G[C]$ for all $u_r \in V(F_j)$ must belong to G_K and G_L simultaneously, which is a contradiction since $V(G_K) \cap V(G_L) = \emptyset$. This shows that g is one-to-one, as claimed.

As noticed before, for each $K \in \mathcal{K}_{\overline{C}}(G)$ such that $g(K) = F_j$, we have

$$f(K) + 1 = \min\{f(J) : J \in \mathcal{I}_G(C_r), u_r \in F_j\}.$$

Now, let K_0 be the root block of G. As argued before, $K_0 \notin \mathcal{K}_{\overline{C}}(G)$. In other words, $V(K_0) \cap C_{i_0} \neq \emptyset$ for some component $G[C_{i_0}]$ of $G[C]$. Let $u_{i_0} \in F_{j_0}$. Therefore, we have

$$0 \leq \min\{f(J) : J \in \mathcal{I}_G(C_r), u_r \in F_{j_0}\} \leq f(K_0) = 0 \quad \text{(since } K_0 \in \mathcal{I}_G(C_{i_0})\text{)}.$$

This implies that $g(K) = F_{i_0}$ for any K would imply $f(K) = -1$, which is not possible. Hence, the image of the function g is a subset of $\{F_1, F_2, \ldots, F_l\} \setminus \{F_{i_0}\}$. This, along with the fact that g is one-to-one, therefore implies that $|\mathcal{K}_{\overline{C}}(G)| \leq l - 1$. ♦

Claim D. $|\mathcal{K}_C(G) \cup \mathcal{K}_4(G)| \leq |\mathcal{K}(G[C])| - n_0(G[C])$.

Proof of Claim D. Assume that $K \in \mathcal{K}(G)$ is a block of $\mathcal{K}_C(G) \cup \mathcal{K}_4(G)$. Then, $V(K)$ contains at least two vertices, say, $u, v \in C$. Therefore, $uv \in E(G)$. So, assume $L \in \mathcal{K}(G[C])$ to be the block such that $u, v \in V(L)$. Then, $V(L) \subset V(K)$. Thus, every block $K \in \mathcal{K}_C(G) \cup \mathcal{K}_4(G)$ can be associated with a block $L \in \mathcal{K}(G[C])$ such that $|V(L)| \geq 2$. Moreover, by the structure of a block graph, this association is one-to-one. This implies that $|\mathcal{K}_C(G) \cup \mathcal{K}_4(G)| \leq |\mathcal{K}(G[C])| - n_0(G[C])$. ♦

To compute $|\mathcal{K}(G)|$ now, we have from the above Claims A, B, C and D that

$$\begin{aligned} |\mathcal{K}(G)| &\leq |\mathcal{K}_C(G) \cup \mathcal{K}_4(G)| + |\mathcal{K}_{\overline{C}}(G)| + |\mathcal{K}_2(G)| + |\mathcal{K}_3(G)| \\ &\leq |\mathcal{K}(G[C])| - n_0(G[C]) + l - 1 + |V_2| + 2(k - l). \end{aligned} \tag{1}$$

Therefore, using Equation (1), we have the following.

For ID-codes:

$$\begin{aligned} |\mathcal{K}(G)| &\leq |\mathcal{K}(G[C])| - n_0(G[C]) + l - 1 + |V_2| + 2(k - l) \\ &\leq |C| - k + l - 1 + |C| - n_0(G[C]) + 2(k - l) \quad \text{[using Corollary 3.4 and Lemma 3.6(1)]} \\ &= 2|C| + k - l - n_0(G[C]) - 1 \\ &\leq 2|C| + k - n_0(G[C]) - 2. \end{aligned}$$

Now, $k - n_0(G[C])$ is the total number of components of $G[C]$ of order at least 2. Any such component must contain at least 3 vertices of the code C (or else, if a component contained exactly

two vertices, they would not be closed-separated in G by C). Therefore, $3(k - n_0(G[C])) \leq |C| - n_0(G[C])$. Therefore, $3|\mathcal{K}(G)| \leq 7|C| - n_0(G[C]) - 6 \leq 7|C| - 6$ and, hence, the result holds.

For LD-codes:

$$\begin{aligned}|\mathcal{K}(G)| &\leq |\mathcal{K}(G[C])| - n_0(G[C]) + l - 1 + |V_2| + 2(k-l)\\ &\leq |C| - k + l - 1 + |C| + 2(k-l) \qquad \text{[using Corollary 3.4 and Lemma 3.6(2)]}\\ &= 2|C| + k - l - 1\\ &\leq 3|C| - 2.\end{aligned}$$

Finally, for OLD-codes, we have $n_0(G[C]) = 0$. Hence,

$$\begin{aligned}|\mathcal{K}(G)| &\leq |\mathcal{K}(G[C])| + l - 1 + |V_2| + 2(k-l)\\ &\leq |C| - 2k + l - 1 + |C| + 2(k-l) \qquad \text{[using Lemmas 3.6(3) and 3.10(2)]}\\ &= 2|C| - l - 1\\ &\leq 2|C| - 2, \quad \text{using } l \geq 1.\end{aligned}$$

This proves the theorem. □

Recall that a block in a graph G (not necessarily a block graph) is a maximal complete subgraph of G. Then, the lower bounds in Theorem 3.15 in terms of the number of blocks in a graph fail for graphs in general. For example, consider the split graph G with its vertex set $V(G) = \{v_1, ..., v_k\} \cup \{u_X : X \subseteq \{1, ..., k\} \text{ and } X \neq \emptyset\}$. The vertices $v_1, \ldots, v_k$ induce a clique, whereas the vertices u_X induce an independent set. Moreover, there is an edge between the vertices v_i and u_X if and only if $i \in X$. This graph has an identifying code of size $2k$ (the clique with the vertices corresponding to the singletons), but the number of blocks in G is 2^k. Similar examples exist to show that the LD- and OLD-numbers of some split graphs that are not block-graphs violate the lower bounds in Theorem 3.15.

Note that, for any tree G, we have $|\mathcal{K}(G)| = |E(G)| = |V(G)| - 1$. Thus, for trees (which are particular block graphs with each block being of order 2), Theorem 3.15 provides the same lower bound $\left(\frac{3(|V(G)|+1)}{7}\right)$ for ID-numbers as was given in [21]. The lower bounds for LD-numbers of trees given in Theorems 3.11 and 3.15 are the same; and which, in turn, are the same as that given in [7]. The lower bound for OLD-numbers of trees given by Theorem 3.15 is the same when $|V(G)|$ is even and one short when $|V(G)|$ is odd as the lower bound given in [9] $\left(\text{which is } \left\lceil \frac{|V(G)|}{2} \right\rceil + 1\right)$. However, for $|V(G)|$ odd, this lower bound for OLD-numbers given by Theorem 3.15 is tight for block graphs in general (see Figure 9c and E.g. (3) below). The following are examples of extremal block graphs described in Propositions 3.12, 3.13 and 3.14 whose code numbers attain the lower bounds in Theorem 3.15.

E.g. (1) For $k = 3$ in Proposition 3.12, the graph G is a 1-corona of a P_3 (with $|\mathcal{K}(G)| = 5$) and is an extremal example whose ID-number ($= 3$) attains the bound in Theorem 3.15.

E.g. (2) In the proof of Proposition 3.13, the graph G' obtained by deleting all the edges u_iu_{i+1} of G is an example whose LD-number ($= k$) attains the bound in Theorem 3.15 (note that $|\mathcal{K}(G')| = 3k - 2$).

E.g. (3) Finally, the graph Z described in Proposition 3.14(1) (with $|\mathcal{K}(G)| = 4$) also serves as an example whose OLD-number ($= 3$) and attains the bound given in Theorem 3.15.

Apart from the above examples, there are infinite families of trees reaching the three bounds in Theorem 3.15 (see [21] for ID-codes, [7] for LD-codes and [9] for OLD-codes).

4. Conclusion

Block graphs form a subclass of chordal graphs for which all three considered identification problems can be solved in linear time [34]. In this paper, we complemented this result by presenting lower and upper bounds for all three codes. We gave bounds using both the number of vertices as it has been done for several other classes of graphs and also using the parameter $|\mathcal{K}(G)|$ of the number of blocks of G that is more fitting for block graphs. In particular, we verified a conjecture from [35] (Conjecture 1.1) concerning an upper bound for $\gamma^{ID}(G)$, and also proved the conjecture on the LD-number [29] (Conjecture 1.2) for the special case of block graphs. Moreover, we addressed the questions to find block graphs where the provided lower and upper bounds are attained.

The structural properties of block graphs have enabled us to prove interesting bounds for the three considered identification problems. It would be interesting to see whether other structured classes can be studied in a similar way. It would also be interesting to prove Conjecture 1.2 for a larger class of graphs, for example for all chordal graphs.

Acknowledgments

Works by Dipayan Chakraborty and Florent Foucaud were partially supported by the French government initiative IDEX-ISITE CAP 20-25 (ANR-16-IDEX-0001), the International Research Center "Innovative Transport and Production Systems" I-SITE CAP 20-25 and the ANR GRALMECO project (ANR -21- CE48-0004).

References

[1] Jean D, Lobstein A. Watching systems, identifying, locating-dominating and discriminating codes in graphs: a bibliography. *Published electronically at https://dragazo.github.io/bibdom/main.pdf*, 2012.

[2] Rényi A. On random generating elements of a finite Boolean algebra. *Acta Scientiarum Mathematicarum Szeged*, 1961. **22**:75–81.

[3] Rao N. Computational complexity issues in operative diagnosis of graph-based systems. *IEEE Transactions on Computers*, 1993. **42**(4):447–457. doi:10.1109/12.214691.

[4] Moret BME, Shapiro HD. On Minimizing a Set of Tests. *SIAM Journal on Scientific and Statistical Computing*, 1985. **6**(4):983–1003. doi:10.1137/090606.

[5] Chlebus BS, Nguyen SH. On Finding Optimal Discretizations for Two Attributes. In: Proceedings of the First International Conference on Rough Sets and Current Trends in Computing, volume 1424. Springer Berlin Heidelberg, Berlin, Heidelberg, 1998 pp. 537–544. doi:10.1007/3-540-69115-4_74.

[6] Karpovsky MG, Chakrabarty K, Levitin LB. On a new class of codes for identifying vertices in graphs. *IEEE transactions on information theory*, 1998. **44**(2):599–611. doi:10.1109/18.661507.

[7] Slater PJ. Domination and location in acyclic graphs. *Networks*, 1987. **17**(1):55–64. doi:10.1002/net.3230170105.

[8] Slater PJ. Dominating and reference sets in a graph. *Journal of Mathematical and Physical Sciences*, 1988. **22**(4):445–455.

[9] Seo SJ, Slater PJ. Open neighborhood locating dominating sets. *Australasian Journal of Combinatorics*, 2010. **46**:109–120.

[10] Henning MA, Yeo A. Total domination in graphs. Springer, 2013. doi:10.1007/978-1-4614-6525-6.

[11] Charon I, Hudry O, Lobstein A. Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard. *Theoretical Computer Science*, 2003. **290**(3):2109–2120. doi:10.1016/S0304-3975(02)00536-4.

[12] Foucaud F, Mertzios GB, Naserasr R, Parreau A, Valicov P. Identification, location-domination and metric dimension on interval and permutation graphs. II. Algorithms and complexity. *Algorithmica*, 2017. **78**(3):914–944. doi:10.1007/s00453-016-0184-1.

[13] Foucaud F. Decision and approximation complexity for identifying codes and locating-dominating sets in restricted graph classes. *J. Discrete Algorithms*, 2015. **31**:48–68. doi:10.1016/j.jda.2014.08.004.

[14] Pandey A. Open Neighborhood Locating-Dominating Set in Graphs: Complexity and Algorithms. In: Proceedings of the 14th International Conference on Information Technology (ICIT 2015). 2015 pp. 1–6. doi:10.1109/ICIT.2015.14.

[15] Bertrand N, Charon I, Hudry O, Lobstein A. Identifying and locating-dominating codes on chains and cycles. *European Journal of Combinatorics*, 2004. **25**(7):969–987. doi:10.1016/j.ejc.2003.12.013.

[16] Gravier S, Moncel J. On graphs having a $V\setminus\{x\}$ set as an identifying code. *Discrete Mathematics*, 2007. **307**(3-5):432–434. doi:10.1016/j.disc.2005.09.035.

[17] Argiroffo GR, Bianchi SM, Lucarini Y, Wagler AK. Polyhedra associated with identifying codes. *Discrete Applied Mathematics*, 2018. **245**:16–27. doi:10.1016/j.dam.2017.06.005.

[18] Argiroffo GR, Bianchi SM, Lucarini Y, Wagler AK. Polyhedra associated with locating-dominating, open locating-dominating and locating total-dominating sets in graphs. *Discrete Applied Mathematics*, 2022. **322**:465–480. doi:10.1016/j.dam.2022.06.025.

[19] Argiroffo GR, Bianchi SM, Wagler AK. Study of identifying code polyhedra for some families of split graphs. In: International Symposium on Combinatorial Optimization. Springer, 2014 pp. 13–25. doi:10.1007/978-3-319-09174-7_2.

[20] Foucaud F, Mertzios GB, Naserasr R, Parreau A, Valicov P. Identification, location-domination and metric dimension on interval and permutation graphs. I. Bounds. *Theoretical Computer Science*, 2017. **668**:43–58. doi:10.1016/j.tcs.2017.01.006.

[21] Bertrand N, Charon I, Hudry O, Lobstein A. 1-identifying codes on trees. *Australas. J Comb.*, 2005. **31**:21–36.

[22] Foucaud F, Gravier S, Naserasr R, Parreau A, Valicov P. Identifying codes in line graphs. *Journal of Graph Theory*, 2013. **73**(4):425–448. doi:10.1002/jgt.21686.

[23] Rall DF, Slater PJ. On location-domination numbers for certain classes of graphs. *Congressus Numerantium*, 1984. **45**:97–106.

[24] Bousquet N, Lagoutte A, Li Z, Parreau A, Thomassé S. Identifying codes in hereditary classes of graphs and VC-dimension. *SIAM Journal on Discrete Mathematics*, 2015. **29**(4):2047–2064. doi:10.1137/14097879X.

[25] Balbuena C, Foucaud F, Hansberg A. Locating-dominating sets and identifying codes in graphs of girth at least 5. *The Electronic Journal of Combinatorics*, 2015. **22**(2):P2.15. doi:10.37236/4562.

[26] Foucaud F, Lehtilä T. Revisiting and improving upper bounds for identifying codes. *SIAM Journal on Discrete Mathematics*, 2022. doi:10.1137/22M148999X

[27] Foucaud F, Perarnau G. Bounds on identifying codes in terms of degree parameters. *The Electronic Journal of Combinatorics*, 2012. **19**(1):P32. doi:10.37236/2036.

[28] Foucaud F, Henning MA. Location-domination and matching in cubic graphs. *Discrete Mathematics*, 2016. **339**(4):1221–1231. doi:10.1016/j.disc.2015.11.016.

[29] Garijo D, González A, Márquez A. The difference between the metric dimension and the determining number of a graph. *Applied Mathematics and Computation*, 2014. **249**:487–501. doi:10.1016/j.amc.2014.10.034.

[30] Henning MA, Yeo A. Distinguishing-Transversal in Hypergraphs and Identifying Open Codes in Cubic Graphs. *Graphs and Combinatorics*, 2014. **30**:909–932. doi:10.1007/s00373-013-1311-2.

[31] Harary F. A Characterization of Block-Graphs. *Canadian Mathematical Bulletin*, 1963. **6**(1):1–6. doi:10.4153/CMB-1963-001-x.

[32] Howorka E. On metric properties of certain clique graphs. *Journal of Combinatorial Theory, Series B*, 1979. **27**(1):67–74. doi:10.1016/0095-8956(79)90069-8.

[33] Bandelt HJ, Mulder HM. Distance-hereditary graphs. *Journal of Combinatorial Theory, Series B*, 1986. **41**(2):182–208. doi:10.1016/0095-8956(86)90043-2.

[34] Argiroffo GR, Bianchi SM, Lucarini Y, Wagler AK. Linear-time algorithms for three domination-based separation problems in block graphs. *Discrete Applied Mathematics*, 2020. **281**:6–41. doi:10.1016/j.dam.2019.08.001.

[35] Argiroffo GR, Bianchi SM, Lucarini Y, Wagler AK. On the identifying code number of block graphs. In: Proceedings of ICGT 2018, Lyon, France. 2018 .

[36] Chakraborty D, Foucaud F, Parreau A, Wagler AK. On Three Domination-Based Identification Problems in Block Graphs. In: Bagchi A, Muthu R (eds.), Algorithms and Discrete Applied Mathematics - 9th International Conference, CALDAM 2023, Gandhinagar, India, February 9-11, 2023, Proceedings, volume 13947 of *Lecture Notes in Computer Science*. Springer, 2023 pp. 271–283. doi:10.1007/978-3-031-25211-2_21.

[37] Foucaud F, Guerrini E, Kovše M, Naserasr R, Parreau A, Valicov P. Extremal graphs for the identifying code problem. *European Journal of Combinatorics*, 2011. **32**(4):628–638. doi:10.1016/j.ejc.2011.01.002.

[38] Foucaud F, Ghareghani N, Roshany-Tabrizi A, Sharifani P. Characterizing extremal graphs for open neighbourhood location-domination. *Discrete Applied Mathematics*, 2021. **302**:76–79. doi:10.1016/j.dam.2021.06.006.

[39] Erdős P. Some combinatorial, geometric and set theoretic problems in measure theory. In: Measure Theory Oberwolfach 1983. Springer, 1984 pp. 321–327.